Introduction to Symmetry Analysis

Variational Symmetries for the Equation Governing Flexural Waves

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Noether’s Theorem

**Theorem 15.1.** Let \( L[x, y, y_1, \ldots, y_p] \) be a differential function. The action integral

\[
S = \int L[x, y, y_1, \ldots, y_p] \, dx^1 \, dx^2 \cdots dx^n
\]  

(15.28)

is invariant under the Lie–Bäcklund group (15.3) with infinitesimals \((\xi^j, \eta^i)\) \((j = 1, \ldots, n, i = 1, \ldots, m)\) if and only if

\[
\mu^i(E_i L) + D_{j_1}(L \xi^{j_1} + \theta^{j_1}) = D_{j_1} \beta^{j_1},
\]  

(15.29)

where \(\mu^i = \eta^i - y^i_\alpha \xi^\alpha\) and \(\int \beta^i \, dA_j = 0\). The vector \(\theta^{j_1}\) is

\[
\theta^{j_1} = \left[ \mu^i \left( \frac{\partial L}{\partial y^i_{j_1}} + \sum_{k=2}^p (-1)^{k-1} D_{j_2 \ldots j_k} \frac{\partial L}{\partial y^i_{j_1 \ldots j_k}} \right) \right]
+ \sum_{\lambda=2}^{p-1} \left[ D_{j_2 \ldots j_\lambda} \mu^i \left( \frac{\partial L}{\partial y^i_{j_1 \ldots j_\lambda}} + \sum_{k=\lambda}^{p-1} (-1)^{k-\lambda+1} D_{j_{\lambda+1} \ldots j_{k+1}} \frac{\partial L}{\partial y^i_{j_1 \ldots j_\lambda \ldots j_{k+1}}} \right) \right]
+ D_{j_2 \ldots j_p} \mu^i \left( \frac{\partial L}{\partial y^i_{j_1 \ldots j_p}} \right), \quad j_1 = i, \ldots, n
\]  

(15.30)
The condition (15.29) is met if \( y \) is a solution of the generalized Euler–Lagrange system

\[
E_i L = \frac{\partial L}{\partial y^i} + \sum_{k=1}^p (-1)^k D_{j_1 j_2 \ldots j_k} \frac{\partial L}{\partial y^i_{j_1 j_2 \ldots j_k}} = 0
\]  

(15.31)

and if

\[
D_j (L \dot{\xi}^j + \theta^j - \beta^j) = 0
\]  

(15.32)

holds on solutions of (15.31). The combination

\[
\Gamma^j = L \dot{\xi}^j + \theta^j - \beta^j
\]  

(15.33)

is a conserved vector for the system (15.31), and (15.32) is a conservation law.
Dispersion of Flexural Waves

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A simple experiment can be performed to demonstrate the dispersive nature of flexural waves:

- An aluminum bar of cross-section 1/4 x 1/2 inch and length 6 ft is vertically inserted in sand so that 4.48 ft (136.5 cm) is allowed to freely vibrate in a horizontal direction.

- The sand acts as a non-reflective boundary so that any flexural waves incident upon the sand will be completely absorbed.

- PCB accelerometers (1 gram) are attached at positions A and B

- Aluminum bar is excited with an impulse, using PCB hammer with plastic tip, at points 1 and 2

- The time signals measured by the accelerometers are displayed on an oscilloscope (one that can store a trace)
Dispersion Relation

The equation of motion for flexural (bending) waves in a beam is

\[
\frac{\partial^2 y}{\partial x^2} + c_L^2 \frac{\partial^4 y}{\partial x^4} = 0
\]

where \( c_L = \sqrt{\frac{E}{\rho}} \) is the speed of a quasi-longitudinal wave (\( E \) is Young's modulus and \( \rho \) is the mass density). This equation of motion is a fourth-order differential equation such that the solutions are not of the form \( y(x,t) = f(x-ct) + g(x+ct) \). In addition, the flexural wave speed is dispersive

\[
c = \sqrt{\omega \xi c_L}
\]

as evidenced by the fact that the wave speed is proportional to the square root of frequency. Thus, higher frequency flexural waves will travel faster than lower frequency flexural waves.

A force pulse contains many (almost infinite) frequency components
Hammer impact at Point 1

- Force pulse is very clean at location B
- Pulse disperses by the time it reaches location A --- higher frequency waves travel faster and arrive first --- lower frequency waves travel slower and arrive later
Hammer impact at Point 2

- Initial force pulse is very clean at location A
- Pulse disperses by the time it reaches location B --- higher frequency waves travel faster and arrive first --- lower frequency waves travel slower and arrive later
- Flexural waves reflect from free end of beam and travel back down to location A --- dispersion is even more evident in reflected signal
Solutions

In dimensioned form the equation is

\[ y_{\tau \tau} + \left( \frac{EI}{\rho A} \right) y_{\chi \chi \chi \chi} = 0 \]

where \( E \) is Young's modulus, \( I \) is the moment of inertia about the neutral axis of bending, \( A \) is the area and \( \rho \) is the density of the material.

Take the origin of coordinates at the center of the rod. Even solutions are of the form

\[ y[\chi, \tau] = e^{i\omega \tau} (ACosh[k\chi] + BCos[k\chi]) \]

and odd solutions are

\[ y[\chi, \tau] = e^{i\omega \tau} (CSinh[k\chi] + DSin[k\chi]) \]
The wave speed is

\[ c = \frac{\omega}{k} \]

The frequency and wave number cannot be selected independently. They are related by

\[ \left( -\omega^2 + \left( \frac{EI}{\rho A} \right) k^4 \right) \chi[\tau] = 0 \]

thus

\[ \omega^2 = \left( \frac{EI}{\rho A} \right) k^4 \]

and the wave speed is

\[ c = \frac{\omega}{k} = \left( \frac{EI}{\rho A} \right)^{1/2} k \]

The waves are highly dispersive with short wavelengths traveling much faster than long waves.
Nondimensionalize using the characteristic wave number and frequency

\[ x = k_0 \chi \quad t = \omega_0 \tau \]

where

\[ k_0 = \left( \frac{1}{I} \right)^{1/4} \quad \omega_0 = \left( \frac{E}{\rho A} \right)^{1/2} \]

In dimensionless variables the equation becomes

\[ y_{tt} + y_{xxxx} = 0 \]
Solution by Separation of variables.

We wish to solve

\[ y_{tt} + y_{xxxx} = 0 \]

Combine invariance under translation in time and invariance under dilation of the dependent variable

\[ X^a = \frac{\partial}{\partial t} \quad X^b = y \frac{\partial}{\partial y} \]

To form the group operator

\[ X = \frac{l}{\lambda} X^a + X^b = \frac{l}{\lambda} \frac{\partial}{\partial t} + y \frac{\partial}{\partial y} \]

With characteristic equations

\[ \frac{dx}{0} = \lambda dt = \frac{dy}{y} \]

With invariants \( x \) and \( y / e^{\lambda t} \).
We are seeking time-periodic solutions of the equation. Let

\[ \lambda = i\omega \]

The solution is of the form

\[ y(x, t) = e^{i\omega t} G(x) \]

\[ y_{tt} + y_{xxxx} = e^{i\omega t} (G_{xxxx} - \omega^2 G) = 0 \]

The fourth order ODE

\[ G_{xxxx} - \omega^2 G = 0 \]

has the general solution

\[ G(x) = Ae^{\omega^{1/2}x} + Be^{-\omega^{1/2}x} + CSin\left(\omega^{1/2}x\right) + DCos\left(\omega^{1/2}x\right) \]
Even and odd solutions are

\[ y[x, t] = e^{i\omega t} (ACosh[kx] + BCos[kx]) \]

\[ y[x, t] = e^{i\omega t} (CSinh[kx] + DSin[kxs]) \]

where

\[ \omega^2 = k^4 \]

Superposition of solutions for various frequencies and associated wave numbers can be used to match the boundary conditions for a given problem.
Lagrangian

The equation governing flexural waves in a beam is

\[ y_{tt} + y_{xxxx} = 0 \]

Solutions of this equation minimize the action integral

\[ S = \int \left( -\frac{1}{2}y_t^2 + \frac{1}{2}y_{xx}^2 \right) dt dx \]

ie, solutions minimize the volume integral of the difference between kinetic and potential energy. The equation can be generated from the Lagrangian

\[ L = -\frac{1}{2}y_t^2 + \frac{1}{2}y_{xx}^2 \]

Substitute the Lagrangian into the Euler-Lagrange equations

\[ \frac{\partial L}{\partial y} - D_t \left( \frac{\partial L}{\partial y_t} \right) - D_x \left( \frac{\partial L}{\partial y_x} \right) + D_{tt} \left( \frac{\partial L}{\partial y_{tt}} \right) + D_{tx} \left( \frac{\partial L}{\partial y_{tx}} \right) + D_{xx} \left( \frac{\partial L}{\partial y_{xx}} \right) = 0 \]

The result is

\[ -D_t \left( \frac{\partial L}{\partial y_t} \right) + D_{xx} \left( \frac{\partial L}{\partial y_{xx}} \right) = 0 \]

\[ -D_t (y_t) + D_{xx} (y_{xx}) = 0 \]

\[ y_{tt} + y_{xxxx} = 0 \]
Symmetries

The equation is invariant under a four parameter group of translations and dilations plus the infinite-dimensional group corresponding to linear superposition of solutions.

\begin{align*}
X^1 &= \frac{\partial}{\partial t} \\
X^2 &= \frac{\partial}{\partial x} \\
X^3 &= t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} \\
X^4 &= y \frac{\partial}{\partial y} \\
X^5 &= \phi(t, x) \frac{\partial}{\partial y}
\end{align*}

where $\phi(t, x)$ is a solution of $\phi_{tt} + \phi_{xxxx} = 0$. 
Conservation laws generated from symmetries

The relations used to generate components of the conserved vectors corresponding to variational symmetries of this equation are:

\[\theta^t = \mu \left( \frac{\partial L}{\partial y_t} - D_t \frac{\partial L}{\partial y_{tt}} - D_x \frac{\partial L}{\partial y_{tx}} \right) + D_t \mu \left( \frac{\partial L}{\partial y_{tt}} \right) + D_x \mu \left( \frac{\partial L}{\partial y_{tx}} \right)\]

\[\theta^x = \mu \left( \frac{\partial L}{\partial y_x} - D_t \frac{\partial L}{\partial y_{xt}} - D_x \frac{\partial L}{\partial y_{xx}} \right) + D_t \mu \left( \frac{\partial L}{\partial y_{xt}} \right) + D_x \mu \left( \frac{\partial L}{\partial y_{xx}} \right)\]

\[\Gamma^t = L \tau + \mu \left( \frac{\partial L}{\partial y_t} - D_t \frac{\partial L}{\partial y_{tt}} - D_x \frac{\partial L}{\partial y_{tx}} \right) + D_t \mu \left( \frac{\partial L}{\partial y_{tt}} \right) + D_x \mu \left( \frac{\partial L}{\partial y_{tx}} \right)\]

\[\Gamma^x = L \xi + \mu \left( \frac{\partial L}{\partial y_x} - D_t \frac{\partial L}{\partial y_{xt}} - D_x \frac{\partial L}{\partial y_{xx}} \right) + D_t \mu \left( \frac{\partial L}{\partial y_{xt}} \right) + D_x \mu \left( \frac{\partial L}{\partial y_{xx}} \right)\]

where the characteristic function is

\[\mu = \eta - \tau y_t - \xi y_x\]
Invariance under translation in time

Infinitesimals

\[ \tau = 1 \]
\[ \xi = 0 \]
\[ \eta = 0 \]

Characteristic function \[ \mu = -y_t \]

Conserved vector

\[ \theta^l = (-y_t)(-y_t) = y_t^2 \]
\[ \theta^x = (-y_t)(-y_{xxx}) - y_{xt}y_{xx} = y_t y_{xxx} - y_{xt}y_{xx} \]
\[ \Gamma^l = L\tau + \theta^l = \left(-\frac{1}{2}y_t^2 + \frac{1}{2}y_{xx}^2\right) + y_t^2 = \frac{1}{2}y_t^2 + \frac{1}{2}y_{xx}^2 \]
\[ \Gamma^x = L\xi + \theta^x = y_t y_{xxx} - y_{xt}y_{xx} \]

Check to see if this vector is indeed conserved!

\[ D_t\Gamma^l + D_x\Gamma^x = D_t\left(\frac{1}{2}y_t^2 + \frac{1}{2}y_{xx}^2\right) + D_x(y_t y_{xxx} - y_{xt}y_{xx}) = \]
\[ y_t y_{tt} + y_{xx}y_{xtt} + y_{xt}y_{xxx} + y_t y_{xxxx} - y_{xxt}y_{xx} - y_{xt}y_{xxx} = \]
\[ y_t y_{tt} + y_t y_{xxxx} = -\mu(y_{tt} + y_{xxxx}) = 0 \]
The quantity
\[ E = \frac{1}{2} y_t^2 + \frac{1}{2} y_{xx} \]
is the energy per unit length of the rod: kinetic energy + strain energy

Note that
\[ \int (D_t \dot{\Gamma} + D_x \Gamma^x) \, dx = \frac{d}{dt} \int_0^L \left( \frac{1}{2} y_t^2 + \frac{1}{2} y_{xx} \right) \, dx + (\Gamma^x |_L - \Gamma^x |_0) = 0 \]

If the bending moment and shear force vanish at the ends of the beam, i.e., the second and third spatial derivatives are zero, then the total energy is conserved.
Invariance under translation in space

Infinitesimals

\[ \tau = 0 \]
\[ \xi = 1 \]
\[ \eta = 0 \]

Characteristic function \( \mu = -y_x \)

Conserved vector

\[ \theta^i = (-y_x)(-y_t) = y_t y_x \]

\[ \theta^x = (-y_x)(-y_{xxx}) + (-y_{xx})(y_{xx}) = y_x y_{xxx} - \frac{1}{2} y_{xx}^2 \]

\[ I^i = L\tau + \theta^i = y_t y_x \]

\[ I^x = L\xi + \theta^x = -\frac{1}{2} y_t^2 + y_x y_{xxx} - \frac{1}{2} y_{xx}^2 \]

Check to see if this vector is indeed conserved!

\[ D_t I^i + D_x I^x = D_t(y_t y_x) + D_x \left( -\frac{1}{2} y_t^2 + y_x y_{xxx} - \frac{1}{2} y_{xx}^2 \right) = \]

\[ y_{tt} y_x + y_t y_{xt} - y_t y_{tx} + y_{xx} y_{xxx} + y_x y_{xxxx} - y_{xx} y_{xxx} = \]

\[ y_x y_{tt} + y_{xx} y_{xxx} = -\mu(y_{tt} + y_{xxxx}) = 0 \]
The quantity

\[ P = y_t y_x \]

can be regarded as the effective “momentum per unit length” of the rod.

Note that

\[
\int (D_t \Gamma' + D_x \Gamma^x) dx = \frac{d}{dt} \int_0^L (y_t y_x) dx + (\Gamma^x |_L - \Gamma^x |_0) = 0
\]

If the acceleration, bending moment and shear force vanish at the ends of the beam, i.e., the time, second and third spatial derivatives are zero, then the total momentum is conserved.
Invariance under translation in $y(x,t)$

Infinitesimals

$\tau = 0$
$\xi = 0$
$\eta = 1$

Characteristic function $\mu = 1$

Conserved vector

$\theta^t = -y_t$

$\theta^x = -y_{xxx}$

$I^t = L\tau + \theta^t = -y_t$

$I^x = L\xi + \theta^x = -y_{xxx}$

Check to see if this vector is indeed conserved!

$D_t I^t + D_x I^x = D_t(-y_t) + D_x(-y_{xxx}) = -y_{tt} - y_{xxxx} = -\mu(y_{tt} + y_{xxxx}) = 0$
The quantity

\[ m = -y_t \]

can be regarded as the effective “mass per unit length” of the rod.

Note that

\[
\int (D_t \Gamma^t + D_x \Gamma^x) \, dx = \frac{d}{dt} \int_0^L (-y_t) \, dx + (\Gamma^x |_L - \Gamma^x |_0) = 0
\]

If the shear force vanishes at the ends of the beam, ie the third spatial derivative is zero, then the total mass is conserved.
Invariance under dilation

Infinitesimals
\[ \tau = 4t \]
\[ \xi = 2x \]
\[ \eta = y \]

Characteristic function
\[ \mu = y - 4ty_t - 2xy_x \]

Conserved vector
\[ \theta^t = (y - 4ty_t - 2xy_x)(-y_{xxx}) + (y_x - 4ty_{tx} - 2y_x - 2xy_xx)(y_{xx}) \]
\[ \theta^x = -xy_t^2 - yy_{xxx} + 4ty_y y_{xxx} + 2xy_x y_{xxx} - 4ty_{tx} y_{xx} - y_x y_{xx} - xy_{xx} \]

Check to see if this vector is indeed conserved!
\[ D_t \Gamma^t + D_x \Gamma^x = \]
\[ -yy_{tt} + 4ty_t y_{tt} + 2xy_x y_{tt} + \]
\[ -yy_{xxxx} + 4ty_t y_{xxxx} + 2xy_x y_{xxxx} = \]
\[ (-y + 4ty_t + 2xy_x)(y_{tt} + y_{xxxx}) = \]
\[ -\mu(y_{tt} + y_{xxxx}) = 0 \]
Integrate along the beam

\[ \int (D_t \Gamma^t + D_x \Gamma^x) \, dx = \frac{d}{dt} \int_0^L (2ty_{xx}^2 - yy_t + 2ty_t^2 + 2xy_x y_t) \, dx + (\Gamma^x |_L - \Gamma^x |_0) = 0 \]

If the acceleration, bending moment and shear force vanish at the ends of the beam, i.e., the time, second and third spatial derivatives are zero, then the time integral above is conserved.

Note that the integral involves the mass momentum and energy per unit length of the rod.

\[ \frac{d}{dt} \int_0^L (2ty_{xx}^2 - yy_t + 2ty_t^2 + 2xy_x y_t) \, dx = \frac{d}{dt} \int_0^L (ym + 2xP + 4tE) \, dx = 0 \]