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AA200 Applied Aerodynamics

Chapter 14 - Viscous free shear flows

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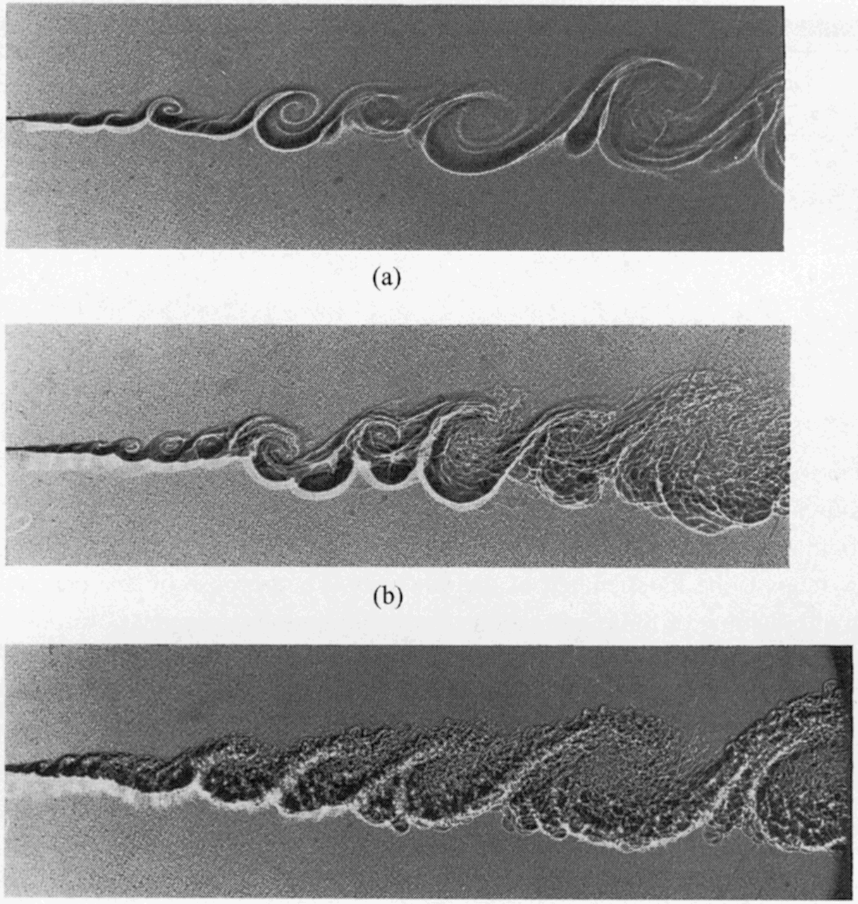
Reynolds number invariance



(a) (b) (c)

Fig. 13.1. Effects of Reynolds number on a plane mixing layer between helium (upper stream) and nitrogen (lower stream) from Brown and Roshko [13.1] and Roshko[13.2]. The Reynolds number in (a) is approximately 1.3×10^4 centimeter^{-1}. The thickness of the layer at the right side of the picture is approximately 2 cm. The speed of the lower stream is 10 m/s. Test-section pressures in atmospheres are: (a) 2, (b) 4, (c) 8. Dynamic pressures in the upper and lower streams are the same: $\rho_1 u_1^2 = \rho_2 u_2^2$.

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Image above: The above photograph shows the turbulence field behind the Horns Rev offshore wind turbines. Horns Rev is located in the North Sea, 14 kilometers west of Denmark. Photographer Christian Steiness. From (http://wattsupwiththat.com/2011/04/28/the-wind-turbine-albedo-effect/).

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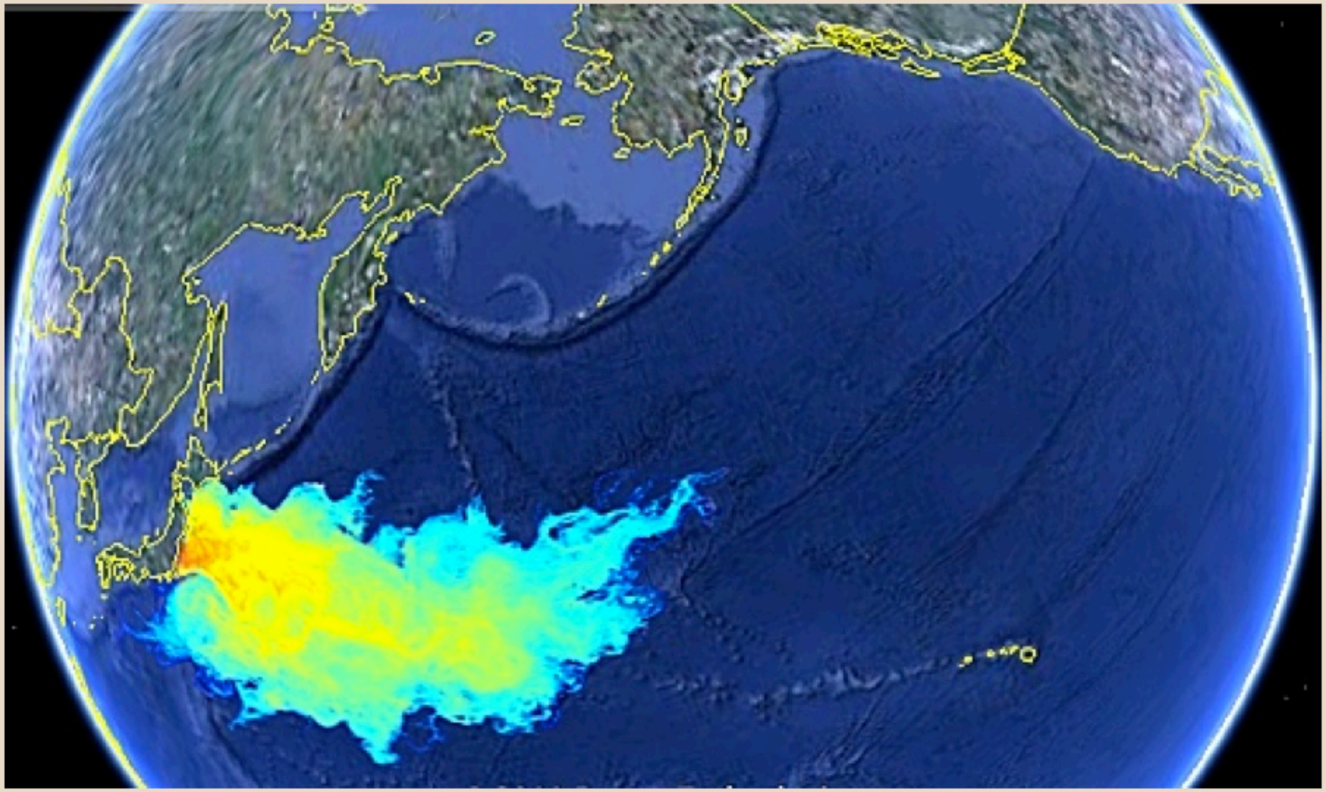
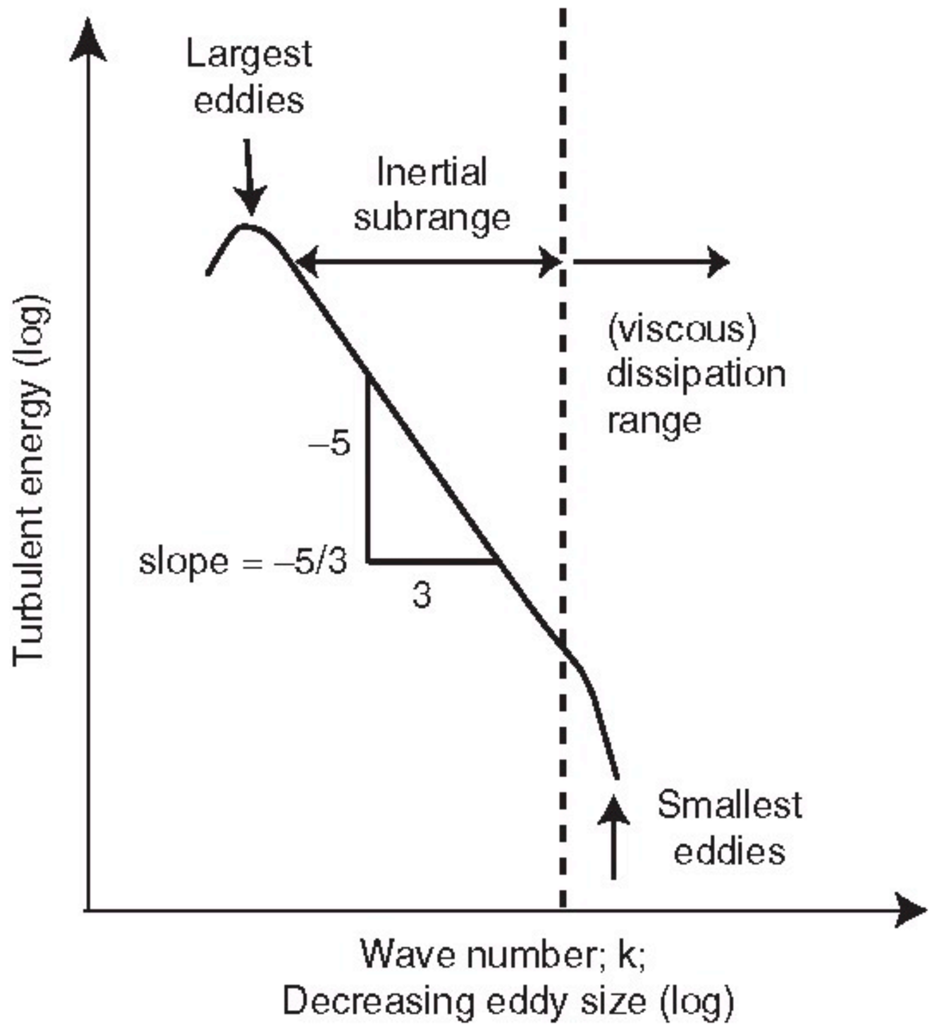


Image above: 8/11/11 simulation of radioactive seawater dispersed from Fukushima nears Hawaii. From (http://www.xydo.com/toolbar/27327691-asr_ltd_-_fukushima_radioactive_seawater_plume_dispersal_simulation). Note - users can zoom and rotate orientation of simulation.

Turbulent kinetic energy spectrum



The Reynolds averaged Navier-Stokes equations

$$\boldsymbol{u} = \bar{\boldsymbol{u}} + \boldsymbol{u}' \quad ; \qquad p = \bar{p} + p' , \tag{14.1}$$

$$\left.\begin{aligned} &\frac{\partial \bar{u}^k}{\partial x^k} = 0 \\ &\frac{\partial \bar{u}^i}{\partial t} + \frac{\partial}{\partial x^j}(\bar{u}^j \bar{u}^i) + \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x^i} - \frac{1}{\rho}\frac{\partial \tau^{ji}}{\partial x^j} - 2\nu \frac{\partial \bar{s}^{ji}}{\partial x^j} = 0 \end{aligned}\right\} \tag{14.2}$$

$$\bar{s}^{ji} = \frac{1}{2}\left(\frac{\partial \bar{u}^i}{\partial x^j} + \frac{\partial \bar{u}^j}{\partial x^i}\right) . \tag{14.3}$$

$$\frac{\tau^{ji}}{\rho} = -\overline{u'^j u'^i} \tag{14.4}$$

$$-\overline{u'^j u'^i} \gg 2\nu s^{ji} \tag{14.5}$$

$$\left.\begin{aligned} &\frac{\partial \bar{u}^j}{\partial x^j} = 0 \\ &\frac{\partial \bar{u}^i}{\partial t} + \frac{\partial}{\partial x^j}(\bar{u}^j \bar{u}^i) + \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x^i} - \frac{1}{\rho}\frac{\partial \tau^{ji}}{\partial x^j} = 0 \end{aligned}\right\} . \tag{14.6}$$

In free shear flows (away from walls) the Reynolds stresses are much larger than the viscous stresses

Integral length and velocity scales

u_0 - *Integral velocity scale characterizing the overall motion*

δ - *Integral length scale characterizing the overall motion*

Turbulent kinetic energy

$$u' = \sqrt{\frac{u'^2 + v'^2 + w'^2}{2}}. \tag{14.7}$$

$$u' \approx u_0 \tag{14.8}$$

Invariant group of the Euler equations

$$\tilde{x}^i = e^s x^i ; \quad \tilde{t} = e^{\left(\frac{s}{k}\right)} t ; \quad \tilde{\bar{u}}^i = e^{s\left(1 - \frac{1}{k}\right)} \bar{u}^i$$

$$\tilde{\tau}^{ki} = e^{s\left(2 - \frac{2}{k}\right)} \tau^{ki} ; \quad \tilde{\bar{p}} = e^{s\left(2 - \frac{2}{k}\right)} \bar{p} \qquad (14.13)$$

The mean momentum equation is invariant for arbitrary s and k

$$\frac{\partial \tilde{\bar{u}}^i}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}^j} \tilde{\bar{u}}^j \tilde{\bar{u}}^i + \frac{\partial \tilde{\bar{p}}}{\partial \tilde{x}^i} - \frac{\partial \tilde{\tau}^{ji}}{\partial \tilde{x}^j} = \left(\frac{\partial \bar{u}^i}{\partial t} + \frac{\partial}{\partial x^j} \bar{u}^j \bar{u}^i + \frac{\partial \bar{p}}{\partial x^i} - \frac{\partial \tau^{ji}}{\partial x^j} \right) e^{a\left(1 - \frac{2}{k}\right)} = 0 \qquad (14.14)$$

One parameter flows

$$\hat{M} = L^m T^{-n} \tag{14.15}$$

Stationary Plane jet - The integral momentum flux, J/ρ is approximately constant at any streamwise position.

$$\frac{J}{\rho} = \int_{-\infty}^{\infty} \tilde{u}^2 d\tilde{y} = e^{a\left(3 - \frac{2}{k}\right)} \int_{-\infty}^{\infty} u^2 dy \tag{14.16}$$

The integral is invariant under dilation only for $k = 2/3$.

Vortex ring - The hydrodynamic impulse, I/ρ, is the conserved integral for this flow, cf. Chapter 11, Section 11.5.1.

$$\frac{I}{\rho} = \frac{3}{2}\int \tilde{u} d\tilde{x} d\tilde{y} d\tilde{z} = e^{a\left(4 - \frac{1}{k}\right)} \frac{3}{2}\int u dx dy dz \tag{14.17}$$

In this case the integral is invariant for $k = 1/4$.

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Temporal similarity rules

$$\frac{dx^i}{x^i} = k\frac{dt}{t} = \left(\frac{k}{k-1}\right)\frac{du^i}{u^i} = \left(\frac{k}{2k-2}\right)\frac{dp}{p} = \left(\frac{k}{2k-2}\right)\frac{d\tau^{ki}}{\tau^{ki}} \qquad (14.18)$$

$$\xi^i = \frac{x^i}{\delta[t]}; \quad P^i = \frac{u^i}{u_0[t]}; \quad P = \frac{p}{u_0[t]^2}; \quad T^{ki} = \frac{\tau^{ki}}{u_0[t]^2} \qquad (14.19)$$

$$\delta[t] \cong M^{\frac{1}{m}}(t-t_0)^k; \quad u_0[t] \cong M^{\frac{1}{m}}(t-t_0)^{k-1} \qquad (14.20)$$

$$k = n/m \qquad (14.21)$$

$$\frac{\bar{u}^i}{u_0[t]} = U^i\left[\frac{\boldsymbol{x}}{\delta[t]}\right]; \quad \frac{\bar{p}}{u_0[t]^2} = P\left[\frac{\boldsymbol{x}}{\delta[t]}\right]; \quad \frac{\tau^{ki}}{u_0[t]^2} = T^{ki}\left[\frac{\boldsymbol{x}}{\delta[t]}\right] \qquad (14.22)$$

Reduced equations

$$\left.\begin{aligned} &\frac{\partial U^j}{\partial \xi^j} = 0 \\ &(k-1)U^i + (U^j - k\xi^j)\frac{\partial U^i}{\partial \xi^j} + \frac{1}{\rho}\frac{\partial P}{\partial \xi^i} - \frac{1}{\rho}\frac{\partial}{\partial \xi^j}(T^{ji}) = 0 \end{aligned}\right\}. \qquad (14.23)$$

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Spatial similarity rules - jets

$$(x - x_0) \cong M^{\frac{1}{m}}(t - t_0)^k \qquad (14.29)$$

$$\delta \cong (x - x_0)\ ; \quad U_0 \cong M^{\frac{1}{n}}(x - x_0)^{\left(1 - \frac{1}{k}\right)} \qquad (14.30)$$

Spatial similarity rules - wakes

$$\frac{D}{\rho} = C_D\left(\frac{1}{2}U_\infty^2\right)(\pi R^2). \qquad (14.31)$$

$$\frac{D}{\rho} \cong \int_A U(U_\infty - U)dA \qquad (14.32)$$

$$\frac{D}{\rho U_\infty} \cong \int_A (U_\infty - U)dA \qquad (14.33)$$

$$x - x_0 = U_\infty(t - t_0) \qquad (14.34)$$

$$\delta \cong M^{\frac{1}{m}} U_\infty^{-k}(x - x_0)^k ; \quad U_0 \cong M^{\frac{1}{m}} U_\infty^{1-k}(x - x_0)^{k-1} \qquad (14.35)$$

Reynolds number scaling

$$R_\delta = \frac{U_0\delta}{\nu} \cong \left(\frac{M^{2/m}}{\nu}\right)(t-t_0)^{2k-1} \tag{14.36}$$

The idea of an eddy viscosity

$$-\overline{u'v'} = \nu_\tau \frac{\partial \bar{u}}{\partial y}.$$

$$\nu_\tau \propto u_0\,\delta.$$

$$Re_\tau = \frac{u_0\delta}{\nu_\tau} \propto \text{constant}$$

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Table 13.1. *Various one-parameter shear flows and the units of the associated governing parameter.*

Flow	Invariant	M	Units	k
	Jetlike flows			
Plane mixing layer	Velocity difference	U_0	LT^{-1}	1
Plane jet	2-D momentum flux	$U_0^2\delta$	L^3T^{-2}	$\frac{2}{3}$
Round jet	3-D momentum flux	$U_0^2\delta^2$	L^4T^{-2}	$\frac{1}{2}$
Radial jet	3-D momentum flux	$U_0^2\delta^2$	L^4T^{-2}	$\frac{1}{2}$
Vortex pair	2-D impulse	$U_0\delta^2$	L^3T^{-1}	$\frac{1}{3}$
Vortex ring	3-D impulse	$U_0\delta^3$	L^4T^{-1}	$\frac{1}{4}$
Plane plume	2-D buoyancy flux	U_0^3	L^3T^{-3}	1
Round plume	3-D buoyancy flux	$U_0^3\delta$	L^4T^{-3}	$\frac{3}{4}$
Plane thermal	2-D buoyancy	$U_0^2\delta$	L^3T^{-2}	$\frac{2}{3}$
Round thermal	3-D buoyancy	$U_0^2\delta^2$	L^4T^{-2}	$\frac{1}{2}$
Line vortex	Circulation	$U_0\delta$	L^2T^{-1}	$\frac{1}{2}$
Diverging channel	Area flux	$U_0\delta$	L^2T^{-1}	$\frac{1}{2}$
Vortex-sheet rollup	Apex $\alpha; n = 1/(2-\alpha/\pi)$	$U_0^2\delta^{2-n}$	$L^{3-n}T^{-1}$	$1/(3-n)$
	Wakelike flows			
Plane wake	(2-D drag)/U_∞	$U_0\delta$	L^2T^{-1}	$\frac{1}{2}$
Round wake	(3-D drag)/U_∞	$U_0\delta^2$	L^3T^{-1}	$\frac{1}{3}$
Plane jet in cross flow	(2-D mom. flux)/U_∞	$U_0\delta$	L^2T^{-1}	$\frac{1}{2}$
Round jet in cross flow	(3-D mom. flux)/U_∞	$U_0\delta^2$	L^3T^{-1}	$\frac{1}{3}$
Plane plume in cross flow	(2-D buoy. flux)/U_∞	U_0^2	L^2T^{-2}	1
Round plume in cross flow	(3-D buoy. flux)/U_∞	$U_0^2\delta$	L^3T^{-2}	$\frac{2}{3}$
Grid turb. initial decay	Saffman invariant	$U_0^2\delta^3$	L^5T^{-2}	$\frac{2}{5}$
Grid turb. initial decay	Loitsianski invariant	$U_0^2\delta^5$	L^7T^{-2}	$\frac{2}{7}$

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Microscale motions - derive the turbulent kinetic energy equation

Momentum and continuity equations

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}\left(u_i u_j + \frac{p}{\rho}\delta_{ij} - 2\nu s_{ij} \right) = 0$$

$$s_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

Kinetic energy equation - project the momentum equation onto the velocity vector

$$u_i \frac{\partial u_i}{\partial t} + u_i \frac{\partial}{\partial x_j}\left(u_i u_j + \frac{p}{\rho}\delta_{ij} - 2\nu s_{ij} \right) = 0$$

$$u_i \frac{\partial u_i}{\partial t} = \frac{\partial}{\partial t}\left(\frac{u_i u_i}{2} \right) \qquad u_i \frac{\partial u_i u_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left(u_j \frac{u_i u_i}{2} \right)$$

$$\frac{\partial}{\partial t}\left(\frac{u_i u_i}{2} \right) + \frac{\partial}{\partial x_j}\left(u_j \frac{u_i u_i}{2} + u_j \frac{p}{\rho} - 2\nu u_i s_{ij} \right) + 2\nu s_{ij} s_{ij} = 0$$

Decompose the flow into a mean and fluctuating part.

$$\frac{\partial\left(\dfrac{\bar{u}_i\bar{u}_i}{2}+\dfrac{\overline{u_i' u_i'}}{2}\right)}{\partial t}+\frac{\partial}{\partial x_j}\left(\frac{\bar{u}_i\bar{u}_i}{2}\bar{u}_j+\frac{\overline{u_i' u_i'}}{2}\bar{u}_j+\overline{u_i' u_j'}\bar{u}_i+\frac{\overline{u_i' u_i' u_j'}}{2}+\bar{u}_j\frac{\bar{p}}{\rho}+\frac{\overline{u_j' p'}}{\rho}-2\nu\left(\bar{u}_i\bar{s}_{ij}+\overline{u_i' s_{ij}'}\right)\right)+2\nu\left(\bar{s}_{ij}\bar{s}_{ij}+\overline{s_{ij}' s_{ij}'}\right)=0$$

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Mean kinetic energy equation - Project the mean momentum equation onto the mean velocity vector

$$\bar{u}_i \frac{\partial \bar{u}_i}{\partial t} + \bar{u}_i \frac{\partial}{\partial x_j}\left(\bar{u}_i \bar{u}_j + \frac{\bar{p}}{\rho}\delta_{ij} + \overline{u_i' u_j'} - 2\nu \bar{s}_{ij} \right) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\bar{u}_i \bar{u}_i}{2} \right) + \frac{\partial}{\partial x_j}\left(\bar{u}_j \frac{\bar{u}_i \bar{u}_i}{2} + \bar{u}_j \frac{\bar{p}}{\rho} + \bar{u}_i \overline{u_i' u_j'} - 2\nu \bar{u}_i \bar{s}_{ij} \right) - \overline{u_i' u_j'} \frac{\partial \bar{u}_i}{\partial x_j} + 2\nu \bar{s}_{ij} \bar{s}_{ij} = 0$$

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Subtract.

$$\frac{\partial\left(\frac{\bar{u}_i\bar{u}_i}{2}+\frac{\overline{u_i'u_i'}}{2}\right)}{\partial t}+\frac{\partial}{\partial x_j}\left(\frac{\bar{u}_i\bar{u}_i}{2}\bar{u}_j+\frac{\overline{u_i'u_i'}}{2}\bar{u}_j+\overline{u_i'u_j'}\bar{u}_i+\frac{\overline{u_i'u_i'u_j'}}{2}+\bar{u}_j\frac{\bar{p}}{\rho}+\frac{\overline{u_j'p'}}{\rho}-2\nu\left(\bar{u}_i\bar{s}_{ij}+\overline{u_i's_{ij}'}\right)\right)+2\nu\left(\bar{s}_{ij}\bar{s}_{ij}+\overline{s_{ij}'s_{ij}'}\right)=0$$

$$-$$

$$\frac{\partial}{\partial t}\left(\frac{\bar{u}_i\bar{u}_i}{2}\right)+\frac{\partial}{\partial x_j}\left(\bar{u}_j\frac{\bar{u}_i\bar{u}_i}{2}+\bar{u}_j\frac{\bar{p}}{\rho}+\bar{u}_i\overline{u_i'u_j'}-2\nu\bar{u}_i\bar{s}_{ij}\right)-\overline{u_i'u_j'}\frac{\partial\bar{u}_i}{\partial x_j}+2\nu\bar{s}_{ij}\bar{s}_{ij}=0$$

$$=$$

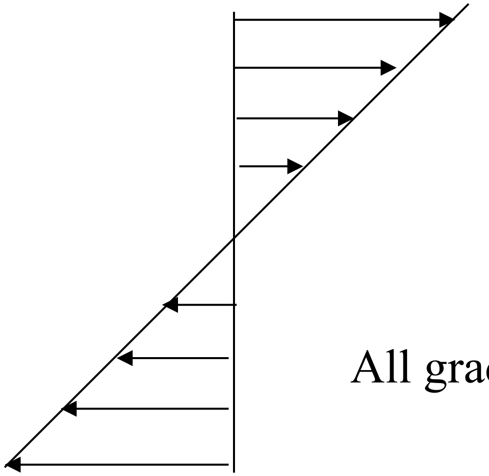
TKE equation

$$\frac{\partial\left(\frac{\overline{u_i'u_i'}}{2}\right)}{\partial t}+\frac{\partial}{\partial x_j}\left(\frac{\overline{u_i'u_i'}}{2}\bar{u}_j+\frac{\overline{u_i'u_i'u_j'}}{2}+\frac{\overline{u_j'p'}}{\rho}-2\nu\left(\overline{u_i's_{ij}'}\right)\right)+\overline{u_i'u_j'}\frac{\partial\bar{u}_i}{\partial x_j}-2\nu\left(\overline{s_{ij}'s_{ij}'}\right)=0$$

The turbulent kinetic energy (TKE) transport equation

$$\frac{\partial\left(\frac{\overline{u_i'u_i'}}{2}\right)}{\partial t}+\frac{\partial}{\partial x_j}\left(\frac{\overline{u_i'u_i'}}{2}\bar{u}_j+\frac{\overline{u_i'u_i'u_j'}}{2}+\frac{\overline{u_j'p'}}{\rho}-2\nu\left(\overline{u_i's_{ij}'}\right)\right)+\overline{u_i'u_j'}\frac{\partial\bar{u}_i}{\partial x_j}-2\nu\left(\overline{s_{ij}'s_{ij}'}\right)=0$$

Consider <u>stationary</u> homogeneous shear flow



$$u=(ky,0,0)$$

All gradients of correlations are zero

$$2\nu\left(\overline{s_{ij}'s_{ij}'}\right)=\overline{u_i'u_j'}\frac{\partial\bar{u}_i}{\partial x_j}$$

Dissipation of TKE equals production of TKE

More generally dissipation of TKE scales with production of TKE

$$\varepsilon \approx \frac{u_0^3}{\delta} \qquad (14.37)$$

Rate-of-strain fluctuations are proportional to the square root of the Reynolds number, ie, they are much larger than mean rates of strain.

$$\sqrt{\overline{s'^{ik} s'^{ki}}} \approx \frac{u_0}{\delta}\left(\frac{u_0 \delta}{2\nu}\right)^{1/2} \qquad (14.38)$$

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Taylor microscale

Let

$$\varepsilon \approx \nu\left(\frac{u_0^2}{\lambda^2}\right). \qquad (14.39)$$

For one parameter flows

$$\frac{\lambda}{\delta} \approx \frac{1}{(R_\delta)^{1/2}} ; \qquad \lambda \approx (\nu(t - t_0))^{1/2} \qquad (14.40)$$

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Kolmogorov microscale

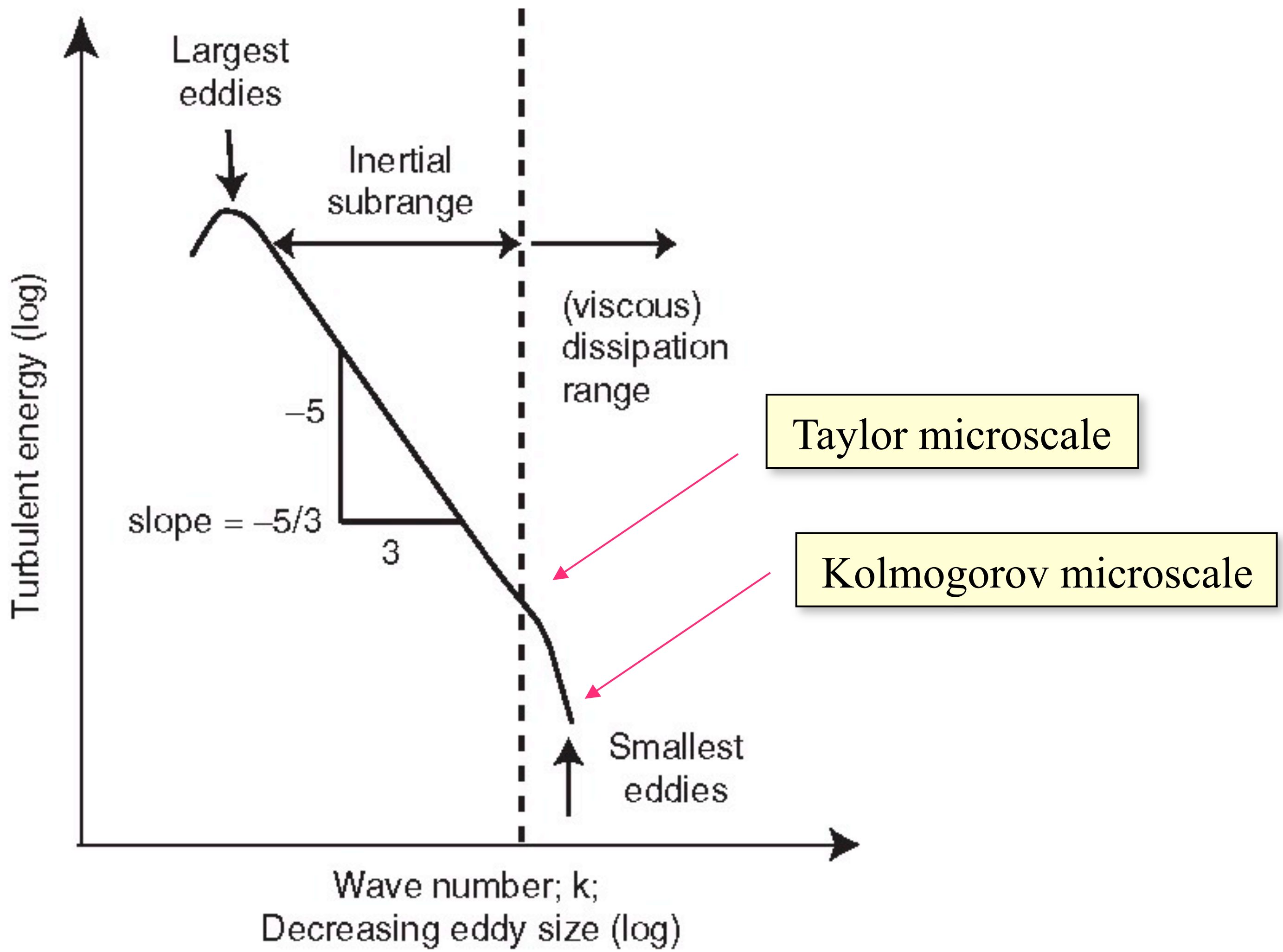
Let

$$\varepsilon \approx \nu\left(\frac{\upsilon^2}{\eta^2}\right) \quad ; \qquad \frac{\upsilon\eta}{\nu} \approx 1 . \tag{14.42}$$

For one parameter flows

$$\frac{\eta}{\delta} = \frac{1}{(R_\delta)^{3/4}} \; ; \qquad \eta = \nu^{\frac{3}{4}} M^{\frac{-1}{2m}} (t - t_0)^{\left(\frac{3}{4} - \frac{k}{2}\right)} \tag{14.43}$$

$$\frac{\upsilon}{u_0} = \frac{1}{(R_\delta)^{1/4}} \; ; \qquad \upsilon = \nu^{\frac{1}{4}} M^{\frac{1}{2m}} (t - t_0)^{\left(\frac{k}{2} - \frac{3}{4}\right)} \tag{14.44}$$


Largest
eddies
Inertial
subrange
(viscous)
dissipation
range
–5
slope = –5/3
3
Turbulent energy (log)
Smallest
eddies
Wave number; k;
Decreasing eddy size (log)
Taylor microscale
Kolmogorov microscale

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Scaling the inertial subrange. Assume

$$M = \varepsilon \approx u_0^3/\delta \qquad (14.46)$$

with units $\hat{u}_0^3/\hat{\delta} = L^2 T^{-3}$ and exponent $k = \frac{3}{2}$.

$$\delta \approx \varepsilon^{1/3}(t-t_0)^{3/2} \; ; \qquad U_0 \approx \varepsilon^{1/3}(t-t_0)^{1/2} \qquad (14.47)$$

$$R_\delta \approx (t-t_0)^2; \qquad \lambda \approx (t-t_0)^{1/2}; \qquad \eta \approx (t-t_0)^0 . \qquad (14.48)$$

The wavenumber of an eddy is essentially the inverse of its length scale.

$$\kappa \approx 1/\delta \, . \qquad (14.49)$$

The TKE as a function of wavenumber should scale as

$$E(\kappa) \approx \frac{u_0^2}{(1/\delta)} \approx \varepsilon^{\frac{3}{2}}(t-t_0)^{\frac{5}{2}} . \qquad (14.50)$$

$$(t-t_0) \approx \frac{\delta^{2/3}}{\varepsilon^{1/3}} = \frac{\kappa^{-2/3}}{\varepsilon^{1/3}} . \qquad (14.51)$$

$$E(\kappa) \approx \varepsilon^{2/3}\kappa^{-5/3} . \qquad (14.52)$$

This is the scaling of TKE first postulated by Kolmogoriv in 1941
and seems to agree with measurements in high Reynolds number flows.

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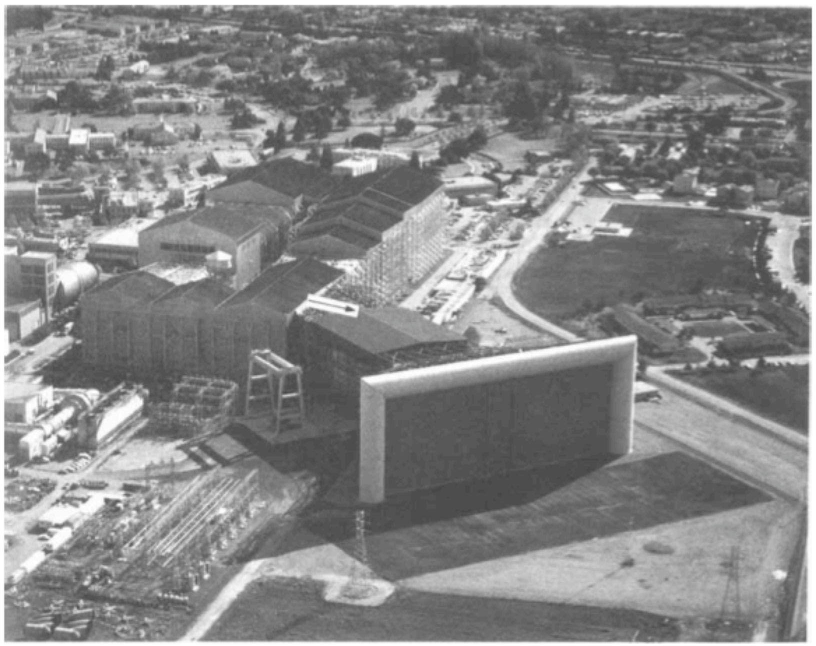

FIGURE 1. An aerial view of the Full-Scale Aerodynamics Facility at NASA Ames Research Center, showing the intake to the 80 × 120 foot test section. The arrow shows our measurement location in the attic.

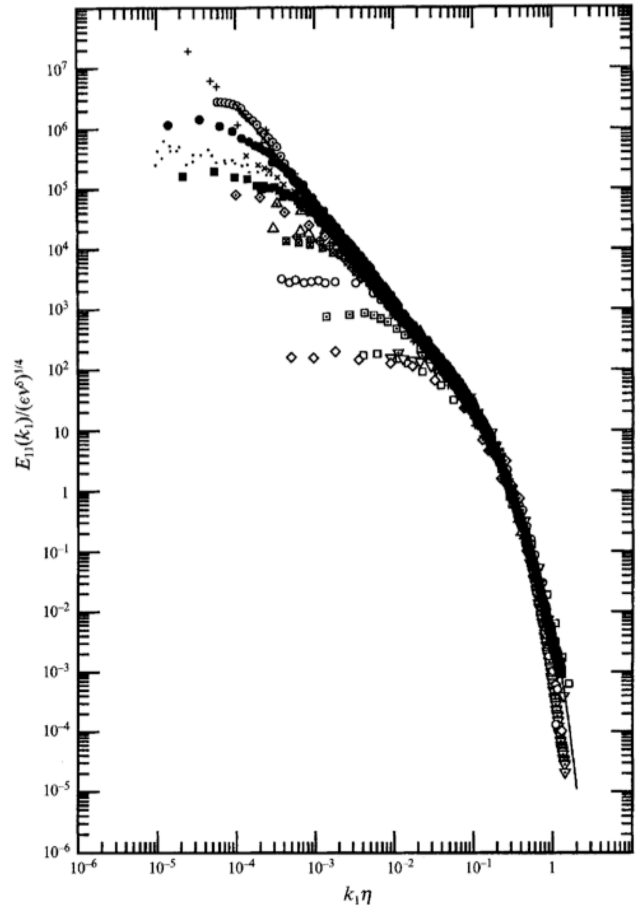

FIGURE 9. Kolmogorov's universal scaling for one-dimensional longitudinal power spectra. The present mid-layer spectra for both free-stream velocities are compared with data from other experiments. This compilation is from Chapman (1979), with later additions. The solid line is from Pao (1965). R_λ: □, 23 boundary layer (Tielman 1967); ◇, 23 wake behind cylinder (Uberoi & Freymuth 1969); ▽, 37 grid turbulence (Comte-Bellot & Corrsin 1971); ▽, 53 channel centreline (Kim & Antonia (DNS) 1991); ⊡, 72 grid turbulence (Comte-Bellot & Corrsin 1971); ○, 130 homogeneous shear flow (Champagne *et al.* 1970); ⊠, 170 pipe flow (Laufer 1954); ⊕, 282 boundary layer (Tielman 1967); ⟐, 308 wake behind cylinder (Uberoi & Freymuth 1969); △, 401 boundary layer (Sanborn & Marshall 1965); △, 540 grid turbulence (Kistler & Vrebalovich 1966); ×, 780 round jet (Gibson 1963); ·, 850 boundary layer (Coantic & Favre 1974); +, ~ 2000 tidal channel (Grant *et al.* 1962); ⊙, 3180 return channel (CAHI Moscow 1991); ●, 1500 boundary layer (present data, mid-layer: $U_e = 50$ m s^{-1}); ■, 600 boundary layer (present data, mid-layer: $U_e = 10$ m s^{-1}).

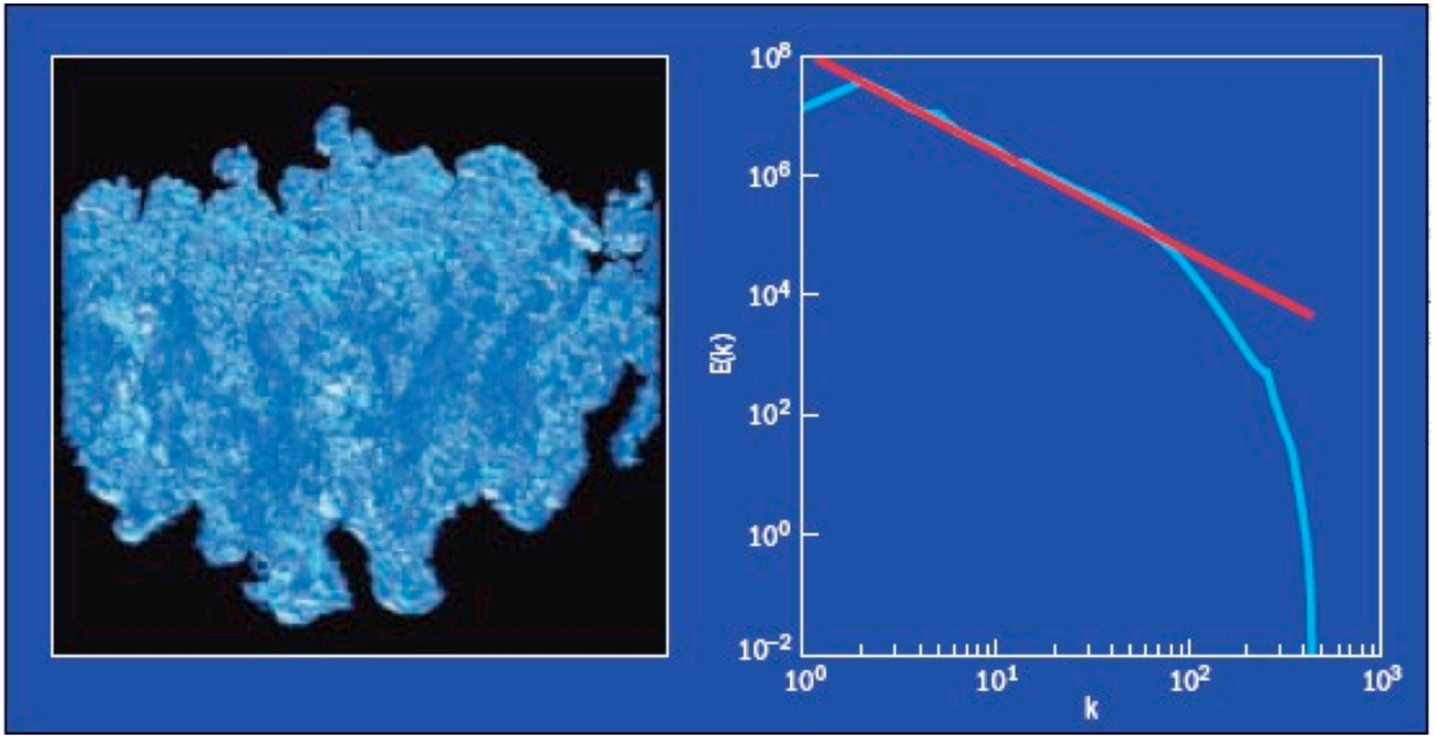

Figure 10. Image of a three-dimensional Rayleigh–Taylor unstable flame in a Type Ia supernova and the computed kinetic energy spectrum (blue curve) exhibiting the classical $k^{5/3}$ decay (red line).

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Scaling of a turbulent vortex ring

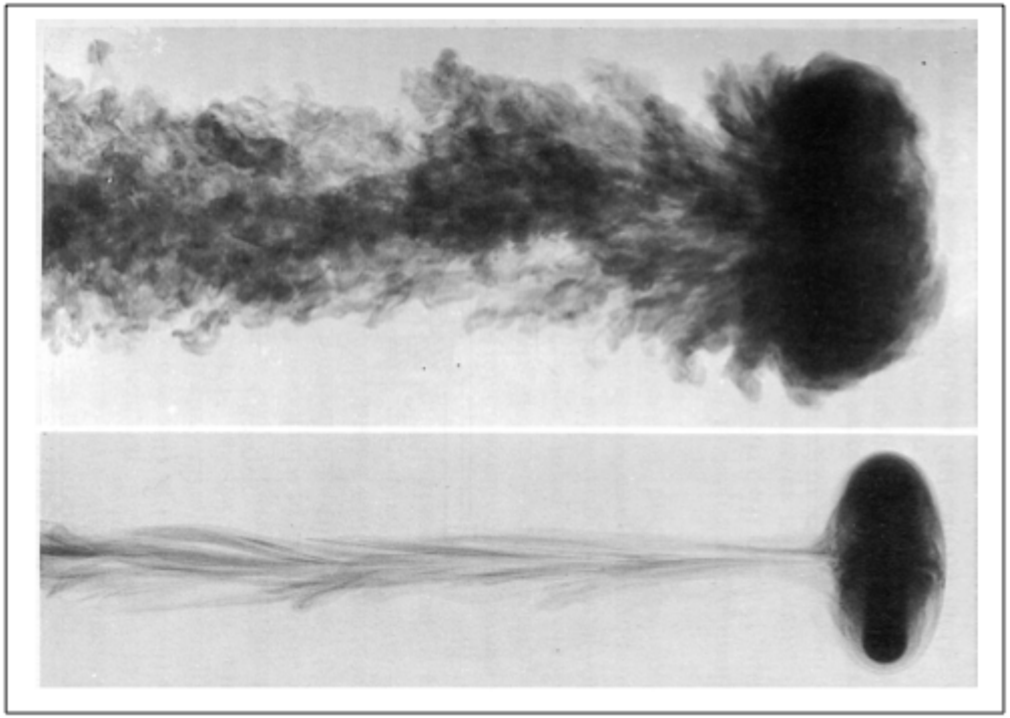

Figure 14.2 *Turbulent and laminar vortex rings produced by an impulsive force from the paper by Glezer and Coles (Ref [14.15]). Top picture initial Reynolds number is* $\Gamma_0 / \nu \cong 27,000$. *Lower picture initial Reynolds number is* $\Gamma_0 / \nu \cong 7,500$.

Experiment to study scaling of a turbulent vortex ring

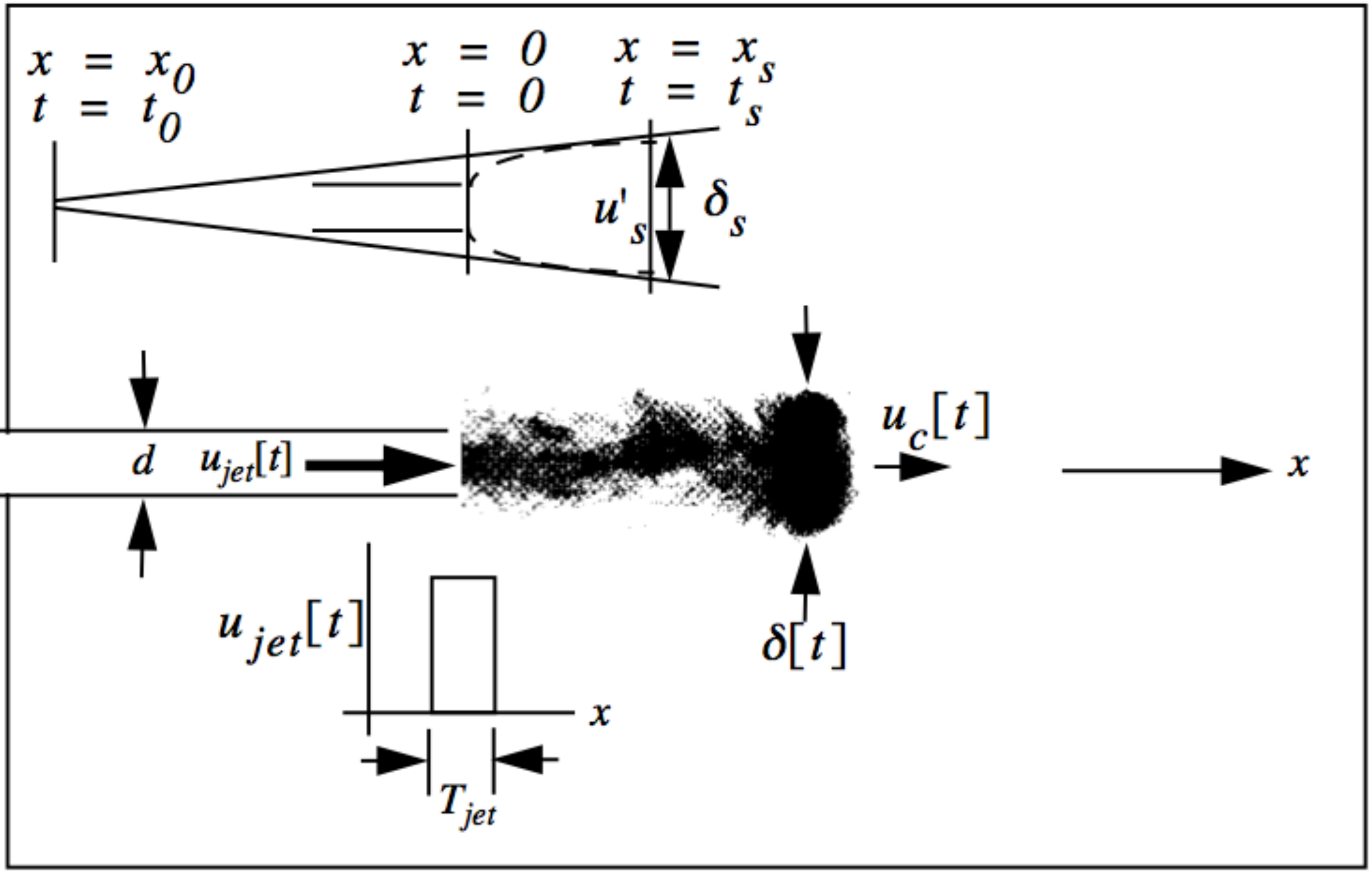

Figure 14.3 Vortex ring apparatus with experimental parameters. The sketch in the upper part of the figure defines parameters used to determine the effective origin of the ring.

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Integral of the motion - the hydrodynamic impulse

$$\frac{3}{2}\int_V u\,dx\,dy\,dz = \int_0^t\int_V \frac{I}{\rho}\delta[x]\delta[y]\delta[z]\delta[t]\,dx\,dy\,dz\,dt = \frac{I}{\rho} \qquad (14.54)$$

$$\delta[t] \cong (I/\rho)^{1/4}(t-t_0)^{1/4} \; ; \; u_0[t] \cong (I/\rho)^{1/4}(t-t_0)^{-3/4} \qquad (14.55)$$

$$\delta \cong (x-x_0) \; ; \quad U_0 \cong (I/\rho)(x-x_0)^{-3} \qquad (14.56)$$

Group

$$\tilde{x}^i = e^a x^i \; ; \; \tilde{t} = e^{4a} t \; ; \; \tilde{\bar{u}}^i = e^{-3a}\bar{u}^i \; ; \; \tilde{\tau}^{ij} = e^{-6a}\tau^{ij} \; ; \; \tilde{\bar{p}} = e^{-6a}\bar{p} \; . \qquad (14.57)$$

$$\frac{U^i}{(I/\rho)^{1/4}(t-t_0)^{-3/4}} = G\left[\frac{\mathbf{x}-\mathbf{x_0}}{(I/\rho)^{1/4}(t-t_0)^{1/4}}\right]. \qquad (14.58)$$

$$\xi = \frac{x-x_0}{(I/\rho)^{1/4}(t-t_0)^{1/4}} \; ; \qquad \eta = \frac{y}{(I/\rho)^{1/4}(t-t_0)^{1/4}} \qquad (14.59)$$

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Streamlines and Particle Paths

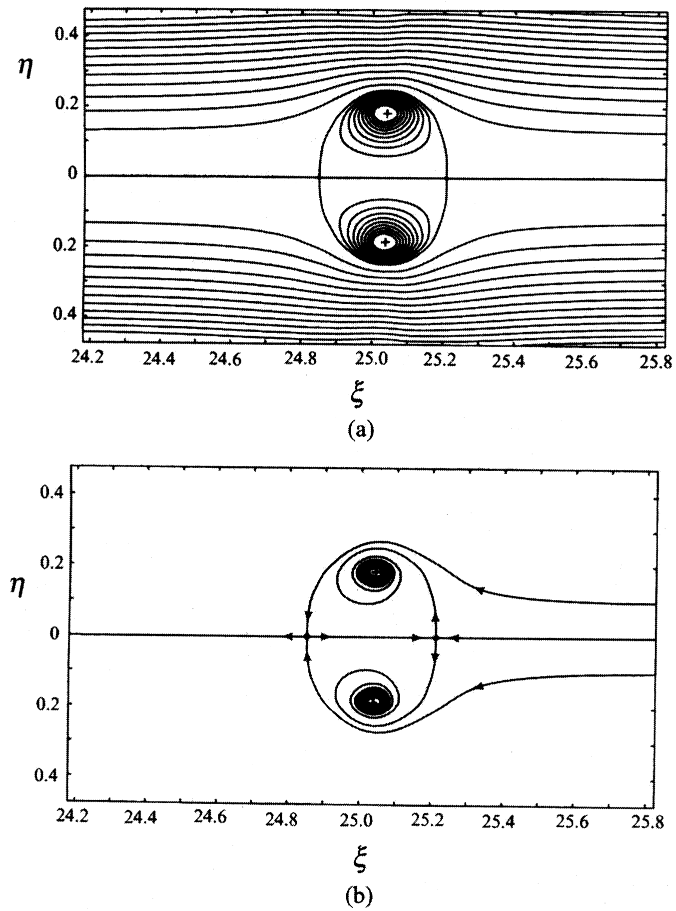

Fig. 13.4. Experimental results from [13.17]: (a) streamline pattern of the ensemble mean velocity field referred to an observer translating to the right with the ring, (b) particle paths of the ensemble mean velocity field.

$$
\left.
\begin{aligned}
\delta[t] &= 0.426(I/\rho)^{1/4}(t-t_0)^{1/4} \\
u_0[t] &= \frac{25.03}{4}(I/\rho)^{1/4}(t-t_0)^{-3/4} = 6.27(I/\rho)^{1/4}(t-t_0)^{-3/4}
\end{aligned}
\right\}
\qquad (14.60)
$$

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Recall that the incompressible Navier-Stokes equations are invariant under a group of arbitrary translations in space.

$$\tilde{x}^j = x^j + a^j[t],$$

$$\tilde{t} = t,$$

$$\tilde{u}^i = u^i + \frac{da^i}{dt},$$

$$\tilde{p} = p - x^j \frac{d^2 a^j}{dt^2} + g[t].$$

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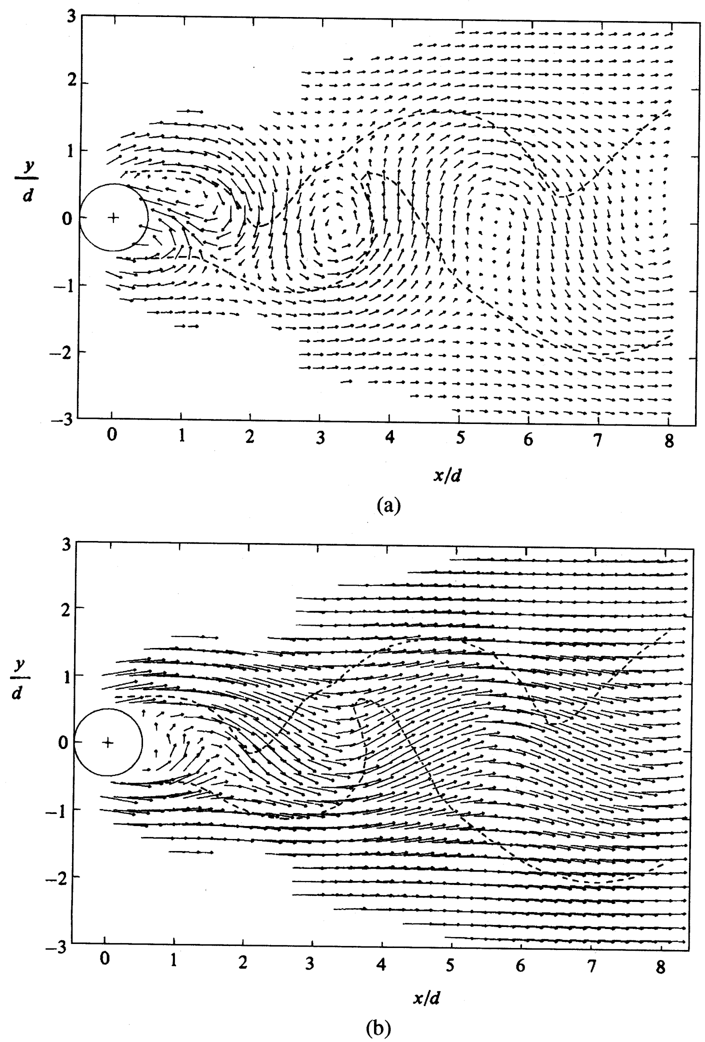

Fig. 11.1. Velocity vector field in the wake of a circular cylinder from Reference [11.6] as viewed by two observers: (a) frame of reference moving downstream at $0.755U_\infty$, (b) frame of reference fixed with respect to the cylinder. The dashed contour roughly corresponds to the instantaneous boundary of turbulence.

Instantaneous flow field in the wake of a circular cylinder as seen by two observers.

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Reduced equations

$$\frac{\partial U^j}{\partial \xi^j} = 0,$$

$$(k-1)U^i + (U^j - k\xi^j)\frac{\partial U^i}{\partial \xi^j} + \frac{1}{\rho}\frac{\partial P}{\partial \xi^i} - \frac{1}{\rho}\frac{\partial}{\partial \xi^j}(T^{ij}) = 0.$$

Particle paths

$$\frac{dx^i}{dt} = u^i[\boldsymbol{x}, t],$$

$$\frac{d\xi^i}{d\tau} = U^i[\boldsymbol{\xi}] - k\xi^i.$$

Frames of reference

$$\tilde{x}^i = x^i - \alpha^i M^{1/m}(t-t_0)^k,$$

$$\tilde{t} = t,$$

$$\tilde{\bar{u}}^i = \bar{u}^i - k\alpha^i M^{1/m}(t-t_0)^{k-1},$$

$$\tilde{\bar{p}} = \bar{p} + x^j k(k-1)\alpha^j M^{1/m}(t-t_0)^{k-2},$$

Particle paths in similarity coordinates do not depend on the observer

$$\tilde{\xi}^i = \xi^i - \alpha^i,$$

$$\tilde{\tau} = \tau,$$

$$\tilde{U}^i = U^i - k\alpha^i,$$

$$\tilde{P} = P + \alpha^j \xi^j k(k-1).$$

$$\frac{d\tilde{\xi}^i}{d\tilde{\tau}} = \frac{d\xi^i}{d\tau},$$

$$\tilde{U}^i[\tilde{\boldsymbol{\xi}}] - k\tilde{\xi}^i = U^i[\boldsymbol{\xi}] - k\xi^i.$$

13.4 Flow past a flat plate of length L is shown in Figure 13.13. Assume an attached laminar Blasius boundary layer over the length of the plate. Show that the drag per unit span of the plate is proportional to $U_\infty^{3/2} L^{1/2}$. How would you expect the turbulence intensity u' to depend on U_∞ and L at a fixed point x in the far wake?

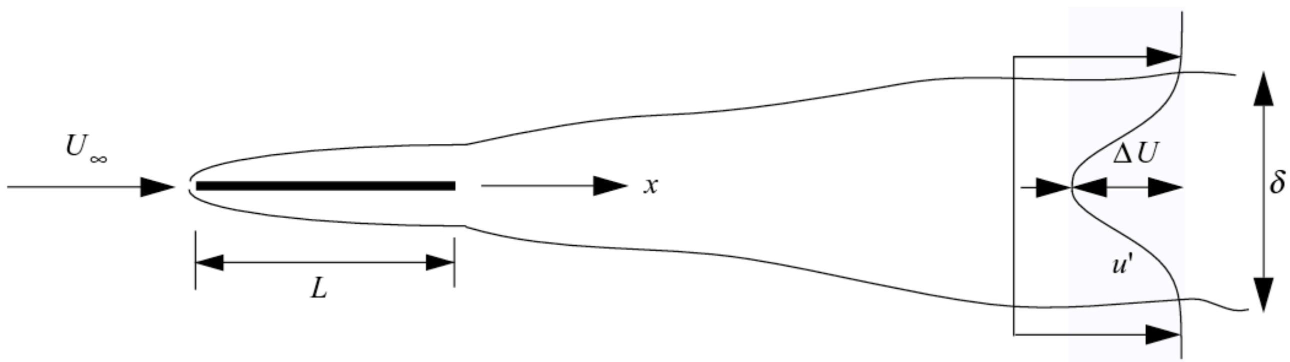

Fig. 13.13.

13.5 Solve the turbulent counterpart of Exercise 13.4. Assume an attached turbulent boundary layer over the length of the plate. The local skin friction coefficient can be taken as $C_f = 0.06(U_\infty x/\nu)^{-1/5}$. How would you expect the turbulence intensity u' to depend on U_∞ and L at a fixed point x in the far wake?

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Derivation of the turbulent kinetic energy equation

Turbulent kinetic energy equation

Momentum and continuity equations

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}\left(u_i u_j + \frac{p}{\rho}\delta_{ij} - 2\nu s_{ij} \right) = 0$$

$$s_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

Kinetic energy equation - project the momentum equation onto the velocity vector

$$u_i \frac{\partial u_i}{\partial t} + u_i \frac{\partial}{\partial x_j}\left(u_i u_j + \frac{p}{\rho}\delta_{ij} - 2\nu s_{ij} \right) = 0$$

$$u_i \frac{\partial u_i}{\partial t} = \frac{\partial \left(\frac{u_i u_i}{2} \right)}{\partial t}$$

$$u_i \frac{\partial u_i u_j}{\partial x_j} = u_i u_j \frac{\partial u_i}{\partial x_j} + u_i u_i \frac{\partial u_j}{\partial x_j} = u_i u_j \frac{\partial u_i}{\partial x_j}$$

$$\frac{\partial \left(\frac{u_i u_i}{2} u_j \right)}{\partial x_j} = u_i u_j \frac{\partial u_i}{\partial x_j} + \frac{u_i u_i}{2} \frac{\partial u_j}{\partial x_j} = u_i u_j \frac{\partial u_i}{\partial x_j}$$

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Viscous and pressure terms

$$u_i \frac{\partial (p/\rho)\delta_{ij}}{\partial x_j} = \frac{\partial (p/\rho) u_i \delta_{ij}}{\partial x_j} + (p/\rho)\frac{\partial u_i \delta_{ij}}{\partial x_j} = \frac{\partial (p/\rho) u_i \delta_{ij}}{\partial x_j}$$

$$\frac{\partial}{\partial x_j}(u_i s_{ij}) = u_i \frac{\partial s_{ij}}{\partial x_j} + s_{ij}\frac{\partial u_i}{\partial x_j} = u_i \frac{\partial s_{ij}}{\partial x_j} + s_{ij}(s_{ij} + w_{ij}) = u_i \frac{\partial s_{ij}}{\partial x_j} + s_{ij} s_{ij}$$

$$s_{ij} s_{ij} = 0$$

$$u_i \frac{\partial s_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}(u_i s_{ij}) - s_{ij} s_{ij}$$

Kinetic energy equation

$$u_i \frac{\partial u_i}{\partial t} + u_i \frac{\partial}{\partial x_j}\left(u_i u_j + \frac{p}{\rho}\delta_{ij} - 2\nu s_{ij} \right) = 0$$

$$\frac{\partial \left(\frac{u_i u_i}{2} \right)}{\partial t} + \frac{\partial \left(\frac{u_i u_i}{2} u_j \right)}{\partial x_j} + \frac{\partial (p/\rho) u_i \delta_{ij}}{\partial x_j} - 2\nu \frac{\partial (u_i s_{ij})}{\partial x_j} + 2\nu s_{ij} s_{ij} = 0$$

$$\frac{\partial \left(\frac{u_i u_i}{2} \right)}{\partial t} + \frac{\partial}{\partial x_j}\left(\frac{u_i u_i}{2} u_j + u_j \frac{p}{\rho} - 2\nu u_i s_{ij} \right) + 2\nu s_{ij} s_{ij} = 0$$

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Insert fluctuations and average

$$\frac{\partial\left(\frac{u_i u_i}{2}\right)}{\partial t}+\frac{\partial}{\partial x_j}\left(\frac{u_i u_i}{2}u_j+u_j\frac{p}{\rho}-2\nu u_i S_{ij}\right)+2\nu s_{ij}s_{ij}=0$$

$$\frac{\partial\left(\frac{\overline{u}_i\overline{u}_i}{2}+\frac{\overline{u_i'u_i'}}{2}\right)}{\partial t}+\frac{\partial}{\partial x_j}\left(\frac{u_i u_i}{2}u_j+u_j\frac{p}{\rho}-2\nu u_i s_{ij}\right)+2\nu s_{ij}s_{ij}=0$$

$$\left(\overline{u}_i+u_i'\right)\left(\overline{u}_i+u_i'\right)\left(\overline{u}_j+u_j'\right)=\left(\overline{u}_i\overline{u}_i+2u_i'\overline{u}_i+u_i'u_i'\right)\left(\overline{u}_j+u_j'\right)$$

$$\left(\overline{u}_i\overline{u}_i\overline{u}_j+2u_i'\overline{u}_i\overline{u}_j+u_i'u_i'\overline{u}_j\right)+\left(\overline{u}_i\overline{u}_i u_j'+2u_i'u_j'\overline{u}_i+u_i'u_i'u_j'\right)$$

$$\left(\frac{\overline{u}_i\overline{u}_i}{2}\overline{u}_j+\frac{\overline{u_i'u_i'}}{2}\overline{u}_j\right)+\left(\overline{u_i'u_j'}\overline{u}_i+\frac{\overline{u_i'u_i'u_j'}}{2}\right)$$

$$\frac{\partial\left(\frac{\overline{u}_i\overline{u}_i}{2}+\frac{\overline{u_i'u_i'}}{2}\right)}{\partial t}+\frac{\partial}{\partial x_j}\left(\left(\frac{\overline{u}_i\overline{u}_i}{2}\overline{u}_j+\frac{\overline{u_i'u_i'}}{2}\overline{u}_j\right)+\left(\overline{u_i'u_j'}\overline{u}_i+\frac{\overline{u_i'u_i'u_j'}}{2}\right)+\overline{u}_j\frac{\overline{p}}{\rho}+\frac{\overline{u_j'p'}}{\rho}-2\nu\left(\overline{u}_i\overline{s}_{ij}+\overline{u_i's_{ij}'}\right)\right)+2\nu\left(\overline{s}_{ij}\overline{s}_{ij}+\overline{s_{ij}'s_{ij}'}\right)=0$$

Project the mean momentum equation onto the mean velocity vector

$$\bar{u}_i \frac{\partial \bar{u}_i}{\partial t} + \bar{u}_i \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} + \bar{u}_i \frac{\partial \overline{u_i' u_j'}}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\frac{\bar{p}\bar{u}_j}{\rho}\right) - 2\nu \bar{u}_i \frac{\partial \bar{s}_{ij}}{\partial x_j} = 0$$

$$\bar{u}_i \frac{\partial}{\partial x_j}\left(\bar{s}_{ij}\right) = \frac{\partial}{\partial x_j}\left(\bar{u}_i \bar{s}_{ij}\right) - \bar{s}_{ij}\bar{s}_{ij}$$

$$\frac{\partial\left(\dfrac{\bar{u}_i \bar{u}_i}{2}\right)}{\partial t} + \bar{u}_j \frac{\partial\left(\dfrac{\bar{u}_i \bar{u}_i}{2}\right)}{\partial x_j} + \bar{u}_i \frac{\partial \overline{u_i' u_j'}}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\frac{\bar{p}\bar{u}_j}{\rho}\right) - 2\nu \frac{\partial}{\partial x_j}\left(\bar{u}_i \bar{s}_{ij}\right) + 2\nu \bar{s}_{ij}\bar{s}_{ij} = 0$$

$$\frac{\partial\left(\frac{\bar{u}_i\bar{u}_i}{2}+\frac{\overline{u_i'u_i'}}{2}\right)}{\partial t}+\frac{\partial}{\partial x_j}\left(\left(\frac{\bar{u}_i\bar{u}_i}{2}\bar{u}_j+\frac{\overline{u_i'u_i'}}{2}\bar{u}_j\right)+\left(\overline{u_i'u_j'}\bar{u}_i+\frac{\overline{u_i'u_i'u_j'}}{2}\right)+\bar{u}_j\frac{\bar{p}}{\rho}+\frac{\overline{u_j'p'}}{\rho}-2\nu\left(\bar{u}_i\bar{s}_{ij}+\overline{u_i's_{ij}'}\right)\right)+2\nu\left(\bar{s}_{ij}\bar{s}_{ij}+\overline{s_{ij}'s_{ij}'}\right)=0$$

$$\frac{\partial\left(\frac{\bar{u}_i\bar{u}_i}{2}\right)}{\partial t}+\bar{u}_j\frac{\partial\left(\frac{\bar{u}_i\bar{u}_i}{2}\right)}{\partial x_j}+\bar{u}_i\frac{\partial\overline{u_i'u_j'}}{\partial x_j}+\frac{\partial}{\partial x_j}\left(\frac{\bar{p}\bar{u}_j}{\rho}\right)-2\nu\frac{\partial}{\partial x_j}\left(\bar{u}_i\bar{s}_{ij}\right)+2\nu\bar{s}_{ij}\bar{s}_{ij}=0$$

$$\frac{\partial\left(\frac{\bar{u}_i\bar{u}_i}{2}+\frac{\overline{u_i'u_i'}}{2}\right)}{\partial t}+\frac{\partial}{\partial x_j}\left(\left(\frac{\bar{u}_i\bar{u}_i}{2}\bar{u}_j+\frac{\overline{u_i'u_i'}}{2}\bar{u}_j\right)+\left(\overline{u_i'u_j'}\bar{u}_i+\frac{\overline{u_i'u_i'u_j'}}{2}\right)+\bar{u}_j\frac{\bar{p}}{\rho}+\frac{\overline{u_j'p'}}{\rho}-2\nu\left(\bar{u}_i\bar{s}_{ij}+\overline{u_i's_{ij}'}\right)\right)+2\nu\left(\bar{s}_{ij}\bar{s}_{ij}+\overline{s_{ij}'s_{ij}'}\right)=0$$

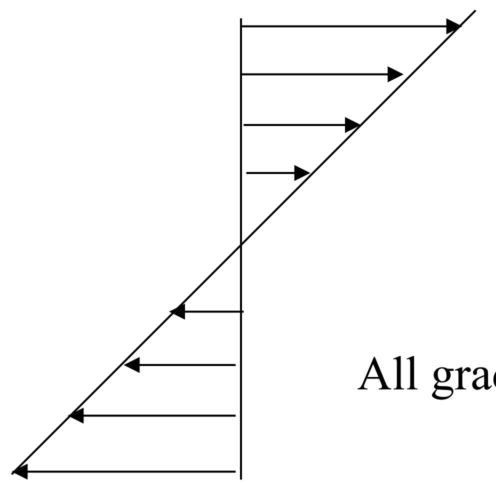
$$\frac{\partial\left(\frac{\bar{u}_i\bar{u}_i}{2}\right)}{\partial t}+\bar{u}_j\frac{\partial\left(\frac{\bar{u}_i\bar{u}_i}{2}\right)}{\partial x_j}+\bar{u}_i\frac{\partial\overline{u_i'u_j'}}{\partial x_j}+\frac{\partial}{\partial x_j}\left(\frac{\bar{p}\bar{u}_j}{\rho}\right)-2\nu\frac{\partial}{\partial x_j}\left(\bar{u}_i\bar{s}_{ij}\right)+2\nu\bar{s}_{ij}\bar{s}_{ij}=0$$

$$\frac{\partial\left(\frac{\overline{u_i'u_i'}}{2}\right)}{\partial t}+\frac{\partial}{\partial x_j}\left(\frac{\overline{u_i'u_i'}}{2}\bar{u}_j+\frac{\overline{u_i'u_i'u_j'}}{2}+\frac{\overline{u_j'p'}}{\rho}-2\nu\left(\overline{u_i's_{ij}'}\right)\right)+\overline{u_i'u_j'}\frac{\partial\bar{u}_i}{\partial x_j}+2\nu\left(\overline{s_{ij}'s_{ij}'}\right)=0$$

The turbulent kinetic energy (TKE) transport equation is

$$\frac{\partial\left(\frac{\overline{u_i'u_i'}}{2}\right)}{\partial t}+\frac{\partial}{\partial x_j}\left(\frac{\overline{u_i'u_i'}}{2}\bar{u}_j+\frac{\overline{u_i'u_i'u_j'}}{2}+\frac{\overline{u_j'p'}}{\rho}-2\nu\left(\overline{u_i's_{ij}'}\right)\right)+\overline{u_i'u_j'}\frac{\partial\bar{u}_i}{\partial x_j}+2\nu\left(\overline{s_{ij}'s_{ij}'}\right)=0$$

Consider homogeneous shear flow



$$u=\left(ky,0,0\right)$$

All gradients of correlations are zero

$$2\nu\left(\overline{s_{ij}'s_{ij}'}\right)=-\overline{u_i'u_j'}\frac{\partial\bar{u}_i}{\partial x_j}$$

Dissipation of TKE equals production of TKE

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Fine scale motions responsible for kinetic energy dissipation.

Recall

$$\varepsilon \propto \frac{u_0^3}{\delta}, \tag{13.40}$$

$$\sqrt{\overline{s'^{ik} s'^{ki}}} \propto \frac{u_0}{\delta}\left(\frac{u_0 \delta}{2\nu}\right)^{1/2} \tag{13.41}$$

Introduce the Taylor microscale

$$\varepsilon \propto \nu\left(\frac{u_0^2}{\lambda^2}\right). \tag{13.42}$$

$$\frac{\lambda}{\delta} \propto \frac{1}{(R_\delta)^{1/2}}, \qquad \lambda \propto (\nu(t - t_0))^{1/2}. \tag{13.43}$$

Introduce the Kolmogorov microscales

$$\varepsilon \propto \nu\left(\frac{\upsilon^2}{\eta^2}\right), \qquad \frac{\upsilon\eta}{\nu} \approx 1. \qquad (13.45)$$

$$\frac{\eta}{\delta} \propto \frac{1}{(R_\delta)^{3/4}}, \qquad \eta \propto \nu^{3/4} M^{-1/2m} (t - t_0)^{3/4 - k/2} \qquad (13.46)$$

$$\frac{\upsilon}{u_0} \propto \frac{1}{(R_\delta)^{1/4}}, \qquad \upsilon \propto \nu^{1/4} M^{1/2m} (t - t_0)^{k/2 - 3/4}. \qquad (13.47)$$

Fine scale velocity gradients

$$\frac{u_0}{\lambda} \propto \frac{\upsilon}{\eta} \propto \nu^{-1/2} M^{1/m} (t - t_0)^{k - 3/2}. \qquad (13.48)$$

Large scale velocity gradients

$$\frac{u_0}{\delta} \propto \frac{1}{t - t_0} \qquad (t > t_0), \qquad (13.44)$$

Turbulent kinetic energy spectrum