Chapter 4

Kinematics of fluid motion

4.1 Elementary flow patterns

Recall the discussion of flow patterns in Chapter 1. The equations for particle paths in a three-dimensional, steady fluid flow are

\[
\frac{dx}{dt} = U(\bar{x}) \quad \frac{dy}{dt} = V(\bar{x}) \quad \frac{dz}{dt} = W(\bar{x}).
\]

(4.1)

Although the position of a particle depends on time as it moves with the flow, the flow pattern itself does not depend on time and the system (4.1) is said to be autonomous. Autonomous systems of differential equations arise in a vast variety of applications in mechanics, from the motions of the planets to the dynamics of pendulums to velocity vector fields in steady fluid flow. A great deal about the flow can be learned by plotting the velocity vector field \( U_i(\bar{x}) \). When the flow pattern is plotted one notices that among the most prominent features are stagnation points also known as critical points that occur where

\[
U_i(\bar{x}_c) = 0.
\]

(4.2)

Quite often the qualitative features of the flow can be almost completely described once the critical points of the flow field have been identified and classified.

4.1.1 Linear flows

If the \( U_i(\bar{x}) \) are analytic functions of \( x \); The velocity field can be expanded in a Taylor
series about the critical point and the result can be used to gain valuable information about
the geometry of the flow field. Retaining just the lowest order term in the expansion of
$U_i(\bar{x})$ the result is a linear system of equations.

$$\frac{dx_i}{dt} = A_{ik} (x_k - x_{ck}) + O \left((x_k - x_{ck})^2\right) + \cdots \quad (4.3)$$

The matrix $A_{ik}$ is the gradient tensor of the velocity field evaluated at the critical point
and $\bar{x}_c$ is the position vector of the critical point.

$$A_{ik} = \left( \frac{\partial U_i}{\partial x_k} \right)_{\bar{x}=\bar{x}_c} \quad (4.4)$$

The linear, local solution is expressed in terms of exponential functions and only a relatively
small number of solution patterns are possible. These are determined by the invariants
of $A_{ik}$. The invariants arise naturally as traces of various powers of $A_{ik}$. They are all
derived as follows. Transform $A_{nm}$

$$B_{ik} = M_{in} A_{nm} \bar{M}_{mk} \quad (4.5)$$

where $M$ is a non-singular matrix and $\bar{M}$ is its inverse. Take the trace of (4.5). The dot
(or inner) product of a vector and a tensor is

$$B_{ii} = M_{in} A_{nm} \bar{M}_{mi} = \bar{M}_{mi} M_{in} A_{nm} = \delta_{mn} A_{nm} = A_{mm} \quad (4.6)$$

The trace is invariant under the affine transformation $M_{ik}$. One can think of the vector
field $U_i$ as if it is imbedded in an $n$-dimensional block of rubber. An affine transformation
is one which stretches or distorts the rubber block without ripping it apart or reflecting it
through itself. For traces of higher powers the proof of invariance is similar.

$$tr \left(B^o\right) =$$

$$M_{jn_1} A_{n_1 m_1} M_{m_1 j_1} M_{j_1 n_2} A_{n_2 m_2} M_{m_2 j_2} \cdots M_{j_{a-1} n_a} A_{n_a m_a} \bar{M}_{m_a j} = \quad (4.7)$$

$$= tr \left(A^o\right)$$

The traces of all powers of the gradient tensor remain invariant under an affine transfor-
mation. Likewise any combination of the traces is invariant.
4.1.2 Linear flows in two dimensions

In two dimensions the eigenvalues of $A_{ik}$ satisfy the quadratic

$$\lambda^2 + P\lambda + Q = 0$$

(4.8)

where $P$ and $Q$ are the invariants

$$P = -A_{ii}$$

$$Q = \text{Det} (A_{ik}) .$$

(4.9)

The eigenvalues of $A_{ik}$ are

$$\lambda = -\frac{P}{2} \pm \frac{1}{2} \sqrt{P^2 - 4Q}$$

(4.10)

and the character of the local flow is determined by the quadratic discriminant

$$D = Q - \frac{P^2}{4} .$$

(4.11)

![Figure 4.1: Classification of linear flows in two dimensions](image)

The various possible flow patterns can be summarized on a cross-plot of the invariants as shown in Figure 4.1. If $D > 0$ the eigenvalues are complex and a spiraling motion can be
expected in the neighborhood of the critical point. Depending on the sign of \( P \) the spiral may be stable or unstable (spiraling in or spiraling out). If \( D < 0 \) the eigenvalues are real and a predominantly straining flow can be expected. In this case the directionality of the local flow is defined by the two eigenvectors of \( A_{ik} \). The case \( P = 0 \) corresponds to incompressible flow for which there are only two possible kinds of critical points, centers with \( Q > 0 \) and saddles with \( Q < 0 \). The line \( Q = 0 \) in Figure 4.1 corresponds to a degenerate case where (4.8) reduces to \( \lambda (\lambda + P) = 0 \). In this instance the critical point becomes a line with trajectories converging from either side of the line.

4.1.3 Linear flows in three dimensions

In three dimensions the eigenvalues of \( A_{ik} \) satisfy the cubic

\[
\lambda^3 + P\lambda^2 + Q\lambda + R = 0
\]  
(4.12)

where the invariants are

\[
P = -tr [A_{ij}] = -A_{ii}
\]

\[
Q = \frac{1}{2} \left( P^2 - tr \left[ A_{ik}A_{kj} \right] \right) = \frac{1}{2} \left( P^2 - A_{ik}A_{ki} \right)
\]  
(4.13)

\[
R = \frac{1}{3} \left( -P^3 + 3PQ - A_{ik}A_{km}A_{mi} \right).
\]

Any cubic polynomial can be simplified as follows. Let

\[
\lambda = \alpha - \frac{P}{3}.
\]  
(4.14)

Then \( \alpha \) satisfies

\[
\alpha^3 + \hat{Q}\alpha + \hat{R} = 0
\]  
(4.15)

where

\[
\hat{Q} = Q - \frac{1}{3} P^2
\]

\[
\hat{R} = R - \frac{1}{3} PQ + \frac{2}{27} P^3.
\]  
(4.16)
Let

\[
a_1 = \left( -\frac{\hat{R}}{2} + \frac{1}{3\sqrt{3}} \left( \hat{Q}^3 + \frac{27}{4} \hat{R}^2 \right)^{1/2} \right)^{1/3}
\]

\[
a_2 = \left( -\frac{\hat{R}}{2} - \frac{1}{3\sqrt{3}} \left( \hat{Q}^3 + \frac{27}{4} \hat{R}^2 \right)^{1/2} \right)^{1/3}
\]  

\( (4.17) \)

The real solution of (4.15) is expressed as

\[
\alpha_1 = a_1 + a_2
\]

and the complex (or remaining real) solutions are

\[
\alpha_2 = -\frac{1}{2} (a_1 + a_2) + \frac{i\sqrt{3}}{2} (a_1 - a_2)
\]

\[
\alpha_3 = -\frac{1}{2} (a_1 + a_2) - \frac{i\sqrt{3}}{2} (a_1 - a_2).
\]  

\( (4.19) \)

When (4.12) is solved for the eigenvalues one is led to the cubic discriminant

\[
D = \frac{27}{4} R^2 + \left( P^3 - \frac{9}{2} P Q \right) R + Q^2 \left( Q - \frac{1}{4} P^2 \right).
\]  

\( (4.20) \)

The surface \( D = 0 \), is depicted in Figure 4.2.

To help visualize the surface in Figure 4.2 it is split down the middle on the plane \( P = 0 \) and the two parts are rotated apart to provide a better view. Note that (4.20) can be regarded as a quadratic in \( R \) and so the surface \( D = 0 \) is really composed of two roots for \( R \) that meet in a cusp. If \( D > 0 \) the point \((P, Q, R)\) lies above the surface and there is one real eigenvalue and two complex conjugate eigenvalues. If \( D < 0 \) all three eigenvalues are real. The invariants can be expressed in terms of the eigenvalues as follows. If the eigenvalues are real

\[
P = - (\lambda_1 + \lambda_2 + \lambda_3)
\]

\[
Q = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3
\]

\[
R = -\lambda_1 \lambda_2 \lambda_3.
\]  

\( (4.21) \)
If the eigenvalues are complex

\[ P = -(2\sigma + b) \]
\[ Q = \sigma^2 + \omega^2 + 2\sigma b \]
\[ R = -b(\sigma^2 + \omega^2). \] (4.22)

The variable \( b \) is the real eigenvalue and \( \sigma \) and \( \omega \) are the real and imaginary parts of the complex conjugate eigenvalues.

### 4.1.4 Incompressible flow

Flow patterns in incompressible flow are characterized by

\[ \nabla \cdot \vec{U} = \frac{\partial U_i}{\partial x_i} = A_{ii} = 0. \] (4.23)

This corresponds to \( P = 0 \). In this case the discriminant is
\[ D = Q^3 + \frac{27}{4} R^2 \]  

(4.24)

and the invariants simplify to

\[ Q = -\frac{1}{2} A_{ik} A_{ki} \]

\[ R = -\frac{1}{3} A_{ik} A_{km} A_{mi}. \]  

(4.25)

The various possible elementary flow patterns for this case can be categorized on a plot of \( Q \) versus \( R \) shown in Figure 4.3.

Figure 4.3: Three-dimensional flow patterns in the plane \( P = 0 \).

Figure 4.1 and Figure 4.3 are cuts through the surface (4.20) at \( R = 0 \) and \( P = 0 \) respectively.
4.1.5 Frames of reference

We introduced the transformation of coordinates between a fixed and moving frame in Chapter 1. Here we briefly revisit the subject again in the context of critical points. For a general smooth flow, the particle path equations (4.1) can be expanded as a Taylor series about any point $\bar{x}_0$ as follows

$$\frac{dx_i}{dt} = U_i|_{\bar{x} = \bar{x}_0} + A_{ik}|_{\bar{x} = \bar{x}_0} (x_k - x_{0k}) + O \left( (x_k - x_{0k})^2 \right) + \cdots \quad (4.26)$$

If a coordinate system is attached to and moves with the particle at $\bar{x}_0$ with the velocity $U_i|_{\bar{x} = \bar{x}_0}$ so that

$$\bar{x}_0' = \bar{x} - \bar{x}_0$$
$$\bar{U}' = \bar{U} - \bar{U}|_{\bar{x} = \bar{x}_0} \quad (4.27)$$

then in that frame of reference the origin of coordinates in effect becomes a critical point (since the velocity is zero there) and the flow pattern that an observer in this coordinate system would see is determined by the second and higher order terms in (4.26).

$$\frac{dx_i'}{dt'} = A_{ik} x_k' + O \left( x_k'^2 \right) + \cdots \quad (4.28)$$

The elementary flow patterns described above are what would be seen locally at an instant by an observer moving with a fluid element. Notice that the velocity gradient tensor referred to either frame is the same. In this way the velocity gradient tensor can be used to infer the geometry of the local flow pattern at any point in an unambiguous, frame-invariant manner.

Categorizing flow patterns using the invariants of the velocity gradient tensor has a long history of applications in fluid mechanics particularly in the kinematic description of flow separation and reattachment near a solid surface. More recently these methods have been used to describe light propagation near complex apertures and to describe changes in the electron charge density field in molecules during the making and breaking of chemical bonds.

4.2 Rate-of-strain and rate-of-rotation tensors

The velocity gradient tensor
\( A_{ij} = \partial U_i / \partial x_j \) \hspace{1cm} (4.29)

can be split into a symmetric and antisymmetric part

\[
A_{ij} = \frac{\partial U_i}{\partial x_j} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) + \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} \right). \hspace{1cm} (4.30)
\]

The symmetric part is the rate-of-strain tensor

\[
S_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right). \hspace{1cm} (4.31)
\]

The anti-symmetric part is the rate-of-rotation tensor or spin tensor

\[
W_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} \right). \hspace{1cm} (4.32)
\]

The vorticity vector \( \bar{\Omega} = \nabla \times \bar{U} \) is related to the velocity gradients by

\[
\Omega_i = \varepsilon_{ijk} \left( \frac{\partial U_k}{\partial x_j} \right) \hspace{1cm} (4.33)
\]

and the spin tensor is related to the vorticity by

\[
W_{ik} = \frac{1}{2} \varepsilon_{ijk} \Omega_j. \hspace{1cm} (4.34)
\]

All local flow patterns can be regarded as a linear sum of a purely rotational motion and a purely straining motion. The balance between these two components determines which of the local flow fields shown in Figure 4.1 or Figure 4.3 will exist at the point. As we move into our studies of compressible flow we shall see that a natural division exists between flows that are irrotational, where the effects of viscosity can often be neglected, and flows that are strain-rate dominated where viscosity plays an important and sometimes dominant role.
CHAPTER 4. KINEMATICS OF FLUID MOTION

4.3 Problems

Problem 1 - The simplest 2-D flows imaginable are given by the linear system

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy.$$ \hspace{1cm} (4.35)

Sketch the corresponding flow pattern for the following cases.

i) \((a, b, c, d) = (1, -1, -1, -1)\)

ii) \((a, b, c, d) = (1, -3, 1, -1)\)

iii) \((a, b, c, d) = (-1, 0, 0, -1)\)

Work out the invariants of the velocity gradient tensor as well as the various components of the rate-of-rotation and rate-of-strain tensors and the vorticity vector. Which flows are incompressible?

Problem 2 - An unforced, damped, pendulum is governed by the second order ODE

$$\frac{d^2\theta}{dt^2} + \beta \frac{d\theta}{dt} + \frac{g}{L} \sin(\theta) = 0.$$ \hspace{1cm} (4.36)

Let \(x = \theta(t)\) and \(y = d\theta(t)/dt\). Use these variables to convert the equation to the canonical form.

$$\frac{dx}{dt} = U(x, y)$$

$$\frac{dy}{dt} = V(x, y).$$ \hspace{1cm} (4.37)

Sketch the "streamlines" defined by (4.37). Locate and categorize any critical points using the methods developed in this chapter. Identify which points are dominated by rotation and which are dominated by the rate-of-strain. You can do this graphically by drawing line segments of the appropriate slope in \((x, y)\) coordinates. The picture of the flow that results is called the phase portrait of the flow in reference to the fact that, for the pendulum, a point in the phase portrait represents the instantaneous relation between the position and velocity of the pendulum. For what value of \(\beta\) can the "flow" defined by the phase portrait be used as a model of an incompressible fluid flow?
Problem 3 - Use (4.14) to reduce (4.12) to (4.15).

Problem 4 - Sketch the flow pattern generated by the 3-D linear system

\[
\frac{dx}{dt} = -y \\
\frac{dy}{dt} = x \\
\frac{dz}{dt} = z. 
\]  

(4.38)

Work out the invariants of the velocity gradient tensor as well as the components of the rate-of-rotation and rate-of-strain tensors and vorticity vector. The vector field plotted in three dimensions is called the phase space of the system of ODEs. In fluid mechanics the phase portrait or phase space is the physical space of the flow.

Problem 5 - Show that

\[
S_{ij} A_{ji} = S_{ij} S_{ji} \tag{4.39}
\]

and is therefore greater than or equal to zero.

Problem 6 - Work out the formulas for the components of the vorticity vector and show that the spin tensor is related to the vorticity vector by

\[
W_{ik} = \frac{1}{2} \varepsilon_{ijk} \Omega_j. \tag{4.40}
\]

Problem 7 - The velocity field given below has been used in the fluid mechanics literature to model a two dimensional separation bubble.

\[
U(x, y) = -y + 3y^2 + x^2y - (2/3)y^3 \\
V(x, y) = -3xy^2. \tag{4.41}
\]

Draw the phase portrait and identify critical points.