AA210A
Fundamentals of Compressible Flow

Chapter 1 - Introduction to fluid flow
1.2 Conservation of mass

\[ \frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \mathbf{U}) = 0 \]

\[ \Delta x \Delta y \Delta z \left( \frac{\partial \rho}{\partial t} \right) + \Delta y \Delta z (\rho U|_x + \Delta x - \rho U|_x) + \\
\Delta x \Delta z (\rho V|_y + \Delta y - \rho V|_y) + \Delta x \Delta y (\rho W|_z + \Delta z - \rho W|_z) = 0 \]
Divide through by the volume of the control volume.

\[
\frac{\partial p}{\partial t} + \frac{\rho U|_{x + \Delta x} - \rho U|_{x}}{\Delta x} + \frac{\rho V|_{y + \Delta y} - \rho V|_{y}}{\Delta y} + \frac{\rho W|_{z + \Delta z} - \rho W|_{z}}{\Delta z} = 0
\]

Let \((\Delta x \to 0, \Delta y \to 0, \Delta z \to 0)\). In this limit (1.4) becomes

\[
\frac{\partial p}{\partial t} + \frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} + \frac{\partial \rho W}{\partial z} = 0
\]

### 1.2.1 Conservation of mass - Incompressible flow

If the density is constant the continuity equation reduces to

\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0.
\]

Note that this equation applies to both steady and unsteady incompressible flow.
1.2.2 Index notation and the Einstein convention

Make the following replacements

\[(x, y, z) \rightarrow (x_1, x_2, x_3)\]
\[(U, V, W) \rightarrow (U_1, U_2, U_3)\]

Using index notation the continuity equation is

\[\frac{\partial \rho}{\partial t} + \sum_{i=1}^{3} \frac{\partial (\rho U_i)}{\partial x_i} = 0\]

Einstein recognized that such sums from vector calculus always involve a repeated index. For convenience he dropped the summation symbol.

\[\frac{\partial \rho}{\partial t} + \frac{\partial (\rho U_i)}{\partial x_i} = 0\]

Coordinate independent form

\[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{U}) = 0\]

\[\nabla \equiv \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)\]
1.3 Particle paths, streamlines and streaklines in 2-D steady flow

The figure below shows the streamlines over a 2-D airfoil.

![Streamlines and Streaklines](image)

*Figure 1.2  Flow over a 2-D lifting wing; (a) streamlines, (b) streaklines.*

The flow is irrotational and incompressible

\[
\nabla \times \vec{U} = 0 \quad \nabla \cdot \vec{U} = 0.
\]
A vector field that satisfies $\nabla \times \overrightarrow{U} = 0$ can always be represented as the gradient of a scalar potential

$$\overrightarrow{U} = \nabla \Phi.$$ 

or

$$(U, V) = \left( \frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y} \right)$$

If the scalar potential is substituted into the continuity equation the result is Laplace's equation.

$$\nabla \cdot \nabla \Phi = \nabla^2 \Phi = 0.$$
A weakly compressible example - flow over a wing flap.

Figure 1.3  Computed streamlines over a wing flap.
The figure below shows the trajectory in space of a fluid element moving under the action of a two-dimensional steady velocity field. The equations that determine the trajectory are:

\[
\begin{align*}
\frac{dx(t)}{dt} &= U(x(t), y(t)) \\
\frac{dy(t)}{dt} &= V(x(t), y(t))
\end{align*}
\]
Formally, these equations are solved by integrating the velocity field in time.

\[
\begin{align*}
  x(t) &= x_0 + \int_0^t U(x(t), y(t))dt \\
  y(t) &= y_0 + \int_0^t V(x(t), y(t))dt
\end{align*}
\]

Along a particle path

\[
  x = F(x_0, y_0, t) ; \quad y = G(x_0, y_0, t).
\]
Eliminate time between the functions F and G to produce a family of lines. These are the streamlines observed in the figures shown earlier.

\[ \psi = \Psi(x, y). \]

The value of a particular streamline is determined by the initial conditions.

\[ \psi_0 = \Psi(x_0, y_0). \]
This situation is depicted schematically below.

\[ x = F(x_0, y_0, t) \]
\[ y = G(x_0, y_0, t) \]
\[ \psi_0 = \Psi(x_0, y_0) = \Psi(x, y) \]

**Figure 1.5** Streamlines in steady flow. The value of a particular streamline is determined by the coordinates of a point on the streamline. This can be regarded as the initial position of a fluid particle that traces out the streamline.
The streamfunction can also be determined by solving the first-order ODE generated by eliminating $dt$ from the particle path equations.

\[
\frac{dy}{dx} = \frac{V(x, y)}{U(x, y)}.
\]

The total differential of the streamfunction is

\[
d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy.
\]
Replace the differentials $dx$ and $dy$. 

\[ d\psi = \left( U(x, y) \frac{\partial \psi}{\partial x} + V(x, y) \frac{\partial \psi}{\partial y} \right) dt. \]

The stream function, can be determined as the solution of a linear, first order PDE.

\[ U \cdot \nabla \psi = U(x, y) \frac{\partial \psi}{\partial x} + V(x, y) \frac{\partial \psi}{\partial y} = 0. \]

This equation is the mathematical expression of the statement that streamlines are parallel to the velocity vector field.
The first-order ODE governing the stream function can be written as

\[-V(x, y)dx + U(x, y)dy = 0.\]

### 1.3.1 The integrating factor

On a streamline

\[\frac{\partial \Psi}{\partial x} dx + \frac{\partial \Psi}{\partial y} dy = 0.\]

What is the relationship between these two equations?
To be a perfect differential the functions $U$ and $V$ have to satisfy the integrability condition

$$\frac{-\partial V}{\partial y} = \frac{\partial U}{\partial x}.$$

For general functions $U$ and $V$ this condition is not satisfied. The equation $-V(x, y)\,dx + U(x, y)\,dy = 0.$ must be multiplied by an integrating factor in order to convert it to a perfect differential.

It was shown by the German mathematician Johann Pfaff in the early 1800’s that an integrating factor $M(x, y)$ always exists.

$$d\psi = -M(x, y)V(x, y)\,dx + M(x, y)U(x, y)\,dy$$

and the partial derivatives are

$$\begin{cases} 
\frac{\partial \psi}{\partial x} = -M(x, y)V(x, y) \\
\frac{\partial \psi}{\partial y} = M(x, y)U(x, y) 
\end{cases}$$
1.3.2 Incompressible flow in 2 dimensions

The flow of an incompressible fluid in 2-D is constrained by the continuity equation

\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]

This is exactly the integrability condition. Continuity is satisfied identically by the introduction of the stream function,

\[
U = \frac{\partial \Psi}{\partial y} ; \quad V = \frac{\partial \Psi}{\partial x}
\]

In this case \(-Vdx+Udy\) is guaranteed to be a perfect differential and one can write.

\[
d\Psi = -Vdx + Udy.
\]

1.3.3 Incompressible, irrotational flow in 2 dimensions

\[
\frac{\partial \Psi}{\partial y} = \frac{\partial \Phi}{\partial x} \\
-\frac{\partial \Psi}{\partial x} = \frac{\partial \Phi}{\partial y}
\]

The Cauchy-Reimann conditions
### 1.3.4 Compressible flow in 2 dimensions

The continuity equation for the steady flow of a compressible fluid in two dimensions is

\[
\frac{\partial}{\partial x}(\rho U) + \frac{\partial}{\partial y}(\rho V) = 0
\]

In this case the required integrating factor is the density and we can write.

\[
d\psi = -\rho V \, dx + \rho U \, dy
\]

The stream function in a compressible flow is proportional to the mass flux and the convergence and divergence of lines in the flow over the flap shown earlier is a reflection of variations of mass flux over different parts of the flow field.
1.4 Particle paths in three dimensions

The figure above shows the trajectory in space traced out by a particle under the action of a general three dimensional unsteady flow,
The equations governing the motion of the particle are:

\[ \frac{dx_i(t)}{dt} = U_i(x_1(t), x_2(t), x_3(t), t) ; \quad i = 1, 2, 3 \]

Formally, these equations are solved by integrating the velocity field.

\[ x_i(t) = x_{i0} + \int_0^t U_i(x_1(t), x_2(t), x_3(t), t) \, dt ; \quad i = 1, 2, 3 \]
1.5 The substantial derivative

The acceleration of a particle is

\[
\frac{d^2 x_i(t)}{dt^2} = \frac{d}{dt} U_i(x_1(t), x_2(t), x_3(t), t) = \frac{\partial U_i}{\partial t} + \frac{\partial U_i}{\partial x_k} \frac{dx_k}{dt}
\]

Insert the velocities. The result is called the substantial or material derivative and is usually denoted by

\[
\frac{DU_i}{Dt} = \frac{\partial U_i}{\partial t} + U_k \frac{\partial U_i}{\partial x_k} = \frac{\partial \vec{U}}{\partial t} + \vec{U} \cdot \nabla \vec{U}
\]

The time derivative of any flow variable evaluated on a fluid element is given by a similar formula. For example the rate of change of density following a fluid particle is

\[
\frac{D \rho}{Dt} = \frac{\partial \rho}{\partial t} + U_k \frac{\partial \rho}{\partial x_k} = \frac{\partial \rho}{\partial t} + \vec{U} \cdot \nabla \rho
\]
1.5.1 Frames of reference

Transformation of position and velocity

\[ x' = x - X(t) \]
\[ y' = y - Y(t) \]
\[ z' = z - Z(t) \]
\[ U' = U - \dot{X}(t) \]
\[ V' = V - \dot{Y}(t) \]
\[ W' = W - \dot{Z}(t) \]

Transformation of momentum

\[ m\ddot{U}' = m\ddot{U} - m\ddot{X}/dt \]

momentum in moving coordinates
momentum in fixed coordinates
Transformation of kinetic energy

**kinetic energy in moving coordinates**

\[
\frac{1}{2} m (U''^2 + V''^2 + W''^2)
\]

**kinetic energy in fixed coordinates**

\[
\frac{1}{2} m (U'^2 + V'^2 + W'^2) = \frac{1}{2} m ((U - \dot{X})^2 + (V - \dot{Y})^2 + (W - \dot{Z})^2).
\]

\[
\frac{1}{2} m (U'^2 + V'^2 + W'^2) = \frac{1}{2} m (U^2 + V^2 + W^2) + \frac{1}{2} m \ddot{X}(\ddot{X} - 2\dot{U}) + \frac{1}{2} m \ddot{Y}(\ddot{Y} - 2\dot{V}) + \frac{1}{2} m \ddot{Z}(\ddot{Z} - 2\dot{W})
\]

\[
k' = k + \frac{1}{2} m \ddot{X}(\ddot{X} - 2\dot{U}) + \frac{1}{2} m \ddot{Y}(\ddot{Y} - 2\dot{V}) + \frac{1}{2} m \ddot{Z}(\ddot{Z} - 2\dot{W}).
\]

Thermodynamic properties such as density, temperature and pressure do not depend on the frame of reference.
1.6 Momentum transport due to convection

Density

\[[\rho] = \frac{M}{L^3}\]

Volume flux in the y direction

\[[V] = \frac{L}{T} = \frac{L^3}{L^2T} = \frac{\text{Volume}}{\text{Area} \cdot \text{Sec}}\]

Control volume surface

Momentum flux

\[[\rho U V] = \frac{M \left( \frac{L}{T} \right)}{L^3} \left( \frac{L}{T} \right) = \frac{M \left( \frac{L}{T} \right)}{L^2T}\]

Outward unit normal vector

\(x\)-momentum per unit volume

\(x\)-momentum convected in the y-direction per unit area per second

\(\text{Volume per unit area per second}\)
The conservation equation for momentum

\[
\begin{align*}
\Delta x \Delta y \Delta z \left( \frac{\partial \rho U}{\partial t} \right) + \Delta y \Delta z (\rho U U_{x} |_{x + \Delta x} - \rho U U_{x} |_{x}) + \\
\Delta x \Delta z (\rho U V_{y} |_{y + \Delta y} - \rho U V_{y} |_{y}) + \Delta x \Delta y (\rho U W_{z} |_{z + \Delta z} - \rho U W_{z} |_{z}) &= \\
\{ \text{the sum of x-component forces acting on the system} \}
\end{align*}
\]

Figure 1.8 Fluxes of x-momentum through a fixed control volume. Arrows denote the velocity component carrying momentum into or out of the control volume.
Divide through by the volume

\[
\frac{\partial \rho U}{\partial t} + \frac{\rho UU|_{x+\Delta x} - \rho UU|_{x}}{\Delta x} + \frac{\rho UV|_{y+\Delta y} - \rho UV|_{y}}{\Delta y} + \frac{\rho UW|_{z+\Delta z} - \rho UW|_{z}}{\Delta z} = \left\{ \text{the sum of } x\text{-component forces acting on the system per unit volume} \right\}
\]

Let \((\Delta x \to 0, \Delta y \to 0, \Delta z \to 0)\). In this limit (1.54) becomes

\[
\frac{\partial \rho U}{\partial t} + \frac{\partial \rho UU}{\partial x} + \frac{\partial \rho UV}{\partial y} + \frac{\partial \rho UW}{\partial z} = \left\{ \text{The sum of } x\text{-component forces per unit volume acting on the control volume} \right\}
\]

In the y and z directions

\[
\frac{\partial \rho V}{\partial t} + \frac{\partial \rho VU}{\partial x} + \frac{\partial \rho VV}{\partial y} + \frac{\partial \rho VW}{\partial z} = \left\{ \text{The sum of } y\text{-component forces per unit volume acting on the control volume} \right\}
\]

\[
\frac{\partial \rho W}{\partial t} + \frac{\partial \rho WU}{\partial x} + \frac{\partial \rho WV}{\partial y} + \frac{\partial \rho WW}{\partial z} = \left\{ \text{The sum of } z\text{-component forces per unit volume acting on the control volume} \right\}
\]
In index notation the momentum conservation equation is

\[
\frac{\partial \rho U_i}{\partial t} + \frac{\partial (\rho U_i U_j)}{\partial x_j} = \left\{ \begin{array}{l}
\text{Sum of the} \\
\text{ith-component forces} \\
\text{per unit volume acting} \\
\text{on the control volume}
\end{array} \right\}; \quad i = 1, 2, 3
\]

Rearrange

\[
\rho \frac{\partial U_i}{\partial t} + \rho U_j \frac{\partial (U_i)}{\partial x_j} + U_i \left( \frac{\partial \rho}{\partial t} + \frac{\partial (\rho U_j)}{\partial x_j} \right) = \left\{ \begin{array}{l}
\text{The sum of} \\
\text{ith-component forces} \\
\text{per unit volume acting} \\
\text{on the control volume}
\end{array} \right\}
\]

\[
\frac{D U_i}{D t} = \left\{ \begin{array}{l}
\text{The sum of} \\
\text{ith-component forces} \\
\text{per unit volume acting} \\
\text{on the control volume}
\end{array} \right\}
\]

In words,

\[
\left\{ \begin{array}{l}
The rate of momentum change \\
of a fluid element
\end{array} \right\} = \left\{ \begin{array}{l}
The vector sum of \\
forces acting \\
on the fluid element
\end{array} \right\}
\]
1.7 Momentum transport due to molecular motion

1.7.1 Pressure

1.7.2 Viscous friction - Plane Couette Flow

Figure 1.9: Build-up to a steady laminar velocity profile for a viscous fluid contained between two parallel plates. At t=0 the upper plate is set into motion at a constant speed $U_\infty$.

Force/Area needed to maintain the motion of the upper plate

$$\frac{F}{A} = \mu \frac{U_\infty}{d} \quad \tau_{xy} = \mu \frac{dU}{dy}$$
1.7.3 A question of signs

1.7.4 Newtonian fluids

1.7.5 Forces acting on a fluid element

Figure 1.10 Pressure and viscous stresses acting in the x-direction
Pressure-viscous-stress force components

\[ F_x = \Delta y \Delta z \left( -P + \tau_{xx} \right)_x + \Delta x \Delta z \left( \tau_{xy} \right)_y + \Delta x \Delta y \left( \tau_{xz} \right)_z \]

\[ F_y = \Delta y \Delta z \left( \tau_{xy} \right)_x + \Delta x \Delta z \left( -P + \tau_{yy} \right)_y + \Delta x \Delta y \left( \tau_{yz} \right)_z \]

\[ F_z = \Delta y \Delta z \left( \tau_{xz} \right)_x + \Delta x \Delta z \left( \tau_{yz} \right)_y + \Delta x \Delta y \left( -P + \tau_{zz} \right)_z \]

Momentum balance in the x-direction

\[ \Delta x \Delta y \Delta z \left( \frac{\partial \rho U}{\partial t} \right)_x = \Delta y \Delta z (\rho U U)_x - \rho U U_x + \Delta x \Delta y (\rho U W)_z - \rho U W_z + \Delta x \Delta y (\rho U V)_y - \rho U V_y + \Delta x \Delta y \left( \tau_{xx} \right)_x + \Delta x \Delta y \left( \tau_{xy} \right)_y + \Delta x \Delta y \left( \tau_{xz} \right)_z \]
Divide by the volume

\[
\frac{\partial \rho U}{\partial t} + \frac{\rho UU|_{x+\Delta x} - \rho UU|_x + (P - \tau_{xx})|_{x+\Delta x} - (P - \tau_{xx})|_x}{\Delta x} + \\
\frac{\rho UV|_{y+\Delta y} - \rho UV|_y - (\tau_{xy}|_{y+\Delta y} - \tau_{xy}|_y)}{\Delta y} + \\
\frac{\rho UW|_{z+\Delta z} - \rho UW|_z - (\tau_{xz}|_{z+\Delta z} - \tau_{xz}|_z)}{\Delta z} = 0
\]

Let \((\Delta x \to 0, \Delta y \to 0, \Delta z \to 0)\). In this limit (1.65) becomes

\[
\frac{\partial \rho U}{\partial t} + \frac{\partial (\rho UU + P - \tau_{xx})}{\partial x} + \frac{\partial (\rho UV - \tau_{xy})}{\partial y} + \frac{\partial (\rho UW - \tau_{xz})}{\partial z} = 0
\]

**x - component**

In the y and z directions

\[
\frac{\partial \rho V}{\partial t} + \frac{\partial (\rho VU - \tau_{xy})}{\partial x} + \frac{\partial (\rho VV + P - \tau_{yy})}{\partial y} + \frac{\partial (\rho VW - \tau_{yz})}{\partial z} = 0
\]

\[
\frac{\partial \rho W}{\partial t} + \frac{\partial (\rho UW - \tau_{xz})}{\partial x} + \frac{\partial (\rho VW - \tau_{yz})}{\partial y} + \frac{\partial (\rho WW + P - \tau_{zz})}{\partial z} = 0
\]
In index notation the equation for conservation of momentum is

\[
\frac{\partial \rho U_i}{\partial t} + \frac{\partial (\rho U_i U_j)}{\partial x_j} + \frac{\partial P}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} = 0; \quad i = 1,2,3.
\]

Coordinate independent form

\[
\frac{\partial \rho U}{\partial t} + \nabla \cdot (\rho U U) + \nabla P - \nabla \cdot \tau = 0.
\]
1.7 Conservation of energy

\[ k = \frac{1}{2}(U^2 + V^2 + W^2) \]

Figure 1.11 Convection of energy into and out of a control volume.
1.8.1 Pressure and viscous work

*Power input to the control volume =* $\vec{F} \cdot \vec{U}$

Fully written out this relation is

\[
\Delta y \Delta z \left\{ \left( -P + \tau_{xx} \right)_{x + \Delta x} - \left( -P + \tau_{xx} \right)_{x} \right\} U + \left( \tau_{xy} \right)_{x + \Delta x} - \tau_{xy} \left\{ \left( -P + \tau_{yy} \right)_{y + \Delta y} - \left( -P + \tau_{yy} \right)_{y} \right\} V + \\
\left( \tau_{xz} \right)_{x + \Delta x} - \tau_{xz} \left\{ \left( -P + \tau_{yz} \right)_{y + \Delta y} - \left( -P + \tau_{yz} \right)_{y} \right\} W \right} + \\
\Delta x \Delta z \left\{ \left( \tau_{xy} \right)_{y + \Delta y} - \tau_{xy} \right\} U + \left( \tau_{yy} \right)_{y + \Delta y} - \left( \tau_{yy} \right)_{y} V + \\
\left( \tau_{yz} \right)_{y + \Delta y} - \tau_{yz} \right\} W \right}\} + \\
\Delta x \Delta y \left\{ \left( \tau_{xz} \right)_{z + \Delta z} - \tau_{xz} \right\} U + \left( \tau_{yz} \right)_{z + \Delta z} - \tau_{yz} \left\{ \left( -P + \tau_{zz} \right)_{z + \Delta z} - \left( -P + \tau_{zz} \right)_{z} \right\} V + \\
\left( -P + \tau_{zz} \right)_{z + \Delta z} - \left( -P + \tau_{zz} \right)_{z} \right\} W \right}\}
The previous equation can be rearranged to read in terms of energy fluxes.

\[
\Delta y \Delta z \left\{ (-PU + \tau_{xx}U + \tau_{xy}V + \tau_{xz}W)_{x+\Delta x} - (-PU + \tau_{xx}U + \tau_{xy}V + \tau_{xz}W)_{x} \right\} + \\
\Delta x \Delta z \left\{ (-PV + \tau_{xy}U + \tau_{yy}V + \tau_{yz}W)_{y+\Delta y} - (-PV + \tau_{xy}U + \tau_{yy}V + \tau_{yz}W)_{y} \right\} + \\
\Delta x \Delta y \left\{ (-PW + \tau_{zx}U + \tau_{zy}V + \tau_{zz}W)_{z+\Delta z} - (-PW + \tau_{zx}U + \tau_{zy}V + \tau_{zz}W)_{z} \right\}
\]

*Figure 1.12 Energy fluxes due to the work done on the control volume by pressure and viscous forces.*
Energy balance.

\[ \Delta x \Delta y \Delta z \left( \frac{\partial \rho(e + k)}{\partial t} \right) = \Delta y \Delta z \left( \rho(e + k) U \right|_{x} - \rho(e + k) U \right|_{x + \Delta x} + \]
\[ \Delta x \Delta z \left( \rho(e + k) V \right|_{y} - \rho(e + k) V \right|_{y + \Delta y} + \]
\[ \Delta x \Delta y \left( \rho(e + k) W \right|_{z} - \rho(e + k) W \right|_{z + \Delta z} + \]
\[ \Delta y \Delta z \left( -(P U + \tau_{xx} U + \tau_{xy} V + \tau_{xz} W) \right|_{x} + \Delta x \right) -(P U + \tau_{xx} U + \tau_{xy} V + \tau_{xz} W) \right|_{x} \]
\[ \Delta x \Delta z \left( -(P V + \tau_{xy} U + \tau_{yy} V + \tau_{yz} W) \right|_{y} + \Delta y \right) -(P V + \tau_{xy} U + \tau_{yy} V + \tau_{yz} W) \right|_{y} \]
\[ \Delta x \Delta y \left( -(P W + \tau_{xz} U + \tau_{zy} V + \tau_{zz} W) \right|_{z} + \Delta z \right) -(P W + \tau_{xz} U + \tau_{zy} V + \tau_{zz} W) \right|_{z} \]
\[ \Delta y \Delta z \left( Q_x \right|_{x} - Q_x \right|_{x + \Delta x} + \Delta z \Delta x \left( Q_y \right|_{y} - Q_y \right|_{y + \Delta y} + \Delta y \Delta z \left( Q_z \right|_{z} - Q_z \right|_{z + \Delta z} \]

\{ Power generation due to sources inside the control volume \}

Divide (1.76) through by the infinitesimal volume \( \Delta x \Delta y \Delta z \) and take the limit \( \Delta x \to 0, \Delta y \to 0, \Delta z \to 0 \). The conservation equation for the energy becomes

\[ \frac{\partial \rho(e + k)}{\partial t} + \frac{\partial (\rho(e + k) U)}{\partial x} + \frac{\partial (\rho(e + k) V)}{\partial y} + \frac{\partial (\rho(e + k) W)}{\partial z} + \]
\[ \frac{\partial (P U - \tau_{xx} U - \tau_{xy} V - \tau_{xz} W)}{\partial x} + \frac{\partial (P V - \tau_{xy} U - \tau_{yy} V - \tau_{yz} W)}{\partial y} + \]
\[ \frac{\partial (P W - \tau_{xz} U - \tau_{zy} V - \tau_{zz} W)}{\partial z} \]
\[ = \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z} \]

\{ Power generation due to sources inside the control volume \}
In index notation the equation for conservation of energy is

\[
\frac{\partial (\rho (e + k))}{\partial t} + \frac{\partial (\rho (e + k) U_j)}{\partial x_j} + \frac{\partial P U_j}{\partial x_j} - \frac{\partial (U_i \tau_{ij})}{\partial x_j} + \frac{\partial Q_j}{\partial x_j} = \{ \text{Power sources} \}. 
\]

Coordinate independent form

\[
\frac{\partial (\rho (e + k))}{\partial t} + \nabla \cdot (\rho (e + k) \bar{U} + P \bar{U} - \bar{\tau} \cdot \bar{U} + \bar{Q}) = \{ \text{Power sources} \}.
\]
1.9 Summary - the equations of motion

Conservation of mass

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} + \frac{\partial \rho W}{\partial z} = 0
\]

Conservation of momentum

\[
\begin{align*}
\frac{\partial \rho U}{\partial t} + \frac{\partial (\rho UU + P - \tau_{xx})}{\partial x} + \frac{\partial (\rho UV - \tau_{xy})}{\partial y} + \frac{\partial (\rho UW - \tau_{xz})}{\partial z} & = 0 \\
\frac{\partial \rho V}{\partial t} + \frac{\partial (\rho VU - \tau_{xy})}{\partial x} + \frac{\partial (\rho VV + P - \tau_{yy})}{\partial y} + \frac{\partial (\rho WV - \tau_{yz})}{\partial z} & = 0 \\
\frac{\partial \rho W}{\partial t} + \frac{\partial (\rho WU - \tau_{xz})}{\partial x} + \frac{\partial (\rho WV - \tau_{yz})}{\partial y} + \frac{\partial (\rho WW + P - \tau_{zz})}{\partial z} & = 0
\end{align*}
\]

Conservation of energy

\[
\begin{align*}
\frac{\partial \rho (e + k)}{\partial t} + \frac{\partial (\rho (e + k) U)}{\partial x} + \frac{\partial (\rho (e + k) U)}{\partial y} + \frac{\partial (\rho (e + k) U)}{\partial z} & + \frac{\partial (PU - \tau_{xx}U - \tau_{xy}V - \tau_{xz}W)}{\partial x} + \frac{\partial (PV - \tau_{xy}U - \tau_{yy}V - \tau_{yz}W)}{\partial y} \\
& + \frac{\partial (PW - \tau_{xz}U - \tau_{xy}V - \tau_{zz}W)}{\partial z} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z} & = \{Power\ sources\}
\end{align*}
\]
Some remarks on the pressure field

Two dimensional steady, inviscid, incompressible flow

Conservation of mass

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

Conservation of momentum

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = -\frac{\partial}{\partial x} \left( \frac{P}{\rho} \right)$$

$$U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} = -\frac{\partial}{\partial y} \left( \frac{P}{\rho} \right)$$

Vorticity

$$\Omega = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}$$
For any steady, inviscid, incompressible, irrotational velocity field the pressure field exists!

\[ \Omega = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} = 0 \]

\[ \frac{\partial}{\partial x} \left( \frac{P}{\rho} \right) = - \left( U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} \right) = -U \frac{\partial U}{\partial x} - V \frac{\partial V}{\partial y} = -\frac{1}{2} \frac{\partial}{\partial x} (U^2 + V^2) \]

\[ \frac{\partial}{\partial y} \left( \frac{P}{\rho} \right) = - \left( U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} \right) = -U \frac{\partial U}{\partial y} - V \frac{\partial V}{\partial y} = -\frac{1}{2} \frac{\partial}{\partial y} (U^2 + V^2) \]

\[ \frac{\partial}{\partial x} \left( \frac{P}{\rho} + \frac{1}{2} (U^2 + V^2) \right) = 0 \]

\[ \frac{\partial}{\partial y} \left( \frac{P}{\rho} + \frac{1}{2} (U^2 + V^2) \right) = 0 \]

\[ \frac{P}{\rho} + \frac{1}{2} (U^2 + V^2) = \text{const} \]

This is the incompressible Bernoulli pressure.
1.10 Problems

Problem 1 - Show that the continuity equation can be expressed as

\[
\frac{1}{\rho} \frac{D\rho}{Dt} + \frac{\partial U_j}{\partial x_j} = 0
\]  

(1.94)

Problem 2 - Use direct measurements from the streamlines in Figure 2.13 to estimate the percent change from the free stream velocity at points A, B, C and D.

![Figure 1.13: Streamlines about a wing in potential flow.](image)

Problem 3 - The general, first order, linear ODE
\[ \frac{dy}{dx} = -g(x) y + f(x) \quad (1.95) \]

can be written as the differential form

\[ (g(x) y - f(x)) \, dx + dy = 0 \quad (1.96) \]

Show that (1.96) can be converted to a perfect differential by multiplying by the integrating factor.

\[ M = e^{\int g(x) \, dx} \quad (1.97) \]

Work out the solution of (1.95) in terms of integrals. What is the solution for the case \( g = \sin(x), \, f = \cos(x) \)? Sketch the corresponding streamline pattern.

**Problem 4 - Solve**

\[ y \frac{\partial \Psi}{\partial x} - x \frac{\partial \Psi}{\partial y} = 0. \quad (1.98) \]

Sketch the resulting streamline pattern.
Problem 5 - Show that the following expression is a perfect differential.

\[- \sin (x) \sin (y) \, dx + \cos (x) \cos (y) \, dy = 0 \quad (1.99)\]

Integrate (1.99) to determine the stream function and sketch the corresponding flow pattern. Work out the substantial derivatives of the velocity components and sketch the acceleration vector field.

Problem 6 - Determine the acceleration of a particle in the 1-D velocity field

\[\ddot{U} = \left( k \frac{x}{t}, 0, 0 \right) \quad (1.100)\]

where \( k \) is constant.

Problem 7 - In a fixed frame of reference a fluid element has the velocity components

\[(U, V, W) = (100, 60, 175) \text{ meters/sec}. \quad (1.101)\]

Suppose the same fluid element is observed in a frame of reference moving at

\[\dot{X} = (25, 110, 90) \text{ meters/sec} \quad (1.102)\]

with respect to the fixed frame. Determine the velocity components measured by the observer in the moving frame. Determine the kinetic energy per unit mass in each frame.
Problem 8 - The stream function of a steady, 2-D compressible flow in a corner is shown in Figure 2.14.

\[
\psi = \frac{xy}{1 + x + y}
\]

Figure 1.14: Streamlines for potential flow in a corner.

Determine plausible expressions for the velocity components and density field. Does a pressure field exist for this flow if it is assumed to be inviscid?

Problem 9 - The expansion into vacuum of a spherical cloud of a monatomic gas such as helium has a well-known exact solution of the equations for compressible isentropic flow. The velocity field is

\[
U = \frac{xt}{t_0^2 + t^2} \quad V = \frac{yt}{t_0^2 + t^2} \quad W = \frac{zt}{t_0^2 + t^2},
\]

(1.103)

The density and pressure are

\[
\frac{\rho}{\rho_0} = \frac{t_0^3}{(t_0^2 + t^2)^{3/2}} \left(1 - \frac{t_0^3}{R_{\text{initial}}^2} \left(1 + \frac{t_0^2}{t_0^2 + t^2} \right)^3 \right)^{3/2}
\]

(1.104)

where \(R_{\text{initial}}\) is the initial radius of the cloud. This problem has served as a model of the expanding gas nebula from an exploding star.

1) Determine the particle paths \((x(t), y(t), z(t))\).

2) Work out the substantial derivative of the density \(D\rho/Dt\).
Problem 10 - A moving fluid contains a passive non-diffusing scalar contaminant. Smoke in a wind tunnel would be a pretty good example of such a contaminant. Let the concentration of the contaminant be $C(x, y, z, t)$. The units of $C$ are

$$mass \ of \ contaminant/\ unit \ mass \ of \ fluid.$$  \hspace{1cm} (1.105)

Derive a conservation equation for $C$.

Problem 11 - Include the effects of gravity in the equations of motion (1.93). You can check your answer with the equations derived in Chapter 5.