Low Reynolds number flow

Take the curl of the incompressible momentum equation

$$\nabla \times \left(\frac{\partial \overline{U}}{\partial t} + \overline{U} \cdot \nabla \overline{U} + \frac{1}{\rho} \nabla P - \nu \nabla^2 \overline{U} \right) = 0$$

The result is the transport equation for the vorticity

$$\frac{\partial \overline{\Omega}}{\partial t} + \overline{U} \cdot \nabla \overline{\Omega} - \overline{\Omega} \cdot \nabla \overline{U} = \nu \nabla^2 \overline{\Omega}$$

If the flow is steady and the velocity is very small the equation reduces to

$$\nabla^2 \overline{\Omega} = 0$$

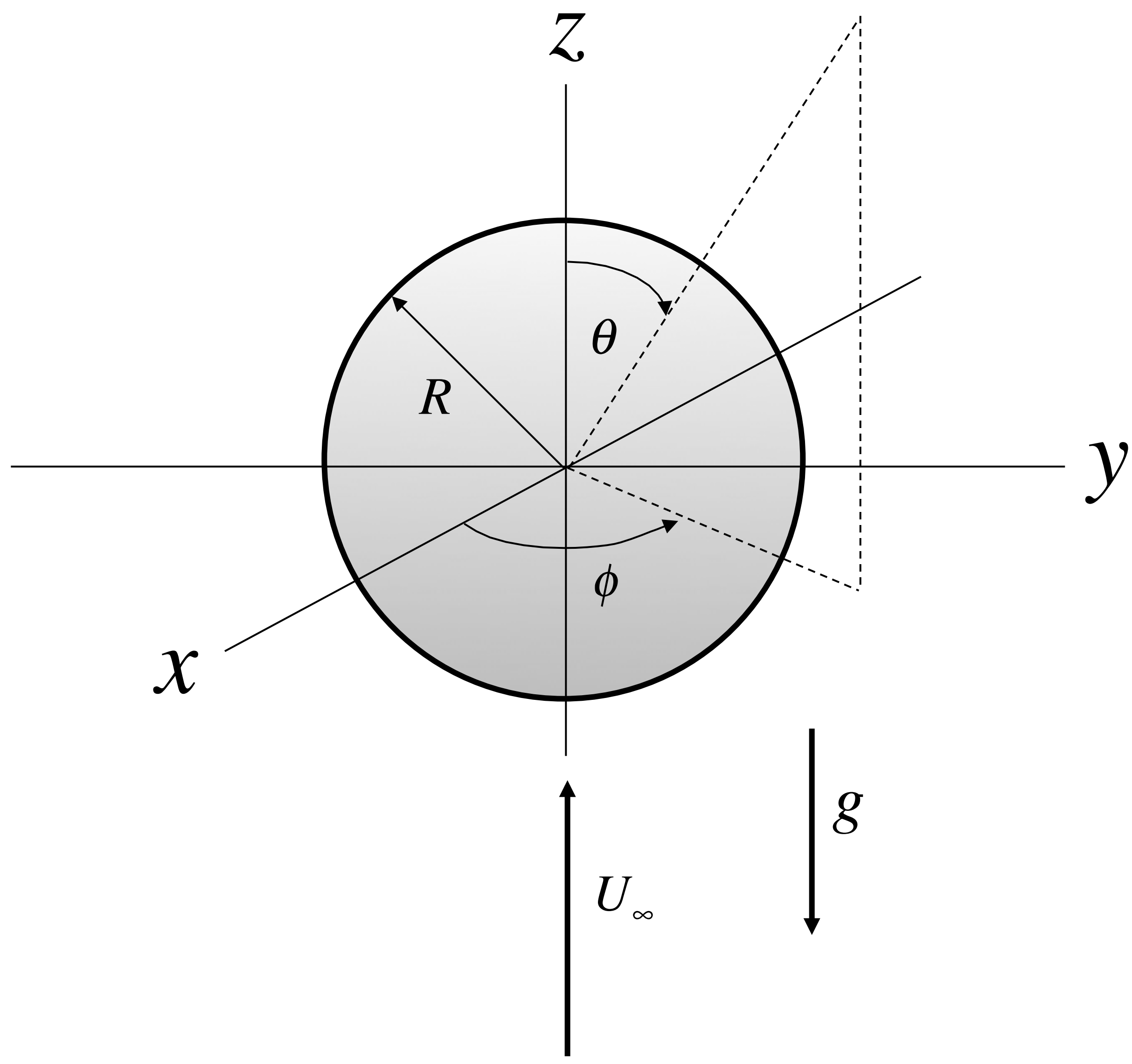
Recall the Poisson equation for the vector potential

$$\nabla^2 \overline{A} = -\overline{\Omega}$$

Low Reynolds number flow is governed by the biharmonic equation

$$\nabla^2 \left(\nabla^2 \overline{A} \right) = 0$$

Viscous flow past a sphere at low Reynolds number



The Stokes stream function

The flow is axisymmetric and best posed in spherical polar coordinates

$$\nabla^2\left(\nabla^2\overline{A}\right)=\left(\frac{\partial^2}{\partial r^2}+\frac{Sin(\theta)}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{Sin(\theta)}\frac{\partial}{\partial\theta}\right)\right)^2\Psi=0$$

Velocities

$$U_r=-\frac{1}{r^2Sin(\theta)}\frac{\partial\Psi}{\partial\theta}$$

$$U_\theta=\frac{1}{rSin(\theta)}\frac{\partial\Psi}{\partial r}$$

Boundary conditions

No-slip condition

$$U_r\left(R,\theta\right)=0$$

$$U_\theta\left(R,\theta\right)=0$$

Uniform flow at infinity

$$\lim_{r\to\infty}\Psi\to-\frac{1}{2}U_\infty r^2Sin^2\left(\theta\right)$$

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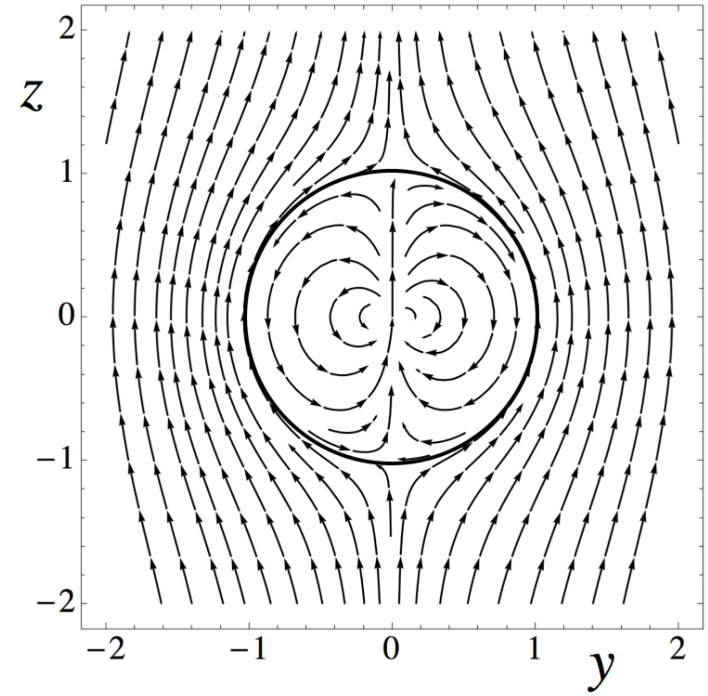


Assume

$$\Psi = f(r)\,Sin^2(\theta)$$

$$f(r) = \frac{a}{r} + br + cr^2 + dr^4$$

Solution

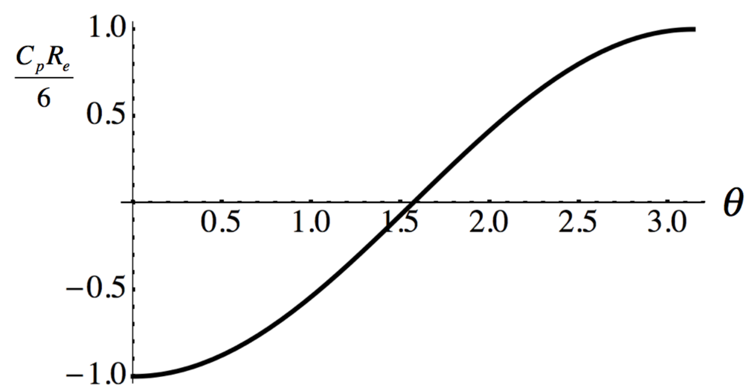
$$\frac{\Psi}{U_\infty R^2} = \left(-\frac{1}{2}\left(\frac{r}{R}\right)^2 + \frac{3}{4}\left(\frac{r}{R}\right) - \frac{1}{4}\left(\frac{r}{R}\right)^{-1}\right) Sin^2(\theta)$$

$$\frac{U_r}{U_\infty} = \left(1 - \frac{3}{2}\left(\frac{r}{R}\right)^{-1} + \frac{1}{2}\left(\frac{r}{R}\right)^{-3}\right) Cos(\theta)$$

$$\frac{U_\theta}{U_\infty} = \left(-1 + \frac{3}{4}\left(\frac{r}{R}\right)^{-1} + \frac{1}{4}\left(\frac{r}{R}\right)^{-3}\right) Sin(\theta)$$

Viscous stress

$$\frac{\tau_{r\theta} R}{\mu U_\infty} = \frac{3}{2}\left(\frac{r}{R}\right)^{-4} Sin(\theta)$$



Pressure

$$\frac{(P - P_\infty)R}{\mu U_\infty} = -\frac{\rho g R^2}{\mu U_\infty}\left(\frac{z}{R}\right) - \frac{3}{2}\left(\frac{r}{R}\right)^2 Cos(\theta)$$

Drag components

$$F_{z_{Pressure}} = -\int_0^{2\pi}\int_0^{\pi}\left(P(R,\theta)Cos(\theta)\right)R^2 Sin(\theta)d\theta d\phi = \frac{4}{3}\pi R^3\rho g + 2\pi\mu RU_\infty$$

$$F_{z_{Viscous}} = \int_0^{2\pi}\int_0^{\pi}\left(\tau_{r\theta}(R,\theta)Sin(\theta)\right)R^2 Sin(\theta)d\theta d\phi = 4\pi\mu RU_\infty$$

$$F_z = \underbrace{\frac{4}{3}\pi R^3\rho g}_{\text{Buoyancy force}} + \underbrace{2\pi\mu RU_\infty}_{\text{Pressure drag}} + \underbrace{4\pi\mu RU_\infty}_{\text{Viscous drag}}$$

Buoyancy force

Pressure drag

Viscous drag

Non buoyant pressure plus viscous drag

$$D_{Stokes} = 6\pi\mu R U_{\infty}$$

Reynolds number

$$R_e = \frac{\rho U_{\infty}\left(2R\right)}{\mu}$$

$$C_D = \frac{D_{Stokes}}{\frac{1}{2}\rho U_{\infty}^2\left(\pi R^2\right)} = \frac{12\pi\mu R U_{\infty}}{\rho U_{\infty}^2\left(\pi R^2\right)} = \frac{12\mu}{\rho U_{\infty} R} = \frac{24}{R_e}$$

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Dissipation of kinetic energy by viscous friction

$$\frac{\varepsilon}{\mu} = \frac{\tau_{ij}}{\mu}\frac{\partial U_i}{\partial x_j} = 2S_{ij}S_{ij}$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

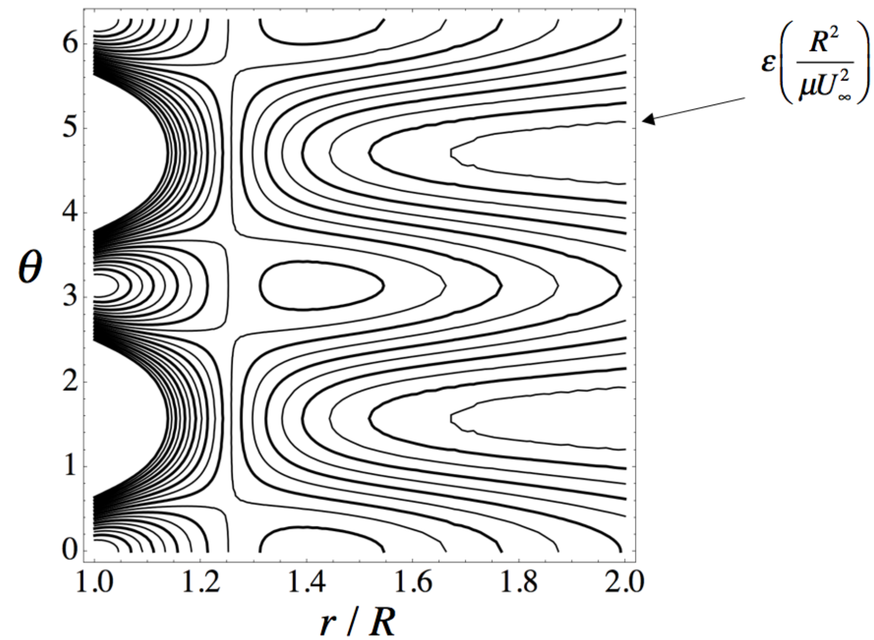
Axisymmetric flow in spherical polar coordinates

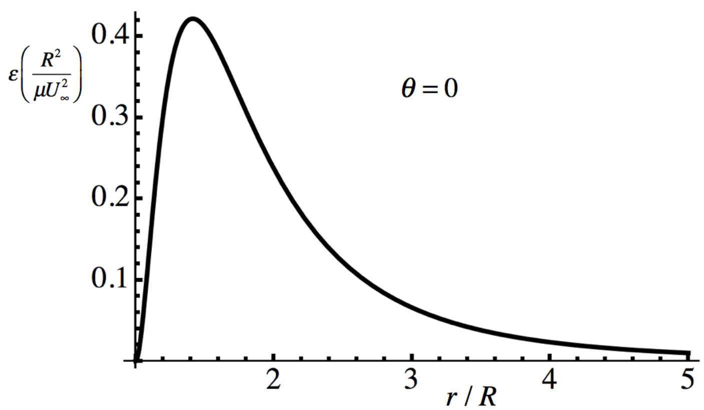
$$\frac{\varepsilon}{\mu} = 2\left(\frac{\partial U_r}{\partial r}\right)^2 + 2\left(\frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r}\right)^2 + 2\left(\frac{U_r}{r} + \frac{U_\theta}{r}Cot(\theta)\right)^2 + \left(\frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r} + \frac{1}{r}\frac{\partial U_r}{\partial \theta}\right)^2$$

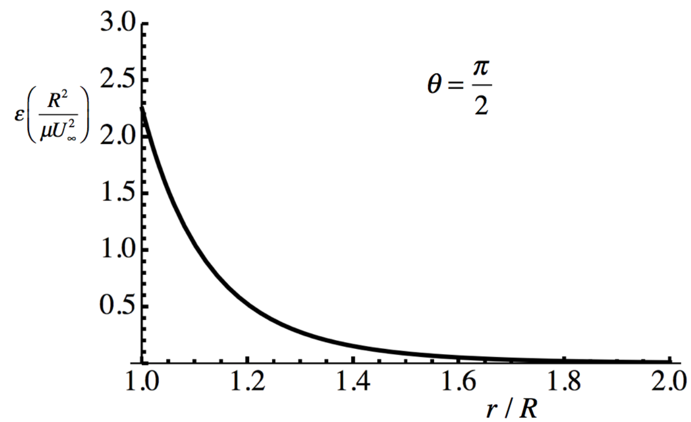
Low Reynolds number flow over a sphere

$$\frac{\varepsilon R^2}{\mu U_\infty^2} = \frac{9}{4}\left(\frac{r}{R}\right)^{-8}\left(3\left(\left(\frac{r}{R}\right)^2 - 1\right)^2 Cos^2(\theta) + Sin^2(\theta)\right)$$

Integrate the kinetic energy dissipation over the flow volume out to infinity







$$\int_0^{2\pi}\int_0^{\pi}\int_R^{\infty}\varepsilon r^2 Sin(\theta)\,dr\,d\theta\,d\phi =$$

$$\frac{9}{4}\mu U^2{}_\infty R\int_0^{2\pi}\int_0^{\pi}\int_1^{\infty}\left(\frac{r}{R}\right)^{-8}\left(3\left(\left(\frac{r}{R}\right)^2-1\right)^2 Cos^2(\theta)+Sin^2(\theta)\right)\left(\frac{r}{R}\right)^2 Sin(\theta)\,d\left(\frac{r}{R}\right)d\theta\,d\phi=(6\pi\mu U_\infty R)U_\infty = D_{Stokes}U_\infty$$