AA218 - Introduction to Symmetry Analysis

Chapter 1 - Introduction to Symmetry

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Course Goal

Learn how to find the symmetries of a differential equation.

Learn how to use those symmetries to solve the equation.

AA218 is fundamentally a course about solving ODEs and PDEs.

What sets symmetry methods apart from other analytical methods for solving differential equations is that they provide a procedure for systematically solving <u>nonlinear</u> equations.

Typical applications of interest

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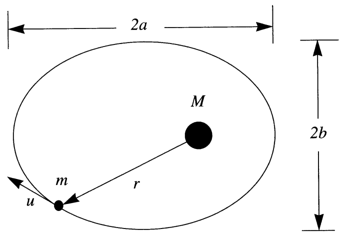
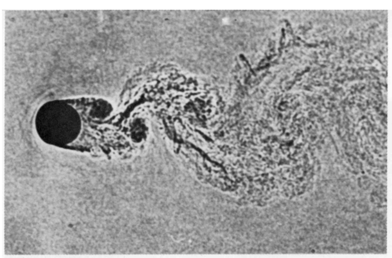
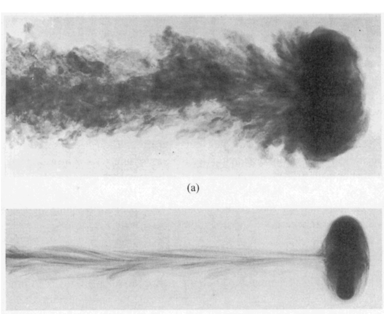
The two-body problem

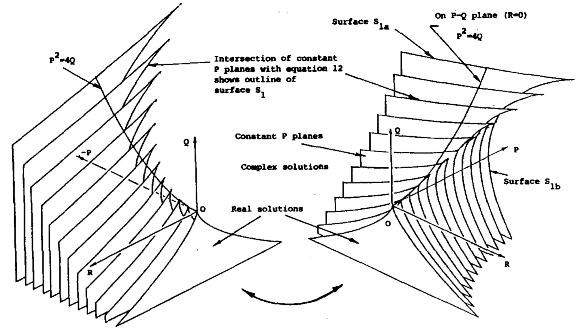


Dimensional analysis

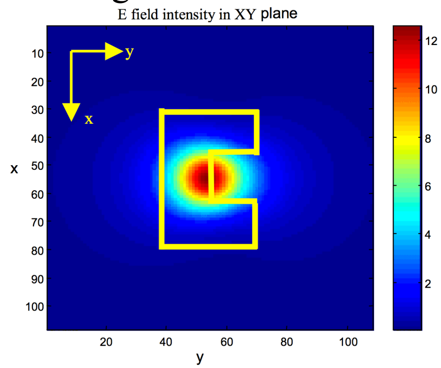




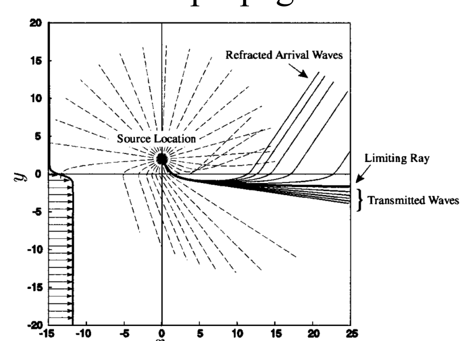
Geometry of 3D vector fields



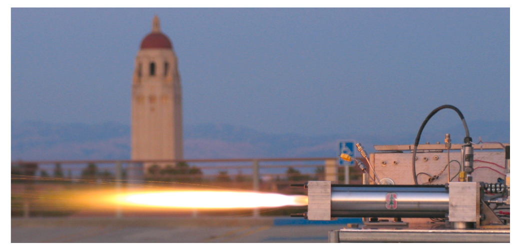
Light transmission



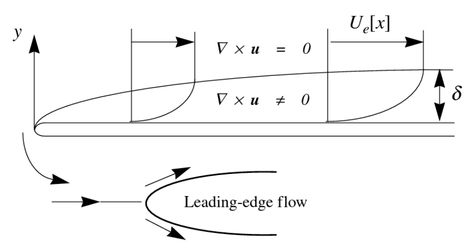
Sound propagation



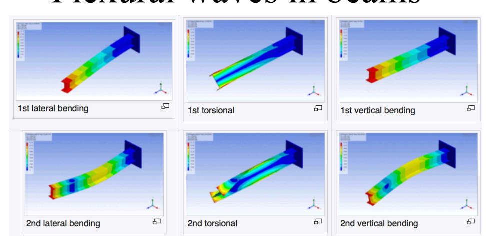
Propulsion



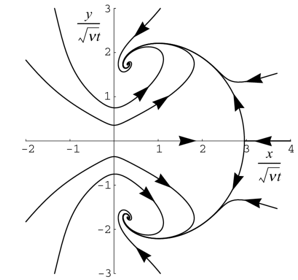
Boundary layers



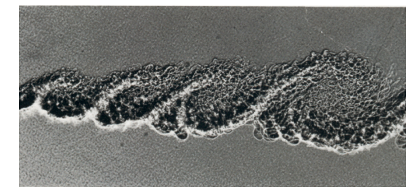
Flexural waves in beams

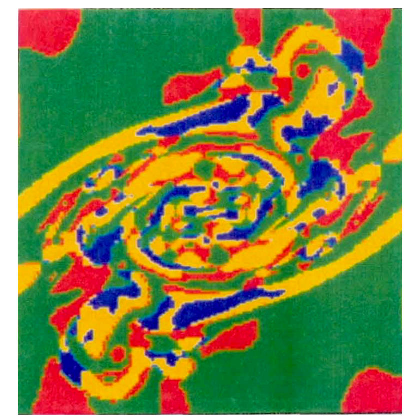


Elliptic curves and flow patterns

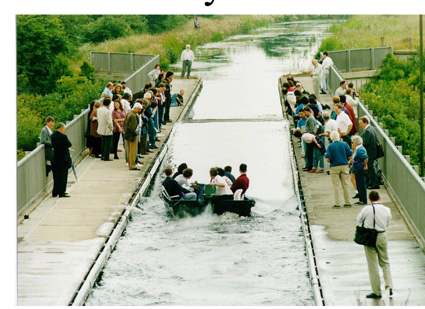


Turbulent flow





Solitary waves



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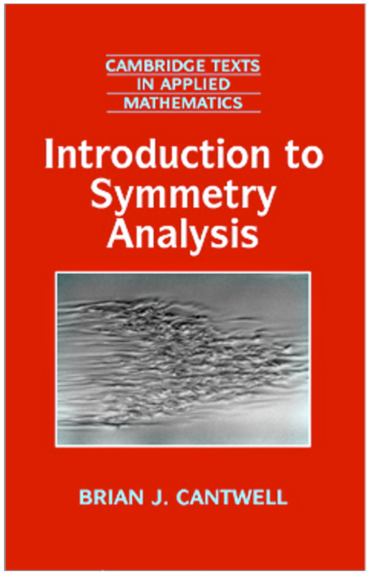

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AA218 - Introduction to Symmetry Analysis

Topics covered
1) Introduction to symmetry
2) Symmetry of functions, dimensional analysis
3) Review of ODEs, first-order PDEs, state-space analysis in 2-D and 3-D
4) Introduction to one-parameter Lie groups, examples of groups
5) Infinitesimal transformations, group operators, Lie series expansion of a function
6) **`IntroToSymmetry.m`** software package
7) Multi-parameter groups, Lie algebras
8) Application to first-order ODE's, integrating factors, differential functions
9) Extended groups, invariance of higher-order ODEs, reduction of order
10) Similarity variables for PDEs, reduction of dimension
11) Invariant groups of the classical equations of mathematical physics
12) Nonlocal groups, use of symmetries to generate solitary wave solutions
13) Noether's theorem and the connection between symmetries and conservation laws

Tentative list of examples.
1) Several examples from dimensional analysis
2) The two-body problem, symmetries in Kepler's laws
3) The geometry of light transmission through apertures
4) Laminar boundary layers
5) Sound propagation through a shear layer
6) Thermal gradient shocks in nonlinear heat conduction
7) Elliptic curves and flow patterns
8) Similarity rules for turbulent shear flows
9) Problems in nonlinear wave propagation, solitary waves
10) Flexural waves in a thin elastic beam

Grading - Homeworks will be assigned each week. Following guidelines set out by the Faculty Senate, grading will be on an S/NC basis. Substantial completion of the homeworks is needed to achieve a satisfactory grade. You will need to work problems to learn the methods of the course which do take practice to master. But during the second half of the course I will reduce the homework load to place more emphasis on a creation of your own making. I will suggest a number of possible projects but I am very open to your suggestions; it could be a problem derived from your research, or some other area of interest you might have from finance, biology, physics, chemistry, control theory, the grid, etc, Symmetry methods can be applied to virtually any field. This does not mean you have to solve some very complex problem. Your project might be exposition of an example from the literature where symmetry methods are applied. It could be something of current interest, such as say, the problem of viral spread or the problem of maintaining a supply chain in the face of disruption, etc.
e-Resources – If you need free access to journal articles through the Stanford system this link, https://library.stanford.edu/using/connecting-e-resources will help you set that up on your home computer.
Course material – Materials for all my courses are available at my website at https://web.stanford.edu/~cantwell/. The folder AA218_Course_Material includes a folder containing a pdf of the course text, *Introduction to Symmetry Analysis*, a folder with pdfs of the lectures, and a folder where homework assignments will be posted. In addition, selected papers related to the course material are also included in a Resources folder.
Software – There is a Mathematica package on my website that can be used to find the symmetries of ODEs and PDEs. It can be downloaded as a zip file or as individual files from: https://web.stanford.edu/~cantwell/SymmetryAnalysisSoftware. Instructions for use of the software can be found in Appendix 4 of the text. You should all be able to access the cluster computers at Stanford remotely using https://cluster-checkout.stanford.edu/. Both Mac and Windows systems should have Mathematica available (thanks to Eric Zelikman for that information).
Suggested prerequisites – Math 53 and Math 131 or equivalent background in differential equations. If you have questions or concerns, please contact me at cantwell.brianj@gmail.com.

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Recommended references

Barenblatt, G.I. 1996. *Scaling, Self-similarity, and Intermediate Asymptotics*. Cambridge Press.

Birkhoff, G.D. 1933. *Aesthetic Measure*. Harvard University Press.

Bluman, G. and Kumei, S. 1989. *Symmetries and Differential Equations*. Applied Mathematical Sciences 81. Springer-Verlag. Also more recent texts by Bluman and Anco 2008, and Bluman, Cheviakov and Anco 2010.

Hydon, P.E. 2000. *Symmetry Methods for Differential Equations: A Beginner's Guide*. Cambridge Press.

Ibragimov, N. 1994-1996. *CRC Handbook of Lie Group Analysis of Differential Equations, Volumes I, II, and III*. CRC Press.

Ibragimov, N.H. 1998. *Elementary Lie Group Analysis and Ordinary Differential Equations*. John Wiley & Sons.

Nussbaum, A. 1971. *Applied Group Theory for Chemists, Physicists and Engineers*. Prentice Hall.

Olver, P. 1986. *Applications of Lie groups to Differential Equations*. Graduate texts in mathematics 107. Springer-Verlag.

Stephani, H. 1989. *Differential Equations: Their Solution Using Symmetries*. Cambridge Press.

Weyl, H. 1952 *Symmetry*. Princeton University Press.

The software *Maplesoft* has an extensive set of functions for finding the symmetries of DEs

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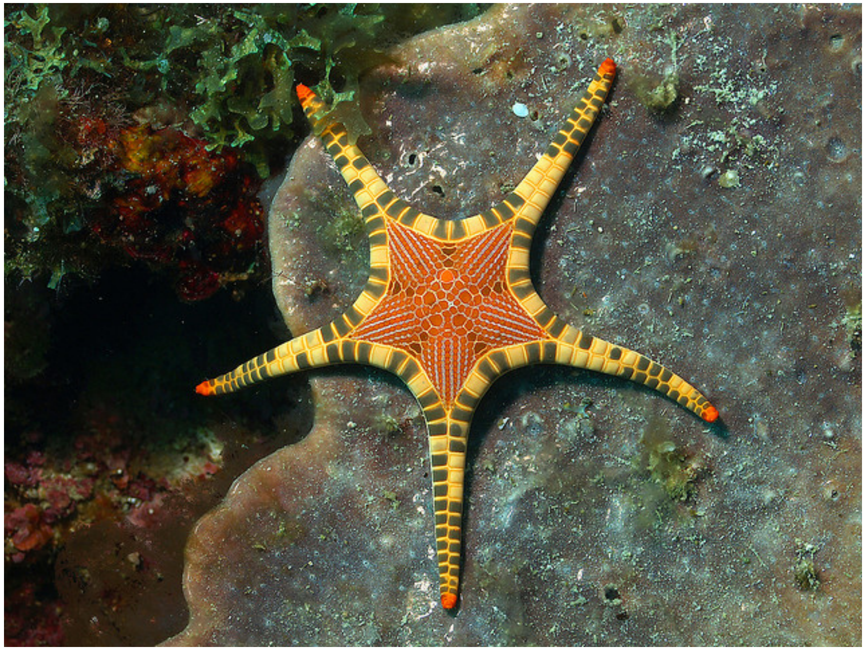
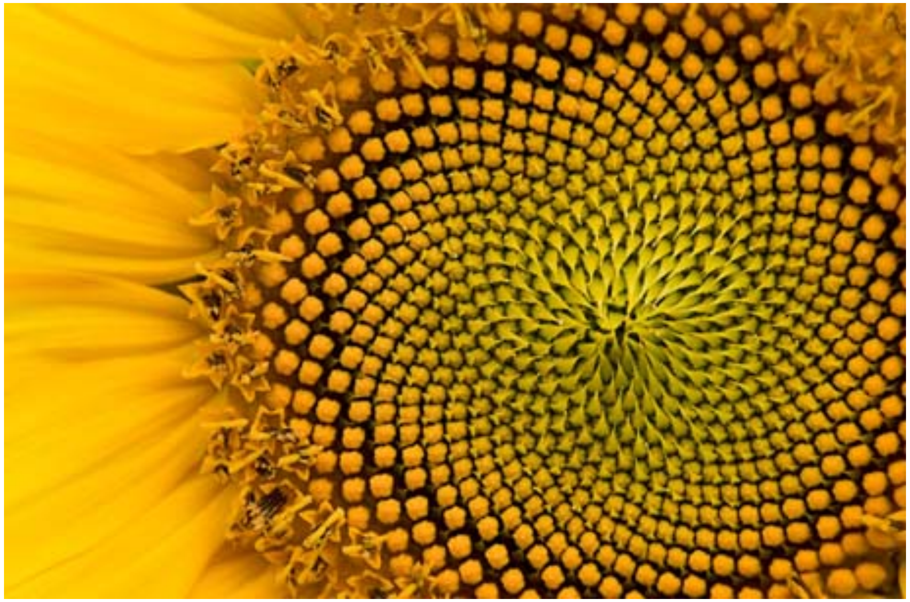
(1.1) Symmetry in Nature



Iconaster Longimanus



Sunflower

Angelina Jolie

Angelina Jolie

Original Image

Right side reflection

Left side reflection

(1.4) The twelve-fold discrete symmetry group of a snowflake

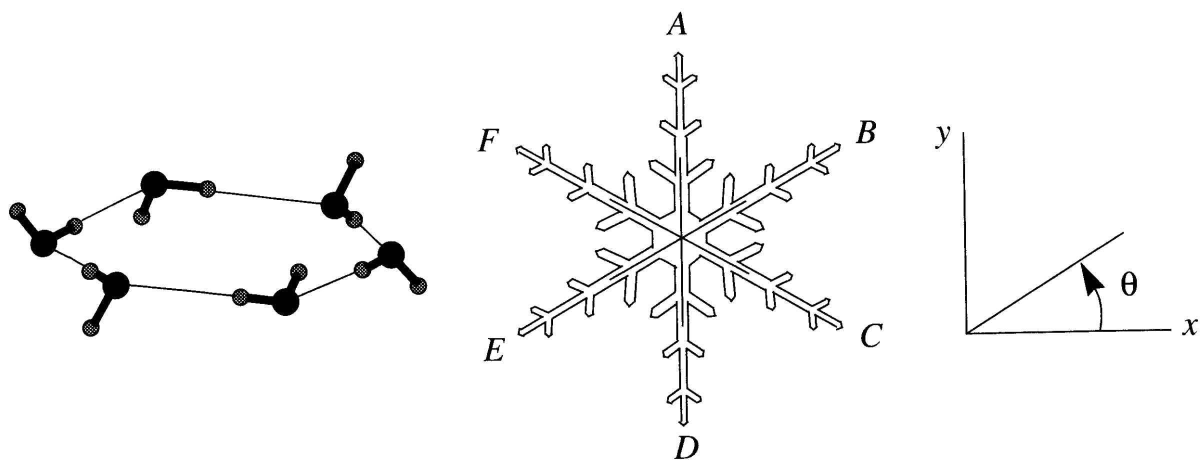


Figure 1.1 Hexagonal structure of ice crystals and snowflakes

Symmetry operations

Suppose we rotate the snowflake by 30°

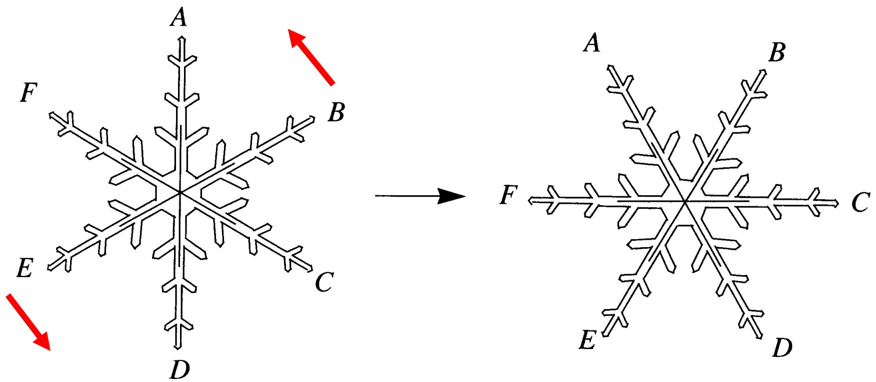


Figure 1.2 Counter-clockwise rotation by 30°

One can tell that the snowflake has been rotated.
Therefore the 30° rotation is not a symmetry operation for the snowflake.

Symmetry operations

Suppose we rotate the snowflake by 30°

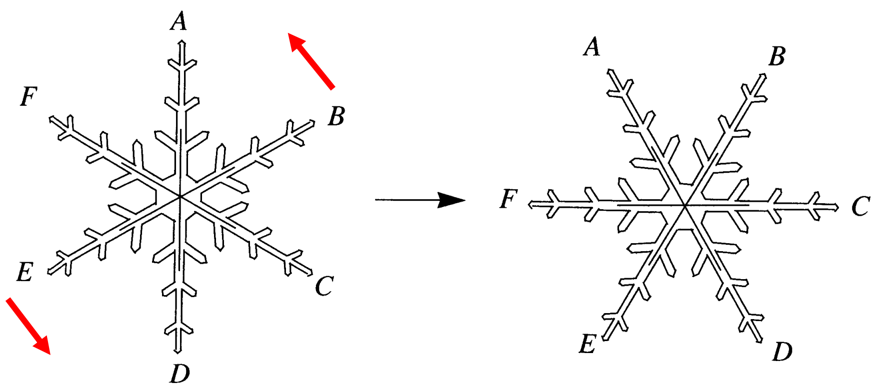


Figure 1.2 Counter-clockwise rotation by 30°

Suppose we rotate the snowflake by 120°

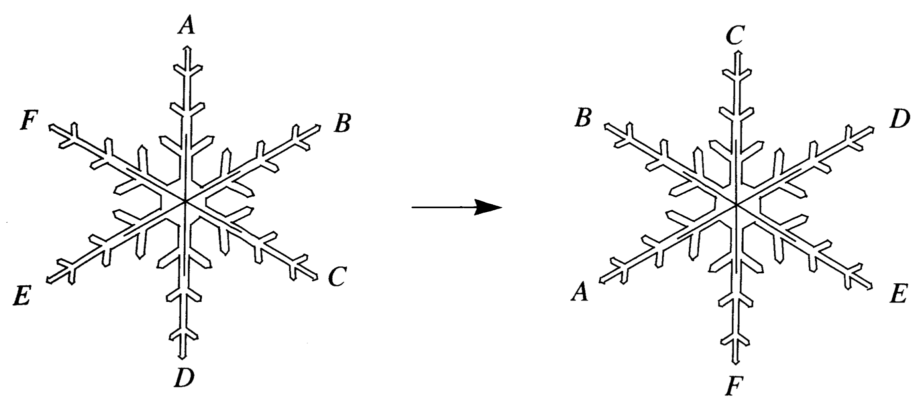


Figure 1.3 Counter-clockwise rotation by 120°

We can express the symmetry properties of the snowflake mathematically as a transformation.

$$\begin{aligned}\tilde{x} &= x\cos\theta - y\sin\theta,\\ \tilde{y} &= x\sin\theta + y\cos\theta.\end{aligned} \tag{1.1}$$

Insert the discrete values 60° , 120° , 180° , 240° , 300° and 360° . The result is a set of six matrices corresponding to the six rotations.

$$\begin{gathered} C_6^1 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \qquad C_6^2 = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \qquad C_6^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \\ C_6^4 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \qquad C_6^5 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{gathered} \tag{1.2}$$

What about reflections?

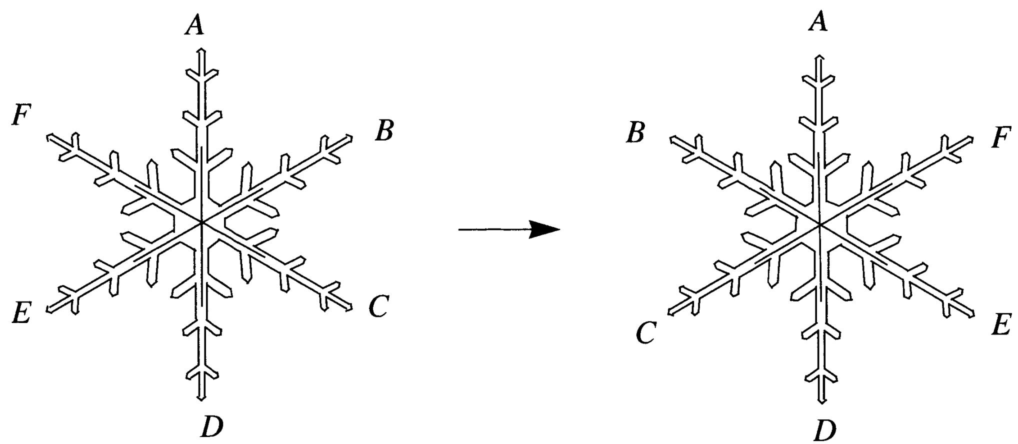


Figure 1.4 Reflection through a vertical axis

The reflection through A-D can be expressed as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} \tag{1.3}$$

Another reflectional symmetry is through axis a-d which splits the angle between A-D and B-E as shown in the figure below.

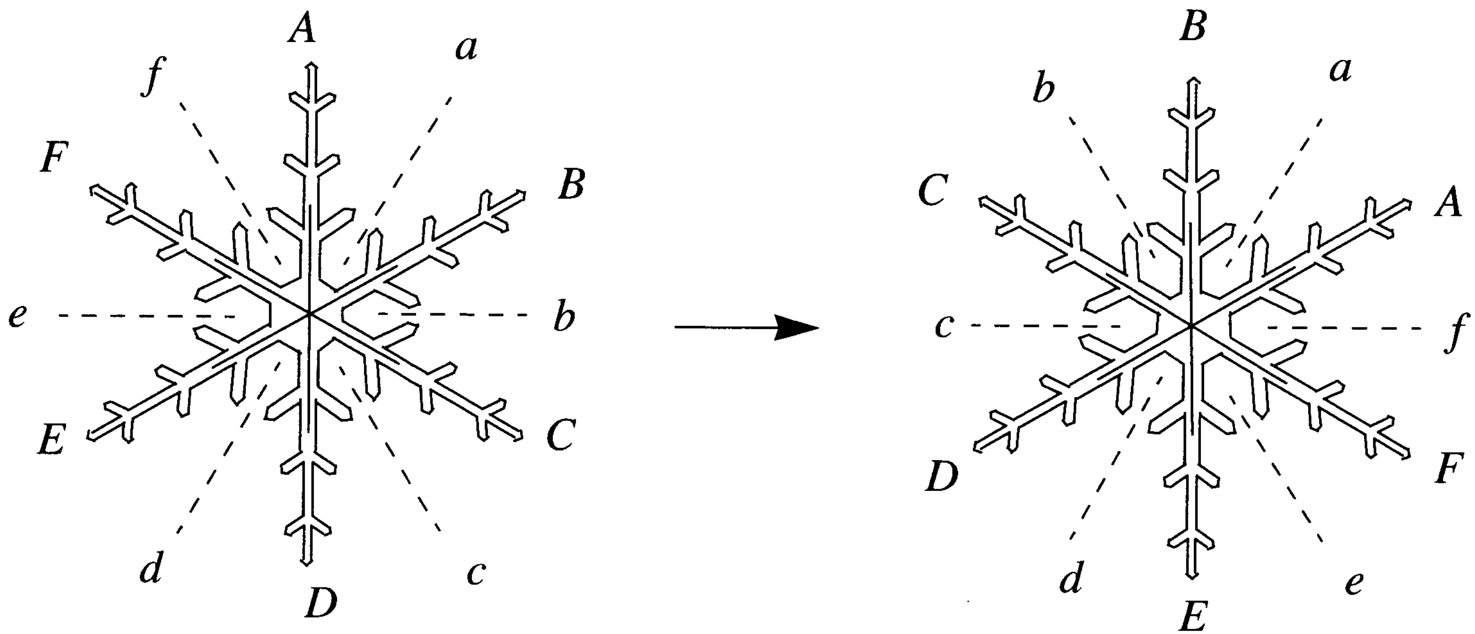


Figure 1.5 Reflection axes of a snowflake

The six reflections are

$$\sigma_v^{AD} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_v^{BE} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \quad \sigma_v^{CF} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$
$$\sigma_v^{be} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_v^{ad} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad \sigma_v^{cf} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \tag{1.4}$$

Group properties

If we combine operations via matrix multiplication, the result is always equal to one of the twelve members of the set. For example.

$$C_6^2\sigma_\nu^{CF} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \sigma_\nu^{BE} \qquad (1.5)$$

Note that commutation leads to a different result

$$\sigma_\nu^{CF} C_6^2 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \sigma_\nu^{AD} \qquad (1.6)$$

The group is *closed* under matrix multiplications

The twelve matrices

$$C_6^1,\, C_6^2,\, C_6^3,\, C_6^4,\, C_6^5,\, E,\, \sigma_\nu^{AD},\, \sigma_\nu^{BE},\, \sigma_\nu^{CF},\, \sigma_\nu^{ad},\, \sigma_\nu^{be},\, \sigma_\nu^{cf}$$

Are said to form a *discrete group* with respect to the operation of multiplication

Every member of the group has an *inverse*. For example

$$C_6^2 C_6^4 = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = E \qquad (1.7)$$

The group is *associative.* For example

$$(\sigma_\nu{}^{AD} C_6^2)\sigma_\nu{}^{CF} = \sigma_\nu{}^{AD}(C_6^2 \sigma_\nu{}^{CF}) \qquad (1.8)$$

Finally there is an *identity element*, *E*, which is a member of the group

Groups may contain subgroups. For example the subset

$$C_6^1, C_6^2, C_6^3, C_6^4, C_6^5, E$$

constitutes a group whereas the subset

$$E, \sigma_\nu^{AD}, \sigma_\nu^{BE}, \sigma_\nu^{CF}$$

does not, since

$$\sigma_\nu^{AD} \sigma_\nu^{BE} = C_6^2$$

is not a member of the subset

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(1.4) The principle of covariance

Definition 1.2 (The principle of covariance). *The equations that govern a physical law must retain the same appearance under certain group transformations of the variables that appear in the equations. This principle incorporates two somewhat distinct ideas.*

(i) Coordinate independence – Physical phenomena must be governed by laws that do not depend on the coordinate system used to describe the phenomena.
(ii) Dimensional homogeneity – Physical phenomena must be described by laws that do not depend on the unit of measure applied to the dimensions of the variables that describe the phenomena.

(1.5) Continuous symmetries of functions and differential equations

One parameter Lie groups in the plane

$$\tilde{x} = F[x, y, s]$$
$$\tilde{y} = G[x, y, s] \qquad (1.9)$$

F and *G* are real analytic functions in the group parameter *s* and so can be expanded in a Taylor series about any value on the open interval that contains *s*.

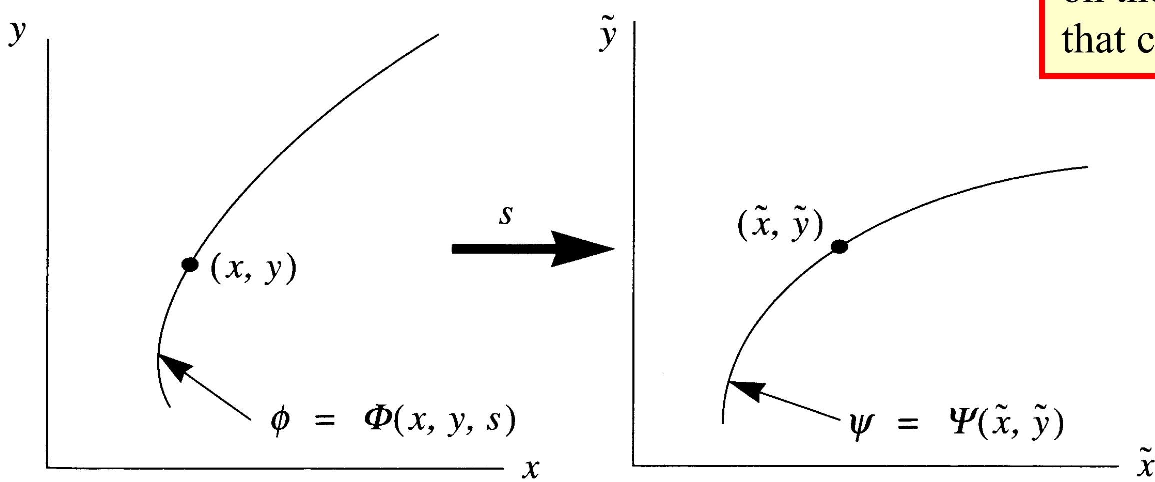
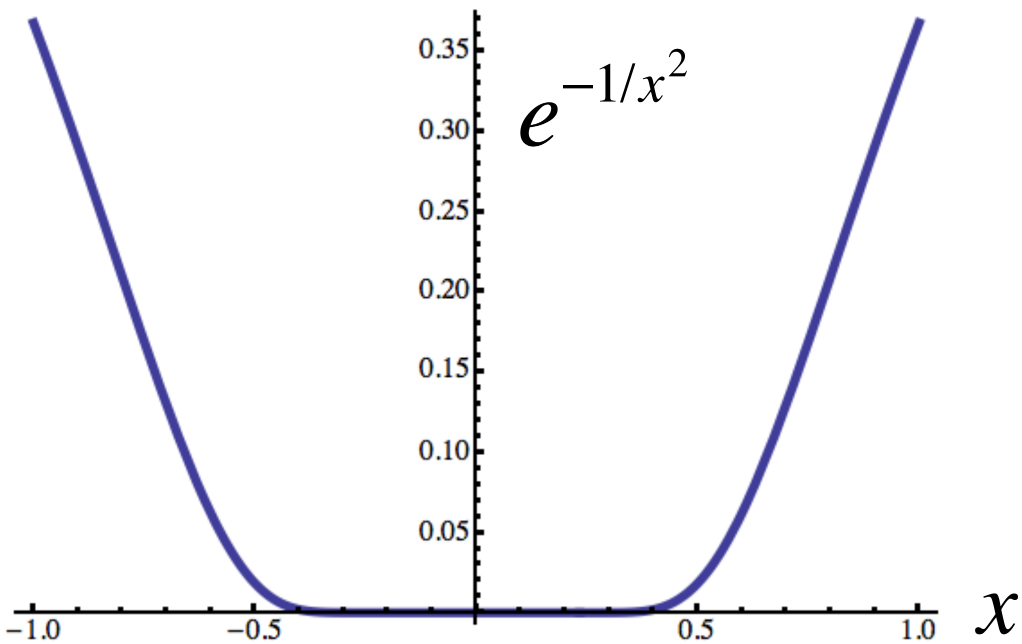


Figure 1.6 Mapping of a point and a curve by the group

At *s=0* the transformation reduces to an identity

$$x = F[x, y, 0]$$
$$y = G[x, y, 0] \qquad (1.10)$$

Sometimes we use the word smooth to describe analytic functions but not all smooth functions are analytic. For example



Derivatives at x = 0 are zero to all orders and so the function cannot be expanded in a Taylor series about $x=0$.

Invariance of functions, ODEs and PDEs under Lie groups

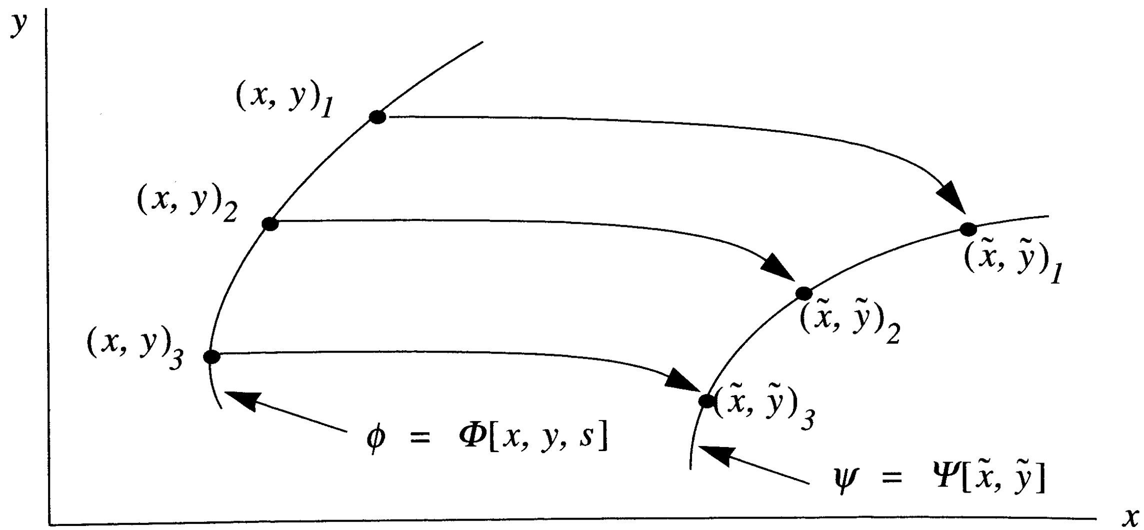


Figure 1.7 Mapping of source to target points by the group

A function is transformed as follows

$$\psi = \Psi[\tilde{x}, \tilde{y}] = \Psi(F[x, y, s], G[x, y, s]) = \Phi[x, y, s] = \phi \quad (1.11)$$

A function is invariant if

$$\Psi[\tilde{x}, \tilde{y}] = \Phi[x, y, s] = \Psi[x, y] \quad (1.12)$$

The symmetry of a first order ODE is analyzed in the tangent space (*x, y, dy/dx*)

Transform the first derivative with the group parameter s constant.

$$\frac{d\tilde{y}}{d\tilde{x}} = \tilde{y}_{\tilde{x}} = \frac{dG}{dF} = \frac{\frac{\partial G}{\partial x}\,dx + \frac{\partial G}{\partial y}\,dy}{\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy} = \frac{G_x + G_y\,\frac{dy}{dx}}{F_x + F_y\,\frac{dy}{dx}} = G_{\{1\}}[x, y, y_x, s] \qquad (1.13)$$

Use the transformation (1.9) and (1.13) to transform an ODE of the form

$$\psi = \Psi\left[\tilde{x}, \tilde{y}, \frac{d\tilde{y}}{d\tilde{x}}\right] \qquad (1.14)$$

Transform a first order ODE as follows.

$$\begin{aligned}\psi &= \Psi\left[\tilde{x}, \tilde{y}, \frac{d\tilde{y}}{d\tilde{x}}\right] \\ &= \Psi\left[F[x, y, s], G[x, y, s], G_{\{1\}}\left[x, y, \frac{dy}{dx}, s\right]\right] \\ &= \Phi\left[x, y, \frac{dy}{dx}, s\right] = \phi.\end{aligned} \tag{1.15}$$

If the equation reads the same in new variables,

$$\Psi\left[\tilde{x}, \tilde{y}, \frac{d\tilde{y}}{d\tilde{x}}\right] = \Phi\left[x, y, \frac{dy}{dx}, s\right] = \Psi\left[x, y, \frac{dy}{dx}\right], \tag{1.16}$$

then the equation is invariant under the group

Example 1.1 Invariance of a first-order ODE under a Lie group

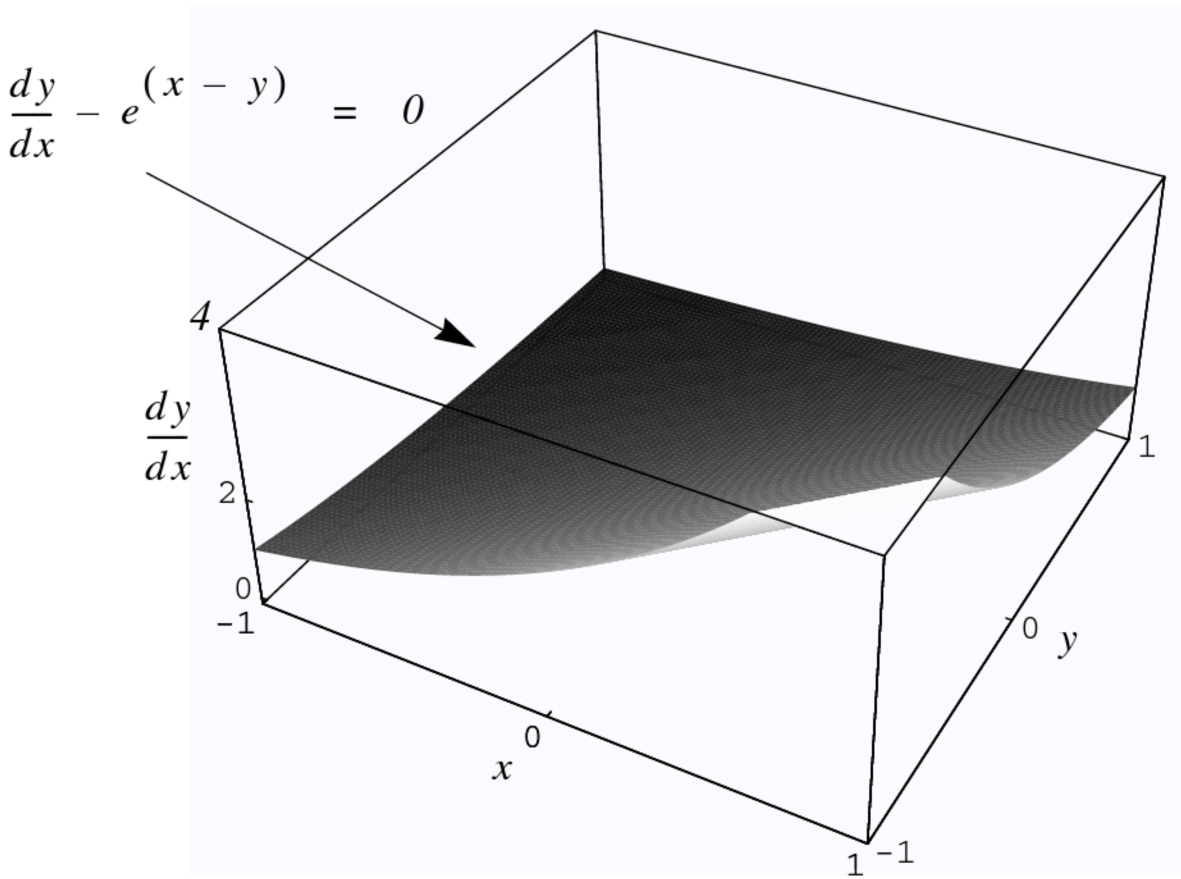


Figure 1.8 The surface defined by a first order ODE

Extended translation group

$$\begin{aligned} \tilde{x} &= x + s, \\ \tilde{y} &= y + s, \\ \frac{d\tilde{y}}{d\tilde{x}} &= \frac{dy}{dx} \end{aligned} \tag{1.17}$$

Transform the equation

$$\begin{aligned} \Psi\left[\tilde{x}, \tilde{y}, \frac{d\tilde{y}}{d\tilde{x}}\right] &= \frac{d\tilde{y}}{d\tilde{x}} - e^{\tilde{x}-\tilde{y}} = \frac{dy}{dx} - e^{(x+s)-(y+s)} \\ &= \frac{dy}{dx} - e^{x-y} = \Psi\left[x, y, \frac{dy}{dx}\right]. \end{aligned} \tag{1.18}$$

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General solution

$$\psi = \Psi[x, y] = e^y - e^x, \tag{1.19}$$

Action of the group on a given solution curve

$$\tilde{\psi} = \Psi[\tilde{x}, \tilde{y}] = e^{\tilde{y}} - e^{\tilde{x}} = e^{y+s} - e^{x+s} = e^s(e^y - e^x). \tag{1.20}$$

The solution curve (1.19) is transformed to

$$\frac{\tilde{\psi}}{e^s} = e^y - e^x. \tag{1.21}$$

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Example 1.2 Invariance of a PDE – Diffusion of heat in a conducting solid

The problem is governed by the linear diffusion equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \tag{1.22}$$

with boundary conditions

$$\begin{aligned} t < 0: &\quad u(0,t) = 0, \quad u(\infty,t) = 0, \\ t \geq 0: &\quad u(0,t) = u_0, \quad u(\infty,t) = 0. \end{aligned} \tag{1.23}$$

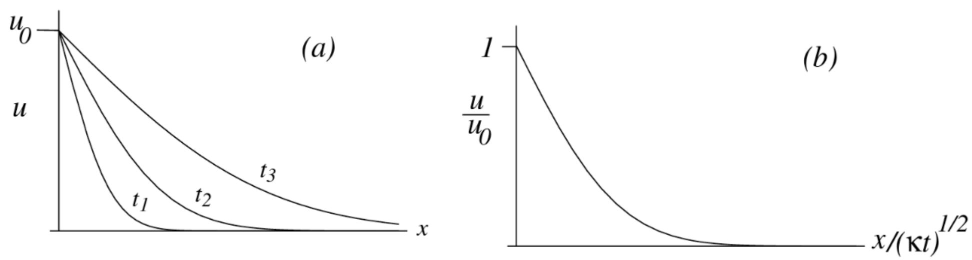

Figure 1.9: *Diffusion of temperature in a semi-infinite solid: (a) physical coordinates, (b) similarity coordinates.*

Test for invariance under a three-parameter dilation group.

$$\tilde{x} = e^{a} x, \qquad \tilde{t} = e^{b} t, \qquad \tilde{u} = e^{c} u. \tag{1.24}$$

The exponential factors out of the derivative

$$\begin{gathered}
\frac{\partial \tilde{u}}{\partial \tilde{x}} = e^{c-a} \frac{\partial u}{\partial x}, \qquad \frac{\partial \tilde{u}}{\partial \tilde{t}} = e^{c-b} \frac{\partial u}{\partial t} \\
\frac{\partial^2 \tilde{u}}{\partial \tilde{t}^2} = e^{c-2b} \frac{\partial^2 u}{\partial t^2}, \qquad \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} = e^{c-2a} \frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial^2 \tilde{u}}{\partial \tilde{x}\, \partial \tilde{t}} = e^{c-a-b} \frac{\partial^2 u}{\partial x\, \partial t}.
\end{gathered} \tag{1.25}$$

The equation transforms as

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} - \kappa \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} = e^{c-b} \frac{\partial u}{\partial t} - \kappa e^{c-2a} \frac{\partial^2 u}{\partial x^2}. \tag{1.26}$$

Invariance holds only if

$$b = 2a$$

This is clear from the following

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} - \kappa \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} = e^{c-2a}\left(\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2}\right) \Rightarrow \begin{array}{l} \dfrac{\partial \tilde{u}}{\partial \tilde{t}} - \kappa \dfrac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} = 0, \\ \dfrac{\partial u}{\partial t} - \kappa \dfrac{\partial^2 u}{\partial x^2} = 0. \end{array} \quad (1.27)$$

The equation reads exactly the same in new variables.

The boundaries at $t = 0, x = 0$

And $x = \infty$ are invariant under the group

The value of the solution on the boundary

$$\tilde{u}(0, \tilde{t}) = u_0 \quad \Rightarrow \quad e^c u(0, e^{2a} t) = u_0. \tag{1.28}$$

is invariant only if

$$c = 0$$

Therefore the problem as a whole is invariant under the one-parameter group

$$\tilde{x} = e^a x, \qquad \tilde{t} = e^{2a} t, \qquad \tilde{u} = u. \tag{1.29}$$

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We can exploit this symmetry to solve the problem

Similarity variables that are invariant under the group are:

$$\frac{\tilde{u}}{u_0} = \frac{u}{u_0}, \qquad \frac{\tilde{x}}{\tilde{t}^{1/2}} = \frac{e^a}{e^a}\frac{x}{t^{1/2}} = \frac{x}{t^{1/2}}. \tag{1.30}$$

We can expect a solution of the form

$$\phi = \Phi\left[\frac{u}{u_0}, \frac{x}{\sqrt{\kappa t}}\right] \tag{1.31}$$

Without loss of generality

$$\frac{u}{u_0} = U\left[\frac{x}{\sqrt{\kappa t}}\right]. \tag{1.32}$$

Substitute the similarity form into the heat equation

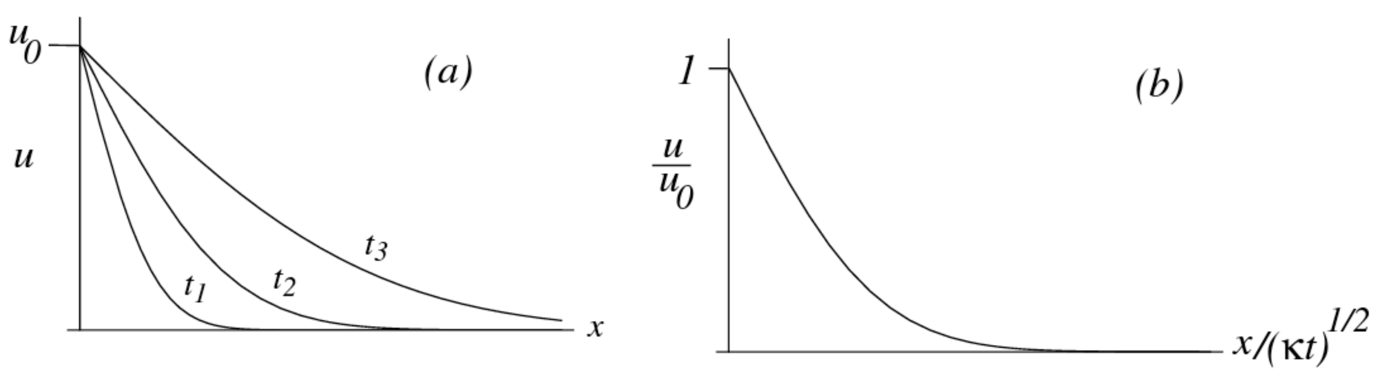
$$U_{\varsigma\varsigma} + \frac{\varsigma}{2} U_{\varsigma} = 0, \qquad U(0) = 1, \qquad U(\infty) = 0 \tag{1.34}$$

where

$$\varsigma = \frac{x}{\sqrt{\kappa t}}$$

The solution is expressed in terms of the complimentary error function

$$U = erfc(\varsigma) = 1 - \frac{1}{\pi} \int_0^{\varsigma} e^{-\varsigma'^2/4} d\varsigma' \tag{1.35}$$



In general, suppose a Lie group of the form

$$
\begin{aligned}
\tilde{x} &= F[x, t, u, s], \\
\tilde{t} &= T[x, t, u, s], \\
\tilde{u} &= G[x, t, u, s]
\end{aligned}
\tag{1.36}
$$

transforms some partial differential equation

$\tilde{u}_{\tilde{t}}$

$$
\Phi[\tilde{x}, \tilde{t}, \tilde{u}, \tilde{u}_{\tilde{x}}, \tilde{u}_{\tilde{x}\tilde{x}}, \tilde{u}_{\tilde{x}\tilde{t}}, \tilde{u}_{\tilde{t}\tilde{t}}, \ldots] = 0. \tag{1.37}
$$

to itself, ie, the untildaed variables satisfy

$$
\Phi[x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots] = 0 \tag{1.38}
$$

Then the transformation can be used to construct one solution from another.

Example 1.3 – Solutions generated from symmetries

The nonlinear PDE

$$u_t + \frac{1}{2}(u_x)^2 - u_{xx} = 0 \tag{1.39}$$

Is invariant under the transformation

$$\begin{aligned} \tilde{x} &= x, \\ \tilde{t} &= t, \\ \tilde{u} &= u - 2\ln\left(1 - f[x,t]e^{u/2}\right), \end{aligned} \tag{1.40}$$

where

$$f_t - f_{xx} = 0$$

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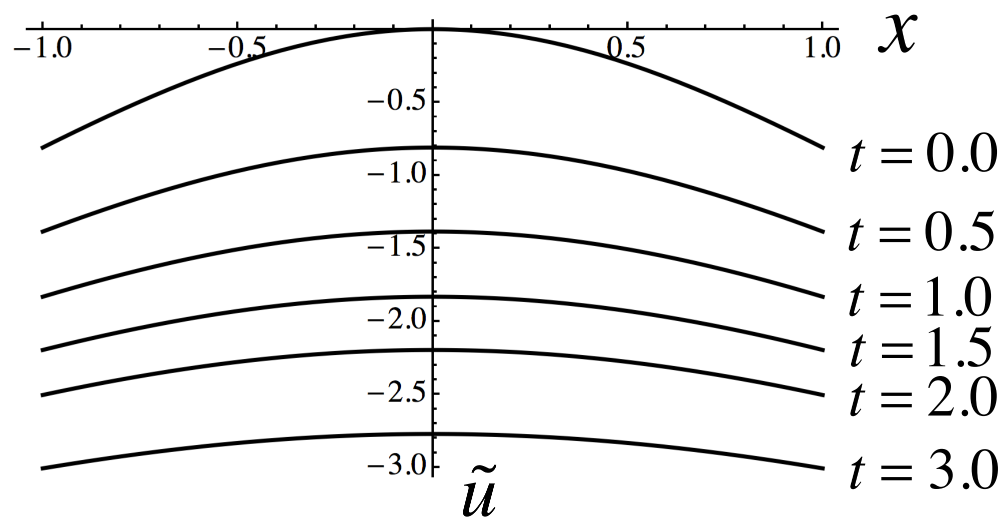
For example, let u be the so-called vacuum solution, $u=0$, and let $f = -t - x^2/2$ then

$$\tilde{u} = \ln\left(\frac{1}{1+t+x^2/2}\right)^2$$

is an exact solution of the given nonlinear equation.



The transformation (1.40) raises some obvious questions: Where in the world does it come from and can it be found through a systematic procedure? Is it a Lie group? The answer to both is YES.

Another important question: How do we prove that this transformation maps the equation to itself?

Approach - Generate transformations of the various terms that appear in the equation and add them together.

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$$\tilde{u}_{\tilde{t}} = D_t\left(u - 2\ln\left[1 - f[x,t]e^{u/2}\right]\right) = u_t + \frac{\left(f_t + \frac{f}{2}u_t\right)2e^{u/2}}{1 - fe^{u/2}}$$

Generate derivatives

$$\tilde{u}_{\tilde{x}} = D_x\left(u - 2\ln\left[1 - f[x,t]e^{u/2}\right]\right) = u_x + \frac{\left(f_x + \frac{f}{2}u_x\right)2e^{u/2}}{1 - fe^{u/2}}$$

$$(\tilde{u}_{\tilde{x}})^2 = (u_x)^2 + \frac{\left(2u_x f_x + f(u_x)^2\right)2e^{u/2}}{1 - fe^{u/2}} + \frac{\left(f_x + \frac{f}{2}u_x\right)^2 4e^{u}}{\left(1 - fe^{u/2}\right)^2}$$

$$\tilde{u}_{\tilde{x}\tilde{x}} = D_{xx}\left(u - 2\ln\left[1 - f[x,t]e^{u/2}\right]\right) =$$

$$u_{xx} + \frac{\left(f_{xx} + f_x u_x + \frac{f}{2}u_{xx} + \frac{f}{4}(u_x)^2\right)2e^{u/2}}{1 - fe^{u/2}} + \frac{\left(f_x + \frac{f}{2}u_x\right)^2 2e^{u}}{\left(1 - fe^{u/2}\right)^2}$$

Add to form the equation. If u is a solution and

$$f_t - f_{xx} = 0$$

Then $\tilde{u}$ is a solution

$$\boxed{\begin{aligned}\tilde{u}_{\tilde{t}} + \frac{1}{2}(\tilde{u}_{\tilde{x}})^2 - \tilde{u}_{\tilde{x}\tilde{x}} &= \left(u_t + \frac{1}{2}(u_x)^2 - u_{xx}\right) + \\ &\quad \frac{(f_t - f_{xx}) + \frac{f}{2}\left(u_t + \frac{1}{2}(u_x)^2 - u_{xx}\right)}{1 - fe^{u/2}}2e^{u/2}\end{aligned}}$$

(1.6) Some Notation Conventions

In group theory we make use of transformations of the following form

$$
\begin{aligned}
\tilde{x}^j &= F^j[\boldsymbol{x}, \boldsymbol{y}, s], & j &= 1, \ldots, n \\
\tilde{y}^i &= G^i[\boldsymbol{x}, \boldsymbol{y}, s], & i &= 1, \ldots, m \\
\tilde{y}^i_j &= G^i_{\{j\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, s], \\
\tilde{y}^i_{j_1 j_2} &= G^i_{\{j_1 j_2\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, s] \\
&\vdots
\end{aligned}
\tag{1.42}
$$

where the partial derivatives are

$$
\tilde{y}^i_j = \frac{\partial \tilde{y}^i}{\partial \tilde{x}^j} \qquad \tilde{y}^i_{j_1 j_2} = \frac{\partial^2 \tilde{y}^i}{\partial \tilde{x}^{j_1}\, \partial \tilde{x}^{j_2}}, \ldots, \tag{1.43}
$$

Notation

Vector of first partial derivatives, $\mathbf{y}_1 = (y^1_1, y^1_2, \ldots, y^1_n, \ldots, y^m_1, y^m_2, \ldots, y^m_n)$

Vector of dependent variables, $\mathbf{y} = (y^1, y^2, \ldots, y^m)$

Vector of independent variables, $\mathbf{x} = (x^1, x^2, \ldots, x^n)$

The ith dependent variable

$$\tilde{y}^i_j = G^i_{\{j\}}[\mathbf{x}, \mathbf{y}, \mathbf{y}_1, s]$$

Group parameter

Differentiation with respect to the jth independent variable

Function that transforms the partial derivative of y^i *with respect to* x^j

Fig. 1.10. Notation for variables, derivatives, and transformations of derivatives.

Einstein used the following notation for partial derivatives. Note the comma

$$y^i_{,j} = \frac{\partial y^i}{\partial x^j} \qquad y^i_{,j_1 j_2} = \frac{\partial^2 y^i}{\partial x^{j_1}\, \partial x^{j_2}} \tag{1.44}$$

We use the Einstein convention on the summation of repeated indices

$$\frac{\partial u^1}{\partial x^1} + \frac{\partial u^2}{\partial x^2} + \frac{\partial u^3}{\partial x^3} = 0 \quad \Rightarrow \quad \frac{\partial u^i}{\partial x^i} = u^i_i = 0. \tag{1.45}$$

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Much of the theory of Lie groups relies on the infinitesimal form of the transformation expanded about small values of the group parameter.

$$\begin{aligned}
\tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x}, \boldsymbol{y}], \\
\tilde{y}^i &= y^i + s\eta^i[\boldsymbol{x}, \boldsymbol{y}], \\
\tilde{y}^i_j &= y^i_j + s\eta^i_{\{j\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1], \\
&\;\vdots
\end{aligned} \tag{1.46}$$

The function that infinitesimally transforms the derivative is of the form

$$\eta^i_{\{j\}} = \eta^i_j + \text{various other terms.} \tag{1.47}$$

Function label

Derivative

Transforming partial derivatives.

Consider the Lie group

$$\tilde{x} = X(x,t,u,s)$$
$$\tilde{t} = T(x,t,u,s)$$
$$\tilde{u} = U(x,t,u,s)$$
$$\frac{\partial \tilde{u}}{\partial \tilde{x}} = ???$$
$$\frac{\partial \tilde{u}}{\partial \tilde{t}} = ???$$

Note that u is a dependent variable

$$u(x,t)$$
$$\tilde{u}(\tilde{x},\tilde{t})$$

The differential in tildaed coordinates is

$$d\tilde{u}(\tilde{x},\tilde{t}) = \frac{\partial \tilde{u}}{\partial \tilde{x}} d\tilde{x} + \frac{\partial \tilde{u}}{\partial \tilde{t}} d\tilde{t}$$

Now work out the differentials at a fixed value of the group parameter s.

$$d\tilde{x} = \left(\frac{\partial X}{\partial x} + \frac{\partial X}{\partial u}\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial X}{\partial t} + \frac{\partial X}{\partial u}\frac{\partial u}{\partial t}\right)dt = \frac{DX}{Dx}dx + \frac{DX}{Dt}dt$$

$$d\tilde{t} = \left(\frac{\partial T}{\partial x} + \frac{\partial T}{\partial u}\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial T}{\partial t} + \frac{\partial T}{\partial u}\frac{\partial u}{\partial t}\right)dt = \frac{DT}{Dx}dx + \frac{DT}{Dt}dt$$

$$d\tilde{u} = \left(\frac{\partial U}{\partial x} + \frac{\partial U}{\partial u}\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial U}{\partial t} + \frac{\partial U}{\partial u}\frac{\partial u}{\partial t}\right)dt = \frac{DU}{Dx}dx + \frac{DU}{Dt}dt$$

and substitute

$$\frac{DU}{Dx}dx + \frac{DU}{Dt}dt = \frac{\partial \tilde{u}}{\partial \tilde{x}}\left(\frac{DX}{Dx}dx + \frac{DX}{Dt}dt\right) + \frac{\partial \tilde{u}}{\partial \tilde{t}}\left(\frac{DT}{Dx}dx + \frac{DT}{Dt}dt\right)$$

Gather terms

$$\left(\frac{DU}{Dx} - \frac{\partial \tilde{u}}{\partial \tilde{x}}\frac{DX}{Dx} - \frac{\partial \tilde{u}}{\partial \tilde{t}}\frac{DT}{Dx}\right)dx + \left(\frac{DU}{Dt} - \frac{\partial \tilde{u}}{\partial \tilde{x}}\frac{DX}{Dt} - \frac{\partial \tilde{u}}{\partial \tilde{t}}\frac{DT}{Dt}\right)dt = 0$$

the differentials dx and dt are independent therefore

$$\frac{DU}{Dx} = \frac{\partial \tilde{u}}{\partial \tilde{x}} \frac{DX}{Dx} + \frac{\partial \tilde{u}}{\partial \tilde{t}} \frac{DT}{Dx}$$
$$\frac{DU}{Dt} = \frac{\partial \tilde{u}}{\partial \tilde{x}} \frac{DX}{Dt} + \frac{\partial \tilde{u}}{\partial \tilde{t}} \frac{DT}{Dt}$$

Solve for $\partial \tilde{u} / \partial \tilde{x}$ and $\partial \tilde{u} / \partial \tilde{t}$.

$$\begin{bmatrix} \frac{DX}{Dx} & \frac{DT}{Dx} \\ \frac{DX}{Dt} & \frac{DT}{Dt} \end{bmatrix} \begin{bmatrix} \frac{\partial \tilde{u}}{\partial \tilde{x}} \\ \frac{\partial \tilde{u}}{\partial \tilde{t}} \end{bmatrix} = \begin{bmatrix} \frac{DU}{Dx} \\ \frac{DU}{Dt} \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \frac{\partial \tilde{u}}{\partial \tilde{x}} \\ \frac{\partial \tilde{u}}{\partial \tilde{t}} \end{bmatrix} = \begin{bmatrix} \frac{DX}{Dx} & \frac{DT}{Dx} \\ \frac{DX}{Dt} & \frac{DT}{Dt} \end{bmatrix}^{-1} \begin{bmatrix} \frac{DU}{Dx} \\ \frac{DU}{Dt} \end{bmatrix}}$$

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Example – dilation group

$$\tilde{x} = e^{a} x$$

$$\tilde{t} = e^{b} t$$

$$\tilde{u} = e^{c} u$$

$$\left(\frac{\partial U}{\partial x} + \frac{\partial U}{\partial u}\frac{\partial u}{\partial x}\right) = \frac{\partial \tilde{u}}{\partial \tilde{x}}\left(\frac{\partial X}{\partial x} + \frac{\partial X}{\partial u}\frac{\partial u}{\partial x}\right) + \frac{\partial \tilde{u}}{\partial \tilde{t}}\left(\frac{\partial T}{\partial x} + \frac{\partial T}{\partial u}\frac{\partial u}{\partial x}\right)$$

$$\left(\frac{\partial U}{\partial t} + \frac{\partial U}{\partial u}\frac{\partial u}{\partial t}\right) = \frac{\partial \tilde{u}}{\partial \tilde{x}}\left(\frac{\partial X}{\partial t} + \frac{\partial X}{\partial u}\frac{\partial u}{\partial t}\right) + \frac{\partial \tilde{u}}{\partial \tilde{t}}\left(\frac{\partial T}{\partial t} + \frac{\partial T}{\partial u}\frac{\partial u}{\partial t}\right)$$

$$\left(0 + e^{c}\frac{\partial u}{\partial x}\right) = \frac{\partial \tilde{u}}{\partial \tilde{x}}\left(e^{a} + 0\frac{\partial u}{\partial x}\right) + \frac{\partial \tilde{u}}{\partial \tilde{t}}\left(0 + 0\frac{\partial u}{\partial x}\right)$$

$$\left(0 + e^{c}\frac{\partial u}{\partial t}\right) = \frac{\partial \tilde{u}}{\partial \tilde{x}}\left(0 + 0\frac{\partial u}{\partial t}\right) + \frac{\partial \tilde{u}}{\partial \tilde{t}}\left(e^{b} + 0\frac{\partial u}{\partial t}\right)$$

$$e^{c}\frac{\partial u}{\partial x} = e^{a}\frac{\partial \tilde{u}}{\partial \tilde{x}}$$

$$e^{c}\frac{\partial u}{\partial t} = e^{b}\frac{\partial \tilde{u}}{\partial \tilde{t}}$$

$$\boxed{\begin{aligned} \frac{\partial \tilde{u}}{\partial \tilde{x}} &= e^{c-a}\frac{\partial u}{\partial x} \\ \frac{\partial \tilde{u}}{\partial \tilde{t}} &= e^{c-b}\frac{\partial u}{\partial t} \end{aligned}}$$

(1.7) Concluding Remarks

(1.8) Exercises

1.1 Work out the 6-member discrete symmetry group of an equilateral triangle. Show that the set of matrices is closed with respect to matrix multiplication, that each member of the set has an inverse, that the matrices are associative, and that the set has an identity element.

1.2 Work out the 24-member discrete rotation group of the cube shown in Figure 1.11. Show with sample calculations that the set of matrices is closed with respect to matrix multiplication, that each member of the set has an inverse, that the matrices are associative, and that the set has an identity element. How many matrices do you get if you include

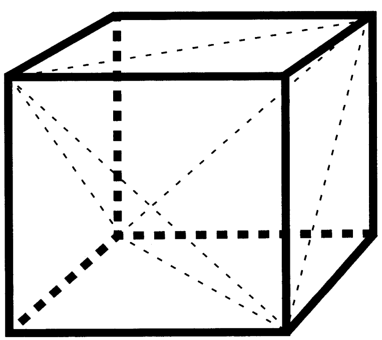

Fig. 1.11.

reflections? Which of these symmetries is shared by a tetrahedron formed by connecting four of the corners of the cube as shown? Show that the tetrahedral group is a subgroup of the cubic group. See Chapter 1 of the reference by Nussbaum [1.32] for a discussion of this and related problems.

1.3 Show that the first-order ODE

$$x\left(\frac{dy}{dx}\right)^2 + y\left(\frac{dy}{dx}\right) + x = 0 \tag{1.48}$$

is invariant under a dilation group. Is it invariant under translation? Plot the surface defined by the equation in (x, y, y_x) coordinates.

1.4 Consider the nonlinear heat equation

$$\frac{\partial T}{\partial t} - \lambda \frac{\partial}{\partial x}\left(T^\beta \frac{\partial T}{\partial x}\right) = 0, \tag{1.49}$$

where T is the temperature. What are the units of λ? Find a two-parameter dilation group that leaves the equation invariant. How is the group connected to β?

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1.5 Transform each of the following equations using the following four-parameter dilation group:

$$\tilde{x}^j = e^a x^j, \qquad \tilde{t} = e^b t, \qquad \tilde{u}^i = e^c u^i, \qquad \tilde{p} = e^d p, \qquad \tilde{\rho} = \rho. \tag{1.50}$$

(i) The incompressible Navier–Stokes equations

$$\frac{\partial u^i}{\partial t} + \frac{\partial}{\partial x^k}\left(u^i u^k + \frac{p}{\rho}\delta^i_k\right) - \nu \frac{\partial u^i}{\partial x^k \, \partial x^k} = 0, \qquad \frac{\partial u^k}{\partial x^k} = 0. \tag{1.51}$$

(ii) The Stokes equations for slow flow,

$$\frac{\partial u^i}{\partial t} + \frac{\partial}{\partial x^i}\left(\frac{p}{\rho}\right) - \nu \frac{\partial u^i}{\partial x^k \, \partial x^k} = 0, \qquad \frac{\partial u^k}{\partial x^k} = 0. \tag{1.52}$$

(iii) The Euler equations for inviscid flow,

$$\frac{\partial u^i}{\partial t} + \frac{\partial}{\partial x^k}\left(u^i u^k + \frac{p}{\rho}\delta^i_k\right) = 0, \qquad \frac{\partial u^k}{\partial x^k} = 0. \tag{1.53}$$

How do the group parameters a, b, c, d have to be related to one another in order for the given equations to be invariant?

1.6 Use the solution (1.41) in (1.40) to initiate a succession of solutions to (1.39). What do these solutions have in common?