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AA200 Applied Aerodynamics

Chapter 13 - Compressible thin airfoil theory

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Lockheed F104 Starfighter

13.1 Compressible potential flow

13.1.1 The full potential equation

Governing equations

$$\nabla(\rho\bar{U}) = 0$$

$$\nabla\left(\frac{\bar{U}\bullet\bar{U}}{2}\right) + \frac{\nabla P}{\rho} = 0 \tag{13.1}$$

$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

The gradient of the isentropic relation is

$$\nabla P = a^2 \nabla\rho. \tag{13.2}$$

Note that

$$\nabla\left(\frac{P}{\rho}\right) = \left(\frac{\gamma-1}{\gamma}\right)\frac{\nabla P}{\rho} \tag{13.3}$$

The momentum equation becomes.

$$\nabla\left(\left(\frac{\gamma}{\gamma-1}\right)\frac{P}{\rho} + \frac{\bar{U}\bullet\bar{U}}{2}\right) = 0 \tag{13.4}$$

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The continuity equation can be written in the form

$$\bar{U} \bullet \nabla a^2 + (\gamma - 1) a^2 \nabla \bullet \bar{U} = 0. \qquad (13.5)$$

Equate the Bernoulli integral to free stream conditions.

$$\frac{a^2}{\gamma - 1} + \frac{\bar{U} \bullet \bar{U}}{2} = \frac{a_\infty^2}{\gamma - 1} + \frac{U_\infty^2}{2} = \frac{a_\infty^2}{\gamma - 1}\left(1 + \frac{\gamma - 1}{2} M_\infty^2\right) = C_p T_t \qquad (13.6)$$

Thus

$$\left(\frac{a^2}{\gamma - 1}\right) = h_t - \frac{\bar{U} \bullet \bar{U}}{2} \qquad (13.7)$$

The continuity equation becomes

Full potential equation

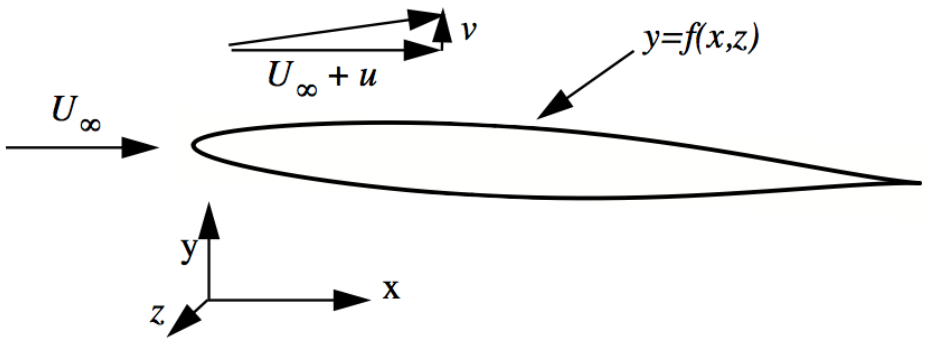
$$(\gamma - 1)\left(h_t - \frac{\bar{U} \bullet \bar{U}}{2}\right) \nabla \bullet \bar{U} - \bar{U} \bullet \nabla\left(\frac{\bar{U} \bullet \bar{U}}{2}\right) = 0 \qquad (13.8)$$

In terms of the velocity potential $\bar{U} = \nabla \Phi$

$$(\gamma - 1)\left(h_t - \frac{\nabla\Phi \bullet \nabla\Phi}{2}\right) \nabla^2 \Phi - \nabla\Phi \bullet \nabla\left(\frac{\nabla\Phi \bullet \nabla\Phi}{2}\right) = 0. \qquad (13.10)$$

13.1.2 The nonlinear small disturbance approximation

Flow past a thin 3-D airfoil



$$
\begin{aligned}
U &= U_\infty + u \\
V &= v \\
W &= w
\end{aligned}
\qquad (13.11)
$$

where

$$u/U_\infty \ll 1 \ , \qquad v/U_\infty \ll 1 \ , \qquad w/U_\infty \ll 1 \ . \qquad (13.12)$$

Similarly the state variables deviate only slightly from freestream values

$$
\begin{aligned}
P &= P_{\infty} + P' \\
T &= T_{\infty} + T' \\
\rho &= \rho_{\infty} + \rho'
\end{aligned}
\qquad (13.13)
$$

and

$$
a = a_{\infty} + a' . \qquad (13.14)
$$

Now substitute this decomposition of variables into 13.8

$$
\begin{aligned}
\frac{\bar{U} \bullet \bar{U}}{2} &= \frac{U_{\infty}^2}{2} + uU_{\infty} + \frac{u^2}{2} + \frac{v^2}{2} + \frac{w^2}{2} \\
\nabla \bullet \bar{U} &= u_x + v_y + w_z
\end{aligned}
\qquad (13.15)
$$

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Various terms are

$$\nabla\left(\frac{\bar{U}\bullet\bar{U}}{2}\right) = (u_xU_\infty + uu_x + vv_x + ww_x, \\ u_yU_\infty + uu_y + vv_y + ww_y, \\ u_zU_\infty + uu_z + vv_z + ww_z) \tag{13.16}$$

$$(\gamma-1)\left(h_t - \frac{\bar{U}\bullet\bar{U}}{2}\right)\nabla\bullet\bar{U} = \\ (\gamma-1)\left(h_t - \left(\frac{U_\infty^2}{2} + uU_\infty + \frac{u^2}{2} + \frac{v^2}{2} + \frac{w^2}{2}\right)\right)u_x + \\ (\gamma-1)\left(h_t - \left(\frac{U_\infty^2}{2} + uU_\infty + \frac{u^2}{2} + \frac{v^2}{2} + \frac{w^2}{2}\right)\right)v_y + \\ (\gamma-1)\left(h_t - \left(\frac{U_\infty^2}{2} + uU_\infty + \frac{u^2}{2} + \frac{v^2}{2} + \frac{w^2}{2}\right)\right)w_z \tag{13.17}$$

$$\bar{U}\bullet\nabla\left(\frac{\bar{U}\bullet\bar{U}}{2}\right) = u_xU_\infty^2 + uu_xU_\infty + vv_xU_\infty + ww_xU_\infty + \\ uu_xU_\infty + u^2u_x + uvv_x + uww_x + \\ vu_yU_\infty + vuu_y + v^2v_y + vww_y + \\ wu_zU_\infty + wuu_z + wvv_z + w^2w_z \quad . \tag{13.18}$$

Neglect terms that are third order in the disturbance velocities and divide through by the freestream speed of sound squared.

$$(\gamma-1)\left(h_t - \frac{\overline{U}\bullet\overline{U}}{2}\right)\nabla\bullet\overline{U} - \overline{U}\bullet\nabla\left(\frac{\overline{U}\bullet\overline{U}}{2}\right) \cong$$

$$(1-M_\infty^2)u_x + v_y + w_z - \frac{(\gamma+1)M_\infty}{a_\infty}uu_x - \tag{13.22}$$

$$\frac{M_\infty}{a_\infty}((\gamma-1)(uv_y + uw_z) + vu_y + wu_z + vv_x + ww_x)$$

Small near Mach one

We can neglect all of the quadratic terms except that involving the derivative of u in the x-direction. The small disturbance equation is

$$(1-M_\infty^2)u_x + v_y + w_z - \frac{(\gamma+1)M_\infty}{a_\infty}uu_x = 0. \tag{13.23}$$

Introduce the disturbance velocity potential $\Phi = U_\infty x + \phi(x, y, z)$.

Transonic small disturbance potential equation

$$\boxed{(1-M_\infty^2)\phi_{xx} + \phi_{yy} + \phi_{zz} = (\gamma+1)\frac{M_\infty}{a_\infty}\phi_x\phi_{xx}} \tag{13.25}$$

13.1.3 Linearized potential flow

For subsonic or supersonic flow not near Mach one the nonlinear small disturbance potential equation reduces to the linear potential equation.

$$\beta^2 \phi_{xx} - (\phi_{yy} + \phi_{zz}) = 0 \qquad (13.26)$$

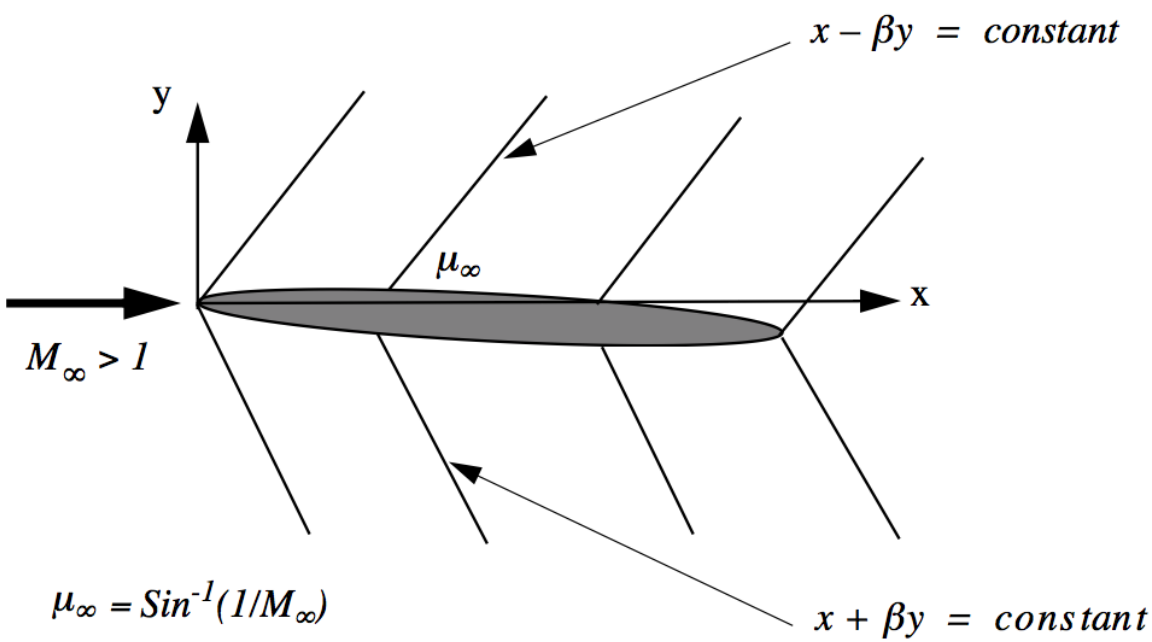
where $\beta = \sqrt{M_{\infty}^2 - 1}$.

In 2-D

$$\beta^2 \phi_{xx} - \phi_{yy} = 0. \qquad (13.27)$$

General solution for supersonic flow

$$\phi(x, y) = F(x - \beta y) + G(x + \beta y). \qquad (13.28)$$



Potential for the upper and lower surfaces

$$\begin{aligned} \phi(x, y) &= F(x - \beta y) \qquad y>0 \\ \phi(x, y) &= G(x + \beta y) \qquad y<0 \end{aligned} \tag{13.29}$$

Let $y = f(x)$ define the coordinates of the upper surface and $y = g(x)$ define the coordinates of the lower surface.

Boundary condition on the upper surface

$$\left.\frac{v}{U}\right|_{y\,=\,f} = \frac{df}{dx}. \tag{13.30}$$

For a thin airfoil this can be approximated by the linearized form

$$\left.\frac{v}{U_\infty}\right|_{y\,=\,0} = \frac{df}{dx} \tag{13.31}$$

This can be written as

$$\left.\frac{\partial \phi(x, y)}{\partial y}\right|_{y = 0} = U_\infty\left(\frac{df}{dx}\right) \qquad (13.32)$$

or

$$F'(x) = -\frac{U_\infty}{\beta}\left(\frac{df}{dx}\right). \qquad (13.33)$$

On the lower surface

$$G'(x) = \frac{U_\infty}{\beta}\left(\frac{dg}{dx}\right). \qquad (13.34)$$

The linearized boundary conditions are valid on thin 2-D wings and thin planar 3-D wings.

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13.1.4 The pressure coefficient

Work out the linearized pressure coefficient

$$C_P = \frac{P - P_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = \frac{2}{\gamma M_\infty^2}\left(\frac{P}{P_\infty} - 1\right). \tag{13.35}$$

The stagnation temperature is constant throughout the flow. The static temperatures at any two points are related by

$$\frac{T}{T_\infty} = 1 + \frac{1}{2C_pT_\infty}(U_\infty^2 - (U^2 + v^2 + w^2)). \tag{13.36}$$

Since the flow is isentropic

$$\frac{P}{P_\infty} = \left(1 + \frac{1}{2C_pT_\infty}(U_\infty^2 - (U^2 + v^2 + w^2))\right)^{\frac{\gamma}{\gamma - 1}} \tag{13.37}$$

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The pressure coefficient is

$$C_P = \frac{2}{\gamma M_\infty^2}\left\{\left(1 + \frac{1}{2C_pT_\infty}(U_\infty^2 - (U^2 + v^2 + w^2))\right)^{\frac{\gamma}{\gamma - 1}} - 1\right\}. \qquad (13.38)$$

The velocity term in this equation is small

$$U_\infty^2 - (U^2 + v^2 + w^2) = -(2uU_\infty + u^2 + v^2 + w^2). \qquad (13.39)$$

The pressure coefficient is approximately

$$C_P \cong -\left(\frac{2u}{U_\infty} + (1 - M_\infty^2)\frac{u^2}{U_\infty^2} + \frac{v^2 + w^2}{U_\infty^2}\right). \qquad (13.40)$$

Note that the binomial expansion has to be carried out to second order.

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For 2-D flows over planar bodies

$$C_P \cong -2\frac{u}{U_\infty}. \tag{13.41}$$

Recall for weak oblique shocks

$$\frac{dU}{U} = -\frac{1}{(M^2-1)^{1/2}}d\theta \qquad \frac{dP}{P} = \frac{\gamma M^2}{(M^2-1)^{1/2}}d\theta \longrightarrow \frac{dP}{P} = -\gamma M^2\frac{dU}{U}$$

For 3-D flows over slender, approximately axisymmetric bodies

$$C_P \cong -\left(\frac{2u}{U_\infty} + \frac{v^2+w^2}{U_\infty^2}\right). \tag{13.42}$$

If the airfoil is a 2-D shape defined by the function y=f(x) the boundary condition at the surface is

$$\frac{df}{dx} = \frac{v}{U_\infty + u} = \tan\theta \qquad (13.43)$$

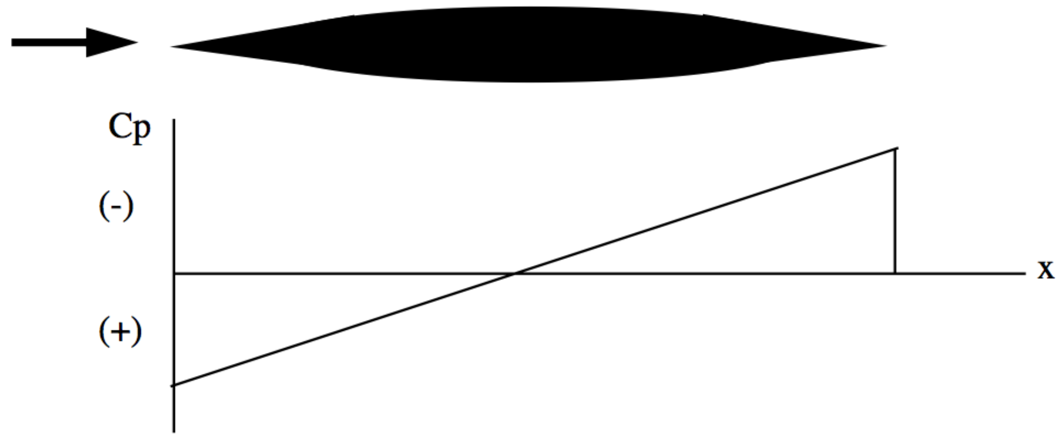
For a thin airfoil

$$\frac{df}{dx} \cong \frac{\phi_y}{U_\infty} \cong \theta . \qquad (13.44)$$

For a thin airfoil in supersonic flow

$$C_{Pwall} = \frac{2}{(M_\infty^2 - 1)^{1/2}}\left(\frac{df}{dx}\right) . \qquad (13.45)$$

13.1.5 Drag coefficient of a thin symmetric airfoil in supersonic flow



Let the y-coordinate of the upper surface of the airfoil be

$$y(x) = A\,Sin\left(\frac{\pi x}{C}\right) \tag{13.46}$$

Where C is the airfoil chord and the thickness to chord ratio is small, $2A/C \ll 1$. The drag integral is

$$D = 2\int_0^C (P - P_\infty) Sin(\alpha) dx \tag{13.47}$$

Since the airfoil is thin the drag coefficient can be written as

$$C_D = \frac{D}{\frac{1}{2}\rho_\infty U_\infty^2 C} = 2\int_0^1 \left(\frac{P - P_\infty}{\frac{1}{2}\rho_\infty U_\infty^2}\right)\left(\frac{dy}{dx}\right) d\left(\frac{x}{C}\right) \qquad (13.49)$$

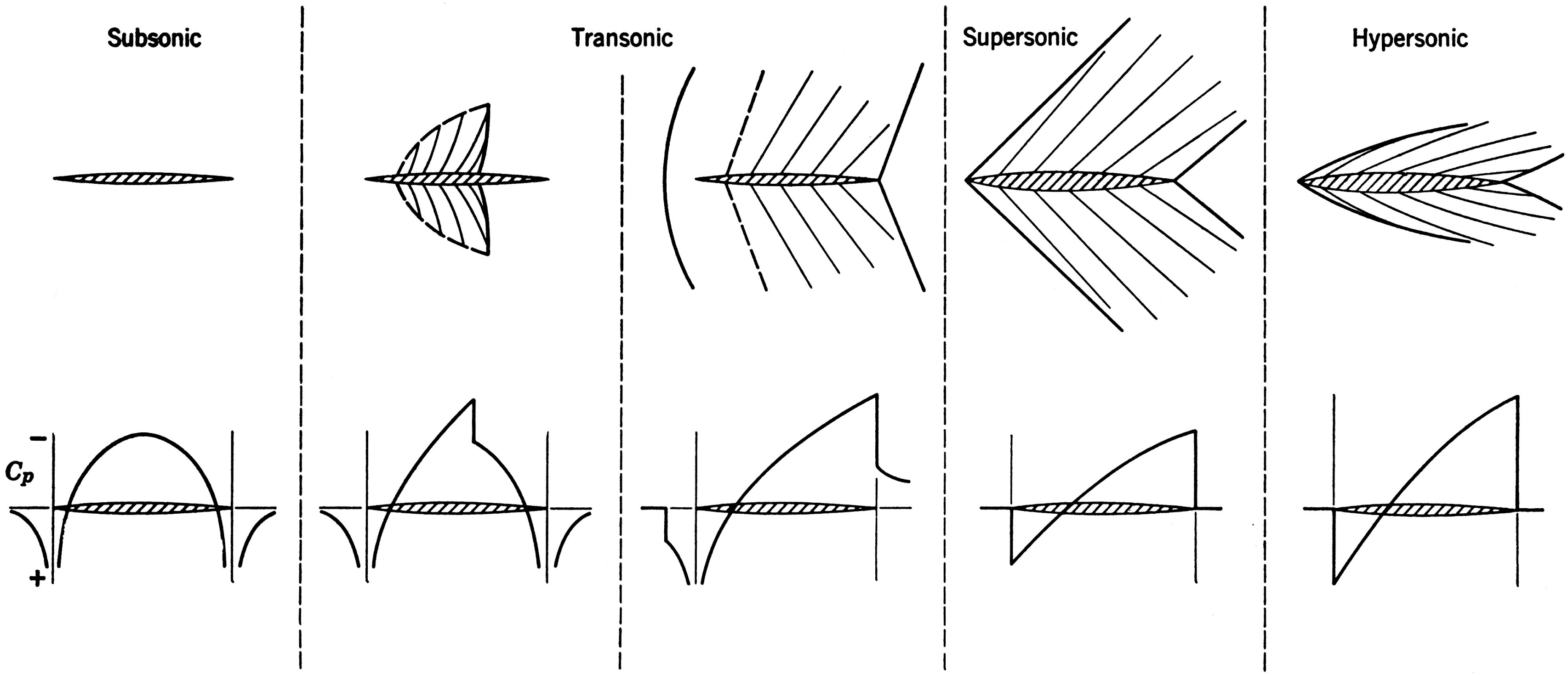
The pressure coefficient is

$$C_P = \frac{P - P_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = \frac{2}{\sqrt{M_\infty^2 - 1}}\left(\frac{dy}{dx}\right) \qquad (13.50)$$

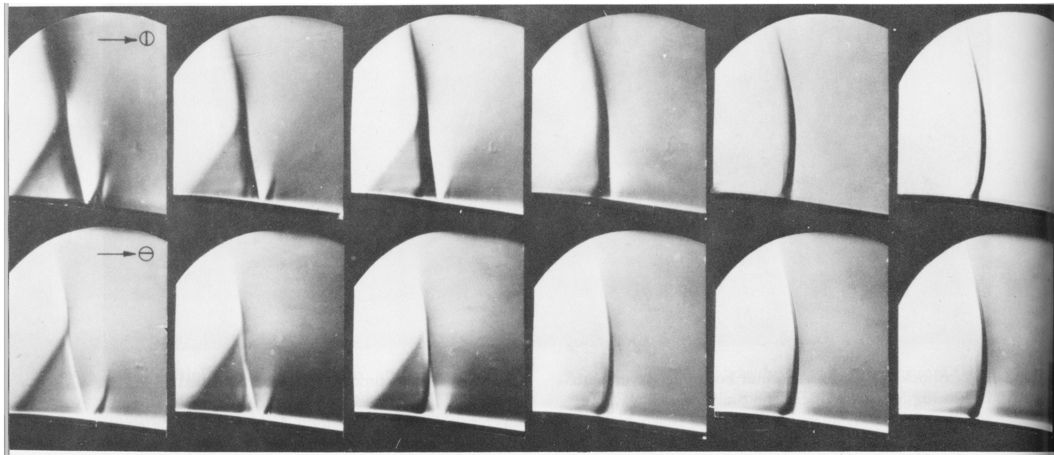
The drag coefficient becomes

$$C_D = \frac{4}{\sqrt{M_\infty^2 - 1}}\int_0^1 \left(\frac{dy}{dx}\right)^2 d\left(\frac{x}{C}\right) = \frac{4A^2\pi}{C^2\sqrt{M_\infty^2 - 1}}\int_0^\pi Cos^2\left(\frac{\pi x}{C}\right) d\left(\frac{\pi x}{C}\right) = \frac{2A^2\pi^2}{C^2\sqrt{M_\infty^2 - 1}} \qquad (13.51)$$

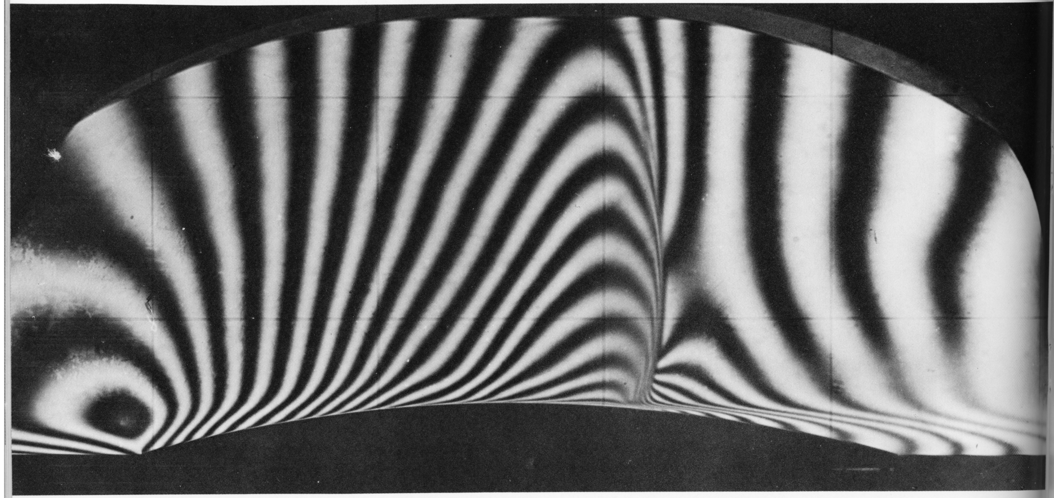
Potential flow pressure distrribution on a symmetric thin airfoil in several flow regimes - subsonic to hypersonic Mach numbers



Shock boundary layer interaction from Van Dyke



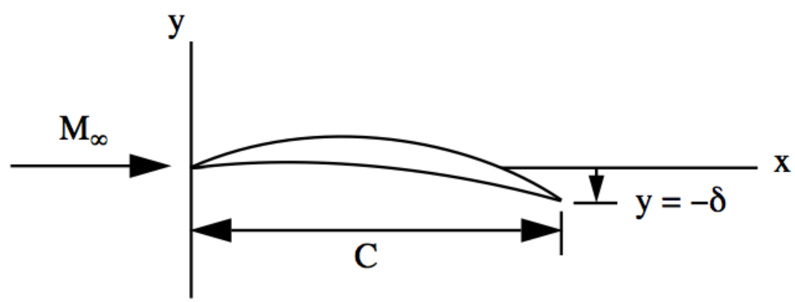
249. Shock waves on a laminar boundary layer becoming turbulent. The local Mach number on a curved plate remains almost fixed at 1.2 or 1.3 as the Reynolds number is doubled, progressing from 1,320,000 at the left to 2,680,000 at the right. As the boundary layer changes from laminar to turbulent ahead of the shock wave, the oblique leg of the lambda shock wave gradually disappears. *Ackeret, Feldmann & Rott 1946*



250. Shock wave in transonic flow over a bump. An infinite-fringe interferogram shows transonic flow over a 7-per-cent-thick circular-arc bump on a channel wall. The local region of supersonic flow terminates in a shock wave that interacts with the turbulent boundary layer on the wall, as in the preceding two photographs. *Délery, Chattot & Le Balleur 1975*

13.1.6 Thin airfoil with lift and camber at a small angle of attack

$$Tan(\alpha) = -\frac{\delta}{C}$$



Upper surface
$$f\left(\frac{x}{C}\right) = A\tau\left(\frac{x}{C}\right) + B\sigma\left(\frac{x}{C}\right) - \frac{\delta x}{C} \tag{13.52}$$

Lower surface
$$g\left(\frac{x}{C}\right) = -A\tau\left(\frac{x}{C}\right) + B\sigma\left(\frac{x}{C}\right) - \frac{\delta x}{C} \tag{13.53}$$

where
$$\tau\left(\frac{x}{C}\right) \quad ; \quad \tau(0) = \tau(1) = 0, \tag{13.54}$$

$$\sigma\left(\frac{x}{C}\right) \quad ; \quad \sigma(0) = \sigma(1) = 0 \tag{13.55}$$

$$Tan(\alpha) = -\frac{\delta}{C} \tag{13.56}$$

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Lift

$$L = \int_0^C (P_{lower} - P_\infty) Cos(\alpha_{lower}) dx - \int_0^C (P_{upper} - P_\infty) Cos(\alpha_{upper}) dx \qquad (13.58)$$

$$Cos(\alpha_{lower}) \cong 1 \qquad Cos(\alpha_{upper}) \cong 1 \qquad \xi = \frac{x}{C}$$

$$C_L = \frac{\cancel{D}\ L}{\frac{1}{2}\rho_\infty U_\infty^2 C} = \int_0^1 C_{P_{lower}} d\xi - \int_0^1 C_{P_{upper}} d\xi \qquad (13.59)$$

$$C_{P_{upper}} = \frac{2}{C\sqrt{M_\infty^2 - 1}}\left(\frac{df}{d\xi}\right) = \frac{2}{C\sqrt{M_\infty^2 - 1}}\left(A\frac{d\tau}{d\xi} + B\frac{d\sigma}{d\xi} - \delta\right) \qquad (13.60)$$

$$C_{P_{lower}} = -\frac{2}{C\sqrt{M_\infty^2 - 1}}\left(\frac{dg}{d\xi}\right) = -\frac{2}{C\sqrt{M_\infty^2 - 1}}\left(-A\frac{d\tau}{d\xi} + B\frac{d\sigma}{d\xi} - \delta\right). \qquad (13.61)$$

$$C_L = \frac{-2}{C\sqrt{M_\infty^2 - 1}}\left(\int_0^1 \frac{df}{d\xi} d\xi + \int_0^1 \frac{dg}{d\xi} d\xi\right) = \frac{-2}{C\sqrt{M_\infty^2 - 1}}\left(\int_0^{-\delta} df + \int_0^{-\delta} dg\right) \qquad (13.62)$$

$$\boxed{C_L = \frac{4}{\sqrt{M_\infty^2 - 1}}\left(\frac{\delta}{C}\right)} \qquad (13.63)$$

Drag

$$D = \int_0^C (P_{upper} - P_\infty) Sin(\alpha_{upper}) dx + \int_0^C (P_{lower} - P_\infty) Sin(-\alpha_{lower}) dx \qquad (13.64)$$

(13.65)

$$C_D = \frac{D}{\frac{1}{2}\rho_\infty U_\infty^2 C} = \int_0^1 \left(\frac{P_{upper} - P_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} \right) (\alpha_{upper}) d\xi + \int_0^1 \left(\frac{P_{lower} - P_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} \right) (-\alpha_{lower}) d\xi$$

$$Sin(\alpha_{upper}) \cong \alpha_{upper} \cong \frac{dy_{upper}}{dx} \qquad Sin(-\alpha_{lower}) \cong -\alpha_{lower} \cong -\frac{dy_{lower}}{dx}$$

$$C_D = \frac{D}{\frac{1}{2}\rho_\infty U_\infty^2 C} = \frac{1}{C}\int_0^1 C_{P_{upper}} \left(\frac{dy_{upper}}{d\xi} \right) d\xi + \frac{1}{C}\int_0^1 C_{P_{lower}} \left(-\frac{dy_{lower}}{d\xi} \right) d\xi \qquad (13.66)$$

$$C_D = \frac{2}{C^2\sqrt{M_\infty^2 - 1}} \left(\int_0^1 \left(A\frac{d\tau}{d\xi} + B\frac{d\sigma}{d\xi} - \delta \right)^2 d\xi + \int_0^1 \left(-A\frac{d\tau}{d\xi} + B\frac{d\sigma}{d\xi} - \delta \right)^2 d\xi \right) \qquad (13.67)$$

$$C_D = \frac{4}{\sqrt{M_\infty^2 - 1}} \left(\left(\frac{A}{C}\right)^2 \int_0^1 \left(\frac{d\tau}{d\xi}\right)^2 d\xi + \left(\frac{B}{C}\right)^2 \int_0^1 \left(\frac{d\sigma}{d\xi}\right)^2 d\xi - \left(\frac{B}{C}\right)\frac{\delta}{C} \cancel{\int_0^1 \left(\frac{d\sigma}{d\xi}\right) d\xi} + \left(\frac{\delta}{C}\right)^2 \int_0^1 d\xi \right) \qquad (13.68)$$

$$\boxed{C_D = \frac{4}{\sqrt{M_\infty^2 - 1}} \left(\left(\frac{A}{C}\right)^2 \int_0^1 \left(\frac{d\tau}{d\xi}\right)^2 d\xi + \left(\frac{B}{C}\right)^2 \int_0^1 \left(\frac{d\sigma}{d\xi}\right)^2 d\xi + \left(\frac{\delta}{C}\right)^2 \right). \qquad (13.70)}$$

13.2 Similarity rules for high speed flight

Inviscid, incompressible flow

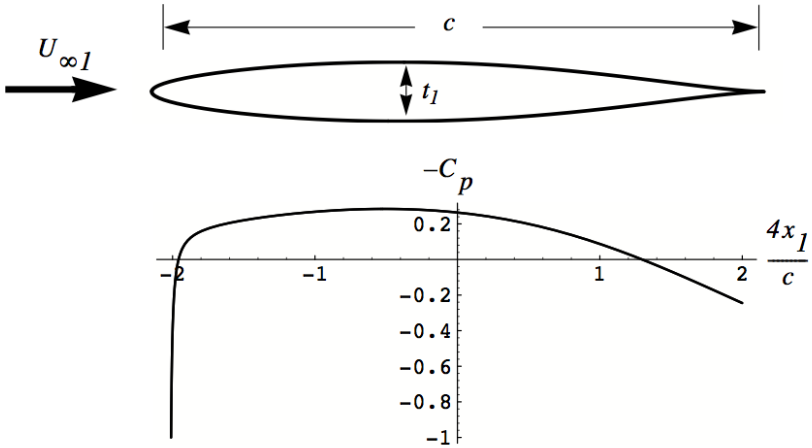


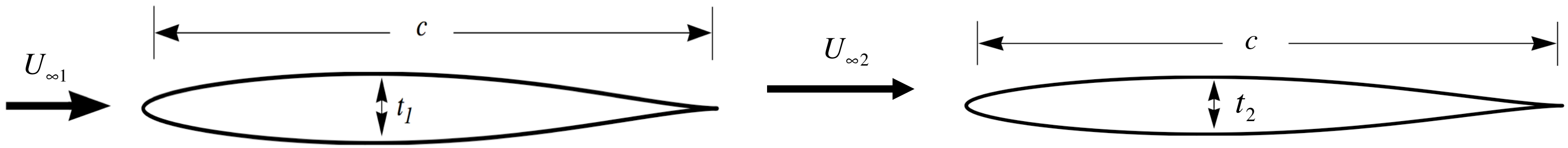
Figure 13.1 Pressure variation over a thin symmetric airfoil in low speed flow.

Governing equation
$$\frac{\partial^2 \phi_1}{\partial^2 x_1} + \frac{\partial^2 \phi_1}{\partial^2 y_1} = 0. \tag{13.75}$$

Pressure
$$P_\infty + \frac{1}{2}\rho_\infty U_{\infty 1}^2 = P_{s1} + \frac{1}{2}\rho_\infty U_{s1}^2. \tag{13.72}$$

Pressure coefficient
$$C_{P1} = \frac{P_s - P_\infty}{\frac{1}{2}\rho_\infty U_{\infty 1}^2}. \tag{13.71}$$

How can we map an incompressible flow to a compressible flow?



$$M_{\infty 1} << 1 \qquad\qquad M_{\infty 2} < 1$$

Equation

$$\frac{\partial^2 \phi_1}{\partial x_1^2} + \frac{\partial^2 \phi_1}{\partial y_1^2} = 0 \qquad\qquad (1 - M_{\infty 2}^2)\frac{\partial^2 \phi_2}{\partial x_2^2} + \frac{\partial^2 \phi_2}{\partial y_2^2} = 0$$

Airfoil shape

$$\frac{y_1}{c} = \tau_1 g[x_1/c] \qquad \tau_1 = t_1/c \qquad\qquad \frac{y_2}{c} = \tau_2 g(x_2/c) \qquad \tau_2 = t_2/c$$

Boundary condition

$$\left(\frac{\partial \phi_1}{\partial y_1}\right)_{y_1 = 0} = U_{\infty 1}\left(\frac{dy_1}{dx_1}\right)_{body} = U_{\infty 1}\tau_1 \frac{dg[x_1/c]}{d(x_1/c)} \qquad\qquad \left(\frac{\partial \phi_2}{\partial y_2}\right)_{y_2 = 0} = U_{\infty 2}\left(\frac{dy_2}{dx_2}\right)_{body} = U_{\infty 2}\tau_2 \frac{dg[x_2/c]}{d(x_2/c)}$$

$$\phi_{1_{x_1 \to \infty}} = 0 \qquad\qquad \phi_{2_{x_2 \to \infty}} = 0$$

Surface pressure

$$C_{P1} = -\frac{2}{U_{\infty 1}}\left(\frac{\partial \phi_1}{\partial x_1}\right)_{y_1 = 0} \qquad\qquad C_{P2} = -\frac{2}{U_{\infty 2}}\left(\frac{\partial \phi_2}{\partial x_2}\right)_{y_2 = 0}$$

Transform variables as follows

$$x_2 = x_1 \quad ; \qquad y_2 = \frac{1}{\sqrt{1-M_{\infty 2}^2}} y_1 \quad ; \qquad \phi_2 = \frac{1}{A}\left(\frac{U_{\infty 2}}{U_{\infty 1}}\right)\phi_1$$

where A is an arbitrary constant

$$(1-M_{\infty 2}^2)\frac{\partial^2 \phi_2}{\partial x_2^2} + \frac{\partial^2 \phi_2}{\partial y_2^2} = 0 \qquad \Rightarrow \qquad \frac{\partial^2 \phi_1}{\partial^2 x_1} + \frac{\partial^2 \phi_1}{\partial^2 y_1} = 0$$

$$\left(\frac{\partial \phi_2}{\partial y_2}\right)_{y_2 = 0} = U_{\infty 2}\left(\frac{dy_2}{dx_2}\right)_{body} = U_{\infty 2}\tau_2 \frac{dg[x_2/c]}{d(x_2/c)} \qquad \Rightarrow \qquad \left(\frac{\partial \phi_1}{\partial y_1}\right)_{y_1 = 0} = U_{\infty 1}\left(\frac{A\tau_2}{\sqrt{1-M_{\infty 2}^2}}\right)\frac{dg[x_1/c]}{d(x_1/c)}$$

$$\phi_{2_{x_2 \to \infty}} = 0 \qquad \qquad \phi_{1_{x_1 \to \infty}} = \phi_{2_{x_2 \to \infty}} = 0$$

The transformation is completed by choosing

$$\frac{t_2}{c} = \frac{\sqrt{1-M_{\infty 2}^2}}{A}\left(\frac{t_1}{c}\right)$$

Pressure coefficient

$$C_{P2} = -\frac{2}{U_{\infty 2}}\left(\frac{\partial \phi_2}{\partial x_2}\right)_{y_2 = 0} \qquad \Rightarrow \qquad C_{P2} = \frac{1}{A} C_{P1}$$

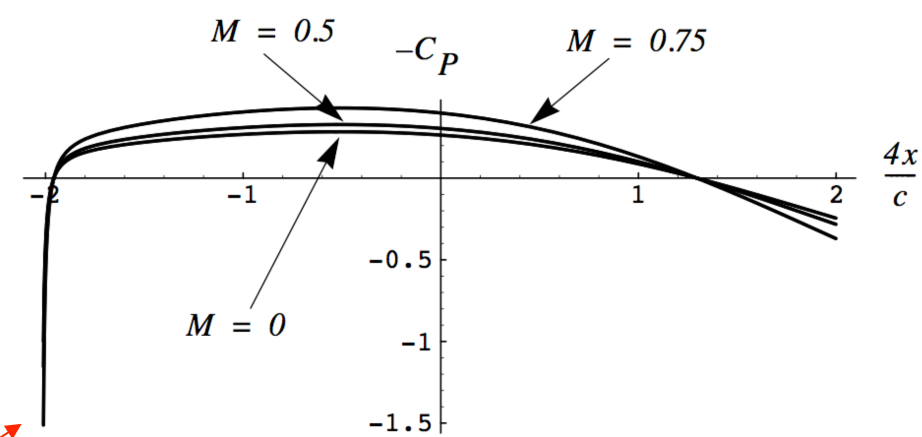
The Prandtl-Glauert rule

Choose $A = \sqrt{1 - M_{\infty 2}^2} \quad \Rightarrow \quad \frac{t_2}{c} = \frac{t_1}{c}$

In this case the airfoils have the same shape and thickness ratio.

The pressure coefficient scales as

$$C_{P2} = \frac{C_{P1}}{\sqrt{1 - M_{\infty 2}^2}}$$



inaccurate at the leading edge

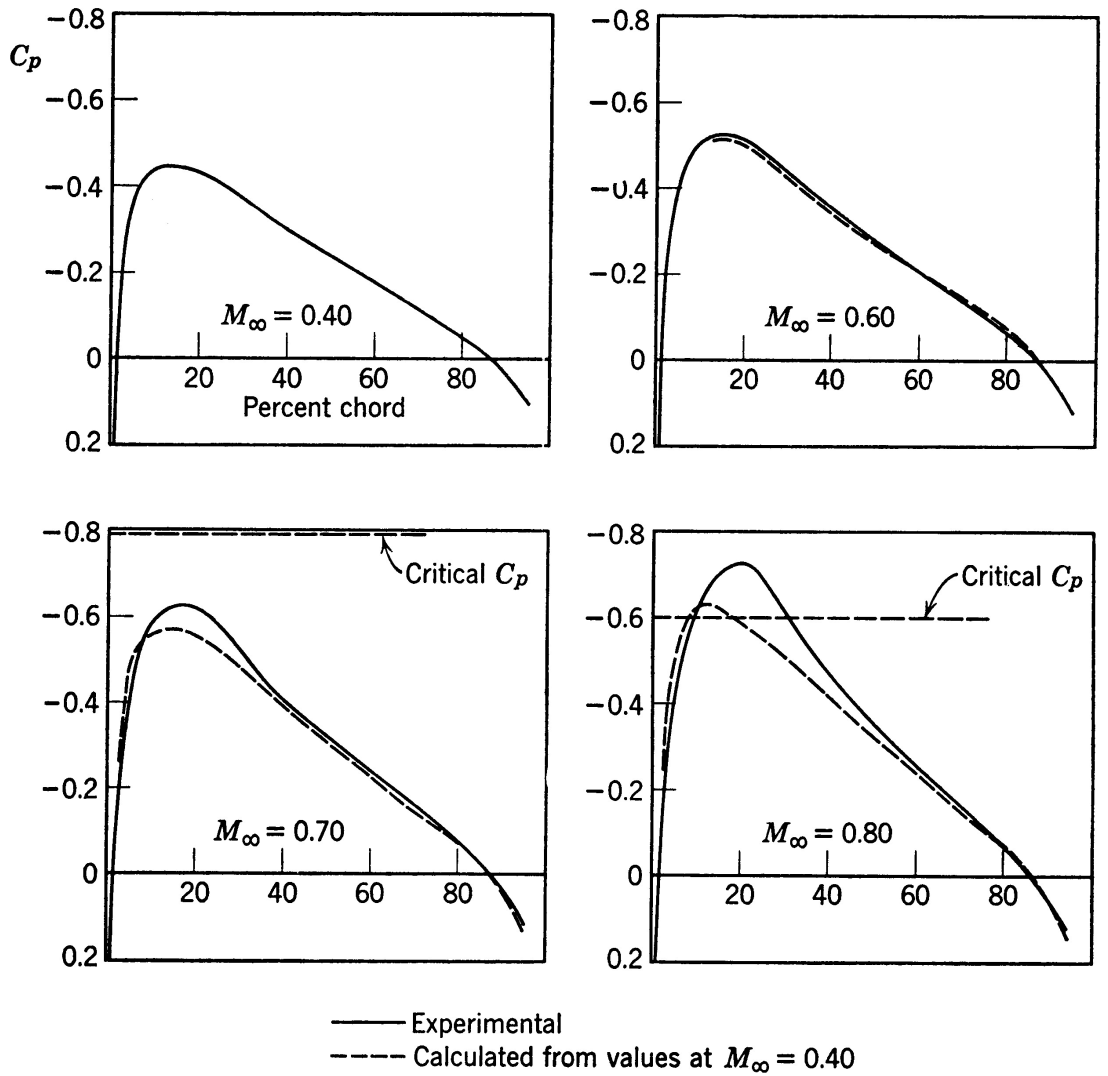


FIG. 10·1 Comparison of Prandtl-Glauert similarity rule with experiment. (Experimental data for NACA 0012 airfoil, taken from *NACA Tech. Note* 2174 by J. L. Amick.)

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Supersonic case - everything is the same with

$$1 - M_\infty^2 \quad \Rightarrow \quad M_\infty^2 - 1$$

Mapping

$$\left(M_{\infty 2}^2 - 1\right)\frac{\partial^2 \phi_2}{\partial x_2^2} - \frac{\partial^2 \phi_2}{\partial y_2^2} = 0 \quad \Rightarrow \quad \frac{\partial^2 \phi_1}{\partial x_1^2} - \frac{\partial^2 \phi_1}{\partial y_1^2} = 0 \qquad M_{\infty 1} = \sqrt{2}$$

Pressure coefficient

$$C_p = \frac{2}{\sqrt{M^2 - 1}}\left(\frac{t}{c}\right)\frac{d\left(y/t\right)}{d\left(x/c\right)}$$

For airfoils with the same dimensionless shape, ie, the same $y/t = f(x/c)$

$$C_p \sim \frac{2}{\sqrt{M^2 - 1}}\left(\frac{t}{c}\right)$$

This is limited to thin airfoils with <u>no shocks</u>.

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Transonic case

$$(1 - M_{\infty 1}^2)\frac{\partial^2 \phi_1}{\partial x_1^2} + \frac{\partial^2 \phi_1}{\partial y_1^2} - \frac{(\gamma_1 + 1)M_{\infty 1}^2}{U_{\infty 1}}\frac{\partial \phi_1}{\partial x_1}\frac{\partial^2 \phi_1}{\partial x_1^2} = 0$$

Transform variables

$$x_2 = x_1 \ ; \qquad y_2 = \frac{\sqrt{1 - M_{\infty 1}^2}}{\sqrt{1 - M_{\infty 2}^2}}y_1 \ ; \qquad \phi_2 = \frac{1}{A}\left(\frac{U_{\infty 2}}{U_{\infty 1}}\right)\phi_1$$

The transonic equation is invariant only if

$$A = \left(\frac{1 + \gamma_2}{1 + \gamma_1}\right)\left(\frac{1 - M_{\infty 1}^2}{1 - M_{\infty 2}^2}\right)\left(\frac{M_{\infty 2}^2}{M_{\infty 1}^2}\right)$$

Pressure coefficient

$$C_{P1} = \left(\frac{1 + \gamma_2}{1 + \gamma_1}\right)\left(\frac{1 - M_{\infty 1}^2}{1 - M_{\infty 2}^2}\right)\left(\frac{M_{\infty 2}^2}{M_{\infty 1}^2}\right)C_{p2}$$

Thickness-to-chord ratio

$$\frac{t_2}{c} = \left(\frac{1 + \gamma_1}{1 + \gamma_2}\right)\left(\frac{1 - M_{\infty 2}^2}{1 - M_{\infty 1}^2}\right)^{\frac{3}{2}}\left(\frac{M_{\infty 1}^2}{M_{\infty 2}^2}\right)\frac{t_1}{c}$$

Other choices of A

$$A = 1 \qquad C_{P2} = C_{P1} \qquad \frac{t_2}{c} = \sqrt{1 - M_{\infty 2}^2}\,\frac{t_1}{c}$$

Cp is constant if thickness is reduced as Mach number is increased

$$A = \left(t_1 / t_2\right) \qquad C_{P2} = \left(\frac{t_2}{t_1}\right) C_{P1} \qquad M_{\infty 2} = const$$

Cp is proportional to thickness/chord for fixed Mach number

Problem – A thin airfoil operating at a small angle of attack and a free stream Mach number $M_\infty = 0.4$ has a lift coefficient $C_L = 0.5$. Estimate the lift coefficient at $M_\infty = 0.6$.

SOLUTION

The lift is proportional to an integral of the pressure coefficient which scales with the Mach number according to the Prandtl-Glauert formula. Thus

$$C_{pM_1} = \frac{C_{pMincomp}}{\left(1 - M_1^2\right)^{1/2}}$$

$$C_{pM_2} = \frac{C_{pMincomp}}{\left(1 - M_2^2\right)^{1/2}}$$

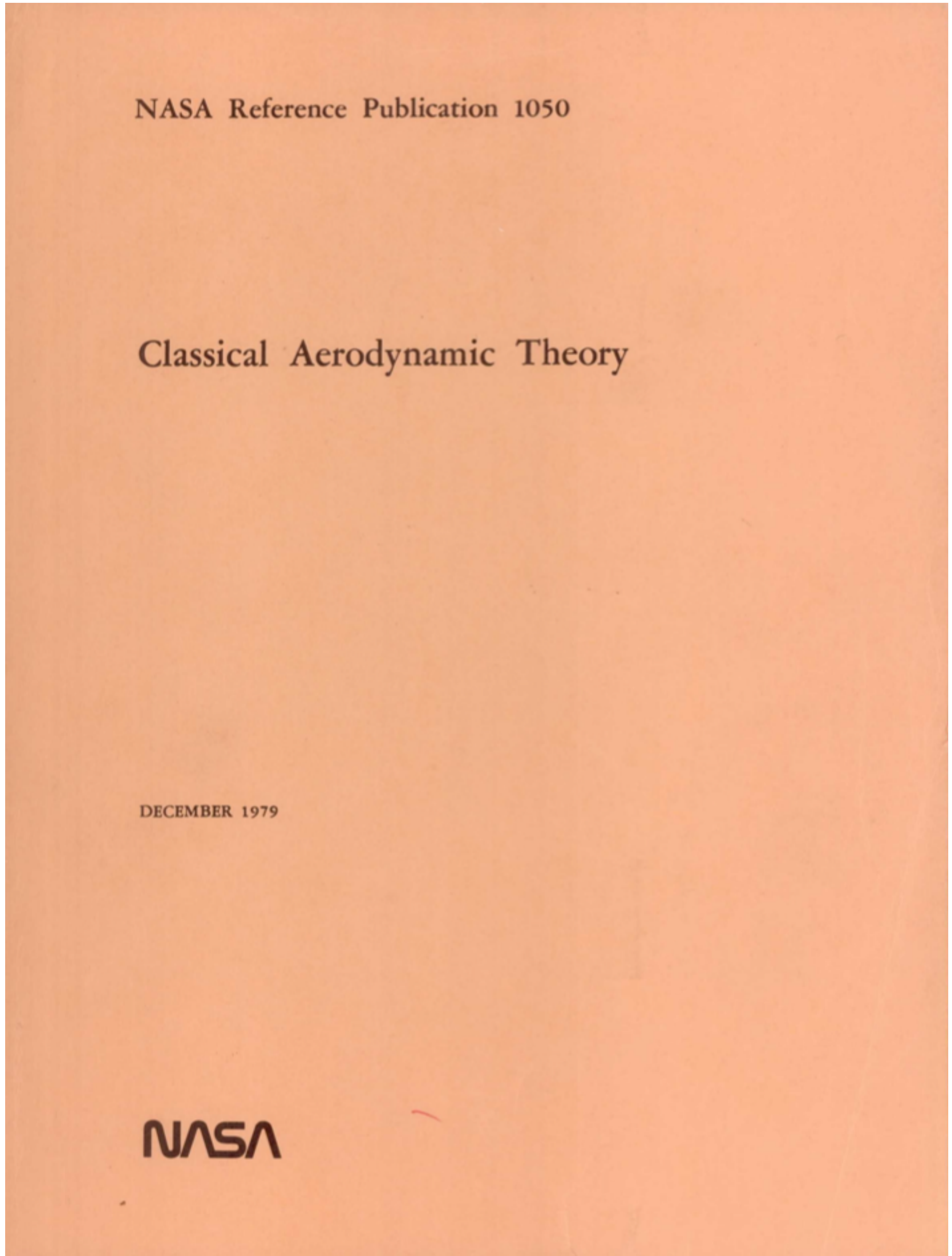
$$\frac{C_{pM_2}}{C_{pM_1}} = \frac{\left(1 - M_1^2\right)^{1/2}}{\left(1 - M_2^2\right)^{1/2}}$$

$$\frac{C_{p0.6}}{C_{p0.4}} = \frac{\left(1 - 0.4^2\right)^{1/2}}{\left(1 - 0.6^2\right)^{1/2}} = 1.14654$$

The lift coefficient should scale the same way.

$$C_{L0.6} = 1.14654 C_{L0.4} = 0.572822$$

This reference can be found on my website



NASA Reference Publication 1050

Classical Aerodynamic Theory

DECEMBER 1979

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13.3 The effect of sweep

Recall the full potential equation

$$(\gamma - 1)\left(h_t - \frac{\bar{U} \bullet \bar{U}}{2}\right)\nabla \bullet \bar{U} - \bar{U} \bullet \nabla\left(\frac{\bar{U} \bullet \bar{U}}{2}\right) = 0 \qquad (13.8)$$

This equation can be written

$$\nabla \bullet \bar{U} - \frac{1}{a^2}\bar{U} \bullet \nabla\left(\frac{\bar{U} \bullet \bar{U}}{2}\right) = 0 \qquad (13.110)$$

Use

$$\nabla \times \bar{U} = 0$$

Ref: R. T. Jones NACA Technical Report 863 1945.

$$\left(1 - \frac{U^2}{a^2}\right)\frac{\partial U}{\partial x} + \left(1 - \frac{V^2}{a^2}\right)\frac{\partial V}{\partial y} + \left(1 - \frac{W^2}{a^2}\right)\frac{\partial W}{\partial z} - 2\left(\frac{UV}{a^2}\frac{\partial U}{\partial y} + \frac{WV}{a^2}\frac{\partial V}{\partial z} + \frac{UW}{a^2}\frac{\partial U}{\partial z}\right) = 0 \qquad (13.111)$$

Flow in a plane normal to the wing is almost 2-D

Neglect

$$\frac{\partial W}{\partial z} << 1 \qquad \frac{\partial U}{\partial z} << 1$$

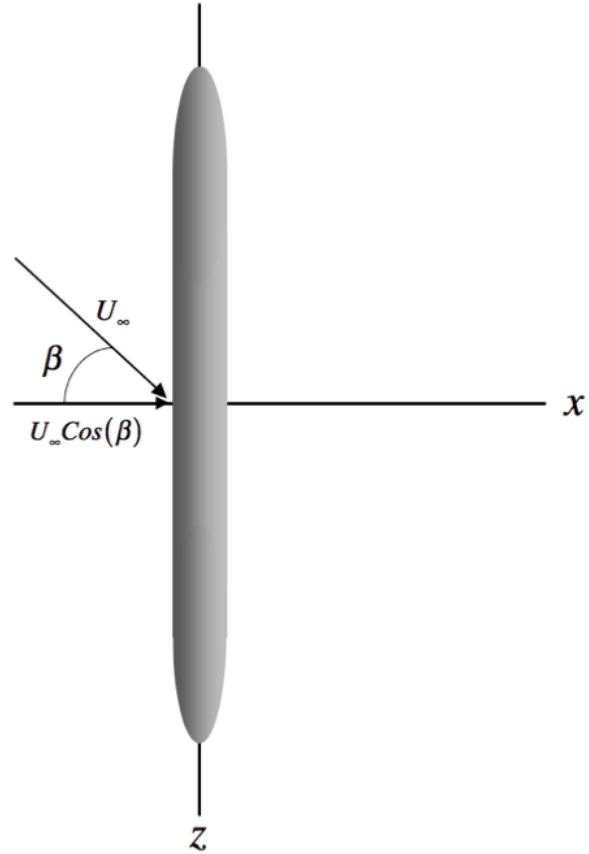

Figure 13.3 Slender wing with free stream approaching at sweep angle β

$$\left(1-\frac{U^2}{a^2}\right)\frac{\partial U}{\partial x}+\left(1-\frac{V^2}{a^2}\right)\frac{\partial V}{\partial y}-2\frac{V}{a}\left(\frac{U}{a}\frac{\partial U}{\partial y}+\frac{W}{a}\frac{\partial V}{\partial z}\right) = 0 \qquad (13.112)$$

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Assume small disturbances

$$U = U_\infty Cos(\beta) + u$$
$$V = v$$
$$W = U_\infty Sin(\beta) + w \qquad (13.113)$$
$$a = a_\infty + a'$$

Neglect cubic terms in the small disturbances

$$\left(1 - \frac{U_\infty^2 Cos^2(\beta)}{a_\infty^2}\right)\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} - 2\frac{V}{a_\infty}\left(\frac{U_\infty Cos(\beta)}{a_\infty}\frac{\partial U}{\partial y} + \frac{U_\infty Sin(\beta)}{a_\infty}\frac{\partial V}{\partial z}\right) = 0 \qquad (13.114)$$

Neglect quadratic terms

$$\frac{V}{a_\infty} << 1 \qquad u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y}$$

$$(1 - M_\infty^2 Cos^2(\beta))\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (13.115)$$

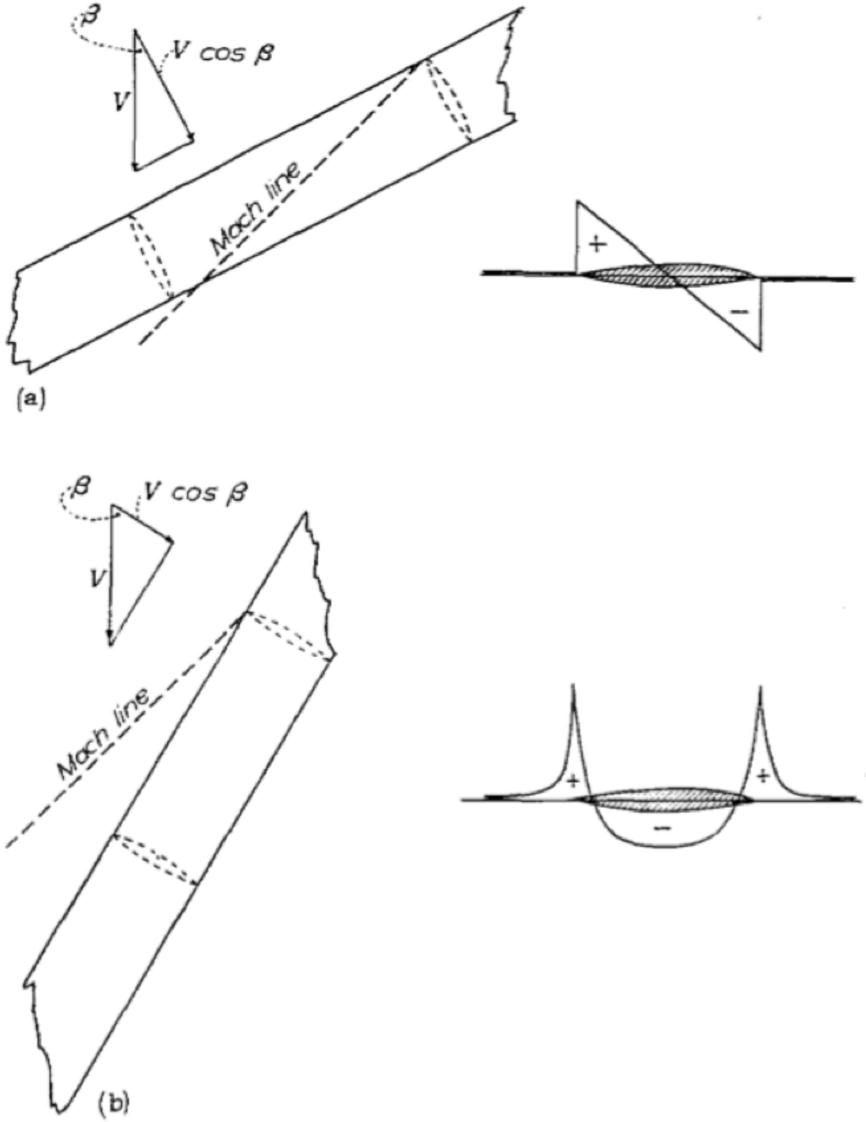
$$(1 - M_\infty^2 Cos^2(\beta))\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (13.115)$$

For the swept wing the disturbance potential is governed by the component of the free stream velocity normal to the leading edge of the wing $U_\infty Cos(\beta)$. If

$M_\infty^2 Cos^2(\beta) > 1$ then the normal flow is supersonic and we would use supersonic theory developed above to relate the pressure coefficient to the slope of the airfoil surface in the (x, y) plane.

If $M_\infty^2 Cos^2(\beta) < 1$ then the normal flow is subsonic and we can use the mapping (13.90) to transform (13.115) to Laplace's equation. The methods of subsonic thin airfoil theory can be used to determine the $M = 0$ disturbance potential ϕ_1. The pressure coefficient of the $M = 0$ solution is used with the Prandtl-Glauert rule (13.99) to determine the pressure coefficient on the swept wing.

$$C_{P_{M_\infty^2 Cos(\beta)}} = \frac{C_{P1}}{\sqrt{1 - M_\infty^2 Cos^2(\beta)}} \qquad (13.116)$$



(a) $V \cos \beta > c$.
(b) $V \cos \beta < c$.

FIGURE 2.—Effect of leading-edge angle on pressure distribution.

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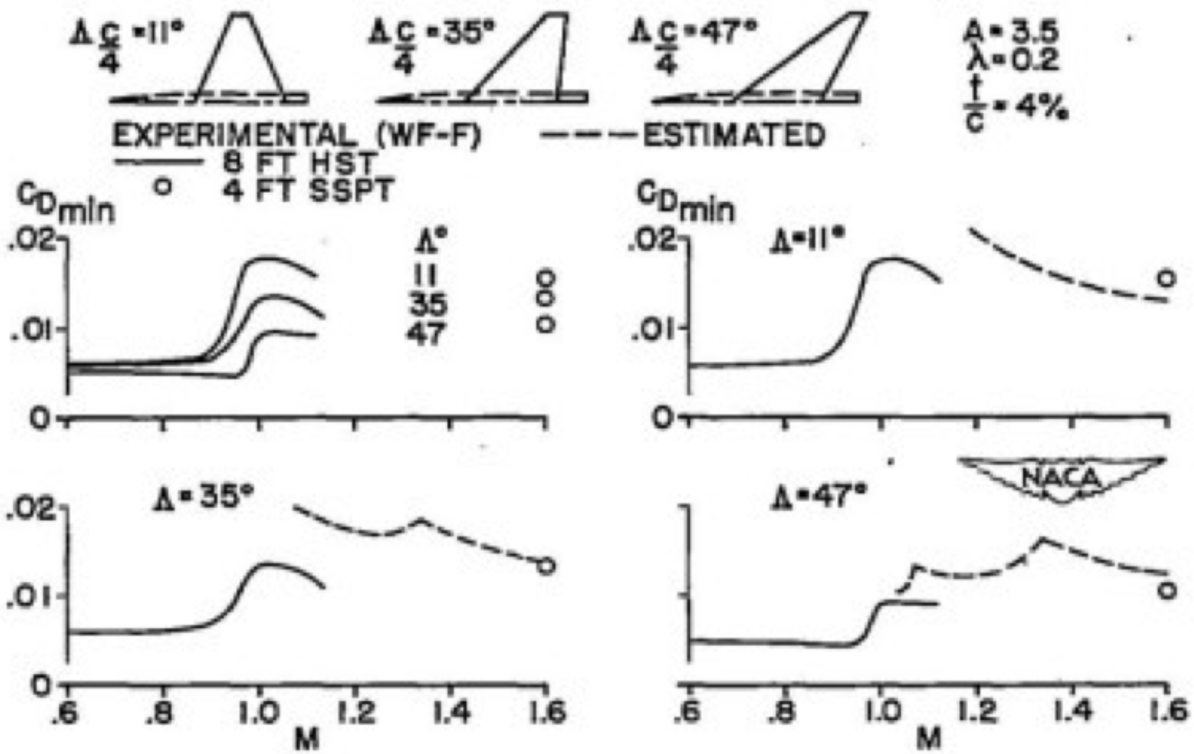

Figure 7.- Effects of sweep angle on minimum-drag characteristics. A = 3.5.

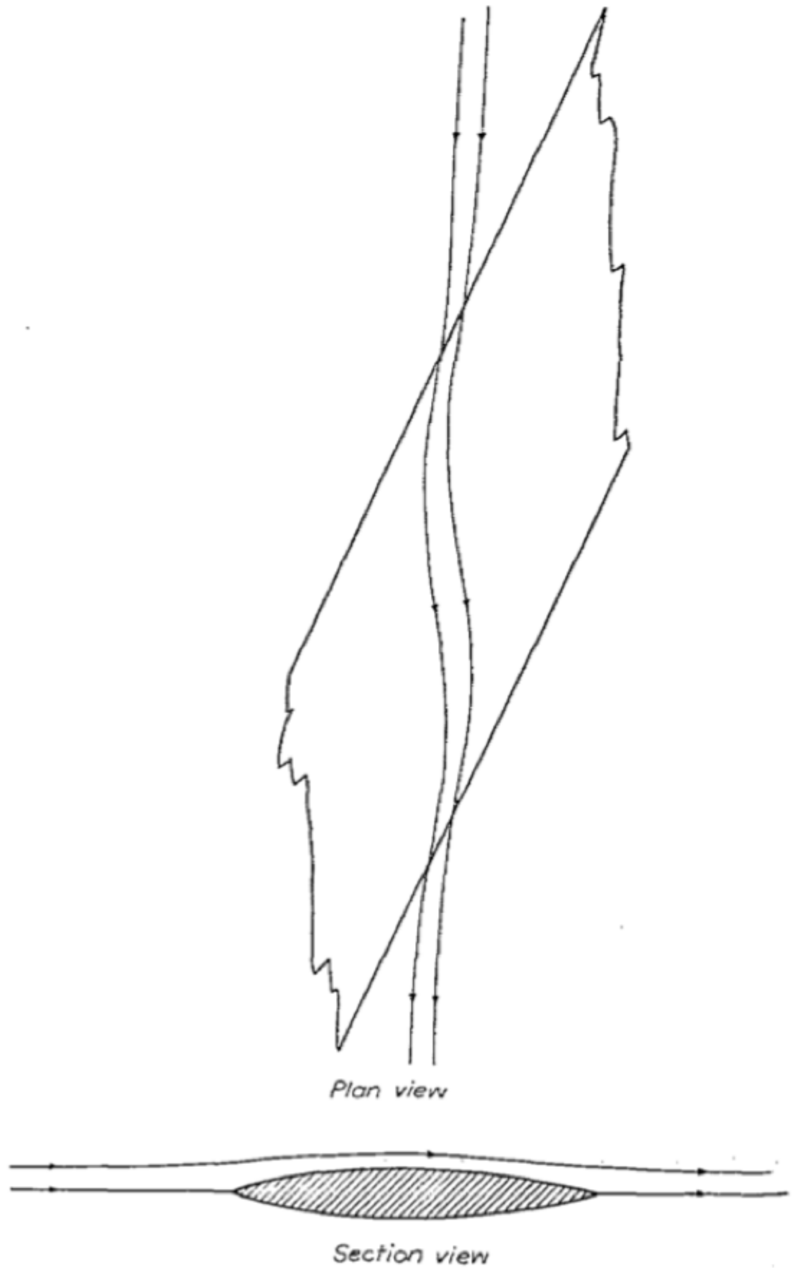

FIGURE 3.—Change in area of stream tube over upper surface of sweptback wing.

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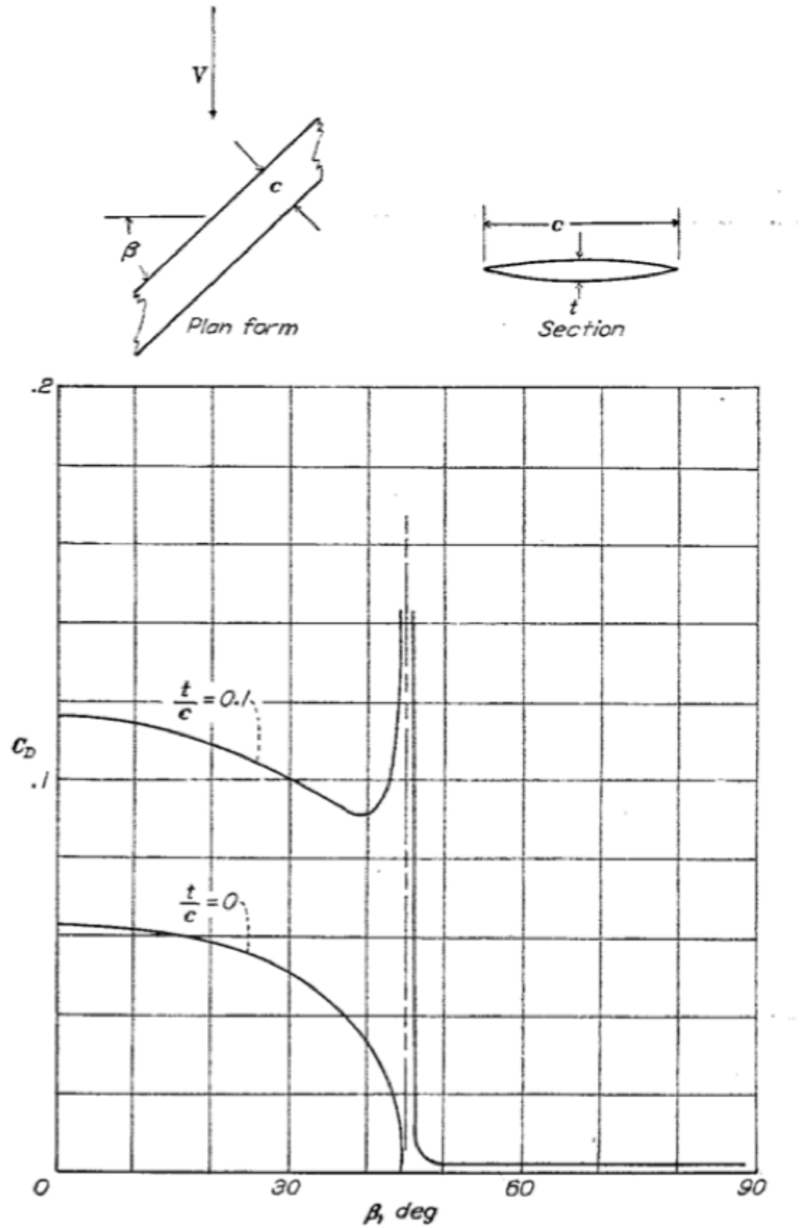


FIGURE 4.—Variation of pressure drag with angle of sweepback for infinite aspect ratio. $M=1.4$; $C_L=0.5$.

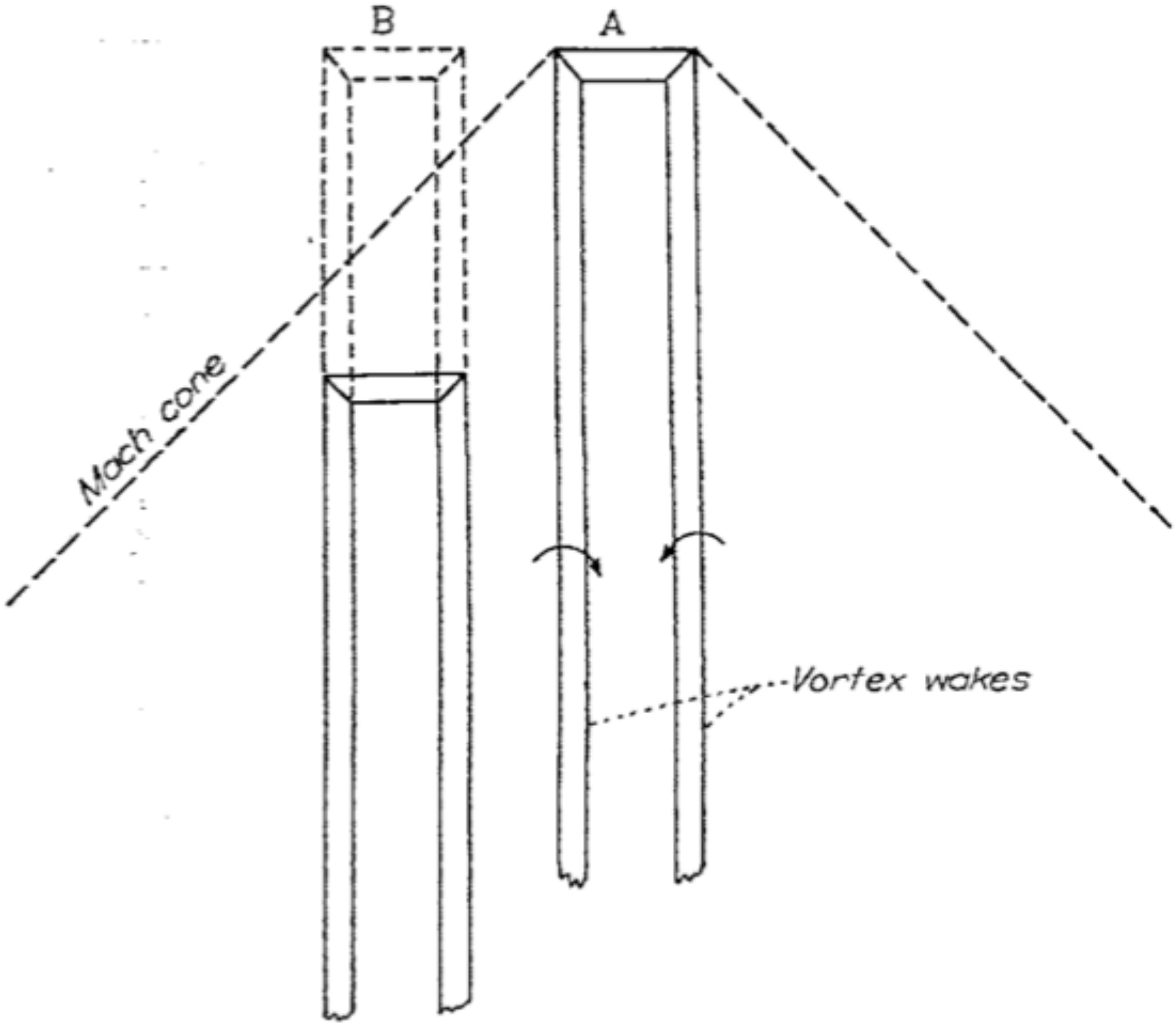


FIGURE 5.—Staggered lifting elements in supersonic flow.

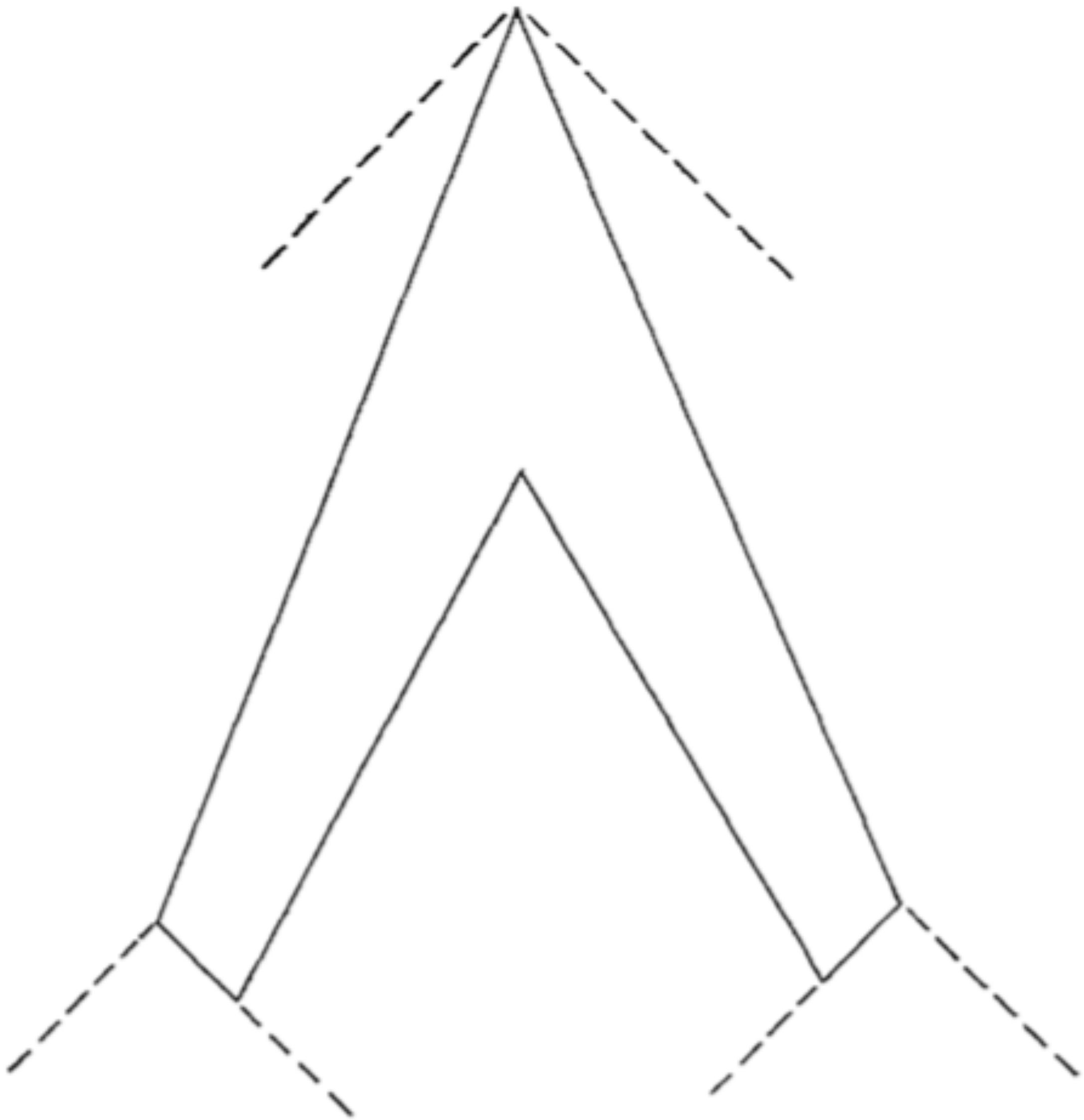


FIGURE 6.—Wing with tips cut away along the Mach lines.

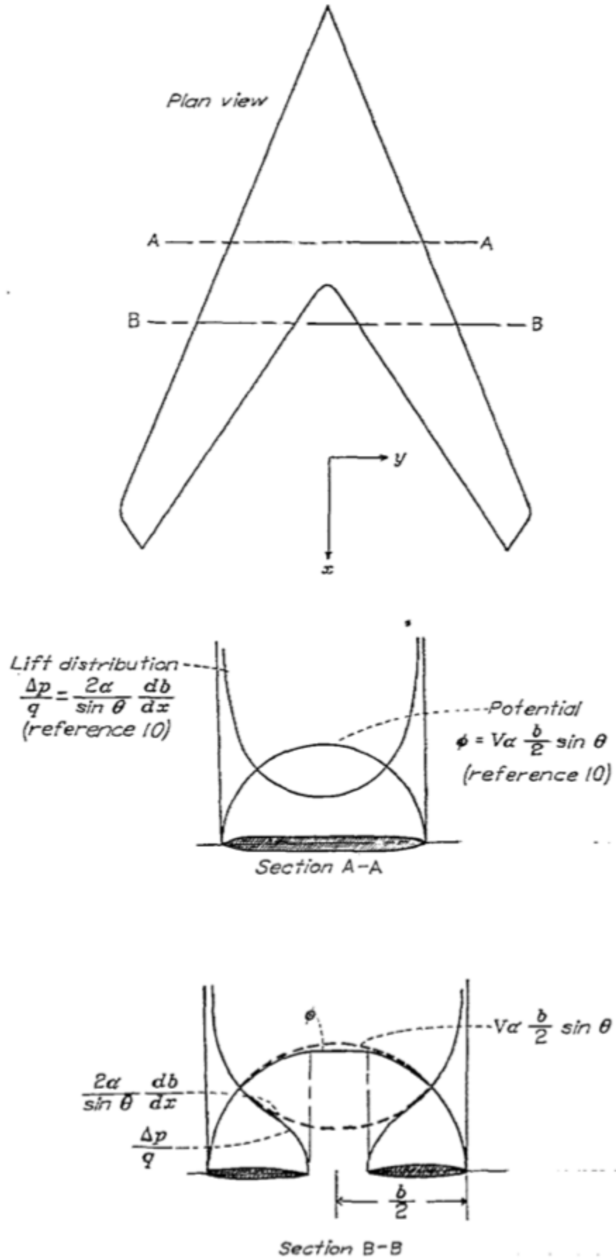
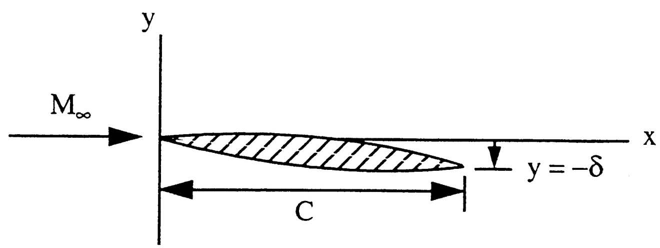


FIGURE 7.—Approximate distribution of lift near vertex of wing with large angle of sweepback.

13.4 Problems

Problem - A thin, 2-D, airfoil is situated in a supersonic stream at Mach number M_∞ and a small angle of attack as shown below.



The y-coordinate of the upper surface of the airfoil is given by the function.

$$f(x) = A\frac{x}{C}\left(1 - \frac{x}{C}\right) - \frac{\delta}{C}x$$

and the y-coordinate of the lower surface is

$$g(x) = -A\frac{x}{C}\left(1 - \frac{x}{C}\right) - \frac{\delta}{C}x$$

where $2A/C \ll 1$ and $\delta/C \ll 1$. Determine the lift and drag coefficients of the airfoil.