STANFORD AERONAUTICS & ASTRONAUTICS

Introduction to Symmetry Analysis

Chapter 10 - Laminar Boundary Layers

Brian Cantwell
Department of Aeronautics and Astronautics
Stanford University

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Problem 10.6

10.6 Consider the buoyancy induced flow produced by a heated flat plate sketched in Figure 10.15. This flow is governed by a coupled system of convection–diffusion equations for the momentum and temperature. Changes in density are related to changes in temperature by a thermal expansion coefficient:

$$\rho - \rho_\infty = \beta(T - T_\infty). \qquad (10.152)$$

If changes in density are small $[(\rho - \rho_\infty)/\rho_\infty \ll 1]$, the fluid behaves incompressibly ($\nabla \cdot \boldsymbol{u} = 0$) with a local body force equal to $(\rho - \rho_\infty)g$. The governing equations are $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$,

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} - \nu\frac{\partial^3\psi}{\partial y^3} = \frac{\beta(T - T_\infty)g}{\rho_\infty},$$
$$\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y} = \kappa\frac{\partial^2 T}{\partial y^2}. \qquad (10.153)$$

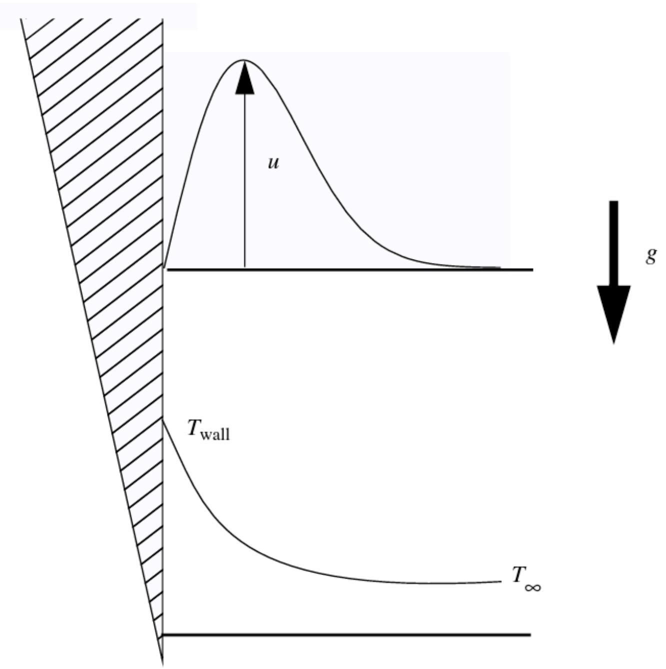

Fig. 10.15.

Suggestion: use $\dfrac{T - T_\infty}{T_w - T_\infty}$ as the appropriate temperature variable

This is OK because of the invariance under translation of the temperature.

(1) Use the package `IntroToSymmetry.m` to work out the infinitesimal groups for this system. Generate the commutator table, and fully characterize the Lie algebra.

(2) Construct similarity variables for the problem depicted above, reduce the governing equations to a pair of coupled ODEs, and solve for the self-similar velocity and temperature profiles.

(3) Show whether a similarity solution exists when the free stream velocity is non zero.

(4) How is the symmetry of the problem changed when the plate is cooled instead of heated?

Ref. Schlichting, *Boundary Layer Theory*

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In this example, we use the package IntroToSymmetry.m to work out the point group of a coupled pair of PDEs, the boundary layer stream function equation with a buoyancy induced forcing term,

ΨxΨxy-ΨxΨyy-β(ginf/rhoinf)(T-Tinf)-nu*Ψyyy=0,

and a convection-diffusion equation for the temperature,

(alpha)Tyy-ΨyTx+ΨxTy=0.

The following command turns off spurious spelling error messages.

```
In[1]:= Off[General::spell]
```

Clear all symbols in the current context.

```
In[2]:= ClearAll[Evaluate[Context[] <> "*"]]
```

First read in the package which is located in User Home Folder/Library/*Mathematica*/Applications/SymmetryAnalysis.

```
In[3]:= Needs["SymmetryAnalysis`IntroToSymmetry`"]
```

Enter the list of independent variables.

```
In[4]:= independentvariables = {"x", "y"};
```

Enter the list of dependent variables.

```
In[5]:= dependentvariables = {"Ψ", "t"};
```

Enter the input equations as strings. Don't include the ==0 at the end. The ambient reference temperature is tinf.

```
In[6]:= equation1 =
  "D[Ψ[x,y],y,y,y]+(1/nu)*beta*(ginf/rhoinf)*(t[x,y]-tinf)+(1/nu)*D[Ψ[x,y],x]*D[Ψ[x,y],y,y]-
   (1/nu)*D[Ψ[x,y],y]*D[Ψ[x,y],x,y]";

In[7]:= equation2 =
  "D[t[x,y],y,y]-(1/alpha)*D[Ψ[x,y],y]*D[t[x,y],x]+(1/alpha)*D[Ψ[x,y],x]*D[t[x,y],y]";

In[8]:= rulesarray =
  {"D[Ψ[x,y],y,y,y]->-(1/nu)*beta*(ginf/rhoinf)*(t[x,y]-tinf)-(1/nu)*D[Ψ[x,y],x]*D[Ψ[x,y],y,
    y]+(1/nu)*D[Ψ[x,y],y]*D[Ψ[x,y],x,y]",
   "D[t[x,y],y,y]->(1/alpha)*D[Ψ[x,y],y]*D[t[x,y],x]-(1/alpha)*D[Ψ[x,y],x]*D[t[x,y],y]"};
```

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The input equations are expressed internally in terms of generic variables, (x1,x2,x3,..., y1,y2,y3,...) using the function StringReplace. When this is done it is important that function names and constant names that may appear in the equation are not modified. Enter these protected names in string format.

```
In[9]:= frozennames = {"nu", "beta", "ginf", "tinf", "rhoinf"};
```

Enter the maximum derivative order of the input equation set.

```
In[10]:= p = 3;
```

The maximum derivative order that the infinitesimals are assumed to depend on is specified by the input parameter r. This parameter is only nonzero when the user is looking for Lie contact groups or Lie-Backlund groups. For the usual case where one is searching for point groups set r=0.

```
In[11]:= r = 0;
```

When searching for Lie-Backlund groups (r=1 or greater) one can, without loss of generality, leave the independent variables untransformed. The corresponding infinitesimals (the xse's) are set to zero by setting xseon=0. If one is searching for point groups then set xseon=1. The choice xseon=1 is also an option when looking for Lie-Backlund groups and this can be useful when looking for contact symmetries.

```
In[12]:= xseon = 1;
```

When searching for Lie-Backlund groups it is necessary to differentiate the input equation to produce derivatives of order p+r and append these higher order differential consequences to the set of rules applied to the invariance condition. This process is carried out automatically when internalrules=1. For point groups the equation or equation system is the only rule or set of rules needed and one sets internalrules=0.

```
In[13]:= internalrules = 0;
```

Now work out the determining equations of the Lie point group that leaves equation1 invariant. The output is available as a table of strings called zdeterminingequations. Notice that rulesarray contains both equations and that therefore both equations in the set are applied to the invariance condition for each equation.

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```
In[14]:= Timing[FindDeterminingEquations[
    independentvariables, dependentvariables, frozennames,
    p, r, xseon, equation1, rulesarray, internalrules]]

The function FindDetermining Equations has begun,
  the memory in use = 92942728, the time used = 1.2857239999999999`

The function FindDeterminingEquations is nearly complete. The invariance condition has been
  created with all rules applied. The final step in the generation of the determining
  equations is to sum together terms in the table of invariance condition terms (called
  infinitesimaltable) that are multiplied by the same combination of products of free y
  derivatives. The result is the table infinitesimaltablesums corresponding to matching
  y-derivative expressions. If the invariance condition is long as it often is this process
  could take a long time since it requires sorting through the table infinitesimaltable
  once for each possible combination of y derivative products. This is the rate limiting
  step in the function FindDeterminingEquations.Virtually all other steps are quite
  fast including the generation of the extended derivatives of the infinitesimals.

The determining equations have been expressed in terms of
  z-variables, the length of zdeterminingequations = 73, the byte count of
  zdeterminingequations = 12328, the memory in use = 94747408, the time used = 1.526876`

FindDeterminingEquations is done. The memory in use = 94748888, the time used = 1.5270979999999998`

FindDeterminingEquations has finished executing. You can look at the output in
  the table zdeterminingequations. Each entry in this table is a determining equation
  in string format expressed in terms of z-variables. Rules for converting between
  z-variables and conventional variables are contained in the table ztableofrules. To
  view the determining equations in terms of conventional variables use the command
  ToExpression[zdeterminingequations]/.ztableofrules. There are two other items the user
  may wish to look at; the equation converted to generic (x1,x2,...,y1,y2,...) variables is
  designated equationgenericvariables and the various derivatives of the equation that appear
  in the invariance condition can be viewed in the table invarconditiontable. Rules for
  converting between z-variables and generic variables are contained in the table ztableofrulesxy.

Out[14]= {0.255383, Null}
```

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In[15]:= equationgenericvariables

```
Out[15]= D[y1[x1,x2],x2,x2,x2]+(1/nu)*beta*(ginf/rhoinf)*(y2[x1,x2]-tinf)+(1/nu)*D[y1[x1,x2],x1]*D[y1[x1,x2],x2,x2]-(1/nu)*D[
  y1[x1,x2],x2]*D[y1[x1,x2],x1,x2]
```

In[16]:= invarconditiontable

Out[16]= $\left\{0, 0, 0, \frac{\text{beta ginf}}{\text{nu rhoinf}}, \frac{y1^{(0,2)}[x1, x2]}{\text{nu}}, -\frac{y1^{(1,1)}[x1, x2]}{\text{nu}}, 0, -\frac{y1^{(0,1)}[x1, x2]}{\text{nu}}, \frac{y1^{(1,0)}[x1, x2]}{\text{nu}}, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\}$

Here are the determining equations in terms of z-variables. Set them equal to a new table called zdeterminingequations1. Remove the semicolon at the end if you want to look at the equations.

In[17]:= zdeterminingequations1 = zdeterminingequations;

Now work out the determining equations of the point group which leaves equation2 invariant.

```
In[18]:= Timing[FindDeterminingEquations[independentvariables, dependentvariables, frozennames, p, r, xseon,
  equation2, rulesarray, internalrules]]
```

```
The function FindDetermining Equations has begun, the memory in use = 95307680, the time used = 1.5480299999999998`

The function FindDeterminingEquations is nearly complete. The invariance condition has been created with all
  rules applied. The final step in the generation of the determining equations is to sum together terms in the
  table of invariance condition terms (called infinitesimaltable) that are multiplied by the same combination
  of products of free y derivatives. The result is the table infinitesimaltablesums corresponding to matching
  y-derivative expressions. If the invariance condition is long as it often is this process could take a long
  time since it requires sorting through the table infinitesimaltable once for each possible combination of
  y derivative products. This is the rate limiting step in the function FindDeterminingEquations.Virtually
  all other steps are quite fast including the generation of the extended derivatives of the infinitesimals.

The determining equations have been expressed in terms of z-variables, the length of zdeterminingequations =
  27, the byte count of zdeterminingequations = 3160, the memory in use = 94482032, the time used = 1.636993`

FindDeterminingEquations is done. The memory in use = 94483176, the time used = 1.637165`

FindDeterminingEquations has finished executing. You can look at the output in the table zdeterminingequations. Each entry in
  this table is a determining equation in string format expressed in terms of z-variables. Rules for converting between
  z-variables and conventional variables are contained in the table ztableofrules. To view the determining equations in terms of
  conventional variables use the command ToExpression[zdeterminingequations]/.ztableofrules. There are two other items the user
  may wish to look at; the equation converted to generic (x1,x2,...,y1,y2,...) variables is designated equationgenericvariables
  and the various derivatives of the equation that appear in the invariance condition can be viewed in the table
  invarconditiontable. Rules for converting between z-variables and generic variables are contained in the table ztableofrulesxy.
```

Out[18]= {0.102888, Null}

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In[19]:= equationgenericvariables

Out[19]= D[y2[x1,x2],x2,x2]-(1/alpha)*D[y1[x1,x2],x2]*D[y2[x1,x2],x1]+(1/alpha)*D[y1[x1,x2],x1]*D[y2[x1,x2],x2]

In[20]:= invarconditiontable

Out[20]= $\left\{0, 0, 0, 0, \frac{y2^{(0,1)}[x1, x2]}{alpha}, -\frac{y2^{(1,0)}[x1, x2]}{alpha}, 0, 0, 0, 0, 0, 0, 0, -\frac{y1^{(0,1)}[x1, x2]}{alpha}, \frac{y1^{(1,0)}[x1, x2]}{alpha}, 0, 0, 1, 0, 0, 0, 0\right\}$

Here are the determining equations in terms of z-variables. Set them equal to a new table called zdeterminingequations2. Remove the semicolon at the end if you want to look at the equations.

In[21]:= zdeterminingequations2 = zdeterminingequations;

Now concatenate these two sets of determining equations together to form the table zdeterminingequations3 that contains the determining equations for the entire equation set.

In[22]:= zdeterminingequations3=Join[
zdeterminingequations1,zdeterminingequations2];

How many determining equations are we dealing with?

In[23]:= Length[zdeterminingequations3]

Out[23]= 100

Here is the correspondence between z-variables and conventional variables.

In[24]:= ztableofrules

Out[24]= {z1 → x, z2 → y, z3 → Ψ[x, y], z4 → t[x, y]}

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Now solve the determining equations in terms of multivariable polynomials of some selected order.

```
In[25]:= Timing[SolveDeterminingEquations[
  independentvariables, dependentvariables, r, xseon, zdeterminingequations3, order = 5]]
```

The variable powertablelength is the number of terms required for each multivariate polynomial used for the infinitesimals. This number is determined by the choice of polynomial order and the number of zvariables. The time needed to solve the determining equations increases as powertable increases. powertablelength = 126

The polynomial expansions have been substituted into the determining equations. It is now time to collect the coefficients of various powers of zvariables into a table called table of coefficientsall. This step uses the function CoefficientList and is a fairly time consuming procedure.

The memory in use = 100922264, The time = 2.085229`

The number of unknown polynomial coefficients = 504

The number of equations for the polynomial coefficients = 4900

Now it we are ready to use the function Solve to find the nonzero polynomial coefficients corresponding to the symmetries of the input equation(s). This can take a while.

The memory in use = 104635840, The time = 2.22211`

Solve has finished.

The function SolveDeterminingEquations is finished executing.

The memory in use = 106679504, The time = 4.111632`

You can look at the output in the tables xsefunctions and etafunctions. Each entry in these tables is an infinitesimal function in string format expressed in terms of z-variables and the group parameters. The output can also be viewed with the group parameters stripped away by looking at the table infinitesimalgroups. In either case you may wish to convert the z-variables to conventional variables using the table ztableofrules.

Keep in mind that this function only finds solutions of the determining equations that are of polynomial form. The determining equations may admit solutions that involve transcendental functions and/or integrals. Note that arbitrary functions may appear in the infinitesimals and that these can be detected by running the package function SolveDeterminingEquations for several polynomial orders. If terms of ever increasing order appear, then an arbitrary function is indicated.

```
Out[25]= {2.46633, Null}
```

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Here are the infinitesimal transformation functions for the independent variables.

```
In[26]:= xsefunctions

Out[26]= {xse1[z1_, z2_, z3_, z4_]=a10 + (a26 + b121)*z1,
 xse2[z1_, z2_, z3_, z4_]=a20 + a21*z1 + a22*z1^2
   + a23*z1^3 + a24*z1^4 + a25*z1^5 + a26*z2}
```

and for the dependent variables.

```
In[27]:= etafunctions

Out[27]= {eta1[z1_, z2_, z3_, z4_]=b10 + b121*z3,
 eta2[z1_, z2_, z3_, z4_]=3*a26*tinf - b121*tinf + (-3*a26 + b121)*z4}

In[28]:= ToExpression[xsefunctions]/.ztableofrules

Out[28]= {a10 + (a26 + b121) x, a20 + a21 x + a22 x^2 + a23 x^3 + a24 x^4 + a25 x^5 + a26 y}

In[29]:= ToExpression[etafunctions]/.ztableofrules

Out[29]= {b10 + b121 Ψ[x, y], 3 a26 tinf - b121 tinf + (-3 a26 + b121) t[x, y]}

In[30]:= Column[infinitesimalgroups /.ztableofrules]

Out[30]=
{{1, 0}, {0, 0}}
{{0, 1}, {0, 0}}
{{0, x}, {0, 0}}
{{0, x^2}, {0, 0}}
{{0, x^3}, {0, 0}}
{{0, x^4}, {0, 0}}
{{0, x^5}, {0, 0}}
{{x, y}, {0, 3 tinf - 3 t[x, y]}}
{{0, 0}, {1, 0}}
{{x, 0}, {Ψ[x, y], -tinf + t[x, y]}}
```

Check the infinitesimals in the determining equations.

```
In[31]:= Simplify[ToExpression[zdeterminingequations3]]

Out[31]= {True, True, True, True, True, True, True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True, True, True, True, True, True}
```

Notice that the infinite dimensional translation group on the wall normal coordinate still comes out, ie, this symmetry is not lost when the temperature equation is coupled to the boundary layer equation. This invariance allows the solution on a flat plate to be mapped to an arbitrary shaped wall (within reason).

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Now generate the commutator table of the Lie algebra. Convert z-variables to conventional variables. Note that in the conversion the functional dependence of the conventional variables is left out. The procedure that creates the commutator table requires that the table of groups be in this form.

```
In[32]:= infinitesimalgroupsxy1 = infinitesimalgroups /. {z1 → x, z2 → y, z3 → Ψ, z4 → t};

In[33]:= infinitesimalgroupsxy =
           Drop[infinitesimalgroupsxy1, {3, 7}];

In[34]:= Column[infinitesimalgroupsxy]

         {{1, 0}, {0, 0}}
         {{0, 1}, {0, 0}}
Out[34]= {{x, y}, {0, -3 t + 3 tinf}}
         {{0, 0}, {1, 0}}
         {{x, 0}, {Ψ, t - tinf}}

In[35]:= MakeCommutatorTable[independentvariables, dependentvariables, infinitesimalgroupsxy]

         MakeCommutatorTable has finished executing. You can look at the output in the table commutatortable.
           To present the output in the most readable form you may want view it as a matrix using
           MatrixForm[commutatortable]. Occasionally the entries in the commutatortable will have terms
           that cancel. To get rid of these terms use the function Simplify before viewing the table.

In[36]:= MatrixForm[Simplify[commutatortable]]
```

Out[36]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

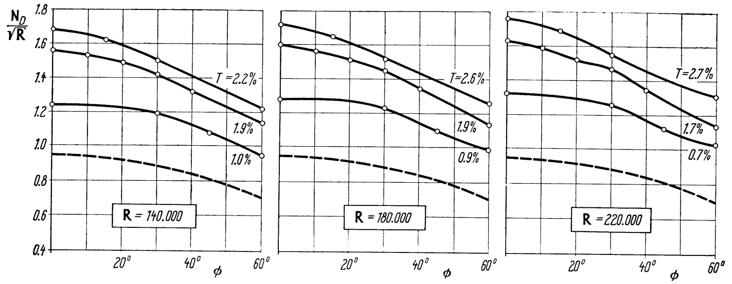


Fig. 12.20. Variation of local Nusselt number N_D on a circular cylinder with turbulence intensity T and angular coordinate ϕ, after J. Kestin, P. F. Maeder and H. H. Sogin [49] (Values of intensity of turbulence T approximate only) — — — Theory after N. Froessling [33]

Unexpectedly, however, the preceding effect is absent on a flat plate at zero incidence. Measurements performed by J. Kestin, P. F. Maeder and H. E. Wang [50b] on a flat plate show no sensitivity to free-stream turbulence in the laminar range. The same result was obtained by A. Edwards and N. Furber [24]. Such results suggest that external turbulence affects the local heat transfer only in the presence of a pressure gradient. The experiments quoted in ref. [50a] provide a certain confirmation of such a supposition. By imposing a pressure gradient artificially on a flat plate, it was found possible to increase the local Nusselt number by increasing the turbulence intensity. A qualitative explanation of this behavior can be obtained with the aid of C. C. Lin's theory described in Chap. XV, as pointed out in ref. [50b]. The effect of free-stream turbulence on heat transfer has been studied also in references [3a, 6a, 34a, 34b, 42a, 64, 77a, 87a, 109] †.

h. Thermal boundary layers in natural flow

Motions which are caused solely by the density gradients created by temperature differences are termed 'natural' as distinct from those 'forced' on the stream by external causes. Such a natural flow exists around a vertical hot plate or around a horizontal hot cylinder. Natural flows also display, in most cases, a boundary-layer structure, particularly if the viscosity and conductivity of the fluid are small.

In the case of a *vertical hot plate*, the pressure in each horizontal plane is equal to the gravitational pressure and is thus constant. The only cause of motion is furnished by the difference between weight and buoyancy in the gravitational field of the earth. The equation of motion is obtained from eqns. (12.36a, b, c) with $dp/dx = 0$ and $\beta = 1/T_\infty$. Neglecting frictional heat we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \tag{12.111}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\,\frac{T_w - T_\infty}{T_\infty}\,\theta\,, \tag{12.112}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = a\,\frac{\partial^2 \theta}{\partial y^2}\,. \tag{12.113}$$

† *Note added in proof:* A comprehensive summary of this subject is given in an article by J. Kestin in "Advances in Heat Transfer", vol. 3, Ac. Press, 1966.

Here $a = k/\varrho\, c_p$ is the thermal diffusivity and $\theta = (T - T_\infty)/(T_w - T_\infty)$ is the dimensionless local temperature. In a theoretical investigation concerning the experimentally determined temperature and velocity field of a case involving natural convection on a vertical hot plate, due to E. Schmidt and W. Beckmann [80], E. Pohlhausen demonstrated that if a stream function is introduced by putting $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, then the resulting partial differential equation for ψ can be reduced to an ordinary differential equation by the similarity transformation

$$\eta = c\,\frac{y}{\sqrt[4]{x}}\,; \qquad \psi = 4\,\nu\,c\,x^{3/4}\,\zeta(\eta)$$

where

$$c = \sqrt[4]{\frac{g\,(T_w - T_\infty)}{4\,\nu^2\,T_\infty}}\,. \tag{12.114}$$

The velocity components now become

$$u = 4\,\nu\,x^{1/2}c^2\,\zeta'\,; \qquad v = \nu\,c\,x^{-1/4}(\eta\,\zeta' - 3\,\zeta)\,,$$

and the temperature distribution is determined by the function $\theta(\eta)$. Equations (12.112), (12.113) and (12.114) lead to the following differential equations

$$\zeta''' + 3\,\zeta\,\zeta'' - 2\,\zeta'^2 + \theta = 0\,, \qquad \theta'' + 3\,\mathsf{P}\,\zeta\,\theta' = 0\,, \tag{12.115a, b}$$

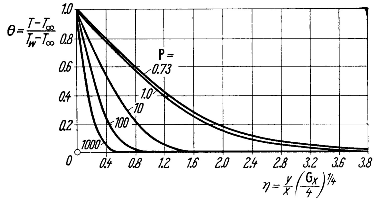


Fig. 12.21. Temperature distribution in the laminar boundary layer on a hot vertical flat plate in natural convection. Theoretical curves, for P = 0·73, after E. Pohlhausen [73] and S. Ostrach [72]

$G_x = \frac{g\,x^3}{\nu^2}\,\frac{T_w - T_\infty}{T_\infty}$ = Grashof number

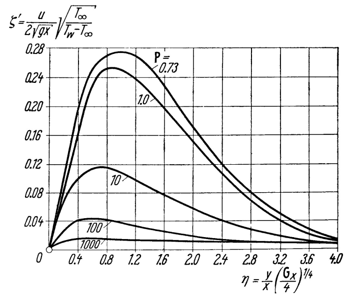


Fig. 12.22. Velocity distribution in the laminar boundary layer on a hot vertical flat plate in natural convection (see also Fig. 12.21)

with the boundary conditions $\zeta = \zeta' = 0$ and $\theta = 1$ at $\eta = 0$ and $\zeta' = 0$, $\theta = 0$ at $\eta = \infty$. Figures 12.21 and 12.22 illustrate the solutions of these equations for various values of P. Figures 12.23 and 12.24 contain a comparison between the calculated velocity and temperature distribution and those measured by E. Schmidt and W. Beckmann [80]. The agreement is seen to be very good. It is seen, further, that the velocity and thermal boundary-layer thickness are proportional to $x^{1/4}$.

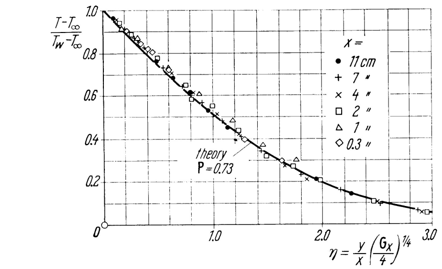


Fig. 12.23. Temperature distribution in the laminar boundary layer on a hot vertical flat plate in natural convection in air, as measured by E. Schmidt and W. Beckmann [80]; x = distance from the lower edge of the plate

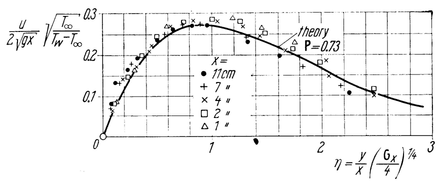


Fig. 12.24. Velocity distribution in the laminar boundary layer on a vertical plate in natural convection in air as measured by E. Schmidt and W. Beckmann [80]

Heat transfer: The quantity of heat $q(x) = -k(\partial T/\partial y)_0$ transferred per unit time and area from the plate to the fluid at section x becomes

$$q(x) = -k\,c\,x^{-1/4}\left(\frac{d\theta}{d\eta}\right)_0 (T_w - T_\infty)\,,$$

with $(\partial\theta/\partial\eta)_0 = -0{\cdot}508$ for $\mathsf{P} = 0{\cdot}733$. The total heat transferred by a plate of length l and width b is $Q = b\int_0^l q(x)\,dx$, and hence

$$Q = \tfrac{4}{3} \times 0{\cdot}508\, b\, l^{3/4}\, c\, k\,(T_w - T_\infty)\,.$$

The mean Nusselt number defined by $Q = b\,k\,\mathsf{N}_m (T_w - T_\infty)$ thus becomes $\mathsf{N}_m = 0{\cdot}677\, c\, l^{3/4}$, or, inserting the value of c from eqn. (12.114):

$$\mathsf{N}_m = 0{\cdot}478\,(\mathsf{G})^{1/4}\,, \qquad (12.116)$$

where

$$\mathsf{G} = \frac{g\,l^3 (T_w - T_\infty)}{\nu^2\, T_\infty} \qquad (12.117)$$

is the Grashof number. It can also be written as $\mathsf{G} = g\,l^3\,\beta\,(T_w - T_\infty)/\nu^2$ in the case of liquids.

The diagram in Fig. 12.25 gives a comparison between theoretical results on free convection with measurements on heated vertical cylinders and flat plates performed by E. R. G. Eckert and T. W. Jackson [18]. When the product $\mathsf{GP} < 10^8$, the flow is laminar, and for $\mathsf{GP} > 10^{10}$ the flow is turbulent. The agreement between theory and experiment is excellent.

E. Pohlhausen's calculations have been extended by H. Schuh [84] to the case of large Prandtl numbers such as exist in oils.

The case of very small Prandtl numbers is treated in a paper by E. M. Sparrow and J. L. Gregg [100]. The limiting cases when $\mathsf{P} \to 0$ and $\mathsf{P} \to \infty$ were examined by E. J. Le Fevre [55], according to whom we may write

$$\frac{\mathsf{N}_m}{(\mathsf{GP}^2)^{1/4}} = 0{\cdot}800 \quad (\mathsf{P} \to 0)\,, \qquad (12.118\text{a})$$

$$\frac{\mathsf{N}_m}{(\mathsf{GP})^{1/4}} = 0{\cdot}670 \quad (\mathsf{P} \to \infty)\,. \qquad (12.118\text{b})$$

Some numerical values for intermediate Prandtl numbers are contained in Table 12.6. Calculations with a temperature-dependent viscosity were performed by T. Hara [39]. The effect of suction or blowing on the rate of heat transfer from a vertical plate in natural convection is described in refs. [25, 98]. Additional classes of similar solutions in natural flows were discussed by K. T. Yang [118]. Thus, temperature distributions on the surface of the plate of the form $T_w - T_\infty = T_1 x^n$ also produce similar solutions, but the differential equation (12.115) is now replaced by

$$\zeta''' + (n+3)\,\zeta\,\zeta'' - 2\,(n+1)\,\zeta'^2 + \theta = 0\,, \qquad (12.119\text{a})$$

$$\theta'' - \mathsf{P}\,[4\,n\,\zeta' - (n+3)\,\zeta\,\theta'] = 0\,. \qquad (12.119\text{b})$$

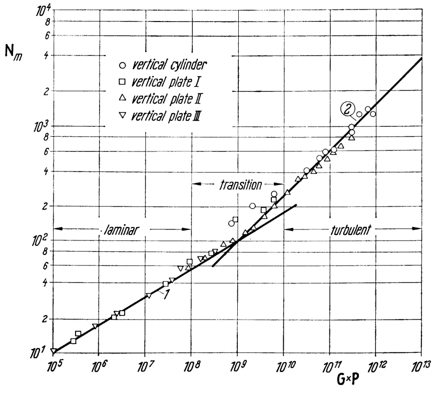
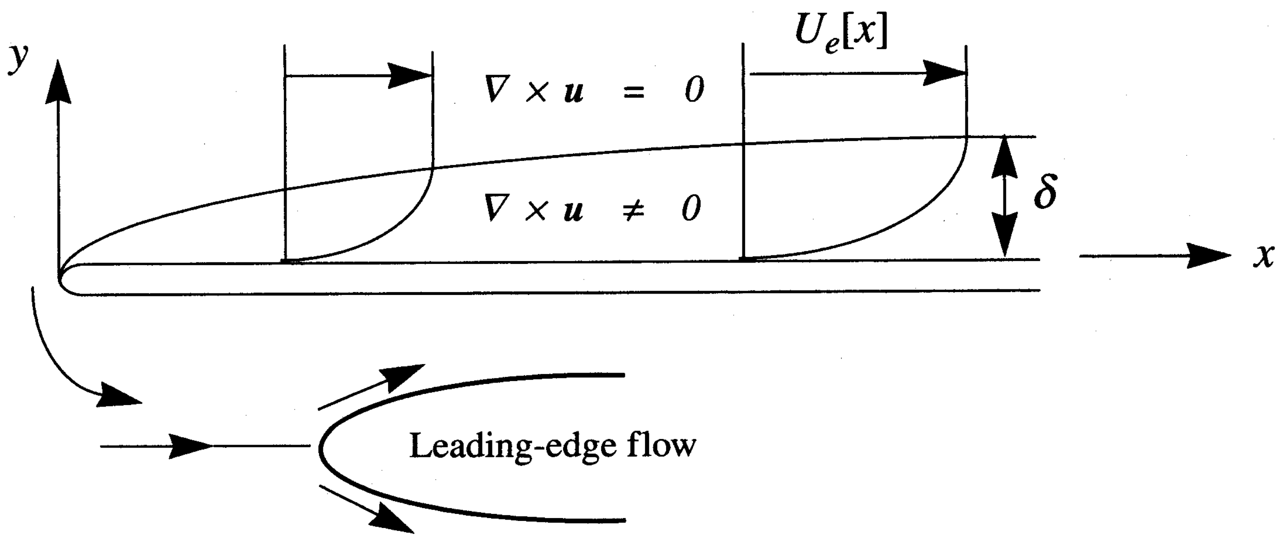


Fig. 12.25. Average Nusselt number for free convection on vertical plates and cylinders, after E. R. G. Eckert and T. W. Jackson [18]

Curve (1) laminar:
$\mathsf{N}_m = 0{\cdot}555\,(\mathsf{GP})^{1/4}$; $\mathsf{GP} < 10^9$

Curve (2) turbulent:
$\mathsf{N}_m = 0{\cdot}0210\,(\mathsf{GP})^{2/5}$; $\mathsf{GP} > 10^9$

Boundary layer on a flat plate



$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2},$$

$$\frac{\partial p}{\partial y} = 0.$$

Assume the static pressure is constant through the boundary layer

$$P_{\text{total}} = P_e[x] + \frac{1}{2}\rho U_e[x]^2.$$

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}.$$

$$\frac{dx}{dt} = \frac{\partial \psi}{\partial y}, \qquad \frac{dy}{dt} = -\frac{\partial \psi}{\partial x}.$$

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} - U_e \frac{dU_e}{dx} - \nu \psi_{yyy} = 0.$$

$$\psi|_{y=0} = 0, \qquad \left.\frac{\partial \psi}{\partial y}\right|_{y=0} = 0, \qquad \left.\frac{\partial \psi}{dy}\right|_{y\to\infty} = U_e[x].$$

Zero pressure gradient case

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} - \nu \psi_{yyy} = 0$$

$$\psi[x, 0] = 0, \qquad \psi_y[x, 0] = 0, \qquad \psi_y[x, \infty] = U_e = \text{constant}.$$

Apply a three parameter dilation group

$$\tilde{x} = e^a x, \qquad \tilde{y} = e^b y, \qquad \psi = e^c \psi,$$

$$\tilde{\psi}_{\tilde{y}} \tilde{\psi}_{\tilde{x}\tilde{y}} - \tilde{\psi}_{\tilde{x}} \tilde{\psi}_{\tilde{y}\tilde{y}} - \nu \tilde{\psi}_{\tilde{y}\tilde{y}\tilde{y}}$$
$$= e^{2c-a-2b} \psi_y \psi_{xy} - e^{2c-a-2b} \psi_x \psi_{yy} - \nu e^{c-3b} \psi_{yyy} = 0.$$

For invariance of the PDE require

$$2c - a - 2b = c - 3b.$$

$$\tilde{x} = e^a x, \qquad \tilde{y} = e^b y, \qquad \tilde{\psi} = e^{a-b} \psi.$$

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Check invariance of the boundary conditions

$$\tilde{y} = 0 \quad \Rightarrow \quad e^{b} y = 0 \quad \Rightarrow \quad y = 0.$$

$$\tilde{\psi}[\tilde{x}, 0] = 0|_{\text{all } \tilde{x}} \quad \Rightarrow \quad e^{a-b}\psi[e^{a}x, 0] = 0 \quad \Rightarrow \quad \psi[x, 0] = 0|_{\text{all } x}$$

$$\tilde{\psi}_{\tilde{y}}[\tilde{x}, 0] = 0|_{\text{all } \tilde{x}} \quad \Rightarrow \quad e^{a-2b}\psi_{y}[e^{a}x, 0] = 0 \quad \Rightarrow \quad \psi_{y}[x, 0] = 0|_{\text{all } x}.$$

$$\tilde{\psi}_{\tilde{y}}[\tilde{x}, \infty] = U_{e}|_{\text{all } x} \quad \Rightarrow \quad e^{a-2b}\psi_{y}[e^{a}x, \infty] = U_{e}|_{\text{all } x}$$

For invariance of the free stream condition require

$$a = 2b$$

$$\psi_{y}[x, \infty] = U_{e}|_{\text{all } x},$$

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The group that leaves the problem as a whole invariant is

$$\tilde{x} = e^{2b}x, \qquad \tilde{y} = e^{b}y, \qquad \tilde{\psi} = e^{b}\psi$$

$$\xi = 2x, \qquad \zeta = y, \qquad \eta = \psi$$

Similarity variables

$$\frac{dx}{2x} = \frac{dy}{y} = \frac{d\psi}{\psi}.$$

$$F = \frac{\psi}{(2\nu U_e x)^{1/2}}, \qquad \alpha = \frac{y}{(2\nu x/U_e)^{1/2}}.$$

$$\omega = \Omega\left[\frac{\psi}{(2\nu U_e x)^{1/2}}, \frac{y}{(2\nu x/U_e)^{1/2}}\right].$$

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We can expect a solution that is invariant under the same group

$$\frac{\psi}{(2\nu U_e x)^{1/2}} = F[\alpha].$$

$$\frac{u}{U_e} = F_\alpha, \qquad \frac{v}{U_e} = \frac{1}{\sqrt{2}}\left(\frac{1}{Re}\right)^{1/2}(F - \alpha F_\alpha).$$

Reynolds number is based on distance from the leading edge

$$R_{ex} = \frac{U_\infty x}{\nu}$$

Vorticity

$$\omega = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \cong -U_\infty\left(\frac{U_\infty}{2\nu x}\right)^{1/2} F_{\alpha\alpha}$$

Derivatives

$$\psi_{xy} = -\frac{U_\infty}{2x}\alpha F_{\alpha\alpha}$$

$$\psi_{yy} = U_\infty\left(\frac{U_\infty}{2\nu x}\right)^{1/2} F_{\alpha\alpha}$$

$$\psi_{yyy} = \frac{U_\infty^2}{2\nu x} F_{\alpha\alpha\alpha}$$

Substitute into the stream function equation and simplify

$$U_\infty F_\alpha\left(-\frac{U_\infty}{2x}\alpha F_{\alpha\alpha}\right) - U_\infty\left(\left(\frac{\nu}{2U_\infty x}\right)^{1/2}(\alpha F_\alpha - F)\right) U_\infty\left(\frac{U_\infty}{2\nu x}\right)^{1/2} F_{\alpha\alpha} = \nu\frac{U_\infty^2}{2\nu x} F_{\alpha\alpha\alpha}$$

$$-F_\alpha(\alpha F_{\alpha\alpha}) + (\alpha F_\alpha - F)F_{\alpha\alpha} = F_{\alpha\alpha\alpha}$$

$$-\alpha F_\alpha F_{\alpha\alpha} - FF_{\alpha\alpha} + \alpha F_\alpha F_{\alpha\alpha} - F_{\alpha\alpha\alpha} = 0$$

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The Blasius equation

$$F_{\alpha\alpha\alpha} + FF_{\alpha\alpha} = 0$$

Boundary conditions

$$F(0) = 0 \qquad F_{\alpha}(0) = 0 \qquad F_{\alpha}(\infty) = 1$$

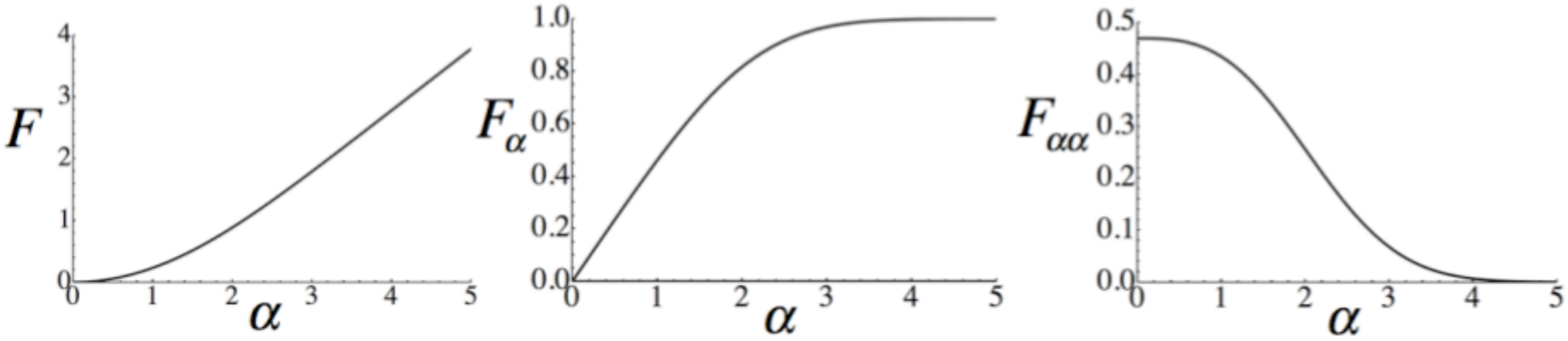

Figure 8.7 Solution of the Blasius equation (8.76) for the streamfunction, velocity and stress (or vorticity) profile in a zero pressure gradient laminar boundary layer.

Friction coefficient

$$C_f = \frac{\tau_w}{(1/2)\rho U_\infty^2} = \frac{0.664}{\sqrt{R_{ex}}}$$

Transverse velocity at the edge of the layer

$$\frac{V_e}{U_\infty} = \frac{0.8604}{\sqrt{R_{ex}}}$$

Boundary layer thickness

$$\frac{\delta_{0.99}}{x} = \frac{4.906}{\sqrt{R_{ex}}} \qquad \frac{\delta^*}{x} = \frac{1.7208}{\sqrt{R_{ex}}} \qquad \frac{\theta}{x} = \frac{0.664}{\sqrt{R_{ex}}}$$

$$\alpha_e = 4.906/\sqrt{2} = 3.469$$

Let $\tau = F_{\alpha\alpha}$

The Blasius equation can be expressed as

$$\frac{d\tau}{\tau} = -F d\alpha$$

$$\frac{\tau}{\tau_w} = e^{-\int_0^{\alpha} F d\alpha}$$

Let $F(\alpha) = \alpha - G(\alpha)$ Then $\lim_{\alpha \to \infty} G(\alpha) = C_1$

$$\left.\frac{\tau}{\tau_w}\right|_{\alpha > \alpha_e} = e^{-\int_0^{\alpha} (\alpha - G(\alpha)) d\alpha} = C_2 e^{C_1 \alpha - \frac{\alpha^2}{2}}$$

Vorticity at the edge of the layer decays exponentially with distance from the wall. This is the justification for dividing the flow into separate regions of rotational and irrotational flow.

Now use the symmetries of the Blasius equation to reduce the problem further

$$F_{\alpha\alpha\alpha} + F F_{\alpha\alpha} = 0$$

$$F[0] = 0, \qquad F_\alpha[0] = 0, \qquad F_\alpha[\infty] = 1.$$

$$\tilde{\alpha} = \alpha + s\xi[\alpha, F],$$

$$\tilde{F} = F + s\eta[\alpha, F].$$

$$\xi = a + b\alpha, \qquad \eta = -bF.$$

Table 10.1.
Commutator table for the Blasius equation.

	X^a	X^b
X^a	0	X^a
X^b	$-X^a$	0

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Begin with the translation group

$$X^a = \frac{\partial}{\partial \alpha}, \qquad X^b = \alpha \frac{\partial}{\partial \alpha} - F \frac{\partial}{\partial F}$$

$$\frac{d\alpha}{1} = \frac{dF}{0} = \frac{dF_\alpha}{0} = \frac{dF_{\alpha\alpha}}{0} = \frac{dF_{\alpha\alpha x}}{0},$$

$$\phi = F, \qquad G = F_\alpha.$$

$$\frac{dG}{d\phi} = \frac{\frac{\partial G}{\partial \alpha} d\alpha + \frac{\partial G}{\partial F} dF + \frac{\partial G}{\partial F_\alpha} dF_\alpha}{\frac{\partial \phi}{\partial \alpha} d\alpha + \frac{\partial \phi}{\partial F} dF} = \frac{F_{\alpha\alpha}}{F_\alpha}$$

$$\frac{d^2G}{d\phi^2} = \left(\frac{F_\alpha F_{\alpha\alpha\alpha} - F_{\alpha\alpha}^2}{F_\alpha^2}\right)\frac{1}{F_\alpha} = \frac{F_\alpha(-F F_{\alpha\alpha}) - F_{\alpha\alpha}^2}{F_\alpha^3},$$

First reduction

$$GG_{\phi\phi} + \phi G_\phi + (G_\phi)^2 = 0.$$

$$G(0) = 0, \qquad G(\infty) = 1,$$

$$\tilde{\phi} = e^{-b}\phi, \qquad \tilde{G} = e^{-2b}G.$$

$$\tilde{G}\tilde{G}_{\tilde{\phi}\tilde{\phi}} + \tilde{\phi}\tilde{G}_{\tilde{\phi}} + (\tilde{G}_{\tilde{\phi}})^2 = e^{-2b}(GG_{\phi\phi} + \phi G_\phi + (G_\phi)^2) = 0.$$

Second reduction – use the dilation group

$$\frac{d\phi}{-\phi} = \frac{dG}{-2G} = \frac{dG_\phi}{-G_\phi}.$$

$$\gamma = \frac{G}{\phi^2} = \frac{F_\alpha}{F^2},$$

$$H(\gamma) = \frac{G_\phi}{\phi} = \frac{F_{\alpha\alpha}}{F F_\alpha}.$$

$$\frac{\left(\frac{DH}{D\phi}\right)}{\left(\frac{D\gamma}{D\phi}\right)} = \frac{dH}{d\gamma} = \frac{H_\phi + H_G \frac{dG}{d\phi} + H_{G_\phi} \frac{dG_\phi}{d\phi}}{\gamma_\phi + \gamma_G \frac{dG}{d\phi}} = \frac{-\frac{G_\phi}{\phi^2} + \frac{1}{\phi}(G_{\phi\phi})}{-2\frac{G}{\phi^3} + \frac{1}{\phi^2} G_\phi}.$$

$$\frac{dH}{d\gamma} = \frac{\gamma H + H + H^2}{2\gamma^2 - \gamma H}.$$

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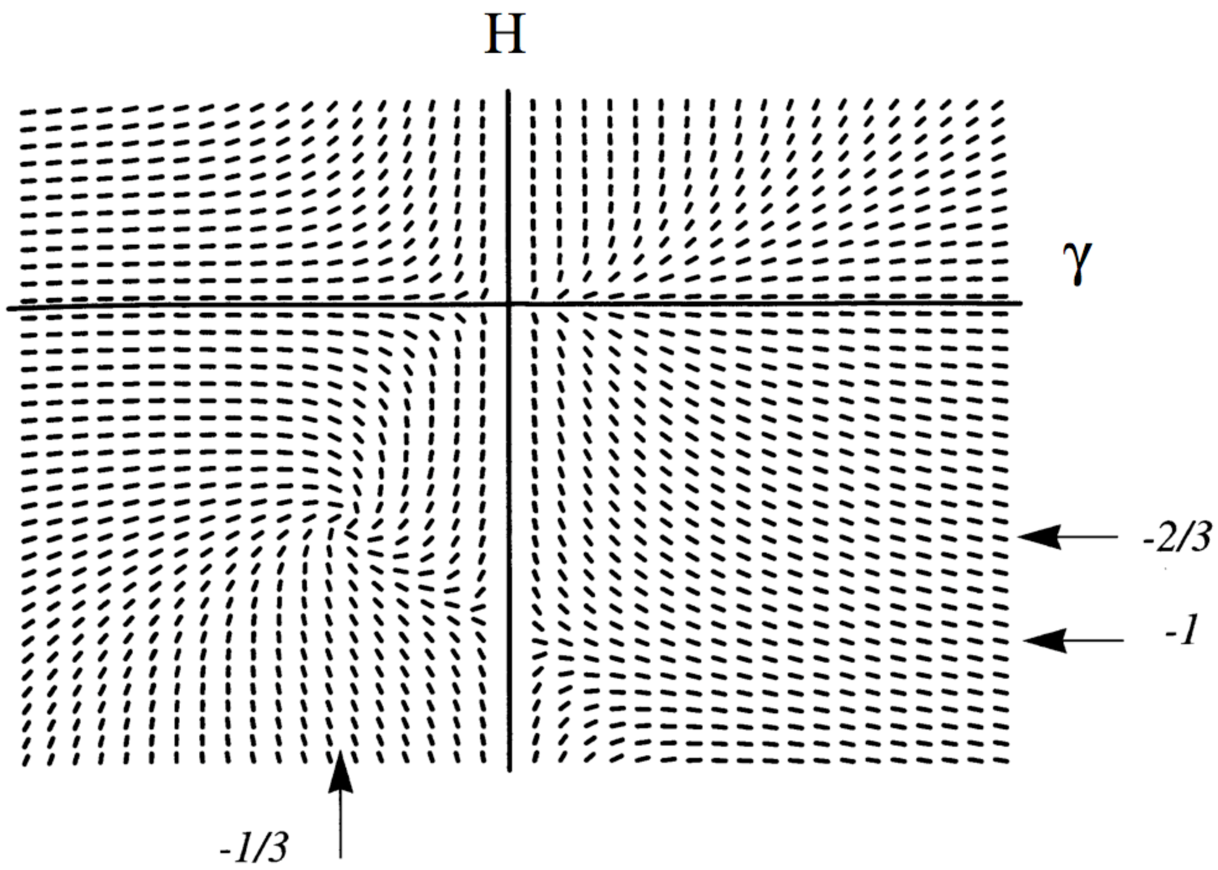
Phase portrait

$$\frac{dH}{ds} = \gamma H + H + H^2,$$

$$\frac{d\gamma}{ds} = 2\gamma^2 - \gamma H$$



Which curve is the correct one ? Consider the flow near the wall.

$$C_f = \frac{\tau_{xy}}{\frac{1}{2}\rho U_e^2}\bigg|_{y=0} = \left(\frac{2\nu}{xU_e}\right)^{1/2} F_{\alpha\alpha}[0].$$

$$\frac{G_\phi}{\phi} = \frac{\tau_0}{FF_\alpha} = \frac{\tau_0}{\phi G}.$$

$$G = (2\tau_0\phi)^{1/2}.$$

$$\lim_{\phi\to 0} H = \frac{1}{2}\left(\frac{2\tau_0}{\phi^3}\right)^{1/2},$$

$$\lim_{\phi\to 0} \gamma = \left(\frac{2\tau_0}{\phi^3}\right)^{1/2}.$$

Near the wall the solution must approach

$$\lim_{\gamma\to\infty}\left(\frac{H}{\gamma}\right) = \frac{1}{2}.$$

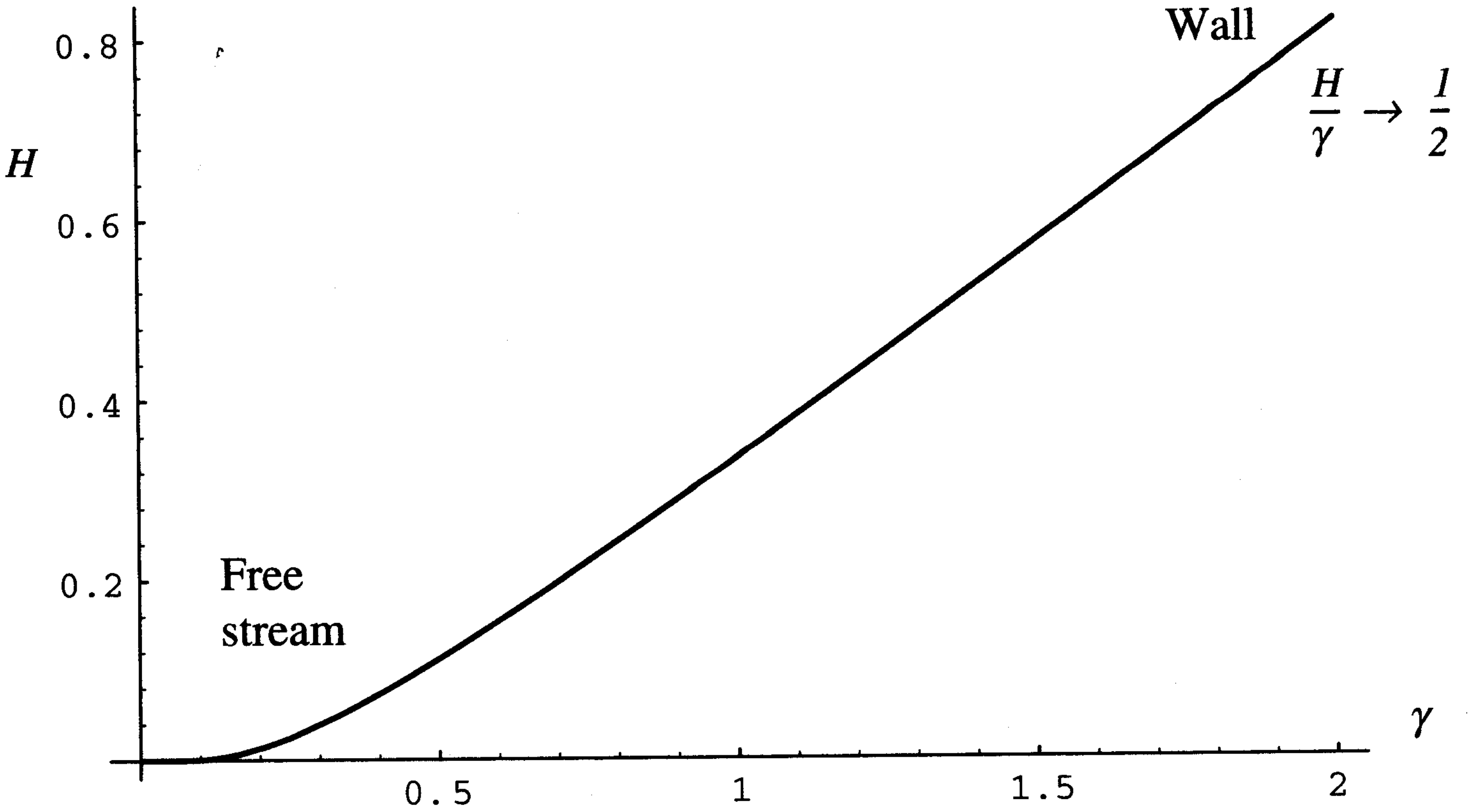


Fig. 10.3. Blasius solution in the phase plane.

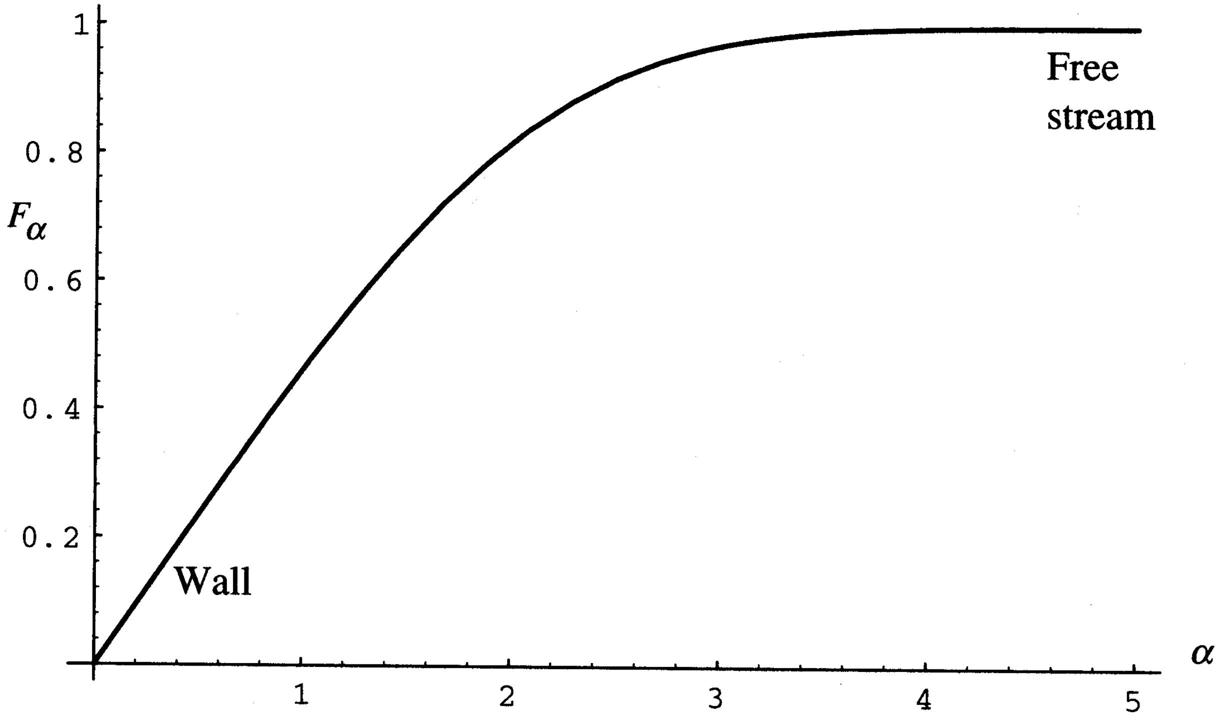


Fig. 10.4. The Blasius velocity profile.

$$C_{f0} = \frac{\tau_0}{\frac{1}{2}\rho U_e^2} = \frac{0.664}{\sqrt{Re}}.$$

Numerical solution

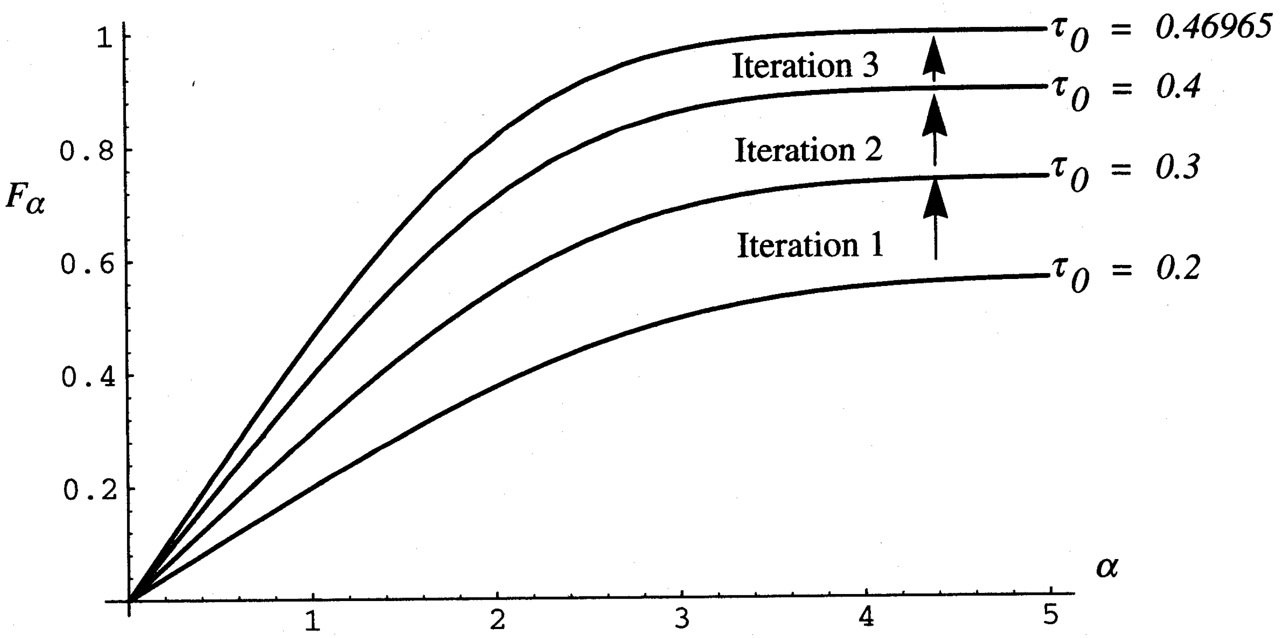


Fig. 10.5. Iteration process leading to the correct match with the free-stream boundary condition $\lim_{\alpha\to\infty} F_\alpha = 1$.

The group can be used to generate the solution in one step!

$$\tilde{\alpha} = e^{b}\alpha,$$
$$\tilde{F} = e^{-b}F,$$
$$\tilde{F}_{\tilde{\alpha}} = e^{-2b}F_{\alpha},$$
$$\tilde{F}_{\tilde{\alpha}\tilde{\alpha}} = e^{-3b}F_{\alpha\alpha}.$$

$$1 = e^{-2b}(0.566067) \quad \Rightarrow \quad b = -0.284557.$$

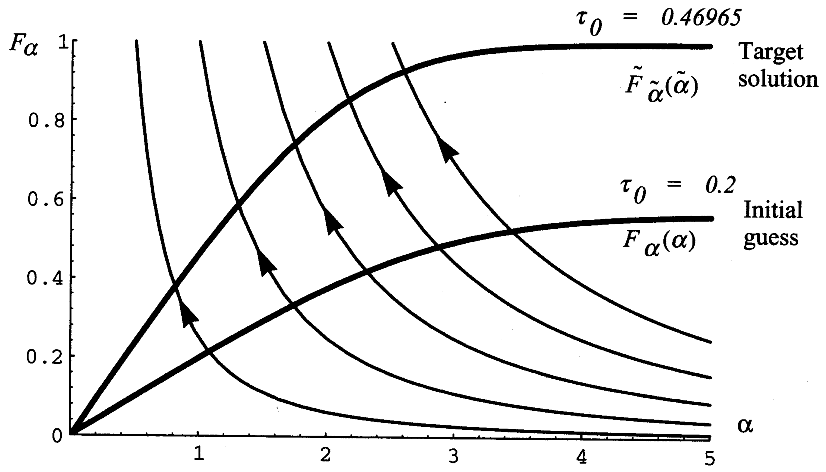


Fig. 10.6. Mapping of an initial guess to the correct solution along the pathlines of the dilation group of the Blasius equation.

$$\sout{0.2 = e^{-3b}F_{\alpha\alpha}[0]} \quad \Rightarrow \quad F_{\alpha\alpha}[0] = 0.46965.$$

$$\tilde{F}_{\tilde{\alpha}\tilde{\alpha}}[0] = e^{-3b}(0.2) \quad \Rightarrow \quad \tilde{F}_{\tilde{\alpha}\tilde{\alpha}}[0] = 0.46965$$

Temperature gradient shocks in nonlinear media

$$\frac{\partial T}{\partial t} = \lambda \frac{\partial}{\partial x}\left(T \frac{\partial T}{\partial x}\right).$$

$$T[x,0] = 0, \qquad x > 0,$$

$$T[0,t] = T_0, \qquad t > 0.$$

Group $\quad \tilde{x} = e^{s} x, \qquad \tilde{t} = e^{2s} t, \qquad \tilde{T} = T$

Similarity variables $\dfrac{dx}{x} = \dfrac{dt}{2t} = \dfrac{dT}{0}.$

$$\frac{T}{T_0} = g[\theta], \qquad \theta = \frac{x}{(2\lambda T_0 t)^{1/2}}.$$

Problem reduces to a 2nd order ODE $\quad g g_{\theta\theta} + \theta g_\theta + (g_\theta)^2 = 0,$

$$g[0] = 1, \qquad g[\infty] = 0.$$

Numerical solution - is the solution unique ?

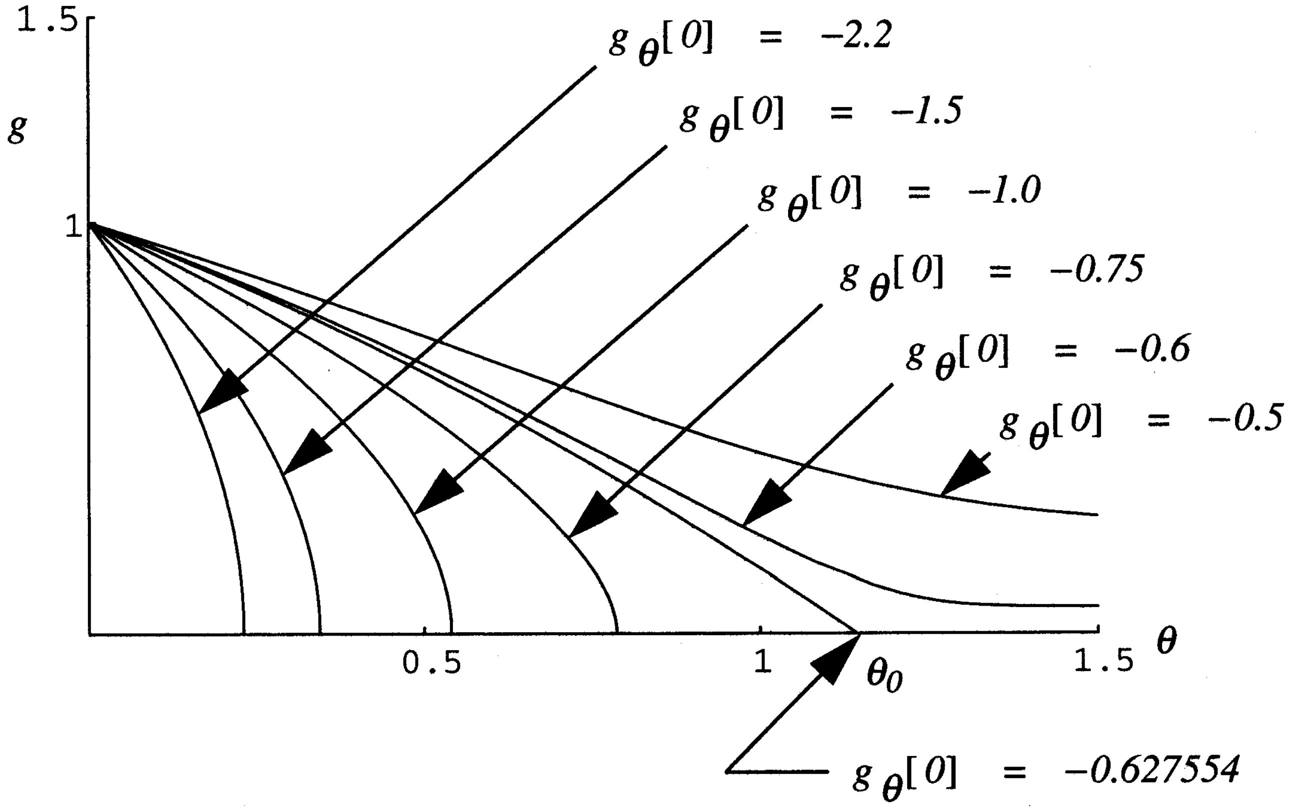


Fig. 10.7. Trial solutions of (10.66) and (10.67) with specified initial slope.

The ODE reduces to an autonomous system

$$gg_{\theta\theta} + \theta g_\theta + (g_\theta)^2 = 0,$$

$$g[0] = 1, \qquad g[\infty] = 0.$$

$$\tilde{\theta} = e^{-b}\theta, \qquad \tilde{g} = e^{-2b} g$$

$$\gamma = \frac{g}{\theta^2},$$

$$H = \frac{g_\theta}{\theta}.$$

$$\frac{dH}{ds} = \gamma H + H + H^2,$$

$$\frac{d\gamma}{ds} = 2\gamma^2 - \gamma H$$

Phase portrait - again

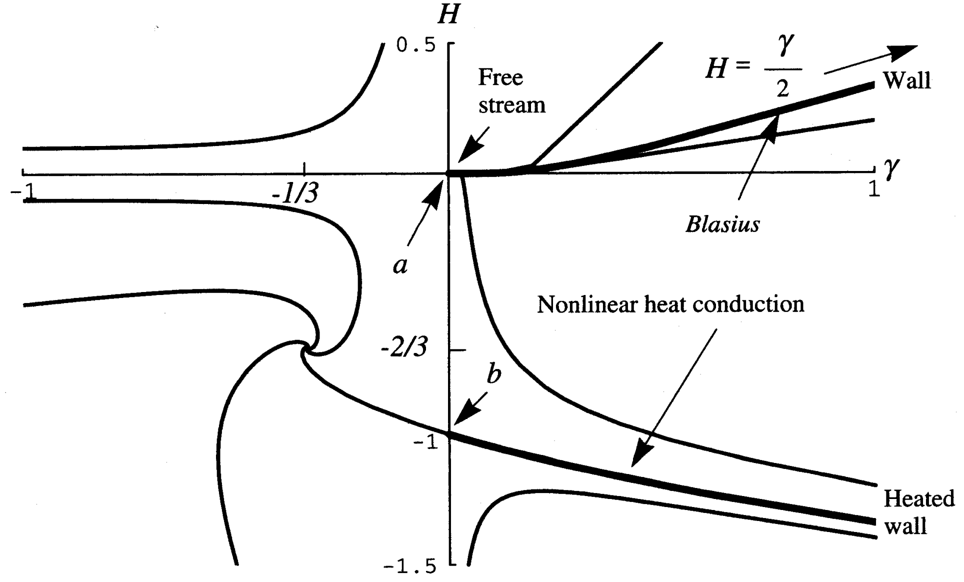


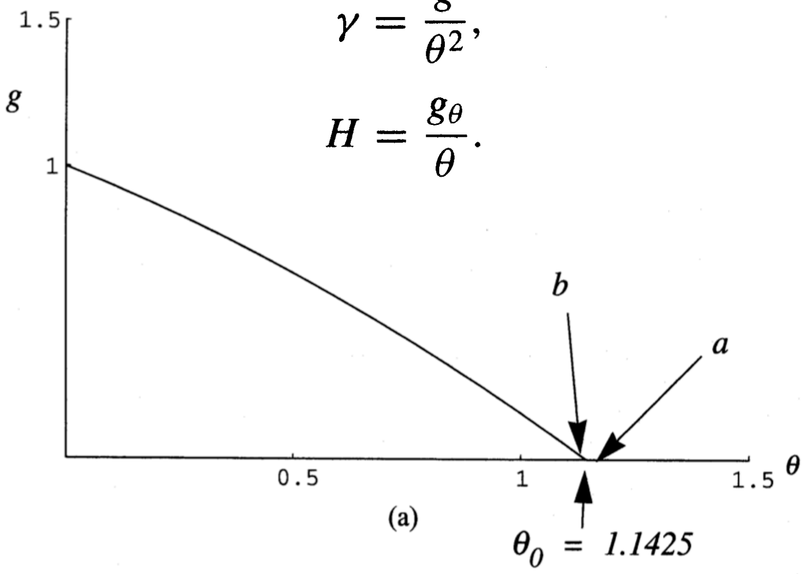
Fig. 10.2. Phase portrait of the Blasius system (10.45).

$$\frac{dH}{ds} = \gamma H + H + H^2,$$

$$\frac{d\gamma}{ds} = 2\gamma^2 - \gamma H$$

$$\gamma = \frac{g}{\theta^2},$$

$$H = \frac{g_\theta}{\theta}.$$



Which trajectory is the correct one ?

At infinity

$$g_\theta|_{g=0}(g_\theta|_{g=0} + \theta_0) = 0,$$

$$g_\theta|_{g=0} = 0, -\theta_0,$$

Two possibilities $(\gamma H) = (0, 0),\qquad (\gamma, H) = (0, -1).$

Flow near the saddle point

$$\frac{d\gamma}{ds} = 2\gamma^2 - \gamma H = A(\gamma, H), \qquad \gamma = \frac{g}{\theta^2},$$

$$\frac{dH}{ds} = \gamma H + H + H^2 = B(\gamma, H). \qquad H = \frac{g_\theta}{\theta}.$$

$$\begin{bmatrix} \frac{d\gamma}{ds} \\ \frac{dH}{ds} \end{bmatrix} = \begin{bmatrix} \frac{\partial A}{\partial \gamma} & \frac{\partial A}{\partial H} \\ \frac{\partial B}{\partial \gamma} & \frac{\partial B}{\partial H} \end{bmatrix}\Bigg|_{(\gamma,H)=(0,-1)} \qquad \begin{bmatrix} \gamma \\ H+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \gamma \\ H+1 \end{bmatrix}.$$

$$e_1 = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}, \qquad e_{-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Near the saddle point

$$\frac{dH}{d\gamma} = -\frac{1}{2}.$$

$$H = -\tfrac{1}{2}\gamma - 1,$$

$$\frac{g_\theta}{\theta} = -\frac{1}{2}\frac{g}{\theta^2} - 1,$$

$$\theta \frac{d}{d\theta}\left(\frac{g}{\theta^2}\right) = -\frac{5}{2}\left(\frac{g}{\theta^2}\right) - 1.$$

$$\int_0^{g/\theta^2} \frac{d\gamma}{\frac{5}{2}\gamma + 1} = -\int_{\theta_0}^{\theta} \frac{d\theta}{\theta},$$

$$\lim_{\theta\to\theta_0} g[\theta] = \frac{2}{5}\left(\frac{\theta_0^{5/2}}{\theta^{1/2}} - \theta^2\right).$$

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Solution is discontinuous in the temperature derivative

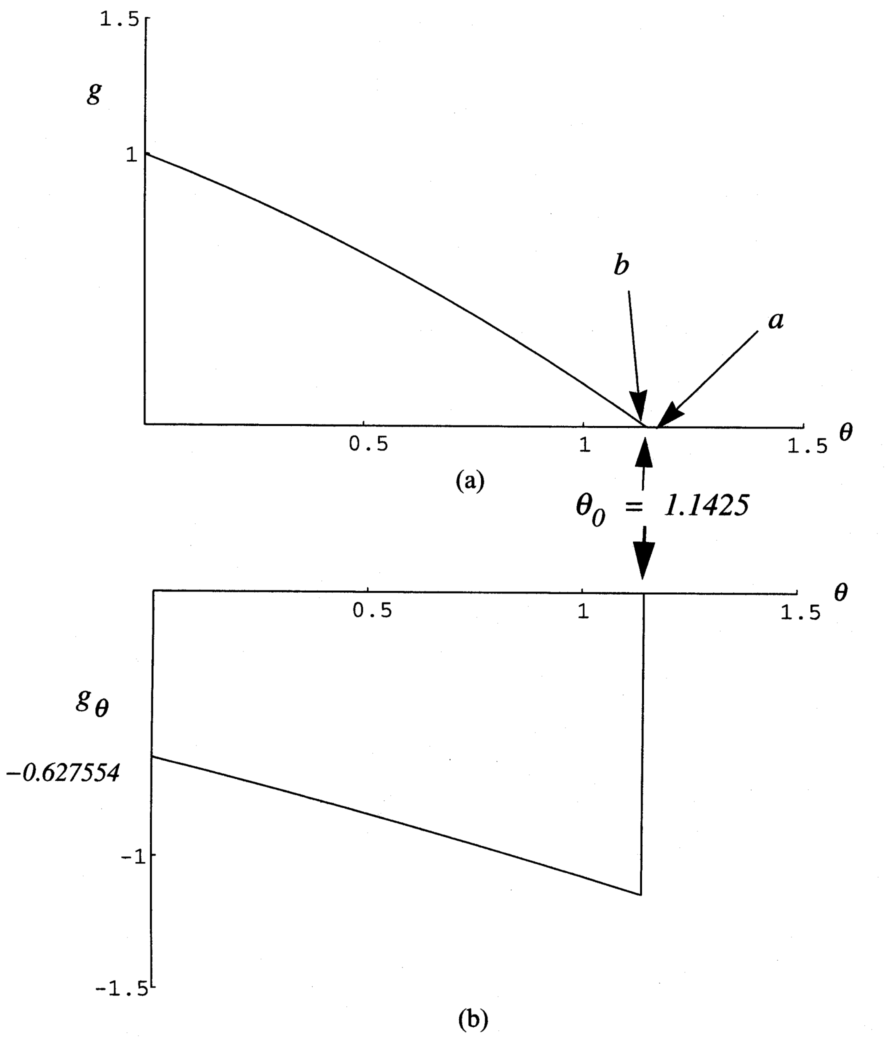

Fig. 10.8. Propagation of a thermal front in a nonlinear medium: (a) self-similar temperature, (b) self-similar temperature gradient.

$$g[\theta_0] = 0,$$
$$g_\theta[\theta_0] = -\theta_0,$$
$$g_{\theta\theta}[\theta_0] = -\tfrac{1}{2}.$$

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Thermal analogy to the Blasius problem

$$g[0] = 0, \qquad g[\infty] = 1.$$

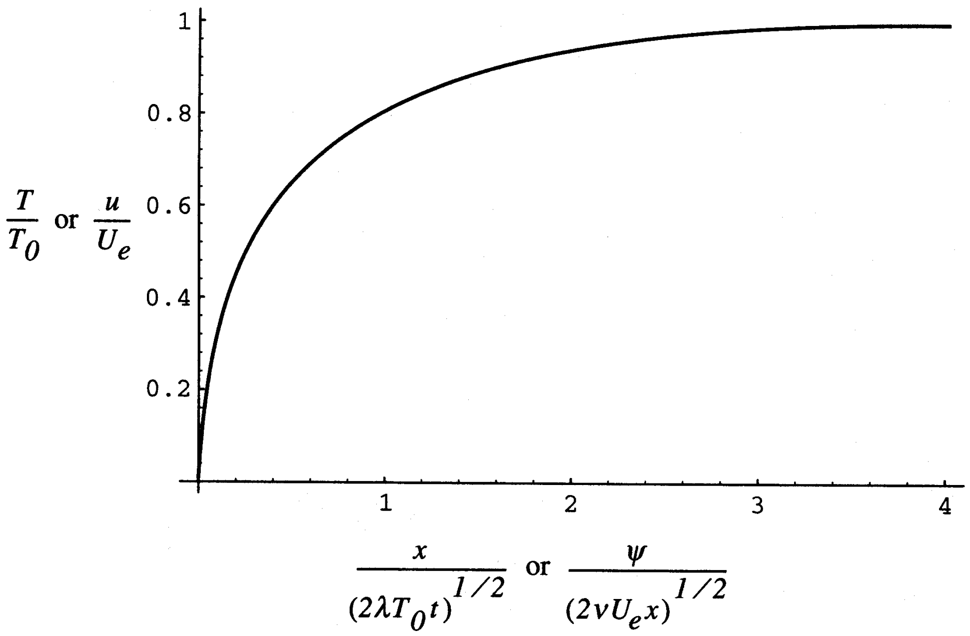

Fig. 10.9. Cooling of a nonlinear medium – analogy with the Blasius boundary layer.

$$\frac{T}{T_0} \rightarrow \frac{u}{U_e},$$

$$\frac{x}{(2\lambda T_0 t)^{1/2}} \rightarrow \frac{\psi}{(2\nu U_e x)^{1/2}}.$$

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General group of the boundary layer equations

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-U_e\frac{dU_e}{dx}-\nu\frac{\partial^3\psi}{\partial y^3}=0.$$

$$\tilde{x}=x+s\xi[x,y,\psi],$$
$$\tilde{y}=y+s\zeta[x,y,\psi],$$
$$\tilde{\psi}=\psi+s\eta[x,y,\psi].$$

Infinitesimals

$$\xi[x,y,\psi]=a\left(\frac{4U_eU_{ex}}{U_{ex}^2+U_eU_{exx}}\right),$$
$$\zeta[x,y,\psi]=-ay+g[x],$$
$$\eta[x,y,\psi]=a\psi+b,$$

In order for the invariance condition to be zero the free stream velocity must satisfy

$$(U_eU_{ex})(U_eU_{ex})_{xx}=(U_eU_{ex})^2.$$

$$U_e=\left(A+Be^{x/L}\right)^{1/2}.$$

The package presents the results as

```
Column[infinitesimalgroupsxy]
{{1, -y (Ue'[x]^2+Ue[x] Ue''[x])/(4 Ue[x] Ue'[x])}, {-Ψ (-Ue'[x]^2-Ue[x] Ue''[x])/(4 Ue[x] Ue'[x])}}
{{0, 1}, {0}}
{{0, x}, {0}}
{{0, x^2}, {0}}
{{0, x^3}, {0}}
{{0, x^4}, {0}}
{{0, 0}, {1}}
```

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Falkner-Skan boundary layers

$$\Omega = \psi_y \psi_{xy} - \psi_x \psi_{yy} - U_e \frac{dU_e}{dx} - \nu \psi_{yyy} = 0 \qquad (10.98)$$

$$\begin{aligned} \tilde{x} &= x + s\xi[x, y, \psi, U_e], \\ \tilde{y} &= y + s\zeta[x, y, \psi, U_e], \\ \tilde{\psi} &= \psi + s\eta[x, y, \psi, U_e], \\ \tilde{U}_e &= U_e + s\theta[x, y, \psi, U_e]. \end{aligned} \qquad (10.99)$$

$$\begin{aligned} \xi(x, y, \psi, U_e) &= a + (b + c)x, \\ \zeta(x, y, \psi, U_e) &= cy + g[x], \\ \eta(x, y, \psi, U_e) &= d + b\psi, \\ \theta(x, y, \psi, U_e) &= (b - c)U_e, \end{aligned} \qquad (10.102)$$

where $\partial U_e / \partial y = 0$ is applied to the invariance condition as a rule.

```
rulesarray=
{"D[Ψ[x,y],y,y,y]->-(1/nu)*Ue[x,y]*D[Ue[x,y],x]-(1/nu)*D[Ψ[x,y],x]*D[Ψ[x,y],y,y]+
(1/nu)*D[Ψ[x,y],y]*D[Ψ[x,y],x,y]",
"D[Ue[x,y],y]->0"};
```

Free stream velocity distribution

$$\frac{dx}{a+(b+c)x} = \frac{dy}{cy+g[x]} = \frac{d\psi}{d+b\psi} = \frac{dU_e}{(b-c)U_e}. \qquad (10.103)$$

$$\frac{dx}{a+(b+c)x} = \frac{dU_e}{(b-c)U_e} \qquad (10.104)$$

Pressure gradient

$$U_e = M(x+x_0)^{\beta}, \qquad (10.105)$$

$$\beta = \frac{b-c}{b+c}, \qquad x_0 = \frac{a}{b+c}, \qquad (10.106)$$

$$\hat{M} = L^{1-\beta}/T. \qquad (10.107)$$

Similarity variables

$$\alpha = \left(\frac{M}{2\nu}\right)^{1/2} \frac{y}{(x+x_0)^{(1-\beta)/2}},$$
$$F = \frac{\psi}{(x+x_0)^{(1+\beta)/2}(2\nu M)^{1/2}}. \tag{10.108}$$

$$F_{\alpha\alpha\alpha} + (1+\beta)FF_{\alpha\alpha} - 2\beta(F_\alpha)^2 + 2\beta = 0 \tag{10.110}$$

$$F[0] = 0, \qquad F_\alpha[0] = 0, \qquad F_\alpha[\infty] = 1. \tag{10.111}$$

Translation group

$$\xi = 1, \qquad \eta = 0, \tag{10.112}$$

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Reduce the order by one

$$\phi = F, \qquad G = F_{\alpha}. \tag{10.113}$$

$$GG_{\phi\phi} + (1+\beta)\phi G_{\phi} + (G_{\phi})^2 + 2\beta\left(\frac{1}{G} - G\right) = 0 \tag{10.116}$$

$$G[0] = 0, \qquad G[\infty] = 1. \tag{10.117}$$

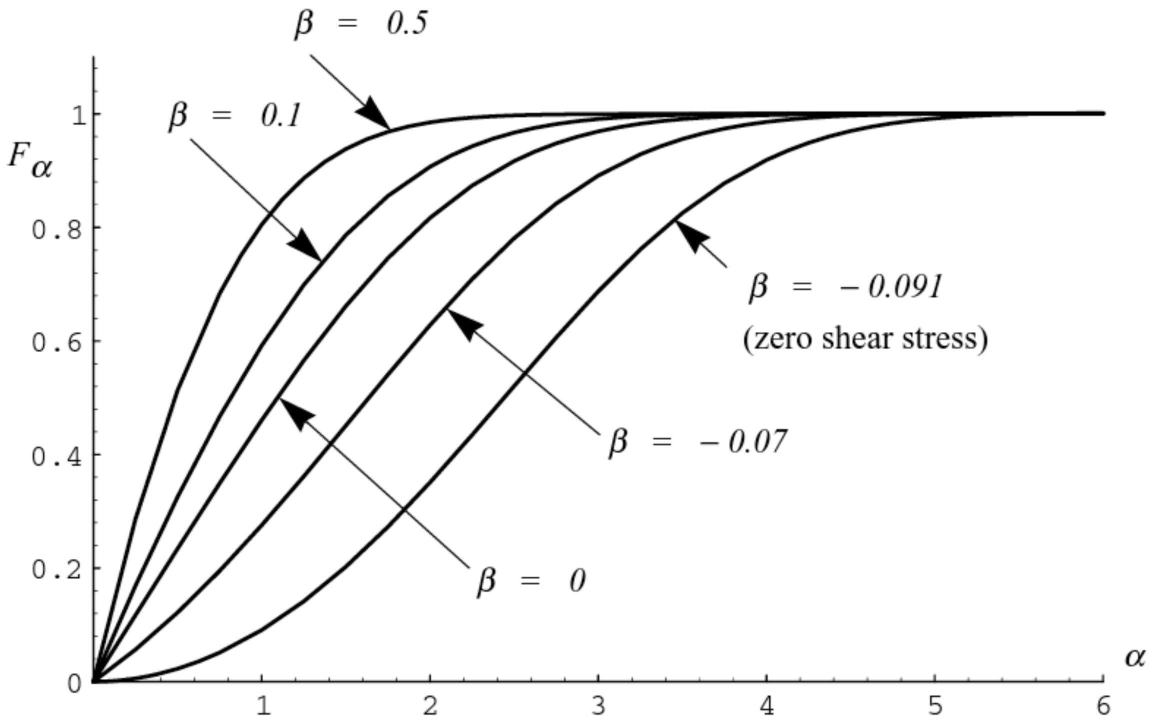

Fig. 10.11. Falkner–Skan velocity profiles.

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The unsteady boundary layer equation

$$\frac{\partial^z \psi}{\partial t \partial y} + \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x\, \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial U_e}{\partial t} - U_e \frac{dU_e}{dx} - \nu \frac{\partial^3 \psi}{\partial y^3} = 0 \quad (10.93)$$

General translation group - infinite dimensional group

$$\tilde{t} = t$$

$$\tilde{x} = x + h[t],$$

$$\tilde{y} = y + g[x, t],$$

$$\tilde{\psi} = \psi,$$

$$\tilde{U}_e = U_e,$$

where $h[t]$ and $g[x, t]$ are arbitrary functions.

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Arbitrary, time dependent translation in y - mapping a general unsteady flow to the flat plate

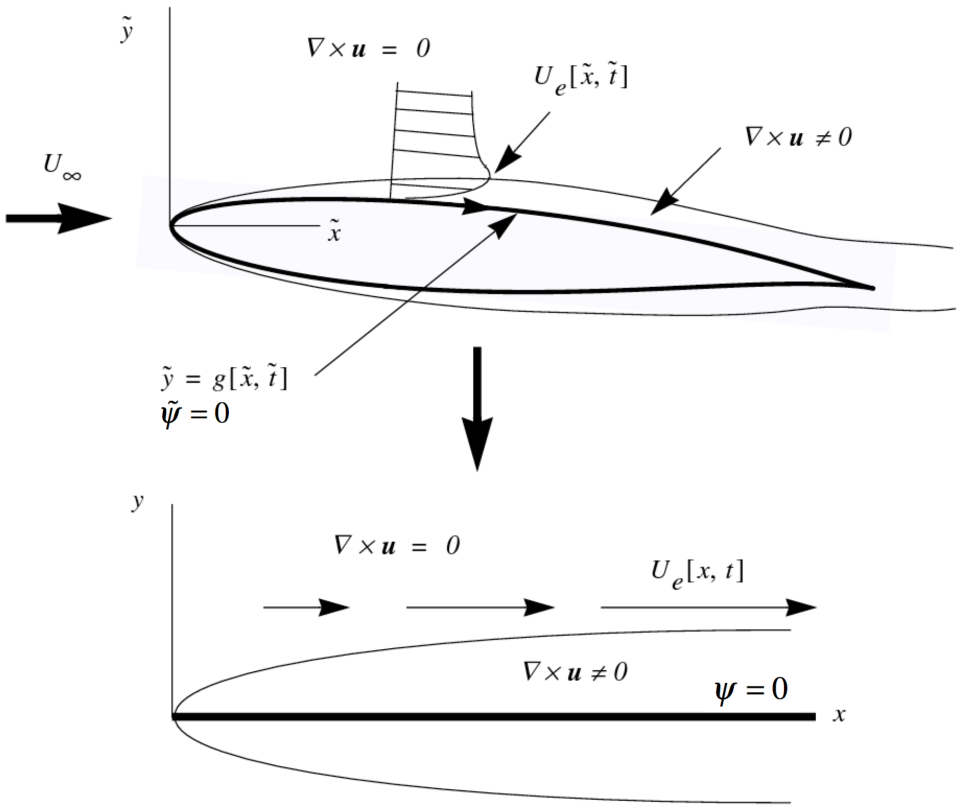

Fig. 10.10. Boundary layer on an unsteady wing transformed to the boundary layer on a flat plate. The function $h[t]$ in (10.94) is assumed to be zero.

$$\tilde{\psi}\left(\tilde{x}, \tilde{y} - g\left(\tilde{x}, \tilde{t}\right), \tilde{t}\right) = \psi\left(x, y, t\right)$$

The discovery that the viscous flow near the wall (the boundary layer) and the potential flow away from the wall can be approximately treated as mathematically distinct regions, and that an arbitrary shape can be mapped to a flat plate within that approximation, represents one of the most important advances in the history of mechanics. Today it is the fundamental basis of the modern theory of aerodynamics.

10.8 Exercises

10.1 In Section 10.1 it is stated that the flow outside the boundary layer is "irrotational ($\nabla \times \boldsymbol{u} = 0$) and therefore unaffected by viscosity." Justify this statement using the incompressible Navier–Stokes equations

$$\frac{\partial u^i}{\partial t} + \frac{\partial}{\partial x^k}\left(u^i u^k + \frac{p}{\rho}\delta^i_k\right) - \nu \frac{\partial u^i}{\partial x^k\,\partial x^k} = 0, \qquad \frac{\partial u^k}{\partial x^k} = 0 \tag{10.140}$$

and the identity $\nabla \times (\nabla \times \boldsymbol{u}) = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla^2 \boldsymbol{u}$.

10.2 Take the boundary-layer variables

$$F = \frac{\psi}{(2\nu U_e x)^{1/2}}, \qquad \alpha = \frac{y}{\left(\frac{2\nu x}{U_e}\right)^{1/2}}, \tag{10.141}$$

and substitute them into the full Navier–Stokes equations, written in terms of the stream function,

$$\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}\right)-\frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}\right)$$
$$-\nu\left(\frac{\partial^4 \psi}{\partial x^4}+2\frac{\partial^4 \psi}{\partial x^{\partial}\partial y^2}+\frac{\partial^4 \psi}{\partial y^4}\right)=0. \qquad (10.142)$$

Use the fact that $Re = U_e x/\nu$ is large to reduce the resulting equation to the Blasius ODE,

$$F_{\alpha\alpha\alpha} + F F_{\alpha\alpha} = 0. \qquad (10.143)$$

10.3 Formulate the nonlinear heat conduction problem

$$\frac{\partial T}{\partial t} = \lambda \frac{\partial}{\partial x}\left(T^{\sigma}\frac{\partial T}{\partial x}\right) \qquad (10.144)$$

with boundary conditions

$$\begin{aligned} T[x,0] &= 0, \qquad x > 0,\\ T[0,t] &= T_0, \qquad t > 0, \end{aligned} \qquad (10.145)$$

for values of σ other than one. Reduce the problem to a first order ODE and set up the phase plane. Describe what happens as σ is varied. Interpret your results physically.

10.4 Consider the case of an instantaneous source that injects a finite amount of heat, E, at a point in a nonlinear medium. The temperature diffuses, and the total heat added is constant. In this case the temperature at the origin decreases with time as the distribution spreads out. The governing equation is

$$\frac{\partial T}{\partial t} = \lambda \frac{\partial}{\partial x}\left(T^{\sigma}\frac{\partial T}{\partial x}\right) \qquad (10.146)$$

with initial source distribution

$$T[x,0] = E\delta[x] \qquad (10.147)$$

and conserved integral

$$\int_{-\infty}^{\infty} T[x,t]\,dx = E. \qquad (10.148)$$