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Introduction to Symmetry Analysis

Chapter 6 - First Order Ordinary Differential Equations

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Example 1.1 Invariance of a first-order ODE under a Lie group

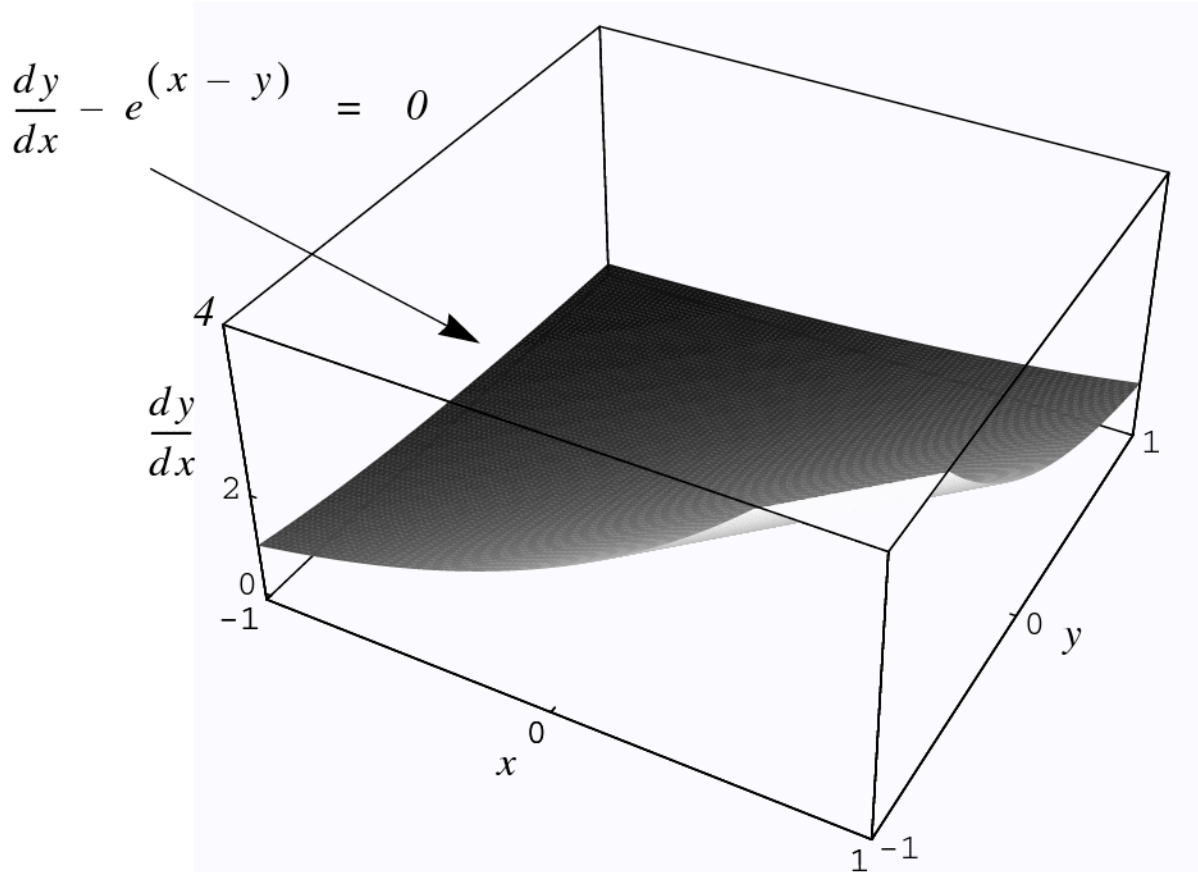

Figure 1.8 The surface defined by a first order ODE

Extended translation group

$$\tilde{x} = x + s,$$
$$\tilde{y} = y + s, \tag{1.17}$$
$$\frac{d\tilde{y}}{d\tilde{x}} = \frac{dy}{dx}$$

Transform the equation

$$\Psi\left[\tilde{x}, \tilde{y}, \frac{d\tilde{y}}{d\tilde{x}}\right] = \frac{d\tilde{y}}{d\tilde{x}} - e^{\tilde{x}-\tilde{y}} = \frac{dy}{dx} - e^{(x+s)-(y+s)}$$
$$= \frac{dy}{dx} - e^{x-y} = \Psi\left[x, y, \frac{dy}{dx}\right]. \tag{1.18}$$

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General solution

$$\psi = \Psi[x, y] = e^{y} - e^{x}, \tag{1.19}$$

Action of the group on a given solution curve

$$\tilde{\psi} = \Psi[\tilde{x}, \tilde{y}] = e^{\tilde{y}} - e^{\tilde{x}} = e^{y+s} - e^{x+s} = e^{s}(e^{y} - e^{x}). \tag{1.20}$$

The solution curve (1.20) is transformed to

$$\frac{\tilde{\psi}}{e^{s}} = e^{y} - e^{x}. \tag{1.21}$$

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6.1 Invariant families

Example 6.1 Rotation group in the plane

$$T^{\text{rot}}: \left\{ \begin{aligned} \tilde{x} &= x\cos[s] - y\sin[s] \\ \tilde{y} &= x\sin[s] + y\cos[s] \end{aligned} \right\}. \qquad (6.1)$$

Group operator

$$X^{\text{rot}} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}. \qquad (6.3)$$

Group invariant

$$\phi = \Phi[x, y] = x^2 + y^2 \qquad (6.4)$$

The family of rays

$$\psi = \Psi[x, y] = \frac{y}{x}, \qquad (6.5)$$

Action of the rotation group on the family of rays

$$\tilde{\psi} = \frac{\tilde{y}}{\tilde{x}} = \frac{\sin[s] + \frac{y}{x}\cos[s]}{\cos[s] - \frac{y}{x}\sin[s]} = G\left(\frac{y}{x}, s\right) = G(\psi, s). \qquad (6.6)$$

Action of the rotation group operator

$$X^{\mathrm{rot}}\,\Psi = -y\frac{\partial}{\partial x}\left(\frac{y}{x}\right) + x\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \left(\frac{y}{x}\right)^2 + 1 = \psi^2 + 1. \qquad (6.7)$$

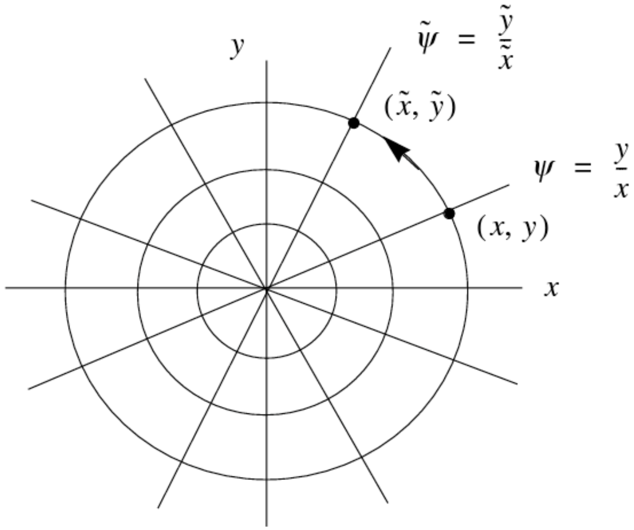

Fig. 6.1. Action of the rotation group on the family of rays.

Example 6.2 Uniform dilation group

$$T^{\text{dil}} : \begin{Bmatrix} \tilde{x} = e^s x \\ \tilde{y} = e^s y \end{Bmatrix} \tag{6.8}$$

Group operator

$$X^{\text{dil}} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{6.9}$$

Group invariant

$$\psi = \Psi[x, y] = \frac{y}{x} \tag{6.10}$$

$$\tilde{\psi} = \Psi[\tilde{x}, \tilde{y}] = \frac{\tilde{y}}{\tilde{x}} = \frac{e^s y}{e^s x} = \frac{y}{x} = \Psi(x, y) = \psi. \tag{6.11}$$

Action of the dilation group on the family of circles

$$\tilde{\phi} = \Phi[\tilde{x}, \tilde{y}] = \tilde{x}^2 + \tilde{y}^2 = e^{2s}(x^2 + y^2) = e^{2s}\Phi[x, y] = G(\phi, s). \tag{6.12}$$

Action of the dilation group operator on the family of circles

$$X^{\text{dil}}\Phi = x\frac{\partial}{\partial x}(x^2+y^2) + y\frac{\partial}{\partial y}(x^2+y^2) = 2(x^2+y^2) = 2\phi. \quad (6.13)$$

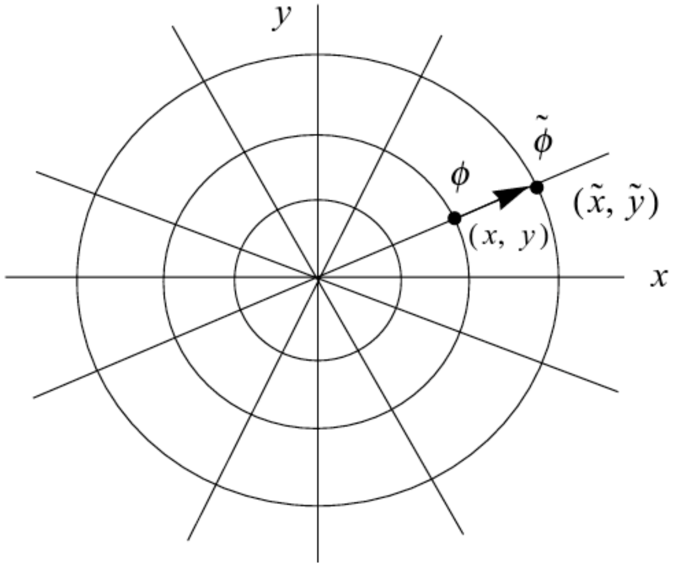

Fig. 6.2. Action of the dilation group on the family of circles.

Evidently the finite condition for a *family* $\psi = \Psi[\boldsymbol{x}]$ to be invariant under a group $\boldsymbol{F}$ is

$$\tilde{\psi} = \Psi[\tilde{\boldsymbol{x}}] = \Psi(\boldsymbol{F}[\boldsymbol{x}, s]) = G[\Psi[\boldsymbol{x}], s] = G[\psi, s] \quad (6.15)$$

The corresponding infinitesimal condition is

$$X\Psi = \Omega[\Psi] \quad (6.16)$$

We can interpret this condition applied to a family in n dimensions as equivalent to

$$X\Gamma = 0$$

where Γ is a single surface in $n + 1$ dimensions.

To see this, let Γ be a function of $n+1$ variables of the form

$$\Gamma[x^1, x^2, x^3, \ldots, x^n, x^{n+1}] = \Psi[x^1, x^2, x^3, \ldots, x^n] - x^{n+1} \tag{6.17}$$

Consider the invariance of Γ under the transformation

$$\begin{aligned} \tilde{x}^j &= F^j[x^1, x^2, x^3, \ldots, x^n, s], \qquad j = 1, 2, \ldots, n \\ \tilde{x}^{n+1} &= x^{n+1} + s, \end{aligned} \tag{6.18}$$

Which is clearly a Lie group

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The function Γ is an invariant single surface under the group (6.18) if and only if

$$\xi^1 \frac{\partial \Gamma}{\partial x^1} + \xi^2 \frac{\partial \Gamma}{\partial x^2} + \cdots + \xi^n \frac{\partial \Gamma}{\partial x^n} + (1) \frac{\partial \Gamma}{\partial x^{n+1}} = 0 \tag{6.19}$$

which becomes

$$X\Psi = 1 \tag{6.20}$$

Thus the family

$$\Psi[x^1, x^2, x^3, \ldots, x^n] = x^{n+1} \tag{6.21}$$

is an invariant family in $(x^1, x^2, x^3, \ldots, x^n)$, or equivalently an invariant single surface in $(x^1, x^2, x^3, \ldots, x^n, x^{n+1})$.

Invariance condition for a family - summary

The finite condition for a family of curves to be invariant under a group is

$$\tilde{\psi} = \Psi[\tilde{\boldsymbol{x}}] = \Psi(\boldsymbol{F}[\boldsymbol{x}, s]) = G[\Psi[\boldsymbol{x}], s] = G[\psi, s],$$

and the corresponding infinitesimal condition is

$$X\Psi = \Omega[\Psi].$$

Without loss of generality we can always choose a once-differentiable function such that the invariance condition becomes

$$X\Psi = 1.$$

This simplification can be illustrated as follows

$$X\Phi[\boldsymbol{x}] = X\Pi[\Psi[\boldsymbol{x}]] = (X\Psi)\frac{d\Pi}{d\Psi} = \Omega[\Psi]\frac{d\Pi}{d\Psi} = 1. \tag{6.22}$$

Choose

$$\Pi = \int \frac{d\Psi}{\Omega[\Psi]}. \tag{6.23}$$

First order ODEs, the Integrating Factor

Consider the first order ordinary differential equation

$$\frac{dy}{dx} = \frac{B[x, y]}{A[x, y]},$$

which we can write as

$$-B[x, y]\, dx + A[x, y]\, dy = 0.$$

The perfect differential of the solution is

$$d\psi = \frac{\partial \Psi}{\partial x}\, dx + \frac{\partial \Psi}{\partial y}\, dy.$$

The solution satisfies the first order linear PDE

$$A[x, y]\frac{\partial \Psi}{\partial x} + B[x, y]\frac{\partial \Psi}{\partial y} = 0.$$

Now, suppose the solution *family* is invariant under the group (ξ, η)

$$\xi[x, y]\frac{\partial \Psi}{\partial x} + \eta[x, y]\frac{\partial \Psi}{\partial y} = 1.$$

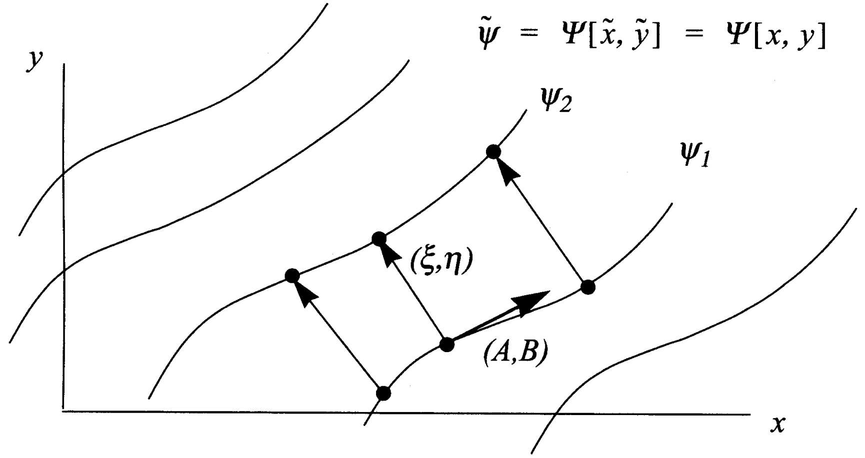

Fig. 6.3. Transformation of points along characteristics by (A, B), and between characteristics by (ξ, η).

We have two simultaneous equations for the partial derivatives of the solution

$$\frac{\partial \Psi}{\partial x} = \frac{-B}{A\eta - B\xi}, \qquad \frac{\partial \Psi}{\partial y} = \frac{A}{A\eta - B\xi}.$$

The integrating factor is

$$M = \frac{1}{A\eta - B\xi}$$

and the perfect differential of the solution is

$$d\psi = \frac{-B}{A\eta - B\xi}\,dx + \frac{A}{A\eta - B\xi}\,dy.$$

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The ODE is solved in the form of a quadrature

$$\psi = \int \frac{-B}{A\eta - B\xi}\,dx\Big|_{y=\text{constant}} + f[y].$$

$$\frac{\partial \Psi}{\partial y} = \frac{\partial}{\partial y}\left(\int \frac{-B}{A\eta - B\xi}\,dx\Big|_{y=\text{constant}}\right) + \frac{df}{dy} = \frac{A}{A\eta - B\xi},$$

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Table 6.1. *Some first-order ODEs and their invariant groups.*

Equation	ξ	η
$y_x = F[y]$	1	0
$y_x = F[x]$	0	1
$y_x = F[ax + by]$	b	$-a$
$y_x = \dfrac{y + xF[x^2 + y^2]}{x - yF[x^2 + y^2]}$	y	$-x$
$y_x = F\left[\dfrac{y}{x}\right]$	x	y
$y_x = x^{k-1}F[y/x^k]$	x	ky
$xy_x = F[xe^{-y}]$	x	1
$y_x = yF[ye^{-x}]$	1	y
$y_x = (y/x) + xF[y/x]$	1	y/x
$xy_x = y + F[y/x]$	x^2	xy
$y_x = \dfrac{y}{x + F[y/x]}$	xy	y^2
$y_x = \dfrac{y}{x + F[y]}$	y	0
$xy_x = y + F[x]$	0	x
$xy_x = \dfrac{y}{\ln[x] + F[y]}$	xy	0
$xy_x = y(\ln[y] + F[x])$	0	xy
$y_x = yF[x]$	0	y

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Example 6.4 (Invariance with respect to a dilation group). Find the general solution of

$$\frac{dy}{dx} = \frac{y}{x} H[xy], \tag{6.43}$$

where H is an arbitrary function. Rearrange (6.43) as

$$-yH[xy]\,dx + x\,dy = 0. \tag{6.44}$$

In the notation adopted above, let

$$A[x, y] = -x, \qquad B[x, y] = -yH[xy]. \tag{6.45}$$

As was just pointed out, we need to find a Lie group that leaves (6.43) invariant. There is really no systematic way to determine such a group. We have to rely on trial and error to transform (6.43). By inspection we can see that (6.43) is invariant under the dilation group

$$\tilde{x} = e^{s} x, \qquad \tilde{y} = e^{-s} y. \tag{6.46}$$

Insert the transformation (6.46) into (6.43):

$$\frac{d\tilde{y}}{d\tilde{x}} = \frac{\tilde{y}}{\tilde{x}} H[\tilde{x}\tilde{y}] \;\Rightarrow\; e^{-2s}\frac{dy}{dx} = e^{-2s}\frac{y}{x} H[xy] \;\Rightarrow\; \frac{dy}{dx} = \frac{y}{x} H[xy]. \tag{6.47}$$

The equation reads the same in the new variables – success: we have found a group that leaves (6.43) invariant. The infinitesimals of (6.46) are

$$\xi = x, \qquad \eta = -y, \tag{6.48}$$

and the integrating factor is

$$M = \frac{1}{A\eta - B\xi} = \frac{1}{xy + xyH[xy]}. \tag{6.49}$$

Therefore the total differential of the solution is

$$d\psi = -\frac{yH[xy]}{xy + xyH[xy]}\, dx + \frac{x}{xy + xyH[xy]}\, dy. \tag{6.50}$$

Finally, the general solution of (6.43) is the family

$$\psi = -\int_{xy} \frac{H(\alpha)}{\alpha(1 + H(\alpha))}\, d\alpha + \ln[y]. \tag{6.51}$$

In essence, ψ is simply the constant of integration of (6.43). Let's demonstrate that (6.51) is in fact an invariant family of (6.48):

$$\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\left(-\int_{xy} \frac{H(\alpha)}{\alpha(1 + H(\alpha))}\, d\alpha + \ln[y]\right)$$
$$= x\left(-\frac{H}{\alpha(1+H)}y\right)_{\alpha=xy} - y\left(-\frac{H}{\alpha(1+H)}x\right)_{\alpha=xy} + y\left(\frac{1}{y}\right) = 1. \tag{6.52}$$

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Example 6.6 (A more complicated case). Find the general solution of

$$\frac{dy}{dx} = \frac{y}{x - f[y]g[y/x]},$$

which we rewrite as

$$\frac{y}{x}\,dx - \left(1 - \frac{f[y]}{x}g[y/x]\right)dy = 0.$$

Let

$$A = \left(1 - \frac{f[y]}{x}g[y/x]\right), \qquad B = \frac{y}{x}.$$

This equation is known to be invariant under the group

$$\xi = \frac{xy}{f[y]}, \qquad \eta = \frac{y^2}{f[y]}$$

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Integrating factor

$$M = -\frac{x}{y^2 g[y/x]}$$

Perfect differential

$$d\psi = -\frac{1}{yg[y/x]}\,dx + \left(\frac{x}{y^2 g[y/x]} - \frac{f[y]}{y^2}\right) dy$$

Exact solution

$$\psi = \Psi[x, y] = \int_{y/x} \frac{1}{\alpha^2 g[\alpha]}\,d\alpha - \int_{y} \frac{f[\alpha]}{\alpha^2}\,d\alpha$$

The solution is an invariant family of the group

$$\left(\frac{xy}{f[y]}\frac{\partial}{\partial x} + \frac{y^2}{f[y]}\frac{\partial}{\partial y}\right)\left(\int_{y/x}\frac{1}{\alpha^2 g[\alpha]}\,d\alpha - \int_{y}\frac{f[\alpha]}{\alpha^2}\,d\alpha\right)$$

$$= \frac{xy}{f[y]}\left(-\frac{1}{\alpha g[\alpha]}\frac{1}{x}\right)_{\alpha=y/x} - \frac{y^2}{f[y]}\left(-\frac{1}{\alpha^2 g[\alpha]}\frac{1}{x}\right)_{\alpha=y/x}$$

$$+ \frac{y^2}{f[y]}\left(\frac{f[\alpha]}{\alpha^2}\right)_{\alpha=y} = \cancel{1}\; -1$$

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Canonical coordinates

Any Lie group can be written in terms of new variables called canonical coordinates such that the transformation is converted to a simple translation in one variable.

The group

$$X = \xi^j[\boldsymbol{x}]\frac{\partial}{\partial x^j}$$

has the associated characteristic equations

$$\frac{dx^1}{\xi^1[\boldsymbol{x}]} = \frac{dx^2}{\xi^2[\boldsymbol{x}]} = \frac{dx^3}{\xi^3[\boldsymbol{x}]} = \cdots = \frac{dx^n}{\xi^n[\boldsymbol{x}]}$$

with invariants

$$r^i = R^i[\boldsymbol{x}], \qquad i = 1, \ldots, n-1.$$

that satisfy the invariance condition

$$\xi^j\frac{\partial R^i}{\partial x^j} = 0, \qquad i = 1, \ldots, n-1.$$

Determine an invariant family such that

$$\xi^j \frac{\partial R^n}{\partial x^j} = 1.$$

In terms of these variables the group is equivalent to the simple translation,

$$\tilde{r}^i = r^i, \qquad i = 1, \ldots, n-1,$$
$$\tilde{r}^n = r^n + s$$

with group operator

$$\boxed{X = \frac{\partial}{\partial r^n}.}$$

The integrals

$$R^i[\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n] = R^i[x^1, x^2, \ldots, x^n], \qquad i = 1, \ldots, n-1,$$
$$R^n[\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n] = R^n[x^1, x^2, \ldots, x^n] + s.$$

are the canonical coordinates . Any Lie group can be expressed as a simple translation using canonical coordinates.

Invariant solutions

Example 6.9 Clairaut's equation

$$x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} + m = 0,$$

This equation is invariant under a one-parameter dilation group.

$$\tilde{x} = e^{2s}x, \qquad \tilde{y} = e^{s}y,$$

$$\xi = 2x, \qquad \eta = y.$$

The equation can be written in the form

$$-\left(y \pm (y^2 - 4mx)^{1/2}\right) dx + 2x\, dy = 0.$$

The invariant group generates the integrating factor

$$M = \frac{1}{A\eta - B\xi} = \frac{1}{\mp 2x(y^2 - 4mx)^{1/2}}$$

and the general solution

$$\psi = \frac{y}{2x} \pm \frac{1}{2}\left(\frac{y^2}{x^2} - \frac{4m}{x}\right)^{1/2}.$$

The solution can be rearranged as follows

$$\left(\psi - \frac{y}{2x}\right)^2 = \left(\frac{y^2}{4x^2} - \frac{m}{x}\right).$$

When this result is expanded, the quadratic terms on both sides cancel leaving the family of straight lines

$$y = \psi x + m/\psi$$

The solution transforms as follows

$$\tilde{y} = \psi \tilde{x} + \frac{m}{\psi} \Rightarrow e^s y = \psi e^{2s} x + \frac{m}{\psi} \Rightarrow y = (\psi e^s)x + \frac{m}{\psi e^s}.$$

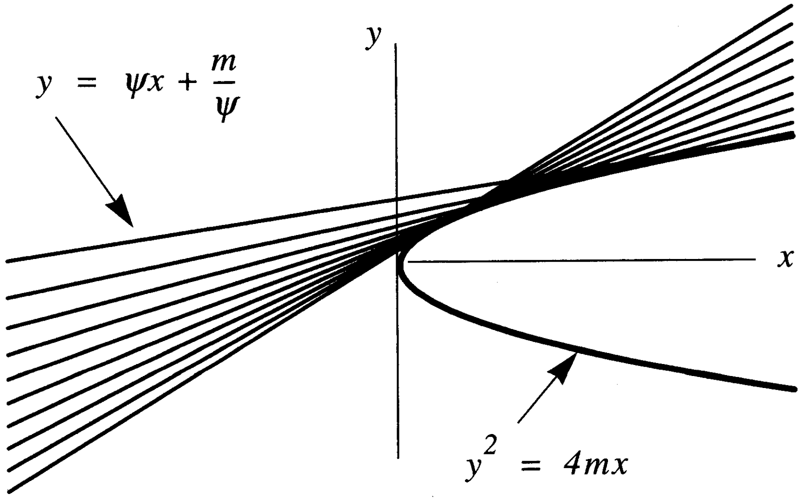

Fig. 6.4. Solution family of the Clairault equation.

An invariant solution can be found as follows. Let

$$\psi_{inv} = y - f(x) = 0$$

The invariance condition is

$$X\psi_{inv} = 2x\frac{\partial \psi_{inv}}{\partial x} + y\frac{\partial \psi_{inv}}{\partial y} = -2xf_x + y = 0$$

When this equation is solved the result is the invariant solution

$$y = \pm 2(mx)^{1/2}$$

6.10 Exercises

6.1 Reconsider the groups studied in Chapter 5, Problem 5.1:

(i) A projective group

$$\tilde{x} = \frac{x}{1 - sy}, \qquad y = \frac{y}{1 - sy}. \tag{6.147}$$

(ii) A hyperbolic group

$$\tilde{x} = x + s, \qquad \tilde{y} = \frac{xy}{x + s}. \tag{6.148}$$

(iii) An arbitrary translation

$$\tilde{x} = x, \qquad \tilde{y} = y + sf[x], \quad f(x) \text{ arbitrary}. \tag{6.149}$$

(iv) A helical transformation

$$\tilde{x} = x\cos[s] - y\sin[s], \quad \tilde{y} = x\sin[s] + y\cos[s], \quad \tilde{z} = z + ms. \tag{6.150}$$

Determine an invariant family for each group.

6.2 Find an integrating factor for each of the following ODEs, and work out the general solution:

$$\frac{dy}{dx} - \frac{y}{x + \sin[x/y]} = 0, \tag{6.151}$$

$$(3x^2 + 2xy - y^2)\, dx + (x^2 - 2xy - 3y^2)\, dy = 0, \tag{6.152}$$

$$\frac{dy}{dx} = \frac{ye^y}{y^3 + 2xe^y}, \tag{6.153}$$

$$x\frac{dy}{dx} + y = x^2, \tag{6.154}$$

$$\frac{dy}{dx} = 4\frac{y}{x} + x^2 \sin[y/x^4]. \tag{6.155}$$

6.3 Revisit Chapter 1, Exercise 1.3. Find an integrating factor, and solve the first-order ODE

$$x\left(\frac{dy}{dx}\right)^2 + y\left(\frac{dy}{dx}\right) + x = 0. \tag{6.156}$$

6.4 Show by direct substitution that (6.99) leaves the family of ellipses (6.92) invariant.

6.5 Show that the first-order ODE

$$\frac{dy}{dx} = \frac{y^3 + x^2y - y - x}{xy^2 + x^3 + y - x} \tag{6.157}$$

is invariant under the rotation group $(\xi, \eta) = (-y, x)$. Sketch the phase portrait and identify critical points. Identify an invariant solution. Use the group to find an integrating factor and work out the solution.

6.6 Beginning with $(R, Q) = (2, -3)$ on $Q^3 + \frac{27}{4}R^2 = 0$, use the chord–tangent construction to identify an infinite sequence of rational roots.

6.7 Can you find a rational root of the equation $Q^3 + \frac{27}{4}R^2 = 1$?