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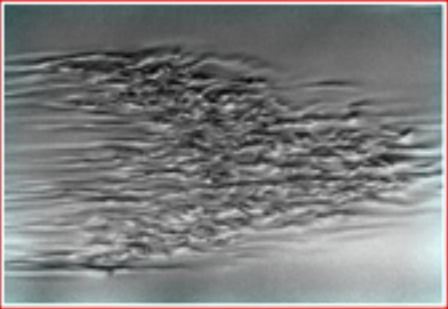
# Introduction to Symmetry Analysis



BRIAN J. CANTWELL

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# Introduction to Symmetry Analysis

Symmetry analysis based on Lie group theory is the most important method for solving nonlinear problems aside from numerical computation. The method can be used to find the symmetries of almost any system of differential equations, and the knowledge of these symmetries can be used to simplify the analysis of physical problems governed by the equations. This text offers a broad, self-contained introduction to the basic concepts of symmetry analysis and is intended primarily for first- and second-year graduate students in science, engineering, and applied mathematics. The book should also be of interest to researchers who wish to gain some familiarity with symmetry methods. The text emphasizes applications, and numerous worked examples are used to illustrate basic concepts.

*Mathematica*® based software for finding the Lie point symmetries and Lie–Bäcklund symmetries of differential equations is included on a CD, along with more than sixty sample notebooks illustrating applications ranging from simple, low-order ordinary differential equations to complex systems of partial differential equations. The notebooks are carefully coordinated with the text and are fully commented, providing the reader with clear, step-by-step instructions on how to work a wide variety of problems. The *Mathematica*® source code for the package is included on the CD.

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# Introduction to Symmetry Analysis

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# *Contents*

# *Author's Preface*

This textbook grew out of the lectures for a course by the same name that I have given at Stanford University since the mid 1980s. The course is designed mainly for first- and second-year graduate students in science, engineering, and applied mathematics, although the material is presented in a form that should be understandable to an upper-level undergraduate with a background in differential equations. The students who come into the course usually have no knowledge of symmetry theory whatsoever and more often than not, they have been imbued with the notion that the only method available for solving nonlinear problems is numerical analysis. By the end of the course they recognize that symmetry analysis provides not an alternative to computation but a complementary analytical approach that is applicable to almost any system of differential equations they are likely to encounter.

The main goal is to teach the methods of symmetry analysis and to instill in the student a sense of confidence in dealing with complex problems. The central theme is that any time one is confronted with a physical problem and a set of equations to solve, the first step is to analyze the problem using dimensional analysis and the second is to use the methods of symmetry analysis to work out the Lie groups (symmetries) of the governing equations. This may or may not produce a simplification, but it will almost always bring clarity to the problem. Knowledge of symmetries provides the user with a certain point of view that enhances virtually any other solution method one may wish to employ. It is my firm belief that any graduate program in science or engineering needs to include a broad-based course on dimensional analysis *and* Lie groups. Symmetry analysis should be as familiar to the student as Fourier analysis, especially when so many unsolved problems are strongly nonlinear.

I have tried to design the book to serve this need and to help the reader become skilled at applying the techniques of symmetry analysis. Therefore, wherever possible I have included the detailed steps leading up to the main theoretical results. Most of the theory is developed in the first half of the book and a large number of relatively short worked examples are included to illustrate the concepts. I have also provided, in Chapters 9 through 16, a number of fully worked problems where the role of symmetry analysis as part of the complete solution of a problem is illustrated. Enough detail is included for the reader to follow each problem from formulation to solution. Although the worked problems are mostly taken from heat conduction, fluid mechanics and nonlinear wave propagation, they are designed to explore many of the different facets of symmetry analysis and therefore should be of general interest. Phase-space methods are established in Chapter 3 and used extensively throughout the rest of the text. The emphasis is on applications, and the exercises provided at the end of each chapter are designed to help the reader practice the material. They range in difficulty from straightforward applications of the theory to challenging research-level problems. Many of the exercises include a reference to the literature where details of the solution can be found.

Some of the exercises involving the identification of Lie symmetries should be worked by hand so that the reader has a chance to practice the Lie algorithm. When this is done, it will become quickly apparent that the calculational effort needed to find symmetries can be huge, even to reach a fairly simple result. To analyze by hand any but the simplest problem, a discouragingly large amount of effort is required. When the subject is approached this way, it is essentially inaccessible to all but the most dedicated workers. This is one of the main reasons why Lie theory was never adopted in the mainstream curricula in science and engineering. It is systematic and powerful but can be very cumbersome! Fortunately, we now live in an era when powerful symbol manipulation software packages are widely available. This allows the vast bulk of the routine effort in group analysis to be automated, bringing the whole subject completely within the reach of an interested student.

Several years ago, I developed a set of *Mathematica*®-based software tools to use in my course. The package is called **`IntroToSymmetry.m`** and has been exercised by about five generations of students working hundreds of problems of varying complexity. So far it seems to work quite well. The package is very good at constructing the list of determining equations of the group for pretty much any system of equations, and it contains limited tools for solving those equations. The main benefit the package brings to the book is a large number of worked examples and an opportunity for the reader to rapidly gain experience by working lots of problems on their own with the aid of the package. Details of the package are described in Appendix 4. The package with its source code is included on a CD along with more than sixty sample notebooks that are carefully coordinated with the examples and exercises in the text. The source code and the sample runs are fully and extensively commented. The sample runs range in complexity from single low-order ODEs, to systems of ODEs, to single PDEs, and large systems of PDEs. Applications to both closed and unclosed systems are illustrated along with the use of built-in *Mathematica* functions for manipulating the results.

There is a good deal more in this book than can be absorbed in one quarter. The course I teach at Stanford covers Chapters 1 to 3 and 5 to 10 with selected examples from Chapters 11 to 13 as well as some of the main results on nonlinear waves in Chapter 16. In a one-semester course I would include the lengthy but relatively self-contained Chapter 14 on Lie–Bäcklund symmetries (also called generalized symmetries). A full two-quarter sequence would include the material on Lagrangian dynamics in Chapters 4 and 15 as well as all of Chapter 16. In the second quarter, I would supplement the book with material from other sources on approximate symmetries and on discrete symmetries with applications to numerical analysis, two important topics that are not covered in the book. In addition, I would add more examples of variational symmetries that are covered briefly in Chapter 15.

My course is introductory in nature and this is the basis of the book title, but for the sake of completeness I have not shied away from including some material of an advanced nature. Appendicies 2 and 3 provide the background needed to understand the infinite order nature of Lie–Bäcklund groups. The development is straightforward but the math is fairly intricate and a little hard to follow. Yet this material underlies the whole treatment of such groups and without these appendicies, there would be a large hole in the development of the theory in Chapter 14. Chapter 9 ends with a rather advanced problem in nonlinear heat conduction and then a brief discussion of nonclassical symmetries, which is an active area of current research.

Chapters 10, 11 and 12 constitute a series of examples, all of which are drawn from heat conduction and fluid mechanics. The examples are intended to show in detail how groups relate to solutions, i.e., to show how symmetry analysis is really used. If the reader does not have a background in fluids, then I recommend three basic references: Van Dyke's collection of flow pictures called *An Album of Fluid Motion*, Batchelor's *An Introduction to Fluid Dynamics* and Liepmann and Roshko's *Elements of Gasdynamics* (complete references are given at the end of Chapters 10 and 12). These provide a good deal of the basic knowledge one may need to get through these examples. Chapter 13 is even more specialized and will probably appeal mainly to someone with a strong interest in turbulence. Yet it is hard to envision a text that does not touch on this important and complex subject where, in the absence of a complete theory, symmetry methods are an essential tool for solving problems.

I would like to acknowledge the fruitful association with my colleague Milton Van Dyke during the times we spent co-teaching his course on similitude when I first came to Stanford in 1978. That course was the predecessor of the one I teach today. I also want to express my appreciation to Nicholas Rott for our shared collaborations in fluid mechanics and for his words of encouragement on this project. Carl Wulfman from the University of the Pacific kindly reviewed an early version of the manuscript, and I thank him for his valuable and timely advice. Substantial parts of the book were developed while I was visiting the University of Notre Dame as the Melchor Chair Professor during the Fall semester of 1998, and I would like to thank Bob Nelson for supporting my visit. I would also like to thank my good friends at ND, Sam Paolucci, Mihir Sen, and Joe Powers, who sat through the course that semester and who provided so many helpful comments and suggestions. Thanks also to Nail Ibragimov for graciously hosting my visit to South Africa in December 1998. I would especially like to thank Stanford graduate students Alison Marsden and Jonathan Dirrenberger for their valuable comments and criticisms of the nearly final text. Finally, I would like to remember my good friend and a great scientist, Tony Perry (1937–2001) who first inspired my interest in the geometry of fluid flow patterns and whose insight was always on the mark. I wish we could meet just one more time in Melbourne to drink a few beers, shed a tear for Collingwood and share a laugh at the world.

*Palo Alto, January 2001*

*This Book is dedicated to my loving family: Ruth, Alice, Kevin, and Tom.*
*(I promise to pick up all those piles of books, papers, photos, etc.,*
*and put them away now.)*

# *Historical Preface*

In his biography of Sophus Lie and Felix Klein, Isaak Yaglom (1988) states:

> It is my firm belief that of all of the general scientific ideas which arose in the 19th century and were inherited by our century, none contributed so much to the intellectual atmosphere of our time as the idea of symmetry.

Few would disagree with that statement. Lie and Klein were the main protagonists in the historical development of the theory of symmetry used widely today. It is a remarkable coincidence how these two late-19th-century mathematicians from such different backgrounds became friends and how their careers both diverged and remained intertwined throughout their lives. Toward the end of the 18th century, one of the main themes of European academic culture, fostered by the age of Enlightenment, was a remarkably free exchange of scholars and ideas across national boundaries. This freedom contributed mightily to the revolution in physics and mathematics that was to come in the 19th and early 20th centuries. It had such a profound effect on the development of the theory of symmetry that to understand the theory and its language one is compelled to understand its history. The story of Klein and Lie, as the reader will see, is virtually the story of the development of modern mathematics, and almost every mathematician whose name is familiar to our experience had an direct or indirect role in their remarkable careers.

## Rise of the Academies

Mathematics and science in Europe lost its provincialism very quickly with the onset of the industrial revolution. By the early 17th century, mathematicians began to work in small groups, communicating their work through books or letters. Networks were created that coordinated and stimulated research. Marin

Mersenne in Paris collected and distributed new results to a number of correspondents, including Fermat, Descartes, Blaise Pascal, and Galileo, keeping them informed of the latest events. John Collins, the librarian of the Royal Society of London, founded in 1660, played this role among British mathematicians. The universities at the time provided relatively little support for research. Instead, state-supported academies, which tended to emphasize science and mathematics, usually with military applications in mind, carried out the most advanced research. At the urging of Jean Colbert, his chief minister, Louis XIV founded the French Academie Royale des Sciences in 1666. The Berlin Academy was founded in 1700, and the St. Petersburg Academy in 1724. After 1700, the movement to found learned societies spread throughout Europe and to the American colonies. At the same time, new journals were created, making possible prompt and, for the first time, wide dissemination of research results. The academy provided a forum for rigorous evaluation by peers, and it afforded scientists protection from political and religious persecution for their ideas. The separation of research from teaching distinguished the academy from the model of university-based science, which developed in the 19th century.

The preeminent mathematicians of the time, among them Leonhard Euler, Jean le Rond d'Alembert, and Joseph-Louis Lagrange, all followed careers in the academies in London, Paris, and St. Petersburg. The academies held meetings on a regular basis, published memoirs, organized scientific expeditions, and administered prize competitions on important scientific questions. One of the most famous of these is the subject of the beautifully written 1995 book *Longitude* by Dava Sobel. This was a £20,000 prize offered by Parliament in 1714 to anyone who could develop an accurate, practical method for determining longitude at sea. The board founded to oversee the prize included the president of the Royal Society as well as professors of mathematics from Oxford and Cambridge. This board was the predecessor of the modern government research and development agency. Over the one hundred years of its existence, the effort to measure longitude spun off other discoveries, including the determination of the mass of the Earth, the distance to the stars, and the speed of light.

During the period from 1700 to 1800, there was free movement of scholars across state boundaries, and the generally apolitical attitude they adopted toward their science contributed to an academic culture almost free of the national chauvinism that was rampant in state politics. Perhaps no one typifies this better than the great Italian–French mathematician Joseph-Louis Lagrange, born in 1736 at Turin, in what was then Sardinia-Piedmont. Lagrange was born into a well-to-do family of French origin on his father's side. His father was treasurer to the king of Sardinia. At 19, Lagrange was teaching mathematics at the artillery school of Turin, where he would later be one of the founders of the Turin academy. By 1761, he was recognized as one of the greatest living mathematicians and was awarded a prize by the Paris Academy of Sciences for a paper on the libration of the moon. In 1766, on the recommendation of the Swiss Leonhard Euler and the French Jean d'Alembert, he was invited by King Frederick II (the Great) of Prussia to become mathematical director of the Berlin Academy. During the next two decades, Lagrange wrote important papers on the three-body problem in celestial mechanics, differential equations, prime-number theory, probability, mechanics, and the stability of the solar system. In his 1770 paper, "Reflections on the Algebraic Resolution of Equations," he ushered in a new period in the theory of equations that would inspire Evariste Galois in his theory of groups four decades later. When Frederick died in 1787, Lagrange moved to Paris at the invitation of Louis XVI and took up residence in the Louvre, where, in 1788 on the eve of the French Revolution, he published his famous *Mécanique Analytique.* Napoleon honored him by making him a senator and a count of the empire. The quiet, unobtrusive mathematician whose career spanned the European continent was revered until his death in Paris in 1813, just as the post-Revolution Napoleonic era was approaching its end.

The French Revolution in 1789 was followed a decade later by the Napoleonic era and the final collapse, after the battle of Austerlitz in 1805, of the Holy Roman Empire, which in the words of Voltaire was "neither holy, nor Roman, nor an empire." This brought an end to the age of Enlightenment and the benevolent despotism of the royalist period, presaging a new era in scientific research and education. The new political order stimulated a rapid spread in scientific interest among all classes of society. Anyone with ability who wished to follow intellectual pursuits was encouraged to do so. There was a great increase in the number of students learning science and mathematics, and this drove an increased demand for teachers. These events forged a new relationship between teaching and research.

New centers of learning were established, and old ones revitalized. The French Revolution provoked a complete rethinking of education in France, and mathematics was given a prominent role. The Ecole Polytechnique was established in 1794 by Gaspard Monge (1746–1818) with Lagrange as its leading mathematician. It prepared students for the civil and military engineering schools of the Republic. Monge believed strongly that mathematics should serve the scientific and technical need of the state. To that end, he devised a syllabus that promoted descriptive geometry, which was useful in the design of forts, gun emplacements, and machines. The Ecole Polytechnique soon began to attract the best scientific minds in France. A similar center of learning was


On the left is a lithograph of Neils Henrik Abel, and on the right a sketch of Evariste Galois by his brother Alfred.

established by Carl Jacobi in Königsberg, Germany, in 1827. The University of Göttingen, founded in 1737 by George II of England in his role as the Elector of the Kingdom of Hanover, was beginning to attract students from all over Europe with a strong faculty in physics and mathematics. There was a fresh atmosphere of scientific excitement and curiosity. The creation of new knowledge flourished.

## Abel and Galois

One of the central problems of mathematics research in the 19th century concerned the theory of equations. Ever since researchers in the 16th century had found rules giving the solutions of cubic and quartic equations in terms of the coefficients of the equations, formulas had unsuccessfully been sought for equations of the fifth and higher degrees. At the center of interest was the search for a formula that could express the roots of a quintic equation in terms of its coefficients using only the operations of addition, subtraction, multiplication, and division, together with the taking of radicals, as had been required for the solution of quadratic, cubic, and quartic equations. By 1770, Lagrange had analyzed all the methods for solving equations of degrees 2, 3, and 4, but he was not able to progress to higher order.

The first proof that the general quintic polynomial is not solvable by radicals was eventually offered in an 1824 paper by the Norwegian mathematician Niels

Henrik Abel (1802–1829). In Abel's time Norway was very provincial, and he was awarded a scholarship from the Norwegian government that enabled him to visit other mathematicians in Germany and France. His talent was recognized by the prominent German engineer and entrepreneur, August Leopold Crelle (1780–1855). Crelle had become very wealthy from the railroad business, and his belief in Abel and in his Swiss co-worker Jacob Steiner prompted Crelle to found the first specialized mathematical journal in Germany. The first volumes were filled with Abel's and Steiner's papers. Abel's work published in Crelle's journal attracted the attention of the famous Carl Jacobi, and because of the efforts of Jacobi and other German scientists, Abel was eventually appointed professor at Berlin University in 1828. Unfortunately, the official notice did not reach Kristiania (now Oslo) until several days after Abel's death from tuberculosis at the age of twenty-seven. Crelle's journal went on to play a major role in the development of German science.

Abel's proof was very limited in that it only asserted the absence of a general formula for the solution of every quintic equation in radicals. It did not indicate the special cases where the equation could be solved. This was taken up by Evariste Galois, who was a great admirer of Abel and who had studied Lagrange's work on the theory of equations and analytic functions while a student at the Lycée Louis-le-Grand in Paris. This well-known school included Robespierre and Victor Hugo among its graduates. Later, it would include the mathematician Charles Hermite (1822–1901), who in 1858 would publish the solution of the quintic equation in terms of elliptic functions.

Galois was born in the town of Bourg-la-Reine near Paris in 1811 and died in a duel over a broken affair with a woman in Paris in 1832. Although his childhood seems to have been quite happy, his late adolescence was marked by the suicide of his father and crushing disappointment at being twice rejected for entry by the Ecole Polytechnique. Like his father, he was a republican in an era when the monarchy was being restored in France. Galois spent much of the last few months of his life in and out of French prisons because of his fiery republican sentiments and for making death threats against the King. By 1830, the new bourgeois king, Louis-Philippe, had been forced to use repressive measures to counter numerous rebellions and attempts on his life. Rumors at the time suggested that Galois had been trapped into the duel. The exact circumstances of his death and whether there was any sort of conspiracy against him will probably never be known. In spite of his difficulties, Galois was able to create his theory of the solution of equations. The mixture of youth, mystery, tragedy, and his towering intellectual achievements make Galois one of the most romantic figures in the history of mathematics.

Galois created the first thorough classification of algebraic equations of the form

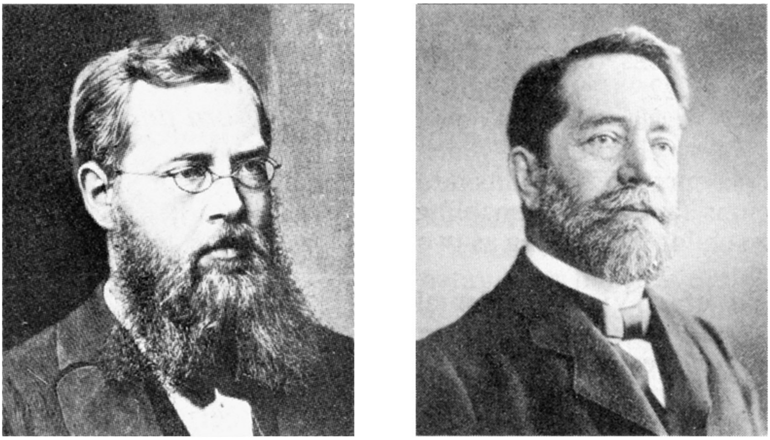
$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0.$$

Galois, like Lagrange, Abel, and Gauss before him, asked: Is it possible to find the general solution of this equation by constructing resolvents (lower-order equations whose roots are rational functions of the roots of the original equation), i.e., can the equation be solved by means of radicals? *Galois theory*, as it is called today, provides a general criterion for the solvability of equations by resolvents, as well as a way to find the solutions. Galois also did extensive work on the integrals of algebraic functions of one variable (Abelian integrals). In addition, he left behind certain results that suggest he may have been a forerunner of Riemann. According to Klein in his *Development of Mathematics in the 19th Century*, Galois, in his farewell letter to his friend Chevalier, spoke of investigations into the "ambiguity of functions"; possibly foreshadowing the idea of Riemann surfaces and multiple connectivity.

Chevalier and Galois' younger brother, Alfred, copied Galois's mathematical papers and sent them to Carl Friedrich Gauss (1777–1855), who was by then the foremost mathematician in Europe, and to Jacobi, but received no response from either. The first to study them carefully was Joseph Liouville (1809–1882), who was professor at the Ecole Polytechnique. He became convinced of their importance and arranged to have them published in 1846, fourteen years after Galois's death. Today, virtually every mathematics department in the world offers a course in Galois theory. Galois introduced the concept of a *group* and defined many of the basic elements of group theory. Camille Jordan (1838–1922) recognized the many and varied applications of Galois's work and was inspired to write the first textbook on Galois theory, published in 1870. He introduced many of the main group-theoretic terms and ideas. In that same year, when he was preoccupied by his book and fascinated by group theory, two young postgraduate students from Berlin came to study with Jordan in Paris. They were Sophus Lie and Felix Klein.

## Lie and Klein

Marius Sophus Lie was born in the vicarage at Eid in Nordfjord, Norway, on December 17, 1842. His father was rector of the parish at Eid, and his church still stands today, a few yards from the sea, next to a memorial to Norway's greatest mathematician. In 1851, the family moved to Moss on the Oslofjord, and he completed his secondary education in Kristiana (Oslo). During his youth, Lie



Sophus Lie on the left and Felix Klein on the right. The photos depict the two men at mid-career.

was strongly encouraged to study for the ministry, and it was not until he was well into his twenties that he began to take a serious interest in mathematics. He published his first paper in Kristiana in 1869, when his talent was beginning to be recognized. On the basis of his paper, he was given a grant by the university to travel to Germany and France, where he was expected to study with eminent mathematicians of the time, develop his talent, and broaden his horizons. Based on the accounts of his friends, Yaglom describes Lie as

> ... quite tall and physically very strong, with an open face and loud laugh; people who knew Lie often said he was their idea of a Viking, distinguished by rare candor and directness, always convivial with anyone who approached him, Lie produced an impression that did not correspond to his inner nature: actually he was very refined and easily hurt.

In 1869, he traveled to Berlin, which then was the center of the mathematical world, dominated by Kummer, Kronecker, and the head of the Berlin school of mathematics and a great proponent of strict mathematical rigor Karl Theodor Wilhelm Weierstrass (1815–1897). There he met the twenty-year-old Felix Klein, and their lifelong friendship began.

Christian Felix Klein was born in Dusseldorf in 1849 into the Prussian family of an official in the government finance department. Following his father's wishes, Klein studied at a classical gymnasium, where the emphasis was on ancient languages with very little attention to mathematics and science. Klein developed an intense dislike for the gymnasium, and this antipathy played an important role in his future views about teaching. After graduating from the gymnasium, Klein entered the university in Bonn, where he met Julius Plücker (1801–1868), who headed the departments of physics and mathematics. In 1866, at the age of seventeen, Klein became Plücker's assistant in the physics department. Two years later, Plücker died and to Klein fell the burden of publishing Plücker's unfinished works. Although this was an arduous and difficult task for someone so young, it contributed significantly to Klein's development as a mathematician. After Plücker's death, Klein lost his post as an assistant, left Bonn, and went to Göttingen, where he became acquainted with Rudolf Friedrich Alfred Clebsch (1833–1872), and then to Berlin, where he met the physicist Wilhelm Weber (1804–1891) and the mathematician Weierstrass.

Klein was known for a very physical way of thinking about mathematics, and his teaching was also characterized by a physical and graphical approach, and therefore a certain lack of rigor. Some of this can be traced back to Klein's rejection of the educational approach of the gymnasium, but much of it was the legacy left behind by Georg Friedrich Bernhard Riemann (1826–1866), whose ideas had profoundly influenced geometry. Riemann had succeeded Peter Gustav Lejuene Dirichlet (1805–1859) in 1859 as professor of mathematics at Göttingen, just as Dirichlet had succeeded the great Gauss four years earlier. Klein revered Riemann's work, but it did not appeal at all to the rigorous Weierstrass, and their relationship was not a warm one. Weierstrass had criticized Riemann and his friend Dirichlet and considered many of their results unproven or incorrect. Eventually, Klein would return to Göttingen to take Riemann's old position in 1886. In any case, Klein's arrival in Berlin in 1869 must have felt to Weierstrass a bit like the second coming of Riemann. Klein made up for the lack of intellectual contact with Weierstrass through his close collaboration with Lie. Although Klein was seven years younger than Lie, his experience publishing Plücker's work had matured him beyond his years. He was gregarious, with many powerful contacts, and he was a superb organizer. Thus, it was Klein who in later years would often provide the help Lie needed to advance his academic career.

## 1870

In February 1870, Lie traveled to Paris, and Klein arrived a few months later. There the students made contact with Camille Jordan and Gaston Darboux (1842–1917), who were then teaching at the Lycée Louis-le-Grande, where Galois had studied nearly half a century earlier. Under the influence of Jordan and Darboux, Klein and Lie continued their research, begun in Berlin, on the so-called *W*-curves, which are homogeneous curves that remain invariant under a certain group. *Homogeneous* curves are curves on which no point differs from any other. In plane Euclidean geometry, they are either straight lines or circles. The homogeneity of curves is related to the existence of a set of isometries that transform a curve into itself and each of its points into other points on the curve. In the case of a straight line, this self-isometry group is the group of translations along the direction of the line. For a circle, it is the group of rotations about the center of the circle. Another plane curve that has almost the same degree of homogeneity is the logarithmic spiral, whose equation in polar coordinates $(r, \varphi)$ is $R = a^{\phi}$. The spiral allows self-similar transformations along itself. These transformations can be written in polar coordinates as $\tilde{R} = a^c R$ and $\tilde{\phi} = \phi + c$. They transform the point $(R, \phi)$ to the point $(\tilde{R}, \tilde{\phi})$, and the spiral onto itself. Lie and Klein posed the problem of finding each curve in the plane that has a group of projective transformations that map the curve into itself. They called such curves *W*-curves.

The search for *W*-curves was important for Lie's further research in that it led him to study one-parameter subgroups of the group of projective transformations, which would later play an important role in the construction of Lie algebras. In addition, he established the idea of an infinitesimal transformation. Their work together also laid the foundation for Klein's later research on the connection between projective geometry and its group of symmetries, which would eventually be the basis for his Erlangen program.

During his stay in Paris, Lie also discovered the concept of a contact transformation. *Contact transformations* are generalized surface mappings in a space that includes points and their tangents. The equations for tangency called contact conditions are preserved under the mapping. Lie's theory of contact transformations turns out to be intimately related to the identification of invariants of the motion in Hamiltonian mechanics.

Lie and Klein's collaboration in Berlin and Paris was motivated by their deep interest in the theory of groups and in the notion of symmetry. Afterward, their areas of scientific interest drifted apart.

## Lie's Arrest

Their stay in Paris was abruptly ended by the outbreak of the Franco-Prussian War on July 18, 1870. Klein left Paris almost immediately for Germany, anticipating possible military service, and Lie left a month later. Lie was in no great hurry, and so he decided to walk to Milan and then hike home across Germany. While walking in a park in the town of Fontainebleau, his nordic looks attracted the attention of the police, and he was arrested as a German spy. The evidence against him included his letters from Klein and papers full of mathematical formulae. He spent a month in prison before Darboux arrived from Paris with his release order. Later, while traveling in Switzerland, he wrote of the experience:

> I have taken things truly philosophically. I think that a mathematician is well suited to be in prison. That doesn't mean that I accepted freedom philosophically. In truth, the sun has never seemed to shine so brightly, the trees have never seemed so green as they did yesterday when, as a free man, I walked to the railway station in Fontainebleau.

Lie had used his time in prison to work on his doctoral thesis, which was finally submitted to the University of Kristiana in June of 1871. In 1874, at the age of 32, Lie married Anna Sophie Birch. The younger Felix Klein had married a short time earlier. Both seem to have enjoyed happy marriages.

## Gauss, Riemann, and the New Geometry

Klein became more interested in discrete groups, their relationship to geometry and the use of groups for the categorization of mathematical objects. Discrete groups of symmetries are also known as crystallographic groups, and the importance of such groups in the study of crystals was well recognized by the end of the 19th century. After returning from France and recovering from typhus, which kept him from military duty, Klein settled in Göttingen, close to Clebsch and Weber. It was there he made his most important scientific achievements.

To understand the context of these achievements, it is necessary to review another of the important threads in 19th-century mathematics. This was the intense interest in fundamental questions in classical geometry. Attention centered on the fifth postulate of Book I of the *Elements*, which Euclid had used to prove the existence of a unique parallel through a point to a given line. There are a number of equivalent ways of stating this postulate. Perhaps the simplest is as follows:

*For each straight line L and point P outside of L there is only one straight line passing through P that does not intersect L.*

This seemingly self-evident statement had troubled Greek, Islamic, and European geometers since antiquity. In contrast to the other of Euclid's postulates, the parallel axiom invokes a global concept of the geometry of space at infinity. Even Euclid avoided using the postulate and managed to prove his first twenty-eight propositions without it. The Italian Jesuit Girolamo Saccheri (1667–1733) attempted to prove the parallel axiom by showing that all possible alternatives produce absurd results. By the end of his efforts, he had, in effect, discovered several of the theorems of a new geometry. However, he refused to accept this and remained devoutly convinced that Euclid's geometry was the only true way to describe space.

One of the implications of the parallel axiom in a Euclidean space is that the sum of the interior angles of a triangle is $\pi$. Independently, Johann Heinrich Lambert (1728–1777) of the Berlin Academy, one of the leading German mathematicians of the time, pursued the same lines followed fifty years earlier by Saccheri. He discovered that axioms used to deny the parallel axiom did not create any inconsistency, but led to the conclusion that a triangle in such a space has an area proportional to the *angle deficit* – the difference between the sum of the interior angles and $\pi$. Ultimately, however, Lambert too remained convinced that Euclidean geometry was the only true geometry. Such was the godlike authority of Euclid in the late 18th century.

Ferdinand Karl Schweikart (1780–1859), who was a professor of law at Kharkov University in the Ukraine and who took a hobbyist's interest in mathematics, wrote a letter to Gauss in 1818 in which he proposed an "astral" geometry in which the sum of the included angles of a triangle is less than $\pi$. He proved that this was true of all such triangles in this geometry and that as the size of the triangle was increased, the vertex angles became smaller and smaller. He further proposed that this geometry could actually exist on some distant stars. Gauss, although he had never published on the subject, was well aware of the possibility of such a geometry and had already planned experimental measurements intended to prove whether or not space was curved. Schweikart's letter to Gauss is generally accepted to be the first written statement that an alternative to Euclidean geometry is possible.

The Hungarian mathematician Janos Bolyai accepted from the outset that geometry branches naturally in two directions depending on whether the fifth postulate is accepted or rejected. He recognized that each branch defines a fully self-consistent system, although an actual proof of consistency came only much later in the work of the Italian mathematician Eugenio Beltrami in 1868. Bolyai published his results in an appendix to a textbook written by his father, Farkas Bolyai, who in 1832 (the year of Galois's death) presented a copy to his lifelong friend, Gauss. Gauss, who had been a student with the elder Bolyai at Göttingen, sent a rather cold reply to his friend that his son had merely reproduced much that Gauss already knew. This had a devastating effect on the younger Bolyai, who eventually became discouraged and never published again. Gauss never published his results, and in his response to Farkas concerning the work of his son, Gauss indicated that he had intended to allow publication of his ideas only after his death. Today, this seems strange to us, but it can be partially understood in the context of Gauss's time.

For many years, Gauss attempted to prove the fifth postulate, but by 1810, he had fully accepted that there were two self-consistent, equally valid geometric systems. In a letter to the astronomer Wilhelm Olbers (1758–1840) in 1817 Gauss stated his views:

> The necessity of our geometry cannot be proved . . . Perhaps in another life we will have different views on the nature of space which are inaccessible to us here. So far geometry has to be regarded as being on a par, not with arithmetic, which exists a priori, but rather with mechanics.

Gauss assigned the geometry of space not to the abstract realm of mathematics or logic, but to physics. This not only contradicted the centuries-old belief in Euclid, but also the teachings of the leading German philosopher and icon, Immanuel Kant (1724–1804). In the *Critique of Pure Reason*, Kant, who greatly admired the work of Isaac Newton, argued that space and time are both a priori forms of human awareness, and since geometry is used to describe space, it too must be unique. Gauss, who was of Nether-Saxon peasant origin, always worried that he could be returned to the poverty of his youth at any time. He feared that if he published such shocking results, no one would understand them and he would be subjected to a storm of criticism by the philosophers and disciples of Kant, whom he held in contempt. Still, Gauss' fear of a backlash can only be a partial explanation for his behavior. He often selfishly held back his best results from publication, including many that would not have caused any stir. So let's salute the brash Janos for having not only the genius to embrace the validity of this new geometry but also the courage to publish his results.

Gauss clearly recognized that the new, non-Euclidean geometry was every bit as legitimate and just as likely to apply to real physical space in the large as was Euclidean geometry. Further, he recognized that the mechanics of Fermat, Newton, and Descartes could be reformulated in a completely self-consistent way in this new geometry. For Gauss, the theorems and postulates of the new geometry had real physical implications for the nature of space, which could only be resolved by experiment, and he expended great effort in an endeavor to measure the curvature of space. In 1816, the government of the Kingdom of Hanover asked Gauss to develop exact geographic maps that could be used in tax administration. In the course of this work, he measured the largest triangle ever before attempted. He set up the vertices on the peaks of Hoher Hagen, Inselsberg, and Brocken in the Harz mountains (the fabled gathering place for witches on Walpurgis night, the eve of May 1). His instrument was the heliotrope, which used concentrated, reflected sunlight to produce bright points of light for sighting. Despite prodigious efforts between 1818 and 1825, he was never able to measure any deviation from $\pi$ of the sum of the angles that was not within experimental error.

The conflict between Gauss and Kant has symbolized the divide between physics and philosophy ever since. The spectacular success of Einstein's theories of special and general relativity, which had a direct lineage to the ideas of Riemann and Gauss, only widened the chasm. The philosophers were arrogantly wrong, and the physicists have never forgiven them for it.

In his 1921 monograph on *The Meaning of Relativity* Albert Einstein wrote:

> I am convinced that the philosophers have had a harmful effect upon the progress of scientific thinking in removing certain fundamental concepts from the domain of empiricism, where they are under our control, to the intangible heights of the *a priori.* For even if it should appear that the universe of ideas cannot be deduced from experience by logical means, but is, in a sense, a creation of the human mind, without which no science is possible, nevertheless this universe of ideas is just as little independent of the nature of our experiences as clothes are of the form of the human body. This is particularly true of our concepts of time and space, which physicists have been obliged by the facts to bring down from the Olympus of the *a priori* in order to adjust them and put them in a serviceable condition.

These sentiments would echo four decades later, when in Volume I of his *Lectures on Physics* Richard Feynman would write:

> Whether or not a thing is measurable is not something to be decided a *priori* by thought alone, but something that can be decided only by experiment. Given the fact that the velocity of light is 186,000 miles/sec, one will find few philosophers who will calmly state that it is self-evident that if light goes 186,000 miles/sec inside a car and the car is going 100,000 miles/sec, that the light also goes 186,000 miles/sec past an observer on the ground. That is a shocking fact to them; the very ones who claim it is obvious find, when you give them a specific fact, that it is not obvious.
>
> . . . if we have a set of "strange" ideas, such as that time goes slower when one moves, and so forth, whether we like them or do not like them is an irrelevant question. The only relevant question is whether the ideas are consistent with what is found experimentally. In other words, the "strange ideas" need only agree with experiment, and the only reason that we have to discuss the behavior of clocks and so forth is to demonstrate that although the notion of the time dilation is strange, it is consistent with the way we measure time.

In Riemann's famous 1854 inaugural lecture at Göttingen, with the aging Gauss in the audience, he clarified the whole field by including hyperbolic, Euclidean, and elliptic geometry in a unified theory of curved manifolds. Riemann was inspired by Gauss's discovery that the curvature of a surface is intrinsic to the surface. In his theory, Euclidean geometry was just one of many geometries, none of which had a preferred status. Riemann elevated the importance of intrinsic concepts in geometry and opened the way to the study of spaces of many dimensions. His work guaranteed that any investigation into the geometric nature of physical space would have to be partly empirical. The belief that fundamental questions about the nature of the world could be answered by a priori reasoning was swept away forever.

## The Erlangen Program

At the end of the 1860s, when Lie and Klein were still together in Berlin, Riemann's famous lecture, delivered in 1854, was finally published. It stimulated an explosion of research in geometry and an expanding list of mathematical topics under its purview. The question of finding a general description of all the geometric systems considered by mathematicians became one of the central questions of the day. Felix Klein, who had actively participated in the revolution, understood the importance of this question better than anyone. Klein, under the influence of Jordan, who taught the importance of the concept of a group and the role of symmetry, decided to find a group-theoretic way of looking at the notion of geometry itself.

In 1872, there was an opening for a professor at the newly organized mathematics department at Erlangen University, and Klein's good friend, the influential Clebsch, recommended him for the post. In Germany at that time, a prospective professor was required to deliver a public lecture to the Academic Board of the university on a subject chosen by the candidate. The decision whether to offer the post to the candidate was then made after the lecture was discussed. The twenty-three-year-old Klein chose a comparative review of recent research in geometry (just as, in a similar situation, eighteen years before in Göttingen, Riemann had spoken *On the hypotheses that lie at the foundations of geometry*). Klein's lecture soon became known as the *Erlangen program*, and it laid out his clear vision for the next era of progress in geometry. Klein defined geometry as the science that studies the properties of figures, which are invariant under transformation by a group. According to Klein, the main difference between Euclidean and hyperbolic geometry is not the correctness or incorrectness of Euclid's fifth postulate, but the difference in the structure of the respective groups of symmetries of Euclidean and hyperbolic space.

Klein's Erlangen years (1872–1875) were extremely productive. In 1872, Clebsch, then thirty-nine years old, suddenly died of diphtheria, and Klein took over *Mathematische Annalen*, the journal founded and headed by Clebsch. Klein became the journal's de facto editor, and in 1876, its formal editor. Under Klein's leadership, *Mathematische Annalen* soon gained a reputation as the world's leading journal of mathematics, greatly enhancing Klein's reputation. As a result, he received an invitation to join the well-known Technische Hochschule in Munich, where he worked for five more years. In 1880, he moved to the geometry department of Leipzig University. During this period, he enjoyed the height of his scientific productivity with substantial contributions to geometry, mechanics, and the theory of functions of a complex variable (theory of automorphic functions). One of the many visitors to Erlangen was the twenty-nine-year-old Swedish mathematician Albert Victor Bäcklund (1845–1922). In 1874, Bäcklund was awarded a six-month travel grant from the Swedish government to pursue studies on the continent. He spent part of the time at Erlangen with Klein. Although there seems to be no written record of their relationship, it is reasonable to assume that his studies with Klein may have helped inspire Bäcklund's later work on geometry. In several papers published between 1875 and 1882, while on the faculty of the University of Lund, he made important contributions to the theory of tangent transformations. He introduced the class of transformations that today bear his name and that have played a key role in modern advances in the theory of nonlinear waves.

Klein worked furiously during 1880 to 1882, developing his theory designed to combine Riemann's geometric approaches with group-theoretic ideas derived from Galois. In 1886, he left Leipzig and moved to Göttingen, where he would remain until the end of his life. Klein's regard for Sophus Lie had only grown since their parting sixteen years earlier, and so he asked Lie to replace him at Leipzig, and Lie readily accepted.

During this intense period working on automorphic functions, Klein became aware of work in the same field published by the young French mathematician Henri Poincaré (1854–1912). Klein developed a strong sense of rivalry toward the brilliant French mathematician, who was developing research along similar lines. To Klein's bitter disappointment, it was Poincaré who discovered the connection between the theory of automorphic functions and non-Euclidean geometry. The stress of this rivalry eventually caused Klein to suffer a nervous breakdown.

From then on, Klein turned more and more to teaching and to academic, organizational, and administrative activities. In 1898, he headed the immense project of publishing Gauss's collected works, which was eventually completed in 1918. If Lie fit the 19th-century myth of the lone scientist creating his work entirely by the force of his own genius, Klein was very much a 20th-century man who recognized the power of collaborative work. He realized that the rapid expansion of knowledge led to a natural human tendency to focus more narrowly in the face of overwhelming information flow and that this was fundamentally changing the nature of mathematics research in the direction of overspecialization. To counter this, he directed work on the *Enzyklopädie der Mathematischen Wissenschaften*. The idea behind this project was to collect in one place all the results and methods obtained up to the early 20th century in pure and applied mathematics and present them from one viewpoint. Unfortunately, this project was hopelessly outpaced by the rapid growth of mathematical results and was never completed.

Klein was the first scientist who fully realized the need for a fundamental reform of the whole system of mathematical education. In 1898, he organized an International Commission on Mathematics Education, which he headed for a number of years. He called for the elimination of the "China wall" separating different mathematical subjects and for decreasing the gap between mathematical education and modern science.

## Lie's Career at Leipzig

The research Lie carried out in the early 1870s brought him wide recognition. Klein had the highest regard for Lie and had extensive contacts in the mathematical world. He used them to help Lie gain a professorship at Norway's only university, in Kristiana. Lie worked in Norway for fourteen years, but he lacked intellectual peers there. So he readily accepted Klein's suggestion, made in 1886, that he replace Klein as professor of geometry at Leipzig University. Lie worked in Leipzig for twelve years, where he published several of his books. These years were very productive scientifically, but not completely satisfying personally. Lie was always reliant on friends to support him, especially toward the end of his Leipzig years, when his most outstanding students, such as Friedrich Engel (1861–1941), Georg Scheffers (1866–1945), Friedrich Schur (1856–1932), Eduard Study (1862–1930), and Felix Hausdorff (1868–1949) moved to different German universities. In Norway, the nature he loved was a source of strength for Lie; in Germany he felt alienated, although his wife and three children were quite happy and had become completely Germanized. Eventually, Lie suffered clinical depression, for which he had to take a cure at a psychiatric clinic in Hanover.

## A Falling Out

During the early 1890s, when both Lie and Klein were suffering from a certain degree of distress in their careers, there occurred an incident that briefly marred their otherwise close friendship. In 1892, Klein decided to republish his Erlangen program and expand on its history, reaching back to the time when they worked together in Paris and Berlin, so he sent the manuscript to Lie for comment. Lie was upset when he perceived Klein to be taking credit for ideas that he felt were solely his own. Then, in Volume 3 of his book published in 1893 with Engel on the *Theory of Transformation Groups*, Lie included a stinging criticism of Klein, ending with "I am no pupil of Klein's nor is the opposite the case, although this might be nearer to the truth." That offensive remark, stated in print, hurt Klein deeply. There is no doubt that both profoundly influenced the field and each had a strong scientific influence on the other even though, objectively, Lie may have had the greater influence on Klein. Klein chose not to respond, but many of his colleagues were enraged, and the brunt of their anger fell on poor Engel, who failed to gain a position at Erlangen as a result. Within a short time, Lie and Klein renewed their friendship and never returned to the incident again.

## Lie's Final Return to Norway

Lie devoted his entire life to the theory of continuous groups, now known as Lie groups, and their relationship to differential equations. The adjective "continuous" in the name of Lie groups underscores that the transformations can be changed continuously by slight alterations of the parameters determining a particular element of the group. One of the crucial points of Lie's theory was that one could assign to each continuous group a much simpler algebraic object, its Lie algebra. Lie analyzed in detail the relationship between Lie groups and Lie algebras. He defined solvable Lie algebras corresponding to so-called solvable Lie groups by analogy to the discrete solvable substitution groups of Galois, and he went on to assign such groups to differential equations. In his equivalent of Galois theory for differential equations, it turns out that only those equations that admit solvable continuous groups are completely integrable. Finally, Lie posed the problem of classifying all simple Lie algebras and Lie groups.

The complete solution of the classification problem for simple Lie groups is now attributed to Eliè-Joseph Cartan (1869–1951), who, while a lecturer at the University of Montpelier, substantially advanced the theory of Lie groups and Lie algebras. Cartan, along with several other young French mathematicians, formed a group that began to publish mathematics under the pseudonym Nicolas Bourbaki, taken from an obscure general of the Franco-Prussian War. The Bourbaki group has since played a major role in the creation of the field of algebraic topology up until the present day.

In 1892 to 1893, the Kazan physicomathematical society created the international Lobachevsky prize on the occasion of N.I. Lobachevsky's centenary. The first prize was awarded in 1898 to Sophus Lie. A review of his work, requested by the society, was written by Felix Klein. The Lobachevsky prize immediately became quite prestigious. The second, third, and fourth recipients were Killing, Hilbert, and Klein. Later recipients include Poincaré, Weyl, Cartan, and, more recently, de Rham, Hopf, and Buseman. In 1898, Lie left Leipzig and returned to his alma mater at Kristiana. He died a short time later, on February 18, 1899.

Sophus Lie with his wife Anna and three children Marie, Dagny, and Herman.

A committee for the publication of Lie's collected mathematical works was created in 1900. The overall editing was done by Friedrich Engel and by the leading Norwegian mathematician of the time, Poul Heegard (1871–1948). The publication of Lie's colossal work took fifteen years to produce and many thousands of pages in fifteen large volumes, excluding some of his work published jointly with Engel.

## After 1900

Klein led the mathematics group at Göttingen with his rare 20th-century vision of collaborative work. From 1886 until his death in 1925, he dedicated himself to turning Göttingen into a world-class center of physics and mathematics. He attracted to Göttingen talented students and teachers from all over the world. Outstanding among these was David Hilbert (1862–1943), regarded by many as the greatest mathematician of the 20th century. From the late 1890s to the 1930s, this outstanding scientific center at Göttingen was dominated by the personalities of Klein and Hilbert.

In 1915, Klein and Hilbert were joined by Amalie Emmy Noether (1882–1935), who had grown up in Erlangen, where Felix Klein a decade before her birth had established his famous program. Noether made major contributions to the theory of groups related to a variational integral and eventually gained recognition as one of the foremost algebraic theorists of her time. Soon, Klein's good friend Hermann Minkowski (1864–1909) came to Göttingen. There followed the arrival of several students and teachers from Breslau University, including the mathematicians Richard Courant and Otto Toplitz (1881–1940) and the future Nobel Prize winner and director of the Göttingen physics institute, Max Born (1882–1970). Born headed the Göttingen school of theoretical physics, which produced such outstanding scientists as the Nobel Prize winner and founder of quantum mechanics Werner Heisenberg (1901–1976) and the American physicist J. Robert Oppenheimer (1904–1967), who would later find themselves on opposite sides working to develop the atomic bomb.

Klein's style of leadership is typified by the following episode. At the International Mathematics Congress in Heidelberg in 1904, Klein listened to a presentation on hydrodynamics by the then relatively unknown German engineer Ludwig Prandtl (1875–1953). Greatly impressed, he immediately invited Prandtl to Göttingen and appointed the twenty-nine-year-old engineer to direct an applied mathematics institute especially founded for him. This was the origin of the world-famous Göttingen school of mechanics.

Klein ascended to very high administrative positions in German science. In 1913, he was elected a corresponding member of the German Academy of Sciences in Berlin and a representative of Göttingen University in the upper chamber of the Prussian parliament. Klein's collected works in three volumes were published by his pupils in the period 1918 to 1924. In addition, a large number of papers dedicated to Klein were published in scientific journals on the occasion of his seventy-fifth birthday. Shortly therafter, Felix Klein died on July 22, 1925. Following his death, Hilbert's student, Richard Courant (1888–1972) became director of the mathematical institute. Fortunately, Klein did not have to witness the rise of the Third Reich and the subsequent destruction of his beloved institute at Göttingen in 1933. Courant and many of the best minds of the institute fled Germany, eventually ending up in the United States in places such as New York University (NYU), Princeton, and Bryn Mawr (Pennsylvania). In 1958, Courant, Stoker, and Friedrichs founded the mathematics institute at NYU, which was later named after Courant on the occasion of his retirement, although it never lost the nickname "Göttingen West."

## The Ariadne Thread

The central importance of Lie groups and Lie algebras has never diminished over the century since Lie's death, but the interest in Lie's methods for solving differential equations has waxed and waned. By the 1930s, through the influence of Hilbert and others, the focus had shifted to functional analysis and transform methods. Questions of solvability and techniques for integration in quadratures receded in importance, and Lie's constructions seemed old-fashioned and cumbersome. Nevertheless, throughout this period, such fields as fluid dynamics continued to progress almost exclusively on the basis of similarity theory and dimensional analysis, but with practically no reference to Lie's work. By the 1940s, digital computers began to appear, and the whole question of solving differential equations had to be completely reexamined in light of the resulting prospect of finding solutions of nonlinear systems for very general and geometrically complex boundary conditions. Lie's ideas fell into obscurity and remained so until the period shortly after World War II.

By the early 1960s, there was a healthy resurgence of interest in Lie's methods, and the field has grown rapidly ever since. Two lines of thought converged to resurrect Lie's methods. Researchers, stimulated by the scientific advances of the postwar years, began to address nonlinear problems more and more often, and it was realized that Lie theory was the only systematic method for analyzing nonlinear equations. At about the same time, workers in physics, fluid dynamics, and other fields began to appreciate the central importance of the symmetries themselves. As Yaglom notes, it was recognized that the symmetries of a differential equation not only determine whether the equation is solvable or not but also describe the symmetries inherent in the physical phenomena modeled by the equation.

In his 1963 *Lectures on Physics* Feynman wrote:

> . . . there is a third suggestion which is a little more technical but which has turned out to be of enormous utility in our study of other physical laws, and that is to *look at the symmetry of the laws* or, more specifically, to look for the ways in which the laws can be transformed and leave the form the same. When we discussed the theory of vectors, we noted that the fundamental laws of motion are not changed when we rotate the coordinate system, and now we learn that they are not changed when we change the space and time variables in a particular way, given by the Lorentz transformation. So this idea of studying the patterns or operations under which the fundamental laws are not changed has proved to be a very useful one.

In Greek mythology, Ariadne, the daughter of Pasiphaë and Minos, the king of Crete, gives the hero Theseus a thread whereby he is able to mark the way of his escape from the labyrinth. In physics, symmetries provide the Ariadne thread that enables us to navigate our way through the infinitely varied and complex labyrinth of natural phenomena.

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# 1

# *Introduction to Symmetry*

## 1.1 Symmetry in Nature

Symmetry is universal, fascinating, and of immense practical importance. As human beings we have evolved a perception of symmetry that lies at the core of our conscious life. Symmetries provide cues that help us relate to our environment and guide our movements through the world. Everyone has a taste for things that are in some way symmetrical or possess a pleasing deviation from perfect symmetry. A highly paid supermodel will often have rather symmetrical facial features. But a perfectly symmetrical face has an unnatural, androgynous look, and rarely is this associated with great beauty or a memorable persona. Perhaps the most perfect object we can imagine is a circle, yet dividing the circumference by the diameter produces the irrational number $\pi$ that we can only symbolize. Perfect, unequivocal, symmetry, like perfect theory, eludes us always.

Objects of the natural world universally exhibit some form of symmetry. Despite an astonishing variety of shapes, all members of the animal kingdom possess body architectures that can be sorted into only about 37 basic types. Almost all animals possess bilateral symmetry; they must eat, and to eat efficiently two hands, grasping symmetrically, are better than one. Animals must move, and to move efficiently it is essential to be balanced about the center of mass. When asymmetric development does occur, it is invaribly associated with some unusual, very specific adaptation, as in the case of the bottom-dwelling flounder with both eyes on the same side of its head. The whorls and spirals of plant organs produced by the response of an expanding growth surface to surrounding mechanical constraints [1.1] have been the subject of scientific inquiry for centuries. The nearly perfect spheres that fill the universe – stars, planets, moons, and the like – are shaped primarily by gravitational forces, which act in a three-dimensional universe where no one direction or position is distinguished from another. Free space is homogeneous and isotropic. We marvel at the incredible variety of delicate geometrical forms associated with the six-sided symmetry of snowflakes or the regular crystalline structure of gems formed over millennia by heat, pressure, and water, their shape a consequence of the forces that act on an atomic scale according to the symmetries of the electronic outer shells that participate in bonding. Anyone who studies fluid mechanics is struck by the aesthetic symmetry of shock wave patterns or bubbly flows or any of the myriad spiral patterns that mark the vortical world that flows over, around, and through us.

There have been many attempts to quantify the relationship between symmetry and beauty. A fine example of this can be found in the fascinating work of George David Birkhoff (1884–1944) [1.2], who was one of the preeminent American mathematicians of the early 20th century and is generally credited with developing the ergodic theorem in the kinetic theory of gases. Birkhoff was originally motivated by the desire to identify what it was that made one musical piece beautiful and another not. He felt that beauty had a universal character and therefore it should be possible to quantify it mathematically, and so he developed what he called the "aesthetic measure." Ultimately he applied this measure to a wide variety of objects – everything from musical pieces to vases to floor tilings. Today such an attempt to categorize music seems naive in view of the vast range of musical technique – everything from guitar "resonant buzz" invented accidentally by country singer Marty Robbins (but claimed by "Spirit in the Sky" Norman Greenbaum) to the patriotic screechings of Jimi Hendrix to the asynchronous beat of Dave Brubeck. No simple measure can cover it all.

Although the use of symmetries to categorize objects is interesting in its own right, that is not the purpose of this text. Our main interest is in the symmetries inherent in the physical laws that govern the natural world. Knowledge of these symmetries will be used to enhance our understanding of complex physical phenomena, to simplify and solve problems, and, ultimately, to deepen our understanding of nature. The primary goal of this text is to develop the methods of symmetry analysis based on Lie groups for the uninitiated reader and to use these methods to find and express the symmetry properties of ordinary differential equations, partial differential equations, integrals, and the solution functions that they govern. The text is directed primarily at first- and second-year graduate students in science and engineering, but it may also be useful to advanced researchers who would like to gain some familiarity with symmetry methods. The student is expected to be familiar with classical approaches to the solution of differential equations, although the early chapters provide much of the required background in terms that should be understandable to an upper-level undergraduate.

## 1.2 Some Background

My first encounter with Lie groups came while browsing in the GALCIT aeronautics library at Caltech in 1975. I ran across the book by Abraham Cohen [1.3], first published in 1911. The first few chapters of this book give a very lucid description of the concept of a Lie group and the idea of invariance under a group. Cohen's book makes interesting reading when one realizes that at the time it was written, Sophus Lie's ideas were still a brand-new development, yet they were seen as important enough to warrant a full-blown textbook treatment. In his 1906 treatise on *The Theory of Differential Equations* Andrew Forsyth devotes several chapters to Lie groups and Bäcklund transformations. It is a fact, however, that shortly thereafter, Lie's ideas fell into obscurity and remained so until soon after World War II. As researchers began to turn more and more often to nonlinear problems and as the inherent importance of symmetries began to be recognized, Lie's ideas gained renewed interest.

The Lie algorithm used to analyze the symmetry of mathematical expressions was developed to an advanced state through the pioneering efforts of Ovsiannikov [1.5] and his students in the Soviet Union. In the United States, Garrett Birkhoff [1.6] at Harvard the son of George Birkhoff played a key role in bringing attention to Lie's ideas by clarifying the relationship between group invariance and dimensional analysis as applied to problems in fluid mechanics. Fluid mechanics, governed as it is by nonlinear equations from which a rich variety of simplified nonlinear and linear approximations can be derived, is an especially fertile source of examples and applications of group theory.

During the same period, new ideas about the role of similarity solutions as approximations to realistic complex physical problems were being developed by Barenblatt and Zel'dovich [1.7] in the Soviet Union. By the late 1960s and early 1970s the whole field was active again, and new applications of group theory were being developed by a number of researchers, including Ibragimov in the Soviet Union [1.8], Bluman and Cole at Caltech [1.9], Anderson, Kumei, and Wulfman at the University of the Pacific [1.10], Chester at Bristol [1.11], Harrison and Estabrook at the Jet Propulsion Laboratory [1.12], and many others. Today group analysis, in one form or another, is the central topic of a number of excellent textbooks, including Hansen [1.13], Ames [1.14], Olver [1.15], Bluman and Kumei [1.16], Rogers and Ames [1.17], Stephani [1.18], and most recently Ibragimov [1.19], Andreev et al. [1.20], Hydon [1.21] and Baumann [1.22]. The valuable collection of results by workers around the world contained in the CRC series edited by Ibragimov [1.23] gives testimony to the achievements of the last half century or so. Today, symmetry analysis constitutes the most important (indeed one might say the only) widely applicable method for finding analytical solutions of nonlinear problems. The Lie algorithm can be applied to virtually any system of ODEs and PDEs. Moreover the procedure is highly systematic and amenable to programming with symbol manipulation software. As a result, sophisticated software tools are now available for analyzing the symmetries of differential equations (References [1.24], [1.25], [1.26]; see also the review of symbolic software for group analysis by Hydon [1.21] and Hereman [1.27]).

## 1.3 The Discrete Symmetries of Objects

For more background on the importance of symmetry, particularly in the early development of modern physics, I would recommend the works of the German–American mathematical physicist Hermann Weyl (1885–1955), who formulated the group-theoretic basis of quantum mechanics. In his monograph [1.28] Weyl writes of the role of symmetry in science and art. Weyl was a student of David Hilbert and a member of the famous group of German mathematicians at the University of Göttingen, which broke up during the Nazi era prior to the start of World War II and later re-formed as the nucleus of the Courant Institute in New York. Finally, one of my favorite readings is Feynman's discussion of the role of symmetry in modern physics, which can be found in Chapter 52 of Volume I of the *Feynman Lectures on Physics* [1.29].

Let's begin with a widely accepted general definition of symmetry usually attributed to Weyl.

***Definition 1.1.*** *An object is symmetrical if one can subject it to a certain operation and it appears exactly the same after the operation. The object is then said to be invariant with respect to the given operation.*

The symmetry properties of an object can usually be expressed in terms of a set of matrices each of which, when used to transform the various points composing the object, leave it unchanged in appearance. To clarify the notion of symmetry and its mathematical description, let's examine the rotational and reflectional symmetry of a snowflake.

### *1.3.1 The Twelvefold Discrete Symmetry Group of a Snowflake*

Transparent ice crystals form around dust particles in the atmosphere when water vapor condenses at temperatures below the freezing point. The water molecule is an isosceles triangle composed of two hydrogen atoms bonded to an oxygen atom at its apex with an angle of $104.5^\circ$ between the bonds. The attraction between the hydrogen atoms of each molecule and the oxygen atoms of other molecules overcomes thermal motions, leading to the formation of

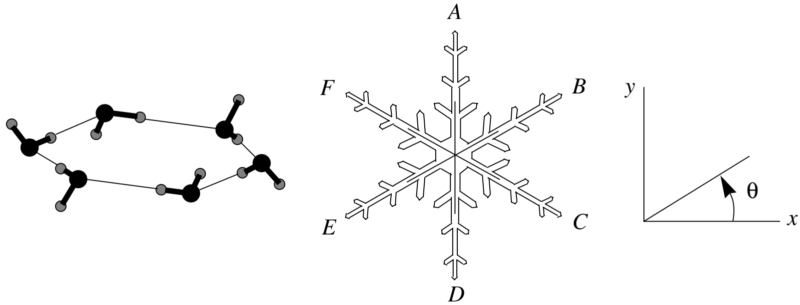


Fig. 1.1. Hexagonal structure of ice crystals and snowflakes.

hydrogen bonds, which link molecules together. The symmetry properties of the water molecule are such that if the formation temperature is below $-14°$C, each molecule bonds to four neighboring molecules in a repeating tetrahedral arrangement with the oxygen atoms at the corners of the tetrahedron. The tetrahedral structure gives rise to hexagonal rings of water molecules as shown in Figure 1.1. These hexagons on the molecular scale are responsible for the hexagonal symmetry of the ice crystal at macroscopic scales.

The exact structure of the ice crystal depends on its temperature history during formation. Thus, because of the infinite variability of atmospheric conditions, the shape of each snowflake is unique.

One final point before we begin: A snowflake is a three-dimensional object with a front and back. Here we wish to study only the planar symmetry of a face-on view, and so we consider the snowflake to be flat, existing entirely in a two-dimensional world. By the way, the tendency for snowflakes to be nearly flat is also explained by the crystal structure at the molecular level, which tends to be composed of relatively weakly bound planar sheets.

Figure 1.1 is my best attempt to sketch a typical snowflake. Overall it looks like a fairly symmetrical six-sided object. However, close inspection reveals a lot of detailed imperfections in my drawing. In order to have a useful discussion of the symmetry properties of the snowflake, we simply must accept the fact that we can't look at it too closely. We have to be willing to gloss over the imperfections and agree that the six corners of the snowflake are indistinguishable. The labels $A$, $B$, $C$, $D$, $E$, $F$ are applied to the corners for reference purposes, but with the convention that the labels do not compromise the property that the corners themselves are indistinguishable.

This seemingly minor point is actually crucial and all-encompassing. It is central to the methods used to test for symmetry. In principle, any real object in all of its detail is completely devoid of symmetry. Therefore it is important to recognize that the symmetries that accrue to an object apply, not to the object itself, but to its abstract representation. The moon is a sphere only when viewed from a perspective that flattens all mountain ranges, mare, rocks, pebbles, etc. Often it is the degree and manner in which a symmetry is broken that is of paramount importance. Galileo's great discovery in the seventeenth century was that the moon is not a smooth sphere but is covered with craters whose dimensions rival the largest geological features found on earth.

So it is the case today that the most important scientific questions are often associated with peeling away symmetries or searching for new symmetries of complex systems in order to reach a deeper understanding of the underlying physics. One often asks: Which parameters in a physical problem are important? Which ones are not? Occasionally, new physics is discovered when the means is found to "fix" a broken symmetry. In the modern era, the most spectacular example of this is the failure of Maxwell's equations to preserve Galilean invariance while preserving invariance under the puzzling Lorentz transformation. This led directly to Einstein's theory of special relativity, the recognition that time and space are connected, and the discovery that the speed of light is a universal invariant for all observers. A more recent example that shook the foundations of particle physics is the famous 1956 discovery by Lee and Yang [1.30], [1.31] that parity is not conserved in beta decay.

#### *1.3.1.1 Symmetry Operations*

Now, let's begin our study of the symmetries of a snowflake.

Suppose we rotate the snowflake by 30° (Figure 1.2). If we close our eyes before the rotation, then open them afterwards, we can see that an operation has been applied to the snowflake. The object is not left invariant, and the 30° rotation does not qualify as a symmetry operation. There are in fact just six rotation angles that leave the snowflake invariant: 60°, 120°, 180°, 240°, 300°, and 360°.

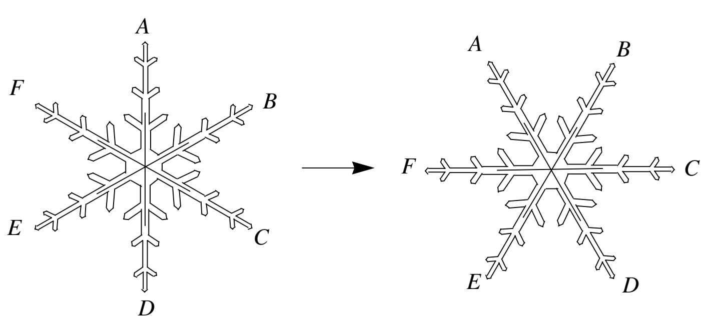


Fig. 1.2. Counterclockwise rotation by 30°.

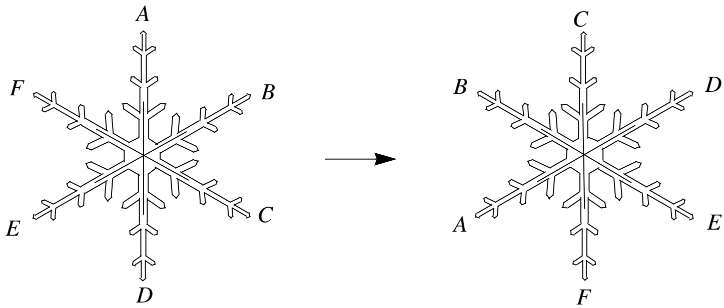


Fig. 1.3. Counterclockwise rotation by 120°.

Now apply a rotation of 120° (Figure 1.3). In this case, there is no way we can tell that the operation has taken place (remember that the labels are not part of the object and tiny details are ignored). The snowflake is invariant, and the rotation by 120° is a symmetry operation. We can express the rotational symmetry of the snowflake mathematically as a transformation

$$\begin{aligned} \tilde{x} &= x\cos\theta - y\sin\theta, \\ \tilde{y} &= x\sin\theta + y\cos\theta. \end{aligned} \tag{1.1}$$

where the $(x, y)$ coordinates are oriented as shown in Figure 1.1 and the parameter of the transformation, $\theta$, can only take on the six discrete values given above. It is convenient (though not necessary) to think of (1.1) as a mapping of points in a given space whose coordinate axes remain fixed, rather than the usual interpretation as a rotation of the coordinate axes themselves. The object moves under the action of the transformation while the reference axes stay fixed. The six rotations are as follows:

$$C_6^1 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad C_6^2 = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \quad C_6^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$C_6^4 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \quad C_6^5 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{1.2}$$

The matrices $C_6^1,\ C_6^2,\ C_6^3,\ C_6^4,\ C_6^5, E$ express the rotational symmetry of *any* hexagonal object with indistinguishable sides and corners.

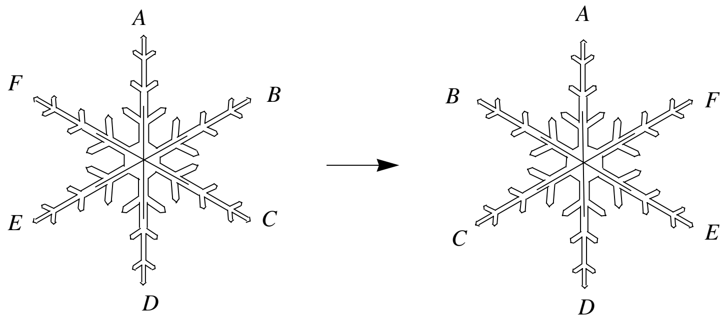


Fig. 1.4. Reflection through a vertical axis.

What about reflections? Reflection through an axis passing through $A$–$D$ leaves the snowflake invariant (Figure 1.4). Recall that we are considering a flat snowflake and so all operations are in the plane of the paper. If we wanted to consider the three-dimensional symmetries of a finite-thickness snowflake, then we would have to include transformations in the $z$-direction, either reflecting points between the front and back or rotating the object out of the plane of the paper.

The reflection through $A$–$D$ can be expressed as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}. \tag{1.3}$$

Another reflectional symmetry is through axis $a$–$d$, which splits the angle between $A$–$D$ and $B$–$E$ as shown in Figure 1.5. Four other symmetry operations are: reflection through axis $B$–$E$, reflection through $C$–$F$ and reflections

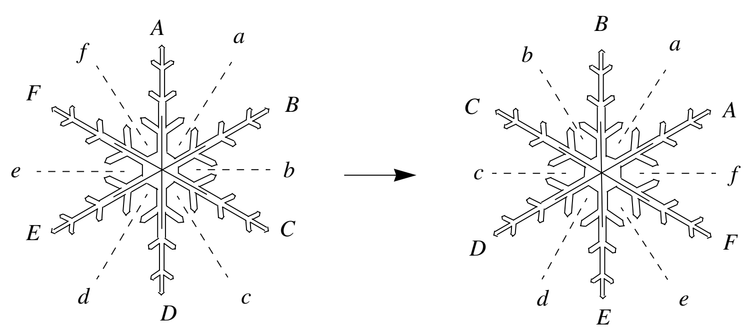


Fig. 1.5. Reflection axes of a snowflake.

through *b–e* and *c–f*. The six reflections are

$$\sigma_v^{AD} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_v^{BE} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \quad \sigma_v^{CF} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix},$$

$$\sigma_v^{be} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_v^{ad} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad \sigma_v^{cf} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}. \tag{1.4}$$

So the complete rotational and reflectional symmetry of a two-dimensional object with hexagonal symmetry is expressed by the twelve matrices in (1.2) and (1.4). The rotations have determinant $+1$ while the reflections have determinant $-1$. One can think of these matrices as the mathematical expression of "hexagonalness." In the above, the choice of symbols and sub- and superscript notation are traditional usages from crystallography.

### *1.3.1.2 Group Properties*

This set of twelve matrices has some very interesting properties. If we combine operations via matrix multiplication, the result is always equal to one of the twelve members of the set. For example:

$$C_6^2 \sigma_v^{CF} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \sigma_v^{BE}. \tag{1.5}$$

Note that commutation of $C_6^2$ and $\sigma_v^{CF}$ leads to a different result:

$$\sigma_v^{CF} C_6^2 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \sigma_v^{AD}. \tag{1.6}$$

These matrices form a group, and the examples (1.5) and (1.6) illustrate the group property of *closure*. The relational operator of the group (matrix multiplication) is not commutative, and so the group is said to be *non-Abelian*. For the most part we will be dealing with continuous transformations called Lie groups for which the group relational operator is commutative. Commutative groups are called *Abelian* after the early 19th-century Norwegian mathematician Niels Henrik Abel (1802–1829).

The twelve matrices $C_6^1$, $C_6^2$, $C_6^3$, $C_6^4$, $C_6^5$, $E$, $\sigma_\nu^{AD}$, $\sigma_\nu^{BE}$, $\sigma_\nu^{CF}$, $\sigma_\nu^{ad}$, $\sigma_\nu^{be}$, $\sigma_\nu^{cf}$ are said to form a *discrete group* with respect to the operation of multiplication. The word discrete refers to the fact that the *group operators* (1.2) and (1.4) involve discontinuous mappings of the snowflake.

Every member of the group has an *inverse*. For example,

$$C_6^2 C_6^4 = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = E. \tag{1.7}$$

The group is *associative*. For example,

$$\left(\sigma_\nu^{AD} C_6^2\right)\sigma_\nu^{CF} = \sigma_\nu^{AD}\left(C_6^2 \sigma_\nu^{CF}\right). \tag{1.8}$$

Finally, there is an *identity element*, $E$, which is a member of the group.

In general a set of objects constitutes a group with respect to a certain operation if it has the property of closure, if it is associative, if each element of it has an inverse, and if there is an identity element. Note that merely referring to a set of objects as a group is not sufficient. The operation used to relate the objects (in this case matrix multiplication) must also be specified.

Groups may contain *subgroups*. In the example of the snowflake discussed here, the subset $C_6^1$, $C_6^2$, $C_6^3$, $C_6^4$, $C_6^5$, $E$ constitutes a group, whereas the subset $E$, $\sigma_\nu^{AD}$, $\sigma_\nu^{BE}$, $\sigma_\nu^{CF}$ does not, since $\sigma_\nu^{AD}\sigma_\nu^{BE} = C_6^2$ is not a member of the subset.

In general, the symmetry properties of faceted objects are described mathematically by discrete groups with finite numbers of matrices. Further considerations of groups of this type would lead us into a study of projection operators, normal modes of vibration, quantum mechanics, molecular spectra, crystallography, solid-state physics, and the like. The book by Nussbaum [1.32] provides a very readable and comprehensive introduction to this topic. Such groups also come up when one considers the discrete symmetries of differential equations and their relationship to continuous Lie symmetries and numerical analysis. See References [1.21], [1.33] and [1.34].

## 1.4 The Principle of Covariance

The equations that describe the laws of physics are formulated according to the self-evident fact that physical phenomena exist outside of their mathematical description. A conservation equation is usually regarded as the most fundamental expression of a physical law. But the form of a conservation equation is a consequence of the symmetry properties of the phenomena that the law governs and of the space where the phenomena reside. Therefore, symmetries are as fundamental as the laws themselves. Often knowledge of a symmetry can be used to construct a conservation law and from that the functional form of the solution to a physical problem. Occasionally a symmetry can be used to construct a complete solution.

As Feynman points out [1.29] the search for symmetries in physical laws is closely connected to the recognition of patterns in the equations that express those laws. This involves the identification of ways in which mathematical expressions can be transformed without a change in their appearance. That the equations should be invariant under certain transformations of variables is one of the cornerstones of modern physics. Central to the whole subject are Lie groups, whose properties make them especially well suited for testing the transformational invariance of mathematical expressions.

***Definition 1.2 (The principle of covariance).*** *The equations that govern a physical law must retain the same appearance under certain group transformations of the variables that appear in the equations. This principle incorporates two somewhat distinct ideas.*

*(i) Coordinate independence – Physical phenomena must be governed by laws that do not depend on the coordinate system used to describe the phenomena.*
*(ii) Dimensional homogeneity – Physical phenomena must be described by laws that do not depend on the unit of measure applied to the dimensions of the variables that describe the phenomena.*

These two notions of phenomenological invariance and dimensional consistency in the equations apply universally across all fields of physics.

Coordinate independence is manifested in the fact that virtually all of the fundamental equations of physics are invariant under simple rotation and translation of the frame of reference. This invariance is a direct consequence of the homogeneity and isotropy of free space. Furthermore, when referred to a uniformly moving observer, a physical law must not explicitly depend on the speed of the observer. In classical mechanics, where the speed of light is infinite and time is absolute, the equations are invariant under a Galilean transformation, whereas in electrodynamics, where the speed of light is finite and the same for all observers, the relativistic transformations of time and space are connected through the Lorentz group. In either case, no particular coordinate system is distinguished as being in any way special in the mathematical formulation of the equations.

The requirement of dimensional homogeneity implies that any physical relationship must be expressible in dimensionless form. The ideas of dimensional homogeneity are expressed in the Buckingham Pi theorem and form the basis of the method of dimensional analysis which is the main subject of Chapter 2. The purpose of the present chapter is to introduce the notion of symmetry and to motivate the subject through a series of examples.

## 1.5 Continuous Symmetries of Functions and Differential Equations

In Chapter 5 one-parameter Lie groups will be formally defined and the main ideas and goals of group theory will be described. The purpose of this chapter is to introduce the subject to the reader for the first time; to present, in general terms, the main ideas underlying Lie group analysis; and to motivate further study. In this context it is useful to introduce a working definition of a Lie group. This will facilitate a general introductory discussion of *how* one investigates the symmetry of mathematical expressions. The larger question of *why* such an investigation is useful is addressed through several examples later in the chapter. For now we simply note that when a symmetry property is identified, it can be exploited to achieve a simplification. If the expression is an ordinary differential equation (ODE), then usually the order of the equation can be reduced. If it is a partial differential equation (PDE), then usually the dependent and independent variables can be combined to accomplish a reduction of dimension. Occasionally a known symmetry can be used to directly construct a solution of a nonlinear equation. In some instances whole classes of solutions can be constructed.

It would be a serious underestimation to regard symmetry analysis as merely a procedure for finding solutions. It is far more. Symmetries provide a systematic means for obtaining an enriched understanding of physical phenomena and the associated equations. Knowledge of the symmetries of a problem often leads to a completely new way of looking at the problem. A good example of this is contained in the paper by Wulfman and Wybourne [1.35], who analyzed the symmetries of a simple undamped harmonic oscillator. The solutions of this system are contained in any elementary textbook on mechanics, yet the result of their analysis led to a completely new understanding of the group origin of periodic time in oscillating systems.

Group theory is especially well suited for analyzing nonlinear systems. Any time a new problem is encountered, the first step should always be to analyze the symmetries of the governing equations. Once the symmetries of the system are known, all the other techniques for attacking the problem such as numerical analysis can be applied more effectively and with a better basic understanding of the problem.

### *1.5.1 One-Parameter Lie Groups in the Plane*

A one-parameter Lie group in two variables is a transformation of the form

$$\begin{aligned} \tilde{x} &= F[x, y, s], \\ \tilde{y} &= G[x, y, s] \end{aligned} \tag{1.9}$$

where $s$ is a scalar parameter whose value defines a one-to-one invertible map from a source space $S : (x, y)$ to a target space $\tilde{S} : (\tilde{x}, \tilde{y})$ as illustrated in Figure 1.6. The functions $F$ and $G$ are smooth analytic functions of the group parameter $s$ and therefore expandable in a Taylor series about any value on the open interval that contains $s$. At $s = 0$ the transformation reduces to an identity:

$$\begin{aligned} x &= F[x, y, 0], \\ y &= G[x, y, 0]. \end{aligned} \tag{1.10}$$

The use of the word *group* in this context may seem odd, in that we tend to think of a group as a collection of objects such as the matrices of the hexagonal group described in the previous section. In fact the transformation (1.9) is exactly that. Each object in the group corresponds to a specific value of the group parameter. The Lie group (1.9) has an infinite number of members corresponding to the infinity of possible values of $s$. Indeed, the key feature of a Lie group, which makes it useful, is the parametric representation of smooth functions on a continuous open interval in $s$. This ensures that the mapping is differentiable and invertible and that the mapping functions can be expanded in a Taylor series about any value of $s$. A differentiable mapping that has a differentiable inverse is called a *diffeomorphism*.

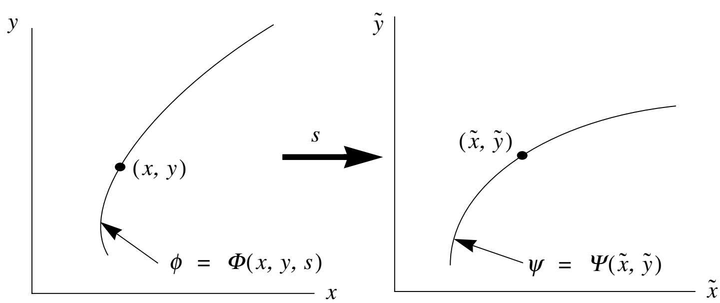


Fig. 1.6. Mapping of a point and a curve from $S$ to $\tilde{S}$ by the group (1.9).

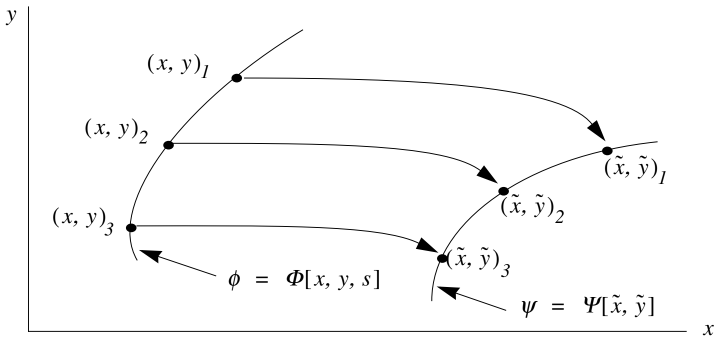


Fig. 1.7. Mapping of source to target points by the group (1.9).

Actually it is somewhat awkward to separate the source and target points into two distinct coordinate systems. The methods of group analysis are based almost entirely on an analytic theory expressed in terms of smooth transformation functions. These functions are expanded about the source point for small values of the group parameter, and the target point is an infinitesimal distance away. Therefore it is more appropriate to think of source and target points as the initial and final points of a transformation within a single coordinate system, as illustrated in Figure 1.7. The theory of infinitesimal transformations will be covered in later chapters. In this introduction we will focus exclusively on examples involving finite transformations.

### *1.5.2 Invariance of Functions, ODEs, and PDEs under Lie Groups*

In Figure 1.7, the curve $\psi$ is transformed to $\phi$ by using the group to change variables in the function $\Psi$:

$$\psi = \Psi[\tilde{x}, \tilde{y}] = \Psi(F[x, y, s], G[x, y, s]) = \Phi[x, y, s] = \phi. \quad (1.11)$$

Different values of the parameter $s$ lead to different curves as the outcome of the transformation. A particularly important case occurs when the resulting function, $\Phi$, reads exactly the same as $\Psi$, with the group parameter $s$ disappearing from the result:

$$\Psi[\tilde{x}, \tilde{y}] = \Phi[x, y, s] = \Psi[x, y]. \quad (1.12)$$

In this case the transformation maps points on the curve $\psi$ to other points on the *same* curve. The curve as a whole is mapped to itself, and the function $\Psi$ is said to be *invariant* under the group (1.9). Invariance under a group constitutes a symmetry property of the function.

The symmetry of a first-order ODE is analyzed by transforming it in the tangent space $(x, y, dy/dx)$. In order to transform a first-order ODE, we need to know how the derivative transforms under (1.9). This is simply a matter of taking differentials and using the definition of the derivative:

$$\frac{d\tilde{y}}{d\tilde{x}} = \tilde{y}_{\tilde{x}} = \frac{dG}{dF} = \frac{\frac{\partial G}{\partial x}\,dx + \frac{\partial G}{\partial y}\,dy}{\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy} = \frac{G_x + G_y\,\frac{dy}{dx}}{F_x + F_y\,\frac{dy}{dx}} = G_{\{1\}}[x, y, y_x, s]. \tag{1.13}$$

All the partial derivatives on the right-hand side of (1.13) are known functions of $[x, y]$ and the group parameter $s$. Together (1.9) and (1.13) constitute a *once-extended* Lie group.

*Note: Subscripts will be used throughout the text to denote differentiation. In addition, the conventional quotient form of the derivative will be used. Subscripts in braces, as in (1.13), will be used to identify a function that transforms a derivative, with the index inside the braces indicating the order of the derivative being transformed. The Einstein convention defined in Reference [1.36] on repeated indices to denote a sum will be adopted. Square brackets [ ] will be used to enclose independent variables and to indicate functional dependence. Some further discussion of notational issues follows in Section 1.6 and again in Chapter 7.*

Consider a first-order ODE

$$\psi = \Psi\left[\tilde{x}, \tilde{y}, \frac{d\tilde{y}}{d\tilde{x}}\right]. \tag{1.14}$$

Generally, we will use capital Greek letters to denote generic function names, aside from the $F$ and $G$ used to name the transformations. Obviously the letter $\Psi$ in (1.14) denotes a different function name than the $\Psi$ in (1.11) and (1.12). If we allow ourselves this slight ambiguity, we can avoid the proliferation of function symbols that would occur every time we wished to prevent a name conflict when introducing a new topic. It will usually be obvious from the context which function name applies to a given configuration of variables. In any case, what is important is the concept, not the symbols used to explain the concept.

The ODE (1.14) is transformed using (1.9) and (1.13) by simply changing variables using the once-extended group in the same spirit as when we transformed a function in (1.11). The result is a new first order ODE,

$$\begin{aligned}\psi &= \Psi\left[\tilde{x}, \tilde{y}, \frac{d\tilde{y}}{d\tilde{x}}\right] \\ &= \Psi\left[F[x, y, s], G[x, y, s], G_{\{1\}}\left[x, y, \frac{dy}{dx}, s\right]\right] \\ &= \Phi\left[x, y, \frac{dy}{dx}, s\right] = \phi. \end{aligned} \tag{1.15}$$

If, after the transformation, the equation reads the same in new variables,

$$\Psi\left[\tilde{x}, \tilde{y}, \frac{d\tilde{y}}{d\tilde{x}}\right] = \Phi\left[x, y, \frac{dy}{dx}, s\right] = \Psi\left[x, y, \frac{dy}{dx}\right], \tag{1.16}$$

then the equation is invariant under the group. As in the case of a function, invariance under the extended group constitutes a symmetry property of the differential equation.

***Example 1.1 (Invariance of a first-order ODE under a Lie group).*** The three-dimensional surface of a first-order ODE in the tangent space $(x, y, dy/dx)$ can be visualized, and an example is shown in Figure 1.8. The simple extended

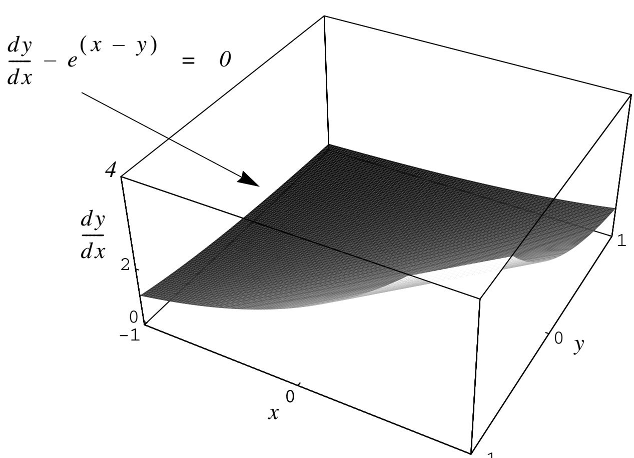


Fig. 1.8. The surface defined by a first-order ODE.

translation group

$$\begin{aligned} \tilde{x} &= x + s, \\ \tilde{y} &= y + s, \\ \frac{d\tilde{y}}{d\tilde{x}} &= \frac{dy}{dx} \end{aligned} \tag{1.17}$$

leaves the equation depicted in Figure 1.8 invariant:

$$\begin{aligned} \Psi\left[\tilde{x}, \tilde{y}, \frac{d\tilde{y}}{d\tilde{x}}\right] &= \frac{d\tilde{y}}{d\tilde{x}} - e^{\tilde{x}-\tilde{y}} = \frac{dy}{dx} - e^{(x+s)-(y+s)} \\ &= \frac{dy}{dx} - e^{x-y} = \Psi\left[x, y, \frac{dy}{dx}\right]. \end{aligned} \tag{1.18}$$

The key ingredient here is that the parameter $s$ vanishes from the transformed equation and the result reads *exactly the same* in the new variables. The equation is transformed to itself. More specifically, a point on the surface shown in Figure 1.8 is transformed to a new point on the *same* surface.

In this example, the equation can be easily integrated to produce the general solution,

$$\psi = \Psi[x, y] = e^y - e^x, \tag{1.19}$$

where $\psi$ is a constant of integration. The action of the group on a given solution curve is to transform it to a new solution curve. This can be seen as follows:

$$\tilde{\psi} = \Psi[\tilde{x}, \tilde{y}] = e^{\tilde{y}} - e^{\tilde{x}} = e^{y+s} - e^{x+s} = e^s(e^y - e^x). \tag{1.20}$$

The solution curve $\tilde{\psi}$ in (1.20) is transformed to

$$\frac{\tilde{\psi}}{e^s} = e^y - e^x. \tag{1.21}$$

While the equation (1.18) is invariant under the group (1.17), a given solution curve is not in that the constant is changed under the transformation. Nevertheless the *family* of solution curves as a whole *is* invariant under the group. A subtle point, but crucial to our later use of groups to solve differential equations.

This is a relatively simple example of an equation where the symmetry can be easily identified and the solution can be found by inspection. As more complex cases are explored in later chapters, the elegant and useful role that such symmetries play in the solution of nonlinear differential equations will be described.

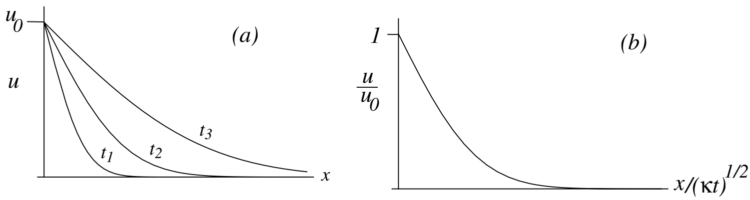


Fig. 1.9. Diffusion of temperature in a semiinfinite solid: (a) physical coordinates, (b) similarity coordinates.

There we shall see exactly how knowledge of a symmetry can be exploited to reduce or solve an ODE. Next we use symmetry analysis to solve a problem governed by a linear second-order PDE.

***Example 1.2 (Invariance of a PDE – Diffusion of heat in a conducting solid).*** Consider the problem of heat conduction in a semiinfinite solid instantaneously heated by holding the temperature on the boundary fixed. The temperature distribution at three successive times is shown in Figure 1.9(a).

The problem is governed by the linear diffusion equation (or heat equation)

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \tag{1.22}$$

with boundary conditions

$$\begin{aligned} t < 0:& \qquad u(0,t) = 0, \quad u(\infty,t) = 0, \\ t \geq 0:& \qquad u(0,t) = u_0, \quad u(\infty,t) = 0. \end{aligned} \tag{1.23}$$

As an ansatz, let's test the invariance of the equations and boundary conditions under a three-parameter dilation (or stretching) group commonly encountered in these types of problems. Let

$$\tilde{x} = e^a x, \qquad \tilde{t} = e^b t, \qquad \tilde{u} = e^c u. \tag{1.24}$$

Because the transformations of variables are completely uncoupled, the transformation of partial derivatives under the dilation group is especially simple. The exponential simply factors out of the derivative:

$$\begin{gathered} \frac{\partial \tilde{u}}{\partial \tilde{x}} = e^{c-a} \frac{\partial u}{\partial x}, \qquad \frac{\partial \tilde{u}}{\partial \tilde{t}} = e^{c-b} \frac{\partial u}{\partial t} \\ \frac{\partial^2 \tilde{u}}{\partial \tilde{t}^2} = e^{c-2b} \frac{\partial^2 u}{\partial t^2}, \qquad \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} = e^{c-2a} \frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial^2 \tilde{u}}{\partial \tilde{x}\, \partial \tilde{t}} = e^{c-a-b} \frac{\partial^2 u}{\partial x\, \partial t}. \end{gathered} \tag{1.25}$$

The equation transforms as

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} - \kappa \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} = e^{c-b} \frac{\partial u}{\partial t} - \kappa e^{c-2a} \frac{\partial^2 u}{\partial x^2}. \tag{1.26}$$

The diffusion coefficient, $\kappa$, is a constant and is left untransformed. Equation (1.22) is invariant under (1.24) if and only if $b = 2a$. This is clear from the following:

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} - \kappa \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} = e^{c-2a} \left( \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} \right) \quad \Rightarrow \quad \begin{aligned} \frac{\partial \tilde{u}}{\partial \tilde{t}} - \kappa \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} &= 0, \\ \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} &= 0. \end{aligned} \tag{1.27}$$

The equation reads exactly the same in new variables. The boundaries at $t = 0$, $x = 0$, and $x = \infty$ are clearly invariant under the group, i.e., the points at 0 and $\infty$ in $x$ and $t$ do not move under the transformation. Now consider the value of $u$ at $x = 0$:

$$\tilde{u}(0, \tilde{t}) = u_0 \quad \Rightarrow \quad e^c u(0, e^{2a} t) = u_0. \tag{1.28}$$

This boundary condition is invariant only if $c = 0$. Note that the range of $t$ and $e^{2a}t$ are the same, 0 to $\infty$. We conclude, therefore that the problem *as a whole* is invariant under the one-parameter group

$$\tilde{x} = e^a x, \qquad \tilde{t} = e^{2a} t, \qquad \tilde{u} = u. \tag{1.29}$$

If the far boundary in $x$ were to be placed at a finite distance, the symmetry of the problem would be broken and, in principle, the group could not be used to simplify the problem. In practice, the group might still be useful to define a solution valid at early time, when the far boundary can still be regarded as effectively infinitely far away. This point is of great importance and resonates with our earlier discussion of the idea that symmetries grow out of approximations. Perfect symmetry is rarely, if ever, achieved in a real physical problem. It is primarily in the abstract representation of that problem that symmetries come forth and that group methods become useful in the solution of the problem. We will come back to this point again and again throughout the text.

The key idea here is that when the governing PDE(s) and boundary conditions of a problem are invariant under a group, the solution is also invariant under the same group. In this case, the solution can be expressed in terms of a reduced set of combinations of the basic variables. These are called *similarity variables* and are themselves invariant under the group. Proof of invariance under a group is essentially a proof of the existence of a similarity solution to the problem.

In Chapter 9 we will see how to systematically construct similarity variables from the knowledge of a group. For the present, it is fairly easy to construct, by inspection, similarity variables that are invariant under (1.29) are

$$\frac{\tilde{u}}{u_0} = \frac{u}{u_0}, \qquad \frac{\tilde{x}}{\tilde{t}^{1/2}} = \frac{e^a}{e^a}\frac{x}{t^{1/2}} = \frac{x}{t^{1/2}}. \tag{1.30}$$

Using the diffusivity to nondimensionalize the similarity variable involving $x$ and $t$, we expect a solution of the form

$$\phi = \Phi\left[\frac{u}{u_0}, \frac{x}{\sqrt{\kappa t}}\right] \tag{1.31}$$

or, without loss of generality,

$$\frac{u}{u_0} = U\left[\frac{x}{\sqrt{\kappa t}}\right]. \tag{1.32}$$

By exploiting the invariance of the problem under the dilation group (1.29), the solution in two independent variables is expressed as a single curve in one similarity variable as shown in Figure 1.9(b).

It may be apparent to the reader at this point that the units of the diffusivity have a great deal to do with the structure of the group (1.24), which leaves the equation invariant. The fact that the solution has the form (1.31) and not, say,

$$\theta = \Theta\left[\frac{u}{u_0}, \frac{x}{\sqrt{t}}, \kappa\right] \tag{1.33}$$

is not an accident, but a direct consequence of the principle of covariance. In its simplest form this is just a statement of the obvious fact that we can't add physical quantities that don't have the same dimensions. When we address the particular equation that may govern a physical phenomenon, invariably the equation will contain dimensioned constants such as the thermal conductivity, viscosity, speed of light, vacuum permittivity, etc., that characterize the physical entities involved. It only takes a little practice with such problems before one quickly recognizes that the dilational invariance of an equation containing a physical constant, is largely determined by the dimensions of that constant and the requirement of covariance.

Continuing with the example we substitute (1.32) into (1.22) and (1.23) leading to

$$U_{\zeta\zeta} + \frac{\zeta}{2}U_\zeta = 0, \qquad U(0) = 1, \qquad U(\infty) = 0, \tag{1.34}$$

where $\zeta = x/\sqrt{\kappa t}$. The symmetry of the problem enables us to reduce the governing PDE to an ODE. The solution of (1.34) is expressed in terms of the complementary error function

$$U = \operatorname{erfc}[\zeta] = 1 - \frac{1}{\pi}\int_0^{\zeta} e^{-\zeta^2/4}\, d\zeta. \tag{1.35}$$

Substitution of (1.35) into the equation and checking boundary conditions confirms the solution.

Let's spend a moment to examine the key assumptions that enabled this problem to be simplified:

(i) The heat is assumed to be added uniformly in an infinitely thin region at the left boundary of the domain.
(ii) The domain is infinite in extent to the right.
(iii) The turning on of the temperature at the boundary is assumed to take place in zero time.

Each of these assumptions removes a length scale or a time scale that, if it were included, would break the dilational invariance of the problem. In effect the near-symmetry of a real problem is idealized to bring into play the exact symmetry of a simplified model of the problem.

Underlying all this is the assumption that, if the ignored scales are sufficiently small or sufficiently large, then their effect on the solution of the problem is small. Barenblatt [1.37] describes a large class of problems where such an assumption either is incorrect or needs to be interpreted very carefully. Such problems often arise in relatively simple geometries where the physics of the problem involves the propagation of some sort of front, the motion of which cannot be predicted by dimensional analysis alone. A fully worked example of this class of problems will be described in considerable detail in Chapter 9.

As we can see from the foregoing examples, Lie theory is all about transformations that leave differential equations invariant. We can often exploit that invariance to generate solutions of nonlinear equations by construction. Suppose a Lie group of the form

$$\begin{aligned}\tilde{x} &= F[x, t, u, s],\\ \tilde{t} &= T[x, t, u, s],\\ \tilde{u} &= G[x, t, u, s]\end{aligned} \tag{1.36}$$

transforms some partial differential equation,

$$\Phi[x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots] = 0 \tag{1.37}$$

to itself, i.e., the variables with tildes satisfy

$$\Phi[\tilde{x}, \tilde{t}, \tilde{u}, \tilde{u}_{\tilde{x}}, \tilde{u}_{\tilde{x}\tilde{x}}, \tilde{u}_{\tilde{x}\tilde{t}}, \tilde{u}_{\tilde{t}\tilde{t}}, \ldots] = 0. \tag{1.38}$$

The implication is that if $u[x, t]$ is a solution of (1.37), then $\tilde{u}[\tilde{x}, \tilde{t}]$ constructed from the group (1.36) is a solution of (1.38). This is so whether the equation is linear or nonlinear. The following example illustrates this procedure.

***Example 1.3 (Solutions generated directly from symmetries).*** The nonlinear PDE

$$u_t + \tfrac{1}{2}(u_x)^2 - u_{xx} = 0 \tag{1.39}$$

is shown in Chapter 16 to be invariant under the transformation

$$\begin{aligned} \tilde{x} &= x, \\ \tilde{t} &= t, \\ \tilde{u} &= u - 2\ln\left(1 - f[x, t]e^{u/2}\right), \end{aligned} \tag{1.40}$$

where $f$ is any solution of the heat equation $f_t - f_{xx} = 0$. The transformation (1.40) immediately raises a whole series of questions: where does it come from, can it be derived systematically, is it a group, where is the group parameter, can it be used to match reasonable boundary conditions, etc.? The complete answer to these questions will have to wait until Chapter 16. For now suffice it to say that it is a group, it can be used to match reasonable boundary conditions, and, most importantly, it can be derived through a systematic process. As a simple example of the application of (1.40), let $u$ be the vacuum solution of (1.39), namely, $u = 0$. If we choose the elementary heat-equation solution $f = -t - x^2/2$, then (1.40) generates

$$\tilde{u} = \ln\left(\frac{1}{1 + t + x^2/2}\right)^2, \tag{1.41}$$

which is an exact solution of (1.39). This is a particularly simple illustration of the use of (1.40). A vast variety of important exact solutions can be generated using (1.40), and further discussion of this facet of symmetry analysis will be given in Chapter 16.

In summary, the transformation group (1.9) together with its extension to derivatives such as (1.13) is a diffeomorphism in the space of dependent variables and independent variables. For a system of differential equations invariant under (1.9), the transformation maps solutions to solutions. The message from these three examples is that symmetry analysis is a very useful technique for generating solutions of both linear and nonlinear differential equations. Indeed, the method can be applied in varying degrees of effectiveness to virtually any problem in mathematical physics.

## 1.6 Some Notation Conventions

One of the biggest headaches in learning group theory is getting used to the notation. Equally important, a consistent, easy-to-understand notation is a must for an introductory text. The problem of notation arises because the object of study is usually a differential equation where the derivatives are treated as independent quantities that must be transformed. In this context it is easy to confuse the label of a transforming function with a subscript that is intended to denote differentiation. Actually, we are getting a little ahead of ourselves here, but it is worthwhile discussing the problem now in order to make the later going a little easier.

The theory of Lie groups makes use of smooth parametric transformations of the form

$$\begin{aligned} \tilde{x}^j &= F^j[\boldsymbol{x}, \boldsymbol{y}, s], && j = 1, \ldots, n \\ \tilde{y}^i &= G^i[\boldsymbol{x}, \boldsymbol{y}, s], && i = 1, \ldots, m \\ \tilde{y}^i_j &= G^i_{\{j\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, s], && \\ \tilde{y}^i_{j_1 j_2} &= G^i_{\{j_1 j_2\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, s] && \\ &\vdots \end{aligned} \tag{1.42}$$

where the partial derivatives are

$$\tilde{y}^i_j = \frac{\partial \tilde{y}^i}{\partial \tilde{x}^j}, \qquad \tilde{y}^i_{j_1 j_2} = \frac{\partial^2 \tilde{y}^i}{\partial \tilde{x}^{j_1}\, \partial \tilde{x}^{j_2}}, \ldots, \tag{1.43}$$

and the dots indicate continuation to higher derivatives up to some unspecified order. The second to last relation in (1.42) is parsed as shown in Figure 1.10. In later chapters we shall see precisely how to work out the functions that transform partial derivatives; the procedure is exactly analogous to that used to derive (1.13). The use of superscripts to label vector components and subscripts

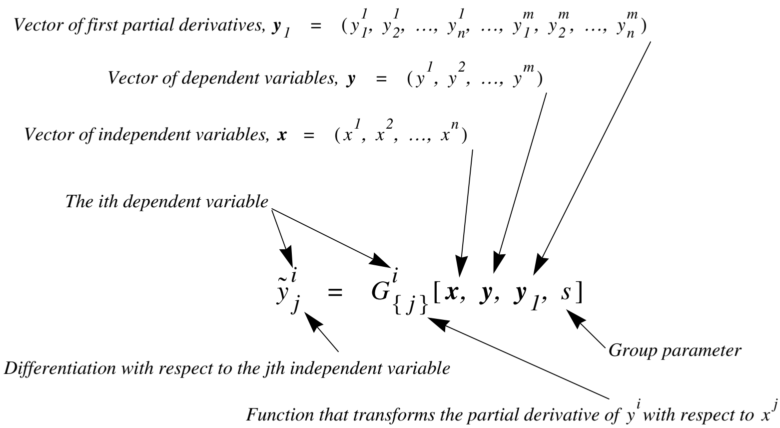


Fig. 1.10. Notation for variables, derivatives, and transformations of derivatives.

to denote differentiation is standard notation and consistent with that used in tensor calculus. In rare circumstances confusion can occur when an expression involves exponents. If and when such a situation arises, parentheses will be used for clarity.

I was tempted to use the comma notation for derivatives introduced by Einstein [1.36]. In this case, derivatives are denoted as follows:

$$y^i_{,j} = \frac{\partial y^i}{\partial x^j}, \qquad y^i_{,j_1 j_2} = \frac{\partial^2 y^i}{\partial x^{j_1}\, \partial x^{j_2}} \tag{1.44}$$

This is the notation adopted by Stephani [1.18] and has the large advantage of being totally unambiguous, enabling uncommaed subscripts to be used as labels. This is especially useful in the context of a theory like general relativity where the dependent variable is a tensor. There are superficial reasons for not adopting this notation: the commas tend to get lost, particularly when viewed on a computer screen, and the differential equations tend to have a busy appearance. But there is a more compelling reason, which has to do with the nature of the group-theoretical point of view. In group theory differential equations are viewed as surfaces in a higher-dimensional (jet) space whose coordinates are the independent variables, the dependent variables, and all the possible derivatives of one with respect to the other (cf. Figure 1.8). In this context derivatives are objects to be manipulated just like common variables. In the end I chose to leave out the commas to promote this point of view.

In one respect I have adopted Einstein's notation: when it comes to summation over repeated indices. For example, the incompressible continuity equation from fluid mechanics is written as follows:

$$\frac{\partial u^1}{\partial x^1} + \frac{\partial u^2}{\partial x^2} + \frac{\partial u^3}{\partial x^3} = 0 \quad \Rightarrow \quad \frac{\partial u^i}{\partial x^i} = u^i_i = 0. \tag{1.45}$$

The use of a subscript in braces to label the function that transforms the derivative may, at first, seem to be a bit overcomplicated, but there is a good reason for it. Much of the theory relies on the infinitesimal form of the transformation where the functions in (1.42) are expanded for small values of the group parameter $s$:

$$\begin{aligned} \tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x}, \boldsymbol{y}], \\ \tilde{y}^i &= y^i + s\eta^i[\boldsymbol{x}, \boldsymbol{y}], \\ \tilde{y}^i_j &= y^i_j + s\eta^i_{\{j\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1], \\ &\vdots \end{aligned} \tag{1.46}$$

The linearized form of the group in (1.46) contains the functions $\eta^i_{\{j\}}$, which infinitesimally transform the first partial derivatives of $y^i$. These functions are expressed in terms of the $\xi^j$, $\eta^j$, and their derivatives. The detailed expressions will be given later, but their form is

$$\eta^i_{\{j\}} = \eta^i_j + \text{various other terms.} \tag{1.47}$$

Without the braces to distinguish the function name on the left and the derivative of $\eta^i$ with respect to $x^j$ on the right, this relation and its higher-order forms would cause endless confusion. An alternative would be to adopt all new function names for the derivative transformations, but this would add a whole set of new symbols to the theory, and, as we shall see later, there are symbols enough already.

There is a price to pay for using subscripts only to denote derivatives. If this convention were applied slavishly everywhere, subscripts would be unavailable for labeling variables and certain expressions would become quite awkward. So, with apologies, subscripts will be used in a few places to label variables where there is no chance of confusion with a derivative.

## 1.7 Concluding Remarks

This chapter provides an introduction to the concept of a group and to the meaning of invariance under a group. The brief sampler of examples presented above is intended to illustrate some of the kinds of problems that are the object of symmetry analysis. In Chapter 2 we will explore the role of groups in dimensional analysis through several applications. Together these two chapters provide the motivation for the rest of the book, and several of the examples are revisited in later chapters.

Following Chapter 2, the theory will be developed fully. In fact, the theory is not all that extensive and can be adequately covered in a few chapters. There are certain concepts from the theory of differential equations that lie at the heart of Lie theory, and so we shall spend Chapter 3 reviewing several topics concerning ODEs, including the method of characteristics, integrating factors, and state-space analysis of linear and nonlinear autonomous systems. Especially important is the relationship between a system of first-order ODEs and its associated first-order PDE, and in Chapter 3 several methods for solving linear and nonlinear first-order PDEs are described. In Chapter 4 the Lagrangian and Hamiltonian formulations of classical dynamics are developed, leading eventually to the derivation of the Hamilton–Jacobi equation. Examples using the methods of Chapter 3 are described in Chapter 4.

Lie groups are formally defined in Chapter 5 along with the infinitesimal form of a group. This leads to the definition of the group operator and the expansion of an analytic function in a Lie series. The fundamental condition for invariance of a function under a group is stated, along with a discussion of the characteristic equations of a group. Finally, the concept of a multiparameter group and its associated Lie algebra is discussed. This comprises almost all the theory needed for application to differential equations. In Chapter 6, group theory is applied to the solution of first-order ODEs. In essence, knowledge of an invariant group can be used to generate an integrating factor, leading to the general solution of the ODE. Here both the power and the limitations of group theory are well illustrated.

The only missing element needed to treat higher-order ODEs and PDEs is the procedure for extending a Lie group to include transformations of derivatives. The formulas are expressed in terms of the total differentiation operator $D$ defined in Appendix 1. This operator is needed to remove some of the notational ambiguities connected with partial differentiation of implicit functions. Extended groups applied to ODEs and PDEs are the main topics of Chapters 8 and 9 respectively. In Chapter 7 a notation is adopted for derivatives and for their transforming functions. At the center of this discussion is the definition of a differential function (Ibragimov [1.38]), which is locally analytic function of independent variables, dependent variables, and derivatives of dependent variables. This is an extremely useful concept, which allows all of the theory of groups applied to functions developed in Chapter 5 to be carried forward intact to the application of extended groups to differential equations in Chapters 8 and 9. For example, the expansion of a PDE in a Lie series leads immediately to the invariance condition of the PDE under an extended group.

The latter part of Chapter 9, along with Chapters 10, 11, and 12, is devoted to the application of point groups to the simplification and solution of PDEs. The examples are drawn mainly from heat conduction and fluid mechanics and are intended to illustrate several of the many facets of the symmetry analysis approach. For this reason a number of the examples are worked out in substantial detail to provide the reader with a complete look at how one searches for the symmetries of a given physical problem with all of its governing equations, boundary conditions, and possibly integral constraints. Once the symmetries are identified, the procedure for generating the reduced problem is carried through to whatever level is feasible. Extensive use is made of the state-space analysis techniques from Chapter 3 to analyze the structure of the reduced problem and its solution.

In Chapter 13 groups are used to develop similarity rules for the growth and decay of turbulent shear flows in simple geometries. The problem here is that the governing equations are unclosed. In this case the goal is not to reach the solution of any one particular problem, but rather to say as much as one can say in the absence of a complete model of turbulent stresses. The result is a general set of relations that bring together and unify a wide range of results normally obtained from dimensional analysis. The method is used to define the parameters for an experiment to measure fine-scale motions in a turbulent vortex ring.

Lie–Bäcklund groups are described in Chapter 14. A Lie–Bäcklund group is a generalization of the concept of a point transformation to that of a higher-order tangent transformation where the mapping of a point can depend on derivatives evaluated at the point. In a few cases Lie–Bäcklund groups lead directly to exact solutions of nonlinear problems. Some of the more complex topics establishing the infinite-order properties of Lie–Bäcklund groups are included in Appendices 2 and 3. Recursion operators for generating higher-order symmetries of a differential function are described through several examples.

The treatment of Lie–Bäcklund groups in Chapter 14 leads naturally to the derivation of the conditions for invariance of an integral of a differential function in Chapter 15. Variational symmetries are discussed, along with the famous theorem due to Emmy Noether that relates the symmetries of an Euler–Lagrange system to conservation laws governing its evolution.

Finally, Chapter 16 is devoted to a discussion of several nonlinear wave equations in the context of what are commonly called Bäcklund transformations. Ordinarily these are viewed as many-valued transformations tied to a particular differential equation or pair of differential equations through an integrability condition. Therefore they are usually viewed as quite distinct from Lie groups and Lie–Bäcklund groups. However, in the several examples described in Chapter 16, well-known Bäcklund transformations are seen to be specializations of nonlocal Lie groups (Lie groups that depend on an integral of the transformed variable) connected to a potential function.

It should be apparent from the examples discussed above that the theory of Lie groups is central to a wide variety of problems encountered in engineering and physics. Yet it is a fact that the subject rarely appears in the core curriculum of the average graduate student, nor is it a topic widely embraced by researchers or faculty. Why is this? There are several reasons. First, many of the most important results that can be derived through the formalism of group analysis were first discovered without it. Sorted by topic and not by method, similarity solutions and results of dimensional analysis abound, scattered throughout the vast body of engineering and science literature. The same goes for the many applications of Lie groups to mechanics, quantum mechanics, and relativity theory. The problem with this is that, without an extraordinary act of lateral thinking, the student usually never gets the whole story and never gains an appreciation of the overarching role of symmetry in the solution of physical problems.

There is a second reason. Lie group theory is essentially a procedure for investigating the structure of differential equations. Thus, one often needs to work out transformations of derivatives, and for derivatives of second order and higher the expressions become extremely long – hugely long if there are many variables. Thus, to analyze any but the simplest differential equations, a discouragingly large amount of effort is required. As far as hand calculations are concerned, the subject is essentially inaccessible to all but the most undaunted workers – those with a strong, direct interest in the subject. An uninitiated student trying to learn group theory for the first time is easily overwhelmed. Homework gets to be an onerous series of repetitive calculations, often just to reach very elementary results. No wonder there is a tendency to shy away!

Fortunately, we now live in an era when powerful symbol manipulation software packages are widely available. This not only allows the vast bulk of the effort in group analysis to be automated, bringing the whole subject completely within the reach of an interested student, but it also opens up new vistas for research. For example, although we have very complete knowledge of the Lie point groups that leave the classical equations of mathematical physics invariant, we know practically nothing of the higher-order Lie–Bäcklund structure of these equations except in a few cases where symmetries can be generated through the use of recursion operators. These are problems that simply could not be attacked by hand because of the sheer bulk of the calculations required.

Appendix 4 describes a software package called **`IntroToSymmetry.m`**, which is provided with the text. The package was developed by the author using *Mathematica*® and requires that the user have this application available. Although this is an appendix, don't treat it as merely a minor addition to the text. The software package is a key tool, which the reader will need in order to work a number of the homework problems. A fair amount of time should be set aside to familiarize oneself with the various functions in the package and the numerous sample runs included on the CD-ROM. The great value of this software package and others like it lies in the fact that one can fairly quickly gain experience in searching for and recognizing symmetries by working out a large number of sample problems that would take ages to work out by hand. Nevertheless it is essential to work out a few problems by hand to gain a basic understanding of Lie's algorithm.

## 1.8 Exercises

1.1 Work out the 6-member discrete symmetry group of an equilateral triangle. Show that the set of matrices is closed with respect to matrix multiplication, that each member of the set has an inverse, that the matrices are associative, and that the set has an identity element.

1.2 Work out the 24-member discrete rotation group of the cube shown in Figure 1.11. Show with sample calculations that the set of matrices is closed with respect to matrix multiplication, that each member of the set has an inverse, that the matrices are associative, and that the set has an identity element. How many matrices do you get if you include

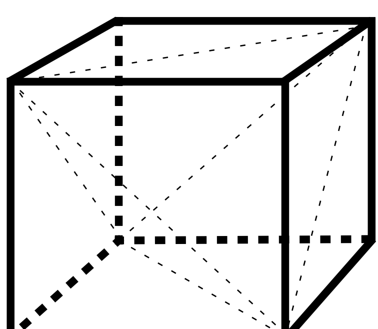

Fig. 1.11.

reflections? Which of these symmetries is shared by a tetrahedron formed by connecting four of the corners of the cube as shown? Show that the tetrahedral group is a subgroup of the cubic group. See Chapter 1 of the reference by Nussbaum [1.32] for a discussion of this and related problems.

1.3 Show that the first-order ODE

$$x\left(\frac{dy}{dx}\right)^2 + y\left(\frac{dy}{dx}\right) + x = 0 \tag{1.48}$$

is invariant under a dilation group. Is it invariant under translation? Plot the surface defined by the equation in $(x, y, y_x)$ coordinates.

1.4 Consider the nonlinear heat equation

$$\frac{\partial T}{\partial t} - \lambda \frac{\partial}{\partial x}\left(T^\beta \frac{\partial T}{\partial x}\right) = 0, \tag{1.49}$$

where $T$ is the temperature. What are the units of $\lambda$? Find a two-parameter dilation group that leaves the equation invariant. How is the group connected to $\beta$?

1.5 Transform each of the following equations using the following four-parameter dilation group:

$$\tilde{x}^j = e^a x^j, \qquad \tilde{t} = e^b t, \qquad \tilde{u}^i = e^c u^i, \qquad \tilde{p} = e^d p, \qquad \tilde{\rho} = \rho. \tag{1.50}$$

(i) The incompressible Navier–Stokes equations

$$\frac{\partial u^i}{\partial t} + \frac{\partial}{\partial x^k}\left(u^i u^k + \frac{p}{\rho}\delta^i_k\right) - \nu \frac{\partial u^i}{\partial x^k \partial x^k} = 0, \qquad \frac{\partial u^k}{\partial x^k} = 0. \tag{1.51}$$

(ii) The Stokes equations for slow flow,

$$\frac{\partial u^i}{\partial t} + \frac{\partial}{\partial x^i}\left(\frac{p}{\rho}\right) - \nu \frac{\partial u^i}{\partial x^k \partial x^k} = 0, \qquad \frac{\partial u^k}{\partial x^k} = 0. \tag{1.52}$$

(iii) The Euler equations for inviscid flow,

$$\frac{\partial u^i}{\partial t} + \frac{\partial}{\partial x^k}\left(u^i u^k + \frac{p}{\rho}\delta^i_k\right) = 0, \qquad \frac{\partial u^k}{\partial x^k} = 0. \tag{1.53}$$

How do the group parameters $a, b, c, d$ have to be related to one another in order for the given equations to be invariant?

1.6 Use the solution (1.41) in (1.40) to initiate a succession of solutions to (1.39). What do these solutions have in common?

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# 2

# *Dimensional Analysis*

Any physical relationship must be expressible in dimensionless form. The implication of this statement is that all of the fundamental equations of physics, all approximations to these equations, and, for that matter, all functional relationships between physical variables must be invariant under a dilation (or stretching) of the dimensions of the variables. This is because the variables are subject to measurement by an observer in terms of units that are selected at the *arbitrary* discretion of the observer. It is clear that a physical event cannot depend on the choice of the unit of measure used to describe the event. It cannot depend on the particular ruler used to measure space, the clock used for time, the scale used to measure mass, or any other standard of measure that might be required, depending on the dimensions that appear in the problem. This principle is the basis for a powerful method of reduction called *dimensional analysis*.

## 2.1 Introduction

A general mathematical relationship between variables is completely devoid of symmetry. However, if the variables describe the properties of a measureable physical system, then the dimensions of the system add a symmetry property to the relationship where none existed before. In effect, assigning dimensions to the variables brings into play the principle of covariance. We can define the notion of dimension as follows.

***Definition 2.1.*** *A dimension is a measurable property of a physical system that can be varied by a dilational transformation of the units of measurement. The value of each variable of the system is proportional to a power monomial function of the fundamental dimensions.*

Often dimensional analysis is carried out without any explicit consideration of the actual equations that may govern a physical phenomenon. Only the variables that affect the problem are considered. Actually, this is a little deceiving. Inevitably, the choice of variables is intimately connected to the phenomenon itself and therefore is always is connected to, and has implications for, the governing equations. In fact the most complex problems in dimensional analysis tend to be filled with ambiguity as regards the choice of variables that govern the phenomenon in question.

## 2.2 The Two-Body Problem in a Gravitational Field

First let's look at a fairly straightforward example that nicely illustrates both the power and the limitations of dimensional analysis. This is the problem of determining the relationship between the mean distance from the sun and the orbital period of the planets. The solution of this problem was published by Johannes Kepler in 1619 and has since been known as Kepler's third law. Kepler, who succeeded Tycho Brahe as the imperial mathematician of the Holy Roman Empire in 1601, was one of the truly outstanding scientists of the Age of Enlightenment. His position gave him access to Brahe's incomparable collection of astronomical data, particularly data for the movement of Mars, collected by a team of astronomers over decades of painstaking work. By 1609 he had published his first two laws (although he did not refer to them as such): that the planets follow elliptical orbits and that the movement of a planet along its orbit traces out equal areas in equal times. A decade later he published his findings that the cube of the distance from the sun divided by the square of the period is a constant. Kepler's accomplishments at the time are all the more remarkable in that they occurred at about the same time that he had to rush to the defense of his mother, who had been indicted as a witch. Only his timely defense in 1620 prevented her from being tortured and burned at the stake. Kepler remained the imperial mathematician for several more years but, through the events leading up to the Thirty Years' War, was eventually forced to find a new patron. He fell ill and died on November 15, 1630. Kepler was the first to provide a dynamical explanation of the movements of the heavens, and his results continued to have an impact long after his death. Newton relied heavily on Kepler's work in developing his theory of gravitation in the 1680s. Today we recognize that the law of equal areas applies to any pair of masses with a radially directed force between them, while the first and third laws apply only to particles that obey an inverse square law, including the motion of satellites and the electrical interactions of charged particles. A fourth law can also be identified that arises from the invariance of the governing equations of the Kepler system under a

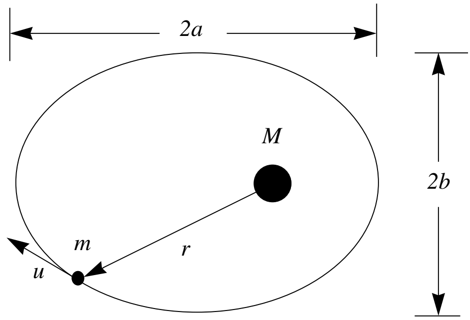


Fig. 2.1. Elliptical orbit of a planet about the sun.

Lie–Bäcklund group. This will be discussed in Chapter 15, where the problem is revisited.

Here we consider the movement of one of the planets about the sun. The orbit is elliptical with major axis $a$, minor axis $b$, and area $A = \pi ab$. The sun lies very close to one of the foci of the ellipse, as shown in Figure 2.1.

The force between the two masses follows the Newtonian law of gravitation,

$$F = -G\frac{Mm}{r^2}, \tag{2.1}$$

where $G$ is the gravitational constant $G = 6.670 \times 10^{-11}$ N-m$^2$/kg$^2$ and the minus sign indicates that the force is attractive. The perturbation of the orbit by all the other planets is ignored. We wish to use dimensional analysis to rediscover Kepler's third law relating the period of the orbit to its size. Data for the solar system are shown in Table 2.1. The mass of the Earth is $5.975 \times 10^{24}$ kg, and the mean diameter is 12742.46 km.

The eccentricity of a planet's orbit is

$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2}. \tag{2.2}$$

The only parameters that can enter the problem are the lengths of the two axes, the two masses, the period, and the gravitational constant, and we have

$$\hat{a} = L, \qquad \hat{b} = L, \qquad \hat{M} = M, \qquad \hat{m} = M, \qquad \hat{T} = T, \qquad \hat{G} = \frac{L^3}{MT^2}. \tag{2.3}$$

The hat over a parameter such as $\hat{a}$ in (2.3) is used to mean "dimensions of." In this problem $M =$ mass, $L =$ length, and $T =$ time are the fundamental

Table 2.1. *The planets and their orbits.*

| Heavenly body | Mass (Earth masses) | Diameter (Earth diameters) | Mean orbit Radius ($10^6$ km) | Eccentricity | Orbital period (years) |
|---|---|---|---|---|---|
| Sun | 332,488.0 | 109.15 | — | — | — |
| Mercury | 0.0543 | 0.38 | 57.9 | 0.2056 | 0.241 |
| Venus | 0.8136 | 0.967 | 108.1 | 0.0068 | 0.615 |
| Earth | 1.0000 | 1.000 | 149.5 | 0.0167 | 1.000 |
| Mars | 0.1069 | 0.523 | 227.8 | 0.0934 | 1.881 |
| Jupiter | 318.35 | 10.97 | 777.8 | 0.0484 | 11.862 |
| Saturn | 95.3 | 9.03 | 1426.1 | 0.0557 | 29.458 |
| Uranus | 14.58 | 3.72 | 2869.1 | 0.0472 | 84.015 |
| Neptune | 17.26 | 3.38 | 4495.6 | 0.0086 | 164.788 |
| Pluto | <0.1 | 0.45 | 5898.9 | 0.2485 | 247.697 |

dimensions. There are six parameters and three fundamental dimensions, and so we can expect the solution to depend on three dimensionless numbers. Two of these are obviously a mass ratio and a length ratio,

$$\Pi_1 = \frac{m}{M}, \qquad \Pi_2 = \frac{b}{a}. \tag{2.4}$$

In view of the dimensions of $G$, it is clear that the third number must involve one of the masses, one of the lengths, and the period. Thus, we can expect a dimensionless variable of the form

$$\Pi_3 = \frac{GMT^2}{a^3}, \tag{2.5}$$

where we have arbitrarily chosen $M$ and $a$ to form $\Pi_3$ rather than $m$ and $b$. According to the principle of covariance one can expect all these variables to be related by a dimensionless function of the form

$$\psi = \Psi(\Pi_1, \Pi_2, \Pi_3). \tag{2.6}$$

Without loss of generality we can solve (2.6) explicitly for $\Pi_3$:

$$\frac{GMT^2}{(r_{\text{mean}})^3} = F\left(\frac{m}{M}, e\right), \tag{2.7}$$

where we have used the eccentricity in place of $b/a$ and a mean radius defined as $r_{\text{mean}} = \sqrt{ab}$. Using the data in Table 2.1, we can plot the values of (2.7) for the various planets in the solar system, as shown in Figure 2.2.

Figure 2.2 provides stunning confirmation of our dimensional analysis result and indicates that the function on the right-hand side of (2.7) is very nearly

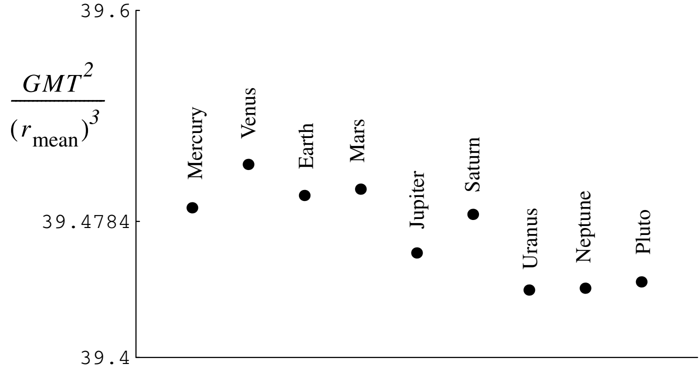


Fig. 2.2. Kepler's third law for the solar system.

constant for all the planets in the solar system. In fact, Kepler's theory tells us that the right-hand side of (2.7) is

$$F\left(\frac{m}{M}, e\right) = 4\pi^2\left(\frac{1}{(1+m/M)(1-e^2)^{3/4}}\right). \tag{2.8}$$

For all the planets $m/M \ll 1$, and for all but Mercury and Pluto $e^2$ is very small. In the limits $m/M \to 0$ and $e \to 0$ the right-hand side of (2.8) approaches the finite limit $4\pi^2 = 39.4784$.

In fact we have made a lucky choice. On purely dimensional grounds, in the absence of Kepler's theory, there is absolutely no reason to select $M$ in the definition of $\Pi_3$; $m$ would have been just as appropriate a choice, but would have produced a highly scattered plot. Dimensional analysis alone provides no information in this matter. The full theory is required. We will return to the Kepler problem three more times in this text. In Chapter 4 we will develop the full theory of the two-body problem, and in Chapters 14 and 15 we will study Lie–Bäcklund symmetries and their connection with conservation laws for the Kepler problem.

## 2.3 The Drag on a Sphere

Next, we will work a problem that also illustrates the power as well as some of the pitfalls of dimensional analysis. This is the problem of viscous flow past a sphere. The previous example involved a rather simple set of basic dimensioned variables, and so it could be worked out by inspection. In the present case that is not quite so easy, and so we will resort to a systematic method of constructing appropriate dimensionless variables. Initially we will make the assumption that the flow is incompressible, and then compressibility effects will be added later. The results will then be compared with experimental data. Uniform flow of a viscous fluid past a sphere is shown in Figure 2.3.

To get started let's assume that the relevant variables of the problem are the drag force $D$, the fluid density $\rho$, the viscosity $\mu$, the freestream flow velocity $U$, and the radius of the sphere, $r$. These variables can be thought of as related to one another through a function of the form

$$\psi_0 = \Psi_0[D, \mu, \rho, U, r], \tag{2.9}$$

where $\psi_0$ is a pure number (i.e., dimensionless), which may be zero.

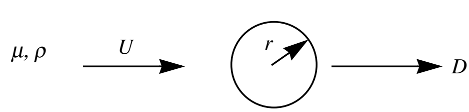


Fig. 2.3. Viscous flow past a sphere.

Considered solely as a mathematical statement, (2.9) has no symmetry. But it is not just an abstraction! It is a physical statement in two respects. First, it states that the drag on the sphere depends only on the selected variables. This is totally at our discretion, and it would be easy to argue that other quantities, for example the speed of sound in the fluid, ought to also play a role. Second, the variables in (2.9) are all measurable properties of a physical system; they have dimensions, and those dimensions are measured in *arbitrarily* chosen units:

$$\hat{D} = \frac{ML}{T^2}, \qquad \hat{\mu} = \frac{M}{LT}, \qquad \hat{\rho} = \frac{M}{L^3}, \qquad \hat{U} = \frac{L}{T}, \qquad \hat{r} = L. \tag{2.10}$$

Because the variables in (2.9) have dimensions, the function $\Psi_0$ cannot be arbitrary. If it were, the constant $\psi_0$ would change whenever the choice of units was changed. In effect, the drag force on the sphere would appear to depend on the choice of the units of measurement, which is impossible. To see this let's suspend the law of covariance for a moment and imagine that the drag relationship (2.9) is

$$0 = D - (\mu + \rho + U + r), \tag{2.11a}$$

or in terms of dimensions

$$0 = \frac{ML}{T^2} - \left(\frac{M}{LT} + \frac{M}{L^3} + \frac{L}{T} + L\right). \tag{2.11b}$$

If we were to change the units of mass from kilograms to grams, then $\mu$ and $\rho$ would both be larger by a factor of $10^3$ while $U$ and $r$ stayed the same. This would increase the term in parentheses in (2.11), but not by this factor. But $D$ also increases by a factor of a thousand; thus the equality (2.11) cannot be maintained when the units are changed. The various terms of the drag relationship (2.11) do not vary together (they do not *covary*) as the units of mass are changed and (2.11) cannot possibly describe the drag on a sphere.

The only way to avoid this problem is to require that the general drag relationship (2.9) satisfies the principle of covariance. Accordingly, (2.9) must be invariant under a three-parameter dilation group

$$\tilde{M} = e^m M, \qquad \tilde{L} = e^l L, \qquad \tilde{T} = e^t T, \tag{2.12}$$

where the group parameters $m$, $l$, and $t$ are arbitrary real numbers. This invariance requirement severely restricts the function $\Psi_0$ and suggests that one can learn something important by searching for a proper invariant form of the drag relationship. We will proceed in steps. Begin by scaling the units of mass using the following one-parameter group:

$$\tilde{M} = e^m M, \qquad \tilde{L} = L, \qquad \tilde{T} = T. \tag{2.13}$$

The effect of this scaling on the variables of the problem is to transform them as follows:

$$\tilde{D} = e^m D, \qquad \tilde{\mu} = e^m \mu, \qquad \tilde{\rho} = e^m \rho, \qquad \tilde{U} = U, \qquad \tilde{r} = r. \tag{2.14}$$

The drag relation (2.9) is required to be independent of the group parameter $m$ and therefore must be of the form

$$\psi_0 = \Psi_1\left[\frac{D}{\rho}, \frac{\rho}{\mu}, U, r\right] \tag{2.15}$$

or something equivalent. That is, (2.15) is not unique. For example, we could have picked $D/\mu$, $\rho/\mu$, $U$, $r$ as the new independent variables. Either choice is invariant under (2.13). We shall return to this point in a moment. The dimensions of the variables remaining in (2.15) are

$$\frac{\hat{D}}{\hat{\rho}} = \frac{L^4}{T^2}, \qquad \frac{\hat{\rho}}{\hat{\mu}} = \frac{T}{L^2}, \qquad \hat{U} = \frac{L}{T}, \qquad \hat{r} = L. \tag{2.16}$$

Now let the units of length be scaled according to

$$\tilde{L} = e^l L, \qquad \tilde{T} = T. \tag{2.17}$$

The effect of this group on the variables in (2.15) is

$$\frac{\tilde{D}}{\tilde{\rho}} = e^{4l}\frac{D}{\rho}, \qquad \frac{\tilde{\rho}}{\tilde{\mu}} = e^{-2l}\frac{\rho}{\mu}, \qquad \tilde{U} = e^l U, \qquad \tilde{r} = e^l r. \tag{2.18}$$

By the principle of covariance, the drag relation (2.15) must be independent of $l$, and a functional form that accomplishes this is

$$\psi_0 = \Psi_3\left[\frac{D}{\rho U^2 r^2}, \frac{\rho U^2}{\mu}, \frac{r}{U}\right]. \tag{2.19}$$

The dimensions of the variables in (2.19) are

$$\frac{\hat{D}}{\hat{\rho}\hat{U}^2\hat{r}^2} = 1, \qquad \frac{\hat{\rho}\hat{U}^2}{\hat{\mu}} = \frac{1}{T}, \qquad \frac{\hat{r}}{\hat{U}} = T. \tag{2.20}$$

Finally, scale the units of time:

$$\tilde{T} = e^t T. \tag{2.21}$$

The effect of this group on the variables in (2.19) is as follows:

$$\frac{\tilde{D}}{\tilde{\rho}\tilde{U}^2\tilde{r}^2} = \frac{D}{\rho U^2 r^2}, \qquad \frac{\tilde{\rho}\tilde{U}^2}{\tilde{\mu}} = e^{-t}\frac{\rho U^2}{\mu}, \qquad \frac{\tilde{r}}{\tilde{U}} = e^{t}\frac{r}{U}. \tag{2.22}$$

The drag relation (2.19) must be independent of $t$, and this leads finally to the dimensionless form

$$\psi_0 = \Psi[C_D, Re] \tag{2.23}$$

where

$$C_D = \frac{D}{\frac{1}{2}\rho U^2(\pi r^2)}, \qquad Re = \frac{\rho U(2r)}{\mu}. \tag{2.24}$$

The first dimensionless variable has the usual interpretation of a drag coefficient. The constants $\frac{1}{2}$ and $\pi$ have been inserted to bring the definition into line with accepted usage where the drag is normalized by the free-stream dynamic pressure and the frontal area of the body. The second dimensionless variable is the Reynolds number, commonly defined in terms of the sphere diameter. The final result (2.23) is invariant under the three-parameter group (2.12), and covariance is satisfied. The experimentally determined relationship (2.23) will be discussed in the next section.

In a way, (2.23) is a remarkable achievement. The drag has been found to depend on only one quantity, not four – a tremendous reduction. Furthermore we were able to reach this simple relationship without ever having to consider the equations of motion for the flow over a sphere. This does not mean we did not do any physics – there is a significant amount of physics in the identification of the relevant variables. In this respect dimensional analysis is deceptively simple. In fact it requires a deep physical understanding of the problem being addressed, and this necessitates an understanding of the governing equations.

It is interesting to seek a further simplification of the problem by considering possible limiting behavior of (2.23). To illustrate this idea we will make use of the well-known exact solution for the drag on a sphere in the limit of small Reynolds number,

$$C_D = \frac{24}{Re}. \tag{2.25}$$

If we restore the dimensioned variables in (2.25) and solve for the drag, the result is

$$\frac{D}{\mu U r} = 6\pi. \tag{2.26}$$

At very low Reynolds number the drag on a sphere is independent of the density of the surrounding fluid – a completely unexpected result and one that could not be determined without knowing the solution (2.25). This amazing result explains a variety of phenomena. It tells us why the atmosphere of Mars, with a surface pressure less than one percent of that of Earth, can support planet-wide dust storms that may take several months to settle out. The density of the atmosphere hardly matters at all; the settling speed of small dust particles is determined almost entirely by the viscosity of the Mars atmosphere, which is 96% cold carbon dioxide at about 200° K. Mariner 9 encountered such a storm when it arrived at the red planet in 1972. At first this was thought to be a major dissapointment, since the surface of the planet was totally obscured, but the optical scattering data obtained over the weeks and months as the dust settled continue to be analyzed today and will remain for a long while as one of the most important sources of data on the composition of Mars [2.2], [2.3].

It is perfectly reasonable to try to extend this result to flow over a circular cylinder, where the drag per unit length has the dimensions

$$\hat{D}_{\text{cylinder}} = \frac{M}{T^2} \tag{2.27}$$

and the drag coefficient is

$$C_{D_{\text{cylinder}}} = \frac{D_{\text{cylinder}}}{\frac{1}{2}\rho U^2 (2r)}. \tag{2.28}$$

The circle in Figure 2.3 is now interpreted as a cylinder extending to infinity. Dimensional analysis leads to a result identical to (2.23), and logic would suggest that, perhaps, in the limit of very small Reynolds number the flow over a cylinder is governed by an equation similar in form to (2.25). Let

$$C_{D_{\text{cylinder}}} = \frac{\psi}{Re}. \tag{2.29}$$

If we restore the dimensioned variables in this relation, the result is

$$\frac{D_{\text{cylinder}}}{\mu U} = \psi, \tag{2.30}$$

which says that the drag on a cylinder is independent of its radius. In this case dimensional reasoning plus a little bit of experience has led us down a garden path to a nonsensical and completely incorrect result. In fact the drag coefficient of a circular cylinder at low Reynolds number depends on $Re \log[C/Re]$. See Reference [2.13].

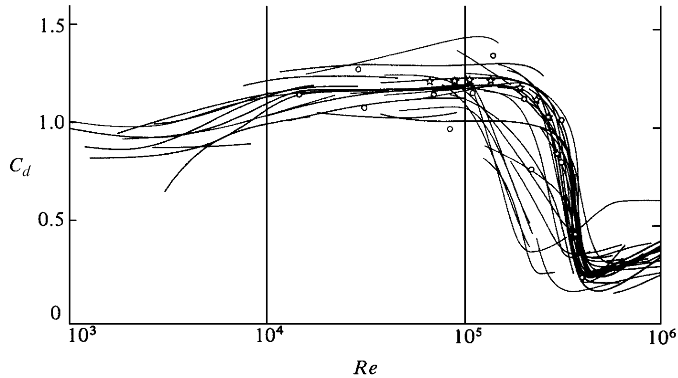


Fig. 2.4. Experimental measurements of the drag on a circular cylinder from [2.1].

### *2.3.1 Some Further Physical Considerations*

Even when dimensional analysis succeeds in producing a physically reasonable result, that result is usually limited in very important ways. Figure 2.4, from Reference [2.1], shows measurements of drag on a circular cylinder versus Reynolds number taken by a variety of investigators. According to (2.23) there should be a single curve of $C_D$ versus $Re$. But one can't help but be struck by the wide variation from one experiment to another depicted in Figure 2.4. Is our analysis wrong?

No, not really, within the confines of the physical variables identified in (2.9). However, it is pretty obvious that important variables have been ignored. The drag on a circular cylinder or a sphere is sensitive to many things. The main dependence is on the Reynolds number, which is successfully identified using dimensional analysis. But in addition, the drag depends on a whole variety of velocity scales and length scales, including surface roughness (measured in terms of a roughness height), the level of freestream turbulent velocity fluctuations, the length scale of turbulent eddies in the freestream, the size of the wind tunnel, the speed of sound (if $U$ is not small enough), etc. A more complete dimensionless description of the problem would be of the form

$$\psi = \Psi\left[\frac{D}{\rho U^2 r^2}, \frac{\rho U r}{\mu}, \frac{\upsilon_1}{U}, \frac{\upsilon_2}{U}, \ldots, \frac{\lambda_1}{r}, \frac{\lambda_2}{r}, \ldots\right], \tag{2.31}$$

where $\upsilon_1, \upsilon_2, \ldots, \lambda_1, \lambda_2, \ldots$ are the neglected velocity and length scales of the problem.

The point of all this is that when we formulated the original problem an implicit assumption was made that these quantities are either infinitesimally small or infinitely large and a finite limit of (2.31) exists as any one of $\upsilon_1, \upsilon_1, \ldots,$

$\lambda_1, \lambda_2 \ldots$ goes to zero or infinity. That is, we assumed that when these variables are asymptotically small or large they have a small effect *and* that the remaining variables provide an adequate description of the physical system. The experience we gained from the Kepler problem, where the limit of (2.8) as $m/M$ and $e$ went to zero was $4\pi^2$, would suggest that such an assumption is justified.

But the lesson of Figure 2.4 is that not all problems are as clean as the Kepler problem. In fact, fluid dynamics presents a wide variety of problems where such an assumption is a close call at best and has to be examined through experiment in each case. The drag law at low Reynolds number, (2.25), is another case in point. Obviously, a finite limit at zero Reynolds number in this relationship does not exist. Only by renormalizing the drag in the form of (2.26) can a finite limit be realized. For an extensive treatment of this issue the reader is referred to the text by Barenblatt [2.4].

This example resonates again with a key point made several times previously. A real physical system in all of its detail is devoid of perfect symmetries. We live in a universe of broken symmetries. In a sense, our mathematical physics, constructed around equations with perfect symmetry and methods that can incorporate only relatively idealized boundary conditions, simply isn't up to the task of fully describing real phenomena in all detail. Nevertheless, by incorporating as exact symmetries those approximate symmetries that play a key role in the phenomena being described, remarkably accurate models of the physical world can be developed. The identification of such symmetries is one of the main objectives of scientific inquiry.

As a final example we consider what happens to the sphere drag problem when the speed of the flow is large and the effects of compressibility are incorporated.

## 2.4 The Drag on a Sphere in High-Speed Gas Flow

Figure 2.5 shows the flow when the speed of the sphere exceeds the speed of sound in the surrounding medium. In this case the pressure disturbance

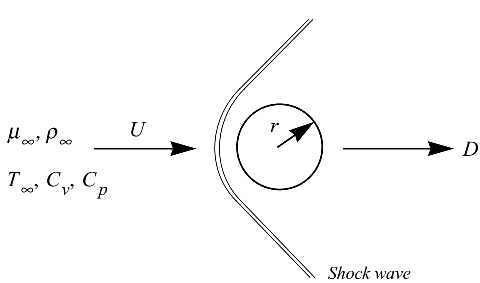


Fig. 2.5. High-speed flow past a sphere.

produced by the sphere is unable to propagate upstream to infinity. In effect, the sphere continually overtakes its own sound field, and the result is a shock wave standing in front of the sphere. Figure 2.5 is intended to illustrate the flow at supersonic speeds. However it is well to recognize that compressibility effects due to the acceleration of the flow about the sphere begin to come into play at subsonic speeds, somewhat below the speed at which shock formation occurs. Moreover, shock waves form on the sphere at freestream speeds well below the speed of sound. We have the same relevant variables that we had before, including the drag force $D$, the fluid density $\rho_\infty$, the viscosity $\mu_\infty$, the freestream flow velocity $U$, and the radius of the sphere, $r$.

At low speed, where the flow is nearly incompressible, the effect of the sphere motion on the internal energy of the fluid is extremely small and mainly confined to slight heating by viscous friction. At high speed, the motion of the sphere can substantially change the internal energy of the gas owing to its compressibility. The kinetic energy of the sphere ratioed to the thermal energy of the surrounding gas becomes an important measure of the degree to which the internal energy of the gas can be changed by the motion of the sphere. Moreover, this ratio is correlated with the strength, shape, and position of the shock and therefore the drag on the sphere. This brings into play the gas temperature and the heat capacities at constant pressure and volume, indicated in Figure 2.5 as additional dimensioned variables governing the drag. Note that the temperature and density of the gas vary throughout the flow, necessitating the use of subscripts to denote freestream values. The dimensions of the relevant variables are

$$\hat{T}_\infty = \Theta, \qquad \hat{C}_p = \frac{M^2}{L^2\Theta}, \qquad \hat{C}_v = \frac{M^2}{L^2\Theta}. \tag{2.32}$$

Now we have one additional fundamental dimension, temperature, and three additional parameters, two of which have the same units. Note that the dimensions of the heat capacities, speed$^2$/temperature, reflect the argument just made comparing the kinetic energy of the motion with the thermal energy of the gas. When we carry through the systematic procedure used in the incompressible case, the result is two additional dimensionless variables:

$$\Pi_1 = \frac{U^2}{C_v T_\infty}, \qquad \Pi_2 = \frac{C_p}{C_v}. \tag{2.33}$$

Note that $C_v T_\infty$ is the internal energy per unit mass of the freestream gas. Finally our drag relation is

$$\psi = \Psi[C_D, Re, M_\infty, \gamma] \tag{2.34}$$

where $\gamma = C_p/C_v$ and $\Pi_1$ is replaced by the usual form of the Mach number,

$$M_\infty = \frac{U}{a_\infty}, \tag{2.35}$$

where the speed of sound is given by

$$a_\infty^2 = \gamma R T_\infty. \tag{2.36}$$

The quantity $R$ is the universal gas constant divided by the molecular weight of the gas ($R = R_u/M_w$), which obeys the ideal gas law $p = \rho R T$. Note that $R$ is related to the heat capacities by $R = C_p - C_v$. Without loss of generality we can write

$$C_D = F[Re, M_\infty, \gamma]. \tag{2.37}$$

Miller and Bailey [2.5] studied the available experimental data for the drag on spheres over a wide range of Reynolds and Mach numbers in air. Interestingly, the most accurate high-Reynolds-number data for Mach numbers between 0.6 and 2.0 turned out to be the 19th-century cannonball measurements by Francis Bashforth [2.6], who was professor of applied mathematics at the Royal Military Academy at Woolwich (near Greenwich), England. In 1947 the Academy was consolidated with the Royal Military Academy at Sandhurst. The Royal Artillery Barracks on Woolwich Common, where many famous British military figures were trained, is now the home of the Royal Artillery Museum.

Bashforth's technique was to measure the successive times when the projectile passed through a series of ten wire screens spaced 150 feet apart and electrically connected to a chronograph consisting of a pair of pens writing on a paper-covered, rotating drum. As the projectile passed through each screen, the current to the chronograph was interrupted, providing a position–time history from which Bashforth could infer the velocity and deceleration of the cannonball. This information could then be used to compile an extensive set of data for the drag coefficient, Mach number, and Reynolds number of spheres. Figure 2.6 (Figure 2 in Miller and Bailey) presents the data of Bashforth, which show the rapid rise in the drag coefficient of a 7.4-cm-diameter sphere through the transonic Mach-number regime.

Figure 2.7 shows their complete compilation of data at various Reynolds numbers and Mach numbers. The most interesting feature of the data in Figure 2.7 is the tendency for the drag coefficient to become essentially independent of Reynolds number for $M_\infty > 1.5$. In this regime, wave drag dominates viscous

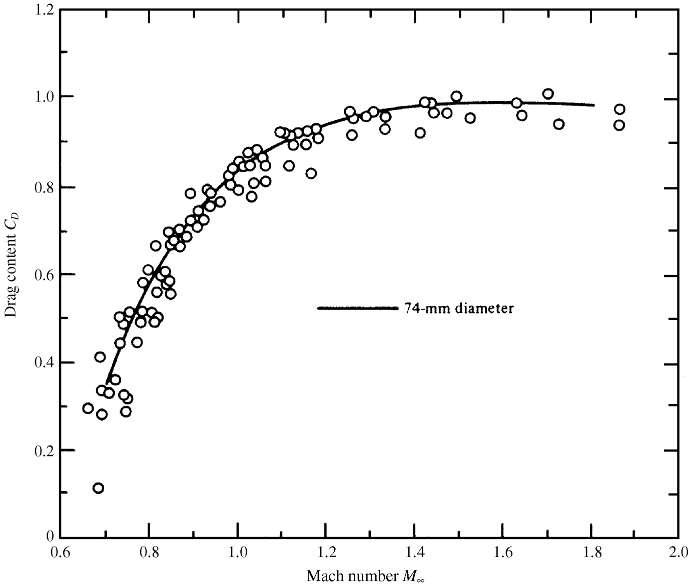


Fig. 2.6. Bashforth's drag data for a 7.4-cm-diameter cannonball from [2.5].

drag. In fact the data suggest that as the Mach number is increased, the drag coefficient approaches

$$C_D \approx 1, \tag{2.38}$$

although there is a slight but systematic decrease above $M_\infty = 2.0$. At low Mach number the drag coefficient shows no sign of reaching an asymptotic value up to the highest Reynolds number measured.

The sphere drag problem beautifully illustrates many of the features of dimensional analysis applied to different parameter regimes. At extremely low Mach number the drag is independent of the fluid density and speed of sound. At high subsonic Mach numbers the drag becomes almost independent of fluid viscosity and at supersonic Mach numbers the drag appears to be almost independent of both the viscosity and speed of sound.

## 2.5 Buckingham's Pi Theorem – The Dimensional-Analysis Algorithm

Finally, let's take a moment to formally state the systematic procedure for generating dimensionless variables. This is one way of stating the well-known Buckingham Pi theorem (Bridgman [2.17]).

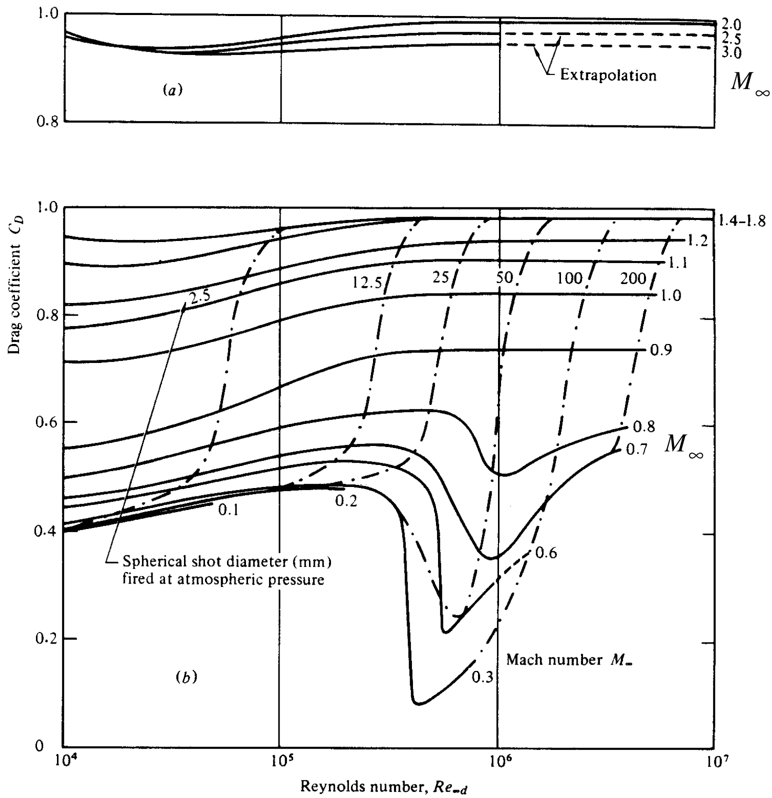


Fig. 2.7. Compilation of sphere drag as a function of Mach number and Reynolds number from Miller and Bailey [2.5].

Dimensional analysis makes use of a simple, purely algorithmic procedure that is extremely general and can be applied to practically any physical problem. The various steps are as follows.

(1) Identify the physical variables relevant to the problem $(a_1, a_2, \ldots, a_\alpha)$.
(2) Determine the fundamental dimensions of each physical variable. The total number of dimensions is $(d_1, d_2, \ldots, d_\beta)\,(\beta \leq \alpha)$. Each variable is a power monomial function of its dimensions,

$$\hat{a}_i = d_1^{k_1} d_2^{k_2} \cdots d_\beta^{k_\beta}. \tag{2.39}$$

where $k_1, k_2, \ldots, k_\beta$ are usually but not always integers.
(3) *Buckingham's Pi Theorem* – A relationship between physical variables $\psi = f[a_1, a_2, \ldots, a_\alpha]$ must be expressible in a form that is invariant under

a $\beta$-parameter dilation group applied to the fundamental dimensions:

$$\tilde{d}_1 = e^{\delta_1} d_1, \qquad \tilde{d}_2 = e^{\delta_2} d_2, \ldots, \qquad \tilde{d}_\beta = e^{\delta_\beta} d_\beta. \tag{2.40}$$

(4) The algorithm for accomplishing step 3 is to apply a one-parameter dilation group to each dimension in succession. New variables are created at each step, which are independent of the dimension being varied. This process is continued until all the dimensions are exhausted. In the final result, the physical problem can only depend on dimensionless variables via a function of the form $\psi = \Psi[\Pi_1, \Pi_2, \ldots, \Pi_\gamma]$. Usually $\gamma = \alpha - \beta$. Occasionally the dimensions of the variables are such that two or more dimensions may be eliminated in a single step. In this case the number of dimensionless variables is larger than $\alpha - \beta$. See Exercise 2.9 for an example. This notion can be quantified by forming the $\beta \times \alpha$ matrix of exponents of the dimensions of the physical variables. The actual count of dimensionless variables is $\alpha$ minus the rank of this matrix. If the rank is less than $\beta$ then two or more dimensions can be combined.

Step 4 is a purely algorithmic process that always leads to a set of dimensionless combinations of the physical variables. The only problem is that any product of these dimensionless variables is also dimensionless, and so the reduced set is not unique and therefore not always recognizable in traditional terms. Changing the order in which various dimensions are subjected to dilation will change the form of the final variables. For example, in the case of sphere drag described above we could have wound up with

$$\phi = \Phi[C_D\, Re, Re] = \Phi\left[\frac{D}{\mu U r}, \frac{\rho U r}{\mu}\right] \tag{2.41}$$

as an equivalent dimensionless form of the drag equation. Note that in this renormalized form the drag law has a finite limit as the Reynolds number goes to zero:

$$\lim_{Re \to 0} \Phi\left[\frac{D}{\mu U r}, \frac{\rho U r}{\mu}\right] = 6\pi. \tag{2.42}$$

The success or failure of dimensional analysis depends entirely on step 1, the choice of the dimensioned physical variables relevant to the problem. This constitutes the art of dimensional analysis. Applied intelligently with a deep knowledge of the problem, very important and profound results can be obtained. Applied blindly, dimensional analysis can easily lead to nonsense.

## 2.6 Concluding Remarks

Although I have tended to emphasize the limitations of dimensional analysis, this should be balanced by the recognition of the great simplification achieved in converting from dimensioned to dimensionless variables. Sphere drag is a great example because, in spite of the fact that the equations governing the flow are perfectly well known and have been for over a century, we are still very far from having an adequate theory for the viscous flow past a sphere. For example, we have no idea of the asymptotic value of $C_D$ as $Re$ approaches infinity at fixed Mach number or as $M_\infty$ approaches infinity at fixed Reynolds number. Nevertheless, dimensional analysis is able to reduce the number of variables in the problem from eight to four – a tremendous accomplishment. Without this all-important tool to organize the experimental data *and our thinking*, rational scientific inquiry into this problem and many others would be utterly impossible.

## 2.7 Exercises

2.1 Under the influence of surface tension, a liquid rises to a height $H$ in a glass tube of diameter $D$ (Figure 2.8). How does $H$ depend on the parameters of the problem?

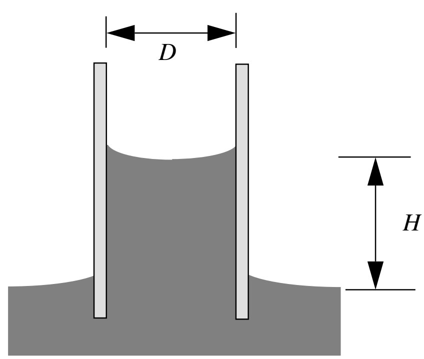


Fig. 2.8.

2.2 Estimate the time of oscillation of a small drop of liquid under its own surface tension.

2.3 When a drop of water strikes a surface at sufficiently low speed, surface tension keeps it round, so it makes a circular spot. As the impact speed is increased, dynamic forces overcome the smoothing effect of surface tension, and the drop becomes unstable and forms a spiky shape as shown in Figure 2.9. (Thanks to Milton Van Dyke for this problem [2.8].) How does the speed at which the impact becomes unstable depend on the properties of the drop? Retain only the essential properties, so that your result involves only a single unknown constant that could be determined from an experiment. Thus you may wish to assume that viscosity is negligible, the properties of the surrounding air are unimportant, etc. See if your result makes sense. For example, does the critical speed depend on the surface tension in the way you would expect?

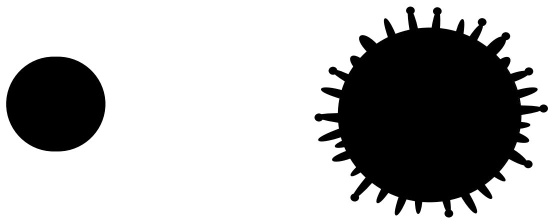

Fig. 2.9.

2.4 Estimate the velocity of fall of a small heavy sphere in a viscous fluid of lower density than the sphere under the influence of gravity. Compare your result with the exact solution. How long does it take the sphere to reach its terminal velocity when dropped from rest?

2.5 Liquid in an open container flows through a long horizontal pipe into a second container as shown in Figure 2.10. How does the time for the liquid level to reach equilibrium depend on the parameters of the problem?

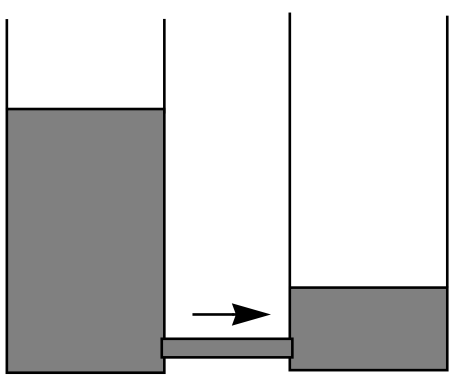

Fig. 2.10.

2.6 Use dimensional analysis to find how the rowing speed depends on the number of oarsmen for racing shells. This problem is discussed by McMahon [2.9] and Barenblatt [2.4]. Use the following asssumptions.

Table 2.2. *Rowing times for 1-, 2-, 4-, and 8-man shells from three previous Olympics. The distance traveled in each case is 2000 m.*

| | Time (s) | | | |
|---|---|---|---|---|
| Olympics | 1 oarsman | 2 oarsmen | 4 oarsmen | 8 oarsmen |
| Atlanta | 404.85 | 376.98 | 356.93 | 342.74 |
| Barcelona | 411.40 | 377.32 | 355.04 | 329.53 |
| Seoul | 409.86 | 381.13 | 363.11 | — |

(i) The boats are geometrically similar.
(ii) The boat weight $W$ per oarsman is constant.
(iii) Each oarsman contributes the same power, $P$.
(iv) The only hindering force is skin friction, and the friction coefficient is constant over the wetted area. The friction coefficient is defined as $c_f = \tau_{\text{wall}}/(\frac{1}{2}\rho U^2)$, where $\tau_{\text{wall}}$ is the wall shear stress.

*Hint.* Find how the volume of the displaced water varies with the number of oarsman and the length of the boat. Equate the expenditure of energy on skin friction to the power supplied by the oarsman. Data for men's rowing over a 2-km course from three recent Olympic summer games are presented in Table 2.2. Plot the data in logarithmic coordinates and compare with your prediction. Notice that in the context of this problem the number of oarsmen is a fundamental dimension.

2.7 Critique the assumptions in Exercise 2.6 – particularly (i), which seems to suggest that the shells get wider as they get longer to accommodate more rowers.

(i) How does the problem work out if the width of the shell is assumed to be constant?
(ii) Suppose the drag is primarily due to the generation of waves and skin friction can be neglected. How will the speed depend on the number of oarsman? Do these results shake your confidence in the solution developed in Exercise 2.6?
(iii) Work the case where the race is carried out by fleas on a lake of honey.

2.8 What is the speed of the wave in a row of falling dominos on a table? Add whatever simplifying assumptions you feel are reasonable, such as perfectly rigid dominos, constant coefficient of friction between the dominos and the table, etc. This problem is the subject of a pair of journal papers by Stronge [2.10] and Stronge and Shu [2.11] as well as a note in the *SIAM Review* Problems and Solutions. The problem was proposed by Daykin [2.12] and solved by McLachlan et al. [2.13].

2.9 Show that if two equal-size elastic spheres are pressed together, the radius of the circle of contact varies as the one-third power of the force between them. How does it vary with the radius of the spheres?

2.10 One of the well-known observations in blood flow is that the viscous shear stress at the wall of an artery is approximately independent of the diameter of the artery. Consider a bifurcation where the flow in one large artery splits into two smaller adjoining arteries of equal size. How are the diameters of the smaller arteries related to the diameter of the large artery?

2.11 Use dimensional analysis to deduce how the weight a man can lift depends on his own weight. Assume that the strength of a muscle varies as its cross-sectional area. See if your result correlates the data in Table 2.3, taken from the 1969 *World Almanac* for the 1968 Senior National AAU weightlifting championships. How much did the heavyweight lifter weigh?

2.12 There is continuing interest in pushing measurements of circular cylinder drag to the highest possible Reynolds numbers. One scheme that has been proposed is to tow a submerged, high-aspect-ratio cylinder behind two nuclear-powered aircraft carriers pulling lines attached to each end of the cylinder. The kinematic viscosity of water is small, the cylinder diameter can be made quite large, and thus high Reynolds numbers ought to be achievable. Assuming only cylinders of a given aspect ratio, say $L/r = 60$, are used, how does the required towing force vary with the Reynolds

Table 2.3. *Total weight lifted for different classes.*

| Class | Body weight (pounds) | Lifted weight (pounds) |
|---|---|---|
| Bantam | 123.5 | 740 |
| Featherweight | 132.25 | 795 |
| Lightweight | 148.75 | 820 |
| Light-heavy | 181.75 | 1025 |
| Middle-heavy | 198.25 | 1055 |
| Heavyweight | ? | 1280 |

number based on cylinder diameter? What force would be required to reach a Reynolds number that exceeds the highest available data ($Re = 10^8$, $C_d = 0.6$)? The maximum towing force available is about $10^8$ N.

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# 3

# *Systems of ODEs and First-Order PDEs – State-Space Analysis*

Before we move on to the main discussion of Lie groups, it is useful to review some basic concepts from the theory of differential equations. A Lie group is always associated with an autonomous set of characteristic ODEs, the integrals of which become new variables in the simplification of physical problems. The purpose of this chapter is to provide the basic analytical tools needed to solve such systems and to review several of the fields where they arise. State-space analysis is introduced in this chapter and used extensively throughout the text to provide a geometrical interpretation of the solutions that we study.

## 3.1 Autonomous Systems of ODEs in the Plane

Let's examine the following autonomous system of ordinary differential equations in two dimensions:

$$\frac{dx}{ds} = \xi[x, y], \qquad \frac{dy}{ds} = \eta[x, y]. \tag{3.1}$$

Solution trajectories of this system are determined by integrating the coupled nonlinear right-hand sides:

$$x = \tilde{x} + \int_{s_0}^{s} \xi[x[\hat{s}], y[\hat{s}]]\, d\hat{s}, \qquad y = \tilde{y} + \int_{s_0}^{s} \eta[x[\hat{s}], y[\hat{s}]]\, d\hat{s}. \tag{3.2}$$

The result is two parametric functions for $x$ and $y$ in terms of the parameter $s$ along a solution trajectory

$$x = F[\tilde{x}, \tilde{y}, s], \qquad y = G[\tilde{x}, \tilde{y}, s], \tag{3.3}$$

where the initial conditions coincide with the initial value of $s$:

$$\tilde{x} = F[\tilde{x}, \tilde{y}, s_0], \qquad \tilde{y} = G[\tilde{x}, \tilde{y}, s_0]. \tag{3.4}$$

In general $s_0$ can be taken to be 0. The term *autonomous* (time independent) used in reference to (3.1) refers to the fact that the functions $\xi[x, y]$ and $\eta[x, y]$ do not depend on the independent variable $s$. The implication of this is that the pattern defined by the vector field $(\xi, \eta)$ is frozen whereas the coordinates of a particle move under the action of this vector field when increasing values of $s$ are inserted into (3.3).

Recalling the discussion in Chapter 1, the parametric functions (3.3) are recognized as being of the same form as a Lie point transformation group.

## 3.2 Characteristics

The solution of (3.1) can also be expressed as a family of curves in $(x, y)$ called *characteristics*. This is accomplished by eliminating $s$ between the functions $F$ and $G$ in (3.3), with the result

$$\psi = \Psi[x, y]. \tag{3.5}$$

The value of a particular characteristic is determined by the initial values $(\tilde{x}, \tilde{y})$:

$$\tilde{\psi} = \Psi[\tilde{x}, \tilde{y}]. \tag{3.6}$$

This situation is depicted schematically in Figure 3.1.

Much more complicated patterns than that depicted in Figure 3.1 are possible, even common, but for now we will take this figure as illustrative of a typical situation. If we take the total differential of $\Psi[x, y]$,

$$d\psi = \frac{\partial \Psi}{\partial x} dx + \frac{\partial \Psi}{\partial y} dy, \tag{3.7}$$

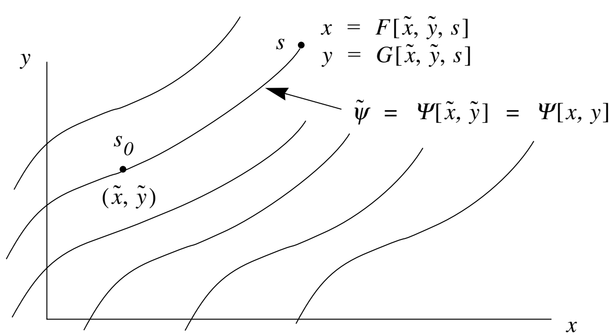


Fig. 3.1. A typical family of characteristic curves.

and then use (3.1) to replace the differentials $dx$ and $dy$ in (3.7), the result is

$$d\Psi = \left( \xi[x, y] \frac{\partial \Psi}{\partial x} + \eta[x, y] \frac{\partial \Psi}{\partial y} \right) ds. \tag{3.8}$$

On a line of constant $\psi = \tilde{\psi}$ the differential is zero, $d\tilde{\psi} = 0$. For nonzero $ds$, (3.8) can only be zero if the expression in parentheses is zero. This condition must be satisfied by $\Psi[x, y]$ if it is to correspond to the family of characteristic curves of $F$ and $G$. In summary, the family of characteristics, $\Psi[x, y]$, can be determined in two ways: either as the solution of a linear first-order PDE

$$\xi[x, y] \frac{\partial \Psi}{\partial x} + \eta[x, y] \frac{\partial \Psi}{\partial y} = 0, \tag{3.9}$$

or by elimination of the parameter $s$ between the parametric functions (3.3) that define the solution trajectories of (3.1).

## 3.3 First-Order Ordinary Differential Equations

The family $\psi = \Psi[x, y]$ is also the set of solution curves of the first order ODE

$$\frac{dy}{dx} = \frac{\eta[x, y]}{\xi[x, y]} \tag{3.10}$$

gotten by dividing $dy/ds$ by $dx/ds$ in (3.1). Equation (3.10) is called the *characteristic equation* of (3.9) and is often written in the form

$$\frac{dx}{\xi[x, y]} = \frac{dy}{\eta[x, y]}. \tag{3.11}$$

The reason for arranging (3.10) in the form (3.11) will become apparent later when we discuss higher-dimensional problems. The correspondence between solutions of the PDE (3.9) and the characteristic ODE (3.11) will play a very important role in our later development of Lie theory.

### *3.3.1 Perfect Differentials*

If we write (3.10) in the suggestive form

$$\eta[x, y]\, dx - \xi[x, y]\, dy = 0, \tag{3.12}$$

there is a temptation to regard (3.12) as a perfect differential (i.e., equal to a total differential $d\psi$) and to try to determine $\Psi[x, y]$ by *quadrature*: integration of the first term in (3.12) with $y$ held fixed or integration of the second term with $x$ held fixed. This is usually incorrect, since in general the integrability condition

$$\frac{\partial \eta}{\partial y} = -\frac{\partial \xi}{\partial x} \tag{3.13}$$

is not satisfied. In other words,

$$\eta[x, y] \neq \frac{\partial \Psi}{\partial x}, \qquad \xi[x, y] \neq -\frac{\partial \Psi}{\partial y}. \tag{3.14}$$

The linear differential form

$$f = \eta[x, y]\, dx - \xi[x, y]\, dy \tag{3.15}$$

(linear in the differentials) is called a *Pfaffian* form after the German mathematician Johann Friedrich Pfaff (1765–1825), who proposed the first method of integrating first-order partial differential equations along the general lines described above. In the language of differential geometry (3.12) is also called a *differential 1-form*. A 1-form defined on an $n$-dimensional differentiable manifold is

$$f = \alpha_j dx^j, \qquad j = 1, \ldots, n, \tag{3.16}$$

where the $\alpha_j$ are the components of a covariant vector field. The exterior derivative of (3.16) is a 2-form,

$$df = \frac{\partial \alpha_j}{\partial x^i}(dx^i \wedge dx^j) \tag{3.17}$$

where the wedge product follows the simple rules

$$\begin{aligned} dx^i \wedge dx^j &= -dx^j \wedge dx^i \qquad (i \neq j), \\ dx^i \wedge dx^j &= 0 \qquad\qquad\qquad (i = j). \end{aligned} \tag{3.18}$$

Using these rules, the 2-form (3.17) can also be written

$$df = -\frac{\partial \alpha_i}{\partial x^j}(dx^i \wedge dx^j) \tag{3.19}$$

Combine (3.17) and (3.19) to form

$$df = \frac{1}{2}\left(\frac{\partial \alpha_j}{\partial x^i} - \frac{\partial \alpha_i}{\partial x^j}\right)(dx^i \wedge dx^j), \tag{3.20}$$

which shows that the coefficients are the components of the $n$-dimensional curl of the vector $\alpha_j$.

For a coordinate transformation in three dimensions, $\tilde{x}^j = F^j[\boldsymbol{x}]$, $j = 1, \ldots, 3$, the wedge product provides the correct formula for the transformation of a differential volume under the coordinate change. Let the $\tilde{x}^j$ be Cartesian coordinates. The differential volume is a parallelepiped of sides $d\tilde{x}^1$, $d\tilde{x}^2$, and $d\tilde{x}^3$:

$$\begin{aligned} dV &= d\tilde{x}^1(d\tilde{x}^2 \wedge d\tilde{x}^3) = \varepsilon_{ijk}\frac{\partial F^i}{\partial x^1}dx^1\frac{\partial F^j}{\partial x^2}dx^2\frac{\partial F^k}{\partial x^3}dx^3 \\ &= J(dx^1 \wedge dx^2 \wedge dx^3), \end{aligned} \tag{3.21}$$

where $\varepsilon_{ijk}$ is the alternating unit tensor (zero if any two indices are equal, one if $i, j, k$ are $1, 2, 3,\ 3, 1, 2$, or $2, 3, 1$, minus one if $i, j, k$ are $1, 3, 2,\ 2, 1, 3$, or $3, 2, 1$). The factor $J$ is the Jacobian of the transformation,

$$J = \frac{\partial(F^1, F^2, F^3)}{\partial(x^1, x^2, x^3)}. \tag{3.22}$$

Although we will not use the language of exterior differential forms in our later development of the theory of Lie groups, one can do so, and the connection is well described by Stephani in [3.1].

### 3.3.2 The Integrating Factor – Pfaff's Theorem

Although (3.12) is usually not a perfect differential, the solution of the system of ODEs (3.1) does exist, and this implies that the vector field defined by the slopes

$$\frac{dy}{dx} = \frac{\eta[x, y]}{\xi[x, y]} \tag{3.23}$$

and that defined by

$$\frac{dy}{dx} = -\frac{\partial\Psi/\partial x}{\partial\Psi/\partial y} \tag{3.24}$$

are identical up to a scalar multiplying factor in the magnitude of the displacement vector along the characteristics. The magnitude of the displacement vector is

$$\nabla\Psi \cdot \nabla\Psi = M^2(\xi^2 + \eta^2) \tag{3.25}$$

where $M$ is some function of position. In other words, the flow *patterns* generated by (3.23) and (3.24) are the same, although the local flow *speeds* are not.

This implies that the partial derivatives in (3.24) and the functions in (3.23) must have a common multiplying factor:

$$\begin{aligned} -\partial\Psi/\partial x &= M[x, y]\eta[x, y], \\ \partial\Psi/\partial y &= M[x, y]\xi[x, y]. \end{aligned} \tag{3.26}$$

This result is called *Pfaff's theorem.*

***Theorem 3.1.*** *An integrating factor $M[x, y]$ always exists that can be used to convert the differential 1-form in two variables, $-\eta[x, y]\,dx + \xi[x, y]\,dy$, to a perfect differential,*

$$d\psi = \frac{\partial\Psi}{\partial x}\,dx + \frac{\partial\Psi}{\partial y}\,dy = -M\eta\,dx + M\xi\,dy, \tag{3.27}$$

*where* $(M\xi, M\eta) = (\partial\Psi/\partial y, -\partial\Psi/\partial x)$.

In fluid mechanics $\psi$ is called the *stream function*. Note that the existence of the integrating factor is only guaranteed in two dimensions. In three or higher dimensions, auxiliary conditions must be satisfied by the appropriate differential 1-form (see section 3.6.2). If the vector field $(M\xi, M\eta)$ happens to be irrotational $[\partial(M\eta)/\partial x - \partial(M\xi)/\partial y = 0]$, then the field can alternatively be described by a scalar potential,

$$d\phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy = M\xi\,dx + M\eta\,dy, \tag{3.28}$$

where $(M\xi, M\eta) = (\partial\Phi/\partial x, \partial\Phi/\partial y)$ with the same integrating factor. In fluid mechanics $\phi$ is called the velocity potential.The stream function and velocity potential satisfy the Cauchy–Riemann conditions

$$\frac{\partial\Phi}{\partial x} = \frac{\partial\Psi}{\partial y}, \qquad \frac{\partial\Phi}{\partial y} = -\frac{\partial\Psi}{\partial x}, \tag{3.29}$$

enabling the powerful tools of complex variables to be brought to bear on the analysis of 2-D irrotational fields.

### *3.3.3 Nonsolvability of the Integrating Factor*

The integrating factor required to turn (3.12) into a perfect differential is usually not known, and, unfortunately, there is no systematic way to determine it short of solving for a particular solution of the original equation. It is instructive to see why this is so. Suppose

$$d\psi = -M\eta\, dx + M\xi\, dy \tag{3.30}$$

is the total differential of a function $\Psi[x, y]$. The function $M[x, y]$ must satisfy the integrability condition,

$$\frac{\partial}{\partial y}(M\eta) = -\frac{\partial}{\partial x}(M\xi). \tag{3.31}$$

This can be expanded and rearranged to produce a first-order PDE for $M[x, y]$:

$$\left(\frac{\xi}{\xi_x + \eta_y}\right)\frac{\partial M}{\partial x} + \left(\frac{\eta}{\xi_x + \eta_y}\right)\frac{\partial M}{\partial y} = -M. \tag{3.32}$$

Without loss of generality, let $M[x, y] = \exp(-\Omega[x, y])$. Then (3.32) becomes the following equation for $\Omega[x, y]$:

$$\left(\frac{\xi}{\xi_x + \eta_y}\right)\frac{\partial \Omega}{\partial x} + \left(\frac{\eta}{\xi_x + \eta_y}\right)\frac{\partial \Omega}{\partial y} = 1. \tag{3.33}$$

The family of solution characteristics of this PDE is a function of three variables,

$$\phi = \Phi[x, y, \omega] = \omega - \Omega[x, y], \tag{3.34}$$

with total differential

$$d\phi = \frac{\partial \Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy + \frac{\partial \Phi}{\partial \omega}d\omega \tag{3.35}$$

and corresponding autonomous system of characteristic equations

$$\frac{dx}{ds} = \frac{\xi}{\xi_x + \eta_y}, \qquad \frac{dy}{ds} = \frac{\eta}{\xi_x + \eta_y}, \qquad \frac{d\omega}{ds} = 1. \tag{3.36}$$

To check, substitute (3.34) into (3.35) and replace differentials using (3.36). This will reproduce equation (3.33). In summary, the solution of (3.33) can be determined by solving the characteristic equations

$$\frac{\xi_x + \eta_y}{\xi}\, dx = \frac{\xi_x + \eta_y}{\eta}\, dy = \frac{d\omega}{1}. \tag{3.37}$$

In order to solve the first equality in (3.37) we have to find a particular solution of the equation $dy/dx = \eta/\xi$. But this is the equation, (3.10), that we set out to solve in the first place!

Thus the integrating factor is almost as far out of reach as the solution of the original equation. Actually things aren't quite that bad. Normally, when one is presented with an equation of the form (3.10), it is the general solution of the equation that is being sought, whereas the integrating factor requires only a particular solution of (3.10) plus a solution of one of the remaining equalities in (3.37) involving $\omega$. Nevertheless, this is still an unsolved problem in general. Perhaps it's just as well. If the integrating factor could always be determined systematically, then, in principle, all nonlinear first-order ODEs would be solvable, this branch of mathematics would be regarded as a closed subject, and we would be forced to go find another line of work. Later we shall see that, while group theory doesn't solve this problem, it does provide a useful strategy for searching for the integrating factor. Often, for the reasons just discussed, the integrating factor is a simpler function than the solution of (3.10), and so a trial-and-error search procedure makes some sense.

### *3.3.4 Examples of Integrating Factors*

***Example 3.1 (Integrating factor for a linear first-order ODE).*** Almost anyone reading this text will have seen this example before. Consider the equation

$$\frac{dy}{dx} = -g[x]y + f[x], \tag{3.38}$$

which can be written as

$$(g[x]y - f[x])\,dx + dy = 0. \tag{3.39}$$

If we multiply (3.39) by

$$e^{\int g dx}, \tag{3.40}$$

the result is a perfect differential

$$dF = e^{\int g dx}(g[x]y - f[x])\,dx + e^{\int g dx}\,dy, \tag{3.41}$$

which can be easily checked by the cross-derivative test. The solution, determined by quadrature, is

$$F[x, y] = ye^{\int^x g(\hat{x})\,d\hat{x}} - \int^x f[\hat{x}]\left(e^{\int^x g[x']\,dx'}\right) d\hat{x} = C. \tag{3.42}$$

Note that (3.38) is also satisfied by any function $\Omega[F]$. The obvious question at

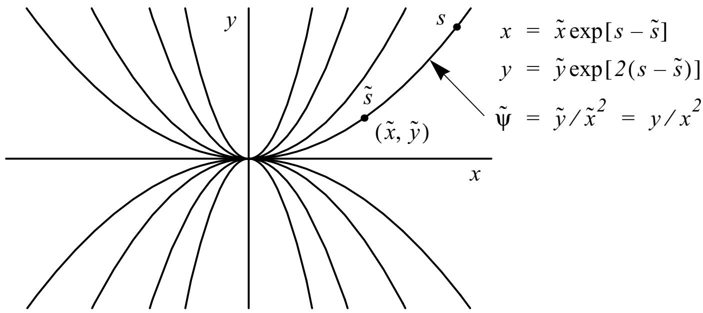


Fig. 3.2. A family of parabolas.

this point is, where does the integrating factor (3.40) come from? The answer will be the main topic of Chapter 6.

***Example 3.2 (A family of parabolas).*** Let $\Psi[x, y]$ be the family of parabolas passing through the origin given by

$$\psi = \Psi[x, y] = y/x^2 \tag{3.43}$$

and illustrated in Figure 3.2. This example will be carried through rather laboriously in order to illustrate as clearly and completely as possible the principles described in the previous section – principles that will come up repeatedly throughout this text.

The partial derivatives of (3.43) are

$$\frac{\partial \Psi}{\partial x} = -\frac{2y}{x^3}, \qquad \frac{\partial \Psi}{\partial y} = \frac{1}{x^2}, \tag{3.44}$$

and the differential of $\Psi$ is

$$d\psi = -\frac{2y}{x^3}\, dx + \frac{1}{x^2}\, dy. \tag{3.45}$$

Comparing cross derivatives of the coefficients in (3.45) confirms, by the usual test, that (3.45) is a perfect differential.

*Method 1 (Quadrature)* We can recover the original family (3.43) from equation (3.45) by quadrature. Integrate the first term in (3.45) with $y$ held fixed:

$$\psi = \Psi[x, y] = -\int \frac{2y}{x^3}\, dx + g[y] = \frac{y}{x^2} + g[y]. \tag{3.46}$$

Taking the partial derivative of (3.46) with respect to $y$ and comparing the result with the coefficient of the second term in (3.45), we see that $dg/dy = 0$. Thus the family (3.43) is recovered up to a constant of integration. Since the constant can be incorporated as a shift in the value of $\psi$, we have recovered the original family of parabolas (3.43).

*Method 2 (Elimination of the parameter along characteristics)* Let's see if we can recover (3.43) in a different way. Along a particular curve $\tilde{\psi} = y/x^2$ defined by initial values $(\tilde{x}, \tilde{y})$, the relationship between the differentials of $x$ and $y$ is

$$0 = -\frac{2y}{x^3}\,dx + \frac{1}{x^2}\,dy. \tag{3.47}$$

Multiply (3.47) by $x^3$:

$$0 = -2y\,dx + x\,dy. \tag{3.48}$$

The result (3.48) can be written as a single first-order ODE:

$$\frac{dy}{dx} = \frac{2y}{x}. \tag{3.49}$$

Equation (3.49) can be separated into two first-order ODEs by introducing a dummy parameter $s$ along solution curves. So a system equivalent to (3.49), in the sense that it governs the same family of curves, is

$$\frac{dy}{ds} = 2y, \qquad \frac{dx}{ds} = x. \tag{3.50}$$

This system is separable, and we can integrate each equation directly:

$$\begin{aligned} x &= F[\tilde{x}, \tilde{y}, s] = \tilde{x}\exp(s - s_0), \\ y &= G[\tilde{x}, \tilde{y}, s] = \tilde{y}\exp[2(s - s_0)]. \end{aligned} \tag{3.51}$$

If we eliminate $s$ between the two parametric equations for $x$ and $y$ in (3.51), the original family of parabolas $\tilde{\psi} = y/x^2$ is recovered as the exact solution of (3.49).

*Erroneous assumption of a perfect differential* O.K., now let's try a third way. Return to equation (3.48) and, for the moment, suspend disbelief and imagine it to be a perfect differential. Let $d\theta = -2y\,dx + x\,dy$. If we attempt to integrate by quadrature as we did equation (3.45), the result, integrating the first term with $y$ held fixed, is $\theta = \Theta[x, y] = -2yx + h[y]$. Differentiating this expression with respect to $y$ and equating the result to the second coefficient in the differential (3.48) leads to the conclusion that $h[y]$ must depend on $x$, which is impossible. Our erroneous assumption has led to an inconsistency. Integration by quadrature became impossible when we multiplied (3.47) by $x^3$, removing the integrating factor needed to ensure a perfect differential.

As pointed out above, when one is presented with a first-order ODE such as (3.10), the integrating factor is usually not known. Furthermore there is no systematic method for finding one. However, it turns out that if a symmetry property of equation (3.10) can be identified, then this can be exploited to construct an integrating factor, which can then be used to convert (3.12) to a perfect differential, leading directly to the general solution of the equation in the form of a quadrature. This point is the main topic of Chapter 6. For now we continue our review of ODEs.

## 3.4 Thermodynamics – The Legendre Transformation

Consider the piston–cylinder combination shown in Figure 3.3. The cylinder contains some undetermined material substance. An infinitesimal amount of heat, $\delta Q$, is added to the system, causing an infinitesimal amount of work, $P\ dV$, to be done by the system on the surroundings and an infinitesimal change in internal energy, $dE$. This balance of energy is stated in the form of the first law of thermodynamics,

$$\delta Q = dE + P\,dV. \tag{3.52}$$

The differential work done by the system is the conventional mechanical work done by a force acting over a distance,

$$P\,dV = \frac{F}{A}\,d(Ax) = F\,dx, \tag{3.53}$$

where $A$ is the cross-sectional area and $F$ is the total force on the piston.

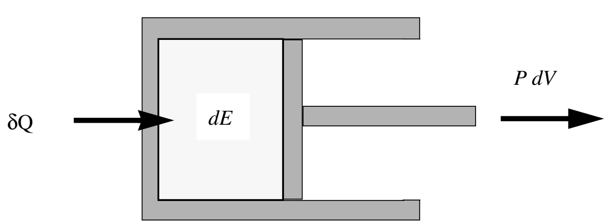


Fig. 3.3. Piston–cylinder combination used to model a thermodynamic system.

It is often convenient to work in terms of intensive variables by dividing through by the mass contained in the cylinder. The first law is then

$$\delta q = de + P dv, \tag{3.54}$$

where $e$ is internal energy per unit mass and $v = 1/\rho$ is the volume per unit mass.

Thermodynamics is only useful if we can determine an equation of state for the substance contained in the cylinder. The equation of state is a functional relationship between the internal energy, specific volume, and pressure $P[e, v]$. Assuming an equation of state can be defined, the first law becomes

$$\delta q = de + P[e, v]\, dv \tag{3.55}$$

According to Pfaff's theorem, such a system must have an integrating factor $M[e, v]$ such that the first law becomes an exact differential:

$$M[e, v]\delta q = M[e, v]\, de + M[e, v]\ P[e, v]\, dv = ds[e, v]. \tag{3.56}$$

In effect, once one accepts the first law and the existence of the function $P[e, v]$, then *two new variables of state are implied*: an integrating factor and an associated integral called the *entropy* (per unit mass), $s[e, v]$. Note that the integrating factor is not unique; in particular, there can be an arbitrary constant scale factor, since a constant times $ds$ is still a perfect differential. This enables the integrating factor to be identified with the sensible temperature of the system, $M[e, v] = 1/T[e, v]$. Thus the first law becomes

$$\frac{\delta q}{T[e, v]} = \frac{de}{T[e, v]} + \frac{P[e, v]}{T[e, v]}\, dv = ds[e, v]. \tag{3.57}$$

This is the famous *Gibbs equation*, usually written

$$T\, ds = de + P\, dv. \tag{3.58}$$

This fundamental equation, which Pfaff's theorem tells us is a perfect differential, is the starting point for virtually all applications of thermodynamics. The Gibbs equation describes states that are in local thermodynamic equilibrium, i.e., states that can be reached through a sequence of reversible steps.

It is often useful to rearrange the Gibbs equation so as to exchange dependent and independent variables. This can be accomplished using the so-called *Legendre transformation*. In this approach, a new variable of state is defined, called the *enthalpy*,

$$h = e + Pv. \tag{3.59}$$

In terms of it, the Gibbs equation becomes

$$ds = \frac{dh}{T} - \frac{v}{T}\,dP. \tag{3.60}$$

Using this simple trick, the pressure has been converted from a dependent variable to an independent variable:

$$ds[h,P] = \frac{dh}{T[h,P]} - \frac{v[h,P]}{T[h,P]}\,dP. \tag{3.61}$$

It is relatively easy to reexpress the Gibbs equation with any two variables selected to be independent. This enables any variable of state to be to be determined as a function of any two others. For example,

$$\begin{aligned} e &= \phi[T,P], \qquad & s &= \zeta[T,v], \\ s &= \xi[e,P], \qquad & h &= \varphi[T,p], \end{aligned} \tag{3.62}$$

and so forth.

The Legendre transformation is an example of a *contact transformation*. It will come up again in Chapter 4, Section 4.3, when we discuss the conversion from a Lagrangian to a Hamiltonian formulation in classical dynamics, and then again in Chapter 14, Section 14.1.1, where Lie contact transformations are discussed.

Thermodynamics goes beyond mere conservation of energy and quantifies the distinction between reversible and irreversible processes that the system may undergo. This is expressed by the second law. For *any* change of a system, not just a reversible change

$$\delta q \le T\,ds, \tag{3.63}$$

where $ds$ is the change in entropy per unit mass. For a reversible change,

$$\delta q = T\,ds. \tag{3.64}$$

For a general substance, an accurate equation of state is not a particularly easy thing to come by, and so most applications tend to focus on approximations based on some sort of idealization. One of the simplest cases is the equation of state for an ideal gas,

$$PV = nR_uT, \tag{3.65}$$

where $n$ is the number of moles contained in the cylinder and $R_u$ is the universal gas constant, $R_u = 8314.510$ J/(kmole-K). A consequence of (3.65) is that the enthalpy and internal energy of an ideal gas depend only on the temperature:

$$dh = C_p \, dT, \qquad de = C_v \, dT, \tag{3.66}$$

where $C_p$ and $C_v$ are the heat capacities at constant pressure and volume, respectively, and are relatively weak functions of temperature. This equation of state is derived from a model that assumes that the material in the cylinder is composed of a large number of randomly moving mass points that can exchange momentum during a collision but otherwise do not interact. It is an excellent approximation for real gases over a wide range of conditions.

## 3.5 Incompressible Flow in Two Dimensions

The flow of an incompressible fluid is constrained by the continuity equation

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0, \tag{3.67}$$

where $U$ and $V$ are the velocity components in the $x$ and $y$ directions respectively. Continuity is satisfied identically by the introduction of a stream function $\psi = \Psi[x, y]$:

$$U = \frac{\partial \Psi}{\partial y}, \qquad V = -\frac{\partial \Psi}{\partial x}. \tag{3.68}$$

The equations for the coordinates $(x[t], y[t])$ of a fluid particle in a steady flow are

$$\frac{dx}{dt} = U[x, y], \qquad \frac{dy}{dt} = V[x, y]. \tag{3.69}$$

If we use (3.68) to replace $U$ and $V$ in (3.69), the result is a Hamiltonian system

$$\frac{dx}{dt} = \frac{\partial \Psi}{\partial y}, \qquad \frac{dy}{dt} = -\frac{\partial \Psi}{\partial x}, \tag{3.70}$$

where the stream function $\Psi$ is the Hamiltonian. It is getting ahead of ourselves to mention the Hamiltonian at this point, since it won't be introduced until the next chapter. However, we can suffer the bit of confusion that may be caused, so as to reinforce a point made later that the Hamiltonian formulation of mechanics is particularly well suited to the description of fields.

Note that the associated PDE $U \, \partial\psi/\partial x + V \, \partial\psi/\partial y = 0$ is identically satisfied. If we compare this system with (3.1), (3.10), and (3.12), then it is clear that the stream function introduced in (3.68) satisfies

$$\frac{\partial \Psi}{\partial x}\,dx + \frac{\partial \Psi}{\partial y}\,dy = 0. \tag{3.71}$$

In this case the form of (3.70) guarantees that (3.12) with $\xi = U$ and $\eta = V$ is a perfect differential, and for incompressible flow one can write

$$d\psi = -V\,dx + U\,dy. \tag{3.72}$$

If $U$ and $V$ are known functions, the stream function is determined by quadrature.

## 3.6 Fluid Flow in Three Dimensions – The Dual Stream Function

The trajectory of a fluid particle in a three-dimensional, unsteady flow is governed by the nonautonomous system

$$\frac{dx}{dt} = u[\boldsymbol{x}, t], \qquad \frac{dy}{dt} = v[\boldsymbol{x}, t], \qquad \frac{dz}{dt} = w[\boldsymbol{x}, t]. \tag{3.73}$$

At a given instant in time $t = t_{\text{fixed}}$ the velocity field is frozen and instantaneous streamlines are determined by integrating the autonomous system

$$\frac{dx}{ds} = u[\boldsymbol{x}, t_{\text{fixed}}], \qquad \frac{dy}{ds} = v[\boldsymbol{x}, t_{\text{fixed}}], \qquad \frac{dz}{ds} = w[\boldsymbol{x}, t_{\text{fixed}}], \tag{3.74}$$

where $s$ is a pseudotime along an instantaneous streamline. The solution trajectories of (3.74) are

$$x = f[\tilde{\boldsymbol{x}}, s;\, t_{\text{fixed}}], \qquad y = g[\tilde{\boldsymbol{x}}, s;\, t_{\text{fixed}}], \qquad z = h[\tilde{\boldsymbol{x}}, s;\, t_{\text{fixed}}], \tag{3.75}$$

where $\tilde{\boldsymbol{x}}$ is the initial coordinate of a particle at $s = 0$, $t = t_{\text{fixed}}$. Elimination of $s$ among these three relations leads to *two* infinite families of integral surfaces, the so-called dual stream-function surfaces (see Reference [3.2] by Lagerstrom)

$$\psi^1 = \Psi^1[\boldsymbol{x};\, t_{\text{fixed}}], \qquad \psi^2 = \Psi^2[\boldsymbol{x};\, t_{\text{fixed}}]. \tag{3.76}$$

These functions are integrals of the first-order PDE

$$\boldsymbol{u} \cdot \nabla \Psi^i = u\frac{\partial \Psi^i}{\partial x} + v\frac{\partial \Psi^i}{\partial y} + w\frac{\partial \Psi^i}{\partial z} = 0, \qquad i = 1, 2, \tag{3.77}$$

with characteristic equations

$$\frac{dx}{u[\boldsymbol{x}, t_{\text{fixed}}]} = \frac{dy}{v[\boldsymbol{x}, t_{\text{fixed}}]} = \frac{dz}{w[\boldsymbol{x}, t_{\text{fixed}}]}. \tag{3.78}$$

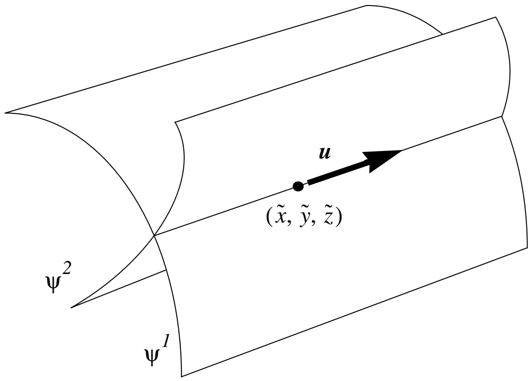


Fig. 3.4. Intersection of dual stream-function surfaces.

Equation (3.77) is derived in the same way as (3.9). Take the total differential of either function in (3.76), and use (3.74) to replace the differentials $dx$, $dy$, and $dz$. A given initial point, $\tilde{\boldsymbol{x}}$, defines two stream surfaces, and the velocity vector through the point lies along the intersection of the surfaces, as shown schematically in Figure 3.4.

Given the dual stream functions, the velocity field can be reconstructed from

$$\boldsymbol{u} = \nabla\Psi^1 \times \nabla\Psi^2. \tag{3.79}$$

### 3.6.1 The Method of Lagrange

The general first-order PDE in three variables of the form

$$U[x, y, z]\frac{\partial z}{\partial x} + V[x, y, z]\frac{\partial z}{\partial y} = W[x, y, z], \tag{3.80}$$

where $z = f[x, y]$, was solved by the great Italian–French mathematician Joseph-Louis Lagrange (1736–1813). One can regard the solution as a surface of constant $\Psi[x, y, z]$. On this surface,

$$\begin{aligned} \frac{D\Psi}{Dx} &= \frac{\partial\Psi}{\partial x} + \frac{\partial\Psi}{\partial z}\frac{\partial z}{\partial x} = 0, \\ \frac{D\Psi}{Dy} &= \frac{\partial\Psi}{\partial y} + \frac{\partial\Psi}{\partial z}\frac{\partial z}{\partial y} = 0. \end{aligned} \tag{3.81}$$

The relations in (3.81) are used to replace $\partial z/\partial x$ and $\partial z/\partial y$ in (3.80) and after some rearrangement,

$$U[x, y, z]\frac{\partial\Psi}{\partial x} + V[x, y, z]\frac{\partial\Psi}{\partial y} + W[x, y, z]\frac{\partial\Psi}{\partial x} = 0, \tag{3.82}$$

which we know from the development in the previous section is solved by solving the characteristic equations,

$$\frac{dx}{U[x, y, z]} = \frac{dy}{V[x, y, z]} = \frac{dz}{W[x, y, z]}. \tag{3.83}$$

Thus any solution of (3.82) is a solution of (3.80), and any solution of (3.80) is a solution of (3.82). The solution trajectories of either equation follow the characteristic equations (3.83). We have already encountered a special case of (3.80) in section 3.3.3, namely, equation (3.32), where $U$ and $V$ depend only on $[x, y]$ and $W = -z$.

Furthermore, we know that there are two independent integrals of (3.83). Let these be $\psi = \Psi[x, y, z]$ and $\phi = \Phi[x, y, z]$. Then the general solution of (3.80) or (3.82) is

$$F[\Psi, \Phi] = \text{constant}. \tag{3.84}$$

An equally general form of the solution of (3.80) is

$$\Phi[x, y, z] = G[\Psi[x, y, z]]. \tag{3.85}$$

Note that if $\Phi$ is a solution of (3.82), then so is any differentiable function $\Omega[\Phi]$. This follows from the fact that

$$\begin{aligned} &U[x, y, z]\frac{\partial\Omega}{\partial x} + V[x, y, z]\frac{\partial\Omega}{\partial y} + W[x, y, z]\frac{\partial\Omega}{\partial x} \\ &= \left(U[x, y, z]\frac{\partial\Phi}{\partial x} + V[x, y, z]\frac{\partial\Phi}{\partial y} + W[x, y, z]\frac{\partial\Phi}{\partial z}\right)\frac{d\Omega}{d\Phi} = 0. \end{aligned} \tag{3.86}$$

***Example 3.3 (Solve $(y+z)\,\partial z/\partial x + (x+z)\partial z/\partial y = x+y$).*** The characteristic equations corresponding to this PDE are

$$\frac{dx}{y+z} = \frac{dy}{x+z} = \frac{dz}{x+y}. \tag{3.87}$$

We need to determine the two integrals of (3.87). These equations can be rewritten in the form

$$\frac{dx+dy+dz}{2x+2y+2z} = \frac{dx-dy}{-(x-y)} = \frac{dx-dz}{-(x-z)}, \tag{3.88}$$

and the two integrals are

$$\begin{aligned} \psi &= \ln[x+y+z] + 2\ \ln[x-y], \\ \phi &= \ln[x-y] - \ln[x-z]. \end{aligned} \tag{3.89}$$

Taking note of (3.86), an equally valid expression of the two integrals is simply

$$\psi = (x + y + z)(x - y)^2,$$
$$\phi = \left(\frac{x - y}{x - z}\right), \tag{3.90}$$

and so the general solution of the given PDE is

$$\frac{x - y}{x - z} = G[(x + y + z)(x - y)^2]. \tag{3.91}$$

where $G$ is arbitrary.

### 3.6.2 The Integrating Factor in Three and Higher Dimensions

Consider the Pfaffian 1-form

$$f = u[x, y, z]\, dx + v[x, y, z]\, dy + w[x, y, z]\, dz. \tag{3.92}$$

Recalling the earlier discussion of integrating factors in two dimensions, we ask: under what circumstances can we say that an integrating factor $M[x, y, z]$ exists such that a perfect differential is defined by

$$d\phi = \frac{\partial \phi}{\partial x}\, dx + \frac{\partial \phi}{\partial y}\, dy + \frac{\partial \phi}{\partial z}\, dz = Mu\, dx + Mv\, dy + Mw\, dz \tag{3.93}$$

From (3.93) we can see that in order for the integrating factor $M$ to exist, the vector $(Mu, Mv, Mw)$ must be the gradient of a scalar,

$$\boldsymbol{g} = \nabla\phi = (Mu, Mv, Mw) = M\boldsymbol{u}, \tag{3.94}$$

i.e., $\phi$ must be a scalar potential function, and the vector field $\boldsymbol{g}$ is irrotational:

$$\nabla \times \boldsymbol{g} = 0, \tag{3.95}$$

or

$$\frac{\partial(Mw)}{\partial y} - \frac{\partial(Mv)}{\partial z} = 0,$$
$$\frac{\partial(Mu)}{\partial z} - \frac{\partial(Mw)}{\partial x} = 0, \tag{3.96}$$
$$\frac{\partial(Mv)}{\partial x} - \frac{\partial(Mu)}{\partial y} = 0.$$

If we expand the derivatives, we have

$$\begin{aligned} w\frac{\partial M}{\partial y} - v\frac{\partial M}{\partial z} + M\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) &= 0,\\ u\frac{\partial M}{\partial z} - w\frac{\partial M}{\partial x} + M\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) &= 0,\\ v\frac{\partial M}{\partial x} - u\frac{\partial M}{\partial y} + M\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) &= 0. \end{aligned} \tag{3.97}$$

Now multiply the first equation by $u$, the second by $v$, and the third by $w$, and add the products. The result is

$$u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + v\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0, \tag{3.98}$$

which is independent of $M$. In vector form, (3.98) and the irrotationality of $\boldsymbol{g}$ produce the result

$$\boldsymbol{g} \cdot (\nabla \times \boldsymbol{g}) = \boldsymbol{Mu} \cdot (\nabla \times (\boldsymbol{Mu})) = M^2(\boldsymbol{u} \cdot (\nabla \times \boldsymbol{u})) = 0. \tag{3.99}$$

In order for an integrating factor to exist for a three-dimensional vector field $\boldsymbol{u}$, it must satisfy the auxiliary condition

$$\boldsymbol{u} \cdot (\nabla \times \boldsymbol{u}) = 0. \tag{3.100}$$

Such a field is called *complex lamellar* (a lamellar field is simply an irrotational field). In fluid mechanics terms, either the velocity field is orthogonal to the vorticity field, or the velocity field is irrotational, in which case $\nabla \times \boldsymbol{u} = 0$. The extension of this result to higher-dimensional spaces is straightforward, and at each level, additional auxiliary conditions are encountered. In general there will be $(n-1)(n-2)/2$ auxiliary conditions for the existence of an integrating factor, where $n$ is the number of dimensions (Kestin [3.4]). Pfaff's theorem guaranteeing the existence of an integrating factor for any two-dimensional field (rotational or irrotational) is a consequence of the fact that for $n = 2$ the number of auxiliary conditions is zero (the vorticity is always orthogonal to the velocity).

### 3.6.3 Incompressible Flow in Three Dimensions

The velocity field of an incompressible flow can be represented as the curl of a vector potential $\boldsymbol{A}$:

$$\boldsymbol{u} = \nabla \times \boldsymbol{A}. \tag{3.101}$$

The representation (3.101) guarantees that the incompressibility condition $\nabla \cdot \boldsymbol{u} = 0$ is satisfied. In two-dimensional flow, only the out-of-plane component of $\boldsymbol{A}$ is nonzero, and it corresponds to the stream function introduced in (3.68) and discussed earlier. If we take the curl of (3.101), use the vector identity

$$\nabla \times \nabla \times \boldsymbol{A} = \nabla(\nabla \cdot \boldsymbol{A}) = \nabla^2 \boldsymbol{A}, \tag{3.102}$$

and assume, without loss of generality, a Coulomb gauge, $\nabla \cdot \boldsymbol{A} = 0$, then the vector potential is related to the vorticity through a Poisson equation

$$\nabla^2 \boldsymbol{A} = -\boldsymbol{\omega}, \tag{3.103}$$

where $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$. The vector potential is easily related to the dual stream functions discussed in the previous section:

$$\boldsymbol{A} = \Psi^1 \nabla \Psi^2 = -\Psi^2 \nabla \Psi^1. \tag{3.104}$$

## 3.7 Nonlinear First-Order PDEs – The Method of Lagrange and Charpit

Lagrange and Charpit developed a method for solving the general, nonlinear, first-order PDE,

$$A[x, y, z, p, q] = 0. \tag{3.105}$$

where, in standard notation,

$$p = \frac{\partial z}{\partial x}, \qquad q = \frac{\partial z}{\partial y}. \tag{3.106}$$

A lucid explanation of the method can be found in the book of Kells [3.3]. The procedure for solving (3.105) is to seek a second equation,

$$B[x, y, z, p, q] = 0 \tag{3.107}$$

which is consistent with (3.105). If such an equation can be found, then it can be used to solve (3.105) and (3.107) for the partial derivatives $p$ and $q$. These may be substituted into

$$dz = p\,dx + q\,dy, \tag{3.108}$$

allowing $z[x, y]$ to be determined by quadrature. Consistency requires that the $p[x, y, z]$ and $q[x, y, z]$ found using this procedure must satisfy the integrability condition

$$\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}. \tag{3.109}$$

Here is how it works. Differentiate $A$ and $B$ partially with respect to $x$ and $y$ to obtain

$$\begin{aligned}
\frac{DA}{Dx} &= \frac{\partial A}{\partial x} + \frac{\partial A}{\partial z}p + \frac{\partial A}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial A}{\partial q}\frac{\partial q}{\partial x} = 0,\\
\frac{DA}{Dy} &= \frac{\partial A}{\partial y} + \frac{\partial A}{\partial z}q + \frac{\partial A}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial A}{\partial q}\frac{\partial q}{\partial y} = 0,\\
\frac{DB}{Dx} &= \frac{\partial B}{\partial x} + \frac{\partial B}{\partial z}p + \frac{\partial B}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial B}{\partial q}\frac{\partial q}{\partial x} = 0,\\
\frac{DB}{Dy} &= \frac{\partial B}{\partial y} + \frac{\partial B}{\partial z}q + \frac{\partial B}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial B}{\partial q}\frac{\partial q}{\partial y} = 0.
\end{aligned} \tag{3.110}$$

Eliminate the three quantities $\partial p/\partial x, \partial q/\partial y$, and $\partial p/\partial y$ (or alternatively, $\partial q/\partial x$) from (3.110). Multiply the first equation by $-\partial B/\partial p$, the second by $-\partial B/\partial q$, the third by $\partial A/\partial p$, and the fourth by $\partial A/\partial q$. Add the four and rearrange to obtain

$$\begin{aligned}
&\left(\frac{\partial A}{\partial p}\right)\frac{\partial B}{\partial x} + \left(\frac{\partial A}{\partial q}\right)\frac{\partial B}{\partial y} + \left(p\frac{\partial A}{\partial p} + q\frac{\partial A}{\partial q}\right)\frac{\partial B}{\partial z}\\
&\qquad -\left(\frac{\partial A}{\partial x} + \frac{\partial A}{\partial z}p\right)\frac{\partial B}{\partial p} - \left(\frac{\partial A}{\partial y} + \frac{\partial A}{\partial z}q\right)\frac{\partial B}{\partial q} = 0.
\end{aligned} \tag{3.111}$$

This is a first-order PDE for the function $B$, which can be found by solving the corresponding characteristic ODEs,

$$\frac{dx}{\partial A/\partial p} = \frac{dy}{\partial A/\partial q} = \frac{dz}{p\frac{\partial A}{\partial p} + q\frac{\partial A}{\partial q}} = \frac{dp}{-\left(\frac{\partial A}{\partial x} + \frac{\partial A}{\partial z}p\right)} = \frac{dq}{-\left(\frac{\partial A}{\partial y} + \frac{\partial A}{\partial z}q\right)}. \tag{3.112}$$

There are four integrals of (3.112), and any one (say the simplest one) involving $p$ and/or $q$ will suffice for the second equation for $B$. The method may not always be successful, since it involves solving at least one first-order ODE for which an integrating factor may not be known.

In general, the solution trajectories of the original PDE, (3.105), follow the characteristic trajectories defined by (3.112). This is particularly easy to see if

$A$ is in the quasilinear form studied in Section 3.6.1. Let

$$A[x, y, z, p, q] = U[x, y, z]p + V[x, y, z]q - W[x, y, z] = 0. \quad (3.113)$$

The system (3.112) becomes

$$\frac{dx}{U} = \frac{dy}{V} = \frac{dp}{-\left(\frac{\partial A}{\partial x} + \frac{\partial A}{\partial z} p\right)} = \frac{dq}{-\left(\frac{\partial A}{\partial y} + \frac{\partial A}{\partial z} q\right)} = \frac{dz}{W}. \quad (3.114)$$

The first two and last expressions in (3.114) do not depend on $p$ or $q$; they are in fact identical to (3.83). In this case, the method of Lagrange and Charpit reduces to the usual system of characteristic ODEs.

***Example 3.4 (Solve $A = z - (\partial z/\partial x)(\partial z/\partial y) = z - pq = 0$).*** The characteristic equations (3.112) are

$$\frac{dx}{q} = \frac{dy}{p} = \frac{dp}{p} = \frac{dq}{q} = \frac{dz}{2pq}. \quad (3.115)$$

The four integrals of this system are

$$\begin{aligned} \psi_1 &= x - q, \\ \psi_2 &= y - p, \\ \psi_3 &= p/q, \\ \psi_4 &= z + 2(x - q)x - (p/q)x^2. \end{aligned} \quad (3.116)$$

If we use $\psi_1$ – that is, we let

$$B(x, y, z, p, q) = x - q = \psi_1, \quad (3.117)$$

and solve for $p$ and $q$ – the result is

$$p = \frac{z}{x - \psi_1}, \qquad q = x - \psi_1, \quad (3.118)$$

and the total differential of $z$ is

$$dz = \left(\frac{z}{x - \psi_1}\right) dx + (x - \psi_1)\, dy. \quad (3.119)$$

The integral of (3.119) is

$$\psi_5 = y - \frac{z}{x - \psi_1}, \quad (3.120)$$

which can be written in the form

$$z - (x - \psi_1)(y - \psi_5) = 0, \tag{3.121}$$

where $\psi_1$ and $\psi_5$ are arbitrary constants. The solution (3.121) is a family of hyperbolae centered at $(x, y) = (\psi_1, \psi_5)$.

The solution of (3.105) expressed in the form

$$F[x, y, z, \psi_1, \psi_2] = 0, \tag{3.122}$$

where $\psi_1$ and $\psi_2$ are constants of integration, is called the *complete* solution of the PDE.

### 3.7.1 The General and Singular Solutions

It is possible to obtain another solution of (3.105) by considering $\psi_1$ and $\psi_2$ to be functions of $x$ and $y$. This leads to a condition by which one of the constants is eliminated. If $\psi_1$ and $\psi_2$ are truly constant, then on $F$,

$$\begin{aligned} \frac{DF}{Dx} &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p = 0, \\ \frac{DF}{Dy} &= \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} q = 0, \end{aligned} \tag{3.123}$$

whereas if $\psi_1$ and $\psi_2$ are treated as functions of $x$ and $y$, then on $F$,

$$\begin{aligned} \frac{DF}{Dx} &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p + \frac{\partial F}{\partial \psi_1}\frac{\partial \psi_1}{\partial x} + \frac{\partial F}{\partial \psi_2}\frac{\partial \psi_2}{\partial x} = 0, \\ \frac{DF}{Dy} &= \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} q + \frac{\partial F}{\partial \psi_1}\frac{\partial \psi_1}{\partial y} + \frac{\partial F}{\partial \psi_2}\frac{\partial \psi_2}{\partial y} = 0. \end{aligned} \tag{3.124}$$

Now, (3.124) reduces to (3.123) if and only if

$$\begin{aligned} \frac{\partial F}{\partial \psi_1}\frac{\partial \psi_1}{\partial x} + \frac{\partial F}{\partial \psi_2}\frac{\partial \psi_2}{\partial x} &= 0, \\ \frac{\partial F}{\partial \psi_1}\frac{\partial \psi_1}{\partial y} + \frac{\partial F}{\partial \psi_2}\frac{\partial \psi_2}{\partial y} &= 0. \end{aligned} \tag{3.125}$$

The condition for (3.125) to be consistent is

$$\begin{vmatrix} \dfrac{\partial \psi_1}{\partial x} & \dfrac{\partial \psi_2}{\partial x} \\ \dfrac{\partial \psi_1}{\partial y} & \dfrac{\partial \psi_2}{\partial y} \end{vmatrix} = 0. \tag{3.126}$$

This is identically satisfied if $\psi_2$ is a function of $\psi_1$:

$$\psi_2 = f[\psi_1], \tag{3.127}$$

where $f$ is an arbitrary once-differentiable function. If (3.127) holds, then (3.125) becomes

$$\frac{\partial F}{\partial \psi_1} + \frac{\partial F}{\partial \psi_2}\frac{df}{d\psi_1} = 0. \tag{3.128}$$

If we eliminate $\psi_2$ from (3.122) using (3.127), the result is the *general* solution,

$$F_1[x, y, z, \psi_1] = 0, \tag{3.129}$$

where $\psi_1$ is permitted to be a function of $(x, y)$ defined by (3.128). This solution is called general because it involves an arbitrary function $f[\psi_1]$. Note that the general solution is independent of the complete solution.

***Example 3.5 (Find the general solution of $z - pq = 0$).*** The complete solution was worked out in Example 3.4:

$$F[x, y, z, \psi_1, \psi_2] = z - (x - \psi_1)(y - \psi_2) = 0, \tag{3.130}$$

where $\psi_1$ and $\psi_2$ are arbitrary constants. The general solution is

$$F_1[x, y, z, \psi_1, \psi_2] = z - (x - \psi_1)(y - f[\psi_1]) = 0, \tag{3.131}$$

where $\psi_1[x, y]$ is defined by

$$(y - f[\psi_1]) + (x - \psi_1)\frac{df}{d\psi_1} = 0. \tag{3.132}$$

A function that satisfies (3.132) is

$$f = \psi_1 = \left(\frac{x + y}{2}\right), \tag{3.133}$$

which leads to the exact solution,

$$z = -\left(\frac{x-y}{2}\right)^2, \tag{3.134}$$

quite different from the complete solution. Another function that satisfies (3.132) is

$$f = \frac{1}{\psi_1}, \qquad \psi_1 = \sqrt{\frac{x}{y}}, \tag{3.135}$$

which produces the solution

$$z = (\sqrt{xy} - 1)^2. \tag{3.136}$$

Equation (3.125) is also satisfied if

$$\frac{\partial F}{\partial \psi_1} = 0, \qquad \frac{\partial F}{\partial \psi_2} = 0. \tag{3.137}$$

If $\psi_1$ and $\psi_2$ can be eliminated between (3.122) and (3.137), the resulting function is the envelope of the family of surfaces represented by (3.122). The equation of this envelope is called the *singular* solution.

***Example 3.6 (Find the singular solution of $z - pq = 0$).*** The complete solution was worked out in Example 3.4:

$$F[x, y, z, \psi_1, \psi_2] = z - (x - \psi_1)(y - \psi_2) = 0, \tag{3.138}$$

where $\psi_1$ and $\psi_2$ are arbitrary constants. The singular solution is found from

$$\begin{aligned} x - \psi_1 &= 0, \\ y - \psi_2 &= 0, \\ z - (x - \psi_1)(y - \psi_2) &= 0. \end{aligned} \tag{3.139}$$

The plane $z = 0$ is the singular solution.

## 3.8 Characteristics in *n* Dimensions

The extension to $n$ dimensions of the theory of characteristics is very straightforward. We are concerned with solving autonomous systems of ordinary differential equations of the form

$$\frac{dx^j}{ds} = \xi^j[\boldsymbol{x}[s]], \qquad j = 1, \ldots, n. \tag{3.140}$$

As before, the solution of (3.140) is written as an integral of the coupled, nonlinear right-hand side,

$$x^j = \tilde{x}^j + \int_{s_0}^{s} \xi^j[\boldsymbol{x}[s]]\, ds. \tag{3.141}$$

Ultimately the result is expressed as a set of parametric functions of $s$ and the initial condition $\tilde{\boldsymbol{x}}$:

$$x^j = F^j[\tilde{\boldsymbol{x}}, s], \qquad j = 1, \ldots, n. \tag{3.142}$$

The system (3.140) generates a vector field in the space $\boldsymbol{x}$, and the solution (3.142) is the trajectory of a particle moving under the action of that field. Note that we can easily map out the family of solution trajectories of (3.140) graphically by simply plotting the vector field (cf. Figure 3.1), although visualizing the field becomes impossible once the number of dimensions exceeds three.

The vector field generated by (3.140) can be expressed in terms of characteristic surfaces in $\boldsymbol{x}$. This is accomplished by combining the parametric functions (3.142) so as to eliminate the parameter $s$. The result is $n-1$ solution surfaces,

$$\psi^i = \Psi^i[\boldsymbol{x}], \qquad i = 1, \ldots, n-1. \tag{3.143}$$

The characteristic surfaces (3.143) are expressed in a form that is intended to distinguish between $\Psi^i$, which is the name of a function (a particular arrangement of variables), and the specific value $\psi^i$ that defines a solution surface. In fact, each function $\Psi^i$ defines an infinite family corresponding to the range of $\psi^i$. Known initial conditions, $\tilde{\boldsymbol{x}}$, determine the values of the $n-1$ surfaces, $\tilde{\psi}^i = \Psi^i[\tilde{x}]$.

It is fair to ask, in what sense does (3.143) represent the solution of the original system (3.140)? Clearly, in moving from (3.142) to (3.143) a certain amount of information is lost, since (3.143) tells us nothing about the value of the parameter $s$ at any point on a solution trajectory.

The differential of (3.143) is

$$d\psi^i = \frac{\partial \Psi^i}{\partial x^j}\, dx^j, \qquad i = 1, \ldots, n-1 \quad (\text{sum over } j = 1, \ldots, n). \tag{3.144}$$

As was explained in Chapter 1, throughout the text it will be understood that, unless otherwise noted, the usual Einstein convention on the summation over repeated indices is used and the extra notation to this effect will be dropped.

Using (3.140) to replace the differentials on the right-hand side of (3.144) yields

$$d\psi^i = \left(\xi^j[\boldsymbol{x}]\frac{\partial \Psi^i}{\partial x^j}\right) ds, \qquad i = 1, \ldots, n-1. \tag{3.145}$$

In order for the function $\Psi^i$ to represent a characteristic solution surface of (3.140), it must satisfy the first-order PDE

$$\xi^j[\boldsymbol{x}]\frac{\partial \Psi^i}{\partial x^j} = 0, \qquad i = 1, \ldots, n-1. \tag{3.146}$$

If (3.146) is satisfied, then, during a small interval $ds$, as a particle moves along a solution trajectory under the action of the vector field (3.140), the particle will remain confined to a set of $n-1$ surfaces of fixed $\Psi_j$. In fact, the trajectory (3.142) of the particle coincides with the curve of *intersection* of the $n-1$ surfaces, whose gradients are all orthogonal to the vector field (3.140), as is clearly seen from the form of (3.146) ($\boldsymbol{\xi}\cdot\nabla\Psi^i = 0$). This is the basis of equation (3.79).

The upshot of all this is that, for any system of $n$ ODEs (3.140), there is an associated $n$-dimensional, first-order, linear PDE (3.146) with $n-1$ integral surfaces whose intersections are the solution trajectories of the original system of ODEs. Conversely, the integral surfaces of any $n$-dimensional, first-order PDE can be expressed in terms of the solution trajectories of an associated system of $n-1$ characteristic ODEs:

$$\frac{dx^1}{\xi^1[\boldsymbol{x}]} = \frac{dx^2}{\xi^2[\boldsymbol{x}]} = \frac{dx^3}{\xi^3[\boldsymbol{x}]} = \cdots = \frac{dx^n}{\xi^n[\boldsymbol{x}]}. \tag{3.147}$$

This correspondence is of crucial importance to our later study of Lie groups.

There is one additional very important point, which was noted in Section 3.6.1 but should be stated again in a general way.

***Theorem 3.2.*** *If* $\Psi$ *is a solution of* $\xi^j\,\partial\Psi/\partial x^j = 0$, *then any bounded, differentiable function* $\Omega[\Psi]$ *is also a solution. This is easily shown by direct substitution*:

$$\xi^j[\boldsymbol{x}]\frac{\partial}{\partial x^j}(\Omega[\Psi]) = \xi^j[\boldsymbol{x}]\frac{\partial \Psi}{\partial x^j}\left(\frac{d\Omega}{d\Psi}\right) = 0. \tag{3.148}$$

*More generally, if* $\Psi^1, \ldots, \Psi^{n-1}$ *are the* $n-1$ *integral surfaces of (3.146), then any bounded differentiable function*

$$\omega = \Omega[\Psi^1, \ldots, \Psi^{n-1}] \tag{3.149}$$

*is also a solution. This follows from the linearity of the operator $\xi^j\ \partial/\partial x^j$, which breaks up into a sum of sums:*

$$\xi^j[\boldsymbol{x}]\frac{\partial\Psi^1}{\partial x^j}\left(\frac{d\Omega}{d\Psi^1}\right)+\xi^j[\boldsymbol{x}]\frac{\partial\Psi^2}{\partial x^j}\left(\frac{d\Omega}{d\Psi^2}\right)+\cdots+\xi^j[\boldsymbol{x}]\frac{\partial\Psi^{n-1}}{\partial x^j}\left(\frac{d\Omega}{d\Psi^{n-1}}\right)=0, \tag{3.150}$$

*each of which is zero. In this respect there is always a certain degree of arbitrariness in the expression of the solution(s) of (3.146) and the system of characteristic equations (3.147).*

One final point: the same procedure developed by Lagrange and described in Section 3.6.1 can be used to show that $\omega = \Omega[\Psi^1, \ldots, \Psi^{n-1}]$ is the general solution of

$$\xi^1[x^1,\ldots,x^{n-1},z]\frac{\partial z}{\partial x^1}+\xi^2[x^1,\ldots,x^{n-1},z]\frac{\partial z}{\partial x^2}+\cdots + \xi^{n-1}[x^1,\ldots,x^{n-1},z]\frac{\partial z}{\partial x^{n-1}}=\xi^n[x^1,\ldots,x^{n-1},z], \tag{3.151}$$

where $(\Psi^1, \ldots, \Psi^{n-1})$ are the independent integrals of

$$\frac{dx^1}{\xi^1[x^1,\ldots,x^{n-1},z]}=\frac{dx^2}{\xi^2[x^1,\ldots,x^{n-1},z]}=\cdots =\frac{dx^{n-1}}{\xi^{n-1}[x^1,\ldots,x^{n-1},z]}=\frac{dz}{\xi^n[x^1,\ldots,x^{n-1},z]}. \tag{3.152}$$

### *3.8.1 Nonlinear First-Order PDEs in n Dimensions*

The method of Lagrange and Charpit described in Section 3.7 can be generalized to $n$-dimensional first-order PDEs in one dependent variable,

$$A[x^1,\ldots,x^n,z,p_1,\ldots,p_n]=0, \tag{3.153}$$

where

$$p_i=\frac{\partial z}{\partial x_i}. \tag{3.154}$$

It should be clear by now that *what sets first-order nonlinear PDEs of the form (3.153) apart from higher-order equations and from systems of equations is* *that the solution is equivalent to the solution of a system of ODEs*. The perfect differential of the solution is

$$dz = p_i\, dx^i. \tag{3.155}$$

We seek a set of equations

$$B^k[x^1, \ldots, x^n, z, p_1, \ldots, p_n] = 0, \qquad k = 1, \ldots, n-1, \tag{3.156}$$

that is consistent with (3.153). If such can be found, then (3.153) and (3.156) can be solved for the partial derivatives $p_i$. These may be substituted into (3.155), allowing $z[\boldsymbol{x}]$ to be determined by quadrature. Differentiate $A$ and $B$ partially with respect to $x^i$ to obtain

$$\begin{aligned} \frac{DA}{Dx^i} &= \frac{\partial A}{\partial x^i} + \frac{\partial A}{\partial z} p_i + \frac{\partial A}{\partial p_i}\frac{\partial p_i}{\partial x^i} = 0, \\ \frac{DB}{Dx^i} &= \frac{\partial B}{\partial x^i} + \frac{\partial B}{\partial z} p_i + \frac{\partial B}{\partial p_i}\frac{\partial p_i}{\partial x^i} = 0. \end{aligned} \tag{3.157}$$

The elimination and multiplication procedure described for two variables in Section 3.7 is used to obtain a first-order PDE for the sought-after equation $B$,

$$\left(\frac{\partial A}{\partial p_i}\right)\frac{\partial B}{\partial x^i} - \left(\frac{\partial A}{\partial x^i} + \frac{\partial A}{\partial z} p_i\right)\frac{\partial B}{\partial p_i} + \left(p_i \frac{\partial A}{\partial p_i}\right)\frac{\partial B}{\partial z} = 0, \tag{3.158}$$

with the corresponding characteristic ODEs,

$$\begin{aligned} &\frac{dx^1}{\partial A/\partial p_1} = \cdots = \frac{dx^n}{\partial A/\partial p_n} = \frac{dz}{p_i(\partial A/\partial p_i)} = \\ &\frac{dp_1}{-\left(\frac{\partial A}{\partial x^1} + \frac{\partial A}{\partial z} p_1\right)} = \cdots = \frac{dp_n}{-\left(\frac{\partial A}{\partial x^n} + \frac{\partial A}{\partial z} p_n\right)}. \end{aligned} \tag{3.159}$$

There are $2n$ integrals of (3.159), and $n-1$ are used together with (3.153) to solve for the partial derivatives of $z[x^1, \ldots, x^n]$. These results will be used in Chapter 4, Section 4.4 [cf. Equation (4.64)] when we consider classical dynamics and the Hamilton–Jacobi equation.

## 3.9 State-Space Analysis in Two and Three Dimensions

In an earlier section it was noted that a great deal about the family of solution paths of a system of the form of (3.140) can be learned by plotting the vector field defined by the functions $\xi^i[\boldsymbol{x}]$. The resulting diagram is called the *phase portrait* of the system. This terminology originates in the application of the technique to a second-order ODE describing a two-degree-of-freedom dynamical system. In canonical form the ODE is equivalent to a pair of first-order ODEs where one dependent variable is the time derivative of the other. In fluid mechanics, the phase portrait is the velocity vector field, and in the following discussion it will occasionally be convenient to call the phase portrait a *flow*. We shall come to some examples shortly.

Graphically solving (3.140) in this way has many applications, especially in the context of two-point boundary value problems, where the solution can often be identified as a particular trajectory in the phase portrait. Quite often, the topography of the phase portrait will suggest an appropriate strategy for integrating the equations. The techniques introduced in this section will be used throughout the text.

### 3.9.1 Critical Points

A key feature of the state-space method is that the qualitative features of the phase portrait, and hence the solution of (3.140), can be almost completely described once the critical points of (3.140) have been identified and classified. Critical points occur where

$$\xi^j[\boldsymbol{x}_c] = 0. \tag{3.160}$$

### 3.9.2 Matrix Invariants

If the $\xi^j[\boldsymbol{x}]$ are analytic functions of $\boldsymbol{x}$, the system (3.140) can be expanded in a Taylor series about the critical point, and the result can be used to gain valuable information about the geometry of the solution. Retaining just the lowest-order term in the expansion of $\xi^j[\boldsymbol{x}]$, the result is

$$\frac{dx^j}{ds} = A^j_k\left(x^k - x^k_c\right) + O\left(\left(x^k - x^k_c\right)^2\right) + \cdots, \tag{3.161}$$

where $A^j_k$ is the gradient tensor of the vector field $\xi^j[\boldsymbol{x}]$, evaluated at the critical point, and $\boldsymbol{x}_c$ is the position vector of the critical point:

$$A^j_k = \left(\frac{\partial \xi^j}{\partial x^k}\right)_{\boldsymbol{x}=\boldsymbol{x}_c}. \tag{3.162}$$

The linear, local solution is expressed in terms of exponential functions, and only a relatively small number of solution patterns are possible. These are determined by the invariants of $A^j_k$.

The invariants arise naturally as traces of various powers of $A^j_k$. They are all derived as follows: Transform $A^j_k$ by

$$B^j_k = M^j_n A^n_m \bar{M}^m_k, \tag{3.163}$$

where $M$ is a nonsingular matrix and $\bar{M}$ is its inverse. Take the trace of (3.163):

$$B^j_j = M^j_n A^n_m \bar{M}^m_j = \bar{M}^m_j M^j_n A^n_m = \delta^m_n A^n_m = A^m_m. \tag{3.164}$$

The trace is invariant under the affine transformation $M^j_k$. One can think of the vector field $\xi^j$ as if it were imbedded in an $n$-dimensional block of rubber. An affine transformation is one that stretches or distorts the rubber block without ripping it apart or reflecting it through itself. For traces of higher powers the proof of invariance is similar to (3.164):

$$\begin{aligned} \text{tr}(B^\alpha) &= M^j_{n_1} A^{n_1}_{m_1} \bar{M}^{m_1}_{j_1} M^{j_1}_{n_2} A^{n_2}_{m_2} \bar{M}^{m_2}_{j_2} \cdots M^{j_{\alpha-1}}_{n_\alpha} A^{n_\alpha}_{m_\alpha} \bar{M}^{m_\alpha}_j \\ &= \text{tr}(A^\alpha). \end{aligned} \tag{3.165}$$

The traces of all powers of the gradient tensor remain invariant under an affine transformation. Likewise any combination of the traces is invariant.

### 3.9.3 Linear Flows in Two Dimensions

In two dimensions the eigenvalues of $A^j_k$ satisfy the quadratic

$$\lambda^2 + P\lambda + Q = 0, \tag{3.166}$$

where $P$ and $Q$ are the matrix invariants

$$P = -A^j_j, \qquad Q = \text{Det}\left(A^j_k\right). \tag{3.167}$$

The eigenvalues are

$$\lambda = -\frac{P}{2} \pm \frac{1}{2}\sqrt{P^2 - 4Q}, \tag{3.168}$$

and the character of the local flow is determined by the quadratic discriminant

$$D = Q - \frac{P^2}{4}. \tag{3.169}$$

If $D > 0$, the eigenvalues are complex and a spiraling motion can be expected. Depending on the sign of $P$, the spiral may be stable or unstable. If $D < 0$, the

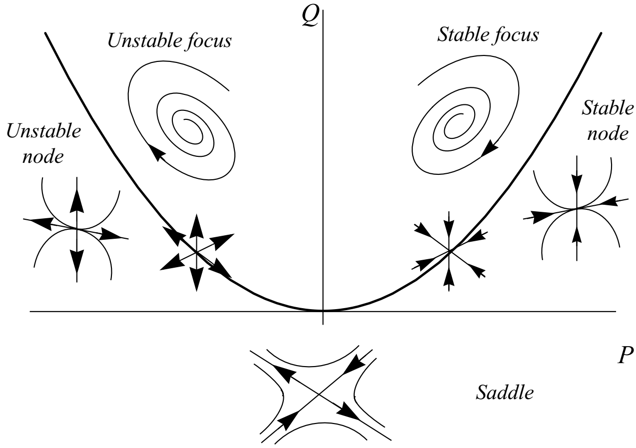


Fig. 3.5. Classification of linear solution trajectories in two dimensions.

eigenvalues are real and a predominantly straining flow can be expected. In this case the directionality of the local flow is defined by the two eigenvectors of $A^j_k$. The various possible flow patterns can be summarized on a crossplot of the invariants shown in Figure 3.5. Categorizing flow patterns using the invariants has a long history of applications in fluid mechanics [3.5], [3.6], [3.7].

***Example 3.7 (Phase portrait of a pair of ODEs).*** Let's look at a particular case. Consider the system

$$\frac{dx}{ds} = 2x^2 - xy, \qquad \frac{dy}{ds} = xy + y + y^2. \tag{3.170}$$

The phase portrait, shown in Figure 3.6, is constructed by evaluating the right-hand sides of (3.170) at each point on a 40 × 40 grid and plotting a line of unit length with the slope determined by the differentials $dx$ and $dy$. The un-normalized line length actually varies considerably over the range covered by the figure and approaches zero in the neighborhood of the critical points. The critical points are clearly identifiable as points where the local slope becomes indeterminate. Solving for the roots of

$$0 = 2x^2 - xy, \qquad 0 = xy + y + y^2, \tag{3.171}$$

we find critical points at $(x_c, y_c) = (0, 0), (0, -1)$ *and* $(-\frac{1}{3}, -\frac{2}{3})$. The critical points located at $(x_c, y_c) = (0, -1)$ and $(-\frac{1}{3}, -\frac{2}{3})$ fit the catagorization of linear flows shown in Figure 3.5. The eigenvectors at the saddle located at $(0, -1)$ give the precise orientation of the trajectories that pass through the saddle.

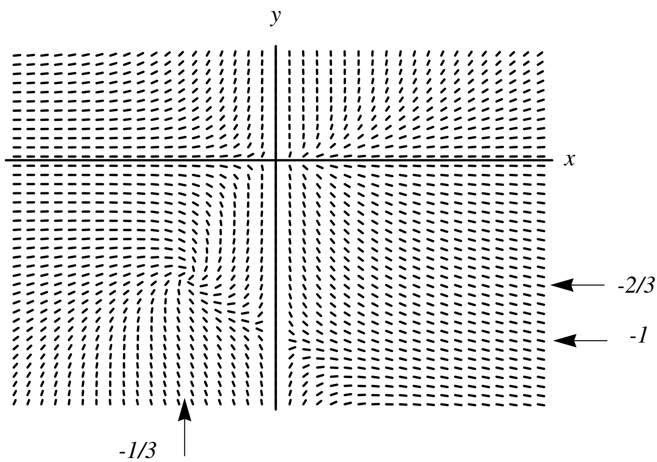


Fig. 3.6. Vector field generated by the system (3.170).

### 3.9.3.1 Nonlinear Critical Points

The critical point at the origin in Example 3.7 is shown in an expanded view in Figure 3.7. Note that this point does not fit the simple linear classification summarized in Figure 3.5. The functions on the right-hand side of (3.170) cannot be linearized near the origin, and the quadratic terms dominate the behavior. In general, when the local linearization (3.161) fails, the phase portrait near a critical point will be determined by nonlinear balances, which can produce quite a complicated picture requiring a much more complicated classification scheme. The critical point in Figure 3.7 is saddle-like to the left of the origin and nodal-like to the right.

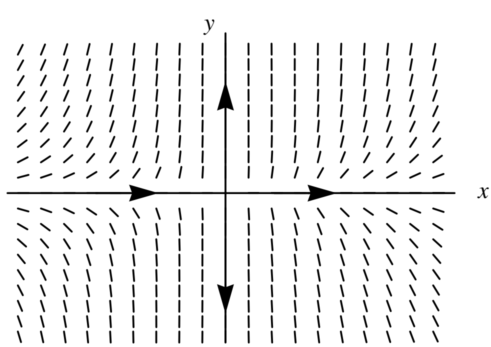


Fig. 3.7. The critical point of (3.170) near the origin.

### 3.9.4 Linear Flows in Three Dimensions

In three dimensions the eigenvalues of $A^j_k$ satisfy the cubic

$$\lambda^3 + P\lambda^2 + Q\lambda + R = 0, \tag{3.172}$$

where the invariants are

$$\begin{aligned} P &= -\mathrm{tr}[A] = -A^j_j, \\ Q &= \tfrac{1}{2}(P^2 - \mathrm{tr}[A^2]) = \tfrac{1}{2}\big(P^2 - A^j_k A^k_j\big), \\ R &= \tfrac{1}{3}(-P^3 + 3PQ - \mathrm{tr}[A^3]) = \tfrac{1}{3}\big(-P^3 + 3PQ - A^j_k A^k_m A^m_j\big). \end{aligned} \tag{3.173}$$

Any cubic can be simplified as follows. Let

$$\lambda = \alpha - \frac{P}{3}. \tag{3.174}$$

Then $\alpha$ satisfies

$$\alpha^3 + \hat{Q}\alpha + \hat{R} = 0, \tag{3.175}$$

where

$$\hat{Q} = Q - \tfrac{1}{3}P^2, \qquad \hat{R} = R - \tfrac{1}{3}PQ + \tfrac{2}{27}P^3. \tag{3.176}$$

The cubic (3.175) was first solved by Scipione del Ferro (1465–1526), who was a professor of arithmetic and geometry at Bologna beginning in 1496. He passed the solution on to a relative, Anton Fior, who challenged the first mathematician of Italy, Niccolo Tartaglia (1500–1557) to a competition – a not uncommon happening in that era. Tartaglia soon solved the problem on his own. On learning of Tartaglia's discovery, another well-known mathematician of the time, Girolamo Cardano, was anxious to include the formula in his book, *The Great Art or the Rules of Algebra*. During a visit to Cardano's house in 1539, Tartaglia revealed the formula on the condition that it not be published. To his suprise and dismay, the formula was included when Cardano finally published the book in 1545. Despite the subterfuge, *The Great Art* is considered a landmark in the history of algebra, and Cardano is credited with the first publication of the solution to the cubic. For more on this fascinating story see the expositions in Stillwell [3.8] and Yaglom [3.9].

Let

$$\begin{aligned} a_1 &= \left(-\frac{\hat{R}}{2} + \frac{1}{3\sqrt{3}}\big(\hat{Q}^3 + \tfrac{27}{4}\hat{R}^2\big)^{1/2}\right)^{1/3}, \\ a_2 &= \left(-\frac{\hat{R}}{2} - \frac{1}{3\sqrt{3}}\big(\hat{Q}^3 + \tfrac{27}{4}\hat{R}^2\big)^{1/2}\right)^{1/3} \end{aligned} \tag{3.177}$$

The real solution of (3.175) is expressed as

$$\alpha_1 = a_1 + a_2, \tag{3.178}$$

and the complex (or remaining real) solutions are

$$\begin{aligned}\alpha_2 &= -\tfrac{1}{2}(a_1 + a_2) + \frac{i\sqrt{3}}{2}(a_1 - a_2),\\ \alpha_3 &= -\tfrac{1}{2}(a_1 + a_2) - \frac{i\sqrt{3}}{2}(a_1 - a_2).\end{aligned} \tag{3.179}$$

Equation (3.178) is still called the *Cardano formula*, although Cardano himself attributed the formula appropriately to Tartaglia. Up until this period in history complex numbers for solving quadratics had been rejected as absurd. But here they are seen to be necessary to express the real solution of the cubic. In spite of this, the general acceptance of complex numbers would not occur for another three hundred years until the work of Cauchy, Bolyai, Gauss, and Hamilton in the first half of the ninteenth century.

Solving (3.172) for the eigenvalues leads to the cubic discriminant

$$D = \tfrac{27}{4}R^2 + \left(P^3 - \tfrac{9}{2}PQ\right)R + Q^2\left(Q - \tfrac{1}{4}P^2\right). \tag{3.180}$$

The Cardano surface $D = 0$ is depicted in Figure 3.8. To help visualize the surface it is split down the middle on the plane $P = 0$ and the two parts are

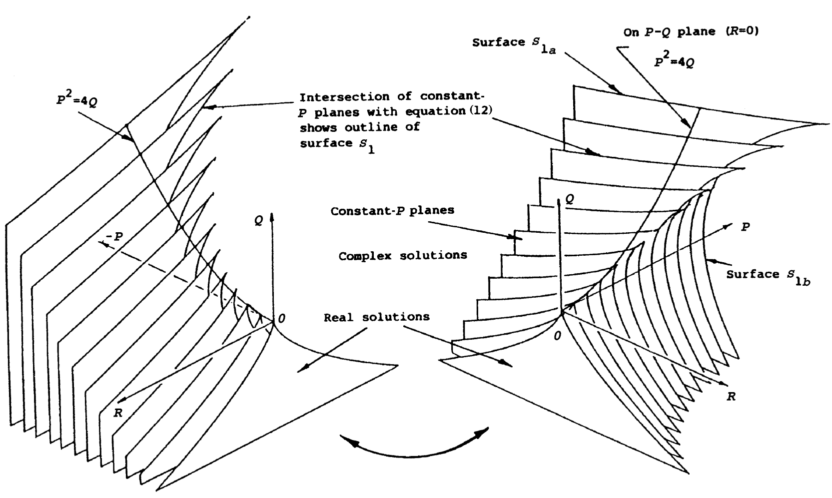


Fig. 3.8. The Cardano surface dividing real and complex eigenvalues in three dimensions (from Reference [3.10]).

rotated away to provide a better view. Note that (3.180) can be regarded as a quadratic in $R$, and so the surface $D = 0$ is really composed of two roots for $R$ that meet in a cusp. A complete road map to the various solutions of the cubic and their associated local vector fields is presented in Reference [3.10].

If $D > 0$, the point $(P, Q, R)$ lies above the surface and there is one real eigenvalue and two complex conjugate eigenvalues. If $D < 0$, all three eigenvalues are real. The invariants can be expressed in terms of the eigenvalues as follows. If the eigenvalues are real,

$$\begin{aligned} P &= -(\lambda^1 + \lambda^2 + \lambda^3), \\ Q &= \lambda^1\lambda^2 + \lambda^1\lambda^3 + \lambda^2\lambda^3, \\ R &= -\lambda^1\lambda^2\lambda^3, \end{aligned} \tag{3.181}$$

and if the eigenvalues are complex,

$$\begin{aligned} P &= -(2\sigma + b), \\ Q &= \sigma^2 + \omega^2 + 2\sigma b, \\ R &= -b(\sigma^2 + \omega^2), \end{aligned} \tag{3.182}$$

where $b$ is the real eigenvalue and $\sigma$ and $\omega$ are the real and imaginary parts of the complex conjugate eigenvalues.

A particularly interesting case occurs when $P = 0$. In this case the discriminant is

$$D = Q^3 + \tfrac{27}{4}R^2, \tag{3.183}$$

and the invariants are

$$Q = -\tfrac{1}{2}A^j_k A^k_j, \qquad R = -\tfrac{1}{3}A^j_k A^k_m A^m_j. \tag{3.184}$$

The various possible critical points in this case can be categorized on a plot of $Q$ versus $R$. Figure 3.9 and Figure 3.5 are cuts through the Cardano surface (3.180) at $P = 0$ and $R = 0$ respectively.

## 3.10 Concluding Remarks

In the following chapters we will approach the problem of solving differential equations using the methods and terminology of Lie groups. Although the terminology may at times seem new and unusual, the mathematical objects of study will quickly be recognized as the same as those reviewed in this chapter. The analytical tools developed here, particularly the solution of first-order PDEs, will be used over and over again as we search for group invariants.

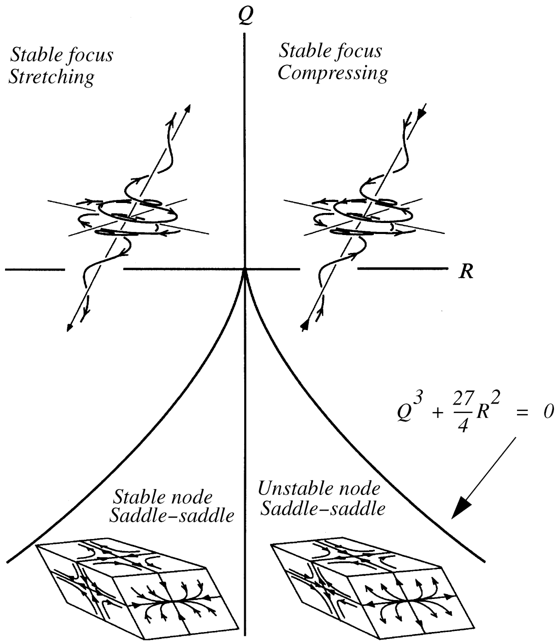


Fig. 3.9. Three-dimensional flow patterns in the plane $P = 0$ (from Reference [3.11]).

Phase space methods will be used throughout the text when we seek to physically understand the solutions that arise from the various applications of symmetry analysis presented.

## 3.11 Exercises

3.1 Consider the two functions

$$x^2 + y^3 + z^4 + u^5 = 1, \qquad x + y^2 + z^3 = 1 \tag{3.185}$$

Let $u$ be defined as a function of $x$, $y$, and $z$ by the first equation, and $z$ be defined as a function of $x$ and $y$ by the second equation. As a consequence, $u$ is a function of $x$ and $y$. Find the first partial derivatives of $u$ with respect to $x$ and $y$.

3.2 Determine whether each of the following expressions is an exact differential:

$$(e^x \cos y)\, dx + (e^x \sin y)\, dy = 0,$$
$$(\cos x \cosh y - \sin x \sinh y)\, dx + (\sin x \sinh y + \cos x \cosh y)\, dy = 0. \tag{3.186}$$

3.3 Prove that the stream function and velocity potential in two-dimensional irrotational flow satisfy the Cauchy-Riemann conditions (3.29).

3.4 Show by the cross-derivative test that (3.41) is an exact differential. Show by substitution that (3.42) is the general solution of (3.38).

3.5 Show that the internal energy and enthalpy of an ideal gas depend only on temperature. First show that the Gibbs equation can be written in the form

$$ds[T, P] = \frac{1}{T}\,de[T, P] + \frac{R}{T}\,dT - \frac{v}{T}\,dP. \tag{3.187}$$

Work out the partial derivatives of the entropy, and show by the cross-derivative test that $\partial e[T, P]/\partial P = 0$.

3.6 Determine the explicit relations (3.62) for the case of an ideal gas. Assume the heat capacities are constant.

3.7 Derive equation (3.79) relating the velocity vector field to the dual stream functions in a three-dimensional flow. Derive (3.104) for the case of incompressible flow.

3.8 Show that $f[x^2 - z^2, x^3 - y^3] = 0$ is a solution of

$$y^2 z\frac{\partial z}{\partial x} + x^2 z\frac{\partial z}{\partial y} = xy^2. \tag{3.188}$$

where $f$ is arbitrary.

3.9 Find a first-order PDE whose integral is

$$z = \alpha x + (\alpha^2 + 1)y + \beta. \tag{3.189}$$

3.10 Solve

$$\frac{\partial z}{\partial x} = x\left(\frac{\partial z}{\partial y}\right)^2. \tag{3.190}$$

3.11 Show that a solution of $F(\partial z/\partial x, \partial z/\partial y) = 0$ is $z = \alpha x + \beta y + \gamma$ where $F(\alpha, \beta) = 0$, and use this result to solve

$$3\left(\frac{\partial z}{\partial x}\right)^2 - 2\left(\frac{\partial z}{\partial y}\right)^2 = \frac{\partial z}{\partial x}\frac{\partial z}{\partial y}. \tag{3.191}$$

3.12 Show that the relation $\partial z/\partial x = \alpha\,\partial z/\partial y$ can be used as a second equation in the solution of $F[z, \partial z/\partial x, \partial z/\partial y] = 0$. Use this to solve

$$z^{2n-2}\left(\left(\frac{\partial z}{\partial x}\right)^2 z^n + \frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\right) = \gamma^2. \tag{3.192}$$

3.13 Show that for an integrating factor to exist for a set of first-order ODEs in $n$ dimensions, $(n-1)(n-2)/2$ auxiliary conditions must be satisfied by the associated vector field. See Kestin ([3.4], p. 469) for a verbal description of the proof.

3.14 Find by inspection an integrating factor for each of the following ODEs, and work out the general solution:

$$x\frac{dy}{dx} + y - x^2 = 0, \tag{3.193}$$

$$(3x^2 + 2xy - y^2)\,dx + (x^2 - 2xy - 3y^2)\,dy = 0, \tag{3.194}$$

$$\frac{dy}{dx} = \frac{ye^y}{y^3 + 2xe^y}. \tag{3.195}$$

3.15 Consider the simple pendulum shown in Figure 3.10.

(i) Use dimensional analysis to determine an expression for the natural frequency of the pendulum.

(ii) Work out the unforced nonlinear equation of motion, assuming that the motion of the mass is damped due to air resistance and that the damping is proportional to the speed of the mass. Show that the equation is invariant under translation in time.

(iii) Convert the second-order equation of motion to an autonomous pair of first-order equations, and sketch the phase portrait of the system. Find and identify all critical points. Consider the mass released from some initial angle. Depending on the amount of damping, the mass may oscillate about the bottom dead-center equilibrium point or it may slowly come to rest without oscillation. Determine the critical damping factor that distinguishes these two cases.

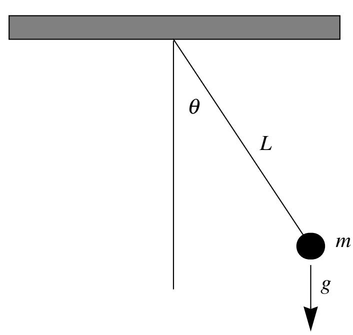


Fig. 3.10.

3.16 Show that the plane $\lambda^3 + P\lambda^2 + Q\lambda + R = 0$ in $(P, Q, R)$ space generates the surface

$$\tfrac{27}{4}R^2 + \left(P^3 - \tfrac{9}{2}PQ\right)R + Q^2\left(Q - \tfrac{1}{4}P^2\right) = 0 \qquad (3.196)$$

when $\lambda$ is allowed to take on all values between minus infinity and plus infinity. Start by showing the result for $R = 0$ and $P = 0$ where the generator is a straight line.

3.17 Rewrite the van der Pol equation

$$y_{tt} + \kappa(y^2 - 1)y_t + y = 0 \qquad (3.197)$$

as an autonomous pair, and characterize any critical points.

3.18 Fully characterize the phase portrait of the autonomous system

$$\begin{aligned} \frac{dx}{dt} &= \frac{3x^4 - 12x^2y^2 + y^4}{(x^2 + y^2)^{3/2}}, \\ \frac{dy}{dt} &= \frac{6x^3y - 10xy^3}{(x^2 + y^2)^{3/2}}. \end{aligned} \qquad (3.198)$$

Consider changing variables to polar coordinates.

3.19 Consider the second-order equation

$$y_{tt} + ay + by^3 = 0. \qquad (3.199)$$

Work out the solution of (3.199) in the phase plane $(y, y_t)$, and completely characterize the critical points for various ranges of the parameters $a$ and $b$.

3.20 Show that solutions of the $p$th-order ODE

$$y_{px} = f(x, y_x, y_{xx}, y_{xxx}, \ldots, y_{(p-1)x}) \qquad (3.200)$$

satisfy the $p$-dimensional first-order PDE

$$\frac{\partial \Psi}{\partial x} + y_x \frac{\partial \Psi}{\partial y} + y_{xx} \frac{\partial \Psi}{\partial y_x} + y_{xxx} \frac{\partial \Psi}{\partial y_{xx}} + \cdots + f \frac{\partial \Psi}{\partial y_{(p-1)x}} = 0. \qquad (3.201)$$

Write down the system of characteristic equations for (3.201), and discuss the closedness of the system.

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# 4

# *Classical Dynamics*

## 4.1 Introduction

The equation for the acceleration of a particle of mass $m$ is described by Newton's law,

$$m\ddot{\boldsymbol{x}} = \boldsymbol{F}. \tag{4.1}$$

This equation can be generalized to a system of equations describing an array of mass points moving under the action of internal forces between particles and external forces applied by outside agents. The mass points might represent any number of different physical situations: a number of celestial bodies interacting though their mutual gravitational attraction, a cloud of charged particles subjected to attractive and repulsive electrostatic forces, the infinity of mass points composing a rigid body or a set of linked rigid bodies held together by internal forces, the field of mass points of a fluid moving under the action of pressure and viscous stress forces and so on.

The solution of (4.1) can be simplified by replacing Newton's laws for the vector acceleration by an equivalent scalar energy relation. This relation is based on the idea that virtual displacements of the mass points under the given system of forces do no net work when the system is displaced from a state of equilibrium. It is called *d'Alembert's principle*, after the 18th-century mathematician Jean le Rond d'Alembert. For a system of $n$ forces in static equilibrium, where $\ddot{\boldsymbol{x}}^i = \boldsymbol{0}$, this principle is stated as

$$\boldsymbol{F}^i \cdot \delta\boldsymbol{x}^i = 0, \qquad \text{Sum over } i = 1, \ldots, n. \tag{4.2}$$

where the $\delta\boldsymbol{x}^i$ are arbitrary vector displacements from equilibrium. This approach can be carried over directly to a dynamical system by replacing $\boldsymbol{F}_i$ with $m\ddot{\boldsymbol{x}}^i - \boldsymbol{F}^i$. d'Alembert's principle applied to the motion of $n$ particles is

$$(m\ddot{\boldsymbol{x}}^i - \boldsymbol{F}^i) \cdot \delta\boldsymbol{x}^i = 0. \qquad \text{Sum over } i = 1, \ldots, n. \tag{4.3}$$

The displacements are independent except for the constraint imposed by (4.3). This enables a set of independent equations to be written that are sufficient to solve for the unknowns in the problem. As the complexity of the problem increases, perhaps involving a mechanism with many individual parts with internal forces of reaction within each part, the problem of determining the equations relating the accelerations to the forces from the geometry of the system becomes more and more unwieldy. Moreover these forces are often not the main object of interest. It is usually the overall motion of a body that matters most, not the motion of every individual element.

d'Alembert's principle can be used to derive a universal system of generating equations called the *Euler–Lagrange equations*. In this approach the main quantities of interest are the scalar kinetic and potential energies of the system expressed in generalized coordinates. The concept of generalized coordinates makes this approach much more powerful for handling complex systems than the basic form of d'Alembert's principle, (4.3). In essence, the position and velocity of the system can be expressed in *any* system of coordinates that fully specifies the energy of the system. The equations of motion are then derived directly from the Euler–Lagrange equations. The formulation of the generalized equations of mechanics is presented in many textbooks; my favorite references are Goldstein [4.1] and Landau and Lifshitz [4.2].

To begin, let's illustrate some of the basic ideas with an example.

***Example 4.1 (A spring–mass system – the undamped harmonic oscillator).*** Consider the undamped (frictionless) spring–mass system shown in Figure 4.1. The equation of motion balancing the acceleration of the mass and the force applied by the spring is

$$m\frac{d^2x}{dt^2} + kx = 0. \tag{4.4}$$

This second-order ordinary differential equation (ODE) can be broken into an autonomous pair of first-order ODEs in which we define a new variable $p$:

$$\frac{dx}{dt} = \frac{p}{m}, \qquad \frac{dp}{dt} = -kx. \tag{4.5}$$

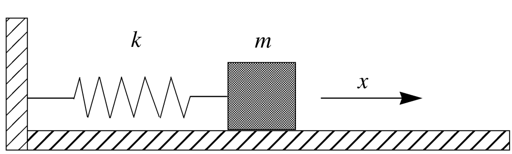


Fig. 4.1. Spring–mass system.

The quantity $p$ is the momentum of the mass at any given instant. Recalling the methods developed in Chapter 3, the characteristics of this pair of ODEs are the integral curves of the first-order partial differential equation (PDE),

$$\frac{p}{m}\frac{\partial H}{\partial x} - kx\frac{\partial H}{\partial p} = 0. \tag{4.6}$$

The function $H(x, p)$ is determined as the constant of integration of the first-order ODE

$$m\frac{dx}{p} = \frac{dp}{-kx}. \tag{4.7}$$

Noting that this is a perfect differential (the cross derivatives are both zero),

$$dH = \frac{p}{m}\,dp + kx\,dx. \tag{4.8}$$

The function

$$H[x, p] = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}kx^2 \tag{4.9}$$

is called the *Hamiltonian* and corresponds to the total energy (kinetic plus potential) of the system. The Hamiltonian is a constant of the motion whose value is set by the initial values of the position and velocity. We shall return to this problem shortly, using a slightly different, more general approach.

## 4.2 Hamilton's Principle

In the most general formulation of mechanics, the equations of motion of a mechanical system are derived using *Hamilton's principle*, also called the *principle of least action*. In this formulation every mechanical system is characterized by a single scalar function of $2n + 1$ independent variables, $L[q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n, t]$, called the *Lagrangian*. The variables $q^1[t], \ldots, q^n[t]$ are called generalized coordinates and can be any set of quantities that completely determine the configuration of the system. The velocities of the system are $\dot{q}^i[t] = dq^i/dt$. At an instant, the state of the system is completely characterized by its position in the space of coordinates and velocities. Earlier we called this the phase space or state space of the system. The study of such systems often makes use of the methods of critical-point analysis described in Chapter 3.

From Newton's laws of motion it is recognized that, if the position and velocity of every point of a mechanical system are specified, then the acceleration of the system at every point is known. The equations for the accelerations, i.e., the equations of motion, constitute conditions for which the integral

$$S = \int_{t_1}^{t_2} L[q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n, t]\, dt \tag{4.10}$$

is an extremum for any choice of the initial and final times of the motion $(t_1, t_2)$. The quantity $S$ is called the *action*. The action can be either a maximum or a minimum, depending on the nature of the problem, but in mechanical systems involving particles moving in a gravitational field, it is most often a minimum, and the phrase *principle of least action* is used. What is important is that, to first order, the integral is invariant under a small perturbation in the path of the system in phase space. Let

$$(q^1[t], \ldots, q^n[t]) \tag{4.11}$$

be the set of functions that correspond to an extremum of the action. Consider the effect of adding a small deviation to this solution. Let

$$\tilde{S} = \int_{t_1}^{t_2} L[\tilde{q}^1, \ldots, \tilde{q}^n, \dot{\tilde{q}}^1, \ldots, \dot{\tilde{q}}^n, t]\, dt, \tag{4.12}$$

where

$$\tilde{q}^i[t] = q^i[t] + \varepsilon \eta^i[t] \tag{4.13}$$

and

$$\tilde{S} = S + \delta S. \tag{4.14}$$

The deviation functions are assumed to be zero at the endpoints of the integration:

$$\eta^i[t_2] = \eta^i[t_1] = 0 \tag{4.15}$$

Substitute this transformation into (4.12):

$$\begin{aligned} S + \delta S = \int_{t_1}^{t_2} L[&q^1[t] + \varepsilon \eta^1[t], \ldots, q^n[t] + \varepsilon \eta^n[t], \\ &\dot{q}^1[t] + \varepsilon \dot{\eta}^1[t], \ldots, \dot{q}^n[t] + \varepsilon \dot{\eta}^n[t], t]\, dt, \end{aligned} \tag{4.16}$$

and expand the integrand in a Taylor series for small $\varepsilon$:

$$\begin{aligned} S + \delta S = &\int_{t_1}^{t_2} L[q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n, t]\, dt \\ &+ \varepsilon \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q^i} \eta^i + \frac{\partial L}{\partial \dot{q}^i} \dot{\eta}^i \right) dt + O(\varepsilon^2). \end{aligned} \tag{4.17}$$

Since the functions $q^i[t]$ correspond to an extremum in the action, the first-order term in the Taylor series, termed the *first variation*, must be zero:

$$\int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q^i} \eta^i + \frac{\partial L}{\partial \dot{q}^i} \dot{\eta}^i \right) dt = 0. \tag{4.18}$$

Now integrate the second term in (4.18) by parts, using

$$\frac{D}{Dt}\left( \frac{\partial L}{\partial \dot{q}^i} \eta^i \right) = \frac{\partial L}{\partial \dot{q}^i} \dot{\eta}^i + \frac{D}{Dt}\left( \frac{\partial L}{\partial \dot{q}^i} \right) \eta^i. \tag{4.19}$$

The integral

$$\int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q^i} - \frac{D}{Dt}\left( \frac{\partial L}{\partial \dot{q}^i} \right) \right) \eta^i \, dt + \int_{t_1}^{t_2} \frac{D}{Dt}\left( \frac{\partial L}{\partial \dot{q}^i} \eta^i \right) dt = 0 \tag{4.20}$$

becomes

$$\int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q^i} - \frac{D}{Dt}\left( \frac{\partial L}{\partial \dot{q}^i} \right) \right) \eta^i \, dt + \left( \frac{\partial L}{\partial \dot{q}^i} \eta^i \right)_{t_1}^{t_2} = 0. \tag{4.21}$$

Since the time interval is arbitrary and the deviation functions are zero at the endpoints, the functions $q^i[t]$ must satisfy

$$\frac{\partial L}{\partial q^i} - \frac{D}{Dt}\left( \frac{\partial L}{\partial \dot{q}^i} \right) = 0. \tag{4.22}$$

These are the famous Euler–Lagrange equations. In the spring–mass problem described above, the Lagrangian is

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2. \tag{4.23}$$

Using (4.22) to produce the equation for the acceleration,

$$m\ddot{x} + kx = 0, \tag{4.24}$$

generates the simple force balance corresponding to Newton's law, $F = ma$. In general, for mechanical systems the Lagrangian and the Hamiltonian are related to one another by

$$L = T - V$$
$$H = T + V$$

where $T$ is the kinetic energy of the system and $V$ is the potential energy.

## 4.3 Hamilton's Equations

Lagrangian dynamics is an extremely general method of analyzing, not just mechanical systems, but much more general problems involving fields (such as the 2-D flow fields discussed in Chapter 3, Section 3.5). In this context it is convenient to reformulate the problem in terms of, not the velocities, but the momenta of the various particles in the system, and in terms of the Hamiltonian instead of the Lagrangian. The generalized momenta are defined as

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \tag{4.25}$$

and the Euler–Lagrange equations are

$$\frac{Dp_i}{Dt} = \frac{\partial L}{\partial q^i} = \dot{p}_i. \tag{4.26}$$

The use of subscript notation for the components $p_i$ of the generalized momenta deserves some explanation. The generalized momentum $\boldsymbol{p}$ is a covariant vector and so the notation is consistent with common vector–tensor notation, but at first sight it would seem to be in violation of the notation we adopted in Chapter 1, in that $p_i$ is not a derivative of $p$. However, in a later section we will develop the Hamilton–Jacobi equation for the action $S$, and there we shall see that $p_i$ is a partial derivative of the action. Thus we will continue to use the subscript notation $p_i$ as a reminder of its derivative origin in (4.25) and in anticipation of its use in the Hamilton–Jacobi equation, where $p_i = \partial S/\partial q^i = S_i$.

In order to express the Euler–Lagrange equations in terms of the generalized momenta instead of the velocities, we make use of the Legendre transformation introduced in the discussion of thermodynamics in Chapter 3, Section 3.4. The total differential of the Lagrangian is

$$dL = \frac{\partial L}{\partial t}\,dt + \frac{\partial L}{\partial q^i}\,dq^i + \frac{\partial L}{\partial \dot{q}^i}\,d\dot{q}^i, \tag{4.27}$$

which, using the definition of generalized momenta, we can write as

$$dL = \frac{\partial L}{\partial t}\,dt + \dot{p}_i\,dq^i + p_i\,d\dot{q}^i. \tag{4.28}$$

Noting that $d(p_i\dot{q}^i) = p_i\,d\dot{q}^i + \dot{q}^i\,dp_i$, the differential (4.28) can be written as

$$d(L - p_i\dot{q}^i) = \frac{\partial L}{\partial t}\,dt + \dot{p}_i\,dq^i - \dot{q}^i\,dp_i. \tag{4.29}$$

Now we use the Legendre transformation to define a new dependent variable that depends on the coordinates and the momenta $(q^1, \ldots, q^n, p_1, \ldots, p_n, t)$ rather than the coordinates and the velocities.

Define the Hamiltonian

$$H[q^1, \ldots, q^n, p_1, \ldots, p_n, t] = p_i \dot{q}^i - L. \tag{4.30}$$

The total differential of the Hamiltonian is

$$dH = \frac{\partial H}{\partial t} dt - \dot{p}_i \, dq^i + \dot{q}^i \, dp_i. \tag{4.31}$$

The partial derivatives of the Hamiltonian are related to the generalized coordinates and momenta as follows:

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}. \tag{4.32}$$

With the Hamiltonian function known, (4.32) comprises the equations of motion of the system in the state space $(q^1, \ldots, q^n, p_1, \ldots, p_n)$. For the spring–mass system described earlier,

$$H[x, p] = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}kx^2, \tag{4.33}$$

and the equations of motion according to (4.32) are

$$\frac{dx}{dt} = \frac{p}{m}, \qquad \frac{dp}{dt} = -kx. \tag{4.34}$$

The system (4.34) is identical to Equation (4.5) that was generated directly from the equation of motion, (4.4).

The total time derivative of the Hamiltonian is

$$\frac{DH}{Dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial q^i}\left(\frac{dq^i}{dt}\right) + \frac{\partial H}{\partial p_i}\left(\frac{dp_i}{dt}\right). \tag{4.35}$$

If we substitute the equations of motion, we get

$$\frac{DH}{Dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q^i} = \frac{\partial H}{\partial t}. \tag{4.36}$$

If the Hamiltonian does not depend explicitly on time, then $DH/Dt = 0$ and the total energy of the system is conserved.

### 4.3.1 Poisson Brackets

Suppose $\Psi[q^1, \ldots, q^n, p_1, \ldots, p_n, t]$ is any function of the generalized coordinates and momenta. The total derivative of $\Psi$ with respect to time is

$$\frac{D\Psi}{Dt} = \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial q^i}\left(\frac{dq^i}{dt}\right) + \frac{\partial \Psi}{\partial p_i}\left(\frac{dp_i}{dt}\right). \tag{4.37}$$

Substitute the equations of motion (4.32) into (4.37):

$$\frac{D\Psi}{Dt} = \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial q^i}\left(\frac{\partial H}{\partial p_i}\right) - \frac{\partial \Psi}{\partial p_i}\left(\frac{\partial H}{\partial q^i}\right). \tag{4.38}$$

The expression

$$\{H, \Psi\} = \frac{\partial \Psi}{\partial q^i}\left(\frac{\partial H}{\partial p_i}\right) - \frac{\partial \Psi}{\partial p_i}\left(\frac{\partial H}{\partial q^i}\right) \tag{4.39}$$

is called the *Poisson bracket* of $H$ with $\Psi$. If $\Psi$ is an integral of the motion, i.e., a conserved quantity, so that $D\Psi/Dt = 0$, then the equation governing $\Psi$ is

$$\frac{\partial \Psi}{\partial t} + \{H, \Psi\} = 0. \tag{4.40}$$

If the integral of the motion does not depend explicitly on time, then it satisfies

$$\{H, \Psi\} = 0. \tag{4.41}$$

The Poisson bracket of the integral and the Hamiltonian is zero.

***Definition 4.1.*** *The Poisson bracket of any two functions, say,* $\Psi[q^1, \ldots, q^n, p_1, \ldots, p_n, t]$ *and* $\Omega[q^1, \ldots, q^n, p_1, \ldots, p_n, t]$*, is written*

$$\{\Omega, \Psi\} = \frac{\partial \Psi}{\partial q^i}\left(\frac{\partial \Omega}{\partial p_i}\right) - \frac{\partial \Psi}{\partial p_i}\left(\frac{\partial \Omega}{\partial q^i}\right) \tag{4.42}$$

*and satisfies the following rules (c is a constant):*

*(1) The Poisson bracket is skew-symmetric:*

$$\{\Omega, \Psi\} = -\{\Psi, \Omega\}. \tag{4.43}$$

*(2) Rules of association:*

$$\begin{aligned} \{\Omega, c\} &= 0, \\ \{\Omega_1 + \Omega_2, \Psi\} &= \{\Omega_1, \Psi\} + \{\Omega_2, \Psi\}, \\ \{\Omega_1\Omega_2, \Psi\} &= \Omega_1\{\Omega_2, \Psi\} + \Omega_2\{\Omega_1, \Psi\}. \end{aligned} \tag{4.44}$$

(3) *The partial derivative of the Poisson bracket with respect to time is*

$$\frac{\partial}{\partial t}\{\Omega, \Psi\} = \left\{\frac{\partial \Omega}{\partial t}, \Psi\right\} + \left\{\Omega, \frac{\partial \Psi}{\partial t}\right\}. \tag{4.45}$$

*If one of the functions is a coordinate or a momentum, then the Poisson bracket reduces to a simple partial derivative:*

$$\{\Omega, q^i\} = \frac{\partial \Omega}{\partial p_i}, \qquad \{\Omega, p_i\} = -\frac{\partial \Omega}{\partial q^i}. \tag{4.46}$$

(4) *Any three functions* $\Psi[q^1, \ldots, q^n, p_1, \ldots, p_n, t]$, $\Omega[q^1, \ldots, q^n, p_1, \ldots, p_n, t]$, *and* $\Phi[q^1, \ldots, q^n, p_1, \ldots, p_n, t]$ *satisfy the* Jacobi identity

$$\{\Psi, \{\Omega, \Phi\}\} + \{\Omega, \{\Phi, \Psi\}\} + \{\Phi, \{\Psi, \Omega\}\} = 0. \tag{4.47}$$

***Theorem 4.1.*** *Poisson's theorem states that if* $\Psi$ *and* $\Omega$ *are any two integrals of the motion, then the Poisson bracket*

$$\{\Psi, \Omega\} = \Theta \tag{4.48}$$

*and* $\Theta$ *is also an integral of the motion. This can be seen from the Jacobi identity as follows. Let* $\Phi = H$ *in* (4.47)*:*

$$\{\Psi, \{\Omega, H\}\} + \{\Omega, \{H, \Psi\}\} + \{H, \{\Psi, \Omega\}\} = 0. \tag{4.49}$$

*Since* $\{\Omega, H\} = 0$ *and* $\{H, \Psi\} = 0$, *then from* (4.49),

$$\{H, \{\Psi, \Omega\}\} = \{H, \Theta\} = 0. \tag{4.50}$$

*Thus* $\Theta$ *is a constant of the motion.*

The conserved elements of a Hamiltonian system, $\Psi, \Omega, \Phi, \Theta, \ldots$, define a vector space. The rules of algebra in this space are given by the skew-symmetry of the composition operator (Poisson bracket) (4.43), the additive properties in (4.44), and the Jacobi identity (4.47). A vector space with these special properties is called a *symplectic* space, and the solution of the Hamiltonian system is said to lie on a symplectic manifold. This odd word comes from the greek *symplektikos* meaning "twining together," from *syn* (together) and *plekein* (to twine). It is an apt description of the solution trajectories of a periodically forced Hamiltonian system, which can be visualized as a family of spiraling curves on a torus in a three-dimensional phase space where the third dimension is the phase angle of the forcing function.

## 4.4 The Hamilton–Jacobi Equation

One can develop an equation for the action itself. If the limits of integration in (4.10) are left indefinite, the action

$$S = \int L\, dt + \text{constant} \tag{4.51}$$

is a function of the coordinates and time, $S[q^1, \ldots, q^n, t]$, that characterizes the motion along some segment of the solution path. We consider the action for paths with a common beginning point at time $t_1$, $q^i[t_1]$, but with variable ending point at time $t_2$. From (4.17) the equation for the change in the action is

$$\delta S = \varepsilon \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q^i} \eta^i + \frac{\partial L}{\partial \dot{q}^i} \dot{\eta}^i \right) dt + O(\varepsilon^2). \tag{4.52}$$

The change in the action between two neighboring paths is

$$\delta S = \varepsilon \left( \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q^i} - \frac{D}{Dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) \right) \eta^i\, dt + \left( \frac{\partial L}{\partial \dot{q}^i} \eta^i \right)_{t_1}^{t_2} \right). \tag{4.53}$$

Since the paths of the system satisy the Euler–Lagrange equations, the integral term in (4.53) is zero. Hence,

$$\delta S = \varepsilon \left( \frac{\partial L}{\partial \dot{q}^i} \eta^i \right)_{t_1}^{t_2}. \tag{4.54}$$

In (4.54) the deviation at the initial point is zero ($\eta^i[t_1] = 0$), and the deviation at the final point is a function of time. So we can write

$$\delta S = \frac{\partial L}{\partial \dot{q}^i}\, \delta q^i, \tag{4.55}$$

where the small parameter $\varepsilon$ has been incorporated in the definition $\delta q^i = \varepsilon \eta^i$. Now replace $\partial L / \partial \dot{q}^i = p_i$, so that

$$\delta S = p_i\, \delta q^i. \tag{4.56}$$

Thus the partial derivatives of the action are the generalized momenta

$$\frac{\partial S}{\partial q^i} = p_i. \tag{4.57}$$

What about the time derivative of the action? From its definition,

$$\frac{DS}{Dt} = L. \tag{4.58}$$

The total time derivative of the action is

$$\frac{DS}{Dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q^i}\frac{dq^i}{dt} = \frac{\partial S}{\partial t} + p_i \dot{q}^i = L. \tag{4.59}$$

Comparing with (4.30), we have

$$\frac{\partial S}{\partial t} = -H. \tag{4.60}$$

Finally, the total differential of the action as a function of the coordinates and the time is

$$dS = -H\, dt + p_i\, dq^i. \tag{4.61}$$

Equation (4.60) is

$$\frac{\partial S}{\partial t} + H[q^1, \ldots, q^n, p_1, \ldots, p_n, t] = 0. \tag{4.62}$$

Recalling that $S$ is a function only of coordinates and time, $S[q^1, \ldots, q^n, t]$, and recognizing that (4.61) determines the partial derivatives of $S$, Equation (4.62) becomes the Hamilton–Jacobi equation,

$$\frac{\partial S}{\partial t} + H\left[q^1, \ldots, q^n, \frac{\partial S}{\partial q^1}, \ldots, \frac{\partial S}{\partial q^n}, t\right] = 0. \tag{4.63}$$

Equation (4.63) is a (usually nonlinear) first-order PDE in $n + 1$ independent variables $(q^1, \ldots, q^n, t)$ governing the action $S$.

The Hamilton–Jacobi equation is an example of a nonlinear first-order PDE of the form $A[t, q^1, \ldots, q^n, S, S_t, p_1, \ldots, p_n] = 0$ that is amenable to the $n$-dimensional method of Lagrange and Charpit described in Chapter 3, Section 3.8.1 [except that (4.63) is simpler in that there is no explicit dependence on $S$]. Using the formula (3.159) in Chapter 3, the characteristic equations of (4.63) are

$$\begin{aligned}\frac{dt}{1} &= \frac{dq^1}{\partial H/\partial p_1} = \cdots = \frac{dq^n}{\partial H/\partial p_n} \\ &= \frac{dS_t}{-\frac{\partial H}{\partial t}} = \frac{dp_1}{-\frac{\partial H}{\partial q^1}} = \cdots = \frac{dp_n}{-\frac{\partial H}{\partial q^n}} = \frac{dS}{\frac{\partial S}{\partial t} + p_i \frac{\partial H}{\partial p_i}}.\end{aligned} \tag{4.64}$$

In summary, the complete Hamilton–Jacobi system generated from (4.63) and (4.64) is

$$\begin{aligned}
\frac{dq^i}{dt} &= \frac{\partial H}{\partial p_i},\\
\frac{dp_i}{dt} &= -\frac{\partial H}{\partial q^i},\\
\frac{dS}{dt} &= \frac{\partial S}{\partial t} + p_i \dot{q}^i,\\
\frac{dS_t}{dt} &= -\frac{dH}{dt}.
\end{aligned} \tag{4.65}$$

The last relation in (4.65) implies

$$\frac{\partial S}{\partial t} + H = \text{constant}, \tag{4.66}$$

which is consistent with (4.60). The first two equations in (4.65) are Hamilton's equations for the characteristics describing the evolution of the system in phase space.

This is not the first time we have seen an equation of Hamilton–Jacobi type. The first-order PDE governing characteristics in $n$ dimensions is

$$\xi^j[\boldsymbol{x}]\frac{\partial \Psi}{\partial x^j} = 0. \tag{4.67}$$

For definiteness assume $\xi^1[\boldsymbol{x}] \neq 0$. In the current notation, with the correspondence

$$\begin{aligned}
\Psi &\to S,\\
x^1 &\to t,\\
x^2, \ldots, x^n &\to q^1, \ldots, q^{n-1},\\
\frac{\partial \Psi}{\partial x^2}, \ldots, \frac{\partial \Psi}{\partial x^n} &\to \frac{\partial S}{\partial q^1}, \ldots, \frac{\partial S}{\partial q^{n-1}},\\
\frac{\xi^2[\boldsymbol{x}]}{\xi^1[\boldsymbol{x}]}, \ldots, \frac{\xi^n[\boldsymbol{x}]}{\xi^1[\boldsymbol{x}]} &\to f^1[t, \boldsymbol{q}], \ldots, f^{n-1}[t, \boldsymbol{q}],
\end{aligned} \tag{4.68}$$

the equation (4.67) takes the form

$$\frac{\partial S}{\partial t} + \sum_{j=1}^{n-1} f^j[t, \boldsymbol{q}]\frac{\partial S}{\partial q^j} = 0 \tag{4.69}$$

with Hamiltonian

$$H[t, \boldsymbol{q}, \boldsymbol{p}] = \sum_{j=1}^{n-1} f^j[t, \boldsymbol{q}]p_j, \tag{4.70}$$

where $p_j = \partial S/\partial q^j$. These ideas will be illustrated in the next section through some examples.

## 4.5 Examples

***Example 4.2 (The harmonic oscillator revisited).*** First let's look at an example of the application of the Hamilton–Jacobi approach. The Hamiltonian for the harmonic oscillator is

$$H[x, p] = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}kx^2. \tag{4.71}$$

The corresponding Hamilton–Jacobi equation is

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2}kx^2 = 0 \tag{4.72}$$

with characteristic equations

$$\frac{dt}{1} = \frac{dx}{p/m} = \frac{dS_t}{0} = \frac{dp}{-kx} = \frac{dS}{\frac{\partial S}{\partial t} + \frac{p^2}{m}} \tag{4.73}$$

Two integrals of (4.73) are

$$\psi_1 = S_t, \qquad \psi_2 = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}kx^2. \tag{4.74}$$

The second integral $H[x, p] = \psi_2$ is the constant total energy of the system, and from (4.60) we know that $\partial S/\partial t = \psi_1 = -H$. We can therefore assume that $S[x, t] = F[x] - Ht$. Now (4.72) can be written as

$$\frac{1}{2m}\left(\frac{\partial F}{\partial x}\right)^2 + \frac{1}{2}kx^2 = H, \tag{4.75}$$

which can be integrated to give

$$F = \sqrt{km}\int\sqrt{\frac{2H}{k} - x^2}\,dx. \tag{4.76}$$

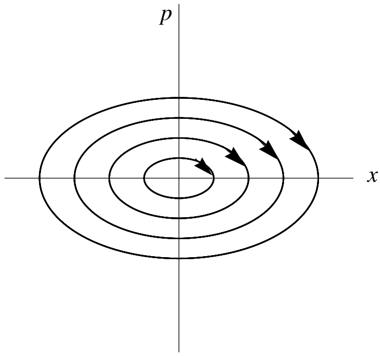


Fig. 4.2. Phase portrait of the undamped harmonic oscillator.

Finally, the action is

$$S[x,t] = \sqrt{km}\left(\frac{x}{2}\right)\sqrt{\frac{2H}{k} - x^2} + \sqrt{km}\left(\frac{H}{k}\right)\tan^{-1}\left[\frac{x}{\sqrt{\frac{2H}{k} - x^2}}\right] - Ht, \tag{4.77}$$

a somewhat more complicated function than the Hamiltonian itself. The system moves along the characteristics of

$$\frac{dx}{dt} = \frac{p}{m}, \qquad \frac{dp}{dt} = -kx, \tag{4.78}$$

which are a set of closed orbits in phase space as shown Figure 4.2. The arrows indicate the direction of increasing time. The period of the motion is $T = 2\pi\sqrt{m/k}$, and the radius of the orbit is proportional to the energy $H$.

***Example 4.3 (The two-body problem).*** Consider the motion of two particles moving under the action of a force field that acts between them (Figure 4.3).

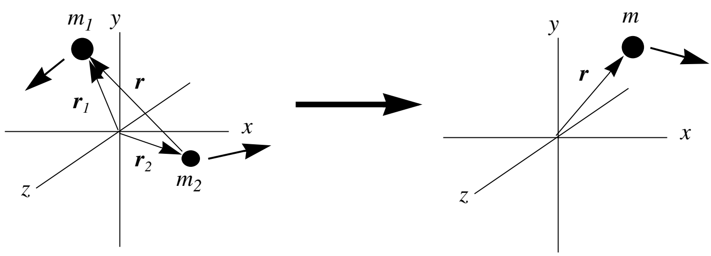


Fig. 4.3. Mapping of the two-body problem to an equivalent one-body problem in center-of-mass coordinates.

The Lagrangian is

$$L = \frac{1}{2}m_1\dot{\boldsymbol{r}}_1^2 + \frac{1}{2}m_2\dot{\boldsymbol{r}}_2^2 - V[|\boldsymbol{r}_1 - \boldsymbol{r}_2|], \tag{4.79}$$

where $\boldsymbol{r}_1 = (x_1, y_1, z_1)$ and $\boldsymbol{r}_2 = (x_2, y_2, z_2)$ are the radius vectors to the mass particles, $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ is the vector joining the particles, and $|\boldsymbol{r}_1 - \boldsymbol{r}_2|$ is the distance between them. To simplify the problem let's set the origin of coordinates at the center of mass of the two points, so that $m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 = 0$. In this system, the two position vectors can be expressed in terms of $\boldsymbol{r}$:

$$\boldsymbol{r}_1 = \frac{m_2\boldsymbol{r}}{m_1 + m_2}, \qquad \boldsymbol{r}_2 = \frac{m_1\boldsymbol{r}}{m_1 + m_2}. \tag{4.80}$$

If we insert these expressions into (4.79), the result is

$$L = \frac{1}{2}m\dot{\boldsymbol{r}}^2 - V[r], \tag{4.81}$$

or, in terms of the coordinates,

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V[\sqrt{x^2 + y^2 + z^2}], \tag{4.82}$$

where $m$ is the reduced mass

$$m = \frac{m_1 m_2}{m_1 + m_2}, \tag{4.83}$$

and the scalar distance $r$ is measured from the center-of-mass origin. By using the reduced mass, the two-body problem is reduced to the motion of a single particle in a spherically symmetric force field. Once the path $\boldsymbol{r}[t]$ has been determined, the motions of the individual particles are obtained by means of (4.80). The equations of motion generated by the Euler–Lagrange equations are

$$\begin{aligned} m\ddot{x} + \frac{x}{r}\left(\frac{\partial V}{\partial r}\right) &= 0, \\ m\ddot{y} + \frac{y}{r}\left(\frac{\partial V}{\partial r}\right) &= 0, \\ m\ddot{z} + \frac{z}{r}\left(\frac{\partial V}{\partial r}\right) &= 0. \end{aligned} \tag{4.84}$$

The Hamiltonian with $\boldsymbol{p} = m\dot{\boldsymbol{r}}$ is

$$H = \frac{1}{2}\frac{\boldsymbol{p}^2}{m} + V[r]. \tag{4.85}$$

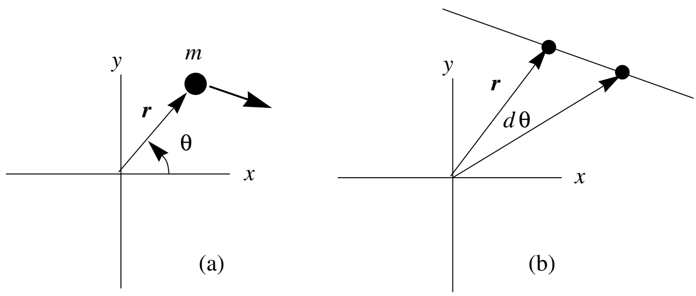


Fig. 4.4. Motion of the reduced-mass particle in cylindrical coordinates.

The motion of the particle actually takes place in a plane, and it is convenient to express the position of the particle in terms of the distance from the center of mass and the angle with respect to some reference axis, as shown in Figure 4.4. In these coordinates the Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V[r], \tag{4.86}$$

and the Hamiltonian is the total energy

$$H = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V[r], \tag{4.87}$$

which is conserved. The equations of motion in cylindrical coordinates simplify to

$$\begin{aligned} m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} &= 0, \\ \frac{d}{dt}(mr^2\dot{\theta}) &= 0. \end{aligned} \tag{4.88}$$

The second of the equations of motion expresses conservation of angular momentum in the center-of-mass system:

$$\Gamma = mr^2\dot{\theta} = \text{constant}. \tag{4.89}$$

Equation (4.89) can be interpreted using the sketch in Figure 4.4(b), which shows the sector swept out by the particle in a small period of time. The area of the sector is $dA = \boldsymbol{r}\cdot\boldsymbol{r}\ d\theta/2$, and $dA/dt = r^2\dot{\theta}/2 = \Gamma/2m$. This result is known as Kepler's second law: the particle sweeps out equal areas in equal times. Note that Kepler's second law applies for any central force field and does not assume anything about the radial dependence of the field.

The easiest way to reach the complete solution of the motion of the particle is to use the two conserved quantities. Use (4.89) to eliminate $\dot{\theta}$ from the Hamiltonian (4.87) and solve for the radial velocity

$$\frac{dr}{dt} = \left(\frac{2}{m}(H - V[r]) - \frac{\Gamma^2}{m^2 r^2}\right)^{1/2}. \tag{4.90}$$

The solution for the radius is expressed implicitly in terms of the time,

$$t = \int_{r_0}^{r} \frac{dr}{\left(\frac{2}{m}(H - V[r]) - \frac{\Gamma^2}{m^2 r^2}\right)^{1/2}}, \tag{4.91}$$

and the angle is determined from conservation of angular momentum,

$$\theta - \theta_0 = \int_{r_0}^{r} \frac{\Gamma\, dr}{r^2\left(2m(H - V[r]) - \frac{\Gamma^2}{r^2}\right)^{1/2}}. \tag{4.92}$$

As the particle moves under the influence of the central field, it is constrained to move in an annular disk between two radii, $r_{\min}$ and $r_{\max}$. The condition for the orbit to be closed is that the angle defined by (4.92) must be an integer multiple of $2\pi$ when the radius, starting at say $r_{\min}$, returns to $r_{\min}$. This only occurs for the case when the potential energy varies as $r^2$ or $1/r$. We shall return to this issue in Chapter 15 in the context of Problem 15.3.

This completes the general two-body problem. In the next example we will look at the important case of an inverse-square-law force field between the particles – the law that governs the heavens.

***Example 4.4 (Kepler's problem – the motion of celestial bodies).*** A very important class of two-body problems is defined by a potential field of the form

$$V = -\frac{\gamma}{r}, \tag{4.93}$$

for which the Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\gamma}{r} \tag{4.94}$$

and the generalized momenta are

$$\begin{aligned} p_1 &= m\dot{r}, \\ p_2 &= mr^2\dot{\theta} = \Gamma. \end{aligned} \tag{4.95}$$

We are considering the interaction between two gravitating bodies, where the constant $\gamma$ is

$$\gamma = Gm_1m_2 \tag{4.96}$$

and the force of attraction varies inversely as the square of the radius. This is the same problem we analyzed in Chapter 2 using dimensional analysis, except that there the final result was reached under an assumption, appropriate to the solar system, that one mass was much larger than the other. The equations of motion in Cartesian coordinates are

$$\begin{aligned} m\ddot{x} + \gamma\frac{x}{r^3} &= 0, \\ m\ddot{y} + \gamma\frac{y}{r^3} &= 0, \\ m\ddot{z} + \gamma\frac{z}{r^3} &= 0. \end{aligned} \tag{4.97}$$

These equations can be cast in terms of cylindrical coordinates in the plane of the motion:

$$\begin{aligned} m\ddot{r} - mr\dot{\theta}^2 + \frac{\gamma}{r^2} &= 0, \\ \frac{d}{dt}(mr^2\dot{\theta}) &= 0. \end{aligned} \tag{4.98}$$

The two-body solution is

$$\begin{aligned} t - t_0 &= \int_{r_0}^{r} \frac{dr}{\left(\frac{2}{m}\left(H + \frac{\gamma}{r}\right) - \frac{\Gamma^2}{m^2r^2}\right)^{1/2}}, \\ \theta - \theta_0 &= \int_{r_0}^{r} \frac{\Gamma\, dr}{r^2\left(2m\left(H + \frac{\gamma}{r}\right) - \frac{\Gamma^2}{r^2}\right)^{1/2}}. \end{aligned} \tag{4.99}$$

The integral relating the angle to the radius can be carried out, leading to

$$-\left(\frac{2Hm}{\Gamma^2}r^2 + \frac{2\gamma m}{\Gamma^2}r - 1\right)^{1/2} = \left(\left(\frac{\gamma m}{\Gamma^2}\right)r - 1\right)\tan[\theta - \theta_0] \tag{4.100}$$

Note that the initial radius $r_0$ does not appear in (4.100). The usual convention that is adopted is that $r_0$ corresponds to $\theta = \theta_0$. That is, $r_0$ satisfies $\frac{2Hm}{\Gamma^2}r_0^2 + \frac{2\gamma m}{\Gamma^2}r_0 - 1 = 0$ where the positive root is selected. This aligns the major axis of the orbit along the horizontal axis of coordinates. After some manipulation, the trajectory of the particle can be written as

$$\left(\frac{2H\Gamma^2}{m\gamma^2}+1\right)^{1/2} r\cos[\theta-\theta_0]-r+\frac{\Gamma^2}{\gamma m}=0. \tag{4.101}$$

This is the equation of a conic section with one focus at the origin and eccentricity

$$e=\left(\frac{2H\Gamma^2}{m\gamma^2}+1\right)^{1/2}. \tag{4.102}$$

The quantity

$$h=\frac{\Gamma^2}{\gamma m} \tag{4.103}$$

is one-half the so-called *latus rectum* of the trajectory of the particle. The equation for the path of the particle in cylindrical coordinates, (4.101), can be cast into Cartesian coordinates, and the result is as follows:

$$\frac{\left(x-\frac{eh}{1-e^2}\right)^2}{\left(\frac{h}{1-e^2}\right)^2}+\frac{y^2}{\left(\frac{h^2}{1-e^2}\right)}=1. \tag{4.104}$$

If the energy $H<0$, so that $e<1$, then the trajectory is an elliptical orbit as shown in Figure 4.5a. The semimajor and semiminor axes of the ellipse are

$$\begin{aligned} a&=\frac{h}{1-e^2}=\frac{\gamma}{-2H},\\ b&=\frac{h}{(1-e^2)^{1/2}}=\frac{\Gamma}{(-2Hm)^{1/2}}, \end{aligned} \tag{4.105}$$

while the apogee and perigee of the orbit are

$$r_{\text{min}}=\frac{h}{1+e},\qquad r_{\text{max}}=\frac{h}{1-e}. \tag{4.106}$$

The elliptical nature of the orbit is the first of Kepler's laws. Note that $r_0=r_{\text{max}}$.

The minimum-energy case $H=-\frac{m\gamma^2}{2\Gamma^2}$ corresponds to $e=0$, for which the orbit is a circle. If the total energy is positive ($H>0$), then $e>1$, the sign of the second term in (4.104) becomes negative, and the trajectory is a hyperbola

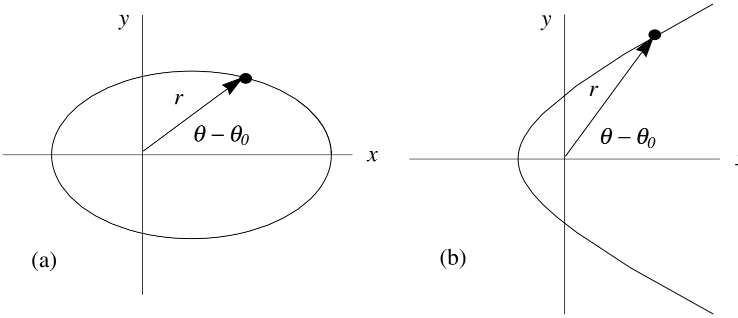


Fig. 4.5. Solutions of the Kepler problem for (a) negative total energy and (b) positive total energy.

as shown in Figure 4.5b. The radius of closest approach (the perihelion) is $r_{\text{min}} = h/(1+e)$. Finally, if $H = 0$, the trajectory is a parabola.

We can use the results of our analysis to develop the complete theory for the relationship between the radius of the orbit and the orbital period, which was treated using dimensional analysis in Chapter 2. For $H < 0$ the period $T$ of the orbit can be derived from the second of Kepler's laws,

$$dA/dt = r^2\dot{\theta} = \text{constant}. \tag{4.107}$$

The area of the orbit is

$$A = r^2\dot{\theta}T = \frac{\Gamma}{2m}T. \tag{4.108}$$

The area of an ellipse is $A = \pi ab$, and so the area of the orbit is

$$A = \frac{\pi\gamma\Gamma}{m^{1/2}(-2H)^{3/2}}. \tag{4.109}$$

Equating (4.108) and (4.109) the orbital period is

$$T = \frac{2\pi\gamma m^{1/2}}{(-2H)^{3/2}}. \tag{4.110}$$

Using $-2H = \gamma/a$, this result can be cast in terms of $r_{\text{mean}} = \sqrt{ab}$ and the gravitational constant $G = 6.670 \times 10^{-11}$ m$^3$/(kg-s$^2$). The result is Kepler's third law,

$$\frac{G(m_1+m_2)T^2}{(r_{\text{mean}})^3} = \frac{4\pi^2}{(1-e^2)^{3/4}}. \tag{4.111}$$

Eventually, in Chapter 15, we will return to this problem again and add a fourth law to the three discussed so far. Moreover, all four will be seen to be intimately connected to the symmetries of the Keplerian system (4.97). The simplest of these can be described now, and it involves the interpretation of (4.111) in terms of the group invariance of the Kepler system. The equations (4.97) are invariant under the dilation group,

$$\tilde{x} = e^{2a}x, \qquad \tilde{y} = e^{2a}y, \qquad \tilde{z} = e^{2a}z, \qquad \tilde{t} = e^{3a}t, \qquad \tilde{m} = m, \quad (4.112)$$

as is (4.111). The group (4.112) shows how to scale time and length to relate any two orbits $\tilde{t}^2/\tilde{r}^3 = t^2/r^3$, as we deduced in (4.111). This invariance is a direct consequence of the units of $G$, the same dimensioned constant that underlies the dimensional-analysis derivation of (4.111) described in Chapter 2, Section 2.2, Equation (2.7).

***Example 4.5 (Damped, linear second-order system).*** Hamiltonian dynamics is generally associated with energy-conserving systems. However it is not difficult to find examples of systems that dissipate energy for which the governing equations have a Hamiltonian analog. As an example, let's reexamine the spring–mass system depicted in Figure 4.1 and add friction proportional to the speed of the mass. The unforced equation of motion with damping is

$$m\frac{d^2x}{dx^2} + \alpha\frac{dx}{dt} + kx = 0, \quad (4.113)$$

where the damping coefficient is $\alpha > 0$. This second-order ODE can be broken into a linear autonomous pair

$$\frac{dx}{dt} = \frac{p}{m}, \qquad \frac{dp}{dt} = -kx - \frac{\alpha}{m}p, \quad (4.114)$$

where $p = m\,dx/dt$ is the same generalized momentum used previously. As it stands, the system (4.114) is not Hamiltonian, that is, it is not in the form

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial p}, \\ \frac{dp}{dt} &= -\frac{\partial H}{\partial x} \end{aligned} \quad (4.115)$$

where $H[x, p]$ is some scalar function of the generalized coordinate and momentum. Equation (4.113) can be easily solved in terms of exponentials:

$$x(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}, \quad (4.116)$$

where the eigenvalues are

$$\begin{aligned}\lambda_1 &= \frac{\alpha}{2m}\left(-1+\sqrt{1-\frac{4km}{\alpha^2}}\right),\\ \lambda_2 &= \frac{\alpha}{2m}\left(-1-\sqrt{1-\frac{4km}{\alpha^2}}\right),\end{aligned} \tag{4.117}$$

and $C_1$ and $C_2$ are arbitrary. The solution is a simple decaying exponential, as shown in the several phase portraits of (4.114) sketched in Figure 4.6. Depending on the value of the quadratic discriminant $\alpha^2 - 4km$, the system may be underdamped as in case (a), overdamped as in case (b), or critically damped as in case (c).

It is possible to put this system in Hamiltonian form, in essence by solving for the function that relates $p$ and $x$ in the phase portrait. We first eliminate $dt$ in the system (4.114) to generate the nonlinear first-order ODE that governs the

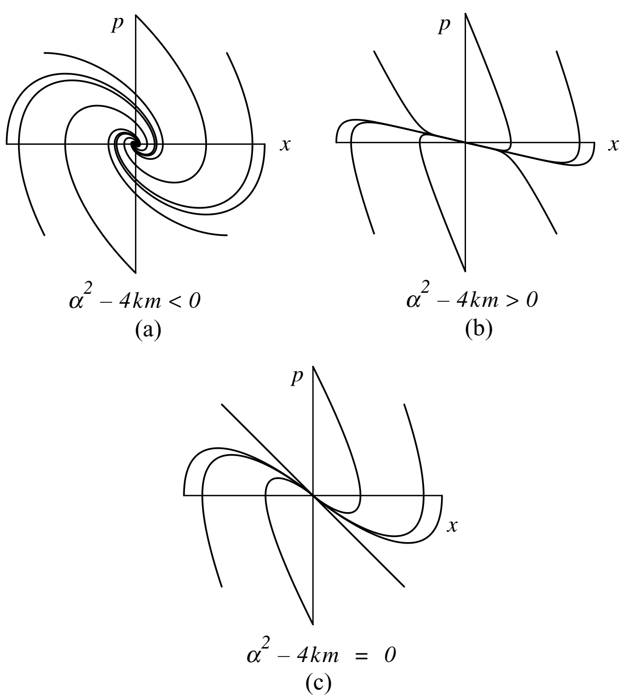


Fig. 4.6. Phase portrait of a damped oscillator.

trajectories in Figure 4.6:

$$pdp + (kmx + \alpha p)\, dx = 0. \tag{4.118}$$

Equation (4.118) is invariant under the dilation group,

$$\tilde{x} = e^a x, \qquad \tilde{p} = e^a p. \tag{4.119}$$

In Chapter 6 we shall learn how to utilize (4.119) to construct an integrating factor for (4.118). For the present, we simply state that as a consequence of this invariance (4.118) has the integrating factor

$$M = \frac{1}{kmx^2 + \alpha xp + p^2}. \tag{4.120}$$

This enables (4.118) to be converted to the integrable form

$$dH = \frac{kmx + \alpha p}{kmx^2 + \alpha xp + p^2}\, dx + \frac{p}{kmx^2 + \alpha xp + p^2}\, dp \tag{4.121}$$

The integral of (4.121) is

$$H = \frac{\alpha}{\sqrt{\alpha^2 - 4km}} \tanh^{-1}\left[\frac{\alpha x + 2p}{x\sqrt{\alpha^2 - 4km}}\right] + \ln\,[kmx^2 + \alpha xp + p^2]^{1/2}. \tag{4.122}$$

This is the sought-after Hamiltonian – a rather more complicated function than (4.9). Note that when $\alpha^2 - 4km$ is negative, the identity $\tanh^{-1}[ix] = i\tan^{-1}[x]$ ensures that the first term in (4.122) is always real. Furthermore one can show that the argument of the logarithm is always positive for positive $k$, $m$, and $\alpha$. The Hamiltonian analog of the system (4.114) is

$$\begin{aligned} \frac{dx}{dt} &= \frac{p}{kmx^2 + \alpha xp + p^2}, \\ \frac{dp}{dt} &= -\left(\frac{kmx + \alpha p}{kmx^2 + \alpha xp + p^2}\right). \end{aligned} \tag{4.123}$$

It must be emphasized that although (4.123) and (4.114) have identical solution trajectories in $(x, p)$ space, the times in the two systems are not at all the same. The system (4.114) describes the motion of the actual physical system in true time, whereas (4.123) describes the same motion (the same $x$, $p$ chart) but in a transformed time [not the same $(x, t)$ or $(p, t)$ chart]. The motion implied by (4.123) is quite unphysical. For example, in the physical system, as $(x, p) \to (0, 0)$, $(dx/dt, dp/dt) \to 0$; the motion comes smoothly to rest. But in the analog system $(dx/dt, dp/dt) \to \infty$, i.e., the analog mass would appear to move faster and faster as the motion decays. This limits the usefullness of Hamiltonian methods for treating this class of problems.

## 4.6 Concluding Remarks

Dynamics problems and the Hamilton–Jacobi equation will come up in a variety of contexts in the following chapters. The discussion of Lie–Bäcklund transformations in Chapter 14 leads naturally to Chapter 15, where a generalization of the variational approach used to derive the Euler–Lagrange system (4.22) is presented. The central point of Chapter 15 is Noether's theorem, showing how to use the symmetries of a generalized Euler–Lagrange system to construct conservation laws for the system. There the Kepler problem will be used again to provide a useful illustration of basic principles.

## 4.7 Exercises

4.1 Show that the Legendre transformation (4.30) can be thought of as a transformation where the locus of points in, say, $(x, y)$ coordinates is replaced by an equation in which the tangent $dy/dx$ at a point and the intercept of the tangent on the $x$-axis are used as independent variables. Show how the curve is constructed as the envelope of a family of straight lines.

4.2 A Foucault pendulum consisting of a heavy mass hanging from a flexible cable is an example of a *spherical pendulum* where the mass is constrained to move on the surface of a sphere of radius $R$. With the attachment of the cable at the center of the sphere and the polar axis downward, the Lagrangian of the system is

$$L = \frac{1}{2} m R^2 \left( \left( \frac{d\theta^2}{dt} \right) + \left( \frac{d\phi^2}{dt} \right) \sin^2 \theta \right) + mgR\cos\theta \qquad (4.124)$$

(i) Work out the equations of motion and identify any conserved quantities.

(ii) Integrate the equations and sketch the 3-D phase portrait. A treatment of perturbations about the motion on a sphere is presented in Reference [4.3].

4.3 Fully integrate the equations of motion for the two-body problem with $V = -\gamma/r^2$. Determine $r_{\min}$ and $r_{\max}$. See Chapter 15, Exercise 15.3.

4.4 Consider the linearly damped oscillator with periodic forcing,

$$\frac{d^2x}{dx^2} + \frac{\alpha}{m}\frac{dx}{dt} + \frac{k}{m}x = A\cos[\Omega t] \tag{4.125}$$

Let the Lagrangian be of the form

$$L = \frac{e^{(\alpha/m)t}}{2}\left(\frac{dx}{dt} + C_1\Omega\sin[\Omega t] - C_2\Omega\cos[\Omega t]\right)^2 - \frac{e^{(\alpha/m)t}}{2}\frac{k}{m}(x - C_1\cos[\Omega t] - C_2\sin[\Omega t])^2. \tag{4.126}$$

Choose $C_1$ and $C_2$ so that Lagrange's equation produces the correct equation of motion, (4.125). Find the Hamiltonian, and construct the appropriate Hamilton–Jacobi equation. Find a complete solution of the Hamilton–Jacobi equation in the form

$$S = \frac{e^{(\alpha/m)t}}{2} f[t](x - C_1\cos[\Omega t] - C_2\sin[\Omega t])^2 \tag{4.127}$$

where $f[t]$ is an unknown function. Show that $f$ satisfies the Ricatti equation,

$$\frac{df}{dt} + \alpha f + f^2 + \frac{k}{m} = 0. \tag{4.128}$$

Integrate this equation, and write down the complete solution for the action. Draw the phase portrait of the system at several times in the cycle. See Reference [4.4] for a discussion of this and other similar problems involving dissipative systems. See Chapter 8 Section 8.11.2 to determine what linear second-order ODE corresponds to (4.128).

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# 5

# *Introduction to One-Parameter Lie Groups*

In Chapter 1 the notion of symmetry and invariance of physical and mathematical objects was developed. A preliminary definition of a one-parameter Lie group was presented and used to illustrate several of the applications that are the main subject matter of this book. In Chapter 2 the role of groups in dimensional analysis was described along with several applications. Chapter 3 presents most of the tools of analysis needed to understand and use group theory. Especially important is the material on linear first-order PDEs and the associated system of characteristic ODEs, that are encountered constantly in group analysis. With all this introductory material out of the way, it is now time to present a formal definition of a one-parameter Lie group and to show how groups are used to identify the symmetry properties of differential equations and the continuous functions that they govern.

## 5.1 The Symmetry of Functions

We begin by considering the symmetry properties of functions. As in the case of the snowflake in Chapter 1, we look at how an object can be transformed without a change in its form or appearance except that now the object is a mathematical expression. As in the case of the snowflake, one maps source points to target points in a given coordinate space. Let's begin with a general definition of the symmetry of mathematical objects.

***Definition 5.1.*** *A mathematical relationship between variables is said to possess a symmetry property if one can subject the variables to a group of transformations and the resulting expression reads the same in the new variables as the original expression. The relationship is said to be invariant under the transformation group.*

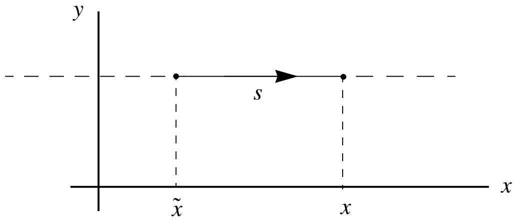


Fig. 5.1. Mapping of points by a translation group.

## 5.2 An Example and a Counterexample

Before we consider a formal definition of Lie groups, it is instructive to look at two simple examples.

### *5.2.1 Translation along Horizontal Lines*

Consider the transformation

$$\begin{aligned} x &= \tilde{x} + s, \\ y &= \tilde{y} \end{aligned} \tag{5.1}$$

(see Figure 5.1). By varying the transformation parameter $s$, we can move continuously and invertibly to any point $(x, y)$ on a horizontal line. For every $s$ there is an inverse $-s$ that restores the point to its original position. The identity element $s = 0$ transforms any point on the line to itself. A small change $\delta s$ produces a small change in $x$, $x \Rightarrow x + \delta x$. The line itself is called a *pathline* of the group (5.1).

Now write the transformation for a second value of $s$, and substitute one into the other to produce a third:

$$\left.\begin{aligned} x &= \tilde{x} + s_1 \\ y &= \tilde{y} \\ \\ \tilde{x} &= \tilde{\tilde{x}} + s_2 \\ \tilde{y} &= \tilde{\tilde{y}} \end{aligned}\right\} \rightarrow \begin{cases} x = \tilde{\tilde{x}} + s_3, \quad s_3 = s_1 + s_2, \\ y = \tilde{\tilde{y}}. \end{cases} \tag{5.2}$$

The operation indicated in (5.2) shows that when the transformation (5.1) is repeated and the two members of the group corresponding to $s_1$ and $s_2$ are composed, the result is a transformation that reads exactly the same as the original transformation with the new group parameter related to the first two by a function that is commutative in $s_1$ and $s_2$. The transformation (5.1) is said to

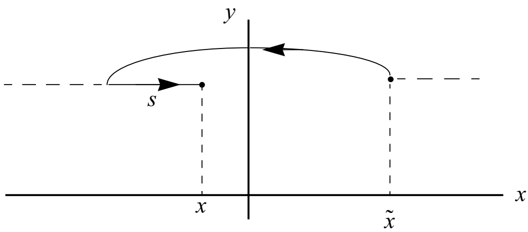


Fig. 5.2. A transformation that is not a group.

be invariant with respect to the binary operation of *composition*. The symmetry of $s_3$ under the exchange of $s_2$ and $s_1$ makes the transformation Abelian; i.e., it doesn't matter in which order the two initial transformations are applied. The transformation (5.1) is a Lie group.

### 5.2.2 A Reflection and a Translation

Now consider the transformation

$$\begin{aligned} x &= -\tilde{x} + s, \\ y &= \tilde{y} \end{aligned} \tag{5.3}$$

shown in Figure 5.2. Here a small change $\delta s$ does not lead to a small change in $x$. There is no identity element: no value of $s$ such that the transformation reduces to $x = \tilde{x}$, $y = \tilde{y}$. The transformation is not invariant under composition:

$$\left.\begin{aligned} x &= -\tilde{x} + s_1 \\ y &= \tilde{y} \end{aligned}\right\} \searrow \quad \left.\begin{aligned} \tilde{x} &= -\tilde{\tilde{x}} + s_2 \\ \tilde{y} &= \tilde{\tilde{y}} \end{aligned}\right\} \nearrow \quad \left\{\begin{aligned} x &= \tilde{\tilde{x}} + s_1 - s_2, \\ y &= \tilde{\tilde{y}}. \end{aligned}\right. \tag{5.4}$$

Note the change in sign when the transformations are composed. The transformation (5.4) is not a Lie group.

## 5.3 One-Parameter Lie Groups

***Definition 5.2.*** *Let the vector* $\boldsymbol{x} = (x^1, x^2, \ldots, x^n)$ *lie in some continuous open set* $D$ *on the* $n$*-dimensional Euclidean manifold* $\mathbb{R}^n$. *Define the transformation*

$$T^s : \{x^j = F^j[\tilde{\boldsymbol{x}}, s],\ \ j = 1, \ldots, n\}. \tag{5.5}$$

*The functions $F^j$ are infinitely differentiable with respect to the real variables $\boldsymbol{x}$ and are analytic functions of the real continuous parameter $s$, which lies in an open interval, $S$.*

*The transformation $T^s$ is a one-parameter Lie group with respect to the binary operation of composition if and only if:*

(i) *There is an* identity *element $s \to s_0$ such that $\tilde{\boldsymbol{x}}$ is mapped to itself:*

$$T^{s_0} : \{\tilde{x}^j = F^j[\tilde{\boldsymbol{x}}, s_0],\ \ j = 1, \ldots, n\}. \tag{5.6}$$

*Note that the identity element can always be arranged to be zero.*

(ii) *For every value of $s$ there is an* inverse *$s \to s_{\text{inv}}$ such that $\boldsymbol{x}$ is returned to $\tilde{\boldsymbol{x}}$:*

$$T^{s_{\text{inv}}} : \{\tilde{x}^j = F^j[\boldsymbol{x}, s_{\text{inv}}],\ \ j = 1, \ldots, n\}. \tag{5.7}$$

(iii) *The binary operation of composition produces a transformation that is a member of the group $T^{s_1} \cdot T^{s_2} = T^{s_3}$ i.e., the group is closed. Consider two members of the group,*

$$T^{s_1} : \{x^j = F^j[\tilde{\boldsymbol{x}}, s_1],\ \ j = 1, \ldots, n\} \tag{5.8}$$

*and*

$$T^{s_2} : \{\tilde{x}^j = F^j[\tilde{\tilde{\boldsymbol{x}}}, s_2],\ \ j = 1, \ldots, n\}. \tag{5.9}$$

*If we compose $T^{s_1}$ and $T^{s_2}$, the result is*

$$T^{s_3} : \{x^j = F^j[\boldsymbol{F}[\tilde{\tilde{\boldsymbol{x}}}, s_2], s_1] = F^j[\tilde{\tilde{\boldsymbol{x}}}, s_3],\ \ j, \ldots, n\}, \tag{5.10}$$

*where $s_3 = \phi[s_1, s_2] \in S$. The function $\phi$ defining the law of composition of $T^s$ is an analytic function of $s_1 \in S$ and $s_2 \in S$ and is commutative* $(s_3 = \phi[s_1, s_2] = \phi[s_2, s_1])$*; thus Lie groups are Abelian.*

(iv) *The group is* associative*: $(T^{s_1} \cdot T^{s_2}) \cdot T^{s_3} = T^{s_1} \cdot (T^{s_2} \cdot T^{s_3})$. Consider three elements of the group,*

$$x^j = F^j[\tilde{\boldsymbol{x}}, s_1], \qquad \tilde{x}^j = F^j[\tilde{\tilde{\boldsymbol{x}}}, s_2], \qquad \tilde{\tilde{x}}^j = F^j[\tilde{\tilde{\tilde{\boldsymbol{x}}}}, s_3]. \tag{5.11}$$

*Composing the first two and then the third leads to*

$$\begin{array}{ccc} x^j = F^j[\tilde{\mathbf{x}}, s_1] & \tilde{x}^j = F^j[\tilde{\tilde{\mathbf{x}}}, s_2] & \tilde{\tilde{x}}^j = F^j[\tilde{\tilde{\tilde{\mathbf{x}}}}, s_3] \\ \searrow \quad \swarrow & & \swarrow \\ x^j = F^j[\tilde{\tilde{\mathbf{x}}}, \phi[s_1, s_2]] & & \\ \searrow & & \\ & x^j = F^j[\tilde{\tilde{\tilde{\mathbf{x}}}}, \varphi[s_1, s_2, s_3]] & \end{array} \tag{5.12}$$

*Composing the second and third and then the first leads to the same result*

$$\begin{array}{ccc} x^j = F^j[\tilde{\mathbf{x}}, s_1] & \tilde{x}^j = F^j[\tilde{\tilde{\mathbf{x}}}, s_2] & \tilde{\tilde{x}}^j = F^j[\tilde{\tilde{\tilde{\mathbf{x}}}}, s_3] \\ \searrow & \searrow \quad \swarrow & \\ & \tilde{x}^j = F^j[\tilde{\tilde{\tilde{\mathbf{x}}}}, \ \phi[s_2, s_3]] & \\ & \swarrow & \\ x^j = F^j[\tilde{\tilde{\tilde{\mathbf{x}}}}, \varphi[s_1, s_2, s_3]] & & \end{array} \tag{5.13}$$

The Abelian nature of the group implies that the composition function $s_4 = \varphi[s_1, s_2, s_3]$ is invariant under any permutation of $s_1$, $s_2$, and $s_3$. Note that here, where there is no possibility of confusion with a derivative, we have used subscripts as labels for the various values of the group parameter.

The four attributes *identity, inverse, closure*, and *associativity* are the same ones that we encountered in discussing the discrete symmetry group of a snowflake in Chapter 1. The main difference is that the group considered in Chapter 1 had a finite number of members and the relational operator of the group (matrix multiplication) was not commutative.

The operation of composition consists in substituting the transformation into itself. The transformation is a group if one can rearrange terms so that the new expression reads the same as the old expression, but in new variables. The parameter of the final transformation must be expressible as a function of the parameters of the two composed functions. At first sight this may seem to be a highly restrictive condition, which only a handful of transformations could possibly satisfy. In fact, while relatively simple transformations are normally used to illustrate most concepts, Lie groups are extremely general and commonly arise as the continuous functions that solve systems of autonomous ODEs such as those discussed in Chapters 1 and 3. The exact correspondence in notation between this chapter and Chapter 3 is intentional.

## 5.4 Invariant Functions

Central to all of the development of symmetry theory is the concept of an invariant function. The example below illustrates the basic idea.

***Example 5.1 (Invariance of a parabola under dilation).*** Transform

$$\Psi[x, y] = y/x^2 \tag{5.14}$$

using the dilation (or stretching) group

$$T^{\text{dil}} : \{x = s\tilde{x},\ \ y = s^n\tilde{y},\ \ s > 0\}. \tag{5.15}$$

The restriction on $s$ in (5.15) arises because $s = 0$ has no inverse. Once a point has been mapped to $(x, y) = (0, 0)$, there is no way to return to the original point by some choice of $s$.

*The important property of a group that makes it so useful is that it is always possible to transform points smoothly and invertibly along the pathlines traced out by the group.*

Note that the identity element of (5.15) is $s = 1$. A more natural way of expressing the dilation group is to write it in the form

$$T^{\text{dil}} : \{x = e^s\tilde{x},\ \ y = e^{ns}\tilde{y}\}. \tag{5.16}$$

Now the parameter $s$ can take on the full range of values from $-\infty$ to $+\infty$, and the identity element is $s = 0$. This is the form of the dilation group that was used in the discussion of dimensional analysis in Chapter 2. Use (5.16) to transform (5.14):

$$\Psi[x, y] = y/x^2 = e^{s(n-2)}(\tilde{y}/\tilde{x}^2). \tag{5.17}$$

For general $n$ the function is not invariant under the group; however, if we set $n = 2$, then

$$\Psi[x, y] = y/x^2 = \tilde{y}/\tilde{x}^2 = \Psi[\tilde{x}, \tilde{y}]. \tag{5.18}$$

The parameter $s$ does not appear in (5.18), and the function is said to be invariant under (5.16) for $n = 2$.

Invariance holds only if the function reads the same, *exactly the same*, when expressed in new variables. The tildes over the variables play the same role that the labels $A$, $B$, $C$, $D$, $E$, $F$ did in the case of the snowflake described in Chapter 1. Such labels are needed as place markers for the mapping from the source point to the target point of the transformation and do not compromise the invariance of the mathematical object being considered (in this case a functional relationship between $x$ and $y$). We can express this notion of invariance in the following definition.

***Definition 5.3.*** *A function* $\Psi[\boldsymbol{x}]$ *is said to be invariant under the Lie group* $T^s : \{x^j = F^j[\tilde{\boldsymbol{x}}, s],\ \ j = 1, \ldots, n\}$ *if and only if*

$$\Psi[\boldsymbol{x}] = \Psi[\boldsymbol{F}[\tilde{\boldsymbol{x}}, s]] = \Psi[\tilde{\boldsymbol{x}}]. \tag{5.19}$$

For invariance, the parameter $s$ must vanish from the transformation so that the function reads the same in the new variables.

## 5.5 Infinitesimal Form of a Lie Group

In the last section we defined one-parameter Lie groups of the form

$$\tilde{x}^j = F^j[\boldsymbol{x}, s], \tag{5.20}$$

where $s$ is the group parameter, which, for present purposes, will be assumed to be defined in such a way that the identity element is $s_0 = 0$. Thus

$$x^j = F^j[\boldsymbol{x}, 0]. \tag{5.21}$$

One may notice that, up to this point, we have used the tilde to denote the source point of the transformation, and in (5.20) we have suddenly switched the role of the tilde to denote the target point of the transformation. Of course it is immaterial which value of $\boldsymbol{x}$ is assigned the tilde, which is merely a distinguishing mark. We are about to develop the infinitesimal theory of groups, which involves expanding the transformation (5.20) about the source point, and for convenience it is simpler to assign the tilde to the target value of the transformation. This avoids having to carry tildes along in all our formulas.

Now expand (5.20) in a Taylor series about $s = 0$:

$$\tilde{x}^j = x^j + s\left[\frac{\partial F^j}{\partial s}\right]_{s=0} + O(s^2) + \cdots, \qquad j = 1, \ldots, n. \tag{5.22}$$

The derivatives of the various $F^j$ with respect to the group parameter $s$ evaluated at $s = 0$ are called the *infinitesimals* of the group and are traditionally denoted by $\xi^j$:

$$\boxed{\xi^j[\boldsymbol{x}] = \left[\frac{\partial}{\partial s} F^j[\boldsymbol{x}, s]\right]_{s=0}, \qquad j = 1, \ldots, n.} \tag{5.23}$$

The vector $\xi^j$ is also called the *vector field* of the group (5.20). The notation of (5.23) is intentionally chosen as a reminder of the vector fields of autonomous ODEs discussed in Chapter 3.

## 5.6 Lie Series, the Group Operator, and the Infinitesimal Invariance Condition for Functions

The condition for invariance of a function given in (5.19) is difficult to apply in practice because of the usually nonlinear dependence of $F^j$ on the group parameter $s$. The condition (5.19) requires that the transformation be substituted into $F^j$ and then rearranged to read like the original function. This can be an extremely tedious procedure for testing the invariance of complicated functions and becomes hopelessly complex when it comes to the testing of differential equations. We need something that is equivalent to (5.19) but much simpler to apply. To this end, substitute (5.20) into the analytic function $\Psi[\tilde{\boldsymbol{x}}]$:

$$\Psi[\tilde{\boldsymbol{x}}] = \Psi[\boldsymbol{F}[\boldsymbol{x}, s]]. \tag{5.24}$$

Now expand (5.24) in a Taylor series about the identity element $s = 0$:

$$\Psi[\tilde{\boldsymbol{x}}] = \Psi[\boldsymbol{x}] + s\left[\frac{\partial \Psi}{\partial s}\right]_{s=0} + \frac{s^2}{2!}\left[\frac{\partial^2 \Psi}{\partial s^2}\right]_{s=0} + \frac{s^3}{3!}\left[\frac{\partial^3 \Psi}{\partial s^3}\right]_{s=0} + \cdots. \tag{5.25}$$

Using the chain rule

$$\left[\frac{\partial \Psi}{\partial s}\right]_{s=0} = \frac{\partial \Psi}{\partial F^j}\left[\frac{\partial F^j}{\partial s}\right]_{s=0} = \xi^j \frac{\partial \Psi}{\partial F^j} \tag{5.26}$$

the expansion (5.25) becomes the *Lie series* representation of the function $\Psi$:

$$\Psi[\tilde{\boldsymbol{x}}] = \Psi[\boldsymbol{x}] + s\left(\xi^j \frac{\partial \Psi}{\partial x^j}\right) + \frac{s^2}{2!}\xi^j \frac{\partial}{\partial x^j}\left(\xi^{j_1}\frac{\partial \Psi}{\partial x^{j_1}}\right) + \frac{s^3}{3!}\xi^j \frac{\partial}{\partial x^j}\left(\xi^{j_1}\frac{\partial}{\partial x^{j_1}}\left(\xi^{j_2}\frac{\partial \Psi}{\partial x^{j_2}}\right)\right) + \cdots. \tag{5.27}$$

where $j_1, j_2 \ldots$ are dummy indices that are summed from 1 to 4.

The condition $\Psi[\tilde{\boldsymbol{x}}] = \Psi[\boldsymbol{x}]$ is satisfied if and only if $\xi^j \partial\Psi/\partial x^j = 0$. We can now state the infinitesimal condition for invariance of a function.

***Theorem 5.1.*** *The analytic function* $\Psi[\boldsymbol{x}]$ *is invariant under the Lie group* $T^s : \{\tilde{x}^j = F^j[\boldsymbol{x}, s],\ j, \ldots, n\}$ *or, equivalently, the infinitesimal group* $\xi^j[\boldsymbol{x}]$,

*$j, \ldots, n$, if and only if $\Psi[\boldsymbol{x}]$ satisfies the condition*

$$\boxed{\xi^j[\boldsymbol{x}]\frac{\partial \Psi}{\partial x^j} = 0.} \tag{5.28}$$

Equation (5.28) is one we encountered in Chapter 3 when we discussed the first-order PDE that governs the $n-1$ characteristic functions $\Psi^i[\boldsymbol{x}]$ that satisfy an autonomous system of $n$ first-order ODEs. In the parlance of group theory the functions $\Psi^i[\boldsymbol{x}]$ are the *invariants* of the group $T^s$. The operator

$$\boxed{X \equiv \xi^j[\boldsymbol{x}]\frac{\partial}{\partial x^j}} \tag{5.29}$$

is called the *group operator* and $X\Psi$ is called the Lie derivative of $\Psi$.

The Lie series (5.27) can be written concisely using the group operator. Any analytic function can be expanded as

$$\boxed{\Psi[\tilde{\boldsymbol{x}}] = \Psi[\boldsymbol{x}] + s(X\Psi) + \frac{s^2}{2!}X(X\Psi) + \frac{s^3}{3!}X(X(X\Psi)) + \cdots.} \tag{5.30}$$

The Lie series (5.30) can be formally written as the exponential map,

$$\Psi[\tilde{\boldsymbol{x}}] = e^{sX}\Psi[\boldsymbol{x}]. \tag{5.31}$$

Sophus Lie's great advance was to replace the complicated, nonlinear finite invariance condition (5.19) by the vastly more useful linear infinitesimal condition (5.28) and to recognize that if a function satisfies the infinitesimal condition then it also satisfies the finite condition. This is the key point that enables Lie theory to be applied usefully to nonlinear problems – one is virtually always working with a linear invariance condition.

Although the theory of Lie groups is a vast discipline with many facets, the development beginning with (5.20) and ending with (5.31) is really the core of the theory required to understand and use groups in applications. This becomes especially evident when we introduce the concept of a differential function in Chapter 7, where ODEs and PDEs are viewed as locally analytic functions in a jet space whose coordinates are independent variables, dependent variables, and the various derivatives of one with respect to the other. In a broad sense, the invariance condition for an ODE or a PDE is no different from the condition defined in Theorem 5.1 for functions; it is just that it is applied in the extended tangent space of differential functions.

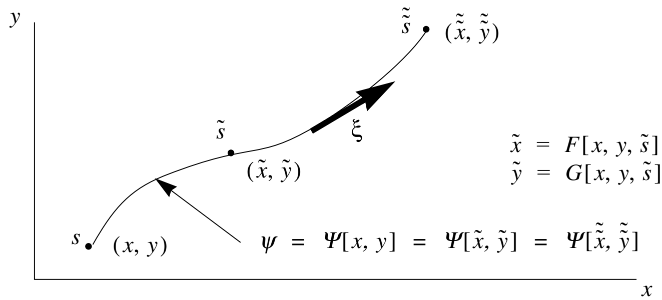


Fig. 5.3. Mapping of points along a single characteristic curve.

### *5.6.1 Group Operators and Vector Fields*

The reader should be aware that there is a tendency in the literature to use the term "group" interchangeably to refer to $\boldsymbol{F}$, $\xi^j$, and the operator $X$. Inasmuch as $X$ is the generator of $\boldsymbol{F}$ via the Lie series (5.30), $X$ is also appropriately termed the vector field of the group $T^s$. Note the similarity between the representation of a group in terms of (5.5), (5.23), and (5.29) and the solution of a system of ODEs discussed in Chapter 3. As was explained in Chapter 3, the solution trajectories of an autonomous system of ODEs can be regarded as the finite form of a Lie group, and the functions on the right-hand side of that system can be regarded as the infinitesimals of a group operator.

To see this, consider several points on one characteristic curve shown in Figure 5.3. Each point on the trajectory can be reached from any other point by a suitable choice of the parameter $s$. Therefore every point has an inverse. Every point can be mapped to itself by a suitable choice of $s$: the group has an identity element. The path by which a point is moved, say up and back along the trajectory, has no effect on the value of $s$ at the final point: the group is associative.

## 5.7 Solving the Characteristic Equation $X\Psi[\boldsymbol{x}] = 0$

As discussed in Chapter 2, the linear first-order PDE $\xi^j[\boldsymbol{x}](\partial\Psi/\partial x^j) = 0$ has an associated system of $n-1$ characteristic first-order ODEs of the form

$$\boxed{\frac{dx^1}{\xi^1[\boldsymbol{x}]} = \frac{dx^2}{\xi^2[\boldsymbol{x}]} = \frac{dx^3}{\xi^3[\boldsymbol{x}]} = \cdots = \frac{dx^n}{\xi^n[\boldsymbol{x}]}} \tag{5.32}$$

with integrals

$$\psi^i = \Psi^i[\boldsymbol{x}], \qquad i = 1, \ldots, n-1, \tag{5.33}$$

which are the invariants of the group. There is an important point here that, in a sense, is so obvious that it needs to be highlighted. Each of the functions $\Psi^i$ represents an infinite family of curves (or surfaces), one for each possible value of $\psi^i$. The family as a whole is invariant under the group $F^j$ with infinitesimals $\xi$. Also, every curve $\psi^i =$ constant is *individually* invariant under the group, i.e., an initial point on a given solution curve is mapped by $\boldsymbol{F}$ to a new point on the *same* curve, as depicted in Figure 5.3. This must be so, since the solution trajectories of the autonomous system

$$\boxed{\frac{dx^j}{ds} = \xi^j[\boldsymbol{x}], \qquad j = 1, \ldots, n,} \tag{5.34}$$

obviously must lie on the invariant surfaces.

It is fruitful to consider transformations that do not leave individual curves invariant but do leave the family as a whole invariant. This point was mentioned in connection with the ODE example 1.1 in Chapter 1 and will come up again in Chapter 6 when we discuss integrating factors for ordinary differential equations.

***Example 5.2 (The rotation group in two dimensions).*** Consider the rotation group

$$T^{rot} : \left\{ \begin{aligned} \tilde{x} &= x\cos[s] - y\sin[s] \\ \tilde{y} &= x\sin[s] + y\cos[s] \end{aligned} \right\}. \tag{5.35}$$

The infinitesimals of the group are $(\xi, \eta) = (-y, x)$, and the invariance condition is

$$-y\frac{\partial\Psi}{\partial x} + x\frac{\partial\Psi}{\partial y} = 0 \tag{5.36}$$

with corresponding characteristic equation

$$\frac{dy}{x} = -\frac{dx}{y}. \tag{5.37}$$

Equation (5.37) is particularly simple in that the two terms are uncoupled. The integral invariant [the integral of (5.36), invariant under (5.35)] is the family of circles

$$\psi = \Psi(x, y) = x^2 + y^2. \tag{5.38}$$

On a given circle $\psi$, we can solve for $y$ as a function of $x$ to obtain $y = \pm(\psi - x^2)^{1/2}$. Differentiating this result gives $dy/dx = -x/(\pm(\psi - x^2)^{1/2}) = -x/y$, which checks with (5.37).

### *5.7.1 Invariant Points*

There may be isolated points in $\boldsymbol{x}$ which are invariant under the group $\boldsymbol{F}$ (or $\xi^j$ or $X$). These points correspond to the roots of

$$\xi^j[\boldsymbol{x}] = 0. \tag{5.39}$$

Invariant points of the group are the critical points of (5.34). See the discussion of critical points in two and three dimensions in Chapter 3.

## 5.8 Reconstruction of a Group from Its Infinitesimals

In Section 5.6 we saw that *any* analytic function $G[\boldsymbol{x}]$ can be expanded in a Lie series in terms of the group operator $X$:

$$G[\tilde{\boldsymbol{x}}] = G[\boldsymbol{x}] + s(XG) + \frac{s^2}{2!}X(XG) + \frac{s^3}{3!}X(X(XG)) + \cdots. \tag{5.40}$$

Let $G[\boldsymbol{x}] = x^j$ for each $x^j$ in turn. Then the Lie series becomes

$$\tilde{x}^j = x^j + s(Xx^j) + \frac{s^2}{2!}X(Xx^j) + \frac{s^3}{3!}X(X(Xx^j)) + \cdots, \qquad j = 1, \dots, n. \tag{5.41}$$

For simple $\xi^j[\boldsymbol{x}]$ the series (5.41) can be summed explicitly. Formally (5.41) can be represented as an exponential map, (see 5.31)

$$\tilde{x}^j = e^{sX}x^j. \tag{5.42}$$

The process by which a source point $\boldsymbol{x}$ is transformed to a target point $\tilde{\boldsymbol{x}}$ is sometimes called *dragging*. In fact the Lie-series form of the finite group given in (5.41) can be used as the basis of a numerical algorithm for solving the characteristic equations (5.34). The procedure is to generate an expansion to whatever order is desired for each of the $x^j$, $j = 1, \dots, n$. These $n$ equations, relating source and target points for a small value of $s$, are then used to approximate the solution at successive points in phase space beginning at a given initial point. It is interesting to compare the accuracy of this procedure with more conventional numerical schemes, such as a fourth-order Runge–Kutta method (see Exercise 5.6).

***Example 5.3 (The rotation group).*** The infinitesimals of the rotation group are $(\xi, \eta) = (-y, x)$, and operating on $x$ we have

$$Xx = -y, \qquad X(Xx) = -x, \qquad X^3 x = y, \qquad X^4 x = x, \ldots. \tag{5.43}$$

The $x$-series (5.41) separates into two terms:

$$\tilde{x} = x\left(1 - \frac{s^2}{2!} + \frac{s^4}{4!} - \cdots\right) - y\left(s - \frac{s^3}{3!} + \frac{s^5}{5!} - \cdots\right) = x\cos[s] - y\sin[s]. \tag{5.44}$$

The terms for the $y$-series are

$$Xy = x, \qquad X(Xy) = -y, \qquad X^3 y = -x, \qquad X^4 y = y, \ldots, \tag{5.45}$$

and the series for $y$ is

$$\tilde{y} = x\left(s - \frac{s^3}{3!} + \frac{s^5}{5!} - \cdots\right) + y\left(1 - \frac{s^2}{2!} + \frac{s^4}{4!} - \cdots\right) = x\sin[s] + y\cos[s]. \tag{5.46}$$

Summing the Lie series produces the finite form of the rotation group.

***Example 5.4 (A dilation group).*** Let $(\xi, \eta) = (x, y)$. Then

$$Xx = x, \qquad X(Xx) = x, \qquad X^3 x = x, \qquad X^4 x = x, \ldots. \tag{5.47}$$

In this case the series has only one term

$$\tilde{x} = x\left(1 + \frac{s^2}{2!} + \frac{s^3}{3!} + \frac{s^4}{4!} + \frac{s^5}{5!} + \cdots\right) = e^s x. \tag{5.48}$$

Note that the exponential form of the dilation group is recovered by this process:

$$Xy = y, \qquad X(Xy) = y, \qquad X^3 y = y, \qquad X^4 y = y, \ldots, \tag{5.49}$$

and the series for $y$ is

$$\tilde{y} = y\left(1 + \frac{s^2}{2!} + \frac{s^3}{3!} + \frac{s^4}{4!} + \frac{s^5}{5!} + \cdots\right) = e^s y. \tag{5.50}$$

As discussed in Chapter 2, dilation groups play the central role in dimensional analysis.

***Example 5.5 (A translation group).*** Let $(\xi, \eta) = (0, 1)$; then

$$\begin{aligned} Xx &= 0, \quad X^n x = 0 \qquad \text{for all } n \\ Xy &= 1, \quad X^n y = 0 \qquad \text{for all } n > 1 \end{aligned} \tag{5.51}$$

In this case the series truncates:

$$\tilde{x} = x, \qquad \tilde{y} = y + s. \tag{5.52}$$

A reminder of the law of covariance may be worthwhile here. The fundamental equations of physics are invariant under translation and rotation of the dependent and independent variables due to the basic homogeneity and isotropy of free space. In addition the equations are invariant under dilation reflecting the dimensional consistency of the equations.

## 5.9 Multiparameter Groups

Multiparameter groups arise when we consider higher-order ODEs and especially when we consider PDEs. The finite form of a multiparameter group may be extremely complex and is rarely required in applications. On the other hand, the infinitesimal generator of a multiparameter group is a simple linear sum of the independent generators of its parameters.

To illustrate the analysis of a multiparameter group, let's look in some detail at the projective group in $n$ dimensions:

$$T^{\text{projn}} : \left\{ \tilde{x}^j = \frac{x^j + a_j + b_{jk} x^k}{1 + c_k x^k}, \ j = 1, 2, \ldots, n \right\}, \tag{5.53}$$

where the sum on the index $k$ is from 1 to $n$. The projective group has the property that it maps straight lines to straight lines in $n$ dimensions. The group has $r = n^2 + 2n$ independent group parameters. Rather than take a series of partial derivatives, which could get rather tedious, let's determine the infinitesimal form of the transformation (5.53) in another way. Let each parameter be replaced by a scale factor proportional to the parameter itself. Let

$$a_j \Rightarrow a_j s, \qquad b_{jk} \Rightarrow b_{jk} s, \qquad c_k \Rightarrow c_k s. \tag{5.54}$$

Substitute (5.54) into (5.53),

$$\tilde{x}^j = \frac{x^j + (a_j + b_{jk} x^k) s}{1 + (c_k x^k) s}, \qquad j = 1, 2, \ldots, n. \tag{5.55}$$

Now assume $s$ is infinitesimally small, and approximate the denominator by the first two terms of a binomial series:

$$\tilde{x}^j = (x^j + (a_j + b_{jk}x^k)s)(1 - (c_k x^k)s), \qquad j = 1, 2, \ldots, n. \quad (5.56)$$

Expand (5.56) and retain only the lowest-order term in $s$:

$$\tilde{x}^j = x^j + (a_j + b_{jk}x^k - c_k x^k x^j)s, \qquad j = 1, 2, \ldots, n. \quad (5.57)$$

The infinitesimals of the projective group are

$$\xi^j(\boldsymbol{x}) = a_j + b_{jk}x^k - c_k x^k x^j, \;\; j = 1, 2, \ldots, n. \quad (5.58)$$

Note that when we deal with multiparameter groups, the scale factors for the parameters appear in the expressions for the infinitesimals. It is essential that they do so in order to keep track of the contribution of each one-parameter group to the generator of the multiparameter group. Each constant in the infinitesimal is a marker for a corresponding finite one-parameter group and group operator. The transformation (5.53) defines $r = n^2 + 2n$ independent one-parameter groups with group operators

$$X^{a_j} = \frac{\partial}{\partial x^j}, \qquad X^{b_{jk}} = x^k \frac{\partial}{\partial x^j}, \qquad X^{c_k} = x^k x^j \frac{\partial}{\partial x^j}. \quad (5.59)$$

### *5.9.1 The Commutator*

The operators of the group considered above have the interesting and useful property that they form a closed set with respect to commutation. The *commutator* of two group operators $X^a$ and $X^b$ is the operator generated as follows:

$$\{X^a, X^b\} = X^a(X^b) - X^b(X^a). \quad (5.60)$$

Let

$$X^a = \alpha^j[\boldsymbol{x}]\frac{\partial}{\partial x^j}, \qquad X^b = \beta^j[\boldsymbol{x}]\frac{\partial}{\partial x^j}. \quad (5.61)$$

The commutator is

$$\begin{aligned}\{X^a, X^b\} &= \alpha^j[\boldsymbol{x}]\frac{\partial}{\partial x^j}\left(\beta^k[\boldsymbol{x}]\frac{\partial}{\partial x^k}\right) - \beta^k[\boldsymbol{x}]\frac{\partial}{\partial x^k}\left(\alpha^j[\boldsymbol{x}]\frac{\partial}{\partial x^j}\right) \\ &= \left(\alpha^j \frac{\partial \beta^k}{\partial x^j}\right)\frac{\partial}{\partial x^k} - \left(\beta^k \frac{\partial \alpha^j}{\partial x^k}\right)\frac{\partial}{\partial x^j}. \quad (5.62)\end{aligned}$$

Note that the second derivatives in the commutator always drop out, so the result is still a first-derivative Lie operator. For example, the first two operators in (5.59) give

$$\{X^{a_j}, X^{b_{j_1k}}\} = X^{a_j}(X^{b_{j_1k}}) - X^{b_{j_1k}}(X^{a_j}) = \frac{\partial}{\partial x^j}\left(x^k \frac{\partial}{\partial x^{j_1}}\right) - x^k \frac{\partial}{\partial x^{j_1}}\left(\frac{\partial}{\partial x^j}\right), \tag{5.63}$$

which reduces to

$$\{X^{a_j}, X^{b_{j_1k}}\} = \delta^j_k \frac{\partial}{\partial x^k} = \begin{cases} X^{a_j} & \text{for} \quad j = k, \\ 0 & \text{otherwise,} \end{cases} \tag{5.64}$$

which is again one of the operators in (5.59). The operators (5.59) form a vector space called a Lie algebra. The implication of this is that one can analyze the nature of a group without having to consider the full nonlinear multiparameter transformation, which may be extremely complicated. Rather, it is sufficient to study its Lie algebra, which is a much simpler object.

## 5.10 Lie Algebras

***Definition 5.2.*** *The infinitesimal generators $X^k$, $k = 1, \ldots, r$, of the $r$, parameter Lie group $T^{a_1,\ldots,a_r} : \{\tilde{x}^j = F^j[x^1, \ldots, x^n; a_1, \ldots, a_r], \ j = 1, \ldots, n\}$ form an $r$-dimensional Lie algebra $\Lambda^r$ with the following properties. Let $X^a, X^b, X^c \in \Lambda^r$, and let $\alpha, \beta$ be real constants. The null algebra is $\Lambda^0$.*

*(i) The Lie algebra $\Lambda^r$ is an $r$-dimensional vector space spanned by the basis set of infinitesimal generators $X^k$, $k = 1, \ldots, r$. Thus*

$$\alpha X^a + \beta X^b = Y, \quad \text{where } Y \in \Lambda^r; \qquad X^a + X^b = X^b + X^a. \tag{5.65}$$

*(ii) The commutator is antisymmetric:*

$$\{X^a, X^b\} = -\{X^b, X^a\}. \tag{5.66}$$

*(iii) The commutator of any two infinitesimal generators of an $r$-parameter Lie group is also an infinitesimal generator that belongs to $\Lambda^r$:*

$$\{X^a, X^b\} = \beta^{ab}_k X^k \qquad (\text{sum over } k = 1, \ldots, r). \tag{5.67}$$

*The coefficients* $\beta_k^{ab}$ *are the structure constants of the Lie algebra* $\Lambda^r$. *Note that*

$$\beta_k^{ab} = -\beta_k^{ba}. \tag{5.68}$$

(iv) *The group operators satisfy the associative rule with respect to addition:*

$$X^a + (X^b + X^c) = (X^a + X^b) + X^c. \tag{5.69}$$

(v) *The group operators satisfy the Jacobi identity,*

$$\{\{X^a, X^b\}, X^c\} + \{\{X^c, X^a\}, X^b\} + \{\{X^b, X^c\}, X^a\} = 0. \tag{5.70}$$

(vi) *It follows from the Jacobi identity that the structure constants defined by the commutation relations (5.67) satisfy*

$$\beta_j^{ab}\beta_k^{jc} + \beta_j^{ca}\beta_k^{jb} + \beta_j^{bc}\beta_k^{ja} = 0 \qquad (\textit{sum over } j = 1, \ldots, r). \tag{5.71}$$

(vii) *The commutator may be expanded as*

$$\{\alpha X^a + \beta X^b, X^c\} = \alpha\{X^a, X^c\} + \beta\{X^b, X^c\}. \tag{5.72}$$

Lie algebras play a central role in modern mathematics. In our applications, we will find that they play a key role in the reduction of higher-order ODEs as well as in the reduction of PDEs where multiparameter groups are the norm. The commutator is used to test the closedness of a given set of group operators, and in the process, additional symmetries belonging to the full Lie algebra are often identified.

### 5.10.1 The Commutator Table

A convenient way to summarize a Lie algebra is to set up a *commutator table* whose entry at position $(a, b)$ is $\{X^a, X^b\}$. From the definition of the commutator one can see that the table will be antisymmetric with zeros on the main diagonal. For $n = 2$ the group (5.53) becomes

$$T^{\mathrm{proj2}}: \left\{\tilde{x} = \frac{x + a_3x + a_4y + a_5}{1 + a_1x + a_2y},\ \tilde{y} = \frac{y + a_6x + a_7y + a_8}{1 + a_1x + a_2y}\right\}. \tag{5.73}$$

Table 5.1. *Commutator table of the two-dimensional projective group.*

| | $X^1$ | $X^2$ | $X^3$ | $X^4$ | $X^5$ | $X^6$ | $X^7$ | $X^8$ |
|---|---|---|---|---|---|---|---|---|
| $X^1$ | 0 | 0 | $-X^1$ | $-X^2$ | $-2X^3-X^7$ | 0 | 0 | $-X^6$ |
| $X^2$ | 0 | 0 | 0 | 0 | $-X^4$ | $-X^1$ | $-X^2$ | $-X^3-2X^7$ |
| $X^3$ | $X^1$ | 0 | 0 | $-X^4$ | $-X^5$ | $X^6$ | 0 | 0 |
| $X^4$ | $X^2$ | 0 | $X^4$ | 0 | 0 | $X^7-X^3$ | $-X^4$ | $-X^5$ |
| $X^5$ | $2X^3+X^7$ | $X^4$ | $X^5$ | 0 | 0 | $X^8$ | 0 | 0 |
| $X^6$ | 0 | $X^1$ | $-X^6$ | $X^3-X^7$ | $-X^8$ | 0 | $X^6$ | 0 |
| $X^7$ | 0 | $X^2$ | 0 | $X^4$ | 0 | $-X^6$ | 0 | $-X^8$ |
| $X^8$ | $X^6$ | $X^3+2X^7$ | 0 | $X^5$ | 0 | 0 | $X^8$ | 0 |

The infinitesimal form of this eight-parameter group is

$$\begin{aligned}\tilde{x} &= x + (-a_1x^2 - a_2xy + a_3x + a_4y + a_5)s,\\ \tilde{y} &= y + (-a_1xy - a_2y^2 + a_6x + a_7y + a_8)s\end{aligned} \tag{5.74}$$

with corresponding one-parameter group operators:

$$\begin{gathered} X^1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \qquad X^2 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}, \qquad X^3 = x\frac{\partial}{\partial x},\\ X^4 = y\frac{\partial}{\partial x}, \qquad X^5 = \frac{\partial}{\partial x},\\ X^6 = x\frac{\partial}{\partial y}, \qquad X^7 = y\frac{\partial}{\partial y}, \qquad X^8 = \frac{\partial}{\partial y}. \end{gathered} \tag{5.75}$$

The commutator table for the group (5.75) is shown in Table 5.1. The structure constants are easily determined from the commutator table. The commutator can be especially useful for finding additional symmetries. For example, if a problem is found to admit the symmetry operators $X^1$, $X^3$, and $X^5$, then, by forming the commutator table of these operators, one finds that it must also admit the operator $X^7$, as can be seen in Table 5.1.

### *5.10.2 Lie Subalgebras*

The Lie algebra $\Lambda^8$ depicted in Table 5.1 contains a number of subalgebras. For example, the operators $X^1$, $X^2$, $X^3$, $X^4$ have the commutator table given in Table 5.2. Several other subalgebras can be identified in Table 5.1, such as $X^1$, $X^3$, $X^5$, $X^7$.

### *5.10.3 Abelian Lie Algebras*

The operators $X^6$, $X^8$ have the commutator table shown in Table 5.3. In this case the commutator is zero in each position. From the definition of the commutator, a

Table 5.2. *Four-parameter subalgebra of* $\Lambda^8$.

| | $X^1$ | $X^2$ | $X^3$ | $X^4$ |
|---|---|---|---|---|
| $X^1$ | 0 | 0 | $-X^1$ | $-X^2$ |
| $X^2$ | 0 | 0 | 0 | 0 |
| $X^3$ | $X^1$ | 0 | 0 | $-X^4$ |
| $X^4$ | $X^2$ | 0 | $X^4$ | 0 |

Lie algebra such that $\{X^a, X^b\} = 0$ for all $a$ and $b$ is Abelian, since the operators must commute. The operators $X^1$, $X^2$ form an Abelian Lie algebra, as do $X^2$, $X^3$ and $X^4$, $X^5$.

### *5.10.4 Ideal Lie Subalgebras*

A subalgebra $\Lambda^q \subset \Lambda^r$ where $q < r$ is called an *ideal subalgebra* of $\Lambda^r$ if, for any $X \in \Lambda^q$ and $Y \in \Lambda^r$, the commutator $\{X, Y\} \in \Lambda^q$. For example, consider the operators $X^5$, $X^6$, $X^7$, $X^8$ with the commutator table shown in Table 5.4. Notice that only $X^6$ and $X^8$ appear among the entries. The operators $X^6$ and $X^8$ form an ideal of the Lie algebra $X^5$, $X^6$, $X^7$, $X^8$. Formally, the null algebra $\Lambda^0$ is an ideal of the Lie algebra $X^6$, $X^8$. Similarly, $X^6$ and $X^8$ form an ideal subalgebra of $X^6$, $X^7$, and $X^8$.

## 5.11 Solvable Lie Algebras

Consider the Lie algebra corresponding to $X^6$, $X^7$, $X^8$ shown in Table 5.5. The sequence of subalgebras ($\Lambda^0$, $\Lambda^1 = X^8$, $\Lambda^2 = X^6, X^8$, $\Lambda^3 = X^6, X^7, X^8$) has the property that each item in the sequence is an ideal of the next item. This is an example of a solvable Lie algebra. The operators $X^3$, $X^4$, $X^5$ provide another example (Table 5.6). In this case the sequence is ($\Lambda^0$, $\Lambda^1 = X^4$ or $\Lambda^1 = X^5$, $\Lambda^2 = X^4, X^5$, $\Lambda^3 = X^3, X^4, X^5$). These examples satisfy the following definition of a solvable Lie algebra.

Table 5.3. *Abelian subalgebra.*

| | $X^6$ | $X^8$ |
|---|---|---|
| $X^6$ | 0 | 0 |
| $X^8$ | 0 | 0 |

Table 5.4. *Example of an ideal subalgebra.*

| | $X^5$ | $X^6$ | $X^7$ | $X^8$ |
|---|---|---|---|---|
| $X^5$ | 0 | $X^8$ | 0 | 0 |
| $X^6$ | $-X^8$ | 0 | $X^6$ | 0 |
| $X^7$ | 0 | $-X^6$ | 0 | $-X^8$ |
| $X^8$ | 0 | 0 | $X^8$ | 0 |

***Definition 5.5.*** *The Lie algebra $\Lambda^q$ is a q-dimensional solvable Lie algebra if there exists a chain of subalgebras*

$$\Lambda^0 \subset \Lambda^1 \subset \Lambda^2 \subset \cdots \subset \Lambda^{q-1} \subset \Lambda^q \tag{5.76}$$

*such that $\Lambda^k$ is a k-dimensional Lie algebra and $\Lambda^{k-1}$ is an ideal subalgebra of $\Lambda^k$ for $k = 1, 2, \ldots, q$. Here $\Lambda^0$ is the null ideal with no operators.*

The condition for solvability is equivalent to the condition that the operators of $\Lambda^q$ can be ordered to form a basis $X^1, \ldots, X^q$ such that

$$\{X^a, X^b\} = \beta_k^{ab} X^k \quad (\text{sum over } k = 1, \ldots, b-1) \qquad \text{for} \quad a < b. \tag{5.77}$$

Note that every two-dimensional Lie algebra is solvable by construction. Consider two operators, $X^a$ and $X^b$. Suppose

$$\{X^a, X^b\} = \beta_a X^a + \beta_b X^b. \tag{5.78}$$

Define two new operators in the Lie algebra $\Lambda^2$. Let

$$\begin{aligned} Y &= \beta_a X^a + \beta_b X^b, \\ Z &= \alpha_a X^a + \alpha_b X^b. \end{aligned} \tag{5.79}$$

Table 5.5. *Example of a solvable Lie algebra.*

| | $X^6$ | $X^7$ | $X^8$ |
|---|---|---|---|
| $X^6$ | 0 | $X^6$ | 0 |
| $X^7$ | $-X^6$ | 0 | $-X^8$ |
| $X^8$ | 0 | $X^8$ | 0 |

Table 5.6. *Another solvable Lie algebra.*

| | $X^3$ | $X^4$ | $X^5$ |
|---|---|---|---|
| $X^3$ | 0 | $-X^4$ | $-X^5$ |
| $X^4$ | $X^4$ | 0 | 0 |
| $X^5$ | $X^5$ | 0 | 0 |

If we form the commutator of $Y$ and $Z$, the result is

$$\begin{aligned}\{Y, Z\} &= (\beta_a X^a + \beta_b X^b)(\alpha_a X^a + \alpha_b X^b) - (\alpha_a X^a + \alpha_b X^b)(\beta_a X^a + \beta_b X^b)\\ &= (\beta_a\alpha_b X^a X^b + \beta_b\alpha_a X^b X^a) - (\beta_a\alpha_b X^b X^a + \beta_b\alpha_a X^a X^b)\\ &= (\beta_a\alpha_b - \beta_b\alpha_a)(X^a X^b - X^b X^a).\end{aligned} \tag{5.80}$$

Thus from (5.78),

$$\{Y, Z\} = (\beta_a\alpha_b - \beta_b\alpha_a)Y. \tag{5.81}$$

The group operators $Y$ and $Z$ form a solvable Lie algebra with the commutator table shown in Table 5.7.

Finally, an example of a three-dimensional Lie algebra that is not solvable is

$$X^1 = \frac{\partial}{\partial x}, \qquad X^2 = x\frac{\partial}{\partial x}, \qquad X^3 = (x)^2\frac{\partial}{\partial x} \tag{5.82}$$

with the commutator table shown in Table 5.8. In this case, all three operators appear in entries in the table, and there is no way to construct a solvable Lie algebra from the given operators.

Table 5.7. *Commutator table of $Y$ and $Z$.*

| | $Y$ | $Z$ |
|---|---|---|
| $Y$ | 0 | $(\beta_a\alpha_b - \beta_b\alpha_a)Y$ |
| $Z$ | $-(\beta_a\alpha_b - \beta_b\alpha_a)Y$ | 0 |

Table 5.8. *Not a solvable Lie algebra.*

| | $X^1$ | $X^2$ | $X^3$ |
|---|---|---|---|
| $X^1$ | $0$ | $X^1$ | $2X^2$ |
| $X^2$ | $-X^1$ | $0$ | $X^3$ |
| $X^3$ | $-2X^2$ | $-X^3$ | $0$ |

We will make use of solvable Lie algebras in Chapter 8 when we consider the reduction of second- and higher-order ODEs. However, it is worthwhile saying a few words in anticipation here. The use of symmetries to reduce the order of an ODE depends critically on the solvability of the Lie algebra of the corresponding multiparameter group that leaves the ODE invariant. The underlying principle is that *at each level of reduction of the order of the equation accomplished using one of its symmetries the transformed equation must inherit the remaining symmetries of the original equation*. This can be realized only if the group symmetries are used in the correct order as dictated by the solvability chain (5.76). This point will be discussed in greater detail in Chapter 8. For a general discussion of solvable Lie algebras see Bluman and Kumei [5.1].

## 5.12 Some Remarks on Lie Algebras and Vector Spaces

Here we follow the development in Yaglom [5.2]. A linear algebra is a finite-dimensional vector space with a set of basis vectors $(\boldsymbol{b}^1, \boldsymbol{b}^2, \ldots, \boldsymbol{b}^n)$. Elements of the vector space are of the form

$$\begin{aligned} \boldsymbol{x} &= x_i \boldsymbol{b}^i, \\ \boldsymbol{y} &= y_i \boldsymbol{b}^i, \end{aligned} \tag{5.83}$$

where the numbers $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ may be real or complex. Products of the basis vectors are themselves expressed in terms of the basis vectors:

$$\boldsymbol{b}^i \boldsymbol{b}^j = c_1^{ij} \boldsymbol{b}^1 + c_2^{ij} \boldsymbol{b}^2 + \cdots + c_n^{ij} \boldsymbol{b}^n = c_k^{ij} \boldsymbol{b}^k. \tag{5.84}$$

The $c_k^{ij}$, $i, j, k = 1, 2, \ldots, n$, are called the *structure constants* of the algebra.

For any two elements of the algebra, multiplication is commutative ($\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \boldsymbol{x}$) if and only if any two basis vectors commute, i.e., if the structure constants are symmetric in the upper indices:

$$c_k^{ij} = c_k^{ji} \qquad \text{for all } i,\ j,\ k. \tag{5.85}$$

Multiplication is anticommutative ($\boldsymbol{x} \cdot \boldsymbol{y} = -\boldsymbol{y} \cdot \boldsymbol{x}$) if and only if any two basis vectors anticommute and the structure constants are antisymmetric in the upper indices:

$$c_k^{ij} = -c_k^{ji} \qquad \text{for all } i,\ j,\ k. \tag{5.86}$$

See for example (5.68). Finally, multiplication is associative if and only if for any three elements $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ the basis vectors are associative $(\boldsymbol{b}^i \cdot \boldsymbol{b}^j) \cdot \boldsymbol{b}^k = \boldsymbol{b}^i \cdot (\boldsymbol{b}^j \cdot \boldsymbol{b}^k)$, or, in terms of the structure constants,

$$c_k^{ij} c_m^{kl} = c_m^{ik} c_k^{jl} \qquad \text{for all } i,\ j,\ l,\ m. \tag{5.87}$$

If $\boldsymbol{b}^1$ is the multiplicative identity of the algebra, then

$$c_j^{1i} = \begin{cases} 0 & \text{if} \quad i \neq j, \\ 1 & \text{if} \quad i = j, \end{cases} \tag{5.88}$$

or, in terms of the Kronecker unit tensor, $c_j^{1i} = \delta_j^i$.

A *null element* $\boldsymbol{n}$ is defined as one such that

$$\boldsymbol{n}^r = 0. \tag{5.89}$$

An idempotent element $\boldsymbol{e}$ is defined as an element whose square coincides with the element itself:

$$\boldsymbol{e}^2 = \boldsymbol{e}. \tag{5.90}$$

Now suppose $\boldsymbol{A} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \ldots \boldsymbol{n}, \boldsymbol{e})$ is an arbitrary associative algebra, in which the multiplication, denoted by "$*$" is not assumed to be commutative or anticommutative. We can use this generic multiplication operator to construct two new multiplications for the algebra, namely, the symmetric multiplication

$$\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x} * \boldsymbol{y} + \boldsymbol{y} * \boldsymbol{x}, \tag{5.91}$$

which is *commutative* ($\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \boldsymbol{x}$), and the skew-symmetric multiplication,

$$\boldsymbol{x} \times \boldsymbol{y} = \boldsymbol{y} * \boldsymbol{x} - \boldsymbol{x} * \boldsymbol{y}, \tag{5.92}$$

which is *anticommutative* ($\boldsymbol{x} \times \boldsymbol{y} = -\boldsymbol{y} \times \boldsymbol{x}$). It is important to recognize that these two procedures for multiplying elements of the algebra are not associative. The structural constants $c_k^{ij}|_\cdot$ and $c_k^{ij}|_\times$ of these new multiplications are connected to the structural constants $c_k^{ij}$ of the (original) multiplication by the formulas

$$\begin{aligned} c_k^{ij}\Big|_* &= c_k^{ij} + c_k^{ji}, \\ c_k^{ij}\Big|_\times &= c_k^{ij} - c_k^{ji}. \end{aligned} \tag{5.93}$$

However, the definitions (5.94) and (5.95) do not exclude associativity altogether: in algebras with these multiplication operations there are certain identities that can be viewed as replacements for a true law of associativity.

Algebras with the commutative multiplication $\cdot$ admit the *Jordan identity*,

$$(\boldsymbol{x}^2 \cdot \boldsymbol{y}) \cdot z = \boldsymbol{x}^2 \cdot (\boldsymbol{y} \cdot z), \tag{5.94}$$

where

$$\boldsymbol{x}^2 = \boldsymbol{x} \cdot \boldsymbol{x}. \tag{5.95}$$

These are called *Jordan algebras* after the great French group theorist, Marie-Ennemond Camille Jordan (1838–1922), whose work brought attention to the significance of the theories of Evariste Galois and who later greatly influenced both Sophus Lie and Felix Klein.

On the other hand, algebras with the anticommutative multiplication $\boldsymbol{x} \times \boldsymbol{y}$ admit the *Jacobi identity*,

$$(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w} + (\boldsymbol{w} \times \boldsymbol{u}) \times \boldsymbol{v} + (\boldsymbol{v} \times \boldsymbol{w}) \times \boldsymbol{u} = 0, \tag{5.96}$$

named after one of the leading German mathematicians of the nineteenth century, Carl Gustav Jacob Jacobi (1804–1851). Algebras with satisfying the Jacobi condition are called *Lie algebras*. See Equation (5.70), where the elements of the algebra are the group operators. See also the rules of algebra governing Poisson brackets in Chapter 4, Section 4.3.1, where the elements of the algebra are the integrals of the motion of a Hamiltonian system.

To quote Yaglom [5.2]:

We have already mentioned the interest and attention which Jordan algebras attract today. But at present they are not nearly as important as *Lie algebras* and *Lie groups*, which constitute two of the central notions of mathematical science. A concept whose importance for science in general is comparable to that of a Lie algebra is not that of a Jordan algebra but that of a Euclidean space.

## 5.13 Concluding Remarks

This completes our formal introduction to Lie groups. From the standpoint of later applications the most important concepts are that of an infinitesimal group, the expansion of a function in a Lie series using the group operator, and the solution of the characteristic equations described in Sections 5.5, 5.6, and 5.7. We will often encounter multiparameter groups in the context of differential equations, and here the most important concept is that of the Lie algebra, the structure of which determines how useful the several groups may be in the reduction and simplification of the associated equation. In the next chapter we will begin to apply groups to the solution of ODEs.

## 5.14 Exercises

5.1 (1) Show by composition that each of the following transformations is a Lie group:

(i) A projective group

$$\tilde{x} = \frac{x}{1 - sy}, \qquad \tilde{y} = \frac{y}{1 - sy}. \tag{5.97}$$

(ii) A hyperbolic group

$$\tilde{x} = x + s, \qquad \tilde{y} = \frac{xy}{x + s}. \tag{5.98}$$

(iii) An arbitrary translation

$$\tilde{x} = x, \qquad \tilde{y} = y + sf[x], \quad f[x] \text{ arbitrary.} \tag{5.99}$$

(iv) A helical transformation

$$\tilde{x} = x\cos[s] - y\sin[s], \qquad \tilde{y} = x\sin[s] + y\cos[s],$$
$$\tilde{z} = z + ms. \tag{5.100}$$

(2) Determine the infinitesimal transformation for each case, and then reconstruct the global transformation by series summation.
(3) Set up the characteristic equations and determine the integral invariants of the group for each case. Find these invariants by elimination of the group parameter.

5.2 Is the transformation

$$\tilde{x} = x - sy, \qquad \tilde{y} = y + sx \tag{5.101}$$

a Lie group?

5.3 The Lorentz transformation of the position and time of a particle moving at speed $u$ is

$$\tilde{x} = \frac{x - ut}{\sqrt{1 - \frac{u^2}{c^2}}}, \qquad \tilde{t} = \frac{t - \frac{ux}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}}, \tag{5.102}$$

where $c$ is the speed of light. Show that the transformation is a group with respect to the parameter $u$. Is it also a group with respect to $c$? Let $u = -\tanh[a]$. Show that the transformation becomes

$$\tilde{x} = x \cosh[a] + t \sinh[a], \qquad \tilde{y} = x \sinh[a] + t \cosh[a]. \tag{5.103}$$

The Lorentz transformation is a kind of "hyperbolic rotation." Determine the infinitesimal transformation, and compare with an ordinary rotation.

5.4 Carefully work out the steps leading from (5.25) to (5.27) for a Lie group in two variables.

5.5 Consider the transformation

$$\tilde{y} = \frac{a + (1 + b)y}{1 + c + dy}. \tag{5.104}$$

Work out the infinitesimal form of the group, and characterize the Lie algebra. Identify the group parameters.

5.6 The following autonomous system of ODEs comes up in the context of a problem involving laminar flame propagation:

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= y - \frac{1}{4}x + \frac{1}{4}x^2. \end{aligned} \tag{5.105}$$

Draw the phase portrait of the system (5.108). Use the Lie series expansion to develop a fourth-order accurate method for solving the equations numerically. Compare your scheme to a standard fourth-order Runge–Kutta method. Solve numerically for $y[x]$ subject to the boundary conditions $y[0]=0$, $y[1]=0$. Use the phase portrait to suggest how to carry out the integration.

5.7 Sum the Lie series to determine the finite transformation corresponding to the group operator,

$$X = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} - \left(\frac{y^2}{4} + \frac{x}{2}\right)z\frac{\partial}{\partial z}. \tag{5.106}$$

Solve the characteristic equations to determine the two invariants of the group, and show that they are invariant under the finite transformation.

5.8 The equations governing inviscid compressible flow of a general fluid (see Chapter 12) are invariant under an eleven-parameter group with operators

$$\begin{gathered}
X^1 = \frac{\partial}{\partial t}, \qquad X^2 = \frac{\partial}{\partial x}, \qquad X^3 = \frac{\partial}{\partial y}, \qquad X^4 = \frac{\partial}{\partial z}, \\
X^5 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}, \\
X^6 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + w\frac{\partial}{\partial v} - v\frac{\partial}{\partial w}, \\
X^7 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + w\frac{\partial}{\partial u} - u\frac{\partial}{\partial w}, \\
X^8 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \qquad X^9 = t\frac{\partial}{\partial y} + \frac{\partial}{\partial u}, \qquad X^{10} = t\frac{\partial}{\partial z} + \frac{\partial}{\partial u}, \\
X^{11} = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},
\end{gathered} \tag{5.107}$$

where $t$, $x$, $y$, $z$ are time and the spatial coordinates, and $u$, $v$, $w$ are the velocity components in the corresponding directions. First, see how many finite groups you can work out by inspection. Identify the nature of each group (translation, rotation, dilation, etc.). Sum the Lie series to work out the finite rotation groups. Note that if you want to generate the full form of the three-parameter 3-D rotation group, you will need to sum the Lie series using a three-term group operator with three independent small parameters. Work out the $11 \times 11$ commutator table for this group, and identify any subalgebras. Identify any solvable subalgebras.

## References

[5.1] Bluman, G. W. and Kumei, S. 1989. *Symmetries and Differential Equations*, Applied Mathematical Sciences **81** Springer-Verlag. See Section 3.4.3.

[5.2] Yaglom, I. M. 1988. *Felix Klein and Sophus Lie: Evolution of the Idea of Symmetry in the Nineteenth Century*. Birkhäuser.

# 6

# *First-Order Ordinary Differential Equations*

In the last chapter we showed that a function $\psi = \Psi[\boldsymbol{x}]$ is invariant under a group $F^j$ (or $\xi^j$) if and only if it is a solution of $\xi^j(\partial\Psi/\partial x^j) = 0$. It was also pointed out that this invariance condition is rather strong, in the sense that it implies that each surface of fixed $\psi$ is individually invariant under the group. An initial point on a given $\psi$ is mapped by $\boldsymbol{F}$ to a new point on the *same* surface.

## 6.1 Invariant Families

To progress further, it is necessary to relax this condition a bit and consider situations in which a family of surfaces is invariant under a group, but an individual surface may be mapped to a new surface within the same family.

***Example 6.1 (The group of rotations in the plane).*** Consider the rotation group

$$T^{\text{rot}}: \left\{ \begin{array}{l} \tilde{x} = x\cos[s] - y\sin[s] \\ \tilde{y} = x\sin[s] + y\cos[s] \end{array} \right\}. \tag{6.1}$$

This is a particular case of the so-called special orthogonal group in two dimensions, in conventional notation SO(2). The term "special" refers to the fact that this is a transformation of the form

$$\begin{aligned} \tilde{x} &= ax - by, \\ \tilde{y} &= cx + dy, \end{aligned} \tag{6.2}$$

where the determinant $ad - bc = 1$, in contrast to the more general affine transformations discussed in connection with critical points in Chapter 3. The infinitesimal group operator corresponding to (6.1) is

$$X^{\text{rot}} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}. \tag{6.3}$$

The family of circles

$$\phi = \Phi[x, y] = x^2 + y^2 \tag{6.4}$$

is clearly invariant under this group. Moreover, since $X^{\text{rot}}\Phi = 0$, each individual circle in the family is invariant.

On the other hand, if we apply the group (6.1) to the family of rays through the origin,

$$\psi = \Psi[x, y] = \frac{y}{x}, \tag{6.5}$$

the result is

$$\tilde{\psi} = \frac{\tilde{y}}{\tilde{x}} = \frac{\sin[s] + \frac{y}{x}\cos[s]}{\cos[s] - \frac{y}{x}\sin[s]} = G\left(\frac{y}{x}, s\right) = G(\psi, s). \tag{6.6}$$

The transformation (6.1) maps a given ray $\psi$ to a new ray $\tilde{\psi}$ in the same family. So while a given ray is not invariant under the group (6.1), the family of rays as a whole is invariant. This situation is shown schematically in Figure 6.1.

If we apply the group operator $X^{\text{rot}}$ to $\Psi$, the result is

$$X^{\text{rot}}\,\Psi = -y\frac{\partial}{\partial x}\left(\frac{y}{x}\right) + x\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \left(\frac{y}{x}\right)^2 + 1 = \psi^2 + 1. \tag{6.7}$$

In this case the group operator produces a result that is a function of $\psi$. If we were to operate with $X^{\text{rot}}$ on the right-hand side of (6.7), the result would again be a function of $\psi$. The point here is to recognize that if $\tilde{\psi}$ were expanded in a Lie series, each term would be a function of $\psi$ multiplied by a power

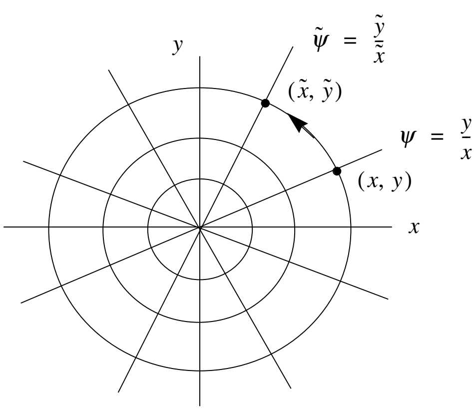


Fig. 6.1. Action of the rotation group on the family of rays.

of $s$. Summing the series would produce (6.6). We shall return to this point in a moment.

***Example 6.2 (The group of uniform dilations).*** Now consider

$$T^{\text{dil}}\colon \left\{ \begin{aligned} \tilde{x} &= e^s x \\ \tilde{y} &= e^s y \end{aligned} \right\} \tag{6.8}$$

with group operator

$$X^{\text{dil}} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{6.9}$$

The family of rays

$$\psi = \Psi[x, y] = \frac{y}{x} \tag{6.10}$$

is clearly invariant under this group, since $X^{\text{dil}}\Psi = 0$. Moreover, each individual ray is invariant. This is clear if we look at the action of the finite form of the group on a ray:

$$\tilde{\psi} = \Psi[\tilde{x}, \tilde{y}] = \frac{\tilde{y}}{\tilde{x}} = \frac{e^s y}{e^s x} = \frac{y}{x} = \Psi(x, y) = \psi. \tag{6.11}$$

The absence of the group parameter $s$ from the last equality in (6.11) implies that the mapping of one point to another takes place along the same ray. See Figure 6.2.

What about the family of circles? In this case

$$\tilde{\phi} = \Phi[\tilde{x}, \tilde{y}] = \tilde{x}^2 + \tilde{y}^2 = e^{2s}(x^2 + y^2) = e^{2s}\Phi[x, y] = G(\phi, s). \tag{6.12}$$

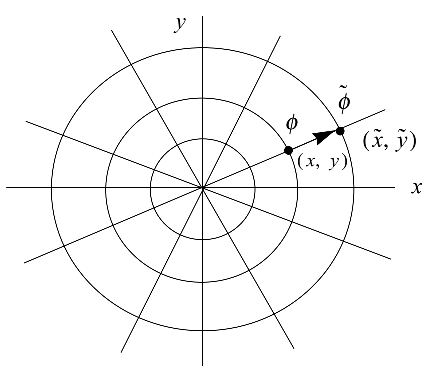


Fig. 6.2. Action of the dilation group on the family of circles.

The circle $\phi$ is mapped to the circle $\tilde{\phi}$. In this case individual circles are not invariant, but the family of circles as a whole is invariant under the dilation group (6.8). If we apply the group operator $X^{\text{dil}}$ to $\Phi$, the result is

$$X^{\text{dil}}\Phi = x\frac{\partial}{\partial x}(x^2 + y^2) + y\frac{\partial}{\partial y}(x^2 + y^2) = 2(x^2 + y^2) = 2\phi. \quad (6.13)$$

The group operator produces a result that is a function of $\phi$. This situation is shown schematically in Figure 6.2.

If we were to operate with $X^{\text{dil}}$ on the right-hand side of (6.13), the result would again be a function of $\phi$. Similarly, each term in the Lie series would be equal to a function of $\phi$ times a power of $s$, and the sum would be the function $G[\phi, s]$, where now

$$\tilde{\phi} = \tilde{x}^2 + \tilde{y}^2 = e^{2s}(x^2 + y^2) = G(x^2 + y^2, s) = G[\phi, s]. \quad (6.14)$$

## 6.2 Invariance Condition for a Family

Evidently the finite condition for a *family* $\psi = \Psi[\boldsymbol{x}]$ to be invariant under a group $\boldsymbol{F}$ is

$$\tilde{\psi} = \Psi[\tilde{\boldsymbol{x}}] = \Psi(\boldsymbol{F}[\boldsymbol{x}, s]) = G[\Psi[\boldsymbol{x}], s] = G[\psi, s], \quad (6.15)$$

where $G$ is some function. The corresponding infinitesimal condition for invariance is

$$X\Psi = \Omega[\Psi]. \quad (6.16)$$

Actually we can interpret the invariance condition (6.16) applied to a family in $n$ dimensions as equivalent to $X\Gamma = 0$, where $\Gamma$ is a single surface in $n + 1$ dimensions.

To see this, let $\Gamma$ be a function of $n + 1$ variables of the form

$$\Gamma[x^1, x^2, x^3, \ldots, x^n, x^{n+1}] = \Psi[x^1, x^2, x^3, \ldots, x^n] - x^{n+1}. \quad (6.17)$$

Consider the invariance of $\Gamma$ under the transformation

$$\begin{aligned} \tilde{x}^j &= F^j[x^1, x^2, x^3, \ldots, x^n, s], \qquad j = 1, 2, \ldots, n, \\ \tilde{x}^{n+1} &= x^{n+1} + s, \end{aligned} \quad (6.18)$$

which can easily be verified to be a group, assuming $F^j$ is a group. The function $\Gamma$ is an invariant single surface under the group (6.18) if and only if it satisfies the invariance condition

$$\xi^1 \frac{\partial \Gamma}{\partial x^1} + \xi^2 \frac{\partial \Gamma}{\partial x^2} + \cdots + \xi^n \frac{\partial \Gamma}{\partial x^n} + (1) \frac{\partial \Gamma}{\partial x^{n+1}} = 0, \tag{6.19}$$

which becomes

$$X\Psi = 1. \tag{6.20}$$

Thus the family

$$\Psi[x^1, x^2, x^3, \ldots, x^n] = x^{n+1} \tag{6.21}$$

is an invariant family in $(x^1, x^2, x^3, \ldots, x^n)$, or equivalently an invariant single surface in $(x^1, x^2, x^3, \ldots, x^n, x^{n+1})$.

***Theorem 6.1.*** *The family $\psi = \Psi[\boldsymbol{x}]$ is invariant under the group $X = \xi^j(\partial/\partial x^j)$ if and only if $X\Psi = \Omega[\Psi]$ for some function $\Omega$. Without loss of generality we can always choose a once differentiable function $\Pi[\Psi]$ such that the family $\phi = \Phi[\boldsymbol{x}] = \Pi[\Psi[\boldsymbol{x}]]$ satisfies $X\Phi = 1$.*

This simplification of the invariance condition can be shown as follows:

$$X\Phi[\boldsymbol{x}] = X\Pi[\Psi[\boldsymbol{x}]] = (X\Psi)\frac{d\Pi}{d\Psi} = \Omega[\Psi]\frac{d\Pi}{d\Psi} = 1. \tag{6.22}$$

Now choose $\Pi[\Psi]$ such that

$$\Pi = \int \frac{d\Psi}{\Omega[\Psi]}. \tag{6.23}$$

The reason that $X\Phi = 1$ is sufficiently general is that a family of curves in some domain of $\boldsymbol{x}$ can be represented in an infinite variety of different, essentially equivalent ways. The functions $\psi = \Psi[\boldsymbol{x}]$ and $\phi = \Phi[\boldsymbol{x}] = \Pi[\Psi[\boldsymbol{x}]]$ represent the same family, although with different values assigned to individual curves of the family.

***Example 6.3 (The rotation group revisited).*** Let's consider the rotation group again with operator (6.3) and invariant function (6.4). We search for an invariant family, beginning with a 3-D surface of the form

$$\Gamma(x, y, \psi) = \Psi[x, y] - \psi, \tag{6.24}$$

which is required to satisfy the invariance condition

$$-y\frac{\partial \Gamma}{\partial x} + x\frac{\partial \Gamma}{\partial y} + \frac{\partial \Gamma}{\partial \psi} = 0, \tag{6.25}$$

or

$$-y\frac{\partial \Psi}{\partial x} + x\frac{\partial \Psi}{\partial y} = 1. \tag{6.26}$$

Recalling the discussion of Lagrange's method for solving first-order PDEs in Chapter 3, Section 3.6.1, the characteristic equations associated with (6.25) are

$$-\frac{dx}{y} = \frac{dy}{x} = \frac{d\psi}{1}. \tag{6.27}$$

The first equality in (6.27) is obviously solved by $\theta = x^2 + y^2$. Now use this result to replace, say, $x$ in the second equality, and integrate the result

$$d\psi = \frac{dy}{\pm\sqrt{\theta - y^2}} \tag{6.28}$$

to produce the invariant family,

$$\psi = \sin^{-1}\left[\frac{y}{\sqrt{\theta}}\right] = \sin^{-1}\left[\frac{y/x}{\sqrt{1 + (y/x)^2}}\right]. \tag{6.29}$$

If we had replaced $y$ instead of $x$ in (6.27), the invariant family would have come out as

$$\psi = \cos^{-1}\left[\frac{x}{\sqrt{\theta}}\right] = \cos^{-1}\left[\frac{x/y}{\sqrt{1 + (x/y)^2}}\right], \tag{6.30}$$

or we could simply take $\psi = \Psi(x, y) = y/x$. In the latter case the rotation group operator gives $X^{\text{rot}}\Psi = \psi^2 + 1$. We can select $\Pi$ such that

$$\Pi[\Psi] = \int \frac{d\Psi}{\Psi^2 + 1} = \tan^{-1}[\Psi] = \tan^{-1}[y/x]. \tag{6.31}$$

Thus

$$\Psi = \tan^{-1}[y/x]. \tag{6.32}$$

The rotation group operator applied to (6.32) gives $X^{\text{rot}}\Psi = 1$.

This example typifies the procedure used to find a family that is invariant under a given group. Once again the main point to keep in mind is that a family of curves may be specified by an infinity of functions. Recall Chapter 3, where it was pointed out that if a given function satisfies a first-order ODE, then any function of that function will satisfy the same ODE.

## 6.3 First-Order ODEs – The Integrating Factor

We are now in a position to use group theory to integrate ODEs. Let $\psi = \Psi[x, y]$ be the characteristic curves of the first-order ODE

$$\frac{dy}{dx} = \frac{B[x, y]}{A[x, y]}, \tag{6.33}$$

which we can write as the Pfaffian 1-form

$$-B[x, y]\, dx + A[x, y]\, dy = 0. \tag{6.34}$$

The decomposition of the right-hand side of (6.33) into a numerator and denominator is essentially a matter of convenience. The differential of $\Psi[x, y]$ is

$$d\psi = \frac{\partial \Psi}{\partial x}\, dx + \frac{\partial \Psi}{\partial y}\, dy. \tag{6.35}$$

On a curve of fixed $\psi$ the differential $d\psi = 0$. Recalling our discussion of integrating factors in Chapter 3, we must be careful not to assume a correspondence between (6.34) and (6.35). What we do know is that $\Psi[x, y]$ satisfies the first-order linear PDE

$$A[x, y]\frac{\partial \Psi}{\partial x} + B[x, y]\frac{\partial \Psi}{\partial y} = 0. \tag{6.36}$$

Now, suppose the *family* $\psi = \Psi[x, y]$ is invariant under a group $X$ with infinitesimals $(\xi, \eta)$. In this case $\Psi[x, y]$ also satisfies the invariance condition

$$\xi[x, y]\frac{\partial \Psi}{\partial x} + \eta[x, y]\frac{\partial \Psi}{\partial y} = 1. \tag{6.37}$$

Here we have a situation where the family of characteristics of the first-order ODE (6.33) is invariant under *two* groups. The first is the trivial group defined by the equation itself with $A$ and $B$ interpreted as infinitesimals with group operator $A(\partial/\partial x) + B(\partial/\partial y)$. The second, nontrivial group is the independent group $(\xi, \eta)$, which leaves the *family* of solutions $\Psi[x, y]$ invariant while mapping each member of the family to a new member. This situation is shown schematically in Figure 6.3.

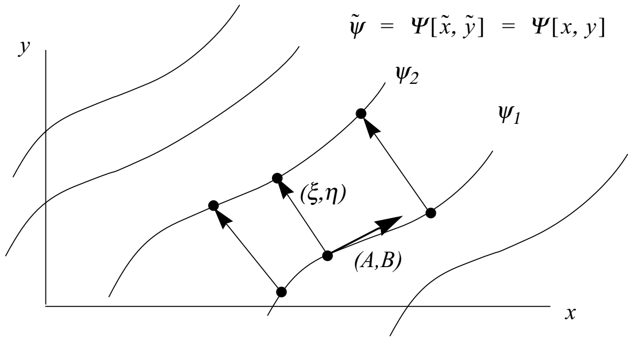


Fig. 6.3. Transformation of points along characteristics by $(A, B)$, and between characteristics by $(\xi, \eta)$.

The group $(A, B)$ moves points that are initially on the characteristic $\psi^1$ to new points on the same characteristic, whereas the group $(\xi, \eta)$ maps the entire characteristic $\psi^1$ to a new characteristic $\psi^2$ for some particular value of the group parameter, $s$. Voila! We have two simultaneous, independent equations for the partial derivatives of $\Psi$. Solving (6.36) and (6.37) produces

$$\frac{\partial \Psi}{\partial x} = \frac{-B}{A\eta - B\xi}, \qquad \frac{\partial \Psi}{\partial y} = \frac{A}{A\eta - B\xi}. \tag{6.38}$$

The function

$$M = \frac{1}{A\eta - B\xi} \tag{6.39}$$

is the sought-after integrating factor, and the total differential of $\psi = \Psi[x, y]$ is

$$d\psi = \frac{-B}{A\eta - B\xi}\, dx + \frac{A}{A\eta - B\xi}\, dy. \tag{6.40}$$

So invariance of the family of solutions of (6.33) under a Lie group leads directly to the solution of the ODE in the form of a quadrature

$$\psi = \int \frac{-B}{A\eta - B\xi}\, dx \bigg|_{y=\text{constant}} + f[y]. \tag{6.41}$$

The function $f[y]$ is determined from

$$\frac{\partial \Psi}{\partial y} = \frac{\partial}{\partial y}\left( \int \frac{-B}{A\eta - B\xi}\, dx \bigg|_{y=\text{constant}} \right) + \frac{df}{dy} = \frac{A}{A\eta - B\xi}, \tag{6.42}$$

and the general solution of (6.33) is complete.

The approach used in this method – i.e., find a second equation involving the partial derivatives, then solve for each and insert in the expression for the perfect differential of the solution – is reminiscent of the method of Lagrange and Charpit for solving nonlinear first-order PDEs described in Chapter 3, Section 3.7. But the spirit is fundamentally different. For a linear equation of the form $A(\partial\Psi/\partial x) + B(\partial\Psi/\partial y) = 0$ the method of Lagrange and Charpit simply reduces to the characteristic equations, $dx/A = dy/B$. No symmetry is involved, and no general principles can be stated. It is a special method that works on particular classes of nonlinear first-order PDEs.

In practice the group that leaves the family of solutions invariant is identified by looking for a group that leaves the ODE itself invariant. This brings us to a final point that is fundamental to what follows.

If an ODE is invariant under a Lie group, the family of solution curves of the ODE is invariant under the group.

The method developed by Lie rests on the principle that symmetries lead to solutions. When Lie's method works, it can be quite spectacular, and several examples will be described in the next few sections. Unfortunately there is no systematic way to determine the invariant group for a given first-order ODE. Nevertheless, according to Pfaff's theorem, the group always exists, for the simple reason that the solution exists. But, in general, its determination requires the particular solution of a system of equations that is equivalent to the original system (6.33).

In Chapter 8 we shall learn how to answer the opposite question; that is, given a group, what is the general form of a first-order ODE that is invariant under the group. This approach allows one to catalog whole classes of ODEs that admit a given group. Table 6.1 contains a list of first-order ODEs and their known groups.

## 6.4 Using Groups to Integrate First-Order ODEs

In Chapter 8 we shall see that in the case of second- and higher-order ODEs the invariant group can usually be determined systematically and that the group can be exploited to accomplish a reduction of order. Sometimes the ODE may admit a multiparameter group enabling a reduction by more than one order, depending on the solvability of the associated Lie algebra. In this section we will illustrate the integration of first-order ODEs for which the invariant group is known *a priori* from Table 6.1 or can be determined by inspection.

Table 6.1. *Some first-order ODEs and their invariant groups.*

| Equation | $\xi$ | $\eta$ |
|---|---|---|
| $y_x = F[y]$ | 1 | 0 |
| $y_x = F[x]$ | 0 | 1 |
| $y_x = F[ax + by]$ | $b$ | $-a$ |
| $y_x = \dfrac{y + xF[x^2 + y^2]}{x - yF[x^2 + y^2]}$ | $y$ | $-x$ |
| $y_x = F\left[\dfrac{y}{x}\right]$ | $x$ | $y$ |
| $y_x = x^{k-1}F[y/x^k]$ | $x$ | $ky$ |
| $xy_x = F[xe^{-y}]$ | $x$ | 1 |
| $y_x = yF[ye^{-x}]$ | 1 | $y$ |
| $y_x = (y/x) + xF[y/x]$ | 1 | $y/x$ |
| $xy_x = y + F[y/x]$ | $x^2$ | $xy$ |
| $y_x = \dfrac{y}{x + F[y/x]}$ | $xy$ | $y^2$ |
| $y_x = \dfrac{y}{x + F[y]}$ | $y$ | 0 |
| $xy_x = y + F[x]$ | 0 | $x$ |
| $xy_x = \dfrac{y}{\ln[x] + F[y]}$ | $xy$ | 0 |
| $xy_x = y(\ln[y] + F[x])$ | 0 | $xy$ |
| $y_x = yF[x]$ | 0 | $y$ |

***Example 6.4 (Invariance with respect to a dilation group).*** Find the general solution of

$$\frac{dy}{dx} = \frac{y}{x}H[xy], \tag{6.43}$$

where $H$ is an arbitrary function. Rearrange (6.43) as

$$-yH[xy]\,dx + x\,dy = 0. \tag{6.44}$$

In the notation adopted above, let

$$A[x, y] = -x, \qquad B[x, y] = -yH[xy]. \tag{6.45}$$

As was just pointed out, we need to find a Lie group that leaves (6.43) invariant. There is really no systematic way to determine such a group. We have to rely on trial and error to transform (6.43). By inspection we can see that (6.43) is invariant under the dilation group

$$\tilde{x} = e^{s}x, \qquad \tilde{y} = e^{-s}y. \tag{6.46}$$

Insert the transformation (6.46) into (6.43):

$$\frac{d\tilde{y}}{d\tilde{x}} = \frac{\tilde{y}}{\tilde{x}}H[\tilde{x}\tilde{y}] \;\Rightarrow\; e^{-2s}\frac{dy}{dx} = e^{-2s}\frac{y}{x}H[xy] \;\Rightarrow\; \frac{dy}{dx} = \frac{y}{x}H[xy]. \tag{6.47}$$

The equation reads the same in the new variables – success: we have found a group that leaves (6.43) invariant. The infinitesimals of (6.46) are

$$\xi = x, \qquad \eta = -y, \tag{6.48}$$

and the integrating factor is

$$M = \frac{1}{A\eta - B\xi} = \frac{1}{xy + xyH[xy]}. \tag{6.49}$$

Therefore the total differential of the solution is

$$d\psi = -\frac{yH[xy]}{xy + xyH[xy]}\,dx + \frac{x}{xy + xyH[xy]}\,dy. \tag{6.50}$$

Finally, the general solution of (6.43) is the family

$$\psi = -\int_{xy} \frac{H(\alpha)}{\alpha(1 + H(\alpha))}\,d\alpha + \ln[y]. \tag{6.51}$$

In essence, $\psi$ is simply the constant of integration of (6.43). Let's demonstrate that (6.51) is in fact an invariant family of (6.48):

$$\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\left(-\int_{xy} \frac{H(\alpha)}{\alpha(1 + H(\alpha))}\,d\alpha + \ln[y]\right)$$
$$= x\left(-\frac{H}{\alpha(1 + H)}y\right)_{\alpha=xy} - y\left(-\frac{H}{\alpha(1 + H)}x\right)_{\alpha=xy} + y\left(\frac{1}{y}\right) = 1. \tag{6.52}$$

The finite transformation (6.46) applied to (6.51) leads to

$$\psi = -\int_{\tilde{x}\tilde{y}} \frac{H(\alpha)}{\alpha(1+H(\alpha))}\, d\alpha + \ln[\tilde{y}] + s = \tilde{\psi} + s. \tag{6.53}$$

The group translates one solution path to another, as we would expect. Finally, any function of (6.51) is also a solution of (6.43).

***Example 6.5 (Integrating factor for a linear first-order ODE).*** Find the solution of

$$\frac{dy}{dx} = -g[x]y + f[x], \tag{6.54}$$

which we can rearrange as

$$-(f[x] - yg[x])\, dx + dy = 0. \tag{6.55}$$

In this case

$$A = 1, \qquad B = f - yg. \tag{6.56}$$

Let's try an arbitrary translation

$$\begin{aligned} \tilde{x} &= x, \\ \tilde{y} &= y + s\,\theta[x]. \end{aligned} \tag{6.57}$$

Substitute (6.57) into (6.54). The equation

$$\frac{d\tilde{y}}{d\tilde{x}} = -g[\tilde{x}]\tilde{y} + f[\tilde{x}] \tag{6.58}$$

becomes

$$\frac{dy}{dx} + s\frac{d\theta}{dx} = -g[x](y + s\theta[x]) + f[x]. \tag{6.59}$$

Equation (6.54) is invariant under the group (6.57) if we choose $\theta[x]$ such that

$$\frac{d\theta}{dx} = -g[x]\theta[x]. \tag{6.60}$$

Thus the appropriate group is the translation (6.57), where

$$\theta[x] = \exp\left[-\int_x g[\alpha]\, d\alpha\right]. \tag{6.61}$$

Note that (6.57) simply expresses the superposition principle for linear equations that states that to any solution of (6.54) one can always add a solution of the homogeneous equation multiplied by an arbitrary amplitude $s$. The infinitesimals of (6.57) are

$$\xi = 0, \qquad \eta = \theta[x]. \tag{6.62}$$

The integrating factor for (6.54) is

$$M[x] = \frac{1}{\theta[x]}, \tag{6.63}$$

and the total differential of the solution is

$$d\psi = \left\{\frac{1}{\theta[x]}(f[x] - yg[x])\right\} dx - \left\{\frac{1}{\theta[x]}\right\} dy. \tag{6.64}$$

Equation (6.64) yields the solution of (6.54) by quadrature:

$$\psi = \int_x \left\{\frac{f[\alpha]}{\theta[\alpha]}\right\} d\alpha - \frac{y}{\theta[x]}. \tag{6.65}$$

The function (6.65) defines the entire family of solution curves of (6.54) for all possible initial conditions (all possible values of $\psi$). As in the last example, the action of the group (6.57) on the family of solutions (6.65) is to effect a translation $\psi = \tilde{\psi} + s$.

***Example 6.6 (A more complicated case).*** Find the general solution of

$$\frac{dy}{dx} = \frac{y}{x - f[y]g[y/x]}, \tag{6.66}$$

which we rewrite as

$$\frac{y}{x}\, dx - \left(1 - \frac{f[y]}{x} g[y/x]\right) dy = 0. \tag{6.67}$$

Let

$$A = \left(1 - \frac{f[y]}{x} g[y/x]\right), \qquad B = \frac{y}{x}. \tag{6.68}$$

Identifying an invariant group by inspection in this case is not so easy. However, this equation is known to be invariant under the group with infinitesimals

$$\xi = \frac{xy}{f[y]}, \qquad \eta = \frac{y^2}{f[y]}. \tag{6.69}$$

The integrating factor is

$$M = -\frac{x}{y^2 g[y/x]}, \tag{6.70}$$

and the total differential is

$$d\psi = -\frac{1}{yg[y/x]}\,dx + \left(\frac{x}{y^2 g[y/x]} - \frac{f[y]}{y^2}\right) dy. \tag{6.71}$$

Integrating by quadrature produces the general solution of (6.66):

$$\psi = \Psi[x, y] = \int_{y/x} \frac{1}{\alpha^2 g[\alpha]}\,d\alpha - \int_{y} \frac{f[\alpha]}{\alpha^2}\,d\alpha. \tag{6.72}$$

Let's demonstrate that (6.72) is in fact an invariant family of (6.69):

$$\begin{aligned}
&\left(\frac{xy}{f[y]}\frac{\partial}{\partial x} - \frac{y^2}{f[y]}\frac{\partial}{\partial y}\right)\left(\int_{y/x} \frac{1}{\alpha^2 g[\alpha]}\,d\alpha - \int_{y} \frac{f[\alpha]}{\alpha^2}\,d\alpha\right) \\
&= \frac{xy}{f[y]}\left(-\frac{1}{\alpha g[\alpha]}\frac{1}{x}\right)_{\alpha=y/x} - \frac{y^2}{f[y]}\left(-\frac{1}{\alpha^2 g[\alpha]}\frac{1}{x}\right)_{\alpha=y/x} \\
&\quad + \frac{y^2}{f[y]}\left(\frac{f[\alpha]}{\alpha^2}\right)_{\alpha=y} = 1
\end{aligned} \tag{6.73}$$

As usual, any function of (6.72) will also be a solution.

## 6.5 Canonical Coordinates

Any Lie group can be written in terms of new variables, called *canonical coordinates*, such that the transformation is converted to a simple translation in one variable. The group

$$\tilde{x}^j = F^j[\boldsymbol{x}, s], \qquad j = 1, \ldots, n, \tag{6.74}$$

with operator

$$X = \xi^j[\boldsymbol{x}]\frac{\partial}{\partial x^j} \tag{6.75}$$

has the associated characteristic equations

$$\frac{dx^1}{\xi^1[\boldsymbol{x}]} = \frac{dx^2}{\xi^2[\boldsymbol{x}]} = \frac{dx^3}{\xi^3[\boldsymbol{x}]} = \cdots = \frac{dx^n}{\xi^n[\boldsymbol{x}]} \tag{6.76}$$

with $n-1$ integrals

$$r^i = R^i[\boldsymbol{x}], \qquad i = 1, \ldots, n-1. \tag{6.77}$$

These functions satisfy the invariance condition

$$\xi^j \frac{\partial R^i}{\partial x^j} = 0, \qquad i = 1, \ldots, n-1. \tag{6.78}$$

Let $r^n = R^n[\boldsymbol{x}]$ be an invariant family chosen such that

$$\xi^j \frac{\partial R^n}{\partial x^j} = 1. \tag{6.79}$$

That is, take $R^n[\boldsymbol{x}]$ to be a family of curves invariant under the group (6.74). In terms of these variables (6.74) is equivalent to the simple translation group

$$\begin{aligned} \tilde{r}^i &= r^i, \qquad i = 1, \ldots, n-1, \\ \tilde{r}^n &= r^n + s \end{aligned} \tag{6.80}$$

with group operator

$$\boxed{X = \frac{\partial}{\partial r^n}.} \tag{6.81}$$

The integrals $(r^1, r^2, \ldots, r^n)$ are the *canonical coordinates*. Any Lie group can be expressed as a simple translation using canonical coordinates. Occasionally, canonical coordinates can be used to reconstruct the finite form of a group from a knowledge of the infinitesimals without having to sum the Lie series. Insert into (6.80) the functional dependence on $x$:

$$\begin{aligned} R^i[\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n] &= R^i[x^1, x^2, \ldots, x^n], \qquad i = 1, \ldots, n-1, \\ R^n[\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n] &= R^n[x^1, x^2, \ldots, x^n] + s. \end{aligned} \tag{6.82}$$

Solving for the $\tilde{x}^j$ in (6.82) leads to the finite transformation (6.74). Many of the results of group theory are expressed in their simplest, most elegant form using canonical coordinates.

***Example 6.7 (Use canonical coordinates to determine the finite form of a group).*** Find the finite transformation corresponding to

$$\xi = \frac{1}{df/dx}, \qquad \eta = \frac{1}{dg/dy}, \tag{6.83}$$

where $f$ and $g$ are arbitrary invertible functions with inverses $f^{-1}$ and $g^{-1}$. The group operator is

$$X = \frac{1}{df/dx}\frac{\partial}{\partial x} + \frac{1}{dg/dy}\frac{\partial}{\partial y} \tag{6.84}$$

with characteristic equations

$$\frac{df}{dx}\,dx = \frac{dg}{dy}\,dy \tag{6.85}$$

and first integral

$$r^1 = R^1[x, y] = f[x] - g[y]. \tag{6.86}$$

The invariant family is found from

$$\frac{df}{dx}\,dx = \frac{dg}{dy}\,dy = \frac{dr^2}{1}, \tag{6.87}$$

which can be solved in several possible ways, e.g., $r^2 = f[x]$ or $r^2 = g[y]$. A choice that keeps $x$ and $y$ on an equal footing is

$$r^2 = \frac{f[x] + g[y]}{2} \tag{6.88}$$

The transformation in canonical variables is

$$\begin{aligned} \tilde{r}^1 &= r^1, \\ \tilde{r}^2 &= r^2 + s, \end{aligned} \tag{6.89}$$

or, in terms of original variables,

$$\begin{aligned} f[\tilde{x}] - g[\tilde{y}] &= f[x] - g[y], \\ f[\tilde{x}] + g[\tilde{y}] &= f[x] + g[y] + s. \end{aligned} \tag{6.90}$$

Solving (6.90) for $\tilde{x}$ and $\tilde{y}$, the transformation in noncanonical coordinates is

$$\begin{aligned} \tilde{x} &= f^{-1}[f[x] + s/2], \\ \tilde{y} &= g^{-1}[g[y] + s/2]. \end{aligned} \tag{6.91}$$

Differentiating (6.91) with respect to the group parameter reproduces the infinitesimals given initially.

***Example 6.8 (A family of ellipses).*** Identify the finite group that leaves the family

$$\psi = \left(\frac{x}{\tau}\right)^2 + \left(\frac{y}{\sigma}\right)^2 - 1 \tag{6.92}$$

invariant. Try an operator of the form discussed in the previous example:

$$X = \left(x - \frac{\tau^2}{2x}\right)\frac{\partial}{\partial x} + \left(y - \frac{\sigma^2}{2y}\right)\frac{\partial}{\partial y}. \tag{6.93}$$

Operating on (6.92), we have

$$X\Psi = \left(x - \frac{\tau^2}{2x}\right)\frac{2x}{\tau^2} + \left(y - \frac{\sigma^2}{2y}\right)\frac{2y}{\sigma^2} = 2\psi, \tag{6.94}$$

which confirms that (6.92) is an invariant family of (6.93). The characteristic equations of the group are

$$\frac{dx}{x - \frac{\tau^2}{2x}} = \frac{dy}{y - \frac{\sigma^2}{2y}}, \tag{6.95}$$

which can be integrated to generate the first canonical coordinate

$$r^1 = \ln\left[\frac{2y^2 - \sigma^2}{2x^2 - \tau^2}\right]^{1/2}. \tag{6.96}$$

The second canonical coordinate is the original invariant family

$$r^2 = \left(\frac{x}{\tau}\right)^2 + \left(\frac{y}{\sigma}\right)^2 - 1. \tag{6.97}$$

Solving

$$\begin{aligned} r^1[\tilde{x}, \tilde{y}] &= r^1[x, y], \\ r^2[\tilde{x}, \tilde{y}] &= r^2[x, y] + s \end{aligned} \tag{6.98}$$

for $\tilde{x}$ and $\tilde{y}$ using (6.96) and (6.97), the finite transformation group in non-canonical coordinates is found to be

$$\begin{aligned} \tilde{x} &= \left(e^{2s}\left(x^2 - \frac{\tau^2}{2}\right) + \frac{\tau^2}{2}\right)^{1/2}, \\ \tilde{y} &= \left(e^{2s}\left(y^2 - \frac{\sigma^2}{2}\right) + \frac{\sigma^2}{2}\right)^{1/2}. \end{aligned} \tag{6.99}$$

This transformation leaves the family of ellipses (6.92) invariant.

## 6.6 Invariant Solutions

In the development of the integrating factor we made use of the invariance of a first-order ODE under two groups: the trivial group defined by the ODE itself, which leaves invariant individual solution curves, and the nontrivial group, which leaves invariant the family of solution curves while mapping each curve to another member of the family. It is often the case that certain solution curves are left invariant by the nontrivial group. Such invariant curves often play an important role in defining the asymptotic behavior of the solution. Let's look at a couple of examples.

***Example 6.9 (The Clairault equation).*** An interesting example illustrating the notion of invariant curves and envelopes is the Clairault equation

$$x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} + m = 0, \tag{6.100}$$

which is invariant under a one-parameter dilation group

$$\begin{aligned} \tilde{x} &= e^{2s}x, & \tilde{y} &= e^{s}y, \\ \xi &= 2x, & \eta &= y. \end{aligned} \tag{6.101}$$

Solving the quadratic (6.100) for the first derivative, the equation can be written in the following form:

$$-\left(y \pm (y^2 - 4mx)^{1/2}\right) dx + 2x\, dy = 0. \tag{6.102}$$

The invariant group (6.101) generates the integrating factor

$$M = \frac{1}{A\eta - B\xi} = \frac{1}{\mp 2x(y^2 - 4mx)^{1/2}} \tag{6.103}$$

and the general solution

$$\psi = \frac{y}{2x} \pm \frac{1}{2}\left(\frac{y^2}{x^2} - \frac{4m}{x}\right)^{1/2}. \tag{6.104}$$

The solution can be rearranged as follows:

$$\left(\psi - \frac{y}{2x}\right)^2 = \left(\frac{y^2}{4x^2} - \frac{m}{x}\right). \tag{6.105}$$

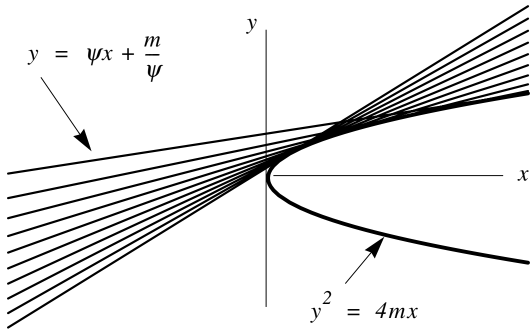


Fig. 6.4. Solution family of the Clairault equation.

When (6.105) is expanded, the quadratic terms on both sides cancel, leaving the family of straight lines

$$y = \psi x + m/\psi \tag{6.106}$$

as the general solution; (6.106) transforms under (6.101) as follows:

$$\tilde{y} = \psi \tilde{x} + \frac{m}{\psi} \quad \Rightarrow \quad e^s y = \psi e^{2s} x + \frac{m}{\psi} \quad \Rightarrow \quad y = (\psi e^s)x + \frac{m}{\psi e^s}. \tag{6.107}$$

As $\psi$ is varied from $-\infty$ to $\infty$, (6.106) generates the pattern shown in Figure 6.4.

An invariant solution of (6.100) can be found as follows. Let $\psi_{inv}$ be expressed as

$$\psi_{inv} = y - f(x) = 0 \tag{6.108}$$

where $f(x)$ is to be determined. The invariance condition on $\psi_{inv}$ is

$$X\psi_{inv} = 2x\frac{\partial \psi_{inv}}{\partial x} + y\frac{\partial \psi_{inv}}{\partial y} = -2xf_x + y = 0 \tag{6.109}$$

Combining (6.108) and (6.109) leads to $f = cx^{1/2}$. The constant is evaluated by substituting $y = cx^{1/2}$ into (6.100). The result is the invariant solution

$$y = \pm 2(mx)^{1/2} \tag{6.110}$$

Note that this is the equation of the envelope that bounds the solution family (6.106).

This idea of combining the equation with the invariance condition to find an invariant solution is often very useful and will come up again at the end of Chapter 9 where nonclassical symmetries are discussed.

Invariant solutions are common in nonlinear problems, and when they occur, they are of great importance in that they often define a universal solution to which the system evolves regardless of initial conditions. The identification of such invariant manifolds is an important step in understanding the nature of nonlinearity. The next example illustrates this type of behavior.

***Example 6.10 (Evolution along the discriminant of a cubic equation).*** This problem comes up in the context of a simple model for the evolution of fine-scale motions in a turbulent flow [6.1]. Solve the autonomous pair

$$\begin{aligned} \frac{dQ}{dt} + 3R &= 0, \\ \frac{dR}{dt} - \frac{2}{3}Q^2 &= 0. \end{aligned} \tag{6.111}$$

The unknowns $Q$ and $R$ are the second and third invariants of the velocity gradient tensor in an incompressible flow, and the equations (6.111) represent a model for the evolution of these quantities following a fluid particle. Eliminating $dt$ between the two equations leads to the Pfaffian 1-form

$$\frac{2}{3}Q^2\, dQ + 3\, R\, dR = 0, \tag{6.112}$$

which is easily integrated to produce the solution

$$D = Q^3 + \frac{27}{4}R^2. \tag{6.113}$$

Recalling the discusssion of 3-D linear flows in Chapter 3, Section 3.9, we recognize (6.113) as the discriminant of a cubic polynomial $\lambda^3 + Q\lambda + R = 0$.

Lines of constant $D$ are shown in Figure 6.5. Equation (6.112) is invariant under a one-parameter dilation group

$$\tilde{Q} = e^{2s} Q, \qquad \tilde{R} = e^{3s} R. \tag{6.114}$$

A given solution is transformed as follows:

$$\tilde{D} = \tilde{Q}^3 + \frac{27}{4}\tilde{R}^2 = e^{6s}\left(Q^3 + \frac{27}{4}R^2\right) = e^{6s} D. \tag{6.115}$$

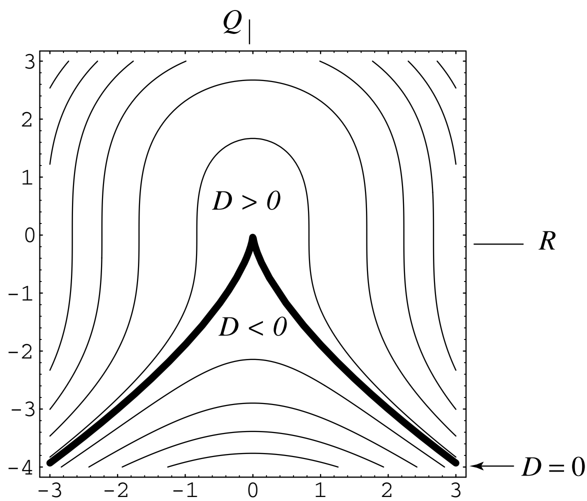


Fig. 6.5. Lines of constant cubic discriminant.

The $D = 0$ curve, shown as a bold line in Figure 6.5, is an invariant solution of (6.111) and is mapped to itself by the group (6.114). To understand the role of this invariant solution in the asymptotic behavior of the full $D \neq 0$ solution, we need to solve for the parametric functions, $Q[t]$ and $R[t]$. With the integral of the motion known, the time evolution of the two invariants will be determined in terms of elliptic functions. The discriminant defines an appropriate time scale for normalizing all the variables in the problem:

$$t_0 = |D|^{1/6}. \tag{6.116}$$

Let

$$q = t_0^2 Q, \qquad r = t_0^3 R, \qquad \tau = \frac{t}{t_0}. \tag{6.117}$$

In the case where $D = 0$, the normalization is carried out using $Q_i$, the initial value of $Q$. In this case, $t_0 = (\text{abs}[Q_i])^{1/2}$ and the normalized discriminant is $q^3 + \frac{27}{4} r^2 = \text{sgn}[D]$, where

$$\text{sgn}[D] = \begin{cases} +1, & D > 0, \\ -1, & D < 0, \\ 0, & D = 0. \end{cases} \tag{6.118}$$

Using this normalization, the first equation in (6.111) becomes,

$$\frac{dq}{d\tau} = \begin{cases} \frac{2}{\sqrt{3}}(\mathrm{sgn}[D] - q^3)^{1/2} & (r < 0), \\ -\frac{2}{\sqrt{3}}(\mathrm{sgn}[D] - q^3)^{1/2}, & (r > 0), \end{cases} \tag{6.119}$$

which is solved in terms of elliptic integrals:

$$\begin{aligned} \sqrt[4]{3}\int_{-\infty}^{q} \frac{d\hat{q}}{(1-\hat{q}^3)^{1/2}} &= F[\alpha, \sin[5\pi/12]] \\ &= \frac{2}{\sqrt[4]{3}}\tau \qquad (D > 0,\ -\infty < q < 1), \\ \sqrt[4]{3}\int_{q}^{-1} \frac{d\hat{q}}{(-1-\hat{q}^3)^{1/2}} &= F[\gamma, \sin[\pi/12]] \\ &= \frac{2}{\sqrt[4]{3}}\tau \qquad (D < 0,\ -\infty < q < -1). \end{aligned} \tag{6.120}$$

The function $F$ is the elliptic integral of the first kind,

$$F[\phi, k] = \int_0^{\phi} \frac{d\hat{\phi}}{(1 - k^2 \sin^2[\hat{\phi}])^{1/2}}, \tag{6.121}$$

and we have

$$\begin{aligned} \cos[\alpha] &= \frac{1 - \sqrt{3} - q[\tau]}{1 + \sqrt{3} - q[\tau]} \qquad (-\infty < q < 1,\ 0 < \alpha < \pi,\ D > 0), \\ \cos[\gamma] &= \frac{1 + \sqrt{3} + q[\tau]}{-1 + \sqrt{3} - q[\tau]} \qquad (-\infty < q < -1,\ 0 < \gamma < \pi,\ D < 0). \end{aligned} \tag{6.122}$$

The solution (6.120) accommodates the sign change indicated in (6.119) by the fact that the function $q[\tau]$ varies smoothly near $r = 0$ for $D > 0$ or $D < 0$. The elliptic integral $F[\phi, k]$ is continued beyond $\phi = \pi/2$ using the relation $F[\phi, k] = K + F[\phi - \pi/2, k]$. When we consider the case $D = 0$, which has a discontinuous first derivative at $r = 0$, we shall see that the sign change is retained and it is necessary to explicitly distinguish between $r > 0$ and $r < 0$ cases. The relationship between $\alpha$ or $\gamma$ and $q[\tau]$ in (6.120) is inverted through the use of Jacobi elliptic functions. We use the cosine amplitude function cn defined by $\cos[\phi] = \mathrm{cn}[F] = \mathrm{cn}[(2/\sqrt[4]{3})\tau]$. Three cases are distinguished.

*Case 1. $D > 0$:*

$$q^{+}(\tau) = \frac{(1-\sqrt{3}) - (1+\sqrt{3})\,\mathrm{cn}[(2/\sqrt[4]{3})\tau]}{1-\mathrm{cn}[(2/\sqrt[4]{3})\tau]}. \tag{6.123}$$

The range of variables in (6.123) is

$$-\infty < q^{+} < 1, \qquad 0 < \tau < \tau_{\text{singular}}, \tag{6.124}$$

where $\tau_{\text{singular}} = 7.28589$. The third invariant is computed from

$$r = \pm\left(\frac{4}{27} - \frac{4}{27}q^{3}\right)^{1/2}. \tag{6.125}$$

*Case 2. $D < 0$:*

$$q^{-}(\tau) = \frac{-(1+\sqrt{3}) + (1-\sqrt{3})\,\mathrm{cn}[(2/\sqrt[4]{3})\tau]}{1-\mathrm{cn}[(2/\sqrt[4]{3})\tau]}. \tag{6.126}$$

The range of variables in (6.126) is

$$-\infty < q^{-} < -1, \qquad 0 < \tau < \tau_{\text{singular}}, \tag{6.127}$$

where $\tau_{\text{singular}} = 4.20654$, and the third invariant is determined from

$$r = \pm\left(-\frac{4}{27} - \frac{4}{27}q^{3}\right)^{1/2}. \tag{6.128}$$

*Case 3. $D = 0$:*

$$\begin{aligned} q^{0}_{r<0}[\tau] &= -\left(\frac{1}{1+(1/\sqrt{3})\tau}\right)^{2} \qquad (0 < \tau < \infty), \\ q^{0}_{r>0}[\tau] &= -\left(\frac{1}{1+(1/\sqrt{3})\tau}\right)^{2} \qquad (0 < \tau < \infty). \end{aligned} \tag{6.129}$$

The third invariant is computed from

$$r = \pm\left(-\frac{4}{27}q^{3}\right)^{1/2} \tag{6.130}$$

The invariants are parameterized by elliptic functions. Note the singularity in$Q[\tau]$ and $R[\tau]$ that develops in finite time for both positive and negative discriminant. When the invariants are plotted parametrically, the result is a pair of curves for $D = \pm 1$ as shown in Figure 6.6.

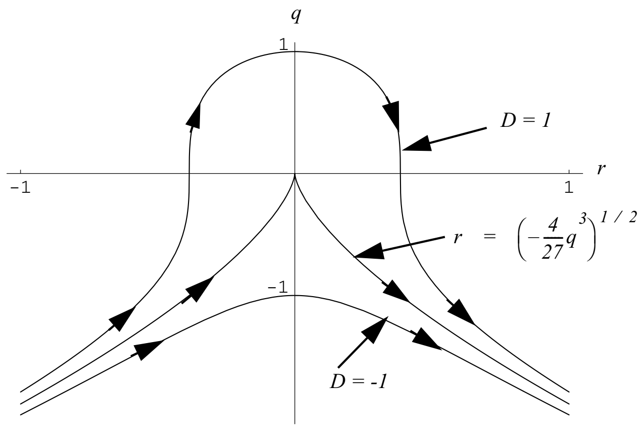


Fig. 6.6. Lines of constant normalized discriminant.

The arrows in Figure 6.6 indicate the direction of increasing time. As the parameter $t$ increases, $r$ increases monitonically, and once $r$ becomes positive, $q$ decreases monotonically. In Figure 6.6 the evolution of the solution is toward the lower right quadrant. So both $q$ and $r$ become infinitely large at a fixed value of $D$. The asymptotic behavior of (6.123) and (6.126) is identical to the second relation in (6.129) with the appropriate value of the singular time inserted in the radical. Thus as time proceeds, the asymptotic solution for any initial value of $q$ or $r$ is

$$q[\tau] = -\left(\frac{1}{1-\tau/\tau_{\text{singular}}}\right)^2, \qquad r[\tau] = \frac{2}{3\sqrt{3}}\left(\frac{1}{1-\tau/\tau_{\text{singular}}}\right)^3. \quad (6.131)$$

In effect, any initial condition evolves asymptotically to the invariant solution at large values of $t$. Note that the "trivial" group that solves (6.111) is fairly complex, being expressed in terms of Jacobi elliptic functions.

## 6.7 Elliptic Curves

The expression for the discriminant (6.113) contains a mixture of quadratic and cubic terms. For constant, nonzero discriminant, this function belongs to a class of functions called *elliptic curves*. The complete classification of cubics, of which elliptic curves are a subset, was described by Isaac Newton in 1695. Elliptic curves have the property that there is a unique tangent everywhere on the curve (hence $D = 0$ is excluded), and they are parameterized by elliptic functions as in (6.123) and (6.126). The curve $D = 0$, which has a cusp at the origin, is parameterized by rational functions as in (6.129). The most familiar example of an elliptic function is the integral of the arc length along an ellipse,

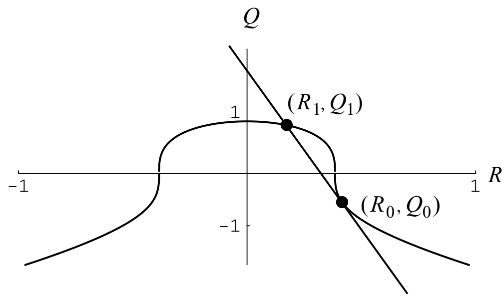


Fig. 6.7. Construction to find rational roots on a curve of constant $D$.

although the ellipse itself is not an elliptic curve, it being parameterized by harmonic functions.

Elliptic curves come up, for example, in the study of the motions that can be executed by mechanical linkages. Some examples can be found in the classic text by Hunt [6.2]. They are also of intrinsic mathematical interest, and an introductory discussion can be found in the article by Ribet and Hayes [6.3]. One of the interesting properties of these functions is that a straight line tangent to a rational root intersects the function at another rational root (this works for $D = 0$ too). This fact can be exploited to create the geometrical construction shown in Figure 6.7, by which all rational roots lying on a curve of constant discriminant can be determined once a single root is known. This is the famous chord–tangent construction used by the Greek mathematician Diophantus of Alexandria in the third century in his studies of number theory.

The cubic discriminant has the same value at both points of intersection in Figure 6.7,

$$Q_1^3 + \frac{27}{4}R_1^2 = Q_0^3 + \frac{27}{4}R_0^2, \tag{6.132}$$

and the straight line is of the form

$$R + aQ + b = 0. \tag{6.133}$$

At $(R_0, Q_0)$ the straight line and line of constant $D$ have the same slope as well as the same coordinates. This is used to evaluate $a$ and $b$, and the equation of the straight line is determined to be

$$R + \left(\frac{2}{9}\frac{Q_0^2}{R_0}\right)Q + \left(-\frac{2}{9}\frac{Q_0^3}{R_0} - R_0\right) = 0. \tag{6.134}$$

Now evaluate (6.134) at $(R_1, Q_1)$, and use it to replace $R_1$ in (6.132). The result is a cubic equation for $Q_1$, which can be factored as

$$(Q_1 - Q_0)^2 \left( Q_1 + \frac{1}{3}\frac{Q_0^4}{R_0^2} + 2Q_0 \right) = 0. \tag{6.135}$$

Two of the roots coincide with the tangent point. The third root, combined with (6.134), leads to the parameterization

$$\begin{aligned} Q_1 &= -\frac{1}{3}\frac{Q_0^4}{R_0^2} - 2Q_0, \\ R_1 &= \frac{2}{27}\frac{Q_0^6}{R_0^3} + \frac{2}{3}\frac{Q_0^3}{R_0} + R_0. \end{aligned} \tag{6.136}$$

It is clear that if $Q_0$ and $R_0$ are rational numbers, then so are $Q_1$ and $R_1$. Repeating the chord–tangent construction at the new root leads to a third rational root, and so on. We shall encounter elliptic curves again in Chapters 11, where we study the geometry of the 3-D flow field of a laminar jet.

## 6.8 Criterion for a First-Order ODE to Admit a Given Group

The various examples in this chapter illustrate both the power and the limitations of group theory. In each case the group had to be stated *a priori*. In later chapters we shall see that for higher-order ODEs and for PDEs the group can be determined through a procedure that is essentially algorithmic. But for first-order ODEs the procedure for finding the group is essentially equivalent to solving the original ODE, and so no systematic procedure exists. Nevertheless, it would be extremely useful to have a test to determine whether a given differential equation is invariant under a given group, which may be known only through its infinitesimals. Such a test can be constructed with the help of the commutator.

We are considering the ODE (6.33)

$$-B\,dx + A\,dy = 0 \tag{6.137}$$

with associated PDE (6.36)

$$A\frac{\partial \Psi}{\partial x} + B\frac{\partial \Psi}{\partial y} = 0 \tag{6.138}$$

and group operator

$$Y = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}. \tag{6.139}$$

The commutator of some general group $X = \xi\, \partial/\partial x + \eta\, \partial/\partial y$ and $Y$ is

$$\{X, Y\} = X(Y) - Y(X). \tag{6.140}$$

When (6.140) is expanded, all second-derivative terms cancel to produce a new operator, which is still of first-order:

$$\{X, Y\} = \left(\xi \frac{\partial A}{\partial x} + \eta \frac{\partial A}{\partial y} - A \frac{\partial \xi}{\partial x} - B \frac{\partial \xi}{\partial y}\right) \frac{\partial}{\partial x} + \left(\xi \frac{\partial B}{\partial x} + \eta \frac{\partial B}{\partial y} - A \frac{\partial \eta}{\partial x} - B \frac{\partial \eta}{\partial y}\right) \frac{\partial}{\partial y}, \tag{6.141}$$

or

$$\{X, Y\} = (XA - Y\xi)\frac{\partial}{\partial x} + (XB - Y\eta)\frac{\partial}{\partial y}. \tag{6.142}$$

If $\psi = \Psi[x, y]$ is the integral of (6.137), then we know that $Y\Psi = 0$. Assume that $\Psi$ is an invariant family of the group $X$; then $X\Psi = \Omega[\Psi]$. The commutator acting on $\Psi$ gives

$$\{X, Y\}\Psi = X(Y\Psi) - Y(X\Psi) = -Y(\Omega[\Psi]) = -\frac{d\Omega}{d\Psi} Y\Psi = 0. \tag{6.143}$$

So the partial derivatives of $\Psi$ satisfy both (6.138) and

$$(XA - Y\xi)\frac{\partial \Psi}{\partial x} + (XB - Y\eta)\frac{\partial \Psi}{\partial y} = 0, \tag{6.144}$$

which means that the two operators cannot be independent. Therefore there must exist a function $\lambda[x, y]$ such that

$$XA - Y\xi = \lambda[x, y]A, \qquad XB - Y\eta = \lambda[x, y]B. \tag{6.145}$$

This is the operator condition for the invariance of a given differential equation under a given group.

***Theorem 6.3.*** *The solution family $\Psi[x, y]$ of the ordinary differential equation $B\,dx - A\,dy = 0$ is an invariant family of the one-parameter group $(\xi, \eta)$ if and only if a function $\lambda[x, y]$ exists such that*

$$\frac{XA - Y\xi}{A} = \lambda[x, y] = \frac{XB - Y\eta}{B}, \tag{6.146}$$

*where $X = \xi\,\partial/\partial x + \eta\,\partial/\partial y$ and $Y = A\,\partial/\partial x + B\,\partial/\partial y$.*

This result gives us a straightforward, systematic procedure for testing given groups against a given equation in a search for one that leaves the equation invariant. Simply form the expressions in (6.146) and see if the equality holds. This can be useful because relatively simple groups often leave large classes of equations invariant.

## 6.9 Concluding Remarks

In this chapter we have applied group methods to the solution of first-order ODEs and derived the invariance condition for an ODE. We will continue this subject to higher-order ODEs in Chapter 8. But first it is necessary to introduce the concept of a differential function in Chapter 7. This notion enables us to use the theory for the invariance of a function defined in Chapter 5, Section 5.6, to define a comparable invariance condition for a differential equation. The same mathematical machinery developed for functions can be carried over, more or less intact, and applied to ODEs and PDEs treated as differential functions.

## 6.10 Exercises

6.1 Reconsider the groups studied in Chapter 5, Problem 5.1:

(i) A projective group

$$\tilde{x} = \frac{x}{1 - sy}, \qquad y = \frac{y}{1 - sy}. \tag{6.147}$$

(ii) A hyperbolic group

$$\tilde{x} = x + s, \qquad \tilde{y} = \frac{xy}{x + s}. \tag{6.148}$$

(iii) An arbitrary translation

$$\tilde{x} = x, \qquad \tilde{y} = y + sf[x], \quad f(x)\text{ arbitrary}. \tag{6.149}$$

(iv) A helical transformation

$$\tilde{x} = x\cos[s] - y\sin[s], \quad \tilde{y} = x\sin[s] + y\cos[s], \quad \tilde{z} = z + ms. \tag{6.150}$$

Determine an invariant family for each group.

6.2 Find an integrating factor for each of the following ODEs, and work out the general solution:

$$\frac{dy}{dx} - \frac{y}{x + \sin[x/y]} = 0, \tag{6.151}$$

$$(3x^2 + 2xy - y^2)\,dx + (x^2 - 2xy - 3y^2)\,dy = 0, \tag{6.152}$$

$$\frac{dy}{dx} = \frac{ye^y}{y^3 + 2xe^y}, \tag{6.153}$$

$$x\frac{dy}{dx} + y = x^2, \tag{6.154}$$

$$\frac{dy}{dx} = 4\frac{y}{x} + x^2 \sin[y/x^4]. \tag{6.155}$$

6.3 Revisit Chapter 1, Exercise 1.3. Find an integrating factor, and solve the first-order ODE

$$x\left(\frac{dy}{dx}\right)^2 + y\left(\frac{dy}{dx}\right) + x = 0. \tag{6.156}$$

6.4 Show by direct substitution that (6.99) leaves the family of ellipses (6.92) invariant.

6.5 Show that the first-order ODE

$$\frac{dy}{dx} = \frac{y^3 + x^2y - y - x}{xy^2 + x^3 + y - x} \tag{6.157}$$

is invariant under the rotation group $(\xi, \eta) = (-y, x)$. Sketch the phase portrait and identify critical points. Identify an invariant solution. Use the group to find an integrating factor and work out the solution.

6.6 Beginning with $(R, Q) = (2, -3)$ on $Q^3 + \frac{27}{4}R^2 = 0$, use the chord–tangent construction to identify an infinite sequence of rational roots.

6.7 Can you find a rational root of the equation $Q^3 + \frac{27}{4}R^2 = 1$?

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# 7

# *Differential Functions and Notation*

In Chapter 6 we concerned ourselves with finding integrating factors for first-order ODEs. In Chapter 8 we will continue our discussion of ODEs and address two fundamental problems. The first is the problem of finding the general ODE $\Psi[x, y, y_x, y_{xx}, \ldots y_{px}] = 0$ which is invariant under a known group. The second is the practically much more important problem of determining the groups that leave a given ODE invariant. The latter problem generally involves the following basic steps:

(1) The point transformation $T^s$ is extended to include transformations of derivatives up to whatever order $p$ may appear in the equation. In practice only the infinitesimal form of the transformation is required. The formulas for these extended transformations become quite long as the order of the derivative being transformed increases.
(2) The differential equation is transformed using the extended group. The transformation of the equation is expressed as a Lie series expanded in terms of the operator $X_{\{p\}}$ of the extended group.
(3) The equation is invariant if and only if $X_{\{p\}}\Psi = 0$ subject to the constraint $\Psi = 0$.
(4) The invariance condition is parsed into a set of linear PDEs for $\xi$ and $\eta$ known as the *determining equations* of the group. For a first-order ODE there is only one determining equation for $\xi$ and $\eta$, which is insufficient to solve for both unknowns. But for second- and higher-order ODEs there are generally two or more determining equations, so that $\xi$ and $\eta$ can usually be determined, enabling the fundamental symmetries of the equation to be identified. While it is possible to write down a higher-order ODE for which the determining equations have no solution (the ODE has no symmetry) this is uncommon and rarely occurs when the ODE arises from an interesting physical problem.

The theoretical basis for the method lies in the fact that the algorithm used to generate the transformation of derivatives is such that the extended transformation automatically inherits the properties of a Lie group. A proof of this will be given in Chapter 8. In particular, the extended transformation is a one-to-one invertible map dependent on a single scalar parameter defined on a continuous open interval. This enables all the results from the group analysis of functions developed in Chapter 5, Sections 5.5 and 5.6 to be carried over, more or less intact, to the group analysis of ODEs and PDEs viewed as *differential functions*. Stated simply, a differential function is a smooth, locally analytic function of variables and derivatives. Examples include differential equations and the functions that transform derivatives. The idea of a differential function is an overarching concept, which provides a framework for the general theory of symmetry analysis and greatly facilitates the treatment of extended point groups as well as their generalization to Lie–Bäcklund groups.

Group theory repeatedly requires the use of the chain rule, and the reader is referred to Appendix 1, where some basic results from calculus are reviewed and the total differentiation operator $D$ is defined. The motivation for using this operator is the need to deal with notational ambiguities that arise when taking partial derivatives of implicit functions.

A notation for derivative and function names was introduced in Chapter 1 and is elaborated further here. I have tried to be as consistent as possible in using the adopted notation throughout the text. It is absolutely essential to use a notation that is precise, concise, and reasonably intuitive. If the notation doesn't work, then the meaning of expressions can be quickly lost in a blizzard of indices. This is especially important when we deal with some of the rather complex formulas that involve differentiation of differential equations with respect to derivatives. This may seem like a strange concept at first, but it is a natural consequence of the treatment of differential equations as analytic functions of derivatives.

## 7.1 Introduction

A differential equation is a locally analytic function of the variables and derivatives that appear in the equation. For example the Blasius equation

$$\frac{d^3y}{dx^3} + y\frac{d^2y}{dx^2} = 0 \tag{7.1}$$

is a function of the form

$$\psi = \Psi\left[x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}\right] \tag{7.2}$$

where each argument after the first in the brackets is a function of $x$. A partial differential equation such as the Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x\,\partial x} = 0 \tag{7.3}$$

is a function in a jet space whose coordinates include all possible first and second derivatives,

$$\Phi\left[t, x, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial t\,\partial t}, \frac{\partial^2 u}{\partial x\,\partial t}, \frac{\partial^2 u}{\partial x\,\partial x}\right] = 0. \tag{7.4}$$

A more complex system with several independent variables, such as the incompressible Navier–Stokes equations

$$\frac{\partial u^i}{\partial t} + u^k\frac{\partial u^i}{\partial x^k} + \frac{\partial p}{\partial x^i} - \nu\frac{\partial^2 u^i}{\partial x^k \partial x^k} = 0, \qquad i = 1, 2, 3, \tag{7.5}$$

is a set of three functions of the form

$$\Theta^i\left[t, x^1, \ldots, x^3, p, u^1, \ldots, u^3, \frac{\partial p}{\partial t}, \frac{\partial u^1}{\partial t}, \ldots, \frac{\partial u^3}{\partial t}, \frac{\partial p}{\partial x^1}, \frac{\partial u^1}{\partial x^1}, \ldots, \frac{\partial u^3}{\partial x^1}, \right.$$
$$\left. \frac{\partial p}{\partial x^2}, \frac{\partial u^1}{\partial x^2}, \ldots, \frac{\partial u^3}{\partial x^2}, \frac{\partial p}{\partial x^3}, \frac{\partial u^1}{\partial x^3}, \ldots, \frac{\partial u^3}{\partial x^3}, \frac{\partial^2 p}{\partial t\,\partial t}, \ldots\ldots\ldots\ldots, \frac{\partial^2 u^3}{\partial x^3\,\partial x^3}\right] = 0,$$
$$i = 1, 2, 3, \tag{7.6}$$

in a jet space that includes all the independent variables, dependent variables, all possible first partial derivatives, and all possible second partial derivatives.

It will often be necessary to differentiate these functions with respect to derivatives. For example, typical partial derivatives of the Blasius equation (7.1) are

$$\frac{\partial \Psi}{\partial y} = \frac{d^2 y}{dx^2}, \qquad \frac{\partial \Psi}{\partial\left(\frac{d^2 y}{dx^2}\right)} = y. \tag{7.7}$$

The derivative of the Burgers equation (7.3) with respect to the spatial first derivative of $u$ is

$$\frac{\partial \Phi}{\partial\left(\frac{\partial u}{\partial x}\right)} = u. \tag{7.8}$$

Differentiating the Navier–Stokes equations with respect to the second spatial derivative of velocity with respect to the third coordinate leads to

$$\frac{\partial \Theta^i}{\partial\left(\frac{\partial^2 u^i}{\partial x^3\,\partial x^3}\right)} = \nu, \tag{7.9}$$

and so forth. The notation in (7.6), (7.7), (7.8), and (7.9) is as clumsy as the wording in the last sentence and does not lend itself to generalization. We really need something better.

### 7.1.1 Superscript Notation for Dependent and Independent Variables

Consider a transformation with $m$ dependent variables

$$\mathbf{y} = (y^i), \qquad i = 1, \ldots, m, \tag{7.10}$$

and $n$ independent variables

$$\mathbf{x} = (x^j), \qquad j = 1, \ldots, n. \tag{7.11}$$

The corresponding Lie point group has the form

$$\begin{aligned} \tilde{x}^j &= F^j[\mathbf{x}, \mathbf{y}, s], \qquad j = 1, \ldots, n, \\ \tilde{y}^i &= G^i[\mathbf{x}, \mathbf{y}, s], \qquad i = 1, \ldots, m. \end{aligned} \tag{7.12}$$

In general, vector components will be denoted with a superscript, consistent with commonly accepted notation for vectors and tensors. There is some possibility here for confusion with an exponent, and where this might occur, parentheses will be used to clarify the taking of a power.

### 7.1.2 Subscript Notation for Derivatives

Subscripts will be used to denote derivatives throughout the text. Otherwise the conventional quotient form of the derivative will be used. In the case of a $p$th-order ODE with one independent variable and one dependent variable we use

$$y_x \equiv \frac{dy}{dx}, \qquad y_{xx} \equiv \frac{d^2 y}{dx^2}, \ldots, \qquad y_{(p-1)x} \equiv \frac{d^{p-1} y}{dx^{p-1}}, \qquad y_{px} \equiv \frac{d^p y}{dx^p}. \tag{7.13}$$

Derivatives expressed in terms of target variables are denoted with a tilde:

$$\tilde{y}_{\tilde{x}} \equiv \frac{d\tilde{y}}{d\tilde{x}}, \qquad \tilde{y}_{\tilde{x}\tilde{x}} \equiv \frac{d^2\tilde{y}}{d\tilde{x}^2}, \ldots, \qquad \tilde{y}_{(p-1)\tilde{x}} \equiv \frac{d^{p-1}\tilde{y}}{d\tilde{x}^{p-1}}, \qquad \tilde{y}_{p\tilde{x}} \equiv \frac{d^p\tilde{y}}{d\tilde{x}^p}. \tag{7.14}$$

The partial derivatives of the transformation functions $F$ and $G$ are denoted

$$\begin{aligned} \frac{\partial F}{\partial x} &= F_x, \qquad & \frac{\partial F}{\partial y} &= F_y, \\ \frac{\partial G}{\partial x} &= G_x, \qquad & \frac{\partial G}{\partial y} &= G_y \end{aligned} \tag{7.15}$$

and

$$\begin{aligned} \frac{\partial^2 F}{\partial x^2} &= F_{xx}, \qquad & \frac{\partial^2 F}{\partial x\, \partial y} &= F_{xy}, \qquad & \frac{\partial^2 F}{\partial y^2} &= F_{yy}, \\ \frac{\partial^2 G}{\partial x^2} &= G_{xx}, \qquad & \frac{\partial^2 G}{\partial x\, \partial y} &= G_{xy}, \qquad & \frac{\partial^2 G}{\partial y^2} &= G_{yy}, \end{aligned} \tag{7.16}$$

and so forth. Generally the variable symbol will be used for the subscript, although it may occasionally be convenient to use the variable index instead. In this case, when dealing with ODEs, the notation

$$y_1 \equiv \frac{dy}{dx}, \qquad y_2 \equiv \frac{d^2y}{dx^2}, \ldots, \qquad y_{p-1} \equiv \frac{d^{p-1}y}{dx^{p-1}}, \qquad y_p \equiv \frac{d^py}{dx^p} \tag{7.17}$$

may be used. If there is any possibility of confusion, the quotient form of the derivative will be employed.

In the case of PDEs with several dependent and independent variables, the partial-derivative notation is as follows:

$$\begin{gathered} y^i_j \equiv \frac{\partial y^i}{\partial x^j}, \qquad y^i_{j_1 j_2} \equiv \frac{\partial^2 y^i}{\partial x^{j_1}\, \partial x^{j_2}}, \qquad y^i_{j_1 j_2 j_3} \equiv \frac{\partial^3 y^i}{\partial x^{j_1}\, \partial x^{j_2}\, \partial x^{j_3}}, \ldots, \\ y^i_{j_1 j_2 j_3 \cdots j_p} \equiv \frac{\partial^p y^i}{\partial x^{j_1}\, \partial x^{j_2}\, \partial x^{j_3} \cdots \partial x^{j_p}} \end{gathered} \tag{7.18}$$

where the indices $j_1$, $j_2$, $j_3 \cdots$ refer to any of the independent variables and the subscripts on these indices are distinguishing labels.

Differential equations will be treated as functions of independent variables, dependent variables, and derivatives of dependent variables. For conciseness define the boldface symbols

$$\boldsymbol{y}_1 \equiv \left(y^i_j\right) = \left(\frac{\partial y^1}{\partial x^1}, \ldots, \frac{\partial y^1}{\partial x^n}, \frac{\partial y^2}{\partial x^1}, \ldots, \frac{\partial y^2}{\partial x^n}, \ldots\ldots, \frac{\partial y^m}{\partial x^1}, \ldots, \frac{\partial y^m}{\partial x^n}\right) \tag{7.19}$$

as the vector of all possible first partial derivatives,

$$\boldsymbol{y}_2 \equiv \left(y^i_{j_1 j_2}\right) = \left(y^1_{11}, y^1_{12}, \ldots, y^1_{nn}, \ldots, y^m_{11}, y^m_{12}, \ldots, y^m_{nn}\right) \tag{7.20}$$

as the vector of all possible second partial derivatives,

$$\boldsymbol{y}_3 \equiv \left(y^i_{j_1 j_2 j_3}\right) \tag{7.21}$$

as the vector of all possible third partial derivatives, and so forth up to $p$th order:

$$\boldsymbol{y}_p \equiv \left(y^i_{j_1 j_2 j_3 \cdots j_p}\right). \tag{7.22}$$

The subscripts in these expressions denote differentiation with respect to any particular combination of independent variables. Where it is convenient, symbolic subscripts may be used to replace numerical subscripts, for example $y_{12} \equiv y_{x^1 x^2}$.

### 7.1.3 Curly-Brace Subscript Notation for Functions That Transform Derivatives

In Chapter 1 we worked out the once extended group in the plane,

$$\begin{aligned} \tilde{x} &= F[x, y, s], \\ \tilde{y} &= G[x, y, s], \\ \tilde{y}_{\tilde{x}} &= G_{\{1\}}[x, y, y_x, s]. \end{aligned} \tag{7.23}$$

A subscript in braces was used to denote the function that transforms the first derivative. This notation provides an efficient and clear association between a derivative and its transforming function while being distinct from the unbracketed subscript, which denotes differentiation.

Later we will work out the transformation for small values of the group parameter. This will have the form

$$\begin{aligned} \tilde{x} &= x + s\xi[x, y], \\ \tilde{y} &= y + s\eta[x, y], \\ \tilde{y}_{\tilde{x}} &= y_x + s\eta_{\{1\}}[x, y, y_x]. \end{aligned} \tag{7.24}$$

A transformation up to second derivatives is of the form

$$\begin{aligned}
\tilde{x} &= x + s\xi[x, y],\\
\tilde{y} &= y + s\eta[x, y],\\
\tilde{y}_{\tilde{x}} &= y_x + s\eta_{\{1\}}[x, y, y_x],\\
\tilde{y}_{\tilde{x}\tilde{x}} &= y_{xx} + s\eta_{\{2\}}[x, y, y_x, y_{xx}].
\end{aligned} \tag{7.25}$$

An equivalent notation is

$$\begin{aligned}
\tilde{x} &= x + s\xi[x, y],\\
\tilde{y} &= y + s\eta[x, y],\\
\tilde{y}_{\tilde{x}} &= y_x + s\eta_{\{x\}}[x, y, y_x],\\
\tilde{y}_{\tilde{x}\tilde{x}} &= y_{xx} + s\eta_{\{xx\}}[x, y, y_x, y_{xx}].
\end{aligned} \tag{7.26}$$

In the case of many variables, a transformation of a second partial derivative would be written as

$$\tilde{y}^i_{j_1 j_2} = G^i_{\{j_1 j_2\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, s], \tag{7.27}$$

and the infinitesimal form would be

$$\tilde{y}^i_{j_1 j_2} = y^i_{j_1 j_2} + s\eta^i_{\{j_1 j_2\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2], \tag{7.28}$$

and so forth. The subscripts on the variable indices $j_1, j_2, j_3, \ldots$ are labels that indicate that each index is distinct and represents any one of the independent variables.

### *7.1.4 The Total Differentiation Operator*

In Appendix 1 the total differentiation operator is defined. This operator is required to overcome certain notational difficulties that arise when taking partial derivatives of functions that depend on functions.

***Definition 7.1.*** *The total differentiation operator with respect to the $j$th independent variable is.*

$$\frac{D}{Dx^j} = D_j = \frac{\partial}{\partial x^j} + y^i_j \frac{\partial}{\partial y^i} + y^i_{j_1 j} \frac{\partial}{\partial y^i_{j_1}} + \cdots + y^i_{j_1 j_2 \cdots j_p j} \frac{\partial}{\partial y^i_{j_1 j_2 \cdots j_p}}. \tag{7.29}$$

*In the case of one dependent and one independent variable (7.29) becomes*

$$D = \frac{D}{Dx} = \frac{\partial}{\partial x} + y_x \frac{\partial}{\partial y} + y_{xx} \frac{\partial}{\partial y_x} + \cdots + y_{(p+1)x} \frac{\partial}{\partial y_{px}}. \tag{7.30}$$

Throughout the literature on group theory, there is a tendency to shorten the name and call it merely the total-derivative operator, thus causing some confusion with the concept of a total differential. This is unfortunate in that, for more than one independent variable, (7.29) defines a *partial-derivative* operator. One could perhaps come up with a more appropriate name and call it, say, the complete partial-derivative operator, but this sounds like an oxymoron. Since current usage is so pervasive, there is probably no way to change it without causing added confusion, and so we will adopt the traditional nomenclature. The usual partial-derivative notation, $\partial\Psi/\partial x^j$ will imply differentiation with respect to the *explicit* dependence of $\Psi$ on $x^j$. Note that in fluid mechanics, $D(\ )/Dt$ is called the substantial derivative and has the physical interpretation of the change with time of some property of a fluid particle as it convects with a flow.

### *7.1.5 Definition of a Differential Function*

Extended Lie point transformations are closed in the space of variables and derivatives up to order $p$. That is, the function that transforms the $p$th derivative contains derivatives no higher than $p$. In later chapters, Lie–Bäcklund transformations are considered, where the transformation of a point can depend on derivatives up to some arbitrary, preselected order. Such transformations are not closed in any finite-dimensional space. To facilitate the development of the theory, we utilize the concept of a space of differential functions (see Ibragimov [7.1], [7.2]).

***Definition 7.2.*** *Let $z$ denote the infinite sequence of variables and derivatives,*

$$z = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots), \tag{7.31}$$

*and let $\langle z\rangle$ denote any finite subsequence of $z$. A* differential function $\Psi[\langle z\rangle]$ *is a locally analytic function of $\langle z\rangle$ (i.e., expandable in a Taylor series about some point $\langle z_0\rangle$). The space of differential functions is denoted by $\mathcal{A}$.*

Differential variables include independent variables $\boldsymbol{x}$, dependent variables $\boldsymbol{y}$, and the vectors of all possible partial derivatives, $\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \ldots$. Typical examples of differential functions are ODEs, PDEs, and the functions that transform derivatives under the action of Lie or Lie–Bäcklund groups.

The space $\mathcal{A}$ is the natural setting for the theory of symmetry analysis and facilitates the extension of the theory for the invariance of functions to include ODEs and PDEs. The concept of a differential function plays a central role in the theory of Lie–Bäcklund transformations discussed in Chapter 14 (see also Appendices 2 and 3).

### 7.1.6 Total Differentiation of Differential Functions

The total differentiation operator defined in (7.29) is straightforwardly extended to infinite order.

***Definition 7.3.*** *The total differentiation operator acting in the space of differential functions is the infinite operator*

$$D_j = \frac{\partial}{\partial x^j} + y^i_j \frac{\partial}{\partial y^i} + y^i_{j_1 j} \frac{\partial}{\partial y^i_{j_1}} + y^i_{j_1 j_2 j} \frac{\partial}{\partial y^i_{j_1 j_2}} + \cdots. \tag{7.32}$$

The operator (7.32) obviously truncates appropriately when applied to a differential function that depends on derivatives up to finite order.

The case of one independent variable and one dependent variable corresponds to ODEs of the form

$$\Psi[\langle z \rangle] = \Psi[x, y, y_x, y_{xx}, y_{xxx}, \ldots, y_{px}]. \tag{7.33}$$

The total differentiation operator acting on $\Psi$ truncates to (7.30):

$$D\Psi = \frac{\partial \Psi}{\partial x} + y_x \frac{\partial \Psi}{\partial y} + y_{xx} \frac{\partial \Psi}{\partial y_x} + y_{xxx} \frac{\partial \Psi}{\partial y_{xx}} + \cdots + y_{(p+1)x} \frac{\partial \Psi}{\partial y_{px}}. \tag{7.34}$$

The case of several variables corresponds to PDEs with $m$ dependent variables and $n$ independent variables treated as differential functions of the form

$$\Psi[\langle z \rangle] = \Psi[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \ldots, \boldsymbol{y}_p]. \tag{7.35}$$

In this case the total differentiation operator acting on $\Psi$ truncates to (7.29):

$$D_j \Psi = \frac{\partial \Psi}{\partial x^j} + y^i_j \frac{\partial \Psi}{\partial y^i} + y^i_{j_1 j} \frac{\partial \Psi}{\partial y^i_{j_1}} + \cdots + y^i_{j_1 \cdots j_p j} \frac{\partial \Psi}{\partial y^i_{j_1 \cdots j_p}}. \tag{7.36}$$

Total differentiation produces a function containing derivatives one order higher than the original differential function.

The operator $D_j$ has the usual properties of a linear differential operator. Let $\Psi[\langle z\rangle]$ and $\Phi[\langle z\rangle]$ be differential functions, and let $a$ and $b$ be constants. Then

$$\begin{aligned} D_j(a\Psi + b\Phi) &= aD_j\Psi + bD_j\Phi, \\ D_j(\Psi\Phi) &= \Psi D_j\Phi + \Phi D_j\Psi. \end{aligned} \tag{7.37}$$

Differential variables differentiate as follows:

$$\begin{aligned} D_j x^\alpha &= \delta_j^\alpha, \\ D_j y^\alpha &= y_j^\alpha, \\ D_{j_1} y_j^\alpha &= y_{j_1 j}^\alpha = D_{j_1} D_j y^\alpha, \\ D_{j_2} y_{j_1 j}^\alpha &= y_{j_1 j_2 j}^\alpha = D_{j_1} D_{j_2} D_j y^\alpha, \\ &\vdots \end{aligned} \tag{7.38}$$

where $\delta_j^\alpha$ is the Kronecker delta. The order of differentiation is immaterial:

$$D_{j_1}(D_{j_2}(\ )) = D_{j_2}(D_{j_1}(\ )). \tag{7.39}$$

## 7.2 Contact Conditions

Extended transformation groups are generated using the contact conditions.

### *7.2.1 One Dependent and One Independent Variable*

Transformations of first derivatives must satisfy the first-order contact condition

$$dy - y_x\, dx = 0. \tag{7.40}$$

Furthermore, this relationship holds in both the source space $(x, y)$ and the target space $(\tilde{x}, \tilde{y})$, so that

$$d\tilde{y} - \tilde{y}_{\tilde{x}}\, d\tilde{x} = 0. \tag{7.41}$$

See Appendix 2. In the parlance of group theory one would say that the transformation (7.12) leaves the differential function

$$\Psi[d\tilde{x}, d\tilde{y}, \tilde{y}_{\tilde{x}}] = d\tilde{y} - \tilde{y}_{\tilde{x}}\, d\tilde{x} \tag{7.42}$$

*invariant* (unchanged in form when expressed in variables without the tilde). This invariance is illustrated in Figure 7.1. A Lie group in two variables acts

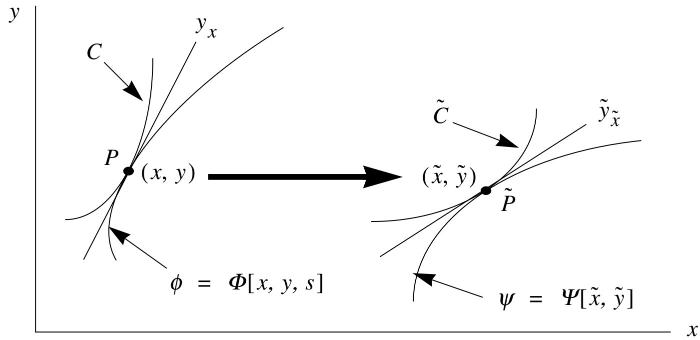


Fig. 7.1. Transformation of curves and tangent curves.

on curves $C$ and $\phi$, which are tangent to one another at a source point $P$. As each point is transformed from $P$ to $\tilde{P}$, the two curves, tangent at the source point $(x, y)$, are mapped to curves $\tilde{C}$ and $\psi$ tangent at the target point $(\tilde{x}, \tilde{y})$.

Lie groups preserve all tangency conditions in both the source and target space. For transformations of all derivatives up to order $p$:

$$\begin{aligned} dy - y_x\,dx &= d\tilde{y} - \tilde{y}_x\,d\tilde{x} = 0, \\ dy_x - y_{xx}\,dx &= d\tilde{y}_{\tilde{x}} - \tilde{y}_{\tilde{x}\tilde{x}}\,d\tilde{x} = 0, \\ &\vdots \\ dy_{(p-1)x} - y_{px}\,dx &= d\tilde{y}_{(p-1)\tilde{x}} - \tilde{y}_{p\tilde{x}}\,d\tilde{x} = 0. \end{aligned} \tag{7.43}$$

So, for example, if two surfaces are in 5th-order contact at some point $(x, y)$ in the source space, then invariance of the contact condition ensures that the mapped surfaces will be in 5th-order contact at the point $(\tilde{x}, \tilde{y})$ in the target space. The proof that the contact conditions are preserved to all orders under a Lie group is given briefly in Chapter 8, where the transformations of derivatives are worked out, and more fully in Appendix 2. In fact, Lie groups preserve tangency up to infinite order. This property, together with the parametric representation of the group, ensures that a Lie point group (7.12), extended to any order of derivative, constitutes a one-to-one, invertible map in the corresponding infinite-order tangent space.

### 7.2.2 Several Dependent and Independent Variables

The requirement that $p$th-order contact be preserved under the transformation from source to target variables holds when considering higher-dimensional systems. The contact conditions in this case are

$$
\begin{aligned}
dy^i - y^i_{j_1}\,dx^{j_1} &= d\tilde{y}^i - \tilde{y}^i_{j_1}\,d\tilde{x}^{j_1} = 0,\\
dy^i_{j_1} - y^i_{j_1 j_2}\,dx^{j_2} &= d\tilde{y}^i_{j_1} - \tilde{y}^i_{j_1 j_2}\,d\tilde{x}^{j_2} = 0,\\
&\vdots\\
dy^i_{j_1\cdots j_{p-1}} - y^i_{j_1\cdots j_p}\,dx^{j_p} &= d\tilde{y}^i_{j_1\cdots j_{p-1}} - \tilde{y}^i_{j_1\cdots j_p}\,d\tilde{x}^{j_p} = 0.
\end{aligned}
\tag{7.44}
$$

Writing the contact condition in terms of variables with the tilde and then replacing the differentials in terms of variables without the tilde leads directly to the transformations of partial derivatives (see Chapter 9, Section 9.1). There is no explicit requirement on derivatives beyond order $p$; however, as noted above, the attributes of a Lie group ensure that the transformations preserve tangency up to infinite order.

## 7.3 Concluding Remarks

The concept of a differential function is an invaluable tool that sets the stage for the next few chapters. In Chapter 8, extended groups in the plane are described and the methodology for generating transformations of derivatives is discussed. Groups in two variables are used to transform ODEs. The result is the Lie-series representation of an ODE, which leads directly to the invariance condition for the ODE. Extended groups involving several dependent and independent variables are discussed in Chapter 9 along with the invariance condition for PDEs. Chapters 10, 11, 12, and 13 are devoted to applications of point groups, mainly to problems in fluid mechanics. In Chapter 14 the concept of a differential function leads naturally to a generalization of Lie point groups to so-called Lie contact and Lie–Bäcklund groups, where the transformation of a point can depend on derivatives at the point. In certain cases such transformations can be used to discover new Lie–Bäcklund symmetries of differential equations that are not equivalent to Lie point symmetries. In the modern theory of symmetry analysis, Lie point and Lie contact transformations are regarded as special cases of Lie–Bäcklund transformations. In Chapter 15 Lie–Bäcklund transformations are used to transform integrals. This leads directly to Noether's theorem relating symmetries of an Euler–Lagrange system to conservation laws for the system. Finally, in Chapter 16 Bäcklund transformations are discussed. These remarkable transformations can be used to generate classes of exact solutions to important problems in nonlinear wave propagation. Bäcklund transformations are generally associated with an integrability condition for the equation in question. For this reason, they are usually regarded as many-valued, and therefore noninvertible, maps. But in the several cases described in Chapter 16, Bäcklund transformations are shown to arise from Lie groups. The complicating twist is that these groups are usually nonlocal in nature.

## 7.4 Exercises

7.1 Confirm the properties of the operator $D$ given in (7.38).

7.2 Carry out the indicated differentiation:

(i) $D_{xx}(y_{xxx} + yy_{xx})$

(ii) $D_x\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x\,\partial x}\right)$, $D_t\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x\,\partial x}\right)$

(iii) $D_{xx}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x\,\partial x}\right)$, $D_{xt}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x\,\partial x}\right)$,

$D_{tt}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x\,\partial x}\right)$

(iv) $D_{x^j}\left(\frac{\partial u^i}{\partial t} + u^k\frac{\partial u^i}{\partial x^k} + \frac{\partial p}{\partial x^i} - \nu\frac{\partial^2 u^i}{\partial x^k\,\partial x^k}\right)$

7.3 In Exercise 7.2(iv) take the trace of the result (i.e., set $i = j$ and sum) to generate the Poisson equation for the pressure $\nabla^2 p = -u^k_i u^i_k$ (note that $\partial u^i/\partial x^i = 0$). Subtract this from the equation generated in (iv) to form the transport equation for the velocity gradient tensor $D_t(u^i_j)$, where $u^i_j = \partial u^i/\partial x^j$.

### References

[7.1] Ibragimov, N. H. 1980. On the theory of Lie–Bäcklund transformation groups. *Math. USSR Sb.* **37**(2):205–226.

[7.2] Ibragimov, N. H. 1995. *CRC Handbook of Lie Group Analysis of Differential Equations, Volume 2*, Section 1.2. CRC Press.

# 8

## *Ordinary Differential Equations*

In Chapter 6 it was shown that knowledge of an invariant group leads immediately to the general solution of a first-order ODE. It was also pointed out that there is no systematic way of finding the group, although the commutator does provide a systematic procedure for carrying out a search. In this chapter the question will be turned around to ask: what is the general form of the ordinary differential equation that is invariant under a given group? But before this question can be addressed, it is necessary to develop the machinery for extending the infinitesimal group to include the transformation of derivatives. To accomplish this, the contact conditions discussed in the last chapter will be used. Finally we will consider the question of finding the groups that leave an ODE of second order or higher invariant and demonstrate how knowledge of the associated Lie algebra can be used to accomplish a reduction of order.

### 8.1 Extension of Lie Groups in the Plane

#### *8.1.1 Finite Transformation of First Derivatives*

First we consider the finite Lie point group in two dimensions,

$$T^s : \begin{Bmatrix} \tilde{x} = F[x, y, s] \\ \tilde{y} = G[x, y, s] \end{Bmatrix}. \tag{8.1}$$

We previewed this discussion in Chapter 1, where the transformation (8.1) was extended to include the first derivative $dy/dx = y_x$. The main requirement that must be satisfied by the extended transformation is that it inherit the properties of a Lie group, thus ensuring that the transformation is an invertible map in the tangent space $(x, y, y_x)$. This is accomplished by requiring that the transformation of the first derivative satisfy the first-order contact condition

$$dy - y_x \, dx = 0. \tag{8.2}$$

This condition holds in both the source coordinates $(x, y)$ and the target coordinates $(\tilde{x}, \tilde{y})$, so that

$$d\tilde{y} - \tilde{y}_{\tilde{x}} \, d\tilde{x} = 0. \tag{8.3}$$

In effect, the transformation of $y_x$ is required to satisfy the definition of the derivative. This may seem like a trivially obvious requirement at this stage, and it is. But later on, we will consider Lie–Bäcklund transformations where the mapping of points can depend on derivatives. In this case, the requirement that the tangent and higher derivatives at the source point be transformed to the tangent and higher derivatives at the target point will restrict, in a nontrivial way, the types of invertible transformations that are possible. More on this topic is discussed in Appendices 2 and 3.

The contact condition (8.3) provides a formula for generating the transformation of the first derivative. First the underlying space is prolonged (the number of variables is increased) from two variables, $(x, y)$, to four, $(x, y, dx, dy)$, by supplementing (8.1) with the transformation of differentials. Once the prolonged transformation has been generated, the contact condition is used to generate the transformation of $y_x$. Once this has been accomplished, the requirement that the contact condition be preserved under the extended group (extended to include transformations of derivatives) will be checked using the condition for invariance of a differential function of the form $\Psi[x, y, y_x, dy, dx]$.

First, we generate the prolonged transformation. Take the differential of each function in (8.1):

$$\begin{aligned} d\tilde{y} &= \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy, \\ d\tilde{x} &= \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy. \end{aligned} \tag{8.4}$$

The combined transformation (8.1) and (8.4) depends on four independent source variables $(x, y, dx, dy)$ mapped to four independent target variables $(\tilde{x}, \tilde{y}, d\tilde{x}, d\tilde{y})$. Only when the contact condition (8.3) is enforced is it assumed that $y$ is a function of $x$. Substitute (8.4) into (8.3), solve for $\tilde{y}_{\tilde{x}}$, and factor $dx$ out of the numerator and denominator. The finite transformation of the first derivative (cf. Chapter 1) is

$$\tilde{y}_{\tilde{x}} = \frac{G_x + G_y y_x}{F_x + F_y y_x} = (DG)(DF)^{-1}, \tag{8.5}$$

where we have used the total differentiation operator defined in Appendix 1. Since the transformation functions depend only on $x$ and $y$, the operator truncates to

$$D(\ ) = \frac{\partial}{\partial x}(\ ) + y_x \frac{\partial}{\partial y}(\ ). \tag{8.6}$$

Thus the once extended finite transformation group is

$$\begin{aligned} \tilde{x} &= F[x, y, s], \\ \tilde{y} &= G[x, y, s], \\ \tilde{y}_{\tilde{x}} &= G_{\{1\}}[x, y, y_x, s], \end{aligned} \tag{8.7}$$

where

$$G_{\{1\}}[x, y, y_x, s] = (DG)(DF)^{-1}. \tag{8.8}$$

As discussed in Chapter 7, functions which transform derivatives will be labelled by a subscript in curly braces. The braces indicate that the subscript is a function label and not a derivative.

### *8.1.2 The Extended Transformation Is a Group*

The extended transformation (8.7) is a Lie group. This is a very important point in that it assures that the transformation including derivatives is invertible. We can demonstrate the group property of (8.7) by composition. Consider the two transformations

$$\begin{aligned} \tilde{x} &= F[x, y, s], \\ \tilde{y} &= G[x, y, s], \\ \tilde{y}_{\tilde{x}} &= \frac{G_x[x, y, s] + G_y[x, y, s]y_x}{F_x[x, y, s] + F_y[x, y, s]y_x} \end{aligned} \tag{8.9}$$

and

$$\begin{aligned} \tilde{\tilde{x}} &= F[\tilde{x}, \tilde{y}, \tilde{s}], \\ \tilde{\tilde{y}} &= G[\tilde{x}, \tilde{y}, \tilde{s}], \\ \tilde{\tilde{y}}_{\tilde{\tilde{x}}} &= \frac{G_{\tilde{x}}[\tilde{x}, \tilde{y}, \tilde{s}] + G_{\tilde{y}}[\tilde{x}, \tilde{y}, \tilde{s}]\tilde{y}_{\tilde{x}}}{F_{\tilde{x}}[\tilde{x}, \tilde{y}, \tilde{s}] + F_{\tilde{y}}[\tilde{x}, \tilde{y}, \tilde{s}]\tilde{y}_{\tilde{x}}}. \end{aligned} \tag{8.10}$$

Composing the two transformations leads to

$$
\begin{aligned}
\tilde{\tilde{x}} &= F[\tilde{x}, \tilde{y}, \tilde{s}] = F[x, y, \phi[s, \tilde{s}]], \\
\tilde{\tilde{y}} &= G[\tilde{x}, \tilde{y}, \tilde{s}] = G[x, y, \phi[s, \tilde{s}]], \\
\tilde{\tilde{y}}_{\tilde{\tilde{x}}} &= \frac{G_{\tilde{x}}[\tilde{x}, \tilde{y}, \tilde{s}] + G_{\tilde{y}}[\tilde{x}, \tilde{y}, \tilde{s}]\tilde{y}_{\tilde{x}}}{F_{\tilde{x}}[\tilde{x}, \tilde{y}, \tilde{s}] + F_{\tilde{y}}[\tilde{x}, \tilde{y}, \tilde{s}]\tilde{y}_{\tilde{x}}} \\
&= \frac{G_{\tilde{x}}[\tilde{x}, \tilde{y}, \tilde{s}] + G_{\tilde{y}}[\tilde{x}, \tilde{y}, \tilde{s}]\left(\frac{G_x[x,y,s]+G_y[x,y,s]y_x}{F_x[x,y,s]+F_y[x,y,s]y_x}\right)}{F_{\tilde{x}}[\tilde{x}, \tilde{y}, \tilde{s}] + F_{\tilde{y}}[\tilde{x}, \tilde{y}, \tilde{s}]\left(\frac{G_x[x,y,s]+G_y[x,y,s]y_x}{F_x[x,y,s]+F_y[x,y,s]y_x}\right)}.
\end{aligned}
\tag{8.11}
$$

The last relation in (8.11) is rearranged to read

$$
\tilde{\tilde{y}}_{\tilde{\tilde{x}}} = \frac{\begin{array}{l}(G_{\tilde{x}}[\tilde{x}, \tilde{y}, \tilde{s}]F_x[x, y, s] + G_{\tilde{y}}[\tilde{x}, \tilde{y}, \tilde{s}]G_x[x, y, s]) \\ \quad + (G_{\tilde{x}}[\tilde{x}, \tilde{y}, \tilde{s}]F_y[x, y, s] + G_{\tilde{y}}[\tilde{x}, \tilde{y}, \tilde{s}]G_y[x, y, s])y_x\end{array}}{\begin{array}{l}(F_{\tilde{x}}[\tilde{x}, \tilde{y}, \tilde{s}]F_x[x, y, s] + F_{\tilde{y}}[\tilde{x}, \tilde{y}, \tilde{s}]G_x[x, y, s]) \\ \quad + (F_{\tilde{x}}[\tilde{x}, \tilde{y}, \tilde{s}]F_y[x, y, s] + F_{\tilde{y}}[\tilde{x}, \tilde{y}, \tilde{s}]G_y[x, y, s])y_x\end{array}}. \tag{8.12}
$$

Differentiating the first and second relations in (8.11) gives the following:

$$
\begin{aligned}
G_x[x, y, \phi[s, \tilde{s}]] &= G_{\tilde{x}}[\tilde{x}, \tilde{y}, \tilde{s}]F_x[x, y, s] + G_{\tilde{y}}[\tilde{x}, \tilde{y}, \tilde{s}]G_x[x, y, s], \\
G_y[x, y, \phi[s, \tilde{s}]] &= G_{\tilde{x}}[\tilde{x}, \tilde{y}, \tilde{s}]F_y[x, y, s] + G_{\tilde{y}}[\tilde{x}, \tilde{y}, \tilde{s}]G_y[x, y, s], \\
F_x[x, y, \phi[s, \tilde{s}]] &= F_{\tilde{x}}[\tilde{x}, \tilde{y}, \tilde{s}]F_x[x, y, s] + F_{\tilde{y}}[\tilde{x}, \tilde{y}, \tilde{s}]G_x[x, y, s], \\
F_y[x, y, \phi[s, \tilde{s}]] &= F_{\tilde{x}}[\tilde{x}, \tilde{y}, \tilde{s}]F_y[x, y, s] + F_{\tilde{y}}[\tilde{x}, \tilde{y}, \tilde{s}]G_y[x, y, s].
\end{aligned}
\tag{8.13}
$$

Comparing (8.13) and the expressions in parentheses in (8.12) shows that the composed transformation is

$$
\begin{aligned}
\tilde{\tilde{x}} &= F[x, y, \phi[s, \tilde{s}]], \\
\tilde{\tilde{y}} &= G[x, y, \phi[s, \tilde{s}]], \\
\tilde{\tilde{y}}_{\tilde{\tilde{x}}} &= \frac{G_x[x, y, \phi[s, \tilde{s}]] + G_y[x, y, \phi[s, \tilde{s}]]y_x}{F_x[x, y, \phi[s, \tilde{s}]] + F_y[x, y, \phi[s, \tilde{s}]]y_s},
\end{aligned}
\tag{8.14}
$$

which is in exactly the same form as the original transformation (8.9). The extended transformation (8.14) is a Lie group, and the function $\phi[s, \tilde{s}]$ defines the rule of composition of the group.

### 8.1.3 Finite Transformation of the Second Derivative

The once extended transformation (8.7) satisfies the second-order contact condition

$$d\tilde{y}_{\tilde{x}} - \tilde{y}_{\tilde{x}\tilde{x}}\, d\tilde{x} = 0. \tag{8.15}$$

The transformation of the second derivative is derived by taking the differentials indicated in (8.15):

$$\tilde{y}_{\tilde{x}\tilde{x}} = \frac{d\tilde{y}_{\tilde{x}}}{d\tilde{x}} = \frac{G_{\{1\}x}\, dx + G_{\{1\}y}\, dy + G_{\{1\}y_x}\, dy_x}{F_x\, dx + F_y\, dy}. \tag{8.16}$$

Factoring $dx$ out of the numerator and denominator of (8.16), the transformation of the second derivative becomes

$$\tilde{y}_{\tilde{x}\tilde{x}} = \frac{G_{\{1\}x} + y_x G_{\{1\}y} + y_{xx} G_{\{1\}y_x}}{F_x + y_x F_y} = DG_{\{1\}}(DF)^{-1} \tag{8.17}$$

The twice extended finite transformation is

$$\begin{aligned} \tilde{x} &= F[x, y, s], \\ \tilde{y} &= G[x, y, s], \\ \tilde{y}_{\tilde{x}} &= G_{\{1\}}[x, y, y_x, s], \\ \tilde{y}_{\tilde{x}\tilde{x}} &= G_{\{2\}}[x, y, y_x, y_{xx}, s], \end{aligned} \tag{8.18}$$

where

$$G_{\{2\}}[x, y, y_x, y_{xx}, s] = DG_{\{1\}}(DF)^{-1}. \tag{8.19}$$

The same procedure used above to prove that the once extended transformation is a Lie group can be applied to (8.18). For higher derivatives, the procedure for proving that the extended transformation is a Lie group is the same as the one just demonstrated, and by induction, the Lie point group (8.1) extended to any order is a group.

### 8.1.4 Finite Transformation of Higher Derivatives

The $(p-1)$th-order extended group is of the form

$$\begin{aligned} \tilde{x} &= F[x, y, s], \\ \tilde{y} &= G[x, y, s], \\ \tilde{y}_{\tilde{x}} &= G_{\{1\}}[x, y, y_x, s], \\ \tilde{y}_{\tilde{x}\tilde{x}} &= G_{\{2\}}[x, y, y_x, y_{xx}, s], \\ &\vdots \\ \tilde{y}_{(p-1)\tilde{x}} &= G_{\{p-1\}}[x, y, y_x, y_{xx}, \ldots, y_{(p-1)x}, s]. \end{aligned} \tag{8.20}$$

The transformation (8.20) is assumed to satisfy the $p$th-order contact condition given by

$$d\left(\tilde{y}_{(p-1)\tilde{x}}\right) - \tilde{y}_{p\tilde{x}}\, d\tilde{x} = 0. \tag{8.21}$$

The transformation of the $p$th-order derivative is determined in the usual way by taking differentials of the $(p-1)$th-order extended transformation

$$\frac{d\left(\tilde{y}_{(p-1)\tilde{x}}\right)}{d\tilde{x}} = \frac{\frac{\partial G_{\{p-1\}}}{\partial x}\, dx + \frac{\partial G_{\{p-1\}}}{\partial y}\, dy + \cdots + \frac{\partial G_{\{p-1\}}}{\partial (y_{(p-1)x})}\, d\left(y_{(p-1)_x}\right)}{\frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy}. \tag{8.22}$$

Factoring $dx$ out of the numerator and denominator of (8.22) and using the subscript notation for derivatives yields

$$\begin{aligned} \tilde{y}_{p\tilde{x}} &= \frac{G_{\{p-1\}x} + y_x G_{\{p-1\}y} + \cdots + y_{px} G_{\{p-1\}y_{(p-1)x}}}{F_x + y_x F_y} \\ &= DG_{\{p-1\}}(DF)^{-1}. \end{aligned} \tag{8.23}$$

The $p$th extended group is

$$\begin{aligned} \tilde{x} &= F[x, y, s], \\ \tilde{y} &= G[x, y, s], \\ \tilde{y}_{\tilde{x}} &= G_{\{1\}}[x, y, y_x, s], \\ \tilde{y}_{\tilde{x}\tilde{x}} &= G_{\{2\}}[x, y, y_x, y_{xx}, s], \\ &\vdots \\ \tilde{y}_{p\tilde{x}} &= G_{\{p\}}[x, y, y_x, y_{xx}, \ldots, y_{px}, s], \end{aligned} \tag{8.24}$$

where

$$G_{\{p\}}[x, y, y_x, y_{xx}, \ldots, y_{px}, s] = DG_{\{p-1\}}(DF)^{-1}. \tag{8.25}$$

The procedure used to derive (8.14) can be repeated as the derivative order increases and by induction, the following theorem can be stated.

***Theorem 8.1.*** *The pth-order extended transformation (8.24) generated using the contact conditions (8.21) is a Lie group. The same holds in the case of groups involving several dependent and independent variables where the extensions involve several partial derivatives.*

This key statement is the theoretical justification for the entire range of applications of extended groups to differential equations.

### 8.1.5 Infinitesimal Transformation of the First Derivative

The infinitesimal form of (8.1) is

$$\begin{aligned}\tilde{x} &= x + s\xi[x, y],\\ \tilde{y} &= y + s\eta[x, y],\end{aligned} \tag{8.26}$$

where

$$\boxed{\xi[x, y] = \left.\frac{\partial F}{\partial s}\right|_{s=0}, \qquad \eta[x, y] = \left.\frac{\partial G}{\partial s}\right|_{s=0},} \tag{8.27}$$

and $s$ is assumed to be small. Substitute $G = y + s\eta$ and $F = x + s\xi$ into (8.5), and carry out the indicated differentiation. The result is

$$\frac{d\tilde{y}}{d\tilde{x}} = \frac{\frac{dy}{dx} + s\left(\eta_x + \eta_y \frac{dy}{dx}\right)}{1 + s\left(\xi_x + \xi_y \frac{dy}{dx}\right)} = \frac{\frac{dy}{dx} + s(D\eta)}{1 + s(D\xi)}. \tag{8.28}$$

Expanding the denominator of the right-hand side of (8.28) in a binomial series for small $s$ and retaining only lowest-order terms in $s$ produces the infinitesimal transformation of the first derivative,

$$\tilde{y}_{\tilde{x}} = y_x + s(D\eta - y_x D\xi). \tag{8.29}$$

The once extended infinitesimal group in the plane is

$$\begin{aligned}\tilde{x} &= x + s\xi[x, y],\\ \tilde{y} &= y + s\eta[x, y],\\ \tilde{y}_{\tilde{x}} &= y_x + s\eta_{\{1\}}[x, y, y_x],\end{aligned} \tag{8.30}$$

where

$$\eta_{\{1\}}[x, y, y_x] = D\eta - y_x D\xi = \eta_x + (\eta_y - \xi_x)y_x - \xi_y (y_x)^2. \tag{8.31}$$

Note the *quadratic* dependence of the infinitesimal (8.31) on $y_x$.

### 8.1.6 Infinitesimal Transformation of the Second Derivative

The infinitesimal form of the twice extended group is derived using the same approach. Substitute $G_{\{1\}} = y_x + s\eta_{\{1\}}$ and $F = x + s\xi$ into (8.19). The result is

$$\tilde{y}_{\tilde{x}\tilde{x}} = \frac{y_{xx} + s\left(D\eta_{\{1\}}\right)}{1 + s(D\xi)}. \tag{8.32}$$

Expanding the denominator of (8.32) in a binomial series and retaining lowest-order terms in the group parameter $s$ gives the infinitesimal transformation of the second derivative,

$$\tilde{y}_{\tilde{x}\tilde{x}} = y_{xx} + s\left(D\eta_{\{1\}} - y_{xx} D\xi\right). \tag{8.33}$$

The twice extended infinitesimal group is

$$\begin{aligned} \tilde{x} &= x + s\xi[x, y], \\ \tilde{y} &= y + s\eta[x, y], \\ \tilde{y}_{\tilde{x}} &= y_x + s\eta_{\{1\}}[x, y, y_x], \\ \tilde{y}_{\tilde{x}\tilde{x}} &= y_{xx} + s\eta_{\{2\}}[x, y, y_x, y_{xx}], \end{aligned} \tag{8.34}$$

where

$$\eta_{\{2\}}[x, y, y_x, y_{xx}] = D\eta_{\{1\}} - y_{xx} D\xi \tag{8.35}$$

and

$$D\eta_{\{1\}} = \frac{\partial \eta_{\{1\}}}{\partial x} + y_x \frac{\partial \eta_{\{1\}}}{\partial y} + y_{xx} \frac{\partial \eta_{\{1\}}}{\partial y_x}, \qquad D\xi = \frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial y} y_x. \tag{8.36}$$

Written out, the infinitesimal transformation for the second derivative is

$$\begin{aligned} \eta_{\{2\}} = {} & \eta_{xx} + (2\eta_{xy} - \xi_{xx}) y_x + (\eta_{yy} - 2\xi_{xy}) y_x^2 \\ & - \xi_{yy} y_x^3 + (\eta_y - 2\xi_x) y_{xx} - 3\xi_y y_x y_{xx}. \end{aligned} \tag{8.37}$$

Note that $\eta_{\{2\}}$ is *linear* in $y_{xx}$. The formula for the infinitesimal transformation of third derivatives is

$$\begin{aligned} \eta_{\{3\}} = {} & \eta_{xxx} + (3\eta_{xxy} - \xi_{xxx}) y_x + (3\eta_{xyy} - 3\xi_{xxy}) y_x^2 \\ & + (\eta_{yyy} - 3\xi_{xyy}) y_x^3 - \xi_{yyy} y_x^4 + (3\eta_{xy} - 3\xi_{xx}) y_{xx} \\ & + (3\eta_{yy} - 9\xi_{xy}) y_x y_{xx} - (6\xi_{yy}) y_x^2 y_{xx} - (3\xi_y) y_{xx}^2 \\ & + (\eta_y - 3\xi_x) y_{xxx} - (4\xi_y) y_x y_{xxx}. \end{aligned} \tag{8.38}$$

Similarly, $\eta_{\{3\}}$ is linear in $y_{xxx}$, and so on. The linearity of higher-order infinitesimals means that when it comes to solving for the invariants of an extended group, the third and higher invariants will merely involve the solution of a linear first-order ODE at each stage. This point will be clarified shortly.

### *8.1.7 Infinitesimal Transformation of Higher-Order Derivatives*

Extensions of the infinitesimal group $(\xi, \eta)$ to higher order are generated by the same expansion procedure. Substitute $G_{\{p-1\}} = y_{(p-1)x} + s\eta_{\{p-1\}}$ and $F = x + s\xi$ into (8.25). The result is

$$\tilde{y}_{\tilde{p}\tilde{x}} = \frac{y_{px} + sD\eta_{\{p-1\}}}{1 + sD\xi}. \tag{8.39}$$

Expand the denominator retaining only lowest-order terms in $s$. The $p$th-order infinitesimal transformation is

$$\tilde{y}_{\tilde{p}\tilde{x}} = y_{px} + s\big(D\eta_{\{p-1\}} - y_{px}D\xi\big), \tag{8.40}$$

and the $p$th-order extended infinitesimal group is

$$\begin{aligned}
\tilde{x} &= x + s\xi[x, y],\\
\tilde{y} &= y + s\eta[x, y],\\
\tilde{y}_{\tilde{x}} &= y_x + s\eta_{\{1\}}[x, y, y_x],\\
\tilde{y}_{\tilde{x}\tilde{x}} &= y_{xx} + s\eta_{\{2\}}[x, y, y_x, y_{xx}],\\
&\vdots\\
\tilde{y}_{(p-1)\tilde{x}} &= y_{(p-1)x} + s\eta_{\{p-1\}}\big[x, y, y_x, y_{xx}, \ldots, y_{(p-1)x}\big],\\
\tilde{y}_{p\tilde{x}} &= y_{px} + s\eta_{\{p\}}[x, y, y_x, y_{xx}, \ldots, y_{px}],
\end{aligned} \tag{8.41}$$

where

$$\boxed{\eta_{\{p\}}[x, y, y_x, y_{xx}, \ldots, y_{px}] = D\eta_{\{p-1\}} - y_{px}D\xi.} \tag{8.42}$$

The total differentiation operator acting on the first term is

$$D\eta_{\{p-1\}} = \frac{\partial\eta_{\{p-1\}}}{\partial x} + y_x\frac{\partial\eta_{\{p-1\}}}{\partial y} + y_{xx}\frac{\partial\eta_{\{p-1\}}}{\partial y_x} + \cdots + y_{px}\frac{\partial\eta_{\{p-1\}}}{\partial y_{(p-1)x}}. \tag{8.43}$$

### 8.1.8 Invariance of the Contact Conditions

The demonstration of invariance of the contact conditions is rather lengthy, and so the main results are presented here, with full details left to Appendix 2. The once extended group (8.7) or (8.30) leaves invariant the contact condition

$$d\tilde{y} - \tilde{y}_{\tilde{x}}\, d\tilde{x} = 0. \tag{8.44}$$

To show this, we consider the prolongation of the infinitesimal form of the once extended group:

$$\begin{aligned}
\tilde{x} &= x + \xi[x, y]s, \\
\tilde{y} &= y + \eta[x, y]s, \\
\tilde{y}_{\tilde{x}} &= y_x + \eta_{\{1\}}[x, y, y_x]s, \\
d\tilde{x} &= dx + (\xi_x\, dx + \xi_y\, dy)s = dx + (d\xi)s, \\
d\tilde{y} &= dy + (\eta_x\, dx + \eta_y\, dy)s = dy + (d\eta)s.
\end{aligned} \tag{8.45}$$

Note that the transformation of $dy_x$ is not required, since it does not appear in the first-order contact condition (8.44). The group operator corresponding to the prolonged group (8.45) is

$$\hat{X}_{\{1\}} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \eta_{\{1\}}\frac{\partial}{\partial y_x} + (d\xi)\frac{\partial}{\partial(dx)} + (d\eta)\frac{\partial}{\partial(dy)}. \tag{8.46}$$

where the ˆ over the $X$ is used to denote a group operator prolonged to include the differentials $d\xi$ and $d\eta$.† We need to show that the contact condition (8.44) is invariant under the group (8.45). Apply the operator (8.46) to (8.44). The result is

$$\hat{X}_{\{1\}}(dy - y_x\, dx) = d\eta - \eta_{\{1\}}\, dx - y_x\, d\xi. \tag{8.47}$$

Writing out the differentials in (8.47) in full and gathering terms (see Appendix 2) leads to

$$\hat{X}_{\{1\}}(dy - y_x\, dx) = (\eta_y - \xi_y y_x)(dy - y_x\, dx) = 0. \tag{8.48}$$

The contact condition (8.44) is invariant under the prolonged group, and we can write with confidence

$$d\tilde{y} - \tilde{y}_{\{\tilde{x}\}}\, d\tilde{x} = dy - y_{\{x\}}\, dx. \tag{8.49}$$

† Throughout the literature on group theory, the phrases "extended group" and " prolonged group" are normally used interchangably to refer to transformations extended to include derivatives such as (8.41) and (8.42). In this text I will make a slight distinction and use "prolonged group" to refer to transformations such as (8.45) that are supplemented by transformations of differentials.

In other words, if (8.44), considered as a function of $x, y, y_x, dx, dy$, were to be expanded in a Lie series in the operator (8.46), the series would truncate to (8.49). This confirms that, when a curve is transformed from the source space $(x, y)$ to the target space $(\tilde{x}, \tilde{y})$, the tangent to a point in $(x, y)$ will be transformed to the tangent of the mapped point in $(\tilde{x}, \tilde{y})$ as illustrated in Figure 7.1. The transformation of points and first derivatives is a one-to-one invertible map. This is a direct consequence of the group property of the extended transformation. Similarly, all higher-order contact conditions are invariant under the extended group, prolonged to include the required differentials.

***Theorem 8.2.*** *The pth-order contact condition $d(y_{(p-1)x}) - y_{px}\,dx = 0$ is preserved by the transformation group (8.1). So to all orders of derivatives,*

$$\begin{aligned} d(\tilde{y}) - \tilde{y}_{\tilde{x}}\,d\tilde{x} &= d(y) - y_x\,dx, \\ d(\tilde{y}_{\tilde{x}}) - \tilde{y}_{\tilde{x}\tilde{x}}\,d\tilde{x} &= d(y_x) - y_{xx}\,dx, \\ &\vdots \\ d\big(\tilde{y}_{(p-1)\tilde{x}}\big) - \tilde{y}_{p\tilde{x}}\,d\tilde{x} &= d\big(y_{(p-1)x}\big) - y_{px}\,dx. \end{aligned} \tag{8.50}$$

The proof of this theorem is provided in Appendix 2.

The reader may wonder why we make such an issue of the contact conditions at this stage, when they seem self-evident. It is in preparation for our later consideration of Lie–Bäcklund transformations, where the contact conditions play a nontrivial role in determining the infinite-order character of such transformations. A much more detailed account of this point can be found in Appendix 3.

## 8.2 Expansion of an ODE in a Lie Series – The Invariance Condition for ODEs

Now that it is understood that a differential equation can be treated as a locally analytic function of variables and derivatives, the invariance condition for an ODE can be defined in exactly the same way that we defined the invariance condition of a function in Chapter 5, Section 5.6. The whole mathematical apparatus – Lie series, invariance condition, and so on – developed for functions carries over intact to the treatment of differential functions (ODEs and PDEs).

***Theorem 8.3.*** *The pth-order ODE $\psi = \Psi[x, y, y_x, y_{xx}, \ldots, y_{px}] = 0$, treated as a differential function of the variables $x, y, y_x, y_{xx}, \ldots, y_{px}$, can be* *expanded in a Lie series as*

$$\Psi[\tilde{x}, \tilde{y}, \tilde{y}_{\tilde{x}}, \tilde{y}_{\tilde{x}\tilde{x}}, \ldots, \tilde{y}_{p\tilde{x}}]$$
$$= \Psi[x, y, y_x, y_{xx}, \ldots, y_{px}] + sX_{\{p\}}\Psi + \frac{s^2}{2!}X_{\{p\}}\big(X_{\{p\}}\Psi\big) + \cdots, \quad (8.51)$$

*where $X_{\{p\}}$ is the operator of the $p$ times extended group,*

$$X_{\{p\}} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \eta_{\{1\}}\frac{\partial}{\partial y_x} + \eta_{\{2\}}\frac{\partial}{\partial y_{xx}} + \cdots + \eta_{\{p\}}\frac{\partial}{\partial y_{px}}. \quad (8.52)$$

*The ODE is invariant if and only if*

$$\boxed{X_{\{p\}}\Psi[x, y, y_x, y_{xx}, \ldots, y_{px}] = 0.} \quad (8.53)$$

The characteristic equations asssociated with (8.53) are

$$\boxed{\frac{dx}{\xi[x, y]} = \frac{dy}{\eta[x, y]} = \frac{dy_x}{\eta_{\{1\}}[x, y, y_x]} = \frac{dy_{xx}}{\eta_{\{2\}}[x, y, y_x, y_{xx}]} = \cdots = \frac{dy_{px}}{\eta_{\{p\}}}} \quad (8.54)$$

with $p + 1$ integral invariants.

### *8.2.1 What Does It Take to Transform a Derivative?*

In order to apply the invariance condition (8.53), it is necessary to generate the infinitesimal transformation functions $\eta_{\{1\}}, \eta_{\{2\}}, \ldots, \eta_{\{p\}}$ for the various derivatives that appear in the differential equation in question. For an equation containing high-order derivatives this is not an attractive prospect. Figure 8.1

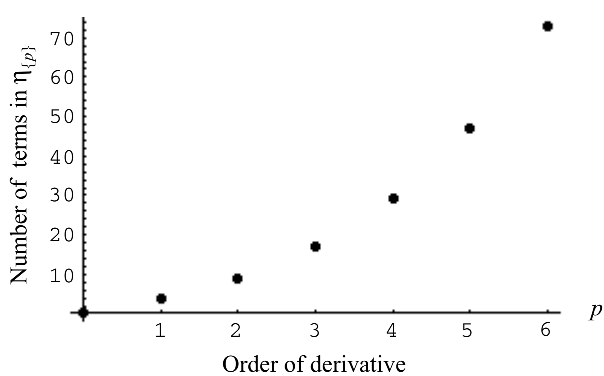


Fig. 8.1. Number of terms in $\eta_{\{p\}}$ versus $p$ for the case of one dependent variable and one independent variable.

illustrates why. The number of terms in $\eta_{\{p\}}$ grows rapidly as the order increases, due to the nested differentiation in the first term on the right-hand, side of (8.42). Additional variables increase the number of terms even more rapidly. The software package **IntroToSymmetry.m** provided with the text automates the generation of these transformations.

## 8.3 Group Analysis of Ordinary Differential Equations

In Chapter 6 it was shown that the knowledge of an invariant group leads immediately to an integrating factor and the general solution of a first-order ODE. It was also pointed out that there is no systematic way of finding the group for a given first-order ODE. In the first part of this chapter, the procedure for generating the transformations of derivatives was developed, and it was shown that the contact conditions are preserved to infinite order by the extended group. The condition for invariance of an ODE was then derived by treating it as a differential function and expanding the ODE in a Lie series using the extended group.

Now we are prepared to ask: what is the general form of the ODE that is invariant under a given group with known infinitesimals $(\xi, \eta)$? For first-order equations this consists in studying the equation in the three dimensional space of differential variables, $(x, y, y_x)$. The ideas presented then generalize easily to second- and higher-order equations by considering higher-order ODEs as differential functions in higher dimensions: $(x, y, y_x, y_{xx})$, $(x, y, y_x, y_{xx}, y_{xxx})$, etc.

## 8.4 Failure to Solve for the Infinitesimals That Leave a First-Order ODE Invariant

In Chapter 3 we discussed the difficulty of finding the integrating factor that will turn a first-order equation into an exact differential. In Chapter 6 we gave this a group interpretation by stating that there is no systematic way to find the group that leaves a given first-order ODE invariant. These results are essentially equivalent, and it is instructive to examine the group approach.

The invariance condition for a first-order ODE $\psi = \Psi[x, y, y_x]$ is

$$X_{\{1\}}\Psi = \xi \frac{\partial \Psi}{\partial x} + \eta \frac{\partial \Psi}{\partial y} + \eta_{\{1\}} \frac{\partial \Psi}{\partial y_x} = 0 \tag{8.55}$$

with characteristic equations

$$\frac{dx}{\xi[x, y]} = \frac{dy}{\eta[x, y]} = \frac{dy_x}{\eta_{\{1\}}[x, y, y_x]}. \tag{8.56}$$

Without loss of generality, let the equation be of the form

$$\Psi[x, y, y_x] = y_x - f[x, y] = 0. \tag{8.57}$$

Differentiating (8.57), substituting the expression for $\eta_{\{1\}}$ into (8.55), and replacing $y_x$ with $f[x, y]$ leads to

$$\eta_x + (\eta_y - \xi_x)f - \xi_y f^2 - \xi\frac{\partial f}{\partial x} - \eta\frac{\partial f}{\partial y} = 0. \tag{8.58}$$

One solution of (8.58) is $f[x, y] = \eta[x, y]/\xi[x, y]$, and so for any $\xi[x, y]$, without loss of generality, we can let

$$\eta[x, y] = \xi[x, y]f[x, y] + g[x, y]. \tag{8.59}$$

Upon substitution of (8.59) into (8.58) and canceling terms, one finds that $g[x, y]$ is the solution of

$$\frac{1}{\partial f/\partial y}\frac{\partial g}{\partial x} + \frac{f[x, y]}{\partial f/\partial y}\frac{\partial g}{\partial y} = g. \tag{8.60}$$

According to the method of Lagrange developed in Chapter 3 Section 3.6, the solution of (8.60) coincides with the characteristics of

$$\frac{\partial f}{\partial y}\,dx = \left(\frac{\partial f}{\partial y}\right)\frac{1}{f[x, y]}\,dy = \frac{dg}{g}. \tag{8.61}$$

But in order to complete the solution for the infinitesimals we have to solve the first equality in (8.61), which is the original problem we set out to solve in the first place. This approach has failed to provide any information about the infinitesimals $(\xi, \eta)$ that will leave a given first-order ODE invariant. In the next section we consider the reverse problem.

## 8.5 Construction of the General First-Order ODE That Admits a Given Group – the Ricatti Equation

Let the integral of the first equality in (8.56) be $\psi^1 = \Psi^1[x, y]$. In general this is a hard problem, since it requires the solution of a first-order nonlinear ODE whose invariant group is not known. However, once $\Psi^1$ has been determined, the second integral of (8.56) can always be determined, as we shall now demonstrate. For the second integral, say $\psi^2 = \Psi^2[x, y, y_x]$, we need to solve

$$\frac{dy_x}{dx} = \frac{\eta_{\{1\}}[x, y, y_x]}{\xi[x, y]} = \frac{\eta_x}{\xi} + \left(\frac{\eta_y}{\xi} - \frac{\xi_x}{\xi}\right)y_x - \frac{\xi_y}{\xi}y_x^2. \tag{8.62}$$

Using $\psi^1 = \Psi^1[x, y]$ to eliminate $y$ in Equation (8.62) leads to a Ricatti equation for $y_x$ of the form

$$\frac{dy_x}{dx} = A[x; \psi^1] + B[x; \psi^1]y_x + C[x; \psi^1](y_x)^2, \tag{8.63}$$

where

$$A[x; \psi^1] = \frac{\eta_x[x, y[x; \psi^1]]}{\xi[x, y[x; \psi^1]]}, \tag{8.64}$$

$$B[x; \psi^1] = \frac{\eta_y[x, y[x; \psi^1]]}{\xi[x, y[x; \psi^1]]} - \frac{\xi_x[x, y[x; \psi^1]]}{\xi[x, y[x; \psi^1]]}, \tag{8.65}$$

$$C[x; \psi^1] = -\frac{\xi_y[x, y[x; \psi^1]]}{\xi[x, y[x; \psi^1]]}. \tag{8.66}$$

The general solution of Equation (8.63) can always be determined if a particular solution can be found. In this case the function

$$f[x; \psi^1] = \frac{\eta[x, y[x; \psi^1]]}{\xi[x, y[x; \psi^1]]} \tag{8.67}$$

satisfies (8.63). This can be seen as follows. Differentiate (8.67):

$$\xi\, df + \xi_x f\, dx + \xi_y f\, dy - \eta_x\, dx - \eta_y\, dy = 0. \tag{8.68}$$

Equation (8.68) can be rearranged to read the same as (8.62):

$$\frac{df}{dx} = \frac{\eta_x}{\xi} + \left(\frac{\eta_y}{\xi} - \frac{\xi_x}{\xi}\right) f - \frac{\xi_y}{\xi} f^2. \tag{8.69}$$

Therefore (8.67) is a solution of (8.62). Now let the complete solution of (8.63) be written as

$$y_x = f[x; \psi^1] + \frac{1}{h[x]}. \tag{8.70}$$

Substitute (8.70) into (8.63). The result is

$$\frac{d(y_x)}{dx} = \frac{df}{dx} - \frac{1}{h^2}\frac{dh}{dx} = A[x; \psi^1] + B[x; \psi^1]\left(f + \frac{1}{h}\right) + C[x; \psi^1]\left(f + \frac{1}{h}\right)^2. \tag{8.71}$$

If (8.69) is subtracted from (8.71), the result is a linear equation for $h(x)$,

$$\frac{dh}{dx} = -(B[x;\psi^1] + 2f[x;\psi^1]C[x;\psi^1])h - C[x;\psi^1], \tag{8.72}$$

which can be solved using the integrating factor derived in Chapter 6, Example 6.5:

$$h[x;\psi^1] = \int_x \left\{ C \exp\left[ \int_{x'} (B+2fC)\, dx'' \right] \right\} dx' - y \exp\left[ \int_x (B+2fC)\, dx' \right]. \tag{8.73}$$

Combining (8.73), (8.70), and (8.67) and replacing $\psi^2$ by its expression in terms of $x$ and $y$, the second integral of (8.56) is finally:

$$\psi^2 = \Psi^2[x, y, y_x] = y_x - \frac{\eta[x, y]}{\xi[x, y]} - \frac{1}{h[x; \Psi^1[x, y]]}. \tag{8.74}$$

The main point here is that once the first integral of (8.56) has been determined, the second integral is determined systematically. Since both integrals are invariant under the same group, the most general first-order ODE invariant under the group $(\xi, \eta)$ is

$$\omega = \Omega[\Psi^1[x, y], \Psi^2[x, y, y_x]], \tag{8.75}$$

where $\Omega$ is an arbitrary function.

***Example 8.1 (General first-order ODE invariant under the Lorentz group).*** The Lorentz group is commonly expressed in terms of hyperbolic functions:

$$T^{\text{Lorentz}} : \left\{ \begin{array}{l} \tilde{x} = x \cosh[s] + y \sinh[s] \\ \tilde{y} = x \sinh[s] + y \cosh[s] \end{array} \right\}. \tag{8.76}$$

The infinitesimals and first extension of the group are

$$\xi = y, \qquad \eta = x, \qquad \eta_{\{1\}} = 1 - y_x^2. \tag{8.77}$$

The once extended invariance condition is

$$X_{\{1\}}\Psi = y\frac{\partial \Psi}{\partial x} + x\frac{\partial \Psi}{\partial y} + \left(1 - y_x^2\right)\frac{\partial \Psi}{\partial (y_x)} = 0 \tag{8.78}$$

with characteristic equations

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dy_x}{1 - y_x^2} \tag{8.79}$$

and first integral

$$\psi^1 = \Psi^1[x, y] = x^2 - y^2. \tag{8.80}$$

The second integral is found by solving

$$\frac{dy}{\pm\sqrt{\psi^1 + y^2}} = \frac{dy_x}{1 - y_x^2}, \tag{8.81}$$

which can be integrated to give

$$\sinh^{-1}[y/\psi^1] = \tanh^{-1}[y_x] + \tanh^{-1}[\psi^2], \tag{8.82}$$

where the constant of integration, $\psi^2$, has been put inside the inverse tanh function for convenience. Using some identities for the hyperbolic functions, Equation (8.82) can be rearranged to read

$$\psi^2 = \Psi^2[x, y, y_x] = \frac{y_x - \frac{y}{x}}{\frac{y}{x}y_x - 1}. \tag{8.83}$$

The most general first-order ODE invariant under the Lorentz group is therefore

$$\omega = \Omega[\psi^1, \psi^1] = \Omega\left[x^2 - y^2, \frac{y_x - \frac{y}{x}}{\frac{y}{x}y_x - 1}\right], \tag{8.84}$$

where $\Omega$ is an arbitrary function. An equally general form is simply

$$y_x = \frac{\frac{y}{x} - \Phi[x^2 - y^2]}{1 - \frac{y}{x}\Phi[x^2 - y^2]}, \tag{8.85}$$

where $\Phi$ is an arbitrary function.

## 8.6 Second-Order ODEs and the Determining Equations of the Group

The general second-order ordinary differential equation

$$\psi = \Psi[x, y, y_x, y_{xx}] = 0 \tag{8.86}$$

is invariant under the twice extended group with infinitesimals $(\xi, \eta, \eta_{\{1\}}, \eta_{\{2\}})$ if and only if

$$X_{\{2\}}\Psi = 0. \tag{8.87}$$

Written out, (8.87) is

$$\xi \frac{\partial \Psi}{\partial x} + \eta \frac{\partial \Psi}{\partial y} + \eta_{\{1\}} \frac{\partial \Psi}{\partial y_x} + \eta_{\{2\}} \frac{\partial \Psi}{\partial y_{xx}} = 0, \tag{8.88}$$

where $\eta_{\{1\}}$ and $\eta_{\{2\}}$ are given in (8.31) and (8.37). In contrast to first-order ODEs, the invariance condition can generally be used to determine the group that leaves a given second- or higher-order equation invariant. This is illustrated by the following example.

### *8.6.1 Projective Group of the Simplest Second-Order ODE*

Let's work out the determining equations of the most elementary second-order ODE,

$$y_{xx} = 0. \tag{8.89}$$

In this case $\psi = \Psi[x, y, y_x, y_{xx}] = y_{xx}$. The invariance condition (8.87) becomes

$$X_{\{2\}}\Psi = \eta_{\{2\}} = 0, \tag{8.90}$$

which, written out fully, is

$$\begin{aligned} &\eta_{xx} + (2\eta_{xy} - \xi_{xx})y_x + (\eta_{yy} - 2\xi_{xy})y_x^2 \\ &\quad - \xi_{yy} y_x^3 + (\eta_y - 2\xi_x)y_{xx} - 3\xi_y y_x y_{xx} = 0. \end{aligned} \tag{8.91}$$

Since (8.91) holds under the restriction $y_{xx} = 0$, the invariance condition takes the form

$$\eta_{xx} + (2\eta_{xy} - \xi_{xx})y_x + (\eta_{yy} - 2\xi_{xy})y_x^2 - \xi_{yy} y_x^3 = 0. \tag{8.92}$$

The quantity $y_x$ can take on arbitrary values limited only by the range of $y_x$ covered by the family of solutions of $y_{xx} = 0$. Thus the only way the invariance condition can be satisfied is if each term in (8.92) is individually zero. Therefore the infinitesimals must satisfy

$$\begin{aligned} \eta_{xx} &= 0, \\ 2\eta_{xy} - \xi_{xx} &= 0, \\ \eta_{yy} - 2\xi_{xy} &= 0, \\ \xi_{yy} &= 0. \end{aligned} \tag{8.93}$$

The system of four linear PDEs in $\xi$ and $\eta$ in (8.93) are called the *determining equations of the group*. This is the system generated by the function **FindDeterminingEquations** in the package **IntroToSymmetry.m** included with the text. In contrast to the case of first-order ODEs, the invariance condition for higher-order ODEs generally contains enough information to find the infinitesimals. One procedure for finding them is described in the next section.

#### *8.6.1.1 Series Solution of the Determining Equations of the Group*

The determining equations (8.93) comprise a (usually overdetermined) set of linear PDEs governing the unknown infinitesimals, and there are no general methods for finding all solutions in all circumstances. A variety of techniques must be used.

The fact that the system is overdetermined tends to highly restrict the classes of solutions that typically arise. For this reason it is always useful to first attempt a power-series solution under the premise that for one or more of the infinitesimals the power series has a good chance of truncating. In the example just considered, we might try a third-order series of the form

$$\begin{aligned}\xi &= a^1 + a^2 x + a^3 y + a^4 x^2 + a^5 xy + a^6 y^2 \\ &\quad + a^7 x^3 + a^8 x^2 y + a^9 xy^2 + a^{10} y^3, \\ \eta &= b^1 + b^2 x + b^3 y + b^4 x^2 + b^5 xy + b^6 y^2 \\ &\quad + b^7 x^3 + b^8 x^2 y + b^9 xy^2 + b^{10} y^3.\end{aligned} \tag{8.94}$$

The two series in (8.94) are substituted into the determining equations (8.93), leading to the following algebraic system for the coefficients:

$$\begin{aligned}\eta_{xx} &= 2b^4 + 6b^7 x + 2b^8 y = 0, \\ 2\eta_{xy} - \xi_{xx} &= 2b^5 + 4b^8 x + 4b^9 y - 2a^4 - 6a^7 x - 2a^8 y = 0, \\ \eta_{yy} - 2\xi_{xy} &= 2b^6 + 2b^9 x + 6b^{10} y - 2a^5 - 2a^8 x - 2a^9 y = 0, \\ \xi_{yy} &= 2a^6 + 2a^9 x + 6a^{10} y = 0.\end{aligned} \tag{8.95}$$

Like powers of $x$ and $y$ are gathered together:

$$\begin{aligned}2b^4 + 6b^7 x + 2b^8 y &= 0, \\ (2b^5 - 2a^4) + (4b^8 - 6a^7)x + (4b^9 - 2a^8)y &= 0, \\ (2b^6 - 2a^5) + (2b^9 - 2a^8)x + (6b^{10} - 2a^9)y &= 0, \\ 2a^6 + 2a^9 x + 6a^{10} y &= 0.\end{aligned} \tag{8.96}$$

The variables $x$ and $y$ are completely independent, and so the only way the system (8.96) can be satisfied is if the coefficients of $x$ and $y$ are each individually zero. The final result is

$$\begin{aligned} a^7 = a^8 = a^9 = a^{10} &= 0, \\ b^7 = b^8 = b^9 = b^{10} &= 0, \\ b^5 &= a^4, \\ b^6 &= a^5, \\ b^4 &= 0, \\ a^6 &= 0, \end{aligned} \tag{8.97}$$

which yields

$$\begin{aligned} \xi &= a^1 + a^2x + a^3y + a^4x^2 + a^5xy, \\ \eta &= b^1 + b^2x + b^3y + a^4xy + a^5y^2 \end{aligned} \tag{8.98}$$

for the infinitesimal group of the equation $y_{xx} = 0$. These are the projective transformations (groups that transform straight lines to straight lines) in the plane with eight independent parameters.

The multivariate polynomial procedure just outlined is essentially the one used by the software package **`IntroToSymmetry.m`** enclosed with this text. As noted earlier the function that finds the determining equations is called **`FindDeterminingEquations`**, and the function that carries out the series solution is called **`SolveDeterminingEquations`**. The choice of the order of the series is up to the user; it might depend on the effort required. When the substitution and solution process is automated, a longer series can be selected; however, one must remember that the number of unknown coefficients grows rapidly with the order of the series and that the solution process is a symbolic one and therefore may be quite slow.

This is a particularly simple example where the series approach leads to the complete solution of the determining equations. But the reader must be aware that it is not at all unusual for the infinitesimals to contain transcendental functions or even arbitrary functions. The following example illustrates this point.

***Example 8.2 (A nonlinear second-order ODE with infinitesimals that involve logarithms).*** Now let's use the tools just developed to work out the symmetries of the nonlinear second-order ODE

$$y_{xx} + \frac{1}{x} y_x + e^y = 0. \tag{8.99}$$

In this case,

$$\psi = \Psi[x, y, y_x, y_{xx}] = y_{xx} + \frac{y_x}{x} + e^y. \tag{8.100}$$

The invariance condition,

$$X_{\{2\}}\Psi = \xi \frac{\partial \Psi}{\partial x} + \eta \frac{\partial \Psi}{\partial y} + \eta_{\{1\}} \frac{\partial \Psi}{\partial y_x} + \eta_{\{2\}} \frac{\partial \Psi}{\partial y_{xx}} = 0, \tag{8.101}$$

is

$$-\frac{\xi}{x^2} y_x + \eta e^y + \frac{\eta_{\{1\}}}{x} + \eta_{\{2\}} = 0, \tag{8.102}$$

which, written out fully, is

$$\begin{aligned} -\frac{\xi}{x^2} y_x + \eta e^y + \frac{1}{x}(\eta_x + (\eta_y - \xi_x) y_x - \xi_y (y_x)^2) \\ + \eta_{xx} + (2\eta_{xy} - \xi_{xx}) y_x + (\eta_{yy} - 2\xi_{xy}) y_x^2 \\ - \xi_{yy} y_x^3 + (\eta_y - 2\xi_x) y_{xx} - 3\xi_y y_x y_{xx} = 0. \end{aligned} \tag{8.103}$$

The relation (8.103) holds under the restriction $y_{xx} = -e^y - y_x/x$. When this rule is applied to (8.103), it takes the form

$$\begin{aligned} -\frac{\xi}{x^2} y_x + \eta e^y + \frac{1}{x}\left(\eta_x + (\eta_y - \xi_x) y_x - \xi_y y_x^2\right) \\ + \eta_{xx} + (2\eta_{xy} - \xi_{xx}) y_x + (\eta_{yy} - 2\xi_{xy}) y_x^2 \\ - \xi_{yy} y_x^3 + (\eta_y - 2\xi_x - 3\xi_y y_x)\left(e^y - \frac{1}{x} y_x\right) = 0, \end{aligned} \tag{8.104}$$

which can be parsed into the final form of the invariance condition,

$$\begin{aligned} \frac{\eta_x}{x} + \eta_{xx} - (\eta_y - 2\xi_x - \eta) e^y \\ + \left(-\frac{\xi}{x^2} + 2\eta_{xy} - \xi_{xx} + \frac{\xi_x}{x} + 3\xi_y e^y\right) y_x \\ + \left(\eta_{yy} - 2\xi_{xy} + \frac{2\xi_y}{x}\right) y_x^2 - \xi_{yy} y_x^3 = 0 \end{aligned} \tag{8.105}$$

Now the same argument used in the last example applies. The quantity $y_x$ can take on arbitrary values limited only by the range of $y_x$ covered by the family of solutions of (8.100). Thus the only way the invariance condition (8.105) can be satisfied is if each coefficient in (8.105) is individually zero. The infinitesimals satisfy the determining equations

$$\begin{aligned}\frac{\eta_x}{x}+\eta_{xx}-(\eta_y-2\xi_x-\eta)e^y&=0,\\-\frac{\xi}{x^2}+2\eta_{xy}-\xi_{xx}+\frac{\xi_x}{x}+3\xi_y e^y&=0,\\\eta_{yy}-2\xi_{xy}+\frac{2\xi_y}{x}&=0,\\\xi_{yy}&=0.\end{aligned}\tag{8.106}$$

The solution of (8.106) is

$$\begin{aligned}\xi&=ax+b(x\ \ln[x]),\\\eta&=a(-2)+b(-2-2\ \ln[x]),\end{aligned}\tag{8.107}$$

with corresponding group operators

$$X^a=x\frac{\partial}{\partial x}-2\frac{\partial}{\partial y},\qquad X^b=x\ \ln[x]\frac{\partial}{\partial x}-2(1+\ln[x])\frac{\partial}{\partial y}.\tag{8.108}$$

See [8.1] for details of the solution procedure.

Transcendental solutions of the determining equations require that a variety of approaches be used in the solution of the determining equations. One procedure is to use the series expansion method first and then add unknown functions to each series. When the expressions are substituted into the determining equations, the result is usually a reduced number of determining equations, which then need to be addressed. The package does not explicitly provide for this, but it is quite easy to use the built-in functions in *Mathematica*® to manipulate and reduce the determining equations in this fashion.

Occasionally, arbitrary functions can appear in the infinitesimals, and these can often be detected by repeating the series expansion method for successively higher orders. If an arbitrary function is present, the highest-order terms in the expansion will continue to have nonzero coefficients as the order is increased.

### 8.6.2 Construction of the General Second-Order ODE That Admits a Given Group

The characteristic equations of (8.88) are

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy_x}{\eta_{\{1\}}} = \frac{dy_{xx}}{\eta_{\{2\}}}. \tag{8.109}$$

The system (8.109) has three invariants:

$$\psi^1 = \Psi^1[x, y], \qquad \psi^2 = \Psi^2[x, y, y_x], \qquad \psi^3 = \Psi^3[x, y, y_x, y_{xx}]. \tag{8.110}$$

The first invariant is found by solving the first equality in (8.109). As noted before, this is a hard problem, since it involves the solution of a first-order ODE that, in general, is nonlinear and for which there is no systematic method for finding a group. The second invariant is found by solving a Ricatti equation as discussed in Section 8.5. If the first invariant is known, the second can always be found. The third invariant is found by solving

$$\frac{dy_{xx}}{dx} = \frac{\eta_{\{2\}}[x, y[x, \psi^1], y_x[x, \psi^1, \psi^2], y_{xx}]}{\xi[x, y[x, \psi^1]]}. \tag{8.111}$$

The linearity of $\eta_{\{2\}}$ with respect to $y_{xx}$ means that (8.111) is linear in $y_{xx}$ and therefore always solvable. The most general second-order ODE that is invariant under the group $(\xi, \eta)$ is

$$\omega = \Omega[\Psi^1[x, y], \Psi^2[x, y, y_x], \Psi^3[x, y, y_x, y_{xx}]], \tag{8.112}$$

where $\Omega$ is an arbitrary function.

Table 8.1 lists a few of the second-order ODEs that can be generated from a known group.

## 8.7 Higher-Order ODEs

The general $p$th-order ordinary differential equation

$$\psi = \Psi[x, y, y_x, y_{xx}, \ldots, y_{px}] = 0 \tag{8.113}$$

is invariant under the $p$ times extended group with infinitesimals $(\xi, \eta, \eta_{\{1\}}, \eta_{\{2\}}, \ldots, \eta_{\{p\}})$ if and only if

$$X_{\{p\}}\Psi = 0, \tag{8.114}$$

Table 8.1. *Some second-order ODEs invariant under a single group.*

| Equation | $\xi$ | $\eta$ |
|---|---|---|
| $y_{xx} = F[y, y_x]$ | 1 | 0 |
| $y_{xx} = F[x, y_x]$ | 0 | 1 |
| $y_{xx} = F[ax + by, y_x]$ | $b$ | $-a$ |
| $y_{xx} = (1 + (y_x)^2)F\left[x^2 + y^2, \dfrac{y - xy_x}{x + yy_x}\right]$ | $y$ | $-x$ |
| $y_{xx} = y_x^3 F\left[y, \dfrac{y - xy_x}{y_x}\right]$ | $y$ | 0 |
| $y_{xx} = F[x, y - xy_x]$ | 0 | $x$ |
| $xy_{xx} = F[y/x, y_x]$ | $x$ | $y$ |
| $y_{xx} = yF[ye^{-x}, y_x/y]$ | 1 | $y$ |
| $y_{xx} = x^{\alpha-2}F[x^{-\alpha}y, x^{1-\alpha}y_x]$ | $x$ | $\alpha y$ |
| $yy_{xx} = y_x^2 + y^2 F\left[x, \dfrac{xy_x}{y} - \ln[y]\right]$ | 0 | $xy$ |
| $x^3 y_{xx} = F[y/x, y - xy_x]$ | $x^2$ | $xy$ |

where the group operator of the $p$ times extended group is

$$X_{\{p\}} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \eta_{\{1\}}\frac{\partial}{\partial y_x} + \eta_{\{2\}}\frac{\partial}{\partial y_{xx}} + \cdots + \eta_{\{p\}}\frac{\partial}{\partial y_{px}}. \quad (8.115)$$

The characteristic equations are

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy_x}{\eta_{\{1\}}} = \frac{dy_{xx}}{\eta_{\{2\}}} = \cdots = \frac{dy_{px}}{\eta_{\{p\}}}. \quad (8.116)$$

### *8.7.1 Construction of the General pth-Order ODE That Admits a Given Group*

The invariance condition (8.114) has $p + 1$ invariants

$$\psi^1 = \Psi^1[x, y], \qquad \psi^2 = \Psi^2[x, y, y_x], \ldots,$$
$$\psi^{p+1} = \Psi^{p+1}[x, y, y_x, \ldots, y_{px}]. \quad (8.117)$$

The first three invariants are found using the procedure just described. All higher invariants share the same property as (8.111); they are determined by the solution of a linear ODE and therefore are always solvable. The most general $p$th-order ODE invariant under the group $(\xi, \eta)$ is

$$\omega = \Omega[\Psi^1[x, y], \Psi^2[x, y, y_x], \ldots, \Psi^{p+1}[x, y, y_x, \ldots, y_{px}]], \quad (8.118)$$

where $\Omega$ is an arbitrary function.

## 8.8 Reduction of Order by the Method of Canonical Coordinates

Lets suppose a $p$th-order ODE $y_{px} = \Psi[x, y, y_x, y_{xx}, \ldots, y_{(p-1)x}]$, where $p \geq 2$, admits a one-parameter Lie group with group operator $X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$. Solving the equations $XR = 0$ and $X\Theta = 1$ produces canonical coordinates $r = R[x, y]$ and $\theta = \Theta[x, y]$. In principle these relations can be inverted to yield the group

$$x = F[r, \theta], \qquad y = G[r, \theta]. \quad (8.119)$$

Now form the derivative

$$\frac{d\theta}{dr} = \frac{\Theta_x + \Theta_y y_x}{R_x + R_y y_x}. \quad (8.120)$$

Solve (8.120) for $y_x$ in the form

$$y_x = \Gamma^1\left[r, \theta, \frac{d\theta}{dr}\right] = H^1\left[r, \theta, \frac{d\theta}{dr}\right]. \quad (8.121)$$

Substituting (8.121) into (8.120) and differentiating again with respect to $r$ leads to

$$y_{xx} = \left(\frac{d^2\theta}{dr^2}\right)\Gamma^2\left[r, \theta, \frac{d\theta}{dr}\right] + \Upsilon^1\left[r, \theta, \frac{d\theta}{dr}\right] = H^2\left[r, \theta, \frac{d\theta}{dr}, \frac{d^2\theta}{dr^2}\right]. \quad (8.122)$$

Similar relations can be derived for higher-order derivatives. At the $p$th-order,

$$\begin{aligned} y_{px} &= \left(\frac{d^p\theta}{dr^p}\right)\Gamma^p\left[r, \theta, \frac{d\theta}{dr}\right] + \Upsilon^{p-1}\left[r, \theta, \frac{d\theta}{dr}, \ldots, \frac{d^{p-1}\theta}{dr^{p-1}}\right] \\ &= H^p\left[r, \theta, \frac{d\theta}{dr}, \ldots, \frac{d^{p-1}\theta}{dr^{p-1}}, \frac{d^p\theta}{dr^p}\right]. \end{aligned} \quad (8.123)$$

Replacing $x$, $y$, and the various derivatives of $y$ in the original equation leads to an equivalent $p$th-order equation expressed in canonical coordinates:

$$\frac{d^p\theta}{dr^p} = \Omega\left[r, \theta, \frac{d\theta}{dr}, \frac{d^2\theta}{dr^2}, \ldots, \frac{d^{p-1}\theta}{dr^{p-1}}\right]. \quad (8.124)$$

But by the definition of canonical coordinates, this equation is invariant under the translation group $\tilde{r} = r$, $\tilde{\theta} = \theta + s$. Therefore (8.124) cannot depend on $\theta$, i.e.,

$$\frac{d^p\theta}{dr^p} = \Omega\left[r, \frac{d\theta}{dr}, \frac{d^2\theta}{dr^2}, \ldots, \frac{d^{p-1}\theta}{dr^{p-1}}\right]. \tag{8.125}$$

If we let

$$\frac{d\theta}{dr} = z[r], \tag{8.126}$$

the original $p$th-order ODE reduces to the system

$$\frac{d^{p-1}z}{dr^{p-1}} = \Omega\left[r, z, \frac{dz}{dr}, \ldots, \frac{d^{p-2}z}{dr^{p-2}}\right], \qquad \frac{d\theta}{dr} = z. \tag{8.127}$$

## 8.9 Reduction of Order by the Method of Differential Invariants

Higher-order invariants can always be generated from the first two without necessarily solving the further equalities in (8.116). For example, we can construct the third invariant in (8.110) from a linear combination of the first two invariants. The function

$$\alpha = \Psi^2[x, y, y_x] - \beta\Psi^1[x, y], \tag{8.128}$$

where $\alpha$ and $\beta$ are constants, is a first-order ordinary differential equation that is clearly invariant under all three operators $X$, $X_{\{1\}}$, and $X_{\{2\}}$. Along a solution trajectory $y[x]$,

$$\frac{D(\Psi^2 - \beta\Psi^1)}{Dx} = \frac{\partial\Psi^2}{\partial x} + y_x\frac{\partial\Psi^2}{\partial y} + y_{xx}\frac{\partial\Psi^2}{\partial y_x} - \beta\left(\frac{\partial\Psi^1}{\partial x} + y_x\frac{\partial\Psi^1}{\partial y}\right) = 0. \tag{8.129}$$

Equation (8.129) must hold for any value of $\beta$, and so

$$\frac{d\Psi^2}{d\Psi^1} = \frac{\frac{\partial\Psi^2}{\partial x} + y_x\frac{\partial\Psi^2}{\partial y} + y_{xx}\frac{\partial\Psi^2}{\partial y_x}}{\frac{\partial\Psi^1}{\partial x} + y_x\frac{\partial\Psi^1}{\partial y}} = \Psi^3[x, y, y_x, y_{xx}] \tag{8.130}$$

is the required third invariant. The right-hand side of (8.130) can always be rearranged so that

$$\Psi^3[x, y, y_x, y_{xx}] = \Omega[\Psi^1[x, y], \Psi^2[x, y, y_x]]. \tag{8.131}$$

Note that (8.131) is essentially the general second-order equation invariant under the group. The problem of solving the general second-order ODE invariant under the group $(\xi, \eta)$ is reduced to the solution of the first-order ODE

$$\frac{d\Psi^2}{d\Psi^1} = \Omega[\Psi^1, \Psi^2] \tag{8.132}$$

plus one integration. In summary, if $\Psi^1[x, y]$ and $\Psi^2[x, y, y_x]$ are invariants of the once extended group $(\xi, \eta, \eta_{\{1\}})$, then $d\Psi^2/d\Psi^1$ is an invariant of the twice extended group $(\xi, \eta, \eta_{\{1\}}, \eta_{\{2\}})$. Similarly, $d(d\Psi^2)/d(\Psi^1)^2$, $d(d(d\Psi^2))/d(\Psi^1)^3, \ldots, d^p\Psi^2/d(\Psi^1)^p$ are invariants of the $p$th extended group.

From a study of the integral curves of (8.132), complete qualitative information about the solutions of the second-order ODE can be found. The state-space method described in Chapter 3 is particularly helpful in this regard. Moreover (8.132) may itself admit a group. This will be the case when the original second-order equation admits two groups with a solvable Lie algebra, so that a complete reduction to quadrature is possible. Recall the discussion in Chapter 5, Section 5.11, where it was pointed out that every two-dimensional Lie algebra is solvable.

All second-order ODEs that admit a two-dimensional Lie algebra and are therefore fully integrable can be put into canonical form and completely catagorized into four basic classes. To learn about this system of classification the reader should consult References [8.1] and [8.2].

## 8.10 Successive Reduction of Order; Invariance under a Multiparameter Group with a Solvable Lie Algebra

If an ODE is invariant under a multiparameter group with a solvable Lie algebra, the reduction procedure uses the groups one at a time to construct level-1 invariants that become new variables for the once reduced equation. The original, say $p$th-order, equation is expressed in terms of an $(p-1)$th-order equation plus a quadrature. The process then turns to a second group in the Lie algebra. The action of this group on the level-1 invariants of the previous group is determined. This group is then used to construct a set of level-2 invariants, which are used to reduce the $(p-1)$th-order, equation to order $p-2$ plus a second quadrature, and so forth. The process works if the Lie algebra is solvable. If the Lie algebra is solvable, then, as the order of the original ODE is reduced, the new ODE inherits the remaining symmetries of the original Lie algebra. The level-1 invariants are invariant *families* of the level-2 group, the level-2 invariants are invariant families of the level-3 group, and so on.

Schematically the situation is as follows. Suppose a $p$th-order equation admits an $r$-parameter group with a $q$-parameter solvable Lie subalgebra. where

$q \leq r$ and $p \geq q$. Then it is possible to reduce the problem to a $p - q$ order equation plus $q$ quadratures. The first two invariants constructed from each group in the subalgebra are as follows:

$$\begin{aligned} &X^{1(x,y)}: && \phi[x, y], G[x, y, y_x] \\ &X^{2(\phi,G)}: && \gamma[\phi, G], H[\phi, G, G_\phi] \\ &X^{3(\gamma,H)}: && \theta[\gamma, H], K[\gamma, H, H_\gamma] \\ &\vdots && \vdots \end{aligned} \tag{8.133}$$

where the superscripts on the group operators indicate that the operator [originally expressed in terms of $(x, y)$] is rewritten in terms of new variables corresponding to a given level of reduction.

For definiteness, let $q = 3$, and let the sequence of ideals be $\Lambda^0$, $\Lambda^1 = X^1$, $\Lambda^2 = X^1,\ X^2$, and $\Lambda^3 = X^1, X^2, X^3$. Begin with the ideal $\Lambda^2$. This has the commutator relation

$$\{X^1, X^2\} = \beta X^1. \tag{8.134}$$

If we operate on the level-1 invariant, the result is

$$\{X^1, X^2\}\phi[x, y] = X^1(X^2\phi) - X^2(X^1\phi), \tag{8.135}$$

or

$$\beta X^1\phi = X^1(X^2\phi) - X^2(X^1\phi). \tag{8.136}$$

But since $\phi$ is an invariant, $X^1\phi = 0$ and we have

$$X^1(X^2\phi) = 0. \tag{8.137}$$

The implication of (8.137) is that

$$X^2\phi = f[\phi]. \tag{8.138}$$

In other words, the level-1 invariant $\phi[x, y]$ is an invariant family of the level-2 group $X^2$.

The same holds for the extensions. If the first extension is worked out, the commutator gives

$$\left\{X^1_{\{1\}}, X^2_{\{1\}}\right\} = \beta X^1_{\{1\}}. \tag{8.139}$$

The same argument just used leads to

$$X^2_{\{1\}}G = g[G]. \tag{8.140}$$

Similarly, all extended level-1 invariants are invariant families for the extended level-2 group and so forth. The same result holds for $\gamma$ and $H$ at the second level, for $\theta$ and $K$ at the third level (if there is one), and so forth. See Bluman and Kumei [8.3] for the general theory. The following examples illustrate the idea.

### *8.10.1 Two-Parameter Group of the Blasius Equation*

Now let's look at a third-order nonlinear equation with a solvable two-dimensional Lie algebra. We consider the Blasius equation,

$$\Psi[x, y, y_x, y_{xx}, y_{xxx}] = y_{xxx} + yy_{xx} = 0. \tag{8.141}$$

This equation comes up in connection with laminar boundary-layer theory and problems in nonlinear heat conduction. We will study it in considerable detail when we get to the topic of boundary layers in Chapter 10.

#### *8.10.1.1 Invariant Group of the Blasius Equation*

First let's determine the group that leaves (8.141) invariant. The invariance condition is

$$\begin{aligned}\xi\frac{\partial\Psi}{\partial x} + \eta\frac{\partial\Psi}{\partial y} + \eta_{\{1\}}\frac{\partial\Psi}{\partial y_x} + \eta_{\{2\}}\frac{\partial\Psi}{\partial y_{xx}} + \eta_{\{3\}}\frac{\partial\Psi}{\partial y_{xxx}}&\\ = \eta y_{xx} + \eta_{\{2\}}y + \eta_{\{3\}} = 0.&\end{aligned} \tag{8.142}$$

Substitute into (8.142) the expressions for the second and third extensions [see Equations (8.37) and (8.38)]. The invariance condition becomes the following rather lengthy expression:

$$\begin{aligned}&\eta y_{xx} + y\eta_{xx} + (2y\eta_{xy} - y\xi_{xx})y_x + (y\eta_{yy} - 2y\xi_{xy})y_x^2\\ &\quad - y\xi_{yy}y_x^3 + (y\eta_y - 2y\xi_x)y_{xx} - 3y\xi_y y_x y_{xx}\\ &\quad + \eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y_x + (3\eta_{xyy} - 3\xi_{xxy})y_x^2\\ &\quad + (\eta_{yyy} - 3\xi_{xyy})y_x^3 - \xi_x^3 - \xi_{yyy}y_x^4 + (3\eta_{xy} - 3\xi_{xx})y_{xx}\\ &\quad + (3\eta_{yy} - 9\xi_{xy})y_x y_{xx} - 6\xi_{yy}y_x^2 y_{xx} - 3\xi_y y_{xx}^2\\ &\qquad + (\eta_y - 3\xi_x)y_{xxx} - 4\xi_y y_x y_{xxx} = 0.\end{aligned} \tag{8.143}$$

Now gather coefficients of various like derivatives of $y$:

$$\begin{aligned}&y\eta_{xx} + \eta_{xxx} + (2y\eta_{xy} - y\xi_{xx} + 3\eta_{xxy} - \xi_{xxx})y_x\\ &\quad + (3\eta_{xyy} - 3\xi_{xxy} + y\eta_{yy} - 2y\xi_{xy})y_x^2\\ &\quad + (\eta_{yyy} - 3\xi_{xyy} - y\xi_{yy})y_x^3 - \xi_{yyy}y_x^4 - 3\xi_y y_{xx}^2\end{aligned}$$

$$
\begin{aligned}
&+(3\eta_{xy} - 3\xi_{xx} + \eta + y\eta_y - 2y\xi_x)y_{xx} \\
&+(3\eta_{yy} - 9\xi_{xy} - 3y\xi_y)y_x y_{xx} \\
&\quad - 6\xi_{yy} y_x^2 y_{xx} + (\eta_y - 3\xi_x)y_{xxx} - 4\xi_y y_x y_{xxx} = 0. \quad (8.144)
\end{aligned}
$$

The invariance condition (8.144) holds under the constraint that $y$ is a solution of the Blasius equation (8.141). To apply this constraint we use the rule $y_{xxx} = -yy_{xx}$ to replace the highest derivative in (8.144) and then regather terms:

$$
\begin{aligned}
&y\eta_{xx} + \eta_{xxx} + (2y\eta_{xy} - y\xi_{xx} + 3\eta_{xxy} - \xi_{xxx})y_x \\
&\quad + (3\eta_{xyy} - 3\xi_{xxy} + y\eta_{yy} - 2y\xi_{xy})y_x^2 \\
&\quad + (\eta_{yyy} - 3\xi_{xyy} - y\xi_{yy})y_x^3 - \xi_{yyy} y_x^4 \\
&\quad + (3\eta_{xy} - 3\xi_{xx} + \eta + y\xi_x)y_{xx} + (3\eta_{yy} - 9\xi_{xy} + y\xi_{xy} + y\xi_y)y_x y_{xx} \\
&\qquad - 6\xi_{yy} y_x^2 y_{xx} - 3\xi_y y_{xx}^2 = 0. \quad (8.145)
\end{aligned}
$$

The last two terms in (8.145) imply that $\xi_{yy} = 0$ and $\xi_y = 0$. Use these rules to simplify (8.145):

$$
\begin{aligned}
&y\eta_{xx} + \eta_{xxx} + (2y\eta_{xy} - y\xi_{xx} + 3\eta_{xxy} - \xi_{xxx})y_x \\
&\qquad + (3\eta_{xyy} + y\eta_{yy})y_x^2 + (\eta_{yyy})y_x^3 \\
&\qquad + (3\eta_{xy} - 3\xi_{xx} + \eta + y\xi_x)y_{xx} + 3\eta_{yy} y_x y_{xx} = 0 \quad (8.146)
\end{aligned}
$$

The last and third to last terms in (8.146) imply $\eta_{yy} = 0$ and $\eta_{yyy} = 0$. Apply these rules to (8.146):

$$
\begin{aligned}
&y\eta_{xx} + \eta_{xxx} + (2y\eta_{xy} - y\xi_{xx} + 3\eta_{xxy} - \xi_{xxx})y_x \\
&\qquad + (3\eta_{xy} - 3\xi_{xx} + \eta + y\xi_x)y_{xx} = 0. \quad (8.147)
\end{aligned}
$$

Finally, the infinitesimals of the Blasius equation satisfy the following set of determining equations:

$$
\begin{aligned}
\xi_{yy} &= 0, \\
\xi_y &= 0, \\
\eta_{yy} &= 0, \\
\eta_{yyy} &= 0, \quad (8.148) \\
y\eta_{xx} + \eta_{xxx} &= 0, \\
2y\eta_{xy} - y\xi_{xx} + 3\eta_{xxy} - \xi_{xxx} &= 0, \\
3\eta_{xy} - 3\xi_{xx} + \eta + y\xi_x &= 0,
\end{aligned}
$$

from which we can conclude

$$\xi = a + bx, \qquad \eta = -by. \tag{8.149}$$

The result of this example points up a common aspect of the group methodology, especially as applied to higher-order equations. It is one of those things that tends to get complicated before it gets simple. Furthermore, the groups that an equation will admit can often be identified by inspection, as is the case here. Occasionally, an approach based on inspection plus the use of the commutator table to identify additional groups can be employed to circumvent the rather lengthy direct procedure for finding infinitesimals by means of the invariance condition. The software package **`IntroToSymmetry.m`** provided with the text and described in Appendix 4 makes the effort involved less of an issue by automating the process.

*8.10.1.2 The Commutator Table*

The fact that the Blasius equation is invariant under a two-parameter group, which we know is always solvable, guarantees that the order of the equation can be reduced by two. The infinitesimal generators of (8.149) are

$$X^a = \frac{\partial}{\partial x}, \qquad X^b = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \tag{8.150}$$

with the commutator table given in Table 8.2.

Clearly $X^a$ is an ideal of the Lie algebra $X^a$, $X^b$. The order in which we use these groups to reduce the Blasius equation is important.

*8.10.1.3 First Reduction*

We begin with the ideal $X^a$. The characteristic equations of the thrice extended operator $X^a_{\{3\}}$ are

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dy_x}{0} = \frac{dy_{xx}}{0} = \frac{dy_{xxx}}{0}, \tag{8.151}$$

Table 8.2.
*Commutator table for the Blasius equation.*

| | $X^a$ | $X^b$ |
|---|---|---|
| $X^a$ | 0 | $X^a$ |
| $X^b$ | $-X^a$ | 0 |

and the first two invariants are

$$\phi = y, \qquad G = y_x. \tag{8.152}$$

By the method of differential invariants, the equation

$$\frac{DG}{D\phi} = \frac{\frac{\partial G}{\partial x}\,dx + \frac{\partial G}{\partial y}\,dy + \frac{\partial G}{\partial y_x}\,dy_x}{\frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy} = \frac{y_{xx}}{y_x} \tag{8.153}$$

is an invariant, as is

$$\frac{D^2G}{D\phi^2} = \left(\frac{y_x y_{xxx} - y_{xx}^2}{y_x^2}\right)\frac{1}{y_x} = \frac{y_x(-yy_{xx}) - y_{xx}^2}{y_x^3}, \tag{8.154}$$

where the Blasius equation has been used to replace the third derivative. Equation (8.154) can be rearranged to read

$$G\frac{D^2G}{D\phi^2} + \phi\frac{DG}{D\phi} + \left(\frac{DG}{D\phi}\right)^2 = 0. \tag{8.155}$$

This is the once reduced Blasius equation.

#### *8.10.1.4 Second Reduction*

Now we determine the action of the group $\tilde{x} = e^b x$, $\tilde{y} = e^{-b} y$ on the new variables $(\phi, G)$,

$$\tilde{\phi} = e^{-b}\phi, \qquad \tilde{G} = e^{-2b}G, \tag{8.156}$$

and on equation (8.155), which we see is invariant: Note that the infinitesimals of (8.156) can be determined directly from the once extended group operator $X^b_{\{1\}}$, i.e., one does not actually need to construct the finite group (8.156).

$$\tilde{G}\frac{D^2\tilde{G}}{D\tilde{\phi}^2} + \tilde{\phi}\frac{D\tilde{G}}{D\tilde{\phi}} + \left(\frac{D\tilde{G}}{D\tilde{\phi}}\right)^2 = e^{-2b}\left(G\frac{D^2G}{D\phi^2} + \phi\frac{DG}{D\phi} + \left(\frac{DG}{D\phi}\right)^2\right) = 0. \tag{8.157}$$

Now solve the characteristic equations of (8.156):

$$\frac{d\phi}{-\phi} = \frac{dG}{-2G} = \frac{dG_\phi}{-G_\phi}. \tag{8.158}$$

The invariants (new variables) at the second stage are

$$\gamma = \frac{G}{\phi^2}, \qquad H = \frac{G_\phi}{\phi}. \tag{8.159}$$

Use the method of differential invariants to generate the second reduction of order:

$$\begin{aligned}\frac{DH}{D\gamma} &= \frac{H_\phi + H_G \frac{dG}{d\phi} + H_{G_\phi} \frac{dG_\phi}{d\phi}}{\gamma_\phi + \gamma_G \frac{dG}{d\phi}} \\ &= \frac{-\frac{G_\phi}{\phi^2} + \frac{1}{\phi} G_{\phi\phi}}{-2\frac{G}{\phi^3} + \frac{1}{\phi^2} G_\phi}.\end{aligned} \tag{8.160}$$

Using the once reduced equation to eliminate the second-derivative term, the right-hand side of (8.160) can be rearranged to read as follows:

$$\begin{aligned}\frac{DH}{D\gamma} &= \frac{-\frac{1}{\phi^2}\left(\frac{dG}{d\phi}\right) + \frac{1}{\phi}\left(-\frac{\phi}{G}\left(\frac{dG}{d\phi}\right) - \frac{1}{G}\left(\frac{dG}{d\phi}\right)^2\right)}{-2\frac{G}{\phi^3} + \frac{1}{\phi^2}\left(\frac{dG}{d\phi}\right)} \\ &= \frac{-\frac{1}{\phi}\left(\frac{dG}{d\phi}\right) - \frac{\phi}{G}\left(\frac{dG}{d\phi}\right) - \frac{1}{G}\left(\frac{dG}{d\phi}\right)^2}{-2\frac{G}{\phi^2} + \frac{1}{\phi}\left(\frac{dG}{d\phi}\right)}.\end{aligned} \tag{8.161}$$

Using (8.159) in (8.161), the Blasius equation is finally reduced to the following first-order ODE:

$$\frac{dH}{d\gamma} = \frac{\gamma H + H + H^2}{2\gamma^2 - H\gamma}. \tag{8.162}$$

This equation was discussed extensively in Chapter 3 Section 3.9.3 in connection with phase-plane techniques. We will return to it again in Chapter 10, where boundary conditions for the laminar boundary layer on a flat plate are discussed.

#### *8.10.1.5 The Solution*

Once the correct solution trajectory of (8.162) is determined for given boundary conditions,

$$H = F_1[\gamma; \tilde{\gamma}, \tilde{H}], \tag{8.163}$$

the solution of the original problem then requires two further integrations. The first is

$$\frac{1}{\phi}\frac{dG}{d\phi} = F_1[G/\phi^2], \tag{8.164}$$

which can be rearranged to read

$$\frac{d(G/\phi^2)}{d\ln[\phi]} = F_1[G/\phi^2] - 2G/\phi^2 \tag{8.165}$$

with solution

$$\frac{G}{\phi^2} = F_2[\phi]. \tag{8.166}$$

The second integration is

$$x = \int \frac{dy}{y^2 F_2[y]} + C, \tag{8.167}$$

where $C$ is a constant of integration. Finally the solution is

$$y = Y[x; \tilde{\gamma}, \tilde{H}, C], \tag{8.168}$$

where the constants $[\tilde{\gamma}, \tilde{H}, C]$ are related to the three boundary conditions of the original problem. This problem is revisited with actual boundary conditions in Chapter 10.

Occasionally we need to demonstrate that a solution is an invariant family under a group $X$ using the condition $X\psi = F(\psi)$. The analysis can be a little subtle and the following example illustrates the procedure in a particular case.

***Example 8.3 (A solution as an invariant family).*** Show that the solution of $y_{xx} = e^{-y_x}$ is an invariant family under the two parameter family of translations in $y$ and $x$. The solution is easy to work out.

$$y - (x - \psi^2)\ln(x - \psi^2) + (x - \psi^2) - \psi^1 = 0 \tag{8.169}$$

where $\psi^1$ and $\psi^2$ are constants of integration corresponding to invariance under translation in $y$ and $x$ respectively. Notice that $\psi^1$ is defined so as to carefully separate the effect of each group on the solution. That is, the $\psi^2$ in the right parentheses is not combined with $\psi^1$ even though that would not change the generality of the solution. It is clear from (8.169) that expressing the solution in this fashion, the effect of a translation in $y$ is to change only $\psi^1$ and a translation in $x$ changes only $\psi^2$. The group operators are

$$X^1 = \frac{\partial}{\partial y}, \qquad X^2 = \frac{\partial}{\partial x} \tag{8.170}$$

Now apply these to the solution. First apply $X^1$ with $\psi^2$ fixed,

$$X^1(y-(x-\psi^2)\ln(x-\psi^2)+(x-\psi^2)-\psi^1)=1-X^1\psi^1=0. \quad (8.171)$$

This affirms that $\psi^1$ is an invariant family. Demonstrating that $\psi^2$ is an invariant family is slightly more difficult since it appears implicitly in the solution. Now apply $X^2$ with $\psi^1$ fixed.

$$\begin{aligned} &X^2(y-(x-\psi^2)\ln(x-\psi^2)+(x-\psi^2)-\psi^1)\\ &=-(1-X^2\psi^2)\ln(x-\psi^2)=0 \end{aligned} \quad (8.172)$$

According to this result $X^2\psi^2=1$ and $\psi^2$ is an invariant family under translation in $x$.

The next example illustrates reduction of order in a second-order problem with a somewhat more complex structure than we have seen before.

***Example 8.4 (Complete reduction of a nonlinear second-order ODE).*** Find the general solution of the following ODE:

$$y_{xx}+xy_x+xy_x^3-y-yy_x^2=0. \quad (8.173)$$

We will work out the various steps involved in some detail to demonstrate the method of reduction. Running the package **`IntroToSymmetry.m`** reveals that this equation admits the rotation group, $(\xi,\eta)=(-y,x)$. The once extended version of this group is

$$X_{\{1\}}=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}+\left(1+y_x^2\right)\frac{\partial}{\partial y_x} \quad (8.174)$$

with characteristic equations

$$\frac{dx}{-y}=\frac{dy}{x}=\frac{dy_x}{1+y_x^2}. \quad (8.175)$$

The first invariant is

$$u=x^2+y^2. \quad (8.176)$$

The second invariant is the solution of

$$\frac{dy_x}{1+y_x^2}=\frac{dx}{-(u-x^2)^{1/2}}, \quad (8.177)$$

which is

$$\tan^{-1}[y_x]+\tan^{-1}\left[\frac{x(u-x^2)^{1/2}}{-u+x^2}\right]=\tan^{-1}\left[\frac{xy_x-y}{yy_x+x}\right]=\tan^{-1}[v]. \quad (8.178)$$

where we have used the identity

$$\tan^{-1}[a] + \tan^{-1}[b] = \tan^{-1}\left[\frac{ab+1}{b-a}\right]. \tag{8.179}$$

The second invariant is

$$v = \frac{xy_x - y}{yy_x + x}, \tag{8.180}$$

Let's use reduction through the method of differential invariants. Differentiate both invariants with respect to $x$:

$$\begin{aligned} \frac{Dv}{Dx} &= \frac{y\left(1+y_x^2\right) - xy_x\left(1+y_x^2\right) + (x^2+y^2)(-xy_x+y)\left(1+y_x^2\right)}{(yy_x+x)^2}, \\ \frac{Du}{Dx} &= 2(yy_x + x) \end{aligned} \tag{8.181}$$

where (8.173) has been used to replace $y_{xx}$.

Now divide out the $Dx$ and rearrange:

$$\frac{Dv}{Du} = \frac{-v(1+u)\left(1+y_x^2\right)}{2(yy_x+x)^2}. \tag{8.182}$$

Our goal at this point is to rearrange the right-hand side of (8.182) so that it is expressed only in terms of the new variables $(u, v)$. Note that

$$(yy_x + x)^2 = \frac{(xy_x - y)^2}{v^2}, \tag{8.183}$$

and so

$$\frac{Dv}{Du} = \frac{-v^3(1+u)\left(1+y_x^2\right)}{2(xy_x - y)^2}. \tag{8.184}$$

Furthermore,

$$xy_x - y = \frac{uv}{vy - x}. \tag{8.185}$$

Thus

$$\frac{Dv}{Du} = \frac{-v^3(1+u)\left(1+y_x^2\right)(vy-x)^2}{2(uv)^2}. \tag{8.186}$$

Noting

$$1 + y_x^2 = \frac{v^2u + u}{(vy - x)^2}, \tag{8.187}$$

we finally have the sought-after reduction to first order,

$$\frac{dv}{du} = \frac{-v(1+u)(1+v^2)}{2u}. \tag{8.188}$$

This equation can be easily separated:

$$\frac{dv}{v(1+v^2)} = \frac{1+u}{2u}\,du, \tag{8.189}$$

which leads to the solution of (8.188),

$$\psi = \frac{v^2}{1+v^2} ue^u. \tag{8.190}$$

But we want the solution of the original ODE. To this end we now solve for $v$:

$$v = \pm\left(\frac{\psi}{ue^u - \psi}\right)^{1/2}. \tag{8.191}$$

Restoring the variables $(x, y)$,

$$\frac{xy_x - y}{yy_x + x} = \pm\left(\frac{\psi}{(x^2+y^2)e^{(x^2+y^2)} - \psi}\right)^{1/2}. \tag{8.192}$$

The left-hand side can be rearranged to read

$$\frac{xy_x - y}{yy_x + x} = \frac{x^2 \frac{d}{dx}\left(\frac{y}{x}\right)}{yy_x + x} = \frac{2u}{1+(y/x)^2}\frac{d}{du}\left(\frac{y}{x}\right). \tag{8.193}$$

Now

$$\frac{d(y/x)}{1+(y/x)^2} = \pm\left(\frac{\psi}{ue^u - \psi}\right)^{1/2}\frac{du}{2u}. \tag{8.194}$$

Integrating,

$$\tan^{-1}[y/x] = \pm\int_{x^2+y^2}\left(\frac{\psi}{ue^u - \psi}\right)^{1/2}\frac{du}{2u}, \tag{8.195}$$

or

$$\frac{y}{x} = \pm\tan\left[\int_{x^2+y^2}\left(\frac{\psi}{ue^u - \psi}\right)^{1/2}\frac{du}{2u}\right] + C. \tag{8.196}$$

The two constants of integration are $\psi$ and $C$. Note that under rotation in $x$ and $y$,

$$\begin{aligned} \tilde{u} &= u, \\ \tilde{y}/\tilde{x} &= y/x + s, \end{aligned} \tag{8.197}$$

which is consistent with the original observation that the equation (and solution family) are invariant under the rotation group.

## 8.11 Group Interpretation of the Method of Variation of Parameters

Now let's use our method to analyze the general linear, inhomogeneous second-order ODE

$$h[x]y_{xx} + g[x]y_x + f[x]y = F[x], \tag{8.198}$$

normally solved using the method of variation of parameters. In the context of group theory, (8.198) is regarded as a differential function of $(x, y, y_x, y_{xx})$:

$$\psi = \Psi[x, y, y_x, y_{xx}] = h[x]y_{xx} + g[x]y_x + f[x]y - F[x]. \tag{8.199}$$

The invariance condition, $X_{\{2\}}\Psi = 0$, written out, is

$$\xi(h_x y_{xx} + g_x y_x + f_x y - F_x) + \eta f + \eta_{\{1\}} g + \eta_{\{2\}} h = 0, \tag{8.200}$$

where the infinitesimals $(\xi, \eta)$ are unknown functions that need to be determined. Now substitute the expressions for the extensions $\eta_{\{1\}}$ and $\eta_{\{2\}}$ [see Equations (8.31) and (8.37)]:

$$\begin{aligned} &\xi(h_x y_{xx} + g_x y_x + f_x y - F_x) + \eta f \\ &\quad + \big(\eta_x + (\eta_y - \xi_x)y_x - \xi_y y_x^2\big)g \\ &\quad + \big(\eta_{xx} + (2\eta_{xy} - \xi_{xx})y_x + (\eta_{yy} - 2\xi_{xy})y_x^2\big)h \\ &\quad + \big(-\xi_{yy}y_x^3 + (\eta_y - 2\xi_x)y_{xx} - 3\xi_y y_x y_{xx}\big)h = 0. \end{aligned} \tag{8.201}$$

At this stage we could rearrange (8.201) so that all the products $y$ of and its various derivatives were gathered together, then proceed to identify and solve the determining equations of the group. However, since $f$, $g$, $h$, and are assumed to be arbitrary functions of $x$, the invariant group must leave $x$ unchanged. Therefore we can safely assume at the outset that $\xi = 0$, and so we need only solve for the one unknown function $\eta(x, y)$. The invariance condition (8.201) becomes

$$\eta f + (\eta_x + \eta_y y_x) g + \left(\eta_{xx} + 2\eta_{xy} y_x + \eta_{yy} y_x^2\right) h + \eta_y y_{xx} h = 0. \quad (8.202)$$

This relationship is satisfied under the condition that (8.198) is satisfied by $y[x]$. Using (8.198) to replace $y_{xx}$, Equation (8.202) becomes

$$(\eta f + \eta_x g + \eta_{xx} h + \eta_y F - y\eta_y f) + (2\eta_{xy} h) y_x + (\eta_{yy} h) y_x^2 = 0. \quad (8.203)$$

Since $y_x$ is arbitrary, each of the expressions in parentheses in (8.203) must be individually zero. Thus for general $f$, $g$, $h$, and $F$ the determining equations of the group reduce to

$$\begin{aligned} \eta_{yy} &= 0, \\ \eta_{xy} &= 0, \\ \eta_y &= 0, \\ h\eta_{xx} + g\eta_x + f\eta &= 0. \end{aligned} \quad (8.204)$$

The first three relations in (8.204) imply that $\eta$ is independent of $y$. Comparing the last relation in (8.204) with the original ODE, we recognize that $\eta[x]$ is an as yet unknown solution of the homogeneous equation. The homogeneous equation has two independent solutions $\theta[x]$ and $\phi[x]$, and so (8.199) is invariant under a two-parameter group with infinitesimals

$$\xi = 0, \qquad \eta = a\theta[x] + b\phi[x]. \quad (8.205)$$

The corresponding finite group is

$$\tilde{x} = x, \qquad \tilde{y} = y + a\theta[x] + b\phi[x]. \quad (8.206)$$

The invariant group (8.206) simply expresses the fact that to a solution of (8.199) one can always add any linear combination of the two solutions of the homogeneous equation. The group operators are

$$X^a = \theta[x]\frac{\partial}{\partial y}, \qquad X^b = \phi[x]\frac{\partial}{\partial y}, \quad (8.207)$$

and the commutator of the group is $\{X^a, X^b\} = 0$. The Lie algebra of a two-parameter group is always solvable, and a consequence of this for the present example is that, once $\theta$ and $\phi$ are known, the solution of (8.198) can be reduced by two orders to quadrature. An analogous situation exists for the first-order linear equation described in Chapter 6, Example 6.5, where the integral of the homogeneous problem is the infinitesimal of the one-parameter translation group that leaves the equation invariant.

### *8.11.1 Reduction to Quadrature*

In the previous section we looked for the group that left (8.199) invariant. Having identified the invariant group (8.206), we now turn to the problem of finding invariants that can be used as new variables leading to a reduction of order. So we look for other surfaces that are invariant under $X^a$ and $X^b$. The characteristic equations of the twice extended infinitesimal operator $X^a_{\{2\}}$ are

$$\frac{dx}{0} = \frac{dy}{\theta} = \frac{dy_x}{\theta_x} = \frac{dy_{xx}}{\theta_{xx}} \tag{8.208}$$

with invariants

$$u = x, \qquad v = \frac{y_x}{\theta_x} - \frac{y}{\theta}, \qquad w = \frac{y_{xx}}{\theta_{xx}} - \frac{y_x}{\theta_x}. \tag{8.209}$$

The third invariant, $w$, is not actually required for what follows. Using the method of differential invariants, we construct the first-order equation,

$$\frac{dv}{du}\left(\frac{\theta_u}{\theta} - \left(\frac{f[u]}{h[u]}\right)\frac{\theta_u}{\theta_u}\right)u - \frac{F[u]}{h[u]\theta_u} = 0. \tag{8.210}$$

Note that $\theta$, $f$, $h$, and $F$ are all functions of $u$ (or $x$). Now let's consider the action of the group

$$\begin{aligned} \tilde{x} &= x, \\ \tilde{y} &= y + b\phi, \\ \tilde{y}_{\tilde{x}} &= y_x + b\phi_x \end{aligned} \tag{8.211}$$

on $u$ and $v$ (the invariants corresponding to the group parameter $a$). These transform as

$$\begin{aligned} \tilde{u} &= u, \\ \tilde{v} &= v + b\left(\frac{W}{\theta\theta_u}\right), \end{aligned} \tag{8.212}$$

where $W$ is the Wronskian

$$W = \theta\phi_u - \theta_u\phi. \tag{8.213}$$

As expected, Equation (8.210) is invariant under the group (8.212):

$$\frac{d\tilde{v}}{d\tilde{u}} + \left(\frac{\theta_{\tilde{u}}}{\theta} - \left(\frac{f[\tilde{u}]}{h[\tilde{u}]}\right)\frac{\theta_{\tilde{u}}}{\theta_{\tilde{u}}}\right)\tilde{u} - \frac{F[\tilde{u}]}{h[\tilde{u}]\theta_{\tilde{u}}}$$
$$= \frac{dv}{du} + \left(\frac{\theta_u}{\theta} - \left(\frac{f[u]}{h[u]}\right)\frac{\theta_u}{\theta_u}\right)u - \frac{F[u]}{h[u]\theta_u} = 0. \quad (8.214)$$

The integrating factor for (8.210) is

$$M = \frac{1}{A\eta - B\xi} = \frac{\theta\theta_u}{W}, \quad (8.215)$$

and the solution is

$$v = \frac{W}{\theta\theta_u}\int_u \frac{f\theta}{hW}\,du' + C_1\frac{W}{\theta\theta_u}, \quad (8.216)$$

where $C_1$, is a constant of integration. Use the expression in (8.209) to replace $v$ in (8.216) and replace $u$ with $x$:

$$y_x = \frac{\theta_x}{\theta}y + \frac{W}{\theta}\int_x \frac{f\theta}{hW}\,dx' + C_1\frac{W}{\theta}. \quad (8.217)$$

We could have carried out this entire analysis in reverse order (using $X^b$ first, then $X^a$), in which case we would have wound up with an alternative form of the solution:

$$y_x = \frac{\phi_x}{\phi}y + \frac{W}{\phi}\int_x \frac{f\phi}{hW}\,dx' + C_2\frac{W}{\phi}, \quad (8.218)$$

where $C_2$ is a constant of integration. Equate (8.218) and (8.217) and solve for $y$:

$$y[x] = \phi\int_x \frac{f\theta}{hW}\,dx' - \theta\int_x \frac{f\phi}{hW}\,dx' + C_2\phi + C_1\theta. \quad (8.219)$$

This is the classical solution of an inhomogeneous second-order linear ODE, usually derived by variation of parameters.

### 8.11.2 Solution of the Homogeneous Problem

Our analysis of the invariance condition (8.203) led to the two-parameter group (8.206) for the inhomogeneous equation. The infinitesimal transformation of $y$ requires that the two independent solutions of the homogeneous equation be known. As a result, this group is of no use in determining a solution of the homogeneous problem and thus the full solution. Let's reexamine the invariance condition (8.203) with $F = 0$:

$$(\eta f + \eta_x g + \eta_{xx} h - \eta_y f y) + (2\eta_{xy} h) y_x + (h\eta_{yy}) y_x^2 = 0. \tag{8.220}$$

Now the determining equations are

$$\begin{aligned} \eta_{yy} &= 0, \\ \eta_{xy} &= 0, \\ \eta f + \eta_x g + \eta_{xx} h - f \eta_y y &= 0 \end{aligned} \tag{8.221}$$

with infinitesimals

$$\xi = 0, \qquad \eta = a\theta[x] + b\phi[x] + cy. \tag{8.222}$$

The finite group corresponding to $c$ is

$$\tilde{x} = x, \qquad \tilde{y} = e^c y. \tag{8.223}$$

Here we have picked up the invariance of the homogeneous equation under an arbitrary stretching of $y$. The commutator table of (8.222) is given in Table 8.3; it can be seen to be a solvable Lie algebra with ideal $X^a$, $X^b$. The characteristic equations of (8.223) are

$$\frac{dx}{0} = \frac{dy}{y} = \frac{dy_x}{y_x} = \frac{dy_{xx}}{y_{xx}} \tag{8.224}$$

with integrals

$$u = x, \qquad v = \frac{y_x}{y}. \tag{8.225}$$

By the method of differential invariants the homogeneous second-order equation reduces to the first-order Ricatti equation

$$\frac{dv}{du} = \frac{f}{h} + \frac{g}{h} v + v^2. \tag{8.226}$$

Table 8.3. *Commutator table for the homogeneous problem.*

| | $X^a$ | $X^b$ | $X^c$ |
|---|---|---|---|
| $X^a$ | 0 | 0 | $X^a$ |
| $X^b$ | 0 | 0 | $X^b$ |
| $X^c$ | $-X^a$ | $-X^b$ | 0 |

Although the Lie algebra in Table 8.3 is solvable, we can't go beyond this point, since the infinitesimals of the remaining two groups require that the solution of the homogeneous equation be known.

We have a few options: The phase-plane method described in Chapter 3 can be used to analyze (8.226) for given functions $f$, $g$, $h$. For example if $f$, $g$, and $h$ are constants, a solution of (8.226) is $v[u] = \text{constant}$. Let $v = \lambda$; then (8.226) becomes

$$\frac{f}{h} + \frac{g}{h}\lambda + \lambda^2 = 0. \tag{8.227}$$

Solving the quadratic for $\lambda$ and solving $\lambda = y_x/y$ leads to the well-known exponential solutions $(\theta, \phi) = (A \exp[\alpha x], B \exp[\beta x])$. All the classical cases of second-order linear ODEs can be studied this way.

## 8.12 Concluding Remarks

In Section 8.10 it was stated that if a $p$th- order equation admits a $q$-parameter solvable Lie algebra then it can be reduced to order $p - q$. It is important to realize that this is not necessarily the end of the story. At each step in the reduction process there exists the possibility that the reduced equation may admit new symmetries. It is therefore useful to search for such symmetries in order to exhaust all the possibilities for reduction of order. An example where this occurs is discussed in Section 6.2 of Hydon [8.7].

Much more could be said about ODEs. However, we will leave the subject at this point and in the next chapter develop the tools of symmetry analysis for PDEs. The most important points to take away from this chapter are that the group symmetries of a second- or higher-order ODE can be found systematically and that the solvability of the Lie algebra determines how useful the symmetries will be in the reduction of the order of the equation. ODEs will continue to pop up throughout the text as we look at examples where the reduction of a PDE leads to an ODE.

## 8.13 Exercises

8.1 Show that $f = \eta/\xi$ is a solution of (8.58).

8.2 Write down the invariance condition and work out the determining equations for

$$y_{xx} - x - ay^2 = 0. \tag{8.228}$$

Show that the determining equations have no solution and therefore the equation has no symmetry.

8.3 A problem of sound propagation through a free shear flow [8.4] leads to the second-order, nonlinear ODE

$$y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 - a^2y^3 = 0. \tag{8.229}$$

Show by inspection that the equation is invariant under a two-dimensional Lie algebra. Use the package **`IntroToSymmetry.m`** to find the symmetries and compare results. Determine the general solution, and draw the phase portrait. Identify any invariant solution trajectories.

8.4 Use the package to work out the infinitesimal group of

$$y_{xx} = a(y)^{-3}. \tag{8.230}$$

Find the general solution, and check invariance.

8.5 Use the package to work out the symmetries of the equation

$$y_{xx} = x^n y^2 \tag{8.231}$$

for general $n$. By hand, work out the symmetries for the following cases:

(i) $n = -5$
(ii) $n = -\frac{15}{7}$
(iii) $n = -\frac{20}{7}$

Compare your results with Chapter 4 in Stephani [8.5].

8.6 Consider the undamped spring mass system shown in Figure 8.2. The equation of motion is

$$m\frac{d^2x}{dt^2} + kx = 0. \tag{8.232}$$

Determine the unforced solution, and draw the phase portrait. Work out the determining equations of the group, and solve for the infinitesimals. Compare your results with the paper by Wulfman and Wybourne [8.4].

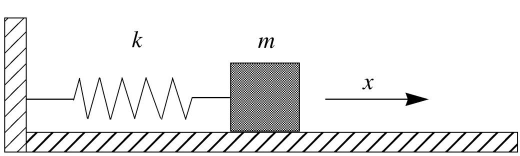


Fig. 8.2.

Carefully study their discussion of the origin of periodic time in the structure of the Lie algebra.

8.7 The spring constant in Exercise 8.6 may be a nonlinear function of $x$. Use group theory to determine the unforced solution for each of the following cases (i) $k[x]=a|x|$; (ii) $k[x]=ax^2$. Draw the phase portrait for each case.

8.8 Use the package **IntroToSymmetry.m** to identify a three-dimensional Lie algebra of

$$yy_{xx} + (y_x)^2 = 1, \tag{8.233}$$

Is the Lie algebra solvable? Obtain the general solution and draw the phase portrait.

8.9 Obtain the general solution of

$$xy^2 y_{xx} - xy_x + y = 0, \tag{8.234}$$

and draw the phase portrait.

8.10 Use the package **IntroToSymmetry.m** to show that the equation

$$y_{xx} - (2/y)y_x^2 - (1/x)y_x - y^2/x = 0 \tag{8.235}$$

admits a five parameter Lie algebra.

8.11 A problem involving wave propagation through an inhomogeneous medium leads to the following rather nasty-looking nonlinear third-order ODE:

$$y_{xxx} + \left(\frac{y_x}{y}\right)y_{xx} - \left(\frac{2}{y_x}\right)(y_{xx})^2 + \frac{y_x}{y^2} = 0. \tag{8.236}$$

(1) Use the package **IntroToSymmetry.m** to find the infinitesimals of the group that leaves the equation invariant. Check the correctness of your result by inspection.

(2) Solve the equation by the following steps:

(i) Use the method of differential invariants to reduce the problem to the solution of a first-order ODE plus two integrations.

(ii) Sketch the phase portrait of the reduced system; locate and identify any critical points.

(iii) Find an integrating factor, and solve the first-order ODE analytically. Use your result to carry out the remaining two integrations and write down the general solution of the equation.

8.12 The following fifth-order nonlinear system comes up in the context of a buoyancy-driven flow in a container:

$$\begin{aligned}
\frac{dy_1}{dt} &= -\sigma y_1 + \sigma r y_2 - \sigma s y_4, \\
\frac{dy_2}{dt} &= -y_2 + y_1 - y_1 y_3, \\
\frac{dy_3}{dt} &= -\omega y_3 + \omega y_1 y_2, \\
\frac{dy_4}{dt} &= -\tau y_4 + y_1 - y_1 y_5, \\
\frac{dy_5}{dt} &= -\omega\tau y_5 + \omega y_1 y_4,
\end{aligned} \tag{8.237}$$

where $\sigma$, $\omega$, $r$, $s$, and $\tau$ are real constants. Use the package **`IntroToSymmetry.m`** to search for invariant groups, and see how far you can reduce the system.

## References

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# 9

## *Partial Differential Equations*

In Chapter 8 we considered group transformations in the plane involving one dependent and one independent variable. This is appropriate for the treatment of ODEs. Now it is time to go on to PDEs and consider groups with several dependent and independent variables. Generally speaking, the ideas presented are elementary extensions of those presented previously. PDEs are treated as differential functions in a space whose coordinates include independent variables, dependent variables, and partial derivatives. The procedure for extending the group to include transformations of partial derivatives is essentially the same as that used in the plane, and the formulas for the group extensions have much the same form when written in terms of the total differentiation operator $D$. As in the two-variable case, the extended transformation in many variables is guaranteed to be a group, thus ensuring that it is is a one-to-one invertible map in the higher-order tangent space of the transformation.

The infinitesimal invariance condition for a system of PDEs is derived in the usual way from the group definition and an expansion of the system in a Lie series. The invariance condition leads to the determining equations of the group. These determining equations form a (usually highly overdetermined) system of linear partial differential equations for the unknown infinitesimals. The infinitesimals and associated Lie algebra define the fundamental symmetries of the system of equations in question. In the vast majority of cases the determining equations can be solved, and over the past century these basic *point* symmetries have been identified for practically all of the important equations of mathematical physics. A very complete collection of results is contained in the compilation edited by Ibragimov [9.1].

More recently there has been a great deal of interest in Lie–Bäcklund transformations. These are higher-order tangent groups in which the transformations of dependent and independent variables can depend on derivatives up to arbitrary order. In contrast to point groups, relatively little is known about the

Lie–Bäcklund structure of many important equations. These is especially so for higher-order transformations of complex systems, where the computational effort needed to identify symmetries is immense and can only be contemplated if the process is automated. These transformations are the main subject of Chapter 14.

## 9.1 Finite Transformation of Partial Derivatives

Consider the finite one-parameter Lie point group, $T^s$, in several variables:

$$T^s : \begin{Bmatrix} \tilde{x}^j &= F^j[\boldsymbol{x}, \boldsymbol{y}, s], \;\; j = 1, \ldots, n \\ \tilde{y}^i &= G^i[\boldsymbol{x}, \boldsymbol{y}, s], \;\; i = 1, \ldots, m \end{Bmatrix}. \tag{9.1}$$

### 9.1.1 Finite Transformation of the First Partial Derivative

The first partial derivative is required to satisfy the first-order contact condition

$$d\tilde{y}^i - \tilde{y}^i_\alpha \, d\tilde{x}^\alpha = 0, \tag{9.2}$$

where the sum is over $\alpha = 1, \ldots, n$. To begin, prolong the group by taking differentials of (9.1). In terms of the total differentiation operator the transformations of differentials are the following:

$$\begin{aligned} d\tilde{x}^\alpha &= (D_\beta F^\alpha)\, dx^\beta, \\ d\tilde{y}^i &= (D_\beta G^i)\, dx^\beta. \end{aligned} \tag{9.3}$$

Substitute (9.3) into (9.2):

$$\left(D_\beta G^i - \tilde{y}^i_\alpha D_\beta F^\alpha\right) dx^\beta = 0. \tag{9.4}$$

The differentials $dx^\beta$ are independent quantities. Therefore, in order for (9.4) to be satisfied, the expression in parentheses must be zero:

$$D_\beta G^i - \tilde{y}^i_\alpha D_\beta F^\alpha = 0, \qquad i = 1, \ldots, m. \tag{9.5}$$

Assume that the determinant of the Jacobian of the transformation is nonzero, $|| D_\beta F^\alpha || \neq 0$. Then the inverse of $D_\beta F^\alpha$ exists such that

$$D_\beta F^\alpha (D_j F^\beta)^{-1} = \delta^\alpha_j. \tag{9.6}$$

Right-multiply both terms in (9.5) by $(D_j F^\beta)^{-1}$:

$$D_\beta G^i (D_j F^\beta)^{-1} - \tilde{y}^i_\alpha D_\beta F^\alpha (D_j F^\beta)^{-1} = 0, \tag{9.7}$$

or

$$D_\beta G^i (D_j F^\beta)^{-1} - \tilde{y}^i_\alpha \delta^\alpha_j = 0. \tag{9.8}$$

Noting that $\tilde{y}^i_j = \tilde{y}^i_\alpha \delta^\alpha_j$, the finite transformation of first partial derivatives is now determined:

$$\tilde{y}^i_j = D_\beta G^i (D_j F^\beta)^{-1}. \tag{9.9}$$

The once extended finite transformation is

$$\begin{aligned} \tilde{x}^j &= F^j[\boldsymbol{x}, \boldsymbol{y}, s], \qquad j = 1, \ldots, n, \\ \tilde{y}^i &= G^i[\boldsymbol{x}, \boldsymbol{y}, s], \qquad i = 1, \ldots, m, \\ \tilde{y}^i_j &= G^i_{\{j\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, s], \end{aligned} \tag{9.10}$$

where $\boldsymbol{y}_1$ is the vector of all possible first partial derivatives and where

$$G^i_{\{j\}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, s) = D_\beta G^i (D_j F^\beta)^{-1}. \tag{9.11}$$

The extended group (9.10) can be shown to be a Lie group using the same approach used in Chapter 8, Section 8.1.2 to prove the group property in the case of one dependent and one independent variable. By induction, the transformation (9.1) extended to all higher partial derivatives is a group.

### *9.1.2 Finite Transformation of Second and Higher Partial Derivatives*

The once extended group (9.10) satisfies the second-order contact condition

$$d\tilde{y}^i_{j_1} - \tilde{y}^i_{j_1\alpha}\, d\tilde{x}^\alpha = 0. \tag{9.12}$$

Take differentials of (9.10):

$$\begin{aligned} d\tilde{x}^\alpha &= (D_\beta F^\alpha)\, dx^\beta, \\ d\tilde{y}^i_{j_1} &= \left(D_\beta G^i_{\{j\}}\right) dx^\beta. \end{aligned} \tag{9.13}$$

Substitute (9.13) into (9.12), and solve for $\tilde{y}^i_{j_1 j_2}$ using the same procedure as in the last section:

$$\tilde{y}^i_{j_1 j_2} = D_\beta G^i_{\{j_1\}} (D_{j_2} F^\beta)^{-1} = G^i_{\{j_1 j_2\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, s], \tag{9.14}$$

where $\boldsymbol{y}_2$ refers to the vector of all possible second partial derivatives.

Similarly the transformation (9.10) is extended to the $p$th derivative by utilizing the contact conditions,

$$\begin{aligned} d(\tilde{y}^i) - \tilde{y}^i_\alpha\, d\tilde{x}^\alpha &= 0, \\ d\big(\tilde{y}^i_{j_1}\big) - \tilde{y}^i_{j_1\alpha}\, d\tilde{x}^\alpha &= 0, \\ &\vdots \\ d\big(\tilde{y}^i_{j_1 j_2 \ldots j_{p-1}}\big) - \tilde{y}^i_{j_1 j_2 \ldots j_p\alpha}\, d\tilde{x}^\alpha &= 0, \end{aligned} \tag{9.15}$$

at successive orders. The $p$th extended finite group is

$$\begin{aligned} \tilde{x}^j &= F^j[\boldsymbol{x}, \boldsymbol{y}, s], \qquad j = 1, \ldots, n, \\ \tilde{y}^i &= G^i[\boldsymbol{x}, \boldsymbol{y}, s], \qquad i = 1, \ldots, m, \\ \tilde{y}^i_{j_1} &= G^i_{\{j_1\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, s], \\ &\vdots \\ \tilde{y}^i_{j_1 j_2 \ldots j_p} &= G^i_{\{j_1 j_2 \ldots j_p\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p, s], \end{aligned} \tag{9.16}$$

where

$$G^i_{\{j_1 j_2 \ldots j_p\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p, s] = D_\beta G^i_{\{j_1 j_2 \ldots j_{p-1}\}}(D_{j_p} F^\beta)^{-1}. \tag{9.17}$$

As usual, $s$ is the group parameter which defines the mapping from the source space

$$(\boldsymbol{x}, \boldsymbol{y}[\boldsymbol{x}], \boldsymbol{y}_1[\boldsymbol{x}], \boldsymbol{y}_2[\boldsymbol{x}], \ldots, \boldsymbol{y}_p[\boldsymbol{x}]) \tag{9.18}$$

to the target space

$$(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}[\tilde{\boldsymbol{x}}], \tilde{\boldsymbol{y}}_1[\tilde{\boldsymbol{x}}], \tilde{\boldsymbol{y}}_2[\tilde{\boldsymbol{x}}], \ldots, \tilde{\boldsymbol{y}}_p[\tilde{\boldsymbol{x}}]). \tag{9.19}$$

Recall the notation adopted in Chapter 7. The indices $j_1 j_2 \ldots j_p$ on the left-hand side of (9.16) refer to differentiation with respect to any possible combination of independent variables, while the same lower indices with curly braces on the right-hand side of (9.16) are part of the function name, i.e., they act as function labels (as do the superscripts $i$).

For example, we might need to differentiate the first extension with respect to one of the independent variables. We would write

$$\frac{\partial G^i_{\{j_1\}}}{\partial x^{j_2}} = G^i_{\{j_1\}_x j_2}, \qquad \text{or just} \qquad \frac{\partial G^i_{\{j_1\}}}{\partial x^{j_2}} = G^i_{\{j_1\} j_2}. \tag{9.20}$$

The expressions in (9.20) would read: take the function, $G^i_{\{j_1\}}$, which defines the transformation of $y^i_{j_1}$ under the once extended group, and differentiate with respect to $x^{j_2}$. Naming functions this way avoids the problem of having to introduce a whole zoo of function names – and, most importantly, *this notation ensures a transparent association between a function name and the derivative it transforms*.

### 9.1.3 Variable Count

The groups considered thus far in this text are point transformations extended to derivatives of order $p$, (9.16). These transformations are closed in the tangent space

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_p) \tag{9.21}$$

with

$$q = n + m + mn + m\left(\frac{(n+1)!}{2!(n-1)!}\right) + m\left(\frac{(n+2)!}{3!(n-1)!}\right) + \cdots + m\left(\frac{(n+p-1)!}{p!(n-1)!}\right) \tag{9.22}$$

dimensions. The indistinguishability of partial derivatives with respect to the order of differentiation is taken into account in (9.22). One can write (9.22) concisely as

$$q = n + m\sum_{k=0}^{p}\frac{(n+k-1)!}{k!(n-1)!}. \tag{9.23}$$

A plot of (9.23) for the case of one dependent variable and two independent variables is shown in Figure 9.1.

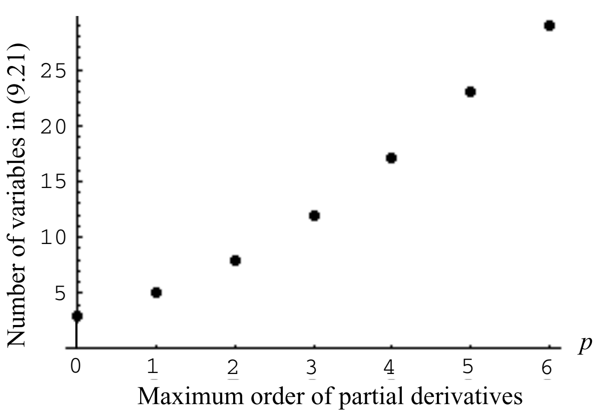


Fig. 9.1. Number of variables versus derivative order for $m = 1$, $n = 2$.

### *9.1.4 Infinitesimal Transformation of First Partial Derivatives*

The infinitesimal transformation corresponding to (9.1) is

$$T^s : \begin{Bmatrix} \tilde{x}^j = x^j + s\xi^j[\boldsymbol{x}, \boldsymbol{y}], & j = 1, \dots, n \\ \tilde{y}^i = y^i + s\eta^i[\boldsymbol{x}, \boldsymbol{y}], & i = 1, \dots, m \end{Bmatrix}, \tag{9.24}$$

generated by expanding (9.1) in a Taylor series about $s = 0$. The infinitesimals are

$$\boxed{\xi^j[\boldsymbol{x}, \boldsymbol{y}] = \left(\frac{\partial F^j}{\partial s}\right)_{s=0}, \qquad \eta^i[\boldsymbol{x}, \boldsymbol{y}] = \left(\frac{\partial G^i}{\partial s}\right)_{s=0}.} \tag{9.25}$$

Substitute $F^\beta = x^\beta + s\xi^\beta$ and $G^i = y^i + s\eta^i$ into (9.9) to produce

$$\tilde{y}^i_j = (D_\beta(y^i + s\eta^i))(D_j(x^\beta + s\xi^\beta))^{-1}. \tag{9.26}$$

Carry out the differentiation indicated in (9.26):

$$\tilde{y}^i_j = \left(y^i_\beta + sD_\beta\eta^i\right)\left(\delta^\beta_j + sD_j\xi^\beta\right)^{-1}. \tag{9.27}$$

The group parameter $s$ is assumed to be small, and so the matrix inverse can be approximated using

$$\left(\delta^\beta_j + sD_j\xi^\beta\right)^{-1} \approx \delta^\beta_j - s(D_j\xi^\beta). \tag{9.28}$$

To derive (9.28) we have used the general exponential form of a matrix. Let $A^\beta_j = D_j\xi^\beta$; then the matrix $\exp(sA^\beta_j) \approx \delta^\beta_j + sA^\beta_j + O(s^2) + \cdots$ has the inverse $\exp(-sA^\beta_j) = (\delta^\beta_j + sA^\beta_j + O(s^2) + \cdots)^{-1} \approx \delta^\beta_j - sA^\beta_j + O(s^2) - \cdots$. Using this result, Equation (9.27) becomes

$$\tilde{y}^i_j = \left(y^i_\beta + sD_\beta\eta^i\right)\left(\delta^\beta_j - sD_j\xi^\beta\right). \tag{9.29}$$

Retaining only lowest-order terms in $s$, the infinitesimal form of the transformation of first partial derivatives is now determined to be

$$\tilde{y}^i_j = y^i_j + s\left(D_j\eta^i - y^i_\beta D_j\xi^\beta\right), \tag{9.30}$$

and the once extended infinitesimal group is

$$\begin{aligned} \tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x}, \boldsymbol{y}], & j &= 1, \dots, n, \\ \tilde{y}^i &= y^i + s\eta^i[\boldsymbol{x}, \boldsymbol{y}], & i &= 1, \dots, m, \\ \tilde{y}^i_j &= y^i_j + s\eta^i_{\{j\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1], \end{aligned} \tag{9.31}$$

where

$$\eta^i_{\{j\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1] = D_j \eta^i - y^i_\beta D_j \xi^\beta. \tag{9.32}$$

### 9.1.5 Infinitesimal Transformation of Second and Higher Partial Derivatives

The transformation of second partial derivatives is generated in the same way. Substitute $F^\beta = x^\beta + s\xi^\beta$ and $G^i_{\{j_1\}} = y^i_{j_1} + s\eta^i_{\{j_1\}}$ into the finite transformation (9.14):

$$\tilde{y}^i_{j_1 j_2} = \left(D_\beta\left(y^i_{j_1} + s\eta^i_{\{j_1\}}\right)\right)\left(D_{j_2}\left(x^\beta + s\xi^\beta\right)\right)^{-1}. \tag{9.33}$$

Carry out the differentiation indicated in (9.33):

$$\tilde{y}^i_{j_1 j_2} = \left(y^i_{j_1\beta} + sD_\beta \eta^i_{\{j_1\}}\right)\left(\delta^\beta_{j_2} + sD_{j_2}\xi^\beta\right)^{-1}. \tag{9.34}$$

Approximating the inverse using (9.28) and retaining only the lowest-order term in $s$ produces the infinitesimal transformation of second partial derivatives,

$$\tilde{y}^i_{j_1 j_2} = y^i_{j_1 j_2} + s\left(D_{j_2}\eta^i_{\{j_1\}} - y^i_{j_1\beta} D_{j_2}\xi^\beta\right). \tag{9.35}$$

The twice extended group is

$$\begin{aligned}
\tilde{x}^j &= x^j + \xi^j[\boldsymbol{x}, \boldsymbol{y}]s, \\
\tilde{y}^i &= y^i + \eta^i[\boldsymbol{x}, \boldsymbol{y}]s, \\
\tilde{y}^i_{j_1} &= y^i_{j_1} + \eta^i_{\{j_1\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1]s, \\
\tilde{y}^i_{j_1 j_2} &= y^i_{j_1 j_2} + \eta^i_{\{j_1 j_2\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2]s,
\end{aligned} \tag{9.36}$$

where

$$\eta^i_{\{j_1 j_2\}} = D_{j_2}\eta^i_{\{j_1\}} - y^i_{j_1\beta} D_{j_2}\xi^\beta. \tag{9.37}$$

Just as a reminder, the total differentiation operator acting on the first extension is

$$D_{j_2}\eta^i_{\{j_1\}} = \frac{\partial \eta^i_{\{j_1\}}}{\partial x^{j_2}} + y^\alpha_{j_2}\frac{\partial \eta^i_{\{j_1\}}}{\partial y^\alpha} + y^\alpha_{\beta j_2}\frac{\partial \eta^i_{\{j_1\}}}{\partial y^\alpha_\beta}. \tag{9.38}$$

One continues to use the contact conditions and the finite form of the transformation at successive orders. For $(p-1)$th derivatives the extended infinitesimal transformation is

$$\begin{aligned}
\tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x}, \boldsymbol{y}], \\
\tilde{y}^i &= y^i + s\eta^i[\boldsymbol{x}, \boldsymbol{y}], \\
\tilde{y}^i_{j_1} &= y^i_{j_1} + s\eta^i_{\{j_1\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1], \\
\tilde{y}^i_{j_1 j_2} &= y^i_{j_1 j_2} + s\eta^i_{\{j_1 j_2\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2], \\
&\vdots \\
\tilde{y}^i_{j_1 j_2 \dots j_{p-1}} &= y^i_{j_1 j_2 \dots j_{p-1}} + s\eta^i_{\{j_1 j_2 \dots j_{p-1}\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \dots, \boldsymbol{y}_{p-1}].
\end{aligned} \tag{9.39}$$

The vector $\boldsymbol{y}_p$ refers to all possible $p$th derivatives. The $p$th-order infinitesimal extension is derived from the $p$th-order finite transformation (9.17). Substitute $F^\beta = x^\beta + s\xi^\beta$ and $G^i_{\{j_1 j_2 \dots j_{p-1}\}} = y^i_{j_1 j_2 \dots j_{p-1}} + s\eta^i_{\{j_1 j_2 \dots j_{p-1}\}}$ into (9.17). Carrying out the differentiation and approximating the matrix inverse using (9.28) gives the infinitesimal transformation of the $p$th partial derivative:

$$\tilde{y}^i_{j_1 j_2 \dots j_p} = y^i_{j_1 j_2 \dots j_p} + s\eta^i_{\{j_1 j_2 \dots j_p\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \dots, \boldsymbol{y}_p], \tag{9.40}$$

where

$$\boxed{\eta^i_{\{j_1 j_2 \dots j_p\}} = D_{j_p}\eta^i_{\{j_1 j_2 \dots j_{p-1}\}} - y^i_{j_1 j_2 \dots j_{p-1}\alpha} D_{j_p}\xi^\alpha.} \tag{9.41}$$

Note that the indices $j_1, j_2, \dots, j_p$ refer to any combination of $p$ of the independent variables. The total differentiation operator appearing in (9.41) is

$$\begin{aligned}
D_{j_p}(\ ) = \frac{\partial(\ )}{\partial x^{j_p}} &+ y^i_{j_p}\frac{\partial(\ )}{\partial y^i} + y^i_{j_1 j_p}\frac{\partial(\ )}{\partial y^i_{j_1}} \\
&+ y^i_{j_1 j_2 j_p}\frac{\partial(\ )}{\partial y^i_{j_1 j_2}} + \cdots + y^i_{j_1 j_2 \dots j_{p-1} j_p}\frac{\partial(\ )}{\partial y^i_{j_1 j_2 \dots j_{p-1}}}.
\end{aligned} \tag{9.42}$$

In Figure 9.1 we plotted the number of variables in the higher-order tangent space defined by the variables in Equation (9.21). This number increases sharply with increasing derivative order $p$. In Figure 9.2 the number of terms in the expression for a given infinitesimal transformation of a $p$th derivative is plotted for the case of one dependent and two independent variables. As can be seen in the figure, this number of terms increases hugely with $p$. This is precisely why group methods have been so slow to be incorporated into the mainstream curricula in science and engineering. Together Figure 9.1 and Figure 9.2 make a compelling argument for the need to automate the procedure for analyzing the group symmetries of ODEs and PDEs.

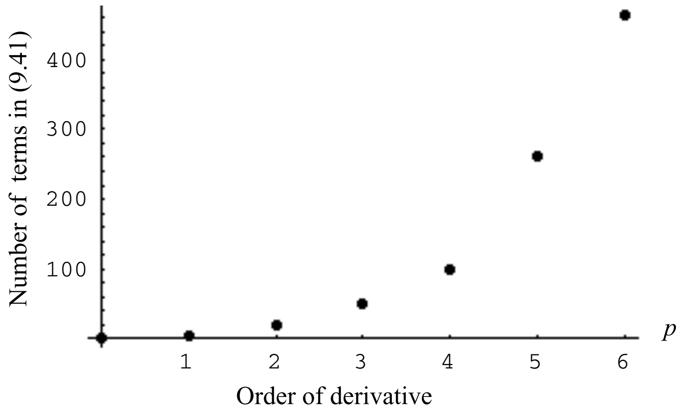


Fig. 9.2. Plot of the number of terms in the $p$th-order infinitesimal for the case $m = 1$, $n = 2$.

### *9.1.6 Invariance of the Contact Conditions*

The transformation (9.31) leaves invariant the first-order contact condition

$$dy^i - y^i_j\, dx^j = 0. \tag{9.43}$$

To prove that (9.43) is invariant under the group (9.1), we use the once extended infinitesimal transformation including the prolongation to include transformations of differentials:

$$\begin{aligned}
\tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x}, \boldsymbol{y}], \qquad j = 1, \dots, n,\\
\tilde{y}^i &= y^i + s\eta^i[\boldsymbol{x}, \boldsymbol{y}], \qquad i = 1, \dots, m,\\
\tilde{y}^i_j &= y^i_j + s\eta^i_{\{j\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1],\\
d\tilde{x}^j &= dx^j + s\left(\frac{\partial \xi^j}{\partial x^\alpha}\, dx^\alpha + \frac{\partial \xi^j}{\partial y^\beta}\, dy^\beta\right) = dx^j + s\, d\xi^j,\\
d\tilde{y}^i &= dy^i + s\left(\frac{\partial \eta^i}{\partial x^\alpha}\, dx^\alpha + \frac{\partial \eta^i}{\partial y^\beta}\, dy^\beta\right) = dy^i + s\, d\eta^i.
\end{aligned} \tag{9.44}$$

The prolonged group operator corresponding to (9.44) is

$$\hat{X}_{\{1\}} = \xi^j \frac{\partial}{\partial x^j} + \eta^i \frac{\partial}{\partial y^i} + \eta^i_{\{j\}} \frac{\partial}{\partial y^i_j} + d\xi^j\, \frac{\partial}{\partial (dx^j)} + d\eta^i\, \frac{\partial}{\partial (dy^i)}, \tag{9.45}$$

where the hat over the operator symbol is used to distinguish the operator of the prolonged group from the conventional operator. Apply this operator to the contact condition (9.43):

$$\hat{X}_{\{1\}}\big(dy^i - y^i_j\, dx^j\big) = d\eta^i - y^i_j\, d\xi^j - \eta^i_{\{j\}}\, dx^j. \tag{9.46}$$

Equation (9.46) can be rearranged to read, after canceling terms,

$$\hat{X}_{\{1\}}\left(dy^i - y^i_j\, dx^j\right) = \left(\frac{\partial \eta^i}{\partial y^\alpha} - y^i_\beta \frac{\partial \xi^\beta}{\partial y^\alpha}\right)\left(dy^\alpha - y^\alpha_j\, dx^j\right) = 0. \quad (9.47)$$

See Appendix 2 for details. The result (9.47) shows that (9.43) is invariant under the group (9.1). That is, the Lie series for (9.43), expanded in terms of $\hat{X}_{\{1\}}$, truncates to

$$d\tilde{y}^i - \tilde{y}^i_j\, d\tilde{x}^j = dy^i - y^i_j\, dx^j. \quad (9.48)$$

When the transformation of $(\boldsymbol{x}, \boldsymbol{y})$ to $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ space is carried out, the tangent to a point in $(\boldsymbol{x}, \boldsymbol{y})$ will be transformed to the tangent of the transformed point in $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$. It follows from the group property of the once-extended transformation that the mapping of source points and first partial derivatives to target points and first partial derivatives is one-to-one and invertible. The same approach is used to prove the invariance of all higher-order contact conditions. Appendix 2 contains the proof of this point for contact conditions of arbitrary order. It is summarized in the following theorem.

***Theorem 9.1.*** *The Lie point group*

$$T^s : \left\{\begin{aligned} \tilde{x}^j &= F^j[\boldsymbol{x}, \boldsymbol{y}, s], \quad j = 1, \ldots, n \\ \tilde{y}^j &= G^i[\boldsymbol{x}, \boldsymbol{y}, s], \quad i = 1, \ldots, m \end{aligned}\right\}, \quad (9.49)$$

*extended to include the transformation of pth derivatives using the algorithm described above preserves the pth-order contact conditions*

$$\begin{aligned} d\tilde{y}^i - \tilde{y}^i_\alpha\, d\tilde{x}^\alpha &= dy^i - y^i_\alpha\, dx^\alpha = 0, \\ d\tilde{y}^i_{j_1} - \tilde{y}^i_{j_1\alpha}\, d\tilde{x}^\alpha &= dy^i_{j_1} - y^i_{j_1\alpha}\, dx^\alpha = 0, \\ &\ \ \vdots \\ d\tilde{y}^i_{j_1 j_2\cdots j_{p-1}} - \tilde{y}^i_{j_1 j_2\cdots j_p\alpha}\, d\tilde{x}^\alpha &= dy^i_{j_1 j_2\cdots j_{p-1}} - y^i_{j_1 j_2\cdots j_p\alpha}\, dx^\alpha = 0. \end{aligned} \quad (9.50)$$

*By induction the extended group preserves tangency to any order.*

The transformation of variables and derivatives is an invertible map. If two curves in the source space $(\boldsymbol{x}, \boldsymbol{y})$ are in contact to some arbitrary order, say $p$, then the two image curves in the target space $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ will also possess $p$th-order tangency. Tangency to infinite order is preserved by the infinitely extended group.

## 9.2 Expansion of a PDE in a Lie Series; Invariance Condition for PDEs

With the infinitesimal transformations of partial derivatives in hand, we are now in a position to generalize our invariance condition to systems of PDEs.

***Theorem 9.2.*** *The pth-order system of partial differential equations* $\psi^i = \Psi^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_p] = 0$ *is a vector of locally analytic functions of the differential variables* $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_p$. *Expand* $\Psi^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_p]$ *in a Lie series*

$$\Psi^i[\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}_1, \tilde{\boldsymbol{y}}_2, \ldots, \tilde{\boldsymbol{y}}_p]$$

$$= \Psi^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_p] + sX_{\{p\}}\Psi^i + \frac{s^2}{2!}X_{\{p\}}\big(X_{\{p\}}\Psi^i\big) + \cdots, \quad (9.51)$$

*where* $X_{\{p\}}$ *is the pth extended group operator*

$$X_{\{p\}} = \xi^j \frac{\partial}{\partial x^j} + \eta^i \frac{\partial}{\partial y^i} + \eta^i_{\{j_1\}} \frac{\partial}{\partial y^i_{j_1}} + \eta^i_{\{j_1 j_2\}} \frac{\partial}{\partial y^i_{j_1 j_2}} + \cdots + \eta^i_{\{j_1 j_2 \ldots j_p\}} \frac{\partial}{\partial y^i_{j_1 j_2 \ldots j_p}}. \quad (9.52)$$

*The system* $\Psi^i$ *is invariant under the group* $(\xi^j, \eta^i)$ *if and only if*

$$\boxed{X_{\{p\}}\Psi^i = 0, \qquad i = 1, \ldots, m.} \quad (9.53)$$

The characteristic equations corresponding to (9.53) are

$$\boxed{\frac{dx^j}{\xi^j} = \frac{dy^i}{\eta^i} = \frac{dy^i_{j_1}}{\eta^i_{\{j_1\}}} = \frac{dy^i_{j_1 j_2}}{\eta^i_{\{j_1 j_2\}}} = \cdots = \frac{dy^i_{j_1 j_2 \ldots j_p}}{\eta^i_{\{j_1 j_2 \ldots j_p\}}}.} \quad (9.54)$$

### *9.2.1 Isolating the Determining Equations of the Group – The Lie Algorithm*

Generally the invariance condition (9.53) contains enough information to determine the unknown infinitesimals $(\xi^j, \eta^i)$ for a given system of PDEs. The strategy for finding the infinitesimals is as follows.

*Step 1.* Note that there is one invariance condition for each equation in the system $\Psi^i = 0$. The invariance condition (9.53) is generally a rather long sum. Each term in (9.53) is of the form $AB$ where $A$ is some partial derivative of $\xi^j$ or $\eta^i$ and $B$ is in general a product of partial derivatives of the various $y^i$. Begin by gathering together terms that multiply the same combinations of derivatives of the $y^i$.

*Step 2.* Attend to those combinations of derivatives of $y^i$ that appear in the original system $\Psi^i$. By assumption, the $y^i$ solve the original system, and so *all* the relations $\Psi^i = 0$ must be imposed on *each* invariance condition. This is commonly done by solving for some isolated derivative in each of the $\Psi^i$ and replacing the corresponding term(s) in the invariance condition(s). The invariance conditions are then rearranged by gathering together common products of derivatives.

*Step 3.* All the coefficients multiplying various combinations of derivatives of the $y^i$ are set equal to zero. These are the *determining equations* of the group. For a system of equations the determining equations from each invariance condition are concatenated together. This is the complete system of determining equations.

*Step 4.* The result of steps 2 and 3 is a (usually) overdetermined set of linear PDEs in the unknown infinitesimals $(\xi^j, \eta^i)$. Initial considerations of these PDEs, many of which may be redundant, permit a number of them to be eliminated, and ultimately relatively few play a role in determining the $(\xi^j, \eta^i)$.

The steps outlined above beginning with the invariance condition will be collectively called the *Lie algorithm*. The infinitesimals $(\xi^j, \eta^i)$ found by solving the determining equations form a Lie algebra. Over the last century most of the equations of mathematical physics have been explored and their point groups identified. In this and the next three chapters the point groups of several well-known equations will be explored.

### *9.2.2 The Classical Point Group of the Heat Equation*

The one-dimensional heat equation written as a differential function is

$$\phi = \Phi[x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}] = u_t - u_{xx} = 0. \tag{9.55}$$

The dimensions of the variables have been scaled so that the diffusivity is one. The infinitesimal form of the extended group that transforms a differential function with one dependent variable and two independent variables is, in general,

$$
\begin{aligned}
\tilde{x} &= x + s\xi[x, t, u], \\
\tilde{t} &= t + s\tau[x, t, u], \\
\tilde{u} &= u + s\eta[x, t, u], \\
\tilde{u}_{\tilde{x}} &= u_x + s\eta_{\{x\}}[x, t, u, u_x, u_t], \\
\tilde{u}_{\tilde{t}} &= u_t + s\eta_{\{t\}}[x, t, u, u_x, u_t], \\
\tilde{u}_{\tilde{x}\tilde{x}} &= u_{xx} + s\eta_{\{xx\}}[x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}], \\
\tilde{u}_{\tilde{x}\tilde{t}} &= u_{xt} + s\eta_{\{xt\}}[x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}], \\
\tilde{u}_{\tilde{t}\tilde{t}} &= u_{tt} + s\eta_{\{tt\}}[x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}].
\end{aligned}
\tag{9.56}
$$

The unknown infinitesimals $\xi$, $\tau$, and $\eta$ are determined from the invariance condition:

$$
\begin{aligned}
X_{\{2\}}\Phi = 0& \\
&= \xi\frac{\partial\Phi}{\partial x} + \tau\frac{\partial\Phi}{\partial t} + \eta\frac{\partial\Phi}{\partial u} + \eta_{\{x\}}\frac{\partial\Phi}{\partial u_x} + \eta_{\{t\}}\frac{\partial\Phi}{\partial u_t} \\
&\quad + \eta_{\{xx\}}\frac{\partial\Phi}{\partial u_{xx}} + \eta_{\{xt\}}\frac{\partial\Phi}{\partial u_{xt}} + \eta_{\{tt\}}\frac{\partial\Phi}{\partial u_{tt}}.
\end{aligned}
\tag{9.57}
$$

Carrying out the indicated differentiation of (9.55) produces the rather compact relation

$$
\eta_{\{t\}} - \eta_{\{xx\}} = 0 \tag{9.58}
$$

as the invariance condition for the one-dimensional heat equation. Remember, the subscripts in (9.58) are function labels, not derivatives. The correspondence between the invariance condition (9.58) and the heat equation itself is a consequence of the linearity of the equation. In (9.58) we have the immediate result that $\eta$ can be any solution of the heat equation. This simply expresses the fact that one can always "translate" a given solution of a linear equation by any function that is also a solution without changing the form of the equation. The required extensions are

$$
\begin{aligned}
\eta_{\{x\}} &= D_x\eta - u_x D_x\xi - u_t D_x\tau, \\
\eta_{\{t\}} &= D_t\eta - u_x D_t\xi - u_t D_t\tau, \\
\eta_{\{xx\}} &= D_x\eta_x - u_{xx} D_x\xi - u_{xt} D_x\tau.
\end{aligned}
\tag{9.59}
$$

Expanding the total differentiation operators with respect to $x$ and $t$, the two first-order infinitesimals are

$$\begin{aligned} \eta_{\{x\}} &= \eta_x + u_x\eta_u - u_x(\xi_x + u_x\xi_u) - u_t(\tau_x + u_x\tau_u), \\ \eta_{\{t\}} &= \eta_t + u_t\eta_u - u_x(\xi_t + u_t\xi_u) - u_t(\tau_t + u_t\tau_u). \end{aligned} \tag{9.60}$$

The second-order infinitesimal is considerably longer, since it depends on derivatives of a first-order infinitesimal:

$$\eta_{\{xx\}} = D_x D_x \eta - 2u_{xx} D_x \xi - 2u_{xt} D_x \tau - u_x D_x D_x \xi - u_t D_x D_x \tau. \tag{9.61}$$

Fully expanded, this yields

$$\begin{aligned} \eta_{\{xx\}} = \eta_{xx} &+ u_{xx}\eta_u - u_{xx}u_t\tau_u - 2u_x u_{xt}\tau_u \\ &- 3u_x u_{xx}\xi_u + u_x^2\eta_{uu} - u_x^2 u_t \tau_{uu} \\ &- u_x^3\xi_{uu} - 2u_{xt}\tau_x - 2u_{xx}\xi_x + 2u_x\eta_{xu} \\ &- 2u_x u_t \tau_{xu} - 2u_x^2\xi_{xu} - u_t\tau_{xx} - u_x\xi_{xx}. \end{aligned} \tag{9.62}$$

Remember, the bracketed subscript refers to the name of the function that transforms a derivative, whereas an unbracketed subscript denotes partial differentiation with respect to the *explicit* dependence on the variable.

It is expressions like (9.60) and (9.62) that prompted our adoption in Chapter 7 of a clear, precise notation for distinguishing derivatives and the functions that transform derivatives. Compare the left-hand side of (9.60) with the first term on the right-hand side. On the left-hand side is $\eta_{\{x\}}$, the name of the function that transforms the first spatial derivative. The first term on the right of (9.62), $\eta_x$, stands for the partial derivative of this function with respect to its explicit dependence on $x$,

$$\eta_x = \frac{\partial \eta}{\partial x}, \tag{9.63}$$

and so forth.

Now form the fully expanded invariance condition (9.58):

$$\begin{aligned} \eta_{\{t\}} - \eta_{\{xx\}} = \eta_t &- \eta_{xx} - u_{xx}\eta_u + u_t\eta_u + u_{xx}u_t\tau_u \\ &- u_t^2\tau_u + 2u_x u_{xt}\tau_u + 3u_x u_{xx}\xi_u - u_x u_t \xi_u \\ &- u_x^2\eta_{uu} + u_x^2 u_t\tau_{uu} + u_x^3\xi_{uu} + 2u_{xt}\tau_x \\ &+ 2u_{xx}\xi_x - 2u_x\eta_{xu} + 2u_x u_t\tau_{xu} + 2u_x^2\xi_{xu} \\ &+ u_t\tau_{xx} + u_x\xi_{xx} - u_t\tau_t - u_x\xi_t = 0. \end{aligned} \tag{9.64}$$

The invariance condition (9.64) holds under the constraint that the function $u[x, t]$ is a solution of the heat equation. Replace $u_{xx}$ by $u_t$ wherever it appears in (9.64). The final form of the invariance condition is

$$\begin{aligned}\eta_{\{t\}} - \eta_{\{xx\}} = (\eta_t - \eta_{xx}) + 2u_x u_{xt}(\tau_u) + 2u_x u_t(\xi_u + \tau_{xu}) \\ + u_x^2(2\xi_{xu} - \eta_{uu}) + u_x^2 u_t(\tau_{uu}) + u_x^3(\xi_{uu}) + 2u_{xt}(\tau_x) \\ + u_t(\tau_{xx} + 2\xi_x - \tau_t) + u_x(\xi_{xx} - \xi_t - 2\eta_{xu}) = 0\end{aligned} \tag{9.65}$$

With the replacement complete, all other derivatives of $u[x, t]$ are not restricted in any way. Therefore, in order for the invariance condition to be satisfied, the various coefficients in (9.65) must be individually zero:

$$\begin{aligned} \eta_t - \eta_{xx} &= 0, & \tau_{uu} &= 0, \\ \tau_u &= 0, & \xi_{uu} &= 0, \\ \xi_u + \tau_{xu} &= 0, & \tau_x &= 0, \\ 2\xi_{xu} - \eta_{uu} &= 0, & \tau_{xx} + 2\xi_x - \tau_t &= 0, \\ \xi_{xx} - \xi_t - 2\eta_{xu} &= 0. \end{aligned} \tag{9.66}$$

These are the *determining equations* of the point group of the heat equation. Four of the equations in (9.66) have only a single term and can be immediately used to simplify some of the other equations.

The determining equations are always linear, regardless of whether the original equation is linear or nonlinear. This can be traced to the expressions for the transformations of derivatives – the group extensions (9.41) – which are linear in the unknown infinitesimals.

#### *9.2.2.1 Series Solution of the Determining Equations*

Let's use the power-series form of the solution of the determining equations discussed in Chapter 8, Section 8.6.1. Initially we will try the third-order series in (9.67). Note that we have adopted a certain convention for numbering the group parameters in (9.67), where the first index in the superscript of the group parameter marks the particular infinitesimal. The *a*-parameters go with the independent variables, and the *b*-parameters go with the dependent variables.

Various powers of the variables are then numbered consecutively. Let

$$\begin{aligned}
\xi &= a^{10} + a^{11}x + a^{12}t + a^{13}u + a^{14}x^2 + a^{15}xt \\
&\quad + a^{16}xu + a^{17}t^2 + a^{18}ut + a^{19}u^2 \\
&\quad + a^{110}x^3 + a^{111}x^2t + a^{112}x^2u + a^{113}xt^2 + a^{114}xtu \\
&\quad + a^{115}xu^2 + a^{116}t^3 + a^{117}t^2u + a^{118}tu^2 + a^{119}u^3, \\
\tau &= a^{20} + a^{21}x + a^{22}t + a^{23}u + a^{24}x^2 + a^{25}xt \\
&\quad + a^{26}xu + a^{27}t^2 + a^{28}ut + a^{29}u^2 \\
&\quad + a^{210}x^3 + a^{211}x^2t + a^{212}x^2u + a^{213}xt^2 + a^{214}xtu \\
&\quad + a^{215}xu^2 + a^{216}t^3 + a^{217}t^2u + a^{218}tu^2 + a^{219}u^3, \\
\eta &= b^{10} + b^{11}x + b^{12}t + b^{13}u + b^{14}x^2 + b^{15}xt \\
&\quad + b^{16}xu + b^{17}t^2 + b^{18}ut + b^{19}u^2 \\
&\quad + b^{110}x^3 + b^{111}x^2t + b^{112}x^2u + b^{113}xt^2 + b^{114}xtu \\
&\quad + b^{115}xu^2 + b^{116}t^3 + b^{117}t^2u + b^{118}tu^2 + b^{119}u^3.
\end{aligned} \tag{9.67}$$

The series (9.67) are substituted into the determining equations (9.66); then like powers of $x$, $t$, and $u$ are gathered together. The variables $x$, $t$, and $u$ are completely independent in the context of the infinitesimals, and so the only way the system can be satisfied is if the coefficients of various power monomials of $x$, $t$, and $u$ are each individually zero. The resulting algebraic system for the coefficients is solved symbolically for the nonzero coefficients. The power-series procedure for solving the determining equations illustrated here is the one implemented in the package **`IntroToSymmetry.m`** enclosed with this text. The function that carries out the series solution is called **`SolvDeterminingEquations`**, and the numbering system used for the coefficients is the same as that used above.

The final result is the classical six-parameter group of the heat equation,

$$\begin{aligned}
\xi &= a^{10} + b^{111}t + a^{24}x + b^{112}(xt), \\
\tau &= a^{20} + a^{24}(2t) + b^{112}(t^2), \\
\eta &= \left( -b^{112}\left(\frac{x^2}{4} + \frac{t}{2}\right) - \frac{b^{111}}{2}x + b^{110}\right)u + g(x,t), \\
&\quad \text{where} \quad g_{xx} - g_t = 0.
\end{aligned} \tag{9.68}$$

The first two infinitesimals, which act on $x$ and $t$, correspond to a subgroup of the projective group in two dimensions discussed in Chapter 5, Section 5.10.1. The last term in the third infinitesimal in (9.68) arises from superposition of solutions permitted by the linearity of the heat equation. It can be deduced using the function **SolveDeterminingEquations** by executing the function for successively increasing powers of the trial polynomials.

The choice of what order polynomial to select for the series depends on the highest power that appears in the infinitesimals. In practice, the order is increased and the solve process is repeated until the power of the result stops changing. Since the substitution and solution process is automated, one can select a reasonably long series; however, one must remember that the number of unknown coefficients in the series grows rapidly and that the process for finding the coefficients is a symbolic one and therefore may be quite slow.

The operators of the six-dimensional Lie algebra $\Lambda^6$ of the infinitesimal group (9.68) are

$$\begin{aligned}
X^1 &= \frac{\partial}{\partial x}, \\
X^2 &= t\frac{\partial}{\partial x} - \frac{xu}{2}\frac{\partial}{\partial u}, \\
X^3 &= x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \\
X^4 &= xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - \left(\frac{x^2}{4} + \frac{t}{2}\right)u\frac{\partial}{\partial u}, \\
X^5 &= \frac{\partial}{\partial t}, \\
X^6 &= u\frac{\partial}{\partial u}.
\end{aligned} \tag{9.69}$$

The commutator table of $\Lambda^6$ is shown in Table 9.1. One of its interesting features is that all six operators can be constructed from just $X^1$, $X^4$, and $X^5$.

Table 9.1. *Commutator table of the point group of the 1-D heat equation.*

| | $X^1$ | $X^2$ | $X^3$ | $X^4$ | $X^5$ | $X^6$ |
|---|---|---|---|---|---|---|
| $X^1$ | $0$ | $-\frac{1}{2}X^6$ | $X^1$ | $X^2$ | $0$ | $0$ |
| $X^2$ | $\frac{1}{2}X^6$ | $0$ | $-X^2$ | $0$ | $-X^1$ | $0$ |
| $X^3$ | $-X^1$ | $X^2$ | $0$ | $2X^4$ | $-2X^5$ | $0$ |
| $X^4$ | $-X^2$ | $0$ | $-2X^4$ | $0$ | $-X^3 + \frac{1}{2}X^6$ | $0$ |
| $X^5$ | $0$ | $X^1$ | $2X^5$ | $X^3 - \frac{1}{2}X^6$ | $0$ | $0$ |
| $X^6$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |

The logic for this is as follows:

$$\begin{aligned} &X^1 \text{ and } X^4 \text{ generate } X^2, \\ &X^1 \text{ and } X^2 \text{ generate } X^6, \\ &X^4 \text{ and } X^5 \text{ generate } X^3, \text{ knowing } X^6. \end{aligned} \tag{9.70}$$

So, for example, invariance under translation in $x$ (operator $X^1$) and under the projective group $X^2$ implies invariance under dilation in $u$ (the operator $X^6$).

It is worth asking whether the fairly elaborate, systematic procedure using the Lie algorithm just outlined is really required to identify all the symmetry groups of the heat equation. Several of the groups in (9.69) can be deduced by inspection, particularly invariance under translation of the independent variables $X^1$ and $X^5$. The operator $X^3$ corresponding to dilation of the independent variables can also be determined by inspection. The form of $X^3$ is expected given the dimensions of the diffusivity ($\hat{\kappa} = L^2T^{-1}$) and the requirement of dimensional homogeneity. The equation places no restriction on the units of $u$, and as a consequence it admits the operator $X^6$, which implies invariance under dilation of $u$ by a parameter that is independent of the parameter that dilates $x$ and $t$. Inspection of the commutator table reveals that these four operators form a subalgebra. However, the remaining two operators $X^2$ and $X^4$ cannot be generated using the commutator, nor can they be found by inspection. The bottom line here is that the Lie algorithm *is* required to fully define the Lie algebra of the heat equation.

This example illustrates a case where the infinitesimal for the dependent variable contains an arbitrary solution of the heat equation – a consequence of the linearity of the PDE. Quite often, when arbitrary functions arise, they can be detected using the package by repeating the series expansion method for successively higher orders. If the highest-order terms in the expansion continue to have nonzero coefficients as the order is increased, the presence of an arbitrary function can be surmised.

A couple of further points should be made here. The main advantage of the Lie algorithm, beginning with the invariance condition and ending with the determining equations, is that it is purely algorithmic, nearly foolproof, and therefore subject to automation. The main disadvantage, especially for higher-order systems with many variables, is that the expressions that are generated can become extraordinarily long, quickly overburdening a pencil-and-paper approach. Even when the method is automated, it is not difficult to pose a problem that brings even the most powerful computer to its knees. The software provided with the text and described in Appendix 4 is designed to generate the determining equations for essentially any system of ODEs or PDEs that is input by the user. In fact, one can use the package on essentially any differential system, including systems that may be either under- or over-determined.

Perhaps the best practical approach for identifying the symmetry groups of a given equation is to use a two-step process plus a little experience. In the first step, inspection is used to identify as many groups as possible, including translations, dilations, and perhaps rotations, with the possible use of the commutator table to enlarge the set of known groups. In the second step, the package **`IntroToSymmetry.m`** is used. First one calls the function **`FindDeterminingEquations`** to generate the determining equations of the group. Then the power-series method is applied using the function **`SolveDeterminingEquations`** in a first attempt to solve the determining equations, all the while checking the results against the known groups determined by inspection. This approach will run into difficulty when the infinitesimals contain arbitrary functions, as is the case with the third infinitesimal of the heat equation. In this case already as noted above one can use the power-series approach repeatedly with increasing orders of the trial polynomial. If the series fails to truncate, then a general function is indicated. This approach will not uncover transcendental functions. In this case, built-in *Mathematica*® functions can be used, together with whatever groups may be known to this point, to reduce the determining equations to as small a set as possible. At that stage special methods may be needed to finish the solution. Even then, all the possible solutions of the determining equations may not have been found. In Chapters 14, 15, and 16 we will look at symmetries that can depend on derivatives and/or integrals. In every case these symmetries represent additional solutions of the same determining equations used to identify point groups.

## 9.3 Invariant Solutions and the Characteristic Function

Once the symmetries of a PDE have been identified, one then turns to an examination of the boundary, initial and perhaps integral conditions that define a physical problem. If a group with operator

$$X = \xi^1 \frac{\partial}{\partial x^1} + \xi^2 \frac{\partial}{\partial x^2} + \cdots \xi^n \frac{\partial}{\partial x^n} + \eta^1 \frac{\partial}{\partial y^1} + \eta^2 \frac{\partial}{\partial y^2} + \cdots + \eta^m \frac{\partial}{\partial y^m} \tag{9.71}$$

can be found that leaves the problem as a whole invariant, then the existence of an invariant solution, $\Omega^i[\boldsymbol{x}, \boldsymbol{y}]$ , that satisfies $X\Omega^i = 0$ can be assumed. The functional form of the invariant solution is constructed from the characteristic equations of the group operator

$$\frac{dx^1}{\xi^1} = \frac{dx^2}{\xi^2} = \cdots = \frac{dx^n}{\xi^n} = \frac{dy^1}{\eta^1} = \frac{dy^2}{\eta^2} = \cdots = \frac{dy^m}{\eta^m} \tag{9.72}$$

with integrals

$$\theta^k = \Theta^k[\boldsymbol{x}, \boldsymbol{y}], \qquad k = 1, \ldots, m + n - 1. \tag{9.73}$$

These integrals become the similarity variables of the problem and the invariant solution is expressed in terms of them. When the similarity variables are substituted into the original system of PDEs the result is a new system in one fewer variables thus achieving a simplification of the problem. The substitution process can be quite difficult if the integrals happen to be complicated functions of the old variables. However, in practice, the groups that find the widest application tend to be elementary dilation and translation groups for which the characteristic equations can be separated. This makes it relatively easy to choose which integrals to use as new independent variables and which to use as dependent variables. Normally the new independent variables would be arranged to involve only the original independent variables although, in principle, that need not be the case and there are situations where one might want to exchange independent and dependent variables. The examples in the next two sections illustrate how to use groups to find invariant solutions.

A slightly different take on this issue can be made by assuming at the outset that, without loss of generality, the invariant solution can be expressed in the explicit form

$$\Omega^i[\boldsymbol{x}, \boldsymbol{y}] = y^i - \Phi^i[\boldsymbol{x}] = 0. \tag{9.74}$$

Written out, the invariance condition is

$$X\Omega^i = \eta^i - \xi^j \left(\frac{\partial \Phi^i}{\partial x^j}\right) = \eta^i - \xi^j \left(\frac{\partial y^i}{\partial x^j}\right) = 0, \qquad i = 1, \ldots, m. \tag{9.75}$$

These are just first order PDEs of the type discussed in Chapter 3 Sections 3.1 and 3.8 and, when collected together, are equivalent to the characteristic equations (9.72). The form (9.75) can sometimes help facilitate the choice of similarity variables.

The combination of infinitesimals and first derivatives in (9.75) comes up quite often in very useful ways and is called the *characteristic function* denoted

$$\mu^i = \eta^i - \xi^j y^i_j. \tag{9.76}$$

Using the characteristic functions, the extended infinitesimals can be written in the convenient form,

$$\begin{aligned} \eta^i_{\{j_1\}} &= D_{j_1}\mu^i + y^i_{j_1\alpha}\xi^\alpha \\ \eta^i_{\{j_1 j_2\}} &= D^2_{j_1 j_2}\mu^i + y^i_{j_1 j_2\alpha}\xi^\alpha \\ &\cdots\cdots \\ \eta^i_{\{j_1 j_2\cdots j_p\}} &= D^p_{j_1 j_2\cdots j_p}\mu^i + y^i_{j_1 j_2\cdots j_p\alpha}\xi^\alpha . \end{aligned} \tag{9.77}$$

where $\alpha$ is a dummy index summed over 1 to $n$. The generation of (9.77) is left as an excercise for the reader. See Chapter 14 Section 14.1.1 and Exercise 14.1. Using (9.77) the extended operator (9.52) can be written in the alternative form

$$\begin{aligned} X_{\{p\}} = \xi^\alpha \frac{\partial}{\partial x^\alpha} + y^i_\alpha \xi^\alpha \frac{\partial}{\partial y^i} + y^i_{j_1\alpha}\xi^\alpha \frac{\partial}{\partial y^i_{j_1}} + \cdots + y^i_{j_1 j_2\cdots j_p\alpha}\xi^\alpha \frac{\partial}{\partial y^i_{j_1 j_2\cdots j_p}} \\ + \mu^i \frac{\partial}{\partial y^i} + D_{j_1}\mu^i \frac{\partial}{\partial y^i_{j_1}} + \cdots + D^p_{j_1 j_2\cdots j_p}\mu^i \frac{\partial}{\partial y^i_{j_1 j_2\cdots j_p}} \end{aligned} \tag{9.78}$$

We will see much more of the characteristic function in Chapter 14 where Lie-Bäcklund transformations are considered and where the interest is in transformations for which $\xi^\alpha = 0$. In the last section of the present chapter the subject of nonclassical symmetries is introduced and there the main interest is in a situation where $\mu^i = 0$. In both cases the discussion is facilitated by considering the extended operator in the form (9.78).

## 9.4 Impulsive Source Solutions of the Heat Equation

Consider a slab of thermally conducting material. Heat is added instantaneously along an infinite vertical strip that divides the slab. As time proceeds, the heat diffuses outward, leading to a decrease in temperature at the center of the strip and an increase further out as the distribution broadens. The physical situation is shown schematically in Figure 9.3.

It is worth noting that the initial condition is not singular. Instead, there is some finite-width, smooth distribution of temperature at $t = 0$. The initial distribution may be symmetric as shown in the figure, or antisymmetric involving insertion of heat on one side of the dividing line and removal on the other, or something more complex. The governing equation is

$$u_t - \kappa u_{xx} = 0 \tag{9.79}$$

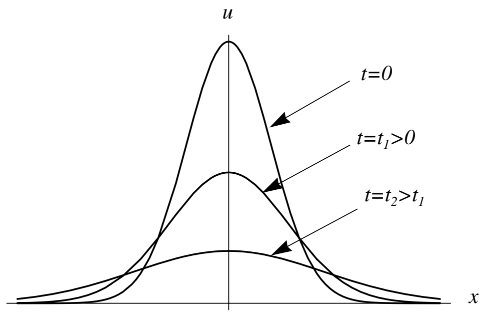


Fig. 9.3. Diffusion of heat from an impulsive source.

with boundary conditions

$$u[\pm\infty, t + t_0] = 0, \qquad t + t_0 \geq 0. \tag{9.80}$$

Both the time $t$ and the time origin parameter $t_0$ are assumed to be positive. The initial condition is specified such that

$$A = \int_0^{\infty} x^{\alpha} u[x, t_0]\, dx. \tag{9.81}$$

At this point we know very little about this integral except that at $t = 0$ it converges for some value of $\alpha$. We don't even know if a solution to this problem exists! To understand this a little further, multiply the heat equation by $x^{\alpha}$ and integrate twice by parts. The result is

$$\begin{aligned} \frac{d}{dt}\left(\int_0^{\infty} x^{\alpha} u\, dx\right) &= \kappa \int_0^{\infty} x^{\alpha} u_{xx}\, dx \\ &= \kappa(x^{\alpha} u_x - \alpha x^{\alpha-1} u)\big|_0^{\infty} + \alpha(\alpha - 1)\left(\kappa \int_0^{\infty} x^{\alpha-1} u\, dx\right). \end{aligned} \tag{9.82}$$

It is clear that the right-hand side of (9.82) is zero when $\alpha$ equals 0 or 1 as long as $u_x$ is zero at $\infty$ and $u$ is even for $\alpha = 0$ and odd for $\alpha = 1$. For these values, the integral is preserved for all time. At this point we can't say anything about other values of $\alpha$.

It is relatively easy to show that this problem is invariant under the three-parameter group of dilations in the dependent and independent variables and translation in time:

$$\tilde{x} = e^{a} x, \qquad \tilde{t} = e^{2a} t + (e^{2a} - 1) t_0, \qquad \tilde{u} = e^{-(1+\alpha)a} u. \tag{9.83}$$

Here the finite groups of the heat equation corresponding to the operators $X^3$ (dilation in $x$ and $t$), $X^5$ (translation in $t$), and $X^6$ (dilation in $u$) have been combined to produce (9.83).

It is quite easy to demonstrate the invariance of the boundary conditions (9.80) under the group (9.83). We know from the previous section that the governing equation is invariant under the group (9.83). Now consider the boundary curves and functions specified on those curves. First the boundary at infinity:

$$\tilde{x} = \infty \quad \Rightarrow \quad e^a x = \infty \quad \Rightarrow \quad x = \infty. \tag{9.84}$$

On the boundary at infinity,

$$\tilde{u} = 0 \quad \Rightarrow \quad e^{-(1+\alpha)a} u = 0 \quad \Rightarrow \quad u = 0. \tag{9.85}$$

It is important that the value of $u$ on the boundary be zero and not, say, a constant. If it were a nonzero constant, then the transformation of $\tilde{u}$ would give

$$u = \text{constant}\; e^{(1+\alpha)a}, \tag{9.86}$$

which changes the value of the constant and breaks the invariance. The boundary in time is treated the same way:

$$\tilde{t} + t_0 = 0 \quad \Rightarrow \quad e^{2a}(t + t_0) = 0 \quad \Rightarrow \quad t + t_0 = 0. \tag{9.87}$$

The integral (9.81) is easily shown to be invariant under the group (9.83):

$$A = \int_0^\infty \tilde{x}^\alpha \tilde{u}\, d\tilde{x} = \int_0^\infty e^{a\alpha} x^\alpha e^{-a(1+\alpha)} u e^a\, dx = \int_0^\infty x^\alpha u\, dx. \tag{9.88}$$

The demonstration that the problem (equation and boundary–initial conditions) is invariant under a group is essentially a proof of the existence of a similarity solution of the problem. This implies that there should exist a solution for which the integral (9.81) converges and is preserved for all $\alpha$.

Now we use the infinitesimal method to generate the similarity variables. The infinitesimal form of the group (9.83) is

$$\tilde{x} = x + sx, \qquad \tilde{t} = t + s(2t + 2t_0), \qquad \tilde{u} = u - s(1+\alpha)u \tag{9.89}$$

with characteristic equations

$$\frac{dx}{x} = \frac{dt}{2t + 2t_0} = \frac{du}{-(1+\alpha)u}. \tag{9.90}$$

The two integrals of (9.82) are

$$\zeta = \frac{x}{(2\kappa(t+t_0))^{1/2}}, \qquad U = \frac{u}{A}(2\kappa(t+t_0))^{(1+\alpha)/2}. \tag{9.91}$$

These become the similarity variables of the problem. The fact that the equation and boundary and initial conditions are invariant under the group (9.83) implies that the solution should be invariant under the same group. Thus we expect the solution to have the form

$$u = A(2\kappa(t+t_0))^{-(1+\alpha)/2}U(\zeta), \tag{9.92}$$

where the parameters $A$ and $\kappa$ have been used to nondimensionalize the similarity variables $\zeta$ and $U$. Substituting the similarity forms into the invariant integral gives

$$\int_0^\infty \zeta^\alpha U[\zeta]\, d\zeta = 1. \tag{9.93}$$

The integral (9.81) is preserved for any $\alpha$ as long as $u$ and $u_x$ go to zero sufficiently fast at infinity.

Now substitute (9.92) into the heat equation. The result is a reduction of the problem to a second-order ODE of Sturm–Liouville type,

$$U_{\zeta\zeta} + \zeta U_\zeta + (1+\alpha)U = 0, \qquad U(\pm\infty) = 0. \tag{9.94}$$

Solving (9.94) for integer values of $\alpha$ and using the integral (9.93) to normalize the solution leads to

$$U[\zeta] = \left(\sqrt{\frac{2}{\pi}}\frac{1}{\alpha!}\right) e^{-\zeta^2/2} H_\alpha[\zeta], \tag{9.95}$$

where the $H_\alpha$ are Hermite polynomials generated by

$$H_\alpha[\zeta] = (-1)^\alpha e^{\zeta^2/2} \frac{d^\alpha}{d\zeta^\alpha}\left(e^{-\zeta^2/2}\right). \tag{9.96}$$

The first six Hermite polynomials are

$$\begin{aligned} H_0 &= 1, & H_1 &= \zeta, \\ H_2 &= \zeta^2 - 1, & H_3 &= \zeta^3 - 3\zeta, \\ H_4 &= \zeta^4 - 6\zeta^2 + 3, & H_5 &= \zeta^5 - 10\zeta^3 + 15\zeta. \end{aligned} \tag{9.97}$$

The solution (9.95) is plotted in Figure 9.4 for the first six values of $\alpha$.

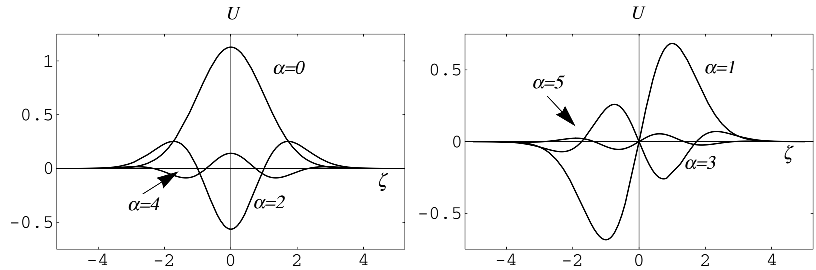


Fig. 9.4. Even and odd similarity solutions of the heat equation.

What about fractional values of $\alpha$? In this case the even and odd solutions of (9.94) can be expressed in terms of confluent hypergeometric functions,

$$\begin{aligned} U_{\text{even}} &= Ce^{-\zeta^2/4}M\left[-\frac{\alpha}{2}, \frac{1}{2}, \frac{\zeta^2}{2}\right], \qquad \alpha = 0, 2, 4, \ldots, \\ U_{\text{odd}} &= C\zeta e^{-\zeta^2/4}M\left[-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, \frac{\zeta^2}{2}\right], \qquad \alpha = 1, 3, 5, \ldots, \end{aligned} \tag{9.98}$$

where

$$\begin{aligned} M[a, b, s] &= 1 + \sum_{k=1}^{\infty} \frac{a_k}{b_k}\frac{s^k}{k!}, \\ a_k &= a(a+1)\cdots(a+k-1), \\ b_k &= b(b+1)\cdots(b+k-1). \end{aligned} \tag{9.99}$$

However, these solutions cannot individually satisfy the integral constraint (9.93). For fractional $\alpha$ and large values of the argument, the asymptotic behavior of the solutions (9.98) is

$$\lim_{\zeta\to\infty} U \approx (\;)\zeta^{\alpha}e^{-\zeta^2/2} + (\;)\zeta^{-\alpha-1} \tag{9.100}$$

In this case the integral does not converge:

$$\int_0^{\infty} \zeta^{\alpha} U[\zeta]\, d\zeta \approx \ln\, \zeta|_0^{\infty} \tag{9.101}$$

Nevertheless it is possible to combine the solutions (9.98) so as to cancel the singularity. This leads to the construction of the parabolic cylinder functions

$U = D_\alpha[\pm\zeta]$, where

$$D_\alpha[\pm\zeta] = \sqrt{\pi}2^{\alpha/2}e^{-\zeta^2/4}\left[\frac{M\left[-\frac{\alpha}{2}, \frac{1}{2}, \frac{\zeta^2}{2}\right]}{\Gamma\left[\frac{1}{2} - \frac{\alpha}{2}\right]} \mp \sqrt{2}\zeta\frac{M\left[-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, \frac{\zeta^2}{2}\right]}{\Gamma\left[-\frac{\alpha}{2}\right]}\right]. \tag{9.102}$$

See Abramowitz and Stegun [9.2]. The gamma function is

$$\Gamma[a] = \int_0^\infty s^{a-1}e^{-s}\,ds. \tag{9.103}$$

The branch cut for defining the parabolic cylinder functions in the complex plane is taken to be along the negative complex axis, $|\text{Arg}[\theta]| < 3\pi/4$. For small values of $\zeta$,

$$\lim_{\zeta\to 0} D_\alpha[\zeta] = \sqrt{\pi}2^{\alpha/2}e^{-\zeta^2/4}\left(\frac{1 - \frac{\alpha}{2!}\zeta^2 + \frac{\alpha(\alpha-2)}{4!}\zeta^4 - \cdots}{\Gamma\left[\frac{1}{2} - \frac{\alpha}{2}\right]} - \sqrt{2}\frac{\zeta - \frac{\alpha-1}{3!}\zeta^3 + \frac{(\alpha-1)(\alpha-3)}{5!}\zeta^5 - \cdots}{\Gamma\left[-\frac{\alpha}{2}\right]}\right), \tag{9.104}$$

and for large $\zeta$

$$\lim_{\zeta\to\infty} D_\alpha[\zeta] = \zeta^\alpha e^{-\zeta^2/4}\left(1 - \frac{\alpha(\alpha-1)}{2\zeta^2} + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{8\zeta^4} - \cdots\right). \tag{9.105}$$

These functions yield solutions for fractional values of $\alpha$ with integrals of the form of (9.93) that converge. However, as can be seen from the combination of even and odd terms in (9.104), they are neither symmetric nor antisymmetric functions.

The solutions (9.92) for various integer $\alpha$ form a complete set of orthogonal functions. Therefore any smooth decaying solution that goes to zero sufficiently fast at infinity can be represented as a series expansion

$$u[x,t] = \sum_{\alpha=0}^{\infty} A_\alpha(2k(t+t_0))^{-(1+\alpha)/2}\left(\left(\sqrt{\frac{2}{\pi}}\frac{1}{\alpha!}\right)e^{-\zeta^2/2}H_\alpha[\zeta]\right). \tag{9.106}$$

The coefficients $A_\alpha$ in the series are determined from the various moments of the initial condition using the orthogonality of the expansion functions. It is clear from (9.106) that regardless of the initial condition, the terms that dominate the large-time, final decay of the temperature are the lowest nonzero modes

$\alpha = 0,\ 1$. This is because these are the modes with the slowest decay. In the next section we will study a remarkable example of nonlinear heat conduction where the fractional-$\alpha$ solutions play a crucial role in a *symmetric* problem and where the final state of decay is not $\alpha = 0$ or $\alpha = 1$.

Before we leave this example, it is worthwhile saying a few words about the initial condition and about the effective origin in time that was included when we incorporated time translation with the group (9.83). The distribution of $u$ at $t = 0$ is

$$u[x, 0] = \sum_{\alpha=0}^{\infty} A_\alpha (2kt_0)^{-(1+\alpha)/2} \left( \sqrt{\frac{2}{\pi}} \frac{1}{\alpha!} \right) e^{-x^2/4kt_0} H_\alpha \left[ \frac{x}{\sqrt{2kt_0}} \right]. \tag{9.107}$$

The parameter $t_0$ enables one to specify an initial distribution that is smooth and infinitely differentiable. In the limit $t_0 \to 0$ the $\alpha = 0$ term in the distribution (9.107) is a useful form of the Dirac delta function. Higher values of $\alpha$ correspond to the various derivatives of the Dirac delta function, and the integral of the $\alpha = 0$ term is the Heaviside function.

## 9.5 A Modified Problem of an Instantaneous Heat Source

Now we consider the problem of diffusion in a nonlinear medium. Actually the motivation for discussing the classical solutions in the previous section was to prepare the student for the fascinating and important nonlinear problem presented here. The type of problem we are about to encounter comes up in a variety of filtration situations where the fluid velocity in a porous medium is governed by Darcy's law,

$$u = -\frac{k}{\mu} \nabla p, \tag{9.108}$$

where $k$ is the permeability of the medium, $\mu$ is the viscosity, and $p$ is the pressure. The continuity equation is

$$\frac{\partial \sigma \rho}{\partial t} + \nabla \cdot \rho \bar{u} = 0, \tag{9.109}$$

where $\sigma$ is the porosity of the medium and $\rho$ is the density of the fluid. Now let

$$\rho = \rho_0 \left( 1 + \frac{p - p_0}{\lambda} \right). \tag{9.110}$$

The compressibility of the fluid is generally very small, with values of $\lambda \approx 10^4$ kg/cm$^2$. When we substitute (9.110) into (9.109) and use (9.108), we find that the pressure satisfies the heat equation,

$$\frac{\partial p}{\partial t} - \kappa \nabla^2 p = 0, \tag{9.111}$$

where the diffusivity of pressure is

$$\kappa = \frac{k\lambda}{\mu\sigma}. \tag{9.112}$$

This equation governs the diffusion of the pressure field associated with the slow movement of oil through sedimentary rock around a well. In most filtration problems, the porous medium is assumed to be incompressible and one is solving a conventional diffusion problem. But in the oil-well problem, the rock medium has the property that, where the pressure is increasing with time, the pores in the rock enlarge, producing one value of the diffusivity, and where the pressure is decreasing with time, the rock has a tendency to collapse, leading to a different value of the diffusivity. Where the time derivative of the pressure is zero, there is a step in the diffusivity, introducing a nonlinearity into the problem. See Barenblatt [9.3] for a complete discussion of the physics of problems of this class. Here we will follow Barenblatt's development. For pedagogical reasons I have filled in most of the mathematical steps in the hope that an interested reader will be able to follow the analysis in full detail and in comparison with the solution of the conventional problem discussed in Section 9.4.

The diffusivity depends nonlinearly on the time derivative of the solution. In Figure 9.5 the diffusivity "shock" is shown propagating along a fixed point in similarity coordinates, $\zeta_a$. In the uniform diffusivity case, the point $x_a$ where

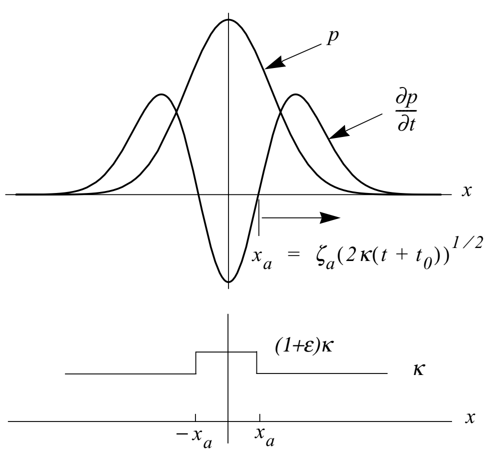


Fig. 9.5. Spatial variation in diffusivity for the modified problem.

$\partial p/\partial t = 0$ is located at $x_a = (2\kappa(t+t_0))^{1/2}$. But in the nonlinear case there is no justification for an assumption of self-similiar behavior. Furthermore there is no reason to assume power-law behavior for the propagation of the diffusivity front. All this will have to be justified *a posteriori* once a solution has been shown to exist and has been determined.

The governing equations are

$$\begin{aligned} p_t - (1+\varepsilon)\kappa p_{xx} = 0, &\qquad p_t \le 0, \\ p_t - \kappa p_{xx} = 0, &\qquad p_t \ge 0, \end{aligned} \tag{9.113}$$

with boundary conditions

$$p[\pm\infty, t+t_0] = 0, \qquad t+t_0 \ge 0. \tag{9.114}$$

Furthermore we recognize that the pressure and pressure gradient must be continuous across the boundary $x = x_a$ (diffusion prevents infinite gradients in pressure from developing anywhere):

$$\begin{aligned} p[x_{a-}, t+t_0] &= p[x_{a+}, t+t_0], \\ p_x[x_{a-}, t+t_0] &= p_x[x_{a+}, t+t_0]. \end{aligned} \tag{9.115}$$

Based on our knowledge of the conventional problem in Section 9.4, we seek a solution where the integral

$$A = \int_0^\infty x^\alpha p[x, t+t_0]\, dx \tag{9.116}$$

is preserved. Following Section 9.4, the governing equation can be integrated by parts as follows:

$$\begin{aligned} \frac{d}{dt}\left(\int_0^\infty x^\alpha p\, dx\right) &= (1+\varepsilon)\kappa \int_0^{x_a} x^\alpha p_{xx}\, dx + \kappa \int_{x_a}^\infty x^\alpha p_{xx}\, dx \\ &= (1+\varepsilon)\kappa (x^\alpha p_x - \alpha x^{\alpha-1} p)|_0^{x_a} \\ &\quad + \alpha(\alpha-1)\left((1+\varepsilon)\kappa \int_0^{x_a} x^{\alpha-1} p\, dx\right) \\ &\quad + \kappa (x^\alpha p_x - \alpha x^{\alpha-1} p)|_{x_a}^\infty + \alpha(\alpha-1)\left(\kappa \int_{x_a}^\infty x^{\alpha-1} p\, dx\right). \end{aligned} \tag{9.117}$$

For $\alpha = 0,\ 1$, corresponding to the first even and odd solutions respectively, the integral becomes

$$\frac{d}{dt}\left(\int_0^\infty x^\alpha p\,dx\right) = \varepsilon\kappa\left(x_a^\alpha p_x\big|_{x=x_a} - \alpha x_a^{\alpha-1} p\big|_{x=x_a}\right). \tag{9.118}$$

In this case the integral for these two values of $\alpha$ is clearly not preserved, and a similarity solution of the problem does not exist for $\varepsilon \neq 0$.

In spite of the fact that no solution exists for $\alpha = 0,\ 1$, we will continue to investigate whether a solution can exist for *some* value of $\alpha$. In fact the problem really boils down to this: given $\varepsilon > 0$, does there exist a value of $\alpha$ that solves the problem defined by (9.113) and (9.114)?

So we push on and assume the existence of a solution that is invariant under the group (9.83), with similarity variables of the same form as in the uniform diffusivity case:

$$\zeta = \frac{x}{(2\kappa(t+t_0))^{1/2}}, \qquad p = A(2\kappa(t+t_0))^{-(1+\alpha)/2} P[\zeta]. \tag{9.119}$$

Upon substitution of these variables, the governing equation in each domain is

$$\begin{aligned} (1+\varepsilon)P^-_{\zeta\zeta} + \zeta P^-_\zeta + (1+\alpha)P^- &= 0, \qquad 0 \le \zeta \le \zeta_\alpha, \\ P^+_{\zeta\zeta} + \zeta P^+_\zeta + (1+\alpha)P^+ &= 0, \qquad \zeta_a \le \zeta \le \infty, \end{aligned} \tag{9.120}$$

where

$$\zeta_a = \frac{x_a}{(2\kappa(t+t_0))^{1/2}}. \tag{9.121}$$

The solution is subject to the far-field condition

$$P^+[\pm\infty] = 0 \tag{9.122}$$

and the integral invariant

$$\int_0^{x_a} \zeta^\alpha P^-[\zeta]\,d\zeta + \int_{x_a}^\infty \zeta^\alpha P^+[\zeta]\,d\zeta = 1. \tag{9.123}$$

Equations (9.120), (9.122), and (9.123) constitute a nonlinear eigenvalue problem for the unknowns $\alpha$ and $\zeta_a$.

At the internal boundary $\zeta = \zeta_a$ the following matching conditions apply:

$$\begin{aligned} P^-[\zeta_a] &= P^+[\zeta_a], \\ P^-_\zeta[\zeta_a] &= P^+_\zeta[\zeta_a], \end{aligned} \tag{9.124}$$

and

$$\zeta_a P^-_\zeta[\zeta_a] + (1+\alpha)P^-[\zeta_a] = 0,$$
$$\zeta_a P^+_\zeta[\zeta_a] + (1+\alpha)P^+[\zeta_a] = 0. \tag{9.125}$$

The last condition comes from the fact that, by assumption, the time derivative of the pressure is zero at the internal boundary and therefore $p_{xx} = 0 \Rightarrow P_{\zeta\zeta} = 0$ at $x_a$. An even solution, valid for arbitrary $\alpha$, inside the internal boundary is

$$\begin{aligned} P^-[\zeta] &= C_1 e^{-\zeta^2/4(1+\varepsilon)}\left(D_\alpha\left[\frac{\zeta}{\sqrt{1+\varepsilon}}\right] + D_\alpha\left[-\frac{\zeta}{\sqrt{1+\varepsilon}}\right]\right) \\ &= C_1\left(\frac{2}{\Gamma\left[\frac{1}{2}-\frac{\alpha}{2}\right]}\right)\sqrt{\pi}2^{\alpha/2}e^{-\zeta^2/2(1+\varepsilon)}M\left[-\frac{\alpha}{2},\frac{1}{2},\frac{\zeta^2}{2(1+\varepsilon)}\right], \\ &\qquad 0 \le \zeta \le \zeta_a, \end{aligned} \tag{9.126}$$

where we have used the parabolic cylinder functions introduced in Section 9.3. A solution beyond the internal boundary that ensures convergence of the integral constraint (9.123) is

$$\begin{aligned} P^+[\zeta] &= C_2 e^{-\zeta^2/4} D_\alpha[\zeta] \\ &= C_2\sqrt{\pi}2^{\alpha/2}e^{-\zeta^2/2}\left(\frac{M\left[-\frac{\alpha}{2},\frac{1}{2},\frac{\zeta^2}{2}\right]}{\Gamma\left[\frac{1}{2}-\frac{\alpha}{2}\right]} - \sqrt{2}\zeta\frac{M\left[-\frac{\alpha}{2}+\frac{1}{2},\frac{3}{2},\frac{\zeta^2}{2}\right]}{\Gamma\left[-\frac{\alpha}{2}\right]}\right). \end{aligned} \tag{9.127}$$

See (9.101). Now apply the first matching condition in (9.124) to eliminate $C_2$:

$$C_1 e^{-\zeta_a^2/4(1+\varepsilon)}\left(D_\alpha\left[\frac{\zeta_a}{\sqrt{1+\varepsilon}}\right] + D_\alpha\left[-\frac{\zeta_a}{\sqrt{1+\varepsilon}}\right]\right) = C_2 e^{-\zeta_a^2/4} D_\alpha[\zeta_a]. \tag{9.128}$$

The second matching condition in (9.124) is automatically satisfied if (9.125) is satisfied. The first relation in (9.125) becomes

$$s_a\left(\left.\frac{dM\left[-\frac{\alpha}{2},\frac{1}{2},s\right]}{ds}\right|_{s=s_a} - M\left[-\frac{\alpha}{2},\frac{1}{2},s_a\right]\right) + \frac{1+\alpha}{2}M\left[-\frac{\alpha}{2},\frac{1}{2},s_a\right] = 0, \tag{9.129}$$

where $s_a = \zeta_a^2/(2(1+\varepsilon))$. Using the identity

$$s\frac{dM[a,b,s]}{ds} + (b-a-s)M[a,b,s] = (b-a)M[a-1,b,s], \tag{9.130}$$

Equation (9.129) becomes

$$M\left[-\frac{\alpha}{2}-1, \frac{1}{2}, \frac{\zeta_a^2}{2(1+\varepsilon)}\right]=0. \tag{9.131}$$

Given the diffusivity ratio $1+\varepsilon$, the matching condition (9.131) provides a relation between $\alpha$ and $\zeta_a$. The second condition in (9.125) yields

$$\zeta_a \frac{dD_\alpha}{d\zeta}\bigg|_{\zeta=\zeta_a} + \left(1+\alpha-\frac{\zeta_\alpha^2}{2}\right)D_\alpha[\zeta_a]=0. \tag{9.132}$$

Now use the following identities for parabolic cylinder functions:

$$\begin{aligned}\frac{dD_\alpha}{d\zeta}+D_{\alpha+1}-\frac{\zeta}{2}D_\alpha&=0,\\ D_{\alpha+2}-\zeta D_{\alpha+1}+(\alpha+1)D_\alpha&=0.\end{aligned} \tag{9.133}$$

Using (9.133) to simplify (9.132) gives

$$D_{\alpha+2}[\zeta_a]=0, \tag{9.134}$$

or, in terms of hypergeometric functions,

$$\frac{M\left[-\frac{\alpha}{2}-1, \frac{1}{2}, \frac{\zeta_a^2}{2}\right]}{\Gamma\left[-\frac{1}{2}-\frac{\alpha}{2}\right]} - \sqrt{2}\zeta_a \frac{M\left[-\frac{\alpha}{2}-\frac{1}{2}, \frac{3}{2}, \frac{\zeta_a^2}{2}\right]}{\Gamma\left[-1-\frac{\alpha}{2}\right]}=0. \tag{9.135}$$

Equations (9.131) and (9.135) provide two equations in two unknowns, allowing one to solve for $\alpha$ and $\zeta_a$ given $\varepsilon$. The results are plotted in Figure 9.6.

The remarkable feature of this problem, the feature that makes it an important problem, is that when full numerical solutions are carried out for a general initial condition, it is found that after an early period of non-self-similar development the solution approaches the one similarity solution that exists for a

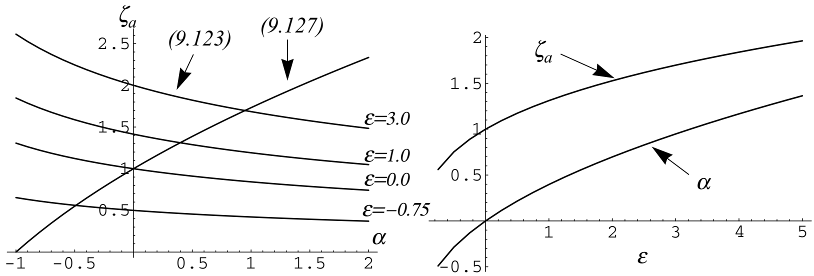


Fig. 9.6. Solution parameters of the modified diffusion problem.

given choice of the material constant $\varepsilon$ (see Barenblatt [9.3], Figures 3.3 and 3.4, and the discussion in Sections 3.2.4 and 3.2.5). By one means or another, symmetry finds a way! The group that leaves the asymptotic solution invariant is the same dilation group of the heat equation we encountered in Section 9.4, Equation (9.83),

$$\tilde{x} = e^a x, \qquad \tilde{t} = e^{2a} t + (e^{2a} - 1)t_0, \qquad \tilde{p} = e^{-(1+\alpha)a} p, \tag{9.136}$$

where the constant $\alpha$ is determined from the solution of the nonlinear eigenvalue problem (9.131) and (9.135).

This is an example of a wide class of important problems in filtration, and further examples can be found in the work of Baikov, Gladkov, and Wiltshire [9.4] and Baikov [9.5].

## 9.6 Nonclassical Symmetries

There is a nonlinear alternative to the Lie algorithm that, in several applications, has led to the identification of new point symmetries of differential equations that do not correspond to classical Lie symmetries. The basic idea is to replace the requirement that the differential equation be invariant under a certain symmetry with a somewhat less restrictive requirement that the equation admit a symmetry over a limited set of solutions of the equation.

In the Lie procedure one solves the $p$th order extended invariance condition

$$X_{\{p\}}\Psi^i = 0, \qquad i = 1, \ldots, m \tag{9.137}$$

for the unknown infinitesimals $(\xi^i[\boldsymbol{x}, \boldsymbol{y}], \eta^i[\boldsymbol{x}, \boldsymbol{y}])$ subject to the constraint imposed by the requirement that $\boldsymbol{y}[\boldsymbol{x}]$ is a solution of the original system of equations. Namely.

$$\Psi^i = 0. \tag{9.138}$$

Equation (9.138) is used to replace derivatives of the $y^i$ that appear in (9.137). When (9.137) is parsed, the result is the set of *linear* determining PDEs for the infinitesimals and the solution of this system is the set of classical point symmetries of the system of equations.

Alternatively, one can search for symmetries that are valid only over some set of invariant solutions of the system of equations. This is accomplished by adding to (9.137) an additional constraint in the form of the invariance condition on a solution. Let the invariant solution be expressed in the form

$$\Omega^i[\boldsymbol{x}, \boldsymbol{y}] = y^i - \Phi^i[\boldsymbol{x}] = 0. \tag{9.139}$$

The invariance condition is

$$X\Omega^i = \eta^i - \xi^j y^i_j = 0, \qquad i = 1, \ldots, m \tag{9.140}$$

Recalling the definition of the characteristic function, $\mu^i = \eta^i - \xi^j y^i_j$, introduced in Section 9.3, the new condition is simply $\mu^i = 0$. The difficulty with this approach is that when (9.140) is is used to replace $m$ of the $y^i_j$ in (9.137) and the equation is parsed, the result is a set of *nonlinear* determining PDEs for the infinitesimals.

Some simplification is possible. Note that the extended operator that we are dealing with is one where $\mu^i = 0$ in (9.86). The implication of this is that if $(\xi^j, \eta^i)$ satisfy (9.86) then $(f\xi^j, f\eta^i)$ also satisfy (9.86) where $f[\boldsymbol{x}, \boldsymbol{y}]$ is any scalar function. This permits one of the $\xi^j$ to be set to unity without loss of generality. Nevertheless, solving the determining equations in this case is much more difficult and only a few examples are known. Note that the set of solutions that admit the symmetry $X$ is determined once $X$ has been identified.

### *9.6.1 A Non-classical Point Group of the Heat Equation*

To illustrate these ideas let's look again at the heat equation

$$u_t - u_{xx} = 0. \tag{9.141}$$

The invariance condition (9.58) after replacing $u_{xx}$ by $u_t$ is given in equation (9.65) and repeated here for convenience.

$$\begin{aligned}\eta_{\{t\}} - \eta_{\{xx\}} = {} & (\eta_t - \eta_{xx}) + 2u_x u_{xt}(\tau_u) + 2u_x u_t(\xi_u + \tau_{xu}) \\ & + u_x^2(2\xi_{xu} - \eta_{uu}) + u_x^2 u_t(\tau_{uu}) + u_x^3(\xi_{uu}) + 2u_{xt}(\tau_x) \\ & + u_t(\tau_{xx} + 2\xi_x - \tau_t) + u_x(\xi_{xx} - \xi_t - 2\eta_{xu}) = 0\end{aligned} \tag{9.142}$$

In the usual approach, with the replacement complete, all other derivatives of $u[x, t]$ are not restricted in any way and in order for (9.142) to be satisfied, the coefficients of various products of derivatives of $u$ are set to zero forming the linear determining equations for the unknown infinitesimals $\xi[x, t, u]$, $\tau[x, t, u]$ and $\eta[x, t, u]$.

Instead we now apply the condition

$$\eta[x, t, u] - \xi[x, t, u]u_x - \tau[x, t, u]u_t = 0. \tag{9.143}$$

Without loss of generality let $\tau = 1$ and make the replacement,

$$u_t = \eta[x,t,u] - \xi[x,t,u]u_x \tag{9.144}$$

in (9.142). The result is

$$\begin{aligned}\eta_{\{t\}} - \eta_{\{xx\}} = (\eta_t - \eta_{xx}) + 2u_x(\eta - \xi u_x)\xi_u + u_x^2(2\xi_{xu} - \eta_{uu})\\ + u_x^3(\xi_{uu}) + (\eta - \xi u_x)2\xi_x + u_x(\xi_{xx} - \xi_t - 2\eta_{xu}) = 0\end{aligned} \tag{9.145}$$

or, with some rearrangement

$$\begin{aligned}\eta_{\{t\}} - \eta_{\{xx\}} = (\eta_t - \eta_{xx} + 2\eta\xi_x)\\ + u_x(2\eta\xi_u - 2\xi\xi_x + \xi_{xx} - \xi_t - 2\eta_{xu})\\ + u_x^2(2\xi_{xu} - \eta_{uu} - 2\xi\xi_u) + u_x^3(\xi_{uu}) = 0\end{aligned} \tag{9.146}$$

In order for (9.146) to be satisfied for arbitrary derivatives of $u$ the coefficients in parentheses must be zero and so the determining equations in this case are

$$\begin{aligned}\eta_t - \eta_{xx} + 2\eta\xi_x &= 0,\\ 2\eta\xi_u - 2\xi\xi_x + \xi_{xx} - \xi_t - 2\eta_{xu} &= 0,\\ 2\xi_{xu} - \eta_{uu} - 2\xi\xi_u &= 0,\\ \xi_{uu} &= 0.\end{aligned} \tag{9.147}$$

The nonlinearity of the system (9.147) precludes any sort of elementary approach to a solution including the power series method used in the linear case. The only reasonable way to make progress is to look for simplifying assumptions that lead to interesting solutions. Let

$$\begin{aligned}\eta &= 0\\ \xi_u &= \xi_t = 0\end{aligned} \tag{9.148}$$

The determining equations reduce to

$$\xi_{xx} - 2\xi\xi_x = 0. \tag{9.149}$$

with the solution

$$\xi = -\sqrt{C}_2 \tanh[\sqrt{C}_2 x + C_1\sqrt{C}_2]. \tag{9.150}$$

Using this procedure we find that the heat equation admits the nonclassical point symmetry

$$X = -\sqrt{C}_2\tanh[\sqrt{C}_2 x + C_1\sqrt{C}_2]\frac{\partial}{\partial x} + \frac{\partial}{\partial t}. \tag{9.151}$$

A couple of questions remain. What sort of solution does this symmetry generate? And, what symmetry arises if we assume $\xi = 1$ instead of $\tau = 1$? These are left as exercises for the reader. Nonclassical symmetries are the subject of considerable research and the reader is referred to the treatments in Hydon [9.8] and Baumann [9.9] as well as the papers of Bluman and Cole [9.10] and more recently Clarkson [9.11]. The package **`IntroToSymmetry.m`** can aid in the search for nonclassical symmetries and several examples are included on the CD.

## 9.7 Concluding Remarks

In Chapters 10, 11, 12 and 13 we will apply the methods developed here to a variety of classical problems in fluid mechanics. The theme is simple, and the same illustrated here in the context of the heat equation; (i) formulate the problem with boundary conditions, (ii) identify the symmetries of the governing equation(s), (iii) find the subgroup that leaves the boundary conditions invariant, (iv) work out new variables based on the invariants of the group, (v) generate the reduced equation(s) and solve. At this point the reader should compare the three versions of the invariance condition developed for PDEs (Section 9.2), ODEs (Chapter 8, Section 8.2) and functions (Chapter 5, Section 5.6). The basic theory in all three cases is fundamentally the same. Actually we will go through the same development one more time in Chapter 14 when we treat Lie–Bäcklund groups and, once again, the theory will be seen to be essentially the same; simply expand a differential function in a Lie series and define the invariance condition so that all higher order terms in the series are zero.

Two questions commonly arise at this point. The first is, does the Lie algorithm find all symmetries of a given system of differential equations and the second is, are all clever reductions related to symmetries? The answer to the first question is clearly no given the discussion of nonclassical symmetries presented in the last section. In fact quite a few different types of symmetries can be identified and one has to get used to the names. Here are a few.

- Point symmetries – These are the symmetries we have dealt with thus far where the infinitesimals depend only on coordinates. They are also called classical symmetries.

- Nonclassical symmetries – See Section 9.7.
- Lie–Bäcklund symmetries – These are symmetries where the infinitesimals can depend on derivatives. See Chapter 14.
- Generalized symmetries – The name preferred by some authors for Lie–Bäcklund symmetries.
- Nonlocal symmetries – These are symmetries where the infinitesimals depend on integrals of the dependent variables. See Chapters 14 and 16.
- Potential symmetries – These are nonlocal symmetries that arise when an equation is expressed in terms of a potential function. See Chapters 14 and 16.
- Hidden symmetries – These are symmetries that arise when an equation is broken into an equivalent system of equations. They are usually nonlocal in nature.
- Variational symmetries – A subset of the symmetries connected with a variational principle that can be used to construct conserved vectors, closely related to Noether symmetries. See Chapter 15.

Despite the variety, the infinitesimals associated with these symmetries are all solutions of the fundamental invariance condition (9.53).

In a sense the second question is really asking whether there are examples of reductions that don't originate from (9.53). Schemes for simplifying equations take an incredible variety of forms and so a simple, safe, answer would be yes, there must be exceptions. But, deep down I believe that all clever reductions probably *are* related to symmetries in one way or another, one just has to look at the problem in the right way. I can't prove this though and given the open-ended nature of the symmetry problem a firm answer to this question is still far out of reach.

## 9.8 Exercises

9.1 Show by composition that the extended transformation (9.10) is a Lie group. Consider the case of one dependent and two independent variables.

9.2 The elliptic equation below has been used by Mahalingham [9.6] to model the effects of streamwise and cross-stream diffusion in a planar, low-speed, nonpremixed jet flame called a Burke–Schumann flame:

$$\phi_{xx} + \phi_{yy} - c\phi_y = 0. \tag{9.152}$$

Using hand calculations only, find the determining equations and solve for the infinitesimal groups of (9.152). Let

$$\phi[x, y] = e^{(c/2)y} f[x, y], \tag{9.153}$$

and convert the equation to the symmetric form

$$f_{xx} + f_{yy} - \frac{c^2}{4} f = 0. \tag{9.154}$$

Use the package **IntroToSymmetry.m** to find the infinitesimals. Compare your results with Reference [9.6].

9.3 Consider the generalized equation for 1-D flow in a porous medium,

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left(\kappa[p]\frac{\partial p}{\partial x}\right). \tag{9.155}$$

Use the package **IntroToSymmetry.m** to help determine the infinitesimal group for each of the following cases:

(i) $\kappa[p]$ arbitrary
(ii) $\kappa[p] = e^p$
(iii) $\kappa[p] = p^n,\ n \neq -\frac{4}{3}$
(iv) $\kappa[p] = p^{-4/3}$

Carefully work out the groups by hand. What is special about $n = -\frac{4}{3}$? Check your result against Ibragimov [9.1] Volume 1 Section 10.2.

9.4 Use the package **IntroToSymmetry.m** to help work out the groups of the 1-D nonlinear wave equation

$$\phi_{tt} - c[\phi]^2\phi_{xx} = 0. \tag{9.156}$$

Determine the infinitesimal group for each of the following cases:

(i) $c[\phi]$ arbitrary
(ii) $c[\phi] = \phi^n,\ n \neq 2$
(iii) $c[\phi] = \phi^2$,

9.5 Use the package **IntroToSymmetry.m** to determine the infinitesimals for each of the following equations:

(i) The axially symmetric wave equation

$$u_{tt} - u_{rr} - \frac{1}{r}u_r = 0. \tag{9.157}$$

(ii) A diffusion equation with shearing convection,

$$u_t + yu_x - u_y - \mu(u_{xx} + u_{yy}) = 0. \tag{9.158}$$

(iii) Laplace's equation in $n$ dimensions,

$$\phi_{x^i x^i} = 0. \tag{9.159}$$

(iv) The $(n+1)$-dimensional axisymmetric Laplace equation

$$\phi_{rr} + \frac{\alpha}{r}\phi_r + \phi_{x^i x^i} = 0, \qquad i = 1, \ldots, n, \tag{9.160}$$

where $\alpha(\alpha - 2) \neq 0$. See Aksenov [9.7] for the solution.

9.6 Use the package **`IntroToSymmetry.m`** to work out the infinitesimal groups of the force-free convection equation

$$\frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x_j} = 0, \qquad i = 1,\ 2,\ 3, \quad j = 1,\ 2, 3. \tag{9.161}$$

9.7 Find the solution of the heat equation corresponding to the nonclassical symmetry (9.151). What symmetry of the heat equation is found using the nonclassical method with $\xi = 1$. See Baumann [9.9] Section 6.3.

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# 10

## *Laminar Boundary Layers*

Fluids exhibit an incredible variety of physical phenomena that, with suitable approximations, can be reduced to the solution of many of the classical equations of mathematical physics. In this chapter we will look at several examples of viscous, nearly parallel, incompressible flow along a wall. The selected problems demonstrate several important aspects of the use of point groups to simplify nonlinear problems. This field is an especially fertile source of interesting and practical problems that can be approached using the techniques of symmetry analysis.

### 10.1 Background

The idea of the boundary layer was one of the seminal developments in fluid mechanics. Until the early 1900s fluid mechanics tended to be treated either by the hydraulic engineer with a purely empirical approach or by the applied mathematician solely interested in theoretical hydrodynamics, which at that time consisted almost entirely of the theory of frictionless flow. There was practically no overlap between the two disciplines, and the difficult problem of viscous flow resisted all attempts at solution except for the discovery of a few exact results for problems posed in highly simplified geometries. This situation began to change though the efforts of Ludwig Prandtl (1875–1953), who became professor of mechanics at the University of Hanover in 1901. Prandtl's goal was to develop a sound theoretical basis for fluid mechanics. At the International Mathematics Congress in Heidelberg in 1904 he gave a paper on his work entitled "Fluid motion with very small friction." In his paper he showed how the flow over a body could be divided into a thin region close to the body where viscous effects are important, which he called the *boundary layer*, and a region outside the boundary layer (essentially all the rest of the flow) where the flow is irrotational and therefore unaffected by viscosity. In the audience was Felix Klein, who invited him to Göttingen, where Klein founded an institute of mechanics and appointed the twenty-nine-year-old Prandtl to be the first director. The first application of Prandtl's boundary-layer concept was in 1908, to the flow past a flat plate, by one of his students, H. Blasius [10.1].

In this chapter we will take a detailed look at the laminar boundary-layer problem in steady plane flow with two independent variables in space $(x, y)$ and two dependent velocity variables $(u, v)$. The pressure outside the boundary layer is governed by the Bernoulli relation, while the pressure within the boundary layer is assumed to be independent of the wall-normal direction and equal to the Bernoulli pressure at the outer edge of the boundary layer.

When viewed in the context of the main subject matter of this book, the boundary-layer problem has it all. The highlights of what we are about to discuss are the following:

- We convert two coupled PDEs to a single PDE by the introduction of a stream function. In this way the underlying structure of the flow in two dimensions is seen to be Hamiltonian.
- The stream-function PDE is reduced to the third-order Blasius ODE through the use of similarity variables derivable from dimensional analysis and/or Lie analysis.
- The Lie group of the Blasius equation contains two symmetry operators with a solvable Lie algebra. See Chapter 8, Section 8.10.1. The Lie algebra is used to decide the order in which the symmetries must be used to reduce the problem to a first-order ODE. Failure to follow this order leads to a dead end.
- The Blasius solution is identified in the phase portrait of the resulting first-order ODE. This last step in the process requires a fairly sophisticated analysis of the asymptotic structure of the solution near the wall.
- The identified solution trajectory must be integrated twice to generate the Blasius velocity profile. The pros and cons are discussed of numerically integrating the third-order Blasius equation by iteration, treating it as a Cauchy initial-value problem. It is shown how knowledge of the Lie group of the equation can be used to map an initial numerical guess to the correct solution in one step.
- A nonlinear heat conduction problem is solved that demonstrates a discontinuous propagating front governed locally by a Hamilton–Jacobi equation. A slight modification of boundary conditions converts this problem to an exact thermal analogy of the Blasius problem.
- Following the discussion of the Blasius problem, a full Lie analysis of the stream-function PDE leads to infinite-dimensional symmetries. These

symmetries can be joined to the boundary-layer solution on a flat plate with a specified free-stream velocity distribution, thereby enabling solutions for an arbitrary wall shape to be determined from the flat plate result.

- The Lie analysis enables one to identify certain classes of free-stream velocity distributions that can be expected to yield similarity solutions. This leads to the famous Falkner–Skan class of boundary layers.

## 10.2 The Boundary-Layer Formulation

We are considering incompressible, viscous, laminar flow at high Reynolds number past a thin flat plate as shown in Figure 10.1. The flow, approaching the leading edge of the plate, splits about the forward stagnation point as shown in the close-up. In this region, the no-slip condition, together with the strong curvature of the plate surface, generates large velocity and pressure gradients for which no simplification of the governing equations is available. However, once the flow reaches well beyond the leading edge, the guiding influence of the plate comes into effect, and the streamwise gradients in velocity become small compared to the transverse velocity gradients. Meanwhile wall-normal gradients in pressure become small, and the thickness of the boundary layer increases through the diffusion of $x$-momentum from the free stream toward the wall.

The local Reynolds number based on distance from the leading edge is

$$Re = \frac{U_e[x]x}{\nu}. \tag{10.1}$$

As the Reynolds number increases, the boundary layer itself becomes more and more slender and the flow near the wall becomes more and more parallel to the wall.

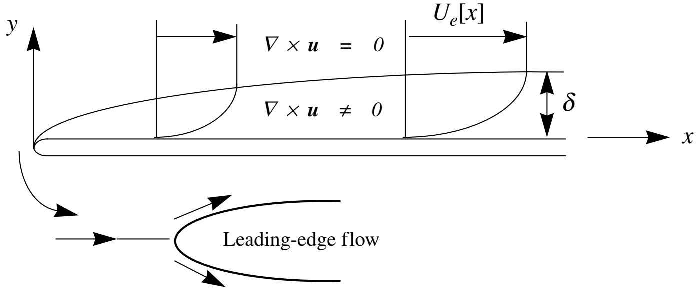


Fig. 10.1. Boundary layer on a flat plate.

In this limit the governing equations of motion reduce to the following set

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$
$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2},\tag{10.2}$$
$$\frac{\partial p}{\partial y}=0.$$

The irrotational flow outside the boundary layer satisfies the incompressible Bernoulli relation,

$$P_{\text{total}}=P_e[x]+\frac{1}{2}\rho U_e[x]^2.\tag{10.3}$$

Introduce the stream function to integrate the continuity equation

$$u=\frac{\partial\psi}{\partial y},\qquad v=-\frac{\partial\psi}{\partial x}.\tag{10.4}$$

Recall that the velocity components are the time derivatives of the position of a fluid particle, and so the equations for particle paths $(x[t],\ y[t])$ have the Hamiltonian structure,

$$\frac{dx}{dt}=\frac{\partial\psi}{\partial y},\qquad \frac{dy}{dt}=-\frac{\partial\psi}{\partial x}.\tag{10.5}$$

The last relation in (10.2) implies that $p=P_e[x]$ and the Bernoulli constant is used to replace the pressure term in the $x$-momentum equation. This leads to the boundary-layer equation for the stream function:

$$\psi_y\psi_{xy}-\psi_x\psi_{yy}-U_e\frac{dU_e}{dx}-\nu\psi_{yyy}=0.\tag{10.6}$$

The premise underlying the boundary-layer formulation is that the velocity distribution in the free stream, $U_e[x]$, is presumed to be a known function. Just how it gets determined is something we will consider later. The remaining boundary conditions needed to solve the problem are that the velocity must satisfy the no-slip condition at the plate surface and that the streamwise velocity must approach the free-stream value at large $y$:

$$\psi|_{y=0}=0,\qquad \left.\frac{\partial\psi}{\partial y}\right|_{y=0}=0,\qquad \left.\frac{\partial\psi}{dy}\right|_{y\to\infty}=U_e[x].\tag{10.7}$$

Let's begin with the zero-pressure-gradient case studied by Blasius [10.1].

## 10.3 The Blasius Boundary Layer

This is the case where the free-stream velocity is constant, $dU_e/dx = 0$, and the governing equation (10.6) reduces to

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} - \nu \psi_{yyy} = 0 \tag{10.8}$$

with boundary conditions

$$\psi[x, 0] = 0, \qquad \psi_y[x, 0] = 0, \qquad \psi_y[x, \infty] = U_e = \text{constant}. \tag{10.9}$$

Begin by checking the invariance of the governing equation under a three-parameter dilation group

$$\tilde{x} = e^a x, \qquad \tilde{y} = e^b y, \qquad \psi = e^c \psi, \tag{10.10}$$

where $a$, $b$, and $c$ are initially independent group parameters. Substitute (10.10) into (10.8). The result is

$$\tilde{\psi}_{\tilde{y}} \tilde{\psi}_{\tilde{x}\tilde{y}} - \tilde{\psi}_{\tilde{x}} \tilde{\psi}_{\tilde{y}\tilde{y}} - \nu \tilde{\psi}_{\tilde{y}\tilde{y}\tilde{y}}$$
$$= e^{2c-a-2b} \psi_y \psi_{xy} - e^{2c-a-2b} \psi_x \psi_{yy} - \nu e^{c-3b} \psi_{yyy} = 0. \tag{10.11}$$

The equation is invariant if and only if the exponents are all equal:

$$2c - a - 2b = c - 3b. \tag{10.12}$$

Note that the invariance relies on the fact that the equation is equal to zero, so that the common exponential term that factors out of the left-hand side can be canceled and the governing equation in untilde'd variables is (10.8). So far we have invariance of the equation under a two-parameter group

$$\tilde{x} = e^a x, \qquad \tilde{y} = e^b y, \qquad \tilde{\psi} = e^{a-b} \psi. \tag{10.13}$$

What about boundary conditions? Before we can conclude that the problem has a similarity solution, we must show that the whole problem is invariant. The boundary curve $y = 0$ is clearly invariant under the group (10.13), since application of the group yields

$$\tilde{y} = 0 \quad \Rightarrow \quad e^b y = 0 \quad \Rightarrow \quad y = 0. \tag{10.14}$$

Note that the fact that the plate extends to infinity is crucial to the invariance. A finite plate would have its length changed by the dilation, thus breaking the symmetry of the problem. This is a theme we have emphasized before. Our abstraction of the problem has a built-in assumption that in a real flat-plate flow, a region exists that is far downstream of the leading edge and far upstream of the end of the plate where it can be argued that the Reynolds number is large and that the real flow should closely approximate the similarity solution treated here.

Now for the remaining boundary conditions; at the wall,

$$\tilde{\psi}[\tilde{x}, 0] = 0|_{\text{all } \tilde{x}} \quad \Rightarrow \quad e^{a-b}\psi[e^a x, 0] = 0 \quad \Rightarrow \quad \psi[x, 0] = 0|_{\text{all } x} \tag{10.15}$$

and

$$\tilde{\psi}_{\tilde{y}}[\tilde{x}, 0] = 0|_{\text{all } \tilde{x}} \quad \Rightarrow \quad e^{a-2b}\psi_y[e^a x, 0] = 0 \quad \Rightarrow \quad \psi_y[x, 0] = 0|_{\text{all } x}. \tag{10.16}$$

The boundary condition at $y \to \infty$ restricts the group:

$$\tilde{\psi}_{\tilde{y}}[\tilde{x}, \infty] = U_e|_{\text{all } x} \quad \Rightarrow \quad e^{a-2b}\psi_y[e^a x, \infty] = U_e|_{\text{all } x} \tag{10.17}$$

Invariance of the outer boundary condition, namely,

$$\psi_y[x, \infty] = U_e|_{\text{all } x}, \tag{10.18}$$

is only satisfied of we require $a = 2b$. The notation $(\,)|_{\text{all } x}$ is intended to convey the fact that the range indicated by $e^a x|_{\text{all } x}$ is the same as $x|_{\text{all } x}$; this validates the removal of the exponential in the function arguments in (10.15), (10.16), and (10.17), completing the invariance argument.

### *10.3.1 Similarity Variables*

Finally, the group that leaves the whole problem invariant is

$$\tilde{x} = e^{2b}x, \qquad \tilde{y} = e^b y, \qquad \tilde{\psi} = e^b \psi \tag{10.19}$$

with infinitesimals

$$\xi = 2x, \qquad \zeta = y, \qquad \eta = \psi \tag{10.20}$$

and characteristic equations

$$\frac{dx}{2x} = \frac{dy}{y} = \frac{d\psi}{\psi}. \tag{10.21}$$

Integrating the characteristic equations and using $U_e$ and $\nu$ to nondimensionalize variables leads to the group invariants

$$F = \frac{\psi}{(2\nu U_e x)^{1/2}}, \qquad \alpha = \frac{y}{(2\nu x/U_e)^{1/2}}. \tag{10.22}$$

Since the equation in (10.8) with boundary conditions (10.9) is invariant under the group (10.19), the solution of the problem must be invariant under the same group. That is, the solution must be some general function of the form

$$\omega = \Omega\left[\frac{\psi}{(2\nu U_e x)^{1/2}}, \frac{y}{(2\nu x/U_e)^{1/2}}\right]. \tag{10.23}$$

With equal generality, we can say that the solution is of the form

$$\frac{\psi}{(2\nu U_e x)^{1/2}} = F[\alpha]. \tag{10.24}$$

Let's consider what line of reasoning would be required to reach the result (10.24) by means of dimensional analysis. The Buckingham Pi algorithm applied to the parameters $(x, y, \psi, \nu, U_e)$ would lead to the dimensionless combinations

$$\Pi_1 = \frac{\psi}{(2\nu U_e x)^{1/2}}, \qquad \Pi_2 = \frac{y}{(2\nu x/U_e)^{1/2}}, \qquad \Pi_3 = \frac{y}{x}. \tag{10.25}$$

The variables $\Pi_1$ and $\Pi_2$ are the ones we just derived, but $\Pi_3$ does not come up in the group approach. The reason is that the group method *starts* with the boundary-layer equations. In effect, it begins with a knowledge of the boundary-layer approximation. In contrast, pure dimensional analysis presupposes nothing about the physics.

The boundary-layer approximation distinguishes physically between the streamwise and cross-stream coordinates. This is evidenced by the invariance of the governing equation under the two-parameter group (10.19) where $a$ and $b$ are independent parameters (the full equations are invariant under only a one-parameter dilation group where $a$ and $b$ are equal). Without the physical understanding derived from Prandtl's experiments and contained in the boundary-layer approximation, there is no logical basis for assuming that the problem cannot depend on $\Pi_3$.

This highlights the fundamental difference between the group-theoretical approach and dimensional analysis. With just the parameters in hand, there is no way to demonstrate that the spatial coordinates $x$ and $y$ are not equivalent; in a sense one has to anticipate the boundary-layer approximation in order to get to the correct dimensional-analysis result. Group theory only begins when all of the science needed to deduce the boundary-layer approximation has been established. When one begins with the boundary-layer equations and constant $U_e$, the variable $y/x$ never comes up. This is clear from the $2x$ that appears in (10.21). Nevertheless, much of what Prandtl was eventually able to deduce came from the clever and repeated use of dimensional analysis.

In the end, dimensional analysis and the group methodology go hand in hand, and the development of a new theory is not a one-step process; hypothesis, theory, experiment, revised hypothesis, revised theory, etc. all play a role in what eventually becomes a refined, validated result. Dimensional analysis alone isn't enough, nor is group theory, and both must be checked by experiment.

In any case, the basic symmetry of the problem eventually finds its way into the solution. The self-similar velocities are

$$\frac{u}{U_e} = F_\alpha, \qquad \frac{v}{U_e} = \frac{1}{\sqrt{2}}\left(\frac{1}{Re}\right)^{1/2}(F - \alpha F_\alpha). \tag{10.26}$$

The appearance of the Reynolds number in the denominator of the expression for the normal velocity component supports the qualitative argument made earlier concerning the flow becoming more and more parallel to the wall as the Reynolds number is increased ($\lim_{Re\to\infty} v/u = 0$). Note however that an assumption of exactly parallel flow would produce an erroneous result. Upon substitution of (10.22) into (10.8) the result is the Blasius equation

$$F_{\alpha\alpha\alpha} + F F_{\alpha\alpha} = 0 \tag{10.27}$$

with boundary conditions

$$F[0] = 0, \qquad F_\alpha[0] = 0, \qquad F_\alpha[\infty] = 1. \tag{10.28}$$

### 10.3.2 Reduction of Order; The Phase Plane

The problem now is to reduce the order of (10.27). To accomplish this we need to work out the symmetries of (10.27). In fact, we already did this in Chapter 8, Section 8.10.1. There we transformed the equation according to

$$\begin{aligned} \tilde{\alpha} &= \alpha + s\xi[\alpha, F], \\ \tilde{F} &= F + s\eta[\alpha, F]. \end{aligned} \tag{10.29}$$

Using the Lie algorithm described in Chapter 8, the infinitesimals were determined to be

$$\xi = a + b\alpha, \qquad \eta = -bF. \tag{10.30}$$

The Blasius equation is invariant under a two-parameter group, which we recall is always guaranteed to have a solvable Lie algebra. This ensures that the order of (10.27) can be reduced by two. To reiterate the results in Chapter 8,

Table 10.1.
*Commutator table for the Blasius equation.*

| | $X^a$ | $X^b$ |
|---|---|---|
| $X^a$ | 0 | $X^a$ |
| $X^b$ | $-X^a$ | 0 |

the infinitesimal generators of (10.30) are

$$X^a = \frac{\partial}{\partial \alpha}, \qquad X^b = \alpha \frac{\partial}{\partial \alpha} - F \frac{\partial}{\partial F} \tag{10.31}$$

with the commutator table given in Table 10.1.

As was pointed out in Chapter 8, Section 8.10 the order in which we use these groups to reduce the Blasius equation is important. We must begin with the ideal $X^a$. The characteristic equations of the thrice extended operator $X^a_{\{3\}}$ are

$$\frac{d\alpha}{1} = \frac{dF}{0} = \frac{dF_\alpha}{0} = \frac{dF_{\alpha\alpha}}{0} = \frac{dF_{\alpha\alpha x}}{0}, \tag{10.32}$$

and the first two invariants are

$$\phi = F, \qquad G = F_\alpha. \tag{10.33}$$

By the method of differential invariants, the equation

$$\frac{dG}{d\phi} = \frac{\frac{\partial G}{\partial \alpha}\, d\alpha + \frac{\partial G}{\partial F}\, dF + \frac{\partial G}{\partial F_\alpha}\, dF_\alpha}{\frac{\partial \phi}{\partial \alpha}\, d\alpha + \frac{\partial \phi}{\partial F}\, dF} = \frac{F_{\alpha\alpha}}{F_\alpha} \tag{10.34}$$

is an invariant, as is

$$\frac{d^2G}{d\phi^2} = \left(\frac{F_\alpha F_{\alpha\alpha\alpha} - F_{\alpha\alpha}^2}{F_\alpha^2}\right)\frac{1}{F_\alpha} = \frac{F_\alpha(-FF_{\alpha\alpha}) - F_{\alpha\alpha}^2}{F_\alpha^3}, \tag{10.35}$$

where the Blasius equation has been used to replace the third derivative. As expected, Equation (10.35) can be rearranged to read entirely in terms of the new invariants:

$$GG_{\phi\phi} + \phi G_\phi + (G_\phi)^2 = 0. \tag{10.36}$$

The boundary conditions on (10.36) are

$$G(0) = 0, \qquad G(\infty) = 1, \tag{10.37}$$

and so the third-order Blasius equation has been reduced to the solution of a second-order system plus the quadrature,

$$\frac{dF}{d\alpha} = G[F]. \tag{10.38}$$

Now we determine the action of the second group $\tilde{\alpha} = e^{b}\alpha$, $\tilde{F} = e^{-b}F$ on the new variables $(\phi, G)$. The appropriate group in new variables is

$$\tilde{\phi} = e^{-b}\phi, \qquad \tilde{G} = e^{-2b}G. \tag{10.39}$$

When (10.39) is used to transform Equation (10.36), the result is

$$\tilde{G}\tilde{G}_{\tilde{\phi}\tilde{\phi}} + \tilde{\phi}\tilde{G}_{\tilde{\phi}} + (\tilde{G}_{\tilde{\phi}})^2 = e^{-2b}(GG_{\phi\phi} + \phi G_{\phi} + (G_{\phi})^2) = 0. \tag{10.40}$$

As expected, the reduced equation inherits the symmetry under $X^b$ of the original Blasius equation.

The reader may wonder why in dealing with ODEs we are not concerned with demonstrating the invariance of the boundary conditions (10.37) and (10.28). In fact they are not invariant; for example, $\tilde{G}[\infty] = 1 \Rightarrow G[\infty] = e^{2b}$. When we apply group analysis to an ODE, the role of boundary conditions recedes somewhat, at least until the final solution is determined. The invariant (multiparameter) group of an ODE transforms the entire solution family to itself while individual solution curves are mapped to different curves [recall Chapter 1, Example 1.1, Equations (1.20) and (1.21)]. Only one curve in the family actually satisfies the given boundary conditions. When the final solution is determined, the boundary conditions are often not invariant under the group of the ODE, in contrast to the invariance of the boundary conditions of the original PDE. It may not seem so now, but this is a highly useful fact. It means that one can often use the group to map an initial guess to the correct solution through a specific choice of the group parameter. We shall return to this point shortly.

Now solve the characteristic equations of (10.39),

$$\frac{d\phi}{-\phi} = \frac{dG}{-2G} = \frac{dG_{\phi}}{-G_{\phi}}. \tag{10.41}$$

The invariants (new variables) at the second stage are

$$\begin{aligned} \gamma &= \frac{G}{\phi^2} = \frac{F_{\alpha}}{F^2}, \\ H(\gamma) &= \frac{G_{\phi}}{\phi} = \frac{F_{\alpha\alpha}}{FF_{\alpha}}. \end{aligned} \tag{10.42}$$

The method of differential invariants is used to generate the twice reduced equation

$$\frac{\left(\frac{DH}{D\phi}\right)}{\left(\frac{D\gamma}{D\phi}\right)} = \frac{dH}{d\gamma} = \frac{H_\phi + H_G \frac{dG}{d\phi} + H_{G_\phi} \frac{dG_\phi}{d\phi}}{\gamma_\phi + \gamma_G \frac{dG}{d\phi}} = \frac{-\frac{G_\phi}{\phi^2} + \frac{1}{\phi}(G_{\phi\phi})}{-2\frac{G}{\phi^3} + \frac{1}{\phi^2} G_\phi}. \quad (10.43)$$

Equation (10.36) is used to eliminate the second-derivative term in (10.43), and the right-hand side of (10.43) can be rearranged to read as follows:

$$\frac{dH}{d\gamma} = \frac{\gamma H + H + H^2}{2\gamma^2 - \gamma H}. \quad (10.44)$$

This equation was discussed extensively in Chapter 3, Example 3.7, in connection with phase-plane techniques.

The phase portrait derived from the autonomous system

$$\begin{aligned} \frac{dH}{ds} &= \gamma H + H + H^2, \\ \frac{d\gamma}{ds} &= 2\gamma^2 - \gamma H \end{aligned} \quad (10.45)$$

is shown in Figure 10.2. See the isocline plot of the same system in Chapter 3, Figure 3.6. The critical points are clearly identifiable as points where the local slope becomes indeterminate. Solving for the roots of

$$\begin{aligned} 2\gamma^2 - \gamma H &= 0, \\ \gamma H + H + H^2 &= 0, \end{aligned} \quad (10.46)$$

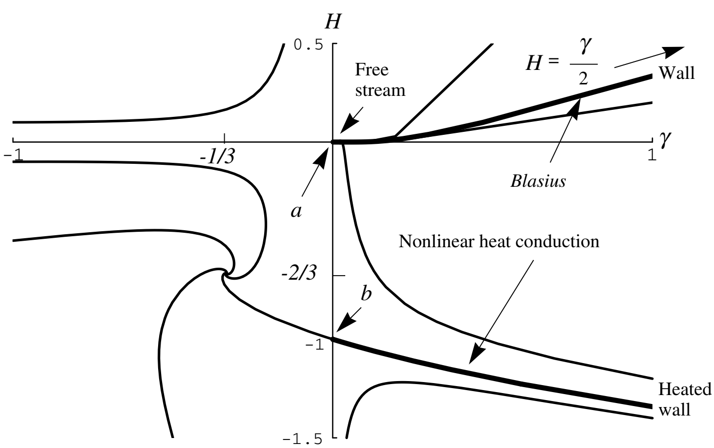


Fig. 10.2. Phase portrait of the Blasius system (10.45).

we find critical points at $(\gamma_c, H_c) = (0,0)$, $(0,-1)$, and $(-\frac{1}{3}, -\frac{2}{3})$. The critical point at $(0,0)$ is nonlinear. The critical point at $(0,-1)$ with invariants $(P, Q) = (0,-1)$ is a saddle, and that at $(-\frac{1}{3}, -\frac{2}{3})$ with invariants $(P, Q) = (\frac{4}{3}, \frac{2}{3})$ is a stable focus. The eigenvectors at the saddle $(0,-1)$ give the precise orientation of the trajectories that pass through the saddle.

Notice that the final solution in the $(\gamma, H)$ plane – $H = \Phi[\gamma]$ or, in terms of earlier variables,

$$\frac{F_{\alpha\alpha}}{F F_\alpha} = \Phi[F_\alpha / F^2] \tag{10.47}$$

– is invariant under the original two-parameter finite group of the Blasius equation,

$$\begin{aligned} \tilde{\alpha} &= e^b \alpha + a, \\ \tilde{F} &= e^{-b} F, \\ \tilde{F}_{\tilde{\alpha}} &= e^{-2b} F_\alpha. \end{aligned} \tag{10.48}$$

But which trajectory in the $(\gamma, H)$ plane is the correct one for the given boundary conditions? We have tracked the boundary conditions of the problem through the first reduction, but the reduced conditions, (10.37), are not very helpful for solving the first-order equation. We need to take a closer look at the problem.

The friction coefficient at the wall is

$$C_f = \left.\frac{\tau_{xy}}{\frac{1}{2}\rho U_e^2}\right|_{y=0} = \left(\frac{2\nu}{x U_e}\right)^{1/2} F_{\alpha\alpha}[0]. \tag{10.49}$$

Recognizing that the stress at the wall cannot be infinite or zero, it can be argued that $F_{\alpha\alpha}$ must be finite at the wall except near the leading edge of the plate, where the boundary-layer approximation breaks down anyway. Let $\tau_0 = F_{\alpha\alpha}[0]$. In the neighborhood of the wall,

$$\frac{G_\phi}{\phi} = \frac{\tau_0}{F F_\alpha} = \frac{\tau_0}{\phi G}. \tag{10.50}$$

Thus near the wall,

$$G = (2\tau_0 \phi)^{1/2}. \tag{10.51}$$

Note that this result is consistent with the boundary condition, $G[0] = 0$.

Now we have enough information to determine the behavior of $H$ and $\gamma$ near the wall:

$$\lim_{\phi\to 0} H = \frac{1}{2}\left(\frac{2\tau_0}{\phi^3}\right)^{1/2},$$
$$\lim_{\phi\to 0} \gamma = \left(\frac{2\tau_0}{\phi^3}\right)^{1/2}. \tag{10.52}$$

Note that in the limit $\phi \to 0$ both $H$ and $\gamma$ become infinite. Moreover, both variables are positive. Thus the solution trajectory lies in the upper right quadrant of Figure 10.2 and asymptotes to

$$\lim_{\gamma\to\infty}\left(\frac{H}{\gamma}\right) = \frac{1}{2}. \tag{10.53}$$

The trajectory of the solution in the $(\gamma, H)$ plane is shown as the bold curve in the upper right quadrant of Figure 10.2. The same solution is shown in more detail in Figure 10.3. Notice that this curve does not depend on the explicit value of $\tau_0$. In fact, this single curve in the $(\gamma, H)$ plane corresponds to an infinity of curves in the $(\phi, G)$ plane and a double infinity of curves in the $(\alpha, F)$ plane, corresponding to a range of values of $\tau_0$. Only one curve in, say, the $(\alpha, F_\alpha)$ plane can satisfy the boundary condition $F_\alpha[\infty] = 1$, and this curve will correspond to the correct value of $\tau_0$.

Note the high degree of tangency as the solution approaches the free stream in Figure 10.3. We shall have more to say about this later, but for now it can be pointed out that this is the main physical feature of the problem that is captured

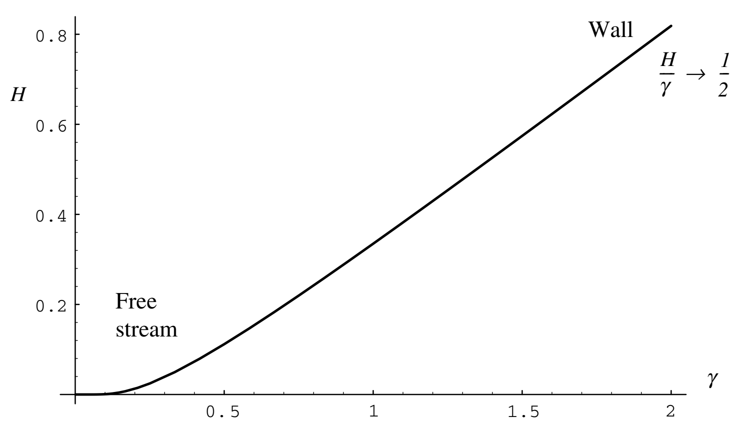


Fig. 10.3. Blasius solution in the phase plane.

by the boundary-layer approximation and that supports the concept of dividing the flow into two distinct regions. Prandtl was right: the boundary layer has a distinct outer edge where the vorticity decays exponentially with distance from the plate.

In summary, we have finally reduced the third-order Blasius equation to a first-order equation plus two quadratures,

$$\frac{G_\phi}{\phi} = H[G/\phi^2] \tag{10.54}$$

and (10.38). At first sight (10.54) may not seem like a simple quadrature, but consider

$$\frac{d(G/\phi^2)}{d\phi} = -\frac{2G}{\phi^3} + \frac{G_\phi}{\phi^2} = \frac{1}{\phi}\left(H[G/\phi^2] - \frac{2G}{\phi^2}\right). \tag{10.55}$$

Once the function $H[\gamma]$ is determined, then the function $G[\phi]$ is determined from

$$\frac{d\phi}{\phi} = \frac{d(G/\phi^2)}{H(G/\phi^2) - 2G/\phi^2} = \frac{d\gamma}{H[\gamma] - 2\gamma}. \tag{10.56}$$

Integrating (10.56) and applying the boundary conditions (10.37), then integrating (10.38) and applying the boundary conditions (10.28), leads to the solution $F[\alpha]$. The Blasius velocity profile is shown in Figure 10.4. Evaluating the second derivative at the wall leads to $\tau_0 = 0.46965$, and the friction coefficient at the wall is

$$C_{f0} = \frac{\tau_0}{\frac{1}{2}\rho U_e^2} = \frac{0.664}{\sqrt{Re}}. \tag{10.57}$$

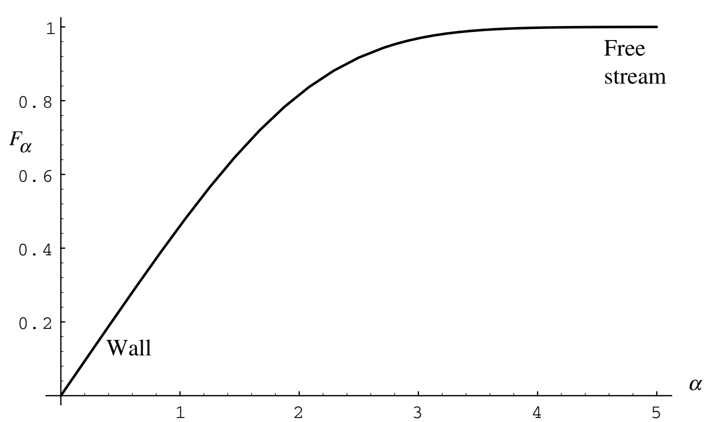


Fig. 10.4. The Blasius velocity profile.

This goes back to the point made above concerning boundary conditions and ODEs. Any curve in the $(\gamma, H)$ plane depicted in Figure 10.2 is invariant under (10.48) and, as just noted, can be mapped to a whole family of geometrically similar curves in the $(\alpha, F)$ plane by arbitrarily choosing the values of the group parameters $(a, b)$. In effect, the group (10.48) maps the complete solution family of the Blasius equation in the $(\alpha, F)$ plane to itself. Of course, only one of these curves actually satisfies the given boundary conditions of the problem, $F[0] = 0$, $F_\alpha[0] = 0$, $F_\alpha[\infty] = 1$. The ability to map one solution curve into another using the group provides a powerful method for finding the desired solution.

### *10.3.3 Numerical Solution of the Blasius Equation as a Cauchy Initial-Value Problem*

The Blasius problem puts into focus an interesting debate about the efficacy of symmetry analysis. The procedure of reducing the problem to first order and then numerically integrating to find the trajectory of the solution in the phase plane, then integrating a second time to find the velocity profile and perhaps a third time to find the stream function, is a lot of work. If all one wants is the velocity profile, then it is a lot easier to numerically integrate the original third-order Blasius equation. In this approach one treats the Blasius problem as a Cauchy initial-value problem with initial conditions $F[0]$, $F_\alpha[0] = 0$, and $F_{\alpha\alpha}[0] = \tau_0$. The constant $\tau_0$ is continuously adjusted until the numerical solution of the initial-value problem matches the free-stream boundary condition, $F_\alpha[\infty] = 1$. A few minutes of iteration on a hand calculator is all that is required to reach the value, $\tau_0 = 0.46965$. The process is illustrated in Figure 10.5.

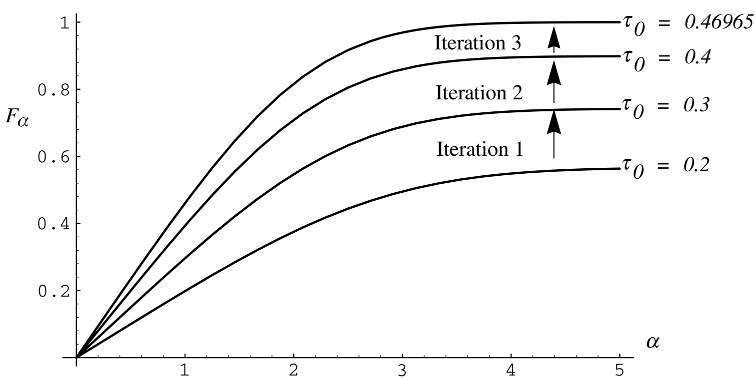


Fig. 10.5. Iteration process leading to the correct match with the free-stream boundary condition $\lim_{\alpha\to\infty} F_\alpha = 1$.

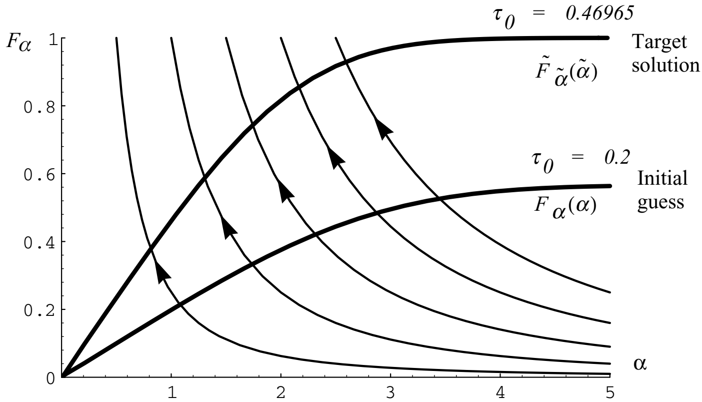


Fig. 10.6. Mapping of an initial guess to the correct solution along the pathlines of the dilation group of the Blasius equation.

So why bother with all the extra work connected with the use of symmetries? One reason becomes clear when it is realized that once the appropriate symmetry group is known, one can directly transform an initial guess to the final solution by correctly choosing the value of the group parameter. The mapping is accomplished using the group

$$\begin{aligned} \tilde{\alpha} &= e^{b}\alpha, \\ \tilde{F} &= e^{-b}F, \\ \tilde{F}_{\tilde{\alpha}} &= e^{-2b}F_{\alpha}, \\ \tilde{F}_{\tilde{\alpha}\tilde{\alpha}} &= e^{-3b}F_{\alpha\alpha}. \end{aligned} \tag{10.58}$$

This procedure is illustrated in Figure 10.6. The lightweight lines in Figure 10.6 are the pathlines of (10.58) along which the solution is mapped. The initial guess using $F_{\alpha\alpha}[0] = 0.2$ produces $\lim_{\alpha\to\infty} F_{\alpha} = 0.566067$. The value of the group parameter needed to scale this result to the correct solution is given by

$$1 = e^{-2b}(0.566067) \quad \Rightarrow \quad b = -0.284557. \tag{10.59}$$

The correct wall stress is also determined immediately from the mapping of $F_{\alpha\alpha}$:

$$0.2 = e^{-3b}F_{\alpha\alpha}[0] \quad \Rightarrow \quad F_{\alpha\alpha}[0] = 0.46965. \tag{10.60}$$

Thus the correct numerical solution is reached in one step.

A broader vision behind symmetry analysis is that it enables one to attack an equation in light of its symmetries – symmetries that approximately describe the real objects modeled by the equation. Symmetry analysis enables one to examine a problem in a general context that can reach far beyond the isolated consideration of a specific set of boundary conditions. Together with phase-space analysis, it provides a means for understanding the whole manifold in which a particular solution is imbedded. Often this leads to connections that might never be revealed through straight numerical analysis. In fact, this is probably the most general criticism that can be leveled at a purely numerical approach: that a broad view of the problem is rarely revealed by a single approximate result. To make this point concrete, we will now consider a problem in nonlinear heat conduction and then relate it to the Blasius problem.

## 10.4 Temperature Gradient Shocks in Nonlinear Diffusion

Now we consider heat conduction in a medium where the thermal diffusivity is proportional to the temperature, $\kappa = \lambda T$. The governing nonlinear heat equation is

$$\frac{\partial T}{\partial t} = \lambda \frac{\partial}{\partial x}\left(T \frac{\partial T}{\partial x}\right). \tag{10.61}$$

This is a reasonable model for heat conduction in gases, although a more realistic model would be $\kappa = \lambda T^{\sigma}$ where $\sigma$ is in the range $0.5 < \sigma < 1$ (see Exercise 10.4). Let's consider the diffusion of heat in a semiinfinite slab where the temperature of the wall is impulsively set to $T_0$. The boundary conditions are

$$\begin{aligned} T[x,0] &= 0, \qquad x > 0, \\ T[0,t] &= T_0, \qquad t > 0. \end{aligned} \tag{10.62}$$

Note that the temperature appears explicitly in the governing equation, making it not invariant under translation in temperature. This prevents us from working the problem in terms of temperature differences; the zero boundary condition at infinity is truly zero. Note that this implies that the diffusivity at infinity is also zero.

This problem is invariant under a one-parameter dilation group

$$\tilde{x} = e^s x, \qquad \tilde{t} = e^{2s} t, \qquad \tilde{T} = T \tag{10.63}$$

with characteristic equations

$$\frac{dx}{x} = \frac{dt}{2t} = \frac{dT}{0}. \tag{10.64}$$

The corresponding similarity form of the solution is

$$\frac{T}{T_0} = g[\theta], \qquad \theta = \frac{x}{(2\lambda T_0 t)^{1/2}}. \tag{10.65}$$

When we substitute (10.65) into (10.61), the result is a nonlinear second-order ODE,

$$g g_{\theta\theta} + \theta g_\theta + (g_\theta)^2 = 0, \tag{10.66}$$

with boundary conditions,

$$g[0] = 1, \qquad g[\infty] = 0. \tag{10.67}$$

### *10.4.1 First Try: Solution of a Cauchy Initial Value Problem – Uniqueness*

Let's try to solve the boundary-value problem (10.66) and (10.67) as a Cauchy initial-value problem. This is the same approach we used to solve the third-order Blasius problem in Section 10.3.3, except that here we are dealing with just a second-order equation and so things should be easier. A trial value of $g_\theta[0]$ is selected, and the solution is computed numerically, beginning at $\theta = 0$ and progressing to large values of $\theta$, in the hope that the solution will approach the far-field boundary condition, $g[\infty] = 0$. If it fails to do so, a new value of $g_\theta[0]$ is selected and the process is repeated. This is an entirely reasonable strategy given the fundamentally diffusive nature of the problem and the expectation that the solution should be smooth everywhere.

Figure 10.7. shows several trial solutions, and as we can see, there is a bit of a problem. For $g_\theta[0] < -0.627554$ there appear to be an infinite number of possible solutions that can match the zero boundary condition at infinity. Furthermore, these solutions reach $g = 0$ at a finite distance from the origin, $\theta_0$, where the temperature gradient appears to be discontinous. Our intuition would suggest that the gradient shouldn't be discontinuous; that perhaps there should be some sort of very thin diffusion layer near $g = 0$ that we simply can't resolve with our numerical method.

Is the solution nonunique? Is it discontinuous? There does seem to be one trajectory that is special, corresponding to $g_\theta[0] = -0.627554$, but before we can conclude anything about the solution, we need an additional piece of information. Where will it come from? Group theory and phase-plane analysis will show the way.

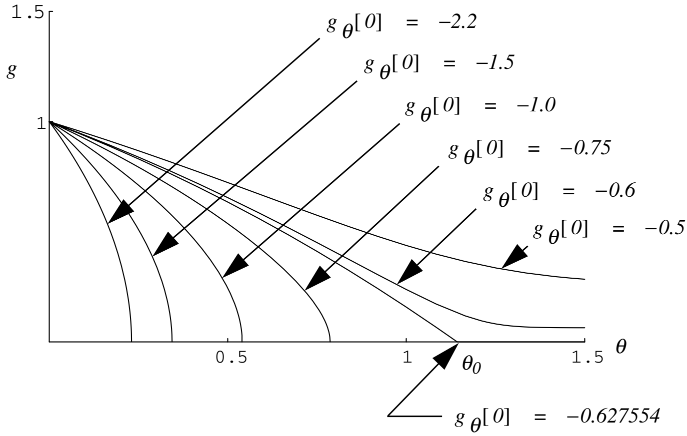


Fig. 10.7. Trial solutions of (10.66) and (10.67) with specified initial slope.

### *10.4.2 Second Try: Solution Using Group Theory*

Note that Equation (10.66) is identical to the second-order equation (10.36), once-reduced from the Blasius equation. Thus we can expect invariance of (10.66) under the group

$$\tilde{\theta} = e^{-b}\theta, \qquad \tilde{g} = e^{-2b} g \tag{10.68}$$

with new variables,

$$\begin{aligned} \gamma &= \frac{g}{\theta^2}, \\ H &= \frac{g_\theta}{\theta}. \end{aligned} \tag{10.69}$$

The reduction to first order of course leads to the same equation, (10.44), and the same phase portrait depicted in Figure 10.2.

But we face the same question we did earlier with the Blasius problem. Which trajectory in the phase portrait corresponds to the solution? Figure 10.2 shows that the solution trajectory must lie wholly in one of the four quadrants; this is ensured by the vectors aligned with the coordinate axes, which prevent any trajectories from crossing the coordinate axes. Furthermore, we know that the solution lies in the range $\theta > 0$, and that somewhere $g > 0$ and therefore $\gamma > 0$. Thus the solution is precluded from lying to the left of the origin in Figure 10.2. In addition, since the temperature in the far field goes to zero, there must be some point where $g_\theta < 0$ and $H < 0$. This implies that the solution must lie in the lower right quadrant in Figure 10.2. That narrows things quite a bit. Now we make the reasonable assumption that far from the hot wall, at the point $\theta_0$ where $g = 0$, the second derivative $g_{\theta\theta}$ is either finite or zero. At that point Equation (10.66) reduces to

$$g_\theta|_{g=0}(g_\theta|_{g=0} + \theta_0) = 0, \tag{10.70}$$

where $\theta_0$ is still to be determined. At the point where the temperature reaches zero, the derivative has two possible values,

$$g_\theta|_{g=0} = 0, -\theta_0, \tag{10.71}$$

and the coordinates in the phase plane corresponding to the physical far boundary of the solution also have two possible values,

$$(\gamma H) = (0, 0), \qquad (\gamma, H) = (0, -1). \tag{10.72}$$

The only solution trajectory in Figure 10.2 that can satisfy this condition is the one that joins the origin with the saddle at $(\gamma, H) = (0, -1)$ and then passes to infinity (i.e., to the heated wall) in the lower right quadrant. This analysis tells us that, indeed, the solution does reach $g = 0$ at some finite distance $\theta_0$ from the heat source and that the temperature gradient is discontinuous there.

The saddle point at $(\gamma, H) = (0, -1)$ contains all the information we need to determine the solution from an initial-value problem, but one that starts at the outer boundary and integrates to the wall. The autonomous pair that determines the phase portrait in Figure 10.2 is

$$\begin{aligned} \frac{d\gamma}{ds} &= 2\gamma^2 - \gamma H = A(\gamma, H), \\ \frac{dH}{ds} &= \gamma H + H + H^2 = B(\gamma, H). \end{aligned} \tag{10.73}$$

If we linearize this system near the saddle in Figure 10.2, the result is

$$\begin{bmatrix} \dfrac{d\gamma}{ds} \\ \dfrac{dH}{ds} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial A}{\partial \gamma} & \dfrac{\partial A}{\partial H} \\ \dfrac{\partial B}{\partial \gamma} & \dfrac{\partial B}{\partial H} \end{bmatrix}\Bigg|_{(\gamma,H)=(0,-1)} \begin{bmatrix} \gamma \\ H+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \gamma \\ H+1 \end{bmatrix}. \tag{10.74}$$

The eigenvalues at the saddle are $\lambda_i = \pm 1$, and the normalized eigenvectors are

$$e_1 = \begin{bmatrix} 1 \\ -\dfrac{1}{2} \end{bmatrix}, \qquad e_{-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{10.75}$$

The slope of the eigenvector emanating from the saddle is

$$\frac{dH}{d\gamma} = -\frac{1}{2}. \tag{10.76}$$

Using the initial values $H[0] = -1$, $H_\gamma[0] = -\frac{1}{2}$, the integration proceeds along the diverging sepratrix shown as a bold line in Figure 10.2.

*10.4.2.1 Asymptotic Solution Near $\theta = \theta_0$*

The slope of the eigenvector at $(\gamma, H) = (0, -1)$ tells us that near $\theta_0$,

$$H = -\tfrac{1}{2}\gamma - 1, \tag{10.77}$$

which we can write as

$$\frac{g_\theta}{\theta} = -\frac{1}{2}\frac{g}{\theta^2} - 1, \tag{10.78}$$

or

$$\theta\frac{d}{d\theta}\left(\frac{g}{\theta^2}\right) = -\frac{5}{2}\left(\frac{g}{\theta^2}\right) - 1. \tag{10.79}$$

Equation (10.79) can be integrated:

$$\int_0^{g/\theta^2} \frac{d\gamma}{\frac{5}{2}\gamma + 1} = -\int_{\theta_0}^{\theta} \frac{d\theta}{\theta}, \tag{10.80}$$

to generate the asymptotic solution near the outer boundary,

$$\lim_{\theta \to \theta_0} g[\theta] = \frac{2}{5}\left(\frac{\theta_0^{5/2}}{\theta^{1/2}} - \theta^2\right). \tag{10.81}$$

The local solution (10.81) can be used to evaluate the derivatives of the full solution at $\theta = \theta_0$. At this boundary we have

$$\begin{aligned} g[\theta_0] &= 0, \\ g_\theta[\theta_0] &= -\theta_0, \\ g_{\theta\theta}[\theta_0] &= -\tfrac{1}{2}. \end{aligned} \tag{10.82}$$

Remarkably, through a combination of group theory and phase-plane analysis, we have discovered all of the needed properties of the solution at the boundary, including the value of the second derivative.

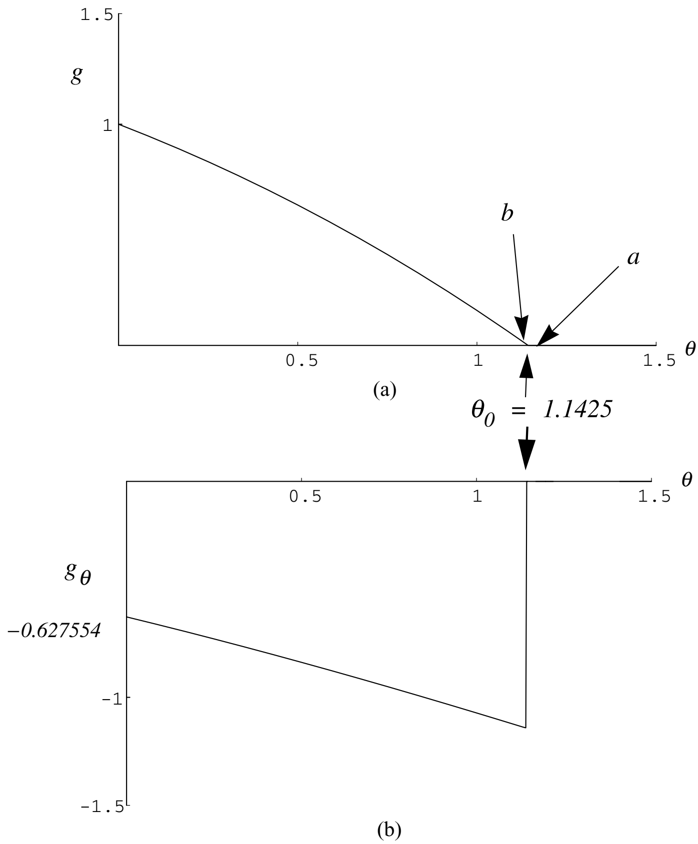


Fig. 10.8. Propagation of a thermal front in a nonlinear medium: (a) self-similar temperature, (b) self-similar temperature gradient.

### *10.4.3 The Solution*

We can now use (10.82), together with the shooting method used in Section 10.4.4 to produce Figure 10.7, to determine the correct and unique solution of the problem. The result is shown in Figure 10.8. The correct trajectory is found by adjusting $g_\theta[0]$ until $g_{\theta\theta} = -\frac{1}{2}$ is reached at $g = 0$. Alternatively, one can adjust $g_\theta[0]$ until $g_\theta[\theta_0] = -\theta_0$ is reached. Both conditions occur together, as indicated by the analysis that led us to (10.82). Only one trajectory satisfies (10.82).

So it does turn out that the solution exhibits behavior that is somewhat surprising in view of the diffusive nature of the problem. Whereas heat conduction in a linear medium produces a temperature field that extends to infinity with an exponential decay, the nonlinear medium produces a temperature field with a finite signal speed terminated by a temperature-gradient *shock* that propagates with a speed proportional to $t^{1/2}$ into the undisturbed field. It is important to recognize that the temperature gradient shock is a true discontinuity in the derivative. A numerical solution of the same problem that ignores the phase plane is almost certain to smooth out the solution, possibly leading to the erroneous conclusion that there exists a thin internal diffusion layer near the shock. In fact the phase portrait suggests exactly how the numerical solution should proceed if high precision is desired. One begins at the saddle and integrates in the direction of the eigenvector emanating along the separatrix leading away from the saddle into the lower right quadrant. This provides the precise direction for the first numerical step away from the saddle; the integration then continues to large values of $\gamma$. In similarity coordinates, one is beginning at the temperature gradient shock and integrating backward toward the heated boundary. Finally, notice how the line joining the origin at $a$ and the saddle at $b$ in the phase portrait, Figure 10.2, is collapsed into the corner of the temperature gradient shock shown in Figure 10.8.

One can make a further physical argument that this is the correct solution by considering a situation where the temperature at infinity is not zero. In that instance, a smooth solution will solve the problem. Two such solutions matching a finite temperature at infinity are shown in Figure 10.7. In the $(\gamma, H)$ plane, these trajectories flow into the critical point at $(\gamma, H) = (0, 0)$. If we imagine a process whereby the temperature at infinity is reduced to zero, the limiting solution as $T_{\text{infinity}} \to 0$ is precisely the discontinuous one we worked out, with $g_\theta[0] = -0.627554$ and $g_\theta[1.1425] = -1.1425$. In the $(\gamma, H)$ plane these solutions flow off to $(\gamma, H) = (0, -\infty)$.

Most materials exhibit a thermal diffusivity $K = \lambda T^\sigma$ where $\sigma < 1$. The case of $\sigma < 1$ is explored in Exercises 10.3 and 10.4 and generally it is found that the propagation speed of the heat remains finite. The practically more important case is where the temperature at infinity is nonzero. In this case the discontinuity disappears but the propagation speed remains finite as shown by the $g_\theta[0] = -0.5, -0.6$ curves in Figure 10.7.

### *10.4.4 Exact Thermal Analogy of the Blasius Boundary Layer*

Now let's change the boundary condition at the wall and consider the situation where the temperature at the wall is suddenly reduced to zero while the temperature of the medium remains finite. In this case the boundary conditions for (10.66) change to

$$g[0] = 0, \qquad g[\infty] = 1. \tag{10.83}$$

The self-similar temperature profile for this case is shown in Figure 10.9.

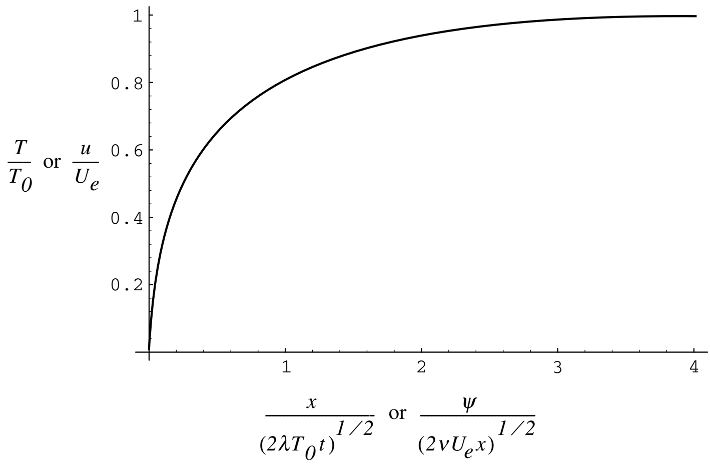


Fig. 10.9. Cooling of a nonlinear medium – analogy with the Blasius boundary layer.

A comparison of these two problems lends some insight into the effect of a nonlinear medium. Pushing heat into the medium creates a situation where the heat flux is in the same direction as the propagation of the disturbance, with the highest diffusivity occurring at the rear. In effect, heat to the rear catches up with heat near the front in a process of nonlinear steepening that produces a finite disturbance region bounded by a temperature gradient shock. Withdrawing heat from the medium causes the direction of heat flux to be opposite to that of the propagation of the disturbance, with the highest diffusivity at the front; heat farther away diffuses more rapidly than heat near the wall. This is a gentler process that produces a disturbance region that spreads out and extends to infinity with no discontinuities, as shown in Figure 10.9. The whole process is quite reminiscent of the flow produced by a piston moving into, or away from, a compressible gas in a tube.

Finally, we point out that this latter problem – governed by Equation (10.61) with similarity variables (10.65) together with the boundary conditions (10.83) – is an exact thermal analog of the Blasius flat-plate boundary-layer problem governed by (10.36) and (10.37), with the following correspondence between variables [10.2]:

$$\begin{aligned} \frac{T}{T_0} &\to \frac{u}{U_e}, \\ \frac{x}{(2\lambda T_0 t)^{1/2}} &\to \frac{\psi}{(2\nu U_e x)^{1/2}}. \end{aligned} \tag{10.84}$$

The velocity (or temperature) is plotted against the stream function (or $x$) in Figure 10.9.

## 10.5 Boundary Layers with Pressure Gradient

Now let's return to the group analysis of the boundary-layer equations with pressure gradient (10.2) and (10.6) and ask: how general is the boundary-layer concept, and how widely applicable are the solutions of the boundary-layer equations? In particular, are we restricted to flat plates? To answer these questions we need to examine the full Lie group of Equation (10.6), repeated here for convenience:

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x\,\partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} - U_e\frac{dU_e}{dx} - \nu\frac{\partial^3 \psi}{\partial y^3} = 0. \tag{10.85}$$

For the moment we regard $U_e$ as a known function of $x$. The infinitesimal transformation is

$$\begin{aligned}\tilde{x} &= x + s\xi[x, y, \psi],\\ \tilde{y} &= y + s\zeta[x, y, \psi],\\ \tilde{\psi} &= \psi + s\eta[x, y, \psi].\end{aligned} \tag{10.86}$$

Using the **`IntroToSymmetry.m`** package leads to the infinitesimals

$$\begin{aligned}\xi[x, y, \psi] &= a\left(\frac{4U_eU_{ex}}{U_{ex}^2 + U_eU_{exx}}\right),\\ \zeta[x, y, \psi] &= -ay + g[x],\\ \eta[x, y, \psi] &= a\psi + b,\end{aligned} \tag{10.87}$$

where $g[x]$ is an arbitrary function and $U_e[x]$ is required to satisfy

$$(U_eU_{ex})(U_eU_{ex})_{xx} = (U_eU_{ex})^2. \tag{10.88}$$

The solution of (10.88) is

$$U_e = \left(A + Be^{x/L}\right)^{1/2}. \tag{10.89}$$

The finite group associated with the group parameter $a$ leads to a similarity solution for an exponentially changing free stream – a highly interesting result.

But there is another, even more interesting, and much more important result contained in (10.87). This is associated with the arbitrary function of $x$ appearing in the infinitesimal transformation of $y$. The corresponding infinite-dimensional finite group is

$$\begin{aligned}\tilde{x} &= x,\\ \tilde{y} &= y + g[x],\\ \tilde{\psi} &= \psi.\end{aligned} \tag{10.90}$$

The invariance of the boundary-layer equations (10.85) under the group (10.90) vastly expands the usefulness of the boundary-layer approximation. The implication of (10.90) is that if

$$\psi[x, y] \tag{10.91}$$

is a solution of the untilde'd boundary-layer equations for flow over a flat plate with some specified $U_e[x]$, then

$$\tilde{\psi}[\tilde{x}, \tilde{y} - g[\tilde{x}]] \tag{10.92}$$

is a solution of the tilde'd equations for flow over a body with the boundary shape $y = 0 \Rightarrow \tilde{y} = g[\tilde{x}]$ and $U_e[\tilde{x}] = U_e[x]$. In effect, the solution on a body with arbitrary shape $\tilde{y} = g[\tilde{x}]$ and free-stream velocity $U_e[\tilde{x}]$ can be directly transformed to the solution on a flat plate with the same $U_e[x]$. Furthermore, the problem of determining the boundary layer on a body of a given shape is, to within the boundary-layer approximation, completely decoupled from the problem of determining $U_e[x]$. This is valid as long as $x$ is measured from the same effective origin and the body geometry is such that the flow remains nearly parallel to the body – the flow doesn't separate.

Incidentally, a more general result holds for the flow over a surface that is moving. The unsteady boundary-layer equation

$$\frac{\partial^z \psi}{\partial t \partial y} + \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x\, \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial U_e}{\partial t} - U_e \frac{dU_e}{dx} - \nu \frac{\partial^3 \psi}{\partial y^3} = 0 \tag{10.93}$$

is invariant under the infinite-dimensional group

$$\begin{aligned}\tilde{x} &= x + h[t],\\ \tilde{y} &= y + g[x, t],\\ \tilde{\psi} &= \psi,\\ \tilde{U}_e &= U_e,\end{aligned} \tag{10.94}$$

where $h[t]$ and $g[x, t]$ are arbitrary functions. This allows the unsteady flow over a moving surface to be mapped to the flow over a flat plate with the same

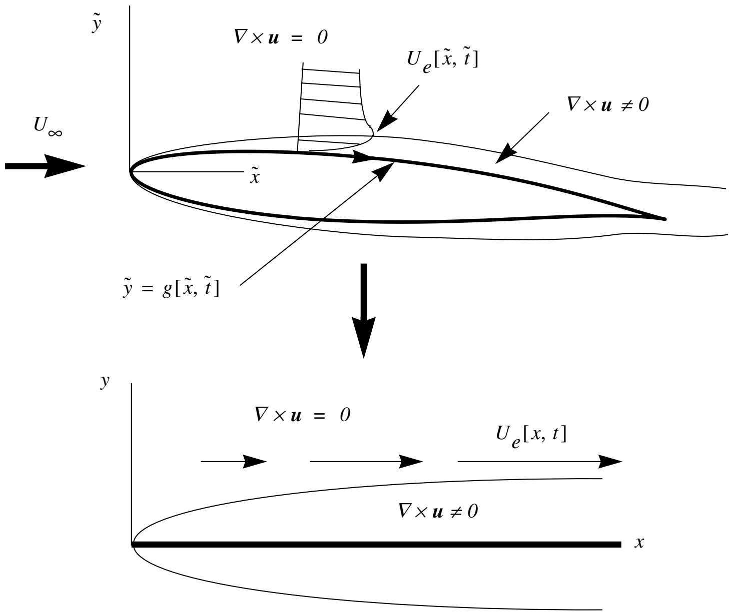


Fig. 10.10. Boundary layer on an unsteady wing transformed to the boundary layer on a flat plate. The function $h[t]$ in (10.94) is assumed to be zero.

$U_e[x, t]$. Further details can be found in the paper by Ma and Hui [10.3], who provide a comprehensive discussion of various groups and similarity solutions of the unsteady boundary-layer equations.

From a practical standpoint the boundary-layer approximation and the symmetries carried with it probably represents the most important theoretical simplification in all of mechanics. It is the basis of a general method for solving the viscous flow over complex body shapes in low-speed flow. The general idea is illustrated in Figure 10.10, where the flow over a 2-D unsteady airfoil is mapped to the flow over a flat plate with the same $U_e[x, t]$.

Of course, for a given shape to give a realistic solution it must be consistent with the basic assumptions of the boundary-layer approximation. In the case of the airfoil depicted in Figure 10.10, accurate treatment of the flow near the leading and trailing edges would require an analysis of the full equations of motion. As Prandtl first showed, at high Reynolds number the flow past a body such as that illustrated in Figure 10.10 divides neatly into two regions. There is an outer flow away from the wall where the motion is irrotational and can be treated as frictionless. The flow in this region is determined by solving Laplace's equation for the velocity potential, $\nabla^2\phi = 0$, from which the velocity is generated using $u = \nabla\phi$. The boundary conditions for the outer problem are that the velocity must match the free stream far from the wing and it must lie parallel to the surface at the surface. In addition, the flow leaving the trailing edge must satisfy the Kutta condition that states that the potential flow leaves the trailing edge of the wing smoothly. The Kutta condition forces the potential solution to mimic the behavior of the actual viscous flow. The real high Reynolds number flow can't negotiate the sharp trailing edge, due to very large viscous dissipation of kinetic energy that would occur if the flow were to go around a sharp corner. This latter requirement in fact implies that there is a net circulation about the wing proportional to the lift on it. The circulation is put into the problem by adding to the irrotational outer flow a potential vortex located on the chord line at the quarter-chord point of the wing. Adjusting the strength of the vortex until the potential flow leaves the trailing edge smoothly yields a reasonably accurate value for the lift on the wing.

However, in a steady flow the drag on the wing is zero – the so-called *d'Alembert's paradox*. To determine the drag one needs to solve for the boundary layer. The boundary-layer solution provides the value of the viscous friction at the wall. The steps needed to solve the flow with a boundary layer are as follows. The potential solution is used to solve for the frictionless-flow velocity at the wall. The potential-flow velocity evaluated at the wall is used as $U_e[x]$ for a subsequent boundary-layer calculation. The boundary-layer calculation defines a modified effective shape for the wing, which is used to repeat the potential-flow calculation, which defines a new $U_e[x]$ for a revised boundary-layer calculation, and so forth. After a few iterations accurate values for both the lift and drag on the wing are obtained.

Finally, what is the justification for dividing the flow into such distinct regions? The main contribution to the vorticity comes from the $y$-derivative of the velocity,

$$\omega \approx \frac{\partial u}{\partial y} = \frac{U_e}{x}\left(\frac{U_e x}{2\nu}\right)^{1/2} F_{\alpha\alpha}. \tag{10.95}$$

Let $\sigma = F_{\alpha\alpha}$. The Blasius equation in terms of $\sigma$ is

$$\frac{d\sigma}{d\alpha} = -F\sigma. \tag{10.96}$$

For constant $U_e$ and large $\alpha$ we have $F \approx \alpha$, and so at the edge of the layer we can approximate the behavior of the vorticity as

$$\sigma \approx e^{-\alpha^2/2}. \tag{10.97}$$

This quantifies the point made earlier in conjunction with Figure 10.3 that at the edge of the boundary layer the vorticity decays exponentially fast. This is the fundamental justification for the separation of the flow into two distinct regions.

## 10.6 The Falkner–Skan Boundary Layers

Finally, we consider other free-stream velocity distributions that lead to similarity solutions in addition to the exponential flow described in (10.89). We again analyze the group properties of the stream-function equation

$$\Omega = \psi_y \psi_{xy} - \psi_x \psi_{yy} - U_e \frac{dU_e}{dx} - \nu \psi_{yyy} = 0 \tag{10.98}$$

using the **`IntroToSymmetry.m`** package. But this time we will use a slightly different approach. Instead of thinking of $U_e[x]$ as an arbitrary function only of $x$, we will initially regard it as a second dependent variable $U_e[x, y]$. Then we will stipulate that $U_e$ must be independent of $y$ by applying the rule $u_{e_y} = 0$ to the invariance condition. In this approach we are analyzing the invariance of a system of one equation in two dependent variables. This may seem futile at first, but as far as symmetry analysis is concerned, the method can be applied to any mathematical expression or set of expressions whatsoever. The fact that the system we are considering is unclosed has no effect at all on how the Lie algorithm is applied and on its ability to reveal symmetries.

The infinitesimal transformation is

$$\begin{aligned} \tilde{x} &= x + s\xi[x, y, \psi, U_e], \\ \tilde{y} &= y + s\zeta[x, y, \psi, U_e], \\ \tilde{\psi} &= \psi + s\eta[x, y, \psi, U_e], \\ \tilde{U}_e &= U_e + s\theta[x, y, \psi, U_e]. \end{aligned} \tag{10.99}$$

Since there is only one equation, there is only one invariance condition $X_{\{3\}}\Omega = 0$. Fully written out, this is

$$\begin{aligned} &\xi \frac{\partial \Omega}{\partial x} + \zeta \frac{\partial \Omega}{\partial y} + \eta \frac{\partial \Omega}{\partial \psi} + \theta \frac{\partial \Omega}{\partial U_e} \\ &+ \eta_{\{x\}} \frac{\partial \Omega}{\partial \psi_x} + \eta_{\{y\}} \frac{\partial \Omega}{\partial \psi_y} + \eta_{\{xx\}} \frac{\partial \Omega}{\partial \psi_{xx}} + \eta_{\{xy\}} \frac{\partial \Omega}{\partial \psi_{xy}} + \eta_{\{yy\}} \frac{\partial \Omega}{\partial \psi_{yy}} \\ &+ \eta_{\{xxx\}} \frac{\partial \Omega}{\partial \psi_{xxx}} + \eta_{\{xxy\}} \frac{\partial \Omega}{\partial \psi_{xxy}} + \eta_{\{xyy\}} \frac{\partial \Omega}{\partial \psi_{xyy}} + \eta_{\{yyy\}} \frac{\partial \Omega}{\partial \psi_{yyy}} \\ &+ \theta_{\{x\}} \frac{\partial \Omega}{\partial U_{e_x}} + \theta_{\{y\}} \frac{\partial \Omega}{\partial U_{e_y}} + \theta_{\{xx\}} \frac{\partial \Omega}{\partial U_{e_{xx}}} + \theta_{\{xy\}} \frac{\partial \Omega}{\partial U_{e_{xy}}} + \theta_{\{yy\}} \frac{\partial \Omega}{\partial U_{e_{yy}}} \\ &+ \theta_{\{xxx\}} \frac{\partial \Omega}{\partial U_{e_{xxx}}} + \theta_{\{xxy\}} \frac{\partial \Omega}{\partial U_{e_{xxy}}} + \theta_{\{xyy\}} \frac{\partial \Omega}{\partial U_{e_{xyy}}} + \theta_{\{yyy\}} \frac{\partial \Omega}{\partial U_{e_{yyy}}} = 0. \end{aligned} \tag{10.100}$$

When the differentiation indicated in (10.100) is carried out, the invariance condition reduces to

$$-\theta\frac{\partial U_e}{\partial x} - \eta_{\{x\}}\psi_{yy} + \eta_{\{y\}}\psi_{xy} + \eta_{\{xy\}}\psi_y - \eta_{\{yy\}}\psi_x - \nu\eta_{\{yyy\}} - \theta_{\{x\}}U_e = 0. \tag{10.101}$$

The resulting infinitesimals generated by the package are

$$\begin{aligned}
\xi(x, y, \psi, U_e) &= a + (b + c)x,\\
\zeta(x, y, \psi, U_e) &= cy + g[x],\\
\eta(x, y, \psi, U_e) &= d + b\psi,\\
\theta(x, y, \psi, U_e) &= (b - c)U_e,
\end{aligned} \tag{10.102}$$

where $dU_e/dy = 0$. The characteristic equations corresponding to (10.102) are

$$\frac{dx}{a + (b + c)x} = \frac{dy}{cy + g[x]} = \frac{d\psi}{d + b\psi} = \frac{dU_e}{(b - c)U_e}. \tag{10.103}$$

First we solve

$$\frac{dx}{a + (b + c)x} = \frac{dU_e}{(b - c)U_e} \tag{10.104}$$

to give

$$U_e = M(x + x_0)^{\beta}, \tag{10.105}$$

where

$$\beta = \frac{b - c}{b + c}, \qquad x_0 = \frac{a}{b + c}, \tag{10.106}$$

and $M$ is a constant of integration with units

$$\hat{M} = L^{1-\beta}/T. \tag{10.107}$$

We find from this analysis that similarity solutions of (10.98) exist for a class of power-law free-stream velocity distributions given by (10.105). This is the well-known Falkner–Skan family of boundary layers [10.4], [10.5], and the exponent $\beta$ is the Falkner–Skan pressure-gradient parameter. The arbitrary translation of $y$ by $g[x]$ can be incorporated at any time, as can the stream-function translation parameter $d$ in (10.102). The parameters $M$ and $\nu$ are used to nondimensionalize the similarity variables derived from the remaining equalities in (10.103),

$$\alpha = \left(\frac{M}{2\nu}\right)^{1/2} \frac{y}{(x+x_0)^{(1-\beta)/2}},$$
$$F = \frac{\psi}{(x+x_0)^{(1+\beta)/2}(2\nu M)^{1/2}}. \tag{10.108}$$

Substitution of (10.108) and (10.105) into the stream-function equation (10.98) yields

$$(x+x_0)^{2\beta-1}(F_\alpha((1+\beta)F-(1-\beta)\alpha F_\alpha)_\alpha$$
$$- F_{\alpha\alpha}((1+\beta)F-(1-\beta)\alpha F_\alpha) - 2\beta - F_{\alpha\alpha\alpha}) = 0. \tag{10.109}$$

Canceling terms produces the Falkner–Skan equation

$$F_{\alpha\alpha\alpha} + (1+\beta)FF_{\alpha\alpha} - 2\beta(F_\alpha)^2 + 2\beta = 0 \tag{10.110}$$

with boundary conditions

$$F[0] = 0, \qquad F_\alpha[0] = 0, \qquad F_\alpha[\infty] = 1. \tag{10.111}$$

Note that $\beta = 0$ reduces (10.110) to the Blasius equation. It is fairly easy to work out the groups of (10.110) based on our experience with the Blasius equation. The constant $2\beta$ breaks the dilational invariance, and so one is left with only the invariance under translation in $\alpha$:

$$\xi = 1, \qquad \eta = 0, \tag{10.112}$$

and so we should be able to reduce the order by one. The new variables are the invariants of (10.112):

$$\phi = F, \qquad G = F_\alpha. \tag{10.113}$$

By the method of differential invariants, the expression

$$\frac{\left(\frac{DG}{D\alpha}\right)}{\left(\frac{D\phi}{D\alpha}\right)} = \frac{dG}{d\phi} = \frac{\frac{\partial G}{\partial \alpha}\,d\alpha + \frac{\partial G}{\partial F}\,dF + \frac{\partial G}{\partial F_\alpha}\,dF_\alpha}{\frac{\partial \beta}{\partial \alpha}\,d\alpha + \frac{\partial \beta}{\partial F}\,dF} = \frac{F_{\alpha\alpha}}{F_\alpha}. \tag{10.114}$$

is an invariant, as is

$$\frac{d^2G}{d\phi^2} = \left(\frac{F_\alpha F_{\alpha\alpha\alpha} - F_{\alpha\alpha}^2}{F_\alpha^2}\right)\frac{1}{F_\alpha}$$
$$= \frac{F_\alpha(-(1+\beta)FF_{\alpha\alpha} + 2\beta(F_\alpha)^2 - 2\beta) - F_{\alpha\alpha}^2}{F_\alpha^3}, \tag{10.115}$$

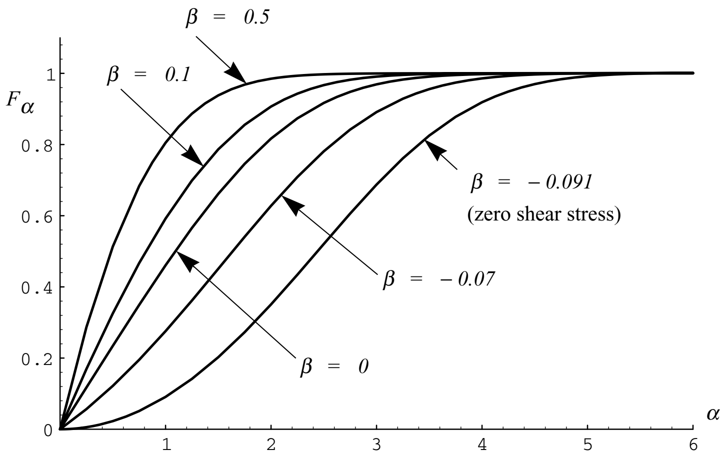


Fig. 10.11. Falkner–Skan velocity profiles.

where the Falkner–Skan equation (10.110) has been used to replace the third derivative. As we expect, equation (10.115) can be rearranged to read entirely in terms of the new variables:

$$GG_{\phi\phi} + (1+\beta)\phi G_\phi + (G_\phi)^2 + 2\beta\left(\frac{1}{G} - G\right) = 0 \qquad (10.116)$$

with the boundary conditions

$$G[0] = 0, \qquad G[\infty] = 1. \qquad (10.117)$$

Several velocity profiles are shown in Figure 10.11.

A particularly interesting case occurs when $\beta = -1$. The pressure gradient term is

$$U_e = \frac{M}{x} \quad \Rightarrow \quad U_e \frac{dU_e}{dx} = -\frac{M^2}{x^3}, \qquad (10.118)$$

and the original variables become

$$\begin{aligned} \alpha &= \left(\frac{2\nu}{|M|}\right)\frac{y}{x+x_0}, \\ F &= \pm\frac{\psi}{(2\nu|M|)^{1/2}}. \end{aligned} \qquad (10.119)$$

In this case the units of the governing parameter, $\hat{M} = L^2/T$, are the same as those of the kinematic viscosity, and so the ratio $|M|/\nu$ is the (constant)

Table 10.2.
*Commutator table for the case $\beta = -1$.*

| | $X^a$ | $X^b$ |
|---|---|---|
| $X^a$ | 0 | 0 |
| $X^b$ | 0 | 0 |

Reynolds number for the $\beta = -1$ flow. The governing equation becomes

$$F_{\alpha\alpha\alpha} \pm (2(F_\alpha)^2 - 2) = 0. \tag{10.120}$$

The quantity $M$ is an area flow rate and can change sign depending on whether the flow is created by a source or a sink. The plus sign corresponds to a source, and the minus sign to a sink. To avoid an imaginary root, the absolute value of $M$ is used to nondimensionalize the stream function in (10.119).

For $\beta = -1$, the second term in (10.116) vanishes, introducing a new symmetry, which produces an equation that is invariant under translation in $\phi$. The corresponding two-parameter group of (10.120) is the two-parameter Abelian group $\xi = a$, $\eta = b$ (corresponding to invariance under translation in $F$ and $\alpha$) with group operators $X^a = \partial/\partial\alpha$ and $X^b = \partial/\partial F$ and solvable Lie algebra given by Table 10.2. The once reduced equation is

$$G^2 G_{\phi\phi} + G G_\phi^2 \pm 2(G^2 - 1) = 0, \tag{10.121}$$

where

$$\phi = F, \qquad G = F_\alpha. \tag{10.122}$$

This equation admits the group

$$\xi = 1, \qquad \eta = 0 \tag{10.123}$$

with invariants

$$\gamma = G, \qquad H = G_\phi. \tag{10.124}$$

From the method of differential invariants or just by inspection

$$\frac{dH}{d\gamma} = \frac{H_\phi + H_G \frac{dG}{d\phi} + H_{G_\phi} \frac{dG_\phi}{d\phi}}{\gamma_\phi + \gamma_G \frac{dG}{d\phi}} = \frac{G_{\phi\phi}}{G_\phi} = \frac{-\frac{(G_\phi)^2}{G} - \frac{2}{G^2} + 2}{G_\phi}. \tag{10.125}$$

Equation (10.121) reduces to

$$\frac{dH}{d\gamma} = \frac{-H^2\gamma \pm (2 - 2\gamma^2)}{\gamma^2 H}. \tag{10.126}$$

Equation (10.126) is invariant under the group

$$\begin{aligned} \tilde{\gamma} &= \gamma, \\ \tilde{H} &= \left(H^2 \mp s\frac{2}{\gamma^2}\right)^{1/2} \end{aligned} \tag{10.127}$$

with group parameters and infinitesimals

$$\xi = 0, \qquad \eta = \mp\frac{1}{\gamma^2 H}, \tag{10.128}$$

where the plus sign corresponds to the sink flow case. Based on the invariance under (10.127), we should expect (10.126) to be fully integrable.

### *10.6.1 Falkner–Skan Sink Flow*

At this point we will restrict ourselves to the case of a sink flow [choose the minus sign in (10.121), (10.126), (10.127) and the plus sign in (10.128)]. See Landau and Lifshitz [10.6] for the case of a source, which is much more complex (see also Exercise 10.5). The flow we are considering is sketched in Figure 10.12.

The negative sign in front of the $F$ in (10.119) ensures that the velocity derived from the stream function is directed in the negative $x$-direction. The first order ODE (10.126) (with the minus sign selected) can be broken into the autonomous pair

$$\begin{aligned} \frac{dH}{ds} &= -H^2\gamma - 2 + 2\gamma^2, \\ \frac{d\gamma}{ds} &= \gamma^2 H \end{aligned} \tag{10.129}$$

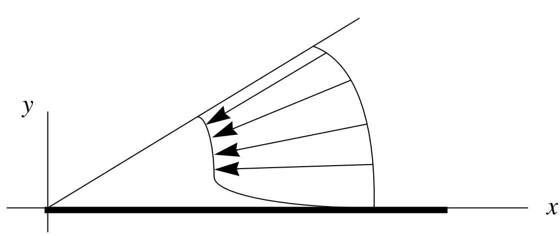


Fig. 10.12. Falkner–Skan sink flow for $\beta = -1$.

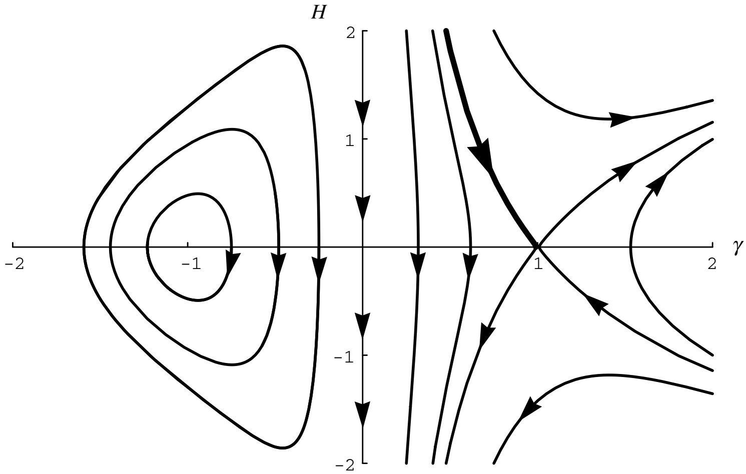


Fig. 10.13. Phase portrait of the Falkner–Skan case $\beta = -1$.

with critical points at $(\gamma, H) = (0 \pm 1)$. The phase portrait of (10.129) is shown in Figure 10.13.

Equation (10.126) is rearranged to read

$$(\gamma H^2 + 2 - 2\gamma^2)\, d\gamma + (\gamma^2 H)\, dH = 0, \tag{10.130}$$

which, by the cross-derivative test, can be shown to be a perfect differential with the integral

$$\psi = 2\gamma - \tfrac{2}{3}\gamma^3 + \tfrac{1}{2}\gamma^2 H^2. \tag{10.131}$$

Recall that

$$\begin{aligned} \gamma &= G = F_\alpha, \\ H &= G_\phi = \frac{F_{\alpha\alpha}}{F_\alpha}. \end{aligned} \tag{10.132}$$

At the edge of the boundary layer,

$$\left.\begin{aligned} \lim_{\alpha\to\infty} F_\alpha &= 1 \\ \lim_{\alpha\to\infty} F_{\alpha\alpha} &= 0 \end{aligned}\right\} \quad \Rightarrow \quad H[1] = 0. \tag{10.133}$$

This allows us to evaluate $\psi$ in (10.131). The result is

$$\psi = \tfrac{4}{3} \tag{10.134}$$

Solving (10.131) for $H$ yields

$$H = \left(\frac{4\gamma}{3} - \frac{4}{\gamma} + \frac{8}{3\gamma^2}\right)^{1/2} \qquad (0 < \gamma < 1), \tag{10.135}$$

where the positive root is recognized to be the physical solution. The solution (10.135) is shown as the thicker trajectory in Figure 10.13. Equation (10.135) can be written as

$$\gamma H = \sqrt{\tfrac{4}{3}}((\gamma - 1)^2(\gamma + 2))^{1/2}. \tag{10.136}$$

In terms of the original variables, we obtain

$$F_{\alpha\alpha} = \sqrt{\tfrac{4}{3}}((F_\alpha - 1)^2(F_\alpha + 2))^{1/2} \tag{10.137}$$

and

$$\alpha = \tanh^{-1}\left[\sqrt{\frac{F_\alpha + 2}{3}}\right] - \tanh^{-1}\left[\sqrt{\frac{2}{3}}\right]. \tag{10.138}$$

The latter result can be solved for the negative of the velocity, $F_\alpha$:

$$F_\alpha = 3\tanh^2\left[\alpha + \tanh^{-1}\sqrt{\tfrac{2}{3}}\right] - 2. \tag{10.139}$$

This is shown plotted in Figure 10.14.

The Falkner–Skan sink flow represents one of the few known exact solutions of the boundary-layer equations. However, the fact that an exact solution exists

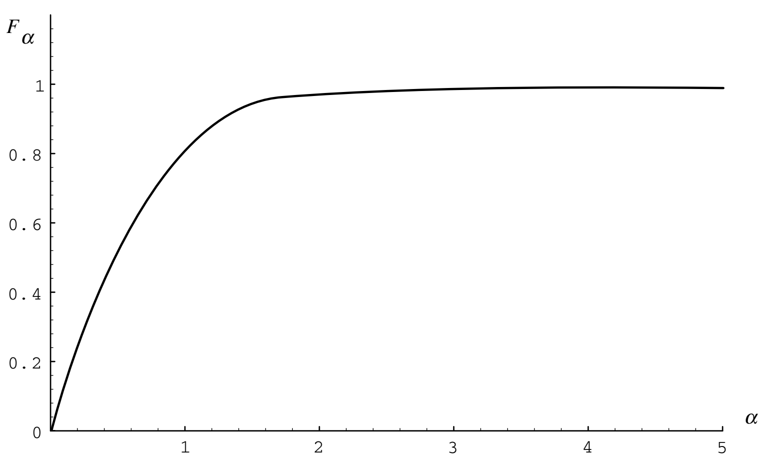


Fig. 10.14. Falkner–Skan sink-flow velocity profile $\beta = -1$.

for the case $\beta = -1$ is no accident. Neither is the fact that this case corresponds to an independent variable of the form $\alpha \approx y/x$ where both coordinate directions are in some sense equivalent. Remember that the essence of the boundary-layer approximation is that the streamwise direction is in a sense "convective" while the transverse direction is "diffusive," producing a flow that is progressively more slender in the $y$-direction as $x$ increases. In the case of the Falkner–Skan sink flow the aspect ratio of the flow is constant. This point will be clarified in Chapter 11, where the group properties of the full Navier–Stokes equations are discussed. There we will study the flow produced by a source or a sink in a diverging channel. It will be seen that the diverging-channel problem is invariant under the dilation group that leaves the full Navier–Stokes equations invariant and that this group is identical to the group (10.102) with the group parameter $b = 0 \Rightarrow \beta = -1$.

In fact the flow in a diverging channel is one of the few know exact solutions of the full Navier–Stokes equations, and at high Reynolds number, the sink version of it asymptotically approaches the exact solution (10.138) of the Falkner–Skan sink flow. Both flows are invariant under the spatially uniform dilation group $\tilde{x} = e^c x$, $\tilde{y} = e^c y$, $\tilde{\psi} = \psi$.

## 10.7 Concluding Remarks

This completes our discussion of laminar boundary layers for now. We shall return to the Falkner–Skan problem again at the end of Chapter 11, after looking at the invariance properties of the full Navier–Stokes equations for incompressible flow.

## 10.8 Exercises

10.1 In Section 10.1 it is stated that the flow outside the boundary layer is "irrotational ($\nabla \times \boldsymbol{u} = 0$) and therefore unaffected by viscosity." Justify this statement using the incompressible Navier–Stokes equations

$$\frac{\partial u^i}{\partial t} + \frac{\partial}{\partial x^k}\left(u^i u^k + \frac{p}{\rho}\delta^i_k\right) - \nu \frac{\partial u^i}{\partial x^k \, \partial x^k} = 0, \qquad \frac{\partial u^k}{\partial x^k} = 0 \tag{10.140}$$

and the identity $\nabla \times (\nabla \times \boldsymbol{u}) = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla^2 \boldsymbol{u}$.

10.2 Take the boundary-layer variables

$$F = \frac{\psi}{(2\nu U_e x)^{1/2}}, \qquad \alpha = \frac{y}{\left(\frac{2\nu x}{U_e}\right)^{1/2}}, \tag{10.141}$$

and substitute them into the full Navier–Stokes equations, written in terms of the stream function,

$$\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)$$
$$-\nu\left(\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^{\partial}\partial y^2}+\frac{\partial^4\psi}{\partial y^4}\right)=0. \tag{10.142}$$

Use the fact that $Re = U_e x/\nu$ is large to reduce the resulting equation to the Blasius ODE,

$$F_{\alpha\alpha\alpha} + FF_{\alpha\alpha} = 0. \tag{10.143}$$

10.3 Formulate the nonlinear heat conduction problem

$$\frac{\partial T}{\partial t} = \lambda\frac{\partial}{\partial x}\left(T^{\sigma}\frac{\partial T}{\partial x}\right) \tag{10.144}$$

with boundary conditions

$$\begin{aligned} T[x,0] &= 0, \qquad x > 0,\\ T[0,t] &= T_0, \qquad t > 0, \end{aligned} \tag{10.145}$$

for values of $\sigma$ other than one. Reduce the problem to a first order ODE and set up the phase plane. Describe what happens as $\sigma$ is varied. Interpret your results physically.

10.4 Consider the case of an instantaneous source that injects a finite amount of heat, $E$, at a point in a nonlinear medium. The temperature diffuses, and the total heat added is constant. In this case the temperature at the origin decreases with time as the distribution spreads out. The governing equation is

$$\frac{\partial T}{\partial t} = \lambda\frac{\partial}{\partial x}\left(T^{\sigma}\frac{\partial T}{\partial x}\right) \tag{10.146}$$

with initial source distribution

$$T[x,0] = E\delta[x] \tag{10.147}$$

and conserved integral

$$\int_{-\infty}^{\infty} T[x,t]\,dx = E. \tag{10.148}$$

This problem is exactly solved by a parabolic distribution of temperature,

$$T = \begin{cases} \dfrac{E^{2/(\sigma+2)}}{(\lambda t)^{1/(\sigma+2)}} \left( \dfrac{\sigma}{2(\sigma+2)} (\theta_0^2 - \theta^2) \right)^{1/\sigma}, & \theta \le \theta_0, \quad \sigma \ne -2, \\ 0, & \theta > \theta_0, \end{cases} \tag{10.149}$$

where

$$\theta = \frac{x}{(E^\sigma \lambda t)^{1/(\sigma+2)}} \tag{10.150}$$

and

$$\theta_0 = \left( \pi^{-1/2} \left( \frac{2\sigma+4}{\sigma} \right)^{1/\sigma} \frac{\Gamma\left(\frac{1}{\sigma} + \frac{3}{2}\right)}{\Gamma\left(\frac{1}{\sigma} + 1\right)} \right)^{\frac{\sigma}{\sigma+2}} \tag{10.151}$$

(Zel'dovich and Kompaneets [10.7]). What group is the solution invariant under? Reduce the problem to a phase plane, and try to identify the solution trajectory, see Ibragimov [10.8] for the solution of the impulsive problem in $n$ dimensions.

10.5 Consider the Falkner–Skan case $\beta = -1$. Show that (10.127) is a group, and show that (10.126) is invariant under (10.127). Use phase-plane analysis to examine the case where the origin is a source, $M > 0$ [cf. Equation (10.129) with a plus sign]. Comment on the existence of the solution. Why is this case more difficult than the sink flow?

10.6 Consider the buoyancy induced flow produced by a heated flat plate sketched in Figure 10.15. This flow is governed by a coupled system of convection–diffusion equations for the momentum and temperature. Changes in density are related to changes in temperature by a thermal expansion coefficient:

$$\rho - \rho_\infty = \beta(T - T_\infty). \tag{10.152}$$

If changes in density are small [$(\rho - \rho_\infty)/\rho_\infty \ll 1$], the fluid behaves incompressibly ($\nabla \cdot \boldsymbol{u} = 0$) with a local body force equal to $(\rho - \rho_\infty)g$. The governing equations are $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$,

$$\begin{aligned} \frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} - \nu\frac{\partial^3\psi}{\partial y^3} &= \frac{\beta(T - T_\infty)g}{\rho_\infty}, \\ \frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y} &= \kappa\frac{\partial^2 T}{\partial y^2}. \end{aligned} \tag{10.153}$$

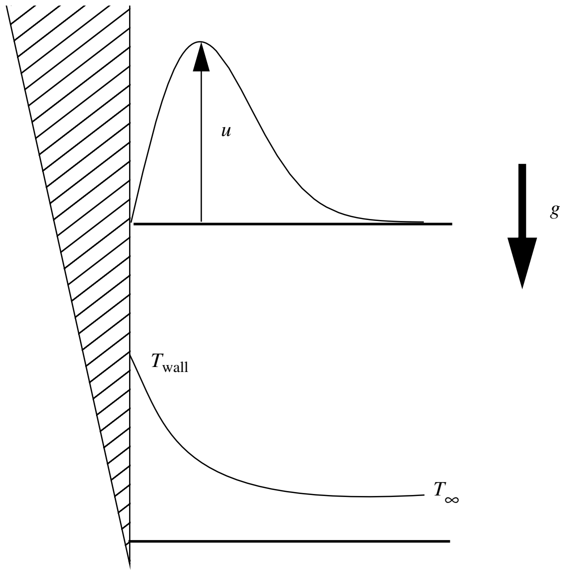


Fig. 10.15.

(1) Use the package **IntroToSymmetry.m** to work out the infinitesimal groups for this system. Generate the commutator table, and fully characterize the Lie algebra.
(2) Construct similarity variables for the problem depicted above, reduce the governing equations to a pair of coupled ODEs, and solve for the self-similar velocity and temperature profiles.
(3) Show whether a similarity solution exists when the free stream velocity is non zero.
(4) How is the symmetry of the problem changed when the plate is cooled instead of heated?

10.7 Consider the group (10.87) and a free-stream velocity distribution of the form

$$U_e = \left(A + Be^{x/L}\right)^{1/2}. \tag{10.154}$$

Determine similarity variables and work out the solution of the governing equation for this case. Think about what sort of wall shape would be needed to generate this free-stream velocity field.

10.8 Show by direct substitution that the unsteady boundary-layer equation

$$\psi_{yt} + \psi_y \psi_{xy} - \psi_x \psi_{yy} - \frac{\partial U_e(x,t)}{\partial t} - U_e(x,t)\frac{\partial U_e(x,t)}{\partial x} - \nu \psi_{yyy} = 0 \tag{10.155}$$

is invariant under the infinite-dimensional group

$$\tilde{x} = x, \qquad \tilde{y} = y + g[x, t], \qquad \tilde{t} = t, \qquad \tilde{\psi} = \psi, \tag{10.156}$$

where $g[x, t]$ is arbitrary. Use the package **`IntroToSymmetry.m`** to work out the invariant groups of (10.155).

(1) Compare your results with the paper by Ma and Hui [10.3]. Work out the unsteady stagnation-point flow discussed in Section 5 of their paper. Why, on the basis of group invariance, does the solution given by Equation (40) in their paper satisfy the full Navier–Stokes equations?

(2) Consider the case where $U_e = Mx^{\beta}$. Use the dilation invariance of the problem to reduce the unsteady boundary-layer equation in $(\psi, x, y, t)$ to a PDE in $(G, r, s)$, where

$$G = \frac{\psi}{\nu^a M^b t^c}, \qquad r = \frac{x}{M^d t^e}, \qquad s = \frac{y}{(\nu t)^{1/2}}. \tag{10.157}$$

Determine $(a, b, c, d, e)$.

(3) Consider a two-parameter dilation group on $G$, $r$, and $s$ of the form

$$\tilde{r} = \exp\left[\frac{2}{1-\beta}b\right] r, \qquad \tilde{s} = e^b s, \qquad \tilde{G} = \exp\left[\frac{1+\beta}{1-\beta}b\right] G. \tag{10.158}$$

Use this group to generate similarity variables and reduce the equation found in part (2) to the steady Falkner–Skan ODE (10.110). Does the group (10.158) leave the equation from part(2) invariant? Why does this reduction work?

## References

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# 11

## *Incompressible Flow*

This chapter is concerned with the application of symmetry analysis to problems of incompressible flow governed by the full Navier–Stokes equations. Two examples are described in considerable detail, illuminating several facets of the method not discussed thus far. Extensive use is made of the three-dimensional state-space methods introduced in Chapter 3. Here, the phase space in question is the space of similarity coordinates. It will be shown how the analysis of symmetries can lead to a description of flow dynamics that is independent of the observer. Fundamental questions are addressed concerning moving frames of reference and the distinction between streamlines and particle particle paths in unsteady flow.

The parameter that governs the dynamics of a viscous flow is the Reynolds number. In the selected examples this is defined in terms of the kinematic viscosity and an integral constant of the motion derived from an overall mass or momentum balance. In the two examples considered, the Reynolds number is constant and the flow field can be represented as a phase portrait in similarity coordinates. Bifurcations in the phase portrait can occur as the Reynolds number is varied. The joining of Lie theory and bifurcation analysis in phase space produces a complete understanding of the Reynolds-number dependence of the space–time structure of the flow.

### 11.1 Invariance Group of the Navier–Stokes Equations

The Navier–Stokes equations governing incompressible flow are

$$\begin{aligned} \frac{\partial u^j}{\partial x^j} &= 0, \quad \text{(sum over } j = 1, 2, 3), \\ \frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} + \frac{\partial p}{\partial x^i} - \nu \frac{\partial^2 u^i}{\partial x^j \partial x^j} &= 0, \qquad i = 1, 2, 3, \\ &\qquad \text{sum over } j = 1, 2, 3 \end{aligned} \tag{11.1}$$

where $p$ is the kinematic pressure (pressure/density) and $\nu$ is the kinematic viscosity (viscosity/density). We transform (11.1) using the following infinitesimal group

$$\begin{aligned}\tilde{x}^i &= x^i + s\xi^i[\boldsymbol{x}, t],\\ \tilde{t} &= t + s\tau[\boldsymbol{x}, t],\\ \tilde{u}^i &= u^i + s\eta^i[\boldsymbol{x}, t],\\ \tilde{p} &= p + s\zeta[\boldsymbol{x}, t].\end{aligned} \tag{11.2}$$

Running the package **`IntroToSymmetry.m`** on (11.1) leads to the following set of group operators:

(1) Invariance under translation in time:

$$X^1 = \frac{\partial}{\partial t}. \tag{11.3}$$

(2) An arbitrary function of time, $g[t]$, added to the pressure:

$$X^2 = g[t]\frac{\partial}{\partial p}. \tag{11.4}$$

(3) Rotation about the $z$-axis:

$$X^3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}. \tag{11.5}$$

(4) Rotation about the $x$-axis:

$$X^4 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + w\frac{\partial}{\partial v} - v\frac{\partial}{\partial w}. \tag{11.6}$$

(5) Rotation about the $y$-axis:

$$X^5 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + w\frac{\partial}{\partial u} - u\frac{\partial}{\partial w}. \tag{11.7}$$

(6) Nonuniform translation in the $x$-direction:

$$X^6 = a[t]\frac{\partial}{\partial x} + \left(\frac{da}{dt}\right)\frac{\partial}{\partial u} - x\left(\frac{d^2a}{dt^2}\right)\frac{\partial}{\partial p}. \tag{11.8}$$

$a[t]$ is an arbitrary, twice differentiable function of time. Simple translation in $x$ corresponds to $a[t] = \text{const}$.

(7) Nonuniform translation in the $y$-direction:

$$X^7 = b[t]\frac{\partial}{\partial y} + \left(\frac{db}{dt}\right)\frac{\partial}{\partial v} - y\left(\frac{d^2b}{dt^2}\right)\frac{\partial}{\partial p}. \tag{11.9}$$

$b[t]$ is an arbitrary, twice differentiable function.

(8) Nonuniform translation in the $z$-direction:

$$X^8 = c[t]\frac{\partial}{\partial z} + \left(\frac{dc}{dt}\right)\frac{\partial}{\partial w} - z\left(\frac{d^2c}{dt^2}\right)\frac{\partial}{\partial p}. \tag{11.10}$$

$c[t]$ is an arbitrary, twice differentiable function.

(9) The one-parameter dilation group of the equation

$$X^9 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w} - 2p\frac{\partial}{\partial p}. \tag{11.11}$$

The finite form of the dilation group corresponding to the infinitesimal operator $X^9$ is

$$\begin{aligned} \tilde{x}^i &= e^s x^i, \\ \tilde{t} &= e^{2s} t, \\ \tilde{u}^i &= e^{-s} u^i, \\ \tilde{p} &= e^{-2s} p. \end{aligned} \tag{11.12}$$

Note that the stretching in all three coordinate directions is the same. This should be compared with the dilation group of the boundary-layer equations discussed in Chapter 10. Intrinsic to the form of the boundary-layer equations is the approximation recognized by Prandtl that distinguishes between convection in the streamwise direction and diffusion in the cross-stream direction. This distinction is expressed by the invariance of the boundary-layer equations under a two-parameter dilation group in the spatial coordinates, whereas the full Navier–Stokes equations admit only the one-parameter dilation group, (11.12). The implication of this is that the boundary-layer equations govern a wider range of similarity solutions (wider in terms of flow geometry) than the full Navier–Stokes equations. This is the main reason why Prandtl's ideas have played such an important role in the advancement of the theory of viscous flow.

If the kinematic viscosity in (11.1) is set equal to zero, the full equations reduce to the incompressible Euler equations, which are invariant under $X^1$ to $X^8$ and a two-parameter dilation group in space and time given by

$$\begin{aligned} \tilde{x}^i &= e^s x^i, \\ \tilde{t} &= e^{s/k} t, \\ \tilde{u}^i &= e^{s(1-1/k)} u^i, \\ \tilde{p} &= e^{s(2-2/k)} p \end{aligned} \tag{11.13}$$

with group parameters $s$ and $k$. The group (11.13) is the main symmetry group governing elementary turbulent shear flows and is discussed further in Chapter 13.

Occasionally, exact solutions of the full Navier–Stokes equations are discovered, and when they are, it is virtually always the case that the solution is invariant under one or more of the above groups. Some of the most interesting solutions are those invariant under the dilation group (11.12) and in the later sections we will describe two famous examples. First, we consider the implications of the invariance under the nonuniform translation groups (11.8), (11.9), and (11.10).

## 11.2 Frames of Reference

Following References [11.1] and [11.2], the finite form of the infinite-dimensional groups corresponding to nonuniform translation in three space directions $X^6$, $X^7$, and $X^8$ can be written concisely as

$$\begin{aligned}\tilde{x}^j &= x^j + a^j[t],\\ \tilde{t} &= t,\\ \tilde{u}^i &= u^i + \frac{da^i}{dt},\\ \tilde{p} &= p - x^j\frac{d^2a^j}{dt^2} + g[t].\end{aligned} \tag{11.14}$$

The arbitrary functions translating the coordinates imply that the Navier–Stokes equations are invariant for all moving observers as long as the observer moves irrotationally. An observer translating and accelerating arbitrarily in three dimensions will sense the same equations of motion as an observer at rest. This invariance implies a great degree of flexibility in the choice of the observer used to view a flow. For example, one may wish to move with a particular fluid element. Or, if some convecting vortical feature happens to be of interest, then one is free to select a frame of reference attached to that feature. This has been used in Figure 11.1 to view the wake of a circular cylinder in a frame where the eddying motions in the wake become apparent. Flow fields are commonly studied this way. However, there is a danger in attaching too much dynamical significance to the flow patterns seen by any specific observer, since the choice of the frame of reference is itself arbitrary and the flow patterns seen by different observers may differ dramatically, as they do in Figure 11.1.

The term added to the pressure in (11.14) represents a spatially uniform effective body force induced by the acceleration of the observer. This force is purely hydrostatic in nature in that it is exactly balanced everywhere by the rate of change of the velocity field (the derivative of the translation term in the transformation of the velocity) and has no dynamical significance; it produces

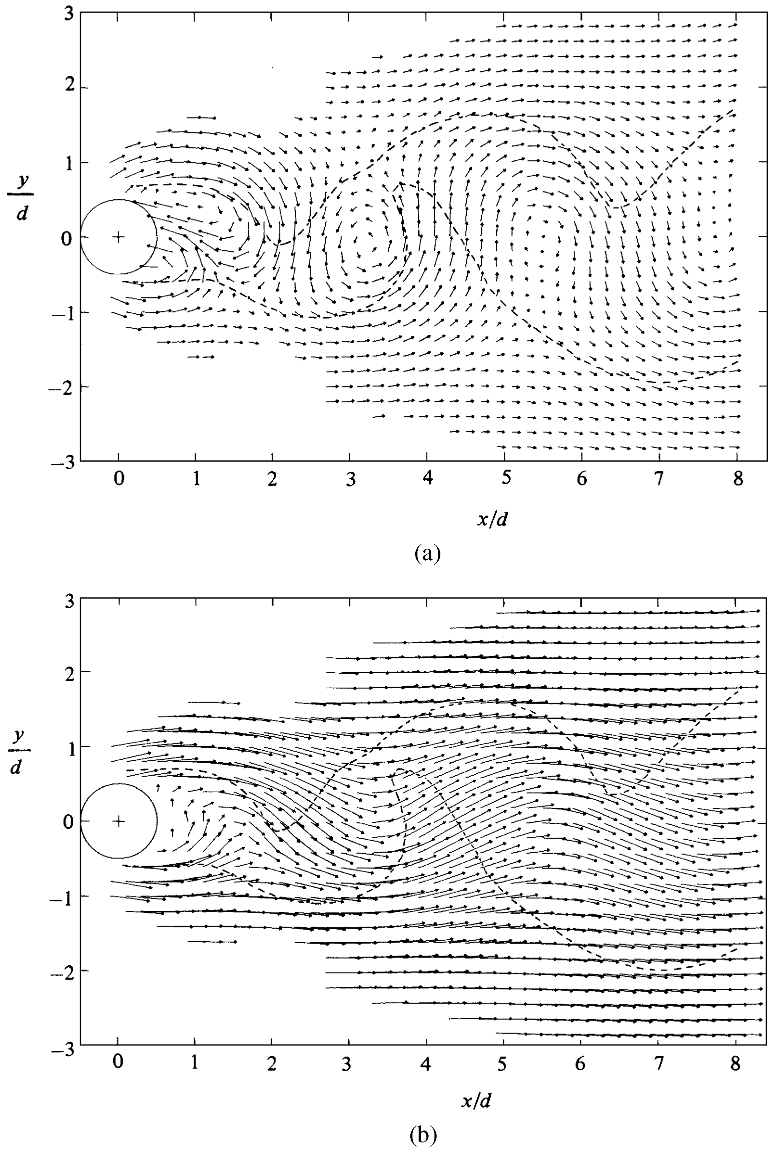


Fig. 11.1. Velocity vector field in the wake of a circular cylinder from Reference [11.6] as viewed by two observers: (a) frame of reference moving downstream at $0.755U_\infty$, (b) frame of reference fixed with respect to the cylinder. The dashed contour roughly corresponds to the instantaneous boundary of turbulence.

no net force on the flow field. The compressible equations of motion, do not admit this symmetry although in one space dimension they are invariant for an accelerating observer moving according to a specific choice of the acceleration function. See Exercise 16.8.

Similarly, the group (11.14) would not be admitted by incompressible problems involving variable density and/or a free boundary. Such flows do involve a rich variety of similarity solutions, and interesting examples can be found in References [11.3] and [11.4].

## 11.3 Two-Dimensional Viscous Flow

The 2-D equations of unsteady incompressible flow are

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} - \nu\frac{\partial^2 u}{\partial y^2} &= 0, \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} - \nu\frac{\partial^2 v}{\partial x^2} - \nu\frac{\partial^2 v}{\partial y^2} &= 0. \end{aligned} \tag{11.15}$$

Introduce the stream function

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}, \tag{11.16}$$

and take the curl of the momentum equation to produce the equation for the vorticity,

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} - \nu\frac{\partial^2 \omega}{\partial x^2} - \nu\frac{\partial^2 \omega}{\partial y^2} = 0, \tag{11.17}$$

where

$$\omega = -\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right). \tag{11.18}$$

If we substitute (11.16) and (11.18) into (11.15), the result is the 2-D unsteady stream-function equation

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) + \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) - \nu\left(\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2\,\partial y^2} + \frac{\partial^4 \psi}{\partial y^4}\right) = 0. \tag{11.19}$$

The stream-function equation is invariant under all of the groups of the Navier–Stokes equations plus the group of uniform rotations (Reference [11.2]) with group operator

$$X^{10} = yt\frac{\partial}{\partial x} - xt\frac{\partial}{\partial y} - \frac{1}{2}(x^2 + y^2)\frac{\partial}{\partial \psi} + \frac{\partial}{\partial \omega}. \tag{11.20}$$

The finite form of the group (11.20) is

$$\begin{aligned} \tilde{x} &= x\cos[\Omega t] - y\sin[\Omega t], \\ \tilde{y} &= x\sin[\Omega t] + y\cos[\Omega t], \\ \tilde{t} &= t, \\ \tilde{\psi} &= \psi - \Omega\frac{x^2 + y^2}{2}, \\ \tilde{\omega} &= \omega + \Omega \end{aligned} \tag{11.21}$$

and represents a transformation to a system of coordinates rotating counter-clockwise at a constant angular velocity $\Omega$. The finite transformation corresponding to nonuniform translation in the $x$ and $y$ directions is

$$\begin{aligned} \tilde{x} &= x + a[t], \\ \tilde{y} &= y + b[t], \\ \tilde{t} &= t, \\ \tilde{\psi} &= \psi - x\frac{db}{dt} + y\frac{da}{dt}. \end{aligned} \tag{11.22}$$

The terms added to the stream function in (11.21) and (11.22) have a simple interpretation as the area flux between the origin of the tilde'd coordinates and a given point, induced by the nonuniform translating and/or uniformly rotating motion of the observer.

The invariance of the stream-function equation under the group of uniform rotations, (11.21), is interesting in that it is not a symmetry of the Navier–Stokes equations in their primitive form (11.1). This is an example of a type of symmetry called a potential symmetry that can arise when the equation is expressed in terms of a potential function. The effect of such a change in the equation is to raise the order of the derivatives that appear, and in the process point symmetries can arise that are *nonlocal symmetries* of the original system (symmetries depending on an integral of the relevant variables; see Bluman and Kumei [11.5]). These are sometimes termed hidden symmetries. We shall have much more to say about nonlocal symmetries in Chapters 14 and 16.

The transformations of the vorticity and stream function by the dilation group (11.12) are

$$\begin{aligned} \tilde{\psi} &= \psi, \\ \tilde{\omega} &= e^{-2s}\omega. \end{aligned} \tag{11.23}$$

Having discussed the basic symmetries of the equations governing incompressible flow, it is now time to see how they can be used to solve problems. In the first example we will examine the steady viscous flow produced by a source of mass at the apex of a diverging channel. This example illustrates how a conserved integral governs the overall motion. Very often for problems set in an infinite or semi infinite domain, invariance of the overall problem under a group boils down to invariance of the conserved integral. The second example considers starting vortex formation in an impulsively started round jet. This problem illustrates the use of the Reynolds number as a bifurcation parameter in phase space. The final example demonstrates the use of symmetries to catagorize the Falkner–Skan boundary layers described in Chapter 10.

## 11.4 Viscous Flow in a Diverging Channel

Figure 11.2 depicts a source (or sink) of mass at the apex of a diverging 2-D channel of half angle $\theta_{1/2}$. This is the so-called *Jeffery–Hamel flow* [11.7], [11.8].

The flow is incompressible, and so we can interpret the source as essentially a generator of area per second. The same area flow per second passes through any cross section at a radius $r$:

$$M = \int_{-\theta_{1/2}}^{\theta_{1/2}} ur\, d\theta, \tag{11.24}$$

where $u$ is the velocity in the radial direction. The first step is to show that this problem is invariant under the group (11.12). The walls extend to infinity and thus are invariant under dilation in $r$ and so in the main, this involves transforming (11.24) using (11.12) to show that the integral remains invariant:

$$\tilde{M} = \int_{-\tilde{\theta}_{1/2}}^{\tilde{\theta}_{1/2}} \tilde{u}\tilde{r}\, d\tilde{\theta} = \int_{-\theta_{1/2}}^{\theta_{1/2}} (e^{-s}u)(e^{s}r)\, d\theta = \int_{-\theta_{1/2}}^{\theta_{1/2}} ur\, d\theta = M. \tag{11.25}$$

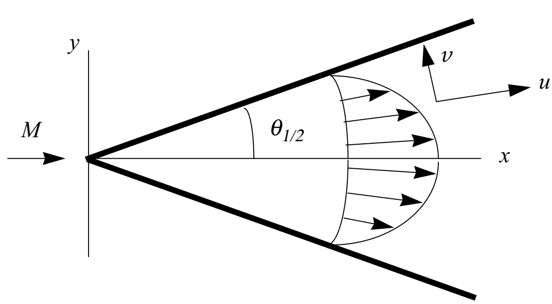


Fig. 11.2. Viscous flow in a diverging channel.

The dimensions of the conserved integral are $[M] = L^2/T$, and the ratio

$$Re = M/\nu \tag{11.26}$$

is the Reynolds number of the flow. The point nature of the source, the invariance of the area flux integral, and the obvious invariance of the semiinfinite channel boundaries under the dilation group (11.12) are the ingredients of a similarity solution – in this case, an exact solution. Integrating the characteristic equations

$$\frac{dx}{x} = \frac{dy}{y} = \frac{d\psi}{0} \tag{11.27}$$

of the group (11.12), (11.23) leads to the similarity variables

$$\begin{aligned} \theta &= \arctan[y/x], \\ G(\theta) &= \frac{\psi}{6\nu}. \end{aligned} \tag{11.28}$$

Note that the group (11.12) produces a similarity solution of the full Navier–Stokes equations that is in the same functional form as the $\beta = -1$ case of the Falkner–Skan boundary layers treated in Chapter 10; both problems are invariant under the group (11.12), and their solutions are closely related to one another. In the case of the Jeffrey–Hamel flow the appropriate similarity variable is the flow angle. The radial velocity component is

$$u = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{6\nu}{r}G_\theta. \tag{11.29}$$

The reason for using the similarity variable in the form of $\theta$ instead of $y/x$ is to simplify some of the later relationships that develop. Note that the $1/r$ decay of the radial velocity is the same as that of the potential flow produced by a source or a sink, although in this viscous case the flow also satisfies the no-slip condition at the wall. When $\psi = 6\nu G[\theta]$ is substituted into the stream-function equation (11.19), the result is the following fourth-order ODE:

$$G_{\theta\theta\theta\theta} + 12G_\theta G_{\theta\theta} + 4G_{\theta\theta} = 0. \tag{11.30}$$

The conserved integral takes the normalized form

$$\frac{M}{6\nu} = \int_{-\theta_{1/2}}^{\theta_{1/2}} G_\theta d\theta = G[\theta_{1/2}] - G[-\theta_{1/2}], \tag{11.31}$$

and the boundary conditions at the wall are

$$G_\theta[\theta_{1/2}] = G_\theta[-\theta_{1/2}] = 0. \tag{11.32}$$

Now let's use group theory to reduce the order of (11.30). Note that the apex of the channel can be a source or a sink of area, depending on the sign of $M$.

The package **IntroToSymmetry.m** is used to search for infinitesimal transformations of (11.30) of the form

$$\begin{aligned}\tilde{\theta} &= \theta + s\xi[\theta, G],\\ \tilde{G} &= G + s\eta[\theta, G].\end{aligned} \tag{11.33}$$

The result is a three-parameter group with infinitesimals

$$\begin{aligned}\xi &= b + c\theta,\\ \eta &= a + c\left(-\tfrac{2}{3}\theta - G\right).\end{aligned} \tag{11.34}$$

The corresponding group operators are

$$X^a = \frac{\partial}{\partial G}, \qquad X^b = \frac{\partial}{\partial \theta}, \qquad X^c = \theta\frac{\partial}{\partial \theta} + \left(-\frac{2}{3}\theta - G\right)\frac{\partial}{\partial G} \tag{11.35}$$

with the commutator table shown in Table 11.1, which is a solvable Lie algebra with ideal $X^a$, $X^b$. In retrospect the invariance under translation in $\theta$ and $G$ is obvious from the fact that the equation does not depend explicitly on either variable. The finite group corresponding to the parameter $c$ can be determined by summing the Lie series. The result is

$$\begin{aligned}\tilde{\theta} &= e^s\theta,\\ \tilde{G} &= e^{-s}G - \tfrac{2}{3}\theta\sinh[s].\end{aligned} \tag{11.36}$$

The solvability of the three-parameter group (11.34) tells us that (11.30) can be reduced three times, to a first-order equation. Whether that final first-order equation can be reduced to quadrature remains an open question unless we can, by inspection or otherwise, identify a group of the ODE. If we can, the

Table 11.1. *Commutator table for the group (11.34).*

| | $X^a$ | $X^b$ | $X^c$ |
|---|---|---|---|
| $X^a$ | 0 | 0 | $-\frac{2}{3}X^a + X^b$ |
| $X^b$ | 0 | 0 | $X^a$ |
| $X^c$ | $\frac{2}{3}X^a - X^b$ | $-X^a$ | 0 |

group can then be used to generate an integrating factor, and the entire problem can be reduced to a series of four quadratures.

Let's begin with the group $X^a$ corresponding to translation in $G$, for which the characteristic equations are

$$\frac{d\theta}{0} = \frac{dG}{1} = \frac{dG_\theta}{0} = \cdots . \tag{11.37}$$

with invariants $\theta$ and $G_\theta$. Let $F = G_\theta$. The once reduced equation becomes

$$F_{\theta\theta\theta} + 12FF_\theta + 4F_\theta = 0. \tag{11.38}$$

At this point we can easily accomplish two more reductions as follows. Equation (11.38) can be written

$$F_{\theta\theta\theta} + 6(F^2)_\theta + 4F_\theta = 0, \tag{11.39}$$

which integrates immediately to

$$F_{\theta\theta} + 6F^2 + 4F = 2C_1. \tag{11.40}$$

If we multiply (11.40) by $F_\theta$, we can integrate again to produce

$$\tfrac{1}{2}(F_\theta)^2 + 2F^3 + 2F^2 = 2C_1F + 2C_2, \tag{11.41}$$

or

$$\tfrac{1}{2}F_\theta = \pm(C_1F + C_2 - F^3 - F^2)^{1/2}. \tag{11.42}$$

The final solution is expressed as

$$2\theta = \pm\int \frac{1}{(C_1F + C_2 - F^3 - F^2)^{1/2}}\, dF + C_3. \tag{11.43}$$

The three constants of integration are determined from the conditions

$$\int_{-\theta_{1/2}}^{\theta_{1/2}} F\, d\theta = \frac{M}{6\nu} \tag{11.44}$$

and

$$F[\theta_{1/2}] = F[-\theta_{1/2}] = 0. \tag{11.45}$$

For the case where the apex is a sink ($M < 0$), the velocity field can be assumed to be symmetric about the centerline. On the centerline one assumes the conditions

$$F(0) = -u_0,\\ F_\theta(0) = 0. \tag{11.46}$$

This enables the constant $C_2$ in (11.43) to be evaluated as

$$C_2 = C_1 u_0 + u_0^2 - u_0^3. \tag{11.47}$$

Using (11.47), the solution can now be expressed as

$$2\theta = \pm \int_{-u_0}^{F} \frac{1}{((u_0 + \hat{F})(C_1 + (u_0 - \hat{F}) - u_0 \hat{F} - (u_0 - \hat{F})^2))^{1/2}}\, d\hat{F}, \tag{11.48}$$

where the integral begins at the centerline. The constants $u_0$ and $C_1$ are evaluated using

$$2\theta_{1/2} = \int_{-u_0}^{0} \frac{1}{((u_0 + \hat{F})(C_1 + (u_0 - \hat{F}) - u_0 \hat{F} - (u_0 - \hat{F})^2))^{1/2}}\, dF,$$

$$\frac{M}{12\nu} = \int_{-u_0}^{0} \frac{1}{((u_0 + \hat{F})(C_1 + (u_0 - \hat{F}) - u_0 \hat{F} - (u_0 - \hat{F})^2))^{1/2}}\, d\hat{F}. \tag{11.49}$$

Note that the square-root singularity that occurs in the denominator as $F \to -u_0$ is easily integrable and presents no problem.

The case where the apex is a source is much more complex, and the assumption of symmetric flow is only valid below a certain critical Reynolds number for a given wedge angle. An extensive discussion can be found in the classic text by Landau and Lifshitz [11.9]. In two papers, Moffat [11.10] and Moffat and Duffy [11.11] present a complete discussion of the rich variety of viscous flow patterns that can occur near a sharp corner. In the latter paper it is shown that the similarity solution of the Jeffery–Hamel flow only exists for a range of wedge angles.

## 11.5 Transition in Unsteady Jets

Transition in fluid flow can occur in a variety of ways, but in general two basic types can be distinguished. The first and most common type is transition to turbulence. The classic case here is the zero-pressure-gradient Blasius boundary layer discussed in Chapter 10. The Reynolds number of this flow increases with distance from the leading edge of the plate:

$$Re_{\text{flat plate}} = \frac{U_e x}{\nu}. \tag{11.50}$$

The streamwise increase in Reynolds number leads to a succession of instabilities, first linear, then nonlinear, that give rise to a chaotic motion that is ultimately turbulent. The flow, which is initially stable and steady, becomes unstable and unsteady as the Reynolds number is increased.

The second type of transition is one where, by parametric variation of the Reynolds number, a flow at low Reynolds number that is steady and stable is replaced abruptly by a new flow that is also steady and stable as the Reynolds number is increased. The classic case here is that of circular Couette flow studied by G.I. Taylor in 1923 [11.12] and more recently by Donald Coles [11.13], who documented a very complex set of stages through which the laminar flow may pass before becoming turbulent. This type of transition has been the subject of intense research in recent years, partly for its own sake, and partly because it is felt that an increased understanding of transition of the second type will lead to an increased understanding of transition to turbulence.

The next example is concerned with the second type of transition. We consider the onset of a starting vortex generated by an impulsively started axisymmetric jet produced by a point force. Like the diverging channel flow, this problem is invariant under the one-parameter dilation group of the Navier–Stokes equations (11.12), and the Reynolds number is a constant in time and space. In this respect, transition in the jet is reminiscent of transition in Couette flow. However, the round jet differs from Couette flow in that, Couette flow involves a bounded steady flow that bifurcates to a new steady flow, whereas the jet involves an unbounded, unsteady, self-similar flow that bifurcates to a topologically distinct unsteady self-similar flow.

### *11.5.1 The Impulse Integral*

Before we treat the specific case of an impulsively started point force, let's examine a control-volume balance of momentum for the flow produced by a compact time-dependent force distribution acting in an infinite domain. The sketch in Figure 11.3 depicts the flow produced after the force distribution, $\boldsymbol{F}(\boldsymbol{x}, t)$, is turned on in an infinite, incompressible, viscous fluid initially at rest. The outline of the jet is shown schematically in Figure 11.3 and roughly delineates the boundary between rotational and irrotational flow. The force and its associated vorticity distribution occupy a finite region inside a spherical control volume $V$ with surface normal vector $n\,dA$. The momentum equation can be integrated exactly over $V$.

Integrating the momentum equation exactly over $V$ leads to a relationship between the force applied at the origin of the flow, the far-field distribution of pressure, and the total hydrodynamic impulse. Recalling the discussion of

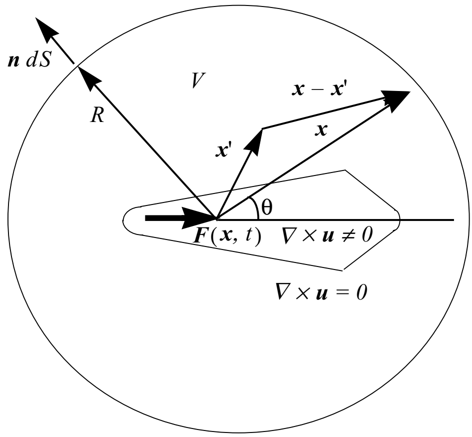


Fig. 11.3. Spherical control volume surrounding a force distribution in a viscous fluid.

incompressible flow in Chapter 3, Section 3.6.3, the vorticity vector $\boldsymbol{\omega}$ is related to the vector potential $\boldsymbol{A}$ through the vector Poisson equation,

$$\boldsymbol{\omega} = -\nabla^2 \boldsymbol{A}. \tag{11.51}$$

The general solution of (11.51) is

$$\boldsymbol{A}[\boldsymbol{x}, t] = \frac{1}{4\pi} \int_V \frac{\boldsymbol{\omega}[\boldsymbol{x}', t]}{|\boldsymbol{x} - \boldsymbol{x}'|}\, d\boldsymbol{x}', \tag{11.52}$$

where $d\boldsymbol{x}'$ is a volume element in $\boldsymbol{V}$. We want to find an expression for the volume-integrated momentum divided by the density, given by

$$\boldsymbol{H}[t] = \int_V \boldsymbol{u}[\boldsymbol{x}, t]\, d\boldsymbol{x}, \tag{11.53}$$

which we can write as a surface integral of the vector potential,

$$\boldsymbol{H}[t] = \int_S \boldsymbol{n} \times \boldsymbol{A}[\boldsymbol{x}, t]\, dS = \boldsymbol{R}^2 \int_S \frac{\boldsymbol{x}}{\boldsymbol{R}} \times \boldsymbol{A}[\boldsymbol{x}, t]\, d\Omega, \tag{11.54}$$

where

$$\boldsymbol{n} = \frac{\boldsymbol{x}}{\boldsymbol{R}} = \boldsymbol{i} \sin\theta \cos\phi + \boldsymbol{j} \sin\theta \sin\phi + \boldsymbol{k} \cos\theta, \tag{11.55}$$

and $d\Omega$ is an infinitesimal solid angle, $d\Omega = \sin\theta\, d\theta\, d\phi$. Substituting the expression for $\boldsymbol{A}$ from (11.52) into (11.54), exchanging the order of integration, and making use of

$$\int_S \frac{\boldsymbol{x}}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d\Omega = \frac{4\pi}{3}\left(\frac{\boldsymbol{x}}{\boldsymbol{R}}\right), \tag{11.56}$$

we obtain

$$\boxed{\boldsymbol{H}[t] = \tfrac{2}{3}\boldsymbol{I}[t],} \tag{11.57}$$

where

$$\boldsymbol{I}[t] = \frac{1}{2}\int_V \boldsymbol{x}\times\boldsymbol{\omega}[\boldsymbol{x},t]\,d\boldsymbol{x} \tag{11.58}$$

is the hydrodynamic impulse of the vorticity distribution [11.14], [11.15]. Saffman [11.15] includes the effect of compressibility in the calculation of the impulse. Note that the integral of the momentum is fully converged as long as one has added up all the contributions from the vorticity-bearing part of the flow. Potential flow motions beyond the vortical region do not contribute to the total momentum.

In order to actually evaluate the impulse integral (11.53), one needs to carry out an integral momentum balance over $V$. The momentum equation is

$$\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{\nabla}\cdot(\boldsymbol{u}\boldsymbol{u}) + \boldsymbol{\nabla}\left(\frac{p}{\rho}\right) - \nu\boldsymbol{\nabla}^2\boldsymbol{u} = \frac{\boldsymbol{F}[\boldsymbol{x},t]}{\rho}. \tag{11.59}$$

The Laplacian in (11.59) can be written in terms of the vorticity using the vector identity

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{u} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \boldsymbol{\nabla}^2\boldsymbol{u} \tag{11.60}$$

and continuity, $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$, so that the momentum equation takes the form

$$\frac{\partial\boldsymbol{u}}{dt} + \boldsymbol{\nabla}\cdot(\boldsymbol{u}\boldsymbol{u}) + \boldsymbol{\nabla}\left(\frac{p}{\rho}\right) + \nu\boldsymbol{\nabla}\times\boldsymbol{\omega} = \frac{\boldsymbol{F}[\boldsymbol{x},t]}{\rho}. \tag{11.61}$$

See Chapter 10 Exercise 10.1. Equation (11.61) is integrated over $V$, and Gauss's theorem is used to convert volume integrals to surface integrals, leading to

$$\frac{d\boldsymbol{H}}{dt} + \int_S\left(\boldsymbol{u}\boldsymbol{u} + \frac{p}{\rho}\boldsymbol{I}\right)\cdot\boldsymbol{n}\,dS + \nu\int_S(\boldsymbol{\omega}\times\boldsymbol{n})\,dS = \int_V \frac{\boldsymbol{F}[\boldsymbol{x},t]}{\rho}\,dV. \tag{11.62}$$

The vorticity on the control-volume surface is zero, and so the integrated momentum balance becomes

$$\frac{d\boldsymbol{H}}{dt} + \int_S \left( \boldsymbol{uu} + \frac{p}{\rho}\boldsymbol{I} \right) \cdot \boldsymbol{n}\, dS = \int_V \frac{\boldsymbol{F}[\boldsymbol{x}, t]}{\rho}\, dV. \tag{11.63}$$

At large values of $r$, the vector potential may be approximated by the first few terms of a multipole expansion:

$$\boldsymbol{A} = \frac{\boldsymbol{q}}{4\pi r} + \frac{\boldsymbol{Q} \cdot \boldsymbol{x}}{4\pi r^3} + O\left( \frac{1}{r^3} \right), \tag{11.64}$$

where $\boldsymbol{q}$ and $\boldsymbol{Q}$ are

$$\boldsymbol{q} = -\int_V \boldsymbol{\omega}[\boldsymbol{x}', t]\, d\boldsymbol{x}', \qquad \boldsymbol{Q} = -\int_V \boldsymbol{\omega}[\boldsymbol{x}', t]\boldsymbol{x}'\, d\boldsymbol{x}'. \tag{11.65}$$

The fact that $\boldsymbol{\omega}$ is divergence-free and localized and that $\boldsymbol{F}[\boldsymbol{x}, t]$ applies no net moment to the fluid implies that $\boldsymbol{q} = 0$. The numerator of the second term in (11.64) can be written

$$\boldsymbol{Q} \cdot \boldsymbol{x} = -\boldsymbol{x} \times \left( \frac{1}{2} \int_V \boldsymbol{x}' \times \boldsymbol{\omega}[\boldsymbol{x}', t]\, d\boldsymbol{x}' \right). \tag{11.66}$$

Thus at large $r$ the vector potential to lowest order is

$$\boldsymbol{A} = \frac{1}{4\pi} \frac{\boldsymbol{I}[t] \times \boldsymbol{x}}{r^3} + O\left( \frac{1}{r^3} \right). \tag{11.67}$$

These results for the far-field vector potential have a direct analogy with magnetostatics. The velocity $\boldsymbol{u}$ is analogous to the magnetic field, and the vorticity $\boldsymbol{\omega}/4\pi$ is analogous to the current density. Jackson [11.16] provides an excellent reference in this connection. Using (11.67), we can estimate the surface integral of the nonlinear term in (11.63) as

$$\int_S (\boldsymbol{uu}) \cdot \boldsymbol{n}\, dS \sim \frac{1}{\boldsymbol{R}^4} \qquad \text{as} \quad R \to \infty. \tag{11.68}$$

Thus as $R \to \infty$ the integral momentum balance reduces to the elegantly simple form

$$\frac{d\boldsymbol{H}}{dt} + \int_S \left( \frac{p}{\rho} \right) \boldsymbol{n}\, dS = \int_V \frac{\boldsymbol{F}[\boldsymbol{x}, t]}{\rho}\, dV \tag{11.69}$$

At large values of $r$ the nonlinear terms become small compared to acceleration and pressure and the momentum equation reduces to

$$\frac{\partial \boldsymbol{u}}{\partial t}+\frac{1}{\rho}\boldsymbol{\nabla} p=0. \tag{11.70}$$

We can use (11.70) to determine an equation for the far-field pressure. Using (11.67) for the vector potential and the fact that $\boldsymbol{\nabla}\cdot(\boldsymbol{x}/r^3)=\boldsymbol{\nabla}\times(\boldsymbol{x}/r^3)=0$, we can write the velocity as the gradient of a scalar. At large $r$,

$$\boldsymbol{u}=-\frac{1}{4\pi}\boldsymbol{\nabla}\left(\frac{\boldsymbol{I}\cdot\boldsymbol{x}}{r^3}\right). \tag{11.71}$$

Substituting (11.71) into (11.70) and solving, we have, to within an additive function of time,

$$\lim_{R\to\infty}\left(\frac{p}{\rho}\right)=\frac{1}{4\pi}\left(\frac{d\boldsymbol{I}}{dt}\right)\cdot\frac{\boldsymbol{x}}{r^3}. \tag{11.72}$$

The surface integral of the pressure in (11.69) can now be evaluated as

$$\boxed{\int_S\left(\frac{p}{\rho}\right)\boldsymbol{n}\,dS=\frac{1}{3}\frac{d\boldsymbol{I}[t]}{dt}.} \tag{11.73}$$

Substituting (11.73) and (11.57) into (11.69) and integrating over time gives

$$\boldsymbol{I}[t]=\int_0^t\int_V\frac{\boldsymbol{F}[\boldsymbol{x},t]}{\rho}\,dx\,dt. \tag{11.74}$$

The function $\boldsymbol{I}[t]$ is the total mechanical impulse applied by the force distribution since the onset of the motion. According to (11.57), two-thirds of the applied impulse is transferred to the momentum of the fluid, and the remaining one-third is removed by the far-field pressure (11.73), which opposes the motion.

Note that this whole analysis of the impulse is exact, regardless of whether the jet under consideration is laminar or turbulent [11.17]. The far-field vector potential is the same in either case. The general ideas presented in this section form the basis for the identification of conserved integrals in a wide variety of laminar and turbulent shear flows, many of which will be discussed in Chapter 13.

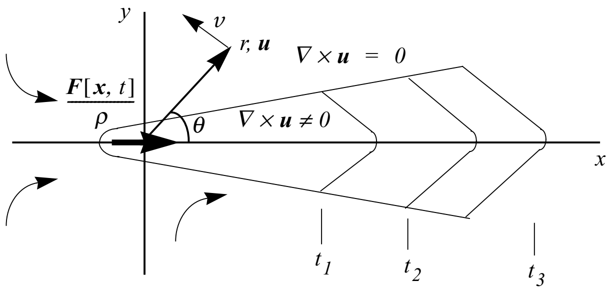


Fig. 11.4. Schematic of the unsteady propagation of a started jet. The boundary schematically delineates the regions of rotational and irrotational flow.

### *11.5.2 Starting-Vortex Formation in an Impulsively Started Jet*

With the discussion of the momentum integral complete, we can begin to consider the problem of starting-vortex formation in the impulsively started laminar jet, following References [11.17] and [11.18]. This flow exemplifies a wide class of problems in low-speed fluid mechanics where the motion is governed by a single integral invariant.

The unsteady flow dynamics in the jet can be completely understood by working out the Reynolds-number dependence of the three-dimensional phase portrait of particle paths in similarity coordinates. In fact it is not really necessary to look at the whole space, since virtually all of the interesting dynamics can be described in terms of the movement and topological changes (bifurcations) of critical points in the phase portrait as the Reynolds number is varied. These ideas can be further refined by considering just the invariants of the critical points and the trajectories of the critical points in the space of invariants. Moving back and forth between the space of the flow (the phase portrait) and the space of critical-point invariants provides a complete picture of the Reynolds-number dependence of the flow.

Figure 11.4 schematically shows the development of the vorticity-bearing region of an unsteady jet at several successive times. The jet is produced by a point force acting impulsively in a fluid that is initially everywhere at rest. The force distribution is of the form

$$\frac{\boldsymbol{F}[\boldsymbol{x}, t]}{\rho} = \frac{J}{\rho} h[t]\delta[x]\delta[y]\delta[z], \tag{11.75}$$

where $h[t]$ is the Heaviside function,

$$h[t] = \begin{cases} 0, & t < 0, \\ 1, & t > 0, \end{cases} \tag{11.76}$$

$\delta[x]$ is the Dirac delta function and J is the amplitude of the force directed along the x-axis. Using (11.57) and (11.73), the impulse integral is

$$\int_V \boldsymbol{u}[\boldsymbol{x}, t]\, dx = \frac{2}{3}\left(\frac{J}{\rho}\right)t, \tag{11.77}$$

indicating that the total momentum of the fluid grows linearly with time.

Now let's show that this problem is invariant under the fundamental dilation group of the Navier–Stokes equations,

$$\begin{aligned} \tilde{x}^i &= e^a x^i, & \tilde{t} &= e^{2a} t, \\ \tilde{u}^i &= e^{-a} u^i, & \frac{\tilde{p}}{\rho} &= e^{-2a}\frac{p}{\rho}, \\ \tilde{\omega} &= e^{-2a}\omega, & \tilde{\psi} &= e^a \psi. \end{aligned} \tag{11.78}$$

Note that $\psi$ in (11.78) is the Stokes stream function with dimensions $\hat{\psi} = L^3/T$. The invariance is confirmed by transforming the impulse integral

$$\int_V \tilde{\boldsymbol{u}}\, d\tilde{\boldsymbol{x}} = \frac{2}{3}\left(\frac{J}{\rho}\right)\tilde{t} \Rightarrow e^{2a}\int_V \boldsymbol{u}\, d\boldsymbol{x} = e^{2a}\frac{2}{3}\left(\frac{J}{\rho}\right)t$$
$$\Rightarrow \int_V \boldsymbol{u}\, d\boldsymbol{x} = \frac{2}{3}\left(\frac{J}{\rho}\right)t. \tag{11.79}$$

The governing parameter of the motion has units $J/\rho = L^4/T^2$. This flow has in common with the diverging channel flow just treated in Section 11.4, the fact that the governing parameter $J/\rho$ has dimensions commensurate with the dimensions of the kinematic viscosity, $\hat{\nu} = L^2/T$. In fact, as long as the force is assumed to act at a point, so that no length scale is introduced, then the *only* two parameters appearing in the jet problem are $J/\rho$ and $\nu$. The infinite nature of the flow, the absence of any walls, and the assumption that the force acts at a point are the ingredients that make the problem invariant under the group (11.78). The natural definition of the jet Reynolds number is

$$Re = \frac{(J/\rho)^{1/2}}{\nu}, \tag{11.80}$$

which is a constant independent of space and time. As was noted earlier, this is in contrast to the boundary layer on a flat plate or, for that matter, the majority of shear flows where the Reynolds number depends on space and time. As in the Jeffrey–Hamel flow, this is the key feature of the problem, which admits a type of transition that does not necessarily lead to turbulence.

Our whole analysis is directed at one basic question: is there a critical Reynolds number for the onset of a starting vortex from the jet, and if so what is it? In the end we shall see that two critical Reynolds numbers are found, and the concept of a starting vortex will be given a precise definition. The main results are derived from an examination of the solution for the creeping flow limit $Re \to 0$ at Reynolds numbers that lie outside the region of validity of the solution. Nevertheless the method of analysis is fundamentally nonlinear and will be used to investigate the structure of a numerical solution of the full nonlinear equations of motion. The creeping solution, which is perfectly symmetric about the equatorial plane $\cos\theta = 0$, is shown to have a suprisingly complex structure when analyzed in terms of nonsteady particle paths. As the Reynolds number is increased, the flow pattern of the jet undergoes a sequence of regular changes, each of which occurs at a specific critical value of the Reynolds number (11.80). In this sense, transition in the unsteady jet is reminiscent of the transition in steady Couette flow studied by Taylor [11.12].

*11.5.2.1 Governing Equations*

The problem is most conveniently formulated in spherical polar coordinates, and the governing equations in in these variables are

$$\frac{1}{r}\frac{\partial(r^2u)}{\partial r} + \frac{1}{\sin\theta}\frac{\partial(v\sin\theta)}{\partial\theta} = 0, \qquad \text{(continuity)},$$

$$\frac{\partial(rv)}{\partial r} - \frac{\partial u}{\partial\theta} = r\omega \qquad \text{(vorticity)},$$

$$\frac{\partial(r\omega)}{\partial t} + \frac{\partial(ru\omega)}{\partial r} + \frac{\partial(v\omega)}{\partial\theta} = \nu\left(\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial(\omega\sin\theta)}{\partial\theta}\right) + \frac{\partial^2(r\omega)}{\partial r^2}\right) \qquad \text{(momentum)}. \tag{11.81}$$

The Stokes stream function is introduced to integrate the continuity equation:

$$u = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad v = \frac{-1}{\sin\theta}\frac{\partial\psi}{\partial r}. \tag{11.82}$$

Note that the dimensions of $\psi$ in (11.82) are $L^3/T$, whereas for the stream function in 2-D flow the units are $L^2/T$. We are primarily concerned with the ODEs governing particle paths,

$$\frac{dr}{dt} = u[r,\theta,t], \qquad \frac{d\theta}{dt} = \frac{v[r,\theta,t]}{r}, \tag{11.83}$$

where the jet is directed along the polar axis and the velocity components, $u$ and $v$, are in the radial and tangential directions respectively. See Figure 11.4. The group (11.78) can be cast in spherical polar coordinates (the transformations of the radius and angle are simply $\tilde{r} = e^a r$, $\tilde{\theta} = \theta$), and the characteristic equations are

$$\frac{dr}{r} = \frac{d\theta}{0} = \frac{dt}{2t} = \frac{du}{-u} = \frac{dv}{-v} = \frac{d(p/\rho)}{-2p/\rho} = \frac{d\omega}{-2\omega} = \frac{d\psi}{\psi}. \tag{11.84}$$

All the relevant similarity variables are generated as the integrals of (11.84):

$$\begin{aligned}
\xi &= r/(\nu t)^{1/2}, \\
\theta &= \theta, \\
U[\xi,\theta] &= \frac{ut^{1/2}}{\nu^{1/2}}, \\
V[\xi,\theta] &= \frac{vt^{1/2}}{\nu^{1/2}}, \\
P[\xi] &= \left(\frac{p}{\rho}\right)\frac{t}{\nu}, \\
\Omega[\xi,\theta] &= \omega t, \\
\Psi[\xi,\theta] &= \frac{\psi}{\nu^{3/2}t^{1/2}}.
\end{aligned} \tag{11.85}$$

Upon substitution of (11.85), the equations of motion (11.81) become

$$\begin{aligned}
\frac{1}{\xi}\frac{\partial(\xi^2 U)}{\partial\xi} + \frac{1}{\sin\theta}\frac{\partial(V\sin\theta)}{\partial\theta} &= 0 \qquad \text{(continuity)}, \\
\frac{\partial(\xi V)}{\partial\xi} - \frac{\partial U}{\partial\theta} &= \xi\Omega \qquad \text{(vorticity)}, \\
\frac{\partial}{\partial\xi}\left(\left(U - \frac{\xi}{2}\right)\xi\Omega\right) + \frac{\partial(V\Omega)}{\partial\theta} &= \frac{1}{\xi}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial(\Omega\sin\theta)}{\partial\theta}\right) \\
&\quad + \frac{\partial^2(\xi\Omega)}{\partial\xi^2} \qquad \text{(momentum)},
\end{aligned} \tag{11.86}$$

and the self-similar velocities are

$$U = \frac{1}{\xi^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad V = \frac{-1}{\sin\theta}\frac{\partial\psi}{\partial\xi}. \tag{11.87}$$

The particle path equations (11.83) become

$$\frac{\partial\xi}{d\tau} = U[\xi,\ \theta; Re] - \frac{\xi}{2}, \qquad \frac{d\theta}{d\tau} = \frac{V[\xi,\ \theta;\ Re]}{\xi}, \tag{11.88}$$

where $\tau = \ln[t]$.

Expressing the particle path equations in terms of similarity variables converts the nonautonomous system (11.83) to an autonomous one, (11.88). Note that, in an unsteady flow, the self-similar velocity vector field is quite different from the vector field of particle paths, and the distinction is evident in the extra term $-\xi/2$ that appears in the radial equation in (11.88). This term adds a radially inward-directed component to the velocity vector field.

The self-similar velocities in (11.88) depend parametrically on the Reynolds number, although at this point it is not quite clear how the Reynolds-number dependence arises, in view of the fact that our normalization has removed the kinematic viscosity altogether from the governing momentum equation in (11.86).

We shall now consider several solutions of the system (11.86) and (11.88). The first will be the classical steady solution, first discovered by Landau in 1944 [11.20] and independently by Squire in 1951 [11.21], which, when recast in unsteady similarity coordinates, forms the boundary condition for the impulsively started jet in the limit $\xi \to 0$ ($t \to \infty$). The second will be the irrotational flow due a dipole of linearly increasing strength located at the origin. This forms the boundary condition for the started jet in the limit $\xi \to \infty$ ($t \to 0$). The key feature of the problem is that, although the reduced system (11.86) does not contain the Reynolds number explicitly, the boundary conditions at $\xi \to 0$ and $\xi \to \infty$ do depend on $Re$. Finally the creeping solution in the limit $Re \to 0$ is examined. In each case the Reynolds number appears in the system (11.88) as a parameter and the possibility of bifurcation in the phase portrait follows.

#### 11.5.2.2 Critical Points in Three Dimensions

Much of the analysis that follows will focus on the various vector field patterns of (11.88) and on the critical points $(\xi_c, \theta_c)$; we have

$$\begin{aligned} U[\xi_c, \theta_c; Re] - \frac{\xi_c}{2} &= 0, \\ \frac{V[\xi_c, \theta_c; Re]}{\xi_c} &= 0, \end{aligned} \tag{11.89}$$

where the parametric dependence of the velocity field on the Reynolds number is indicated. One of the most important aspects of this approach is that structural features of the flow, which are not visible in the streamline pattern, become evident in the pattern of particle trajectories.

The analysis of the critical points of (11.88) is carried out using the theory developed in Chapter 3, Section 3.9.4. For this purpose it is easier to work with the particle path equations in Cartesian coordinates,

$$\frac{dx^i}{dt} = u^i(\boldsymbol{x}, t) \Rightarrow \frac{d\xi^i}{d\tau} = U^i[\boldsymbol{\xi}; Re] - \frac{\xi^i}{2}. \tag{11.90}$$

where $\xi^i = x^i/\sqrt{\nu t}$.

The character of a critical point is determined by expanding the flow in a Taylor series near the critical point and truncating at first order:

$$\frac{d\xi^i}{d\tau} = \left(A^i_j - \frac{1}{2}\delta^i_j\right)\Big|_{\boldsymbol{\xi}=\boldsymbol{\xi}_c} \left(\xi^j - \xi^j_c\right). \tag{11.91}$$

The similarity form of the velocity gradient tensor is

$$a^i_j = \frac{\partial u^i}{\partial x^j} = \frac{1}{t}\frac{\partial U^i}{\partial \xi^j} = \frac{1}{t}A^i_j[\boldsymbol{\xi}]. \tag{11.92}$$

Note that the value of the dimensioned velocity gradient tensor does not depend on $J/\rho$ or $\nu$. Therefore an observer moving at a fixed $\boldsymbol{\xi}$ can use the current value of the velocity gradient as a local clock to determine the global age of the flow, regardless of the flow Reynolds number.

The nature of the critical point is determined by the invariants of the matrix

$$M^i_j = A^i_j - \tfrac{1}{2}\delta^i_j \tag{11.93}$$

in (11.91) evaluated at the critical point. The first invariants of $A$ and $M$ are

$$P_M = \tfrac{3}{2}, \qquad P_A = 0. \tag{11.94}$$

The second and third invariants $(Q, R)$ are expressed in terms of matrix elements

$$\begin{aligned} Q_A &= -\tfrac{1}{2}A^i_k A^k_i, \\ Q_M &= \tfrac{9}{8} - \tfrac{1}{2}M^i_k M^k_i \end{aligned} \tag{11.95}$$

and

$$\begin{aligned} R_A &= -\tfrac{1}{3}A^i_k A^k_j A^j_i, \\ R_M &= -\tfrac{1}{3}M^i_k M^k_j M^j_i - \tfrac{3}{2}Q_M + \tfrac{27}{24}. \end{aligned} \tag{11.96}$$

The invariants of $M$ and $A$ are related to one another as follows

$$\begin{aligned} Q_M &= Q_A + \tfrac{3}{4}, \\ R_M &= R_A + \tfrac{1}{2}Q_A + \tfrac{1}{8}. \end{aligned} \tag{11.97}$$

The discriminant of $A$ is

$$D_A = Q_A^3 + \tfrac{27}{4}R_A^2, \tag{11.98}$$

and the discriminant of $M$ is

$$D_M = Q_M^3 + \tfrac{27}{4}R_M^2 + \tfrac{27}{4}R_M\left(\tfrac{1}{2} - Q_M\right) - \tfrac{9}{16}Q_M^2. \tag{11.99}$$

If $D > 0$, the eigenvalues are complex and vorticity dominates the rate-of-strain. If $D < 0$, the eigenvalues are real and rate-of-strain dominates vorticity. A complete road map to $(P, Q, R)$ space is given in Reference [11.19].

For the axisymmetric flow considered here, the velocity gradient tensor takes the form

$$A^i_j = \begin{bmatrix} \dfrac{\partial U}{\partial \xi} & \dfrac{1}{\xi}\dfrac{\partial U}{\partial \theta} - \dfrac{V}{\xi} & 0 \\ \dfrac{\partial V}{\partial \xi} & \dfrac{1}{\xi}\dfrac{\partial V}{\partial \theta} + \dfrac{U}{\xi} & 0 \\ 0 & 0 & \dfrac{V}{\xi}\cot\theta + \dfrac{U}{\xi} \end{bmatrix}. \tag{11.100}$$

Given the velocity functions, Equation (11.100) is evaluated at the critical point and the invariants are computed. Interestingly, it turns out that *often the values of the invariants can be determined without knowing the velocity functions explicitly*. In general, $Q$, $R$, $\xi_c$, and $\theta_c$ all depend on $Re$ resulting in the possibility of bifurcation in the phase space of particle paths.

### *11.5.2.3 The Limit $\xi \to 0$*

Landau in 1944 [11.20] and, independently, Squire in 1951 [11.21] solved the steady problem of a jet emerging from a point source of momentum that was assumed to have been turned on for all time. The Stokes stream function for this case is

$$\psi = \nu r\left(\frac{2\sin^2\theta}{A[Re] - \cos\theta}\right). \tag{11.101}$$

The constant of integration, $A$, is related to the Reynolds number by considering an integral momentum balance over a sphere of fixed radius, $R$, enclosing the origin as shown in Figure 11.5.

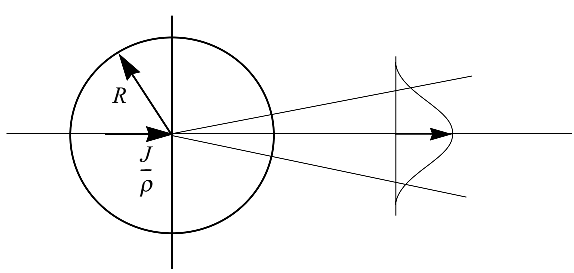


Fig. 11.5. Control volume surrounding the steady round jet.

In this instance the rotational flow of the jet penetrates the control-volume surface, but the solution is known and the integration can be carried through. Balancing forces on the control volume leads to

$$\frac{J}{\rho} = \int_0^{\pi} (u(u\cos\theta - v\sin\theta) - \left(\frac{\tau_{rr}}{\rho}\cos\theta - \frac{P}{\rho}\cos\theta - \frac{\tau_{r\theta}}{\rho}\sin\theta\right) \times 2\pi R^2 \sin\theta \, d\theta. \tag{11.102}$$

The stresses are related to the velocity field by the usual Newtonian relations,

$$\frac{\tau_{rr}}{\rho} = \nu\left(2\frac{\partial u}{\partial r}\right), \qquad \frac{\tau_{r\theta}}{\rho} = \nu\left(r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial \theta}\right). \tag{11.103}$$

The pressure is related to the velocity field by the $r$ and $\theta$ components of the momentum equation. In short, all the terms in (11.102) can be represented explicitly in terms of $r$, $\theta$, and $A$, through repeated use of the Landau–Squire solution (11.101). When the integration is carried out, the result is

$$\frac{Re^2}{16\pi} = A + \frac{4}{3}\left(\frac{A}{A^2 - 1}\right) - \frac{A^2}{2}\ln\left(\frac{A+1}{A-1}\right). \tag{11.104}$$

This relation is plotted in Figure 11.6.

The solution (11.101) is a perfectly steady flow. However, we can put it in the unsteady self-similar form (11.85) by simply multiplying and dividing by $\sqrt{\nu t}$. We obtain the following limiting solution of the impulsively started jet near $\xi = 0$

$$\Psi_0 = \xi\left(\frac{2\sin^2\theta}{A[Re] - \cos\theta}\right). \tag{11.105}$$

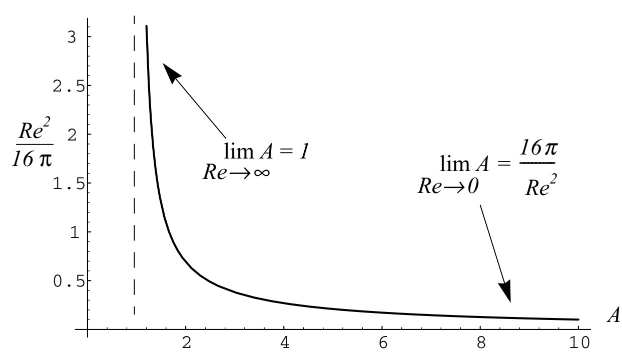


Fig. 11.6. The constant $A$ in the Landau–Squire solution.

#### 11.5.2.4 The Limit $\xi \to \infty$

In the previous section we examined Landau's solution for the steady jet cast in the form of an unsteady self-similar solution to the system (11.86). This solution conserves the flux of momentum from the source at $\xi = 0$, and at first glance there would seem to be no reason to go any further. However, we wish to consider a jet that has been turned on for a finite time and therefore has produced a flow field that contains a finite amount of momentum. The solution (11.105) violates this requirement. The flow at $\xi \to \infty$ must conserve momentum and be irrotational.

We have already worked out the vector potential at infinity when we worked out the impulse integral in Section 11.5.1. The Stokes stream function at infinity is

$$\psi = \frac{1}{4\pi}\left(\frac{J}{\rho}t\right)\frac{\sin^2\theta}{r}. \tag{11.106}$$

Here we replace $r \to \xi\sqrt{\nu t}$ and $J/\rho = Re^2\nu^2$. The limiting solution in the far field is

$$\Psi_\infty = \frac{Re^2}{4\pi}\left(\frac{\sin^2\theta}{\xi}\right). \tag{11.107}$$

The flow at infinity is that of a dipole of linearly increasing strength, $Jt/\rho$. This is also the total impulse applied to the fluid since the initiation of the momentum source (force) at the origin. As indicated in (11.57) and (11.73), two-thirds of this impulse is contained in the motion of the fluid directed along the jet axis, and one-third is lost to opposing unsteady pressure forces that act at infinity.

#### 11.5.2.5 The Limit $Re \to 0$

If one takes the limit of (11.104) as $Re \to 0 (A \to \infty)$, the result is

$$A = 16\pi/Re^2. \tag{11.108}$$

In this limit the solution near $\xi = 0$ becomes symmetric in $\theta$, and one can expect an overall solution of the form

$$\lim_{Re\to 0} \Psi[\xi,\theta] = \frac{Re^2}{16\pi}(\sin^2\theta)g[\xi], \tag{11.109}$$

where the radial function must satisfy

$$\lim_{\xi\to 0} g[\xi] = 2\xi, \qquad \lim_{\xi\to\infty} g[\xi] = \frac{4}{\xi}. \tag{11.110}$$

The corresponding vorticity is of the form

$$\lim_{Re \to 0} \Omega[\xi, \theta] = \frac{Re^2}{16\pi} (\sin^2 \theta) f[\xi]. \tag{11.111}$$

Equations (11.109) and (11.111) are substituted into (11.86), and higher-order terms in the small parameter $Re^2/16\pi$ are neglected. The result is the linear vorticity diffusion equation [the momentum equation in (11.81) with the non-linear terms removed]. Finally we end up with a linear second-order ODE governing the radial vorticity function $f[\xi]$:

$$\xi^2 f_{\xi\xi} + 2\xi \left(1 + \frac{\xi^2}{4}\right) f_\xi + (\xi^2 - 2) f = 0. \tag{11.112}$$

The radial parts of the vorticity function and stream function, $f$ and $g$, are related through the definition of the vorticity,

$$\frac{d}{d\xi}\left(\frac{1}{\xi^2}\frac{d}{d\xi}(\xi g[\xi])\right) = -\frac{f[\xi]}{8}. \tag{11.113}$$

Equations (11.112) and (11.113) are solved using (11.110), leading to the solution of the Stokes creeping jet:

$$\lim_{Re \to 0} \Psi[\xi, \theta] = \frac{Re^2}{16\pi} \sin^2 \theta \left(2\xi - \frac{4}{\sqrt{\pi}} e^{-\xi^2/4} - \left(2\xi - \frac{4}{\xi}\right) \operatorname{erf}[\xi/2]\right). \tag{11.114}$$

#### 11.5.2.6 Particle Paths of the Landau–Squire Jet

Now let's examine the flow pattern of the Landau–Squire solution, $\Psi_0[\xi, \theta]$. Using (11.87), substitute (11.105) into the particle-path equations in similarity coordinates (11.88). The result is

$$\frac{d\xi}{d\tau} = \frac{2}{\xi}\left(\frac{A^2 - 1}{(A - \cos\theta)^2} - 1\right) - \frac{\xi}{2}, \qquad \frac{d\theta}{d\tau} = \frac{2\sin\theta}{\xi^2 (A - \cos\theta)}. \tag{11.115}$$

The system (11.115) has a single critical point on the axis of the jet, located at

$$(\xi_c, \theta_c) = \left(\frac{2^{3/2}}{(A - 1)^{1/2}}, 0\right). \tag{11.116}$$

The relevant gradient tensors evaluated at the critical point are

$$A^i_j = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}, \qquad M^i_j = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix}, \tag{11.117}$$

and the invariants of $M^i_j$ are $(P_M, Q_M, R_M) = (\frac{3}{2}, \frac{9}{16}, \frac{1}{16})$, independent of the Reynolds number. In the terminology of Reference [11.19], the critical point is a stable star node with three real negative eigenvalues, two of which are equal. As the Reynolds number is increased, the radial coordinate of the critical point increases although the invariants remain constant.

#### *11.5.2.7 Particle Paths of the Unsteady Dipole*

The solution for the far field, (11.107), is substituted into (11.87) and (11.88), producing

$$\frac{d\xi}{d\tau} = \frac{Re^2}{2\pi}\frac{\cos\theta}{\xi^3} - \frac{\xi}{2}, \qquad \frac{d\theta}{d\tau} = \frac{Re^2}{2\pi}\frac{\sin\theta}{\xi^4}. \tag{11.118}$$

This system has a critical point on the jet axis at $(\xi_c, \theta_c) = (Re^{1/2}/\pi^{1/4}, 0)$. The two gradient tensors evaluated at the critical point are

$$A^i_j = \begin{bmatrix} -\frac{3}{2} & 0 & 0 \\ 0 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{3}{4} \end{bmatrix}, \qquad M^i_j = \begin{bmatrix} -2 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}, \tag{11.119}$$

and the invariants of $M^i_j$ are $(P_M, Q_M, R_M) = (\frac{3}{2}, -\frac{15}{16}, \frac{1}{8})$. In the parlance of reference [11.19] the critical point is a node–saddle–saddle with three real eigenvalues: one negative, and two positive and equal. Here again, the invariants are independent of the Reynolds number, whereas the position of the critical point moves outward along the axis as the Reynolds number is increased.

We have seen that, while the $\xi_c$-coordinates of the critical points of both $\Psi_0$ and $\Psi_\infty$ increase with $Re$, the values of the invariants at the respective points are independent of $Re$. Thus the stable star node of the Landau–Squire jet remains a stable star node at all Reynolds numbers, and the node–saddle–saddle of the unsteady dipole remains such at all Reynolds numbers. Neither of these flows is subject to transition as we shall define it shortly. In the full solution the flow at intermediate values of $\xi$ must accommodate both the steady Landau–Squire behavior at $\xi \to 0$ and the unsteady dipole behavior at $\xi \to \infty$. If one accepts that the on-axis critical point moves to larger and larger values of $\xi_c$ as the Reynolds number is increased, then this would suggest that the stable node obtained at small Reynolds numbers (small $\xi_c$) could not remain a stable node when the Reynolds number (and thus $\xi_c$) becomes large.

#### 11.5.2.8 Particle Paths in the Low-Reynolds-Number Jet

With that background we now consider particle paths of the low-Reynolds-number solution of the jet. Upon substitution of (11.114) into the particle-path equations (11.88) we have

$$\begin{aligned}\frac{d\xi}{d\tau} &= \frac{Re^2}{2\pi}\frac{\cos\theta}{\xi^2}\left(\frac{\xi}{2} - \frac{1}{\sqrt{\pi}}e^{-\xi^2/4} - \left(\frac{\xi}{2} - \frac{1}{\xi}\right)\operatorname{erf}[\xi/2]\right) - \frac{\xi}{2},\\ \frac{d\theta}{d\tau} &= -\frac{Re^2}{4\pi}\frac{\sin\theta}{\xi^2}\left(\frac{1}{2} + \frac{1}{\xi\sqrt{\pi}}e^{-\xi^2/4} - \left(\frac{1}{2} + \frac{1}{\xi^2}\right)\operatorname{erf}[\xi/2]\right).\end{aligned} \tag{11.120}$$

The critical points of (11.120) now need to be located. This is done by setting the right-hand sides equal to zero and solving for the roots. The zeros of the $\theta$-equation occur at $\theta = 0,\ \pi$ for all $\xi$ and at $\xi = 1.7633$ for all $\theta$, independent of the Reynolds number. However, the zeros of the $\xi$-equation depend on $Re$ as follows:

$$Re^2 = \frac{\pi\xi_c^3}{\left(\frac{\xi_c}{2} - \frac{1}{\sqrt{\pi}}e^{-\xi_c^2/4} - \left(\frac{\xi_c}{2} - \frac{1}{\xi_c}\right)\operatorname{erf}[\xi_c/2]\right)\cos\theta}. \tag{11.121}$$

Equation (11.121) defines a family of curves in the $(\xi_c, \theta_c)$ plane, of which several are drawn in Figure 11.7. The intersections between the zeros of the right-hand sides of (11.120) for a given Reynolds number define the locations of the critical points. This is illustrated in Figure 11.7.

It is clear from Figure 11.7 that there is a critical value of $Re$ between 6 and 8. Below this value there is only one intersection (one critical point) on the axis of the jet, whereas above this $Re$ there are two critical points, one on the axis and one off the axis. Due to the axisymmetry, the off-axis point is in fact a degenerate line of critical points on an azimuthal circle about the jet axis.

The trajectories of the critical points in the space of invariants of both $A^i_j$ and $M^i_j$ are shown in Figure 11.8. The direction arrows on these figures indicate the direction of increasing Reynolds number. The axisymmetry of the on-axis critical point implies that there are always two equal eigenvalues. In the space of invariants, the values of $(R_A, Q_A)$ and $(R_M, Q_M)$ at the critical point are confined to the $R > 0$ branch of the zero discriminant curve, $D_A = 0$ and $D_M = 0$ respectively. See Equations (11.98) and (11.99). The off-axis critical

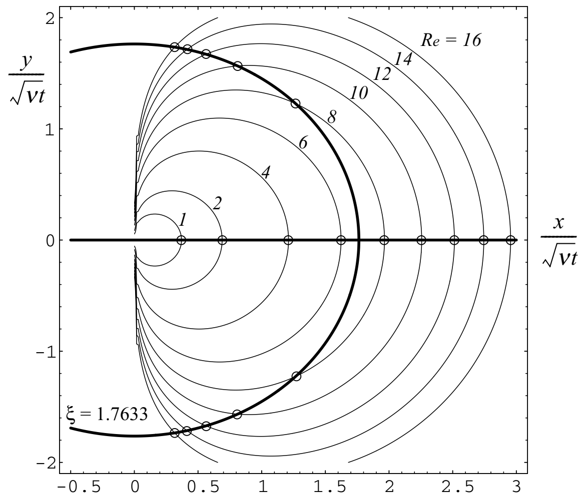


Fig. 11.7. Critical-point locations at several Reynolds numbers for the Stokes jet. The circle has radius 1.7633.

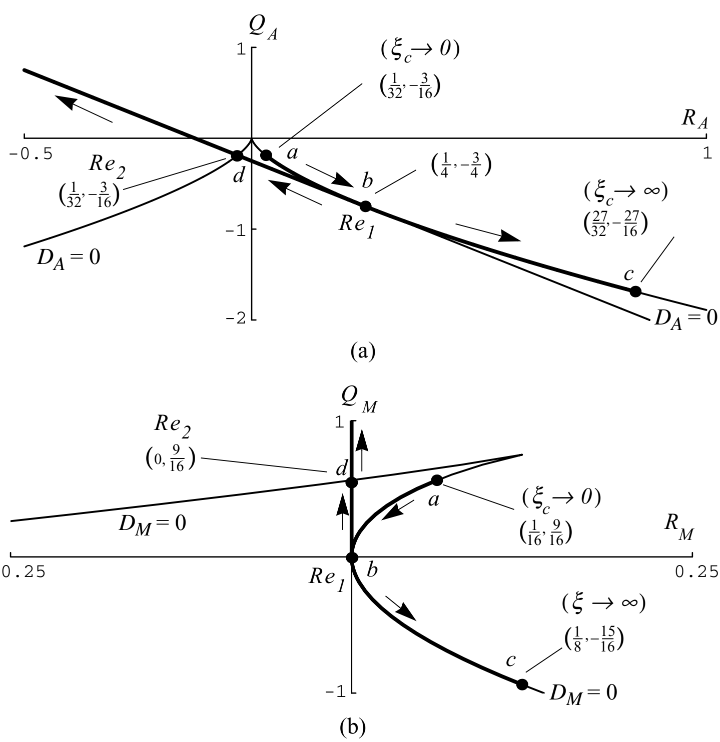


Fig. 11.8. Critical-point trajectories in the round jet: (a) the trajectory in $(Q_A, R_A)$ coordinates at various $Re$; (b) the same trajectory in $(Q_M, R_M)$.

point follows the straight line

$$R_A + \tfrac{1}{2}Q_A + \tfrac{1}{8} = 0, \qquad R_M = 0. \tag{11.122}$$

Note that the vector field of particle paths near the off-axis critical point is intrinsically two-dimensional (third invariant equal to zero), whereas the velocity vector field is not.

If $Re < 6.7806$, there is a single axisymmetric node lying on the axis of the jet, $\theta_c = 0$. In this Reynolds-number range, Equation (11.121) provides a relation between the Reynolds number and the radial coordinate of the on-axis critical point, which moves outward along the axis of the jet as the Reynolds number is increased. When $Re$ exceeds $Re_1 = 6.7806$ (point b in Figure 11.8), the flow bifurcates to an axisymmetric saddle situated on the jet axis and an off-axis node above and below the axis. Actually the off-axis node is a circular line in the azimuthal direction about the axis. We shall return to this point shortly; first we complete the discussion of the jet structure.

Figure 11.9 and Figure 11.10 depict the phase portrait of the jet on a cut at $z/\sqrt{\nu t} = 0$ at three Reynolds numbers in the regimes of interest. The off-axis critical point lies at a radius $\xi_c = 1.7633$ and angle

$$\theta_c = \pm\cos^{-1}\left[\left(\frac{6.7806}{Re}\right)^2\right]. \tag{11.123}$$

As the Reynolds number is increased above $Re_1$, the node moves away from the axis while the radius of the on-axis saddle continues to follow (11.121). The

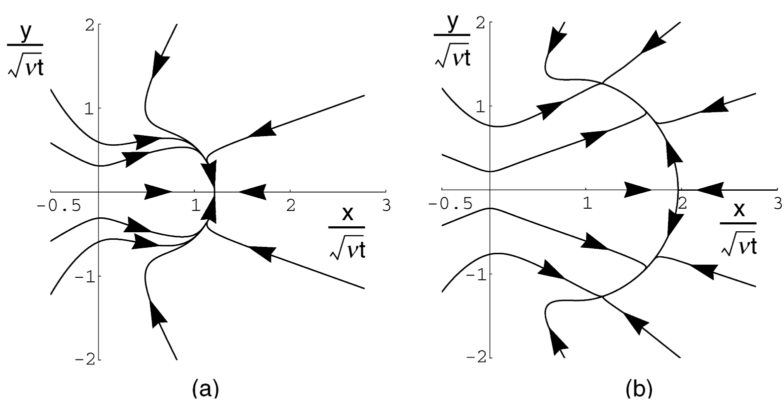


Fig. 11.9. Particle paths for the impulsively started creeping jet at (a) $Re = 4$ and (b) $Re = 8$.

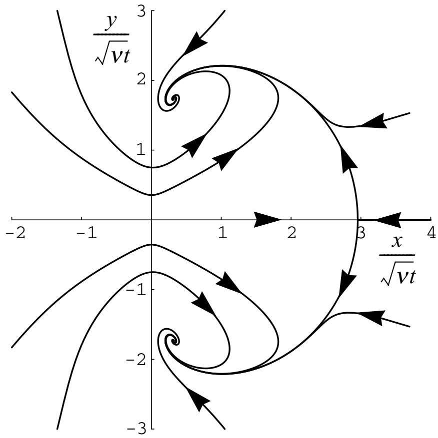


Fig. 11.10. Particle paths for the impulsively started creeping jet at $Re = 16$.

invariants of the off-axis critical point follow:

$$\begin{aligned} P_M|_{\theta_c \neq 0} &= \tfrac{3}{2}, \\ Q_M|_{\theta_c \neq 0} &= 6.8143 \times 10^{-5} Re^4 - 0.14405, \\ R_M|_{\theta_c \neq 0} &= 0. \end{aligned} \tag{11.124}$$

The off-axis node changes to a stable focus when $Q_M|_{\theta_c \neq 0}$ exceeds $\frac{9}{16}$ (Point d in Figure 11.8). This transition from a stable node to a stable focus occurs where the trajectory of $Q_M|_{\theta_c \neq 0}$ crosses the left branch of the $D_M = 0$ line. According to (11.124), this occurs at $Re_2 = 10.09089$. Thus a starting vortex is created. Note that only the first bifurcation involves a change in the actual topology of the phase portrait.

There is an obvious question at this point. What relevance does the behavior of the creeping jet have to the full nonlinear problem? This is answered by recognizing that the trajectory of the invariants in Figure 11.8 comes solely from consideration of boundary conditions, continuity, and axisymmetry and therefore holds for both the linear and nonlinear problems. At low Reynolds number ($\xi_c$ small), $Q_M$ and $R_M$ approach the values for the Landau–Squire solution, whereas at high Reynolds number ($\xi_c$ large) they approach the values for the unsteady dipole. Therefore the sequence of states (on-axis node, followed by saddle plus off-axis node, followed by saddle plus off-axis focus) is preserved in the nonlinear axisymmetric problem, although the critical Reynolds numbers are different from those obtained from the creeping solution.

A numerical solution of the nonlinear problem is reported in References [11.23] and [11.22], where it is noted that the critical Reynolds numbers of the nonlinear jet are $Re_1 = 5.50$ and $Re_2 = 7.545$ respectively. The phase portrait of particle paths in similarity coordinates from this computation is shown in Figure 11.11. The starting vortex from the jet is defined as the rollup of particle paths shown in Figure 11.11c and d. Interestingly, the vorticity decays monitonically through the rollup. There is a pervasive misconception in the fluid mechanics literature that a rollup of fluid such as that shown is accompanied by a local concentration of the vorticity. But there is no peak in the vorticity, and a brief analysis of the invariants of the particle-path equations (11.95) and (11.96) in the neighborhood of the stable focus reveals that it is a

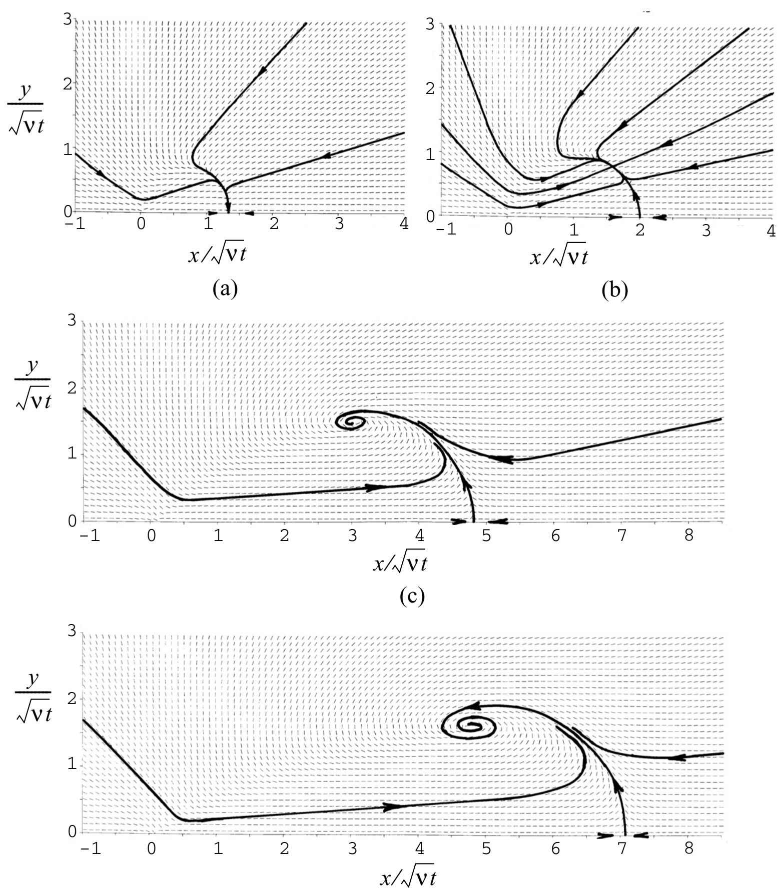


Fig. 11.11. Numerically computed particle paths in the round jet at Reynolds numbers (a) $Re = 4$, (b) $Re = 6$, (c) $Re = 15$, (d) $Re = 25$.

balance between vorticity and strain that determines whether or not the fluid will roll up. If vorticity dominates the rate of strain, then the rollup will occur as in Figure 11.11c and d. If the rate of strain dominates the vorticity, then the flow will tend to exhibit nodal or saddle-like behavior as in Figure 11.11a and b.

#### *11.5.2.9 Invariance of the Vector Field of Particle Paths Relative to a Moving Observer*

In Sections 11.1 and 11.2 we saw that the incompressible Navier–Stokes equations are invariant with respect to any nonuniformly moving observer as long as the observer moves without rotating. [Actually, a constant rotation frequency is permitted in two dimensions according to invariance under the group (11.21) with operator $X^{10}$ given by (11.20)]. On changing observers, the velocity vector field changes dramatically, and this was illustrated in Figure 11.1 where the velocity field in the wake of a circular cylinder was observed in two frames of reference. Now it is time to revisit this issue in the context of the round jet analyzed in the previous sections.

The self-similarity in time of the jet enabled us to reduce the particle-path equations (11.83) to an autonomous system, (11.88). The invariance of the governing equations under the nonuniform translation group (11.14) can be used to show that, the vector field of particle paths in similarity coordinates is the same for all observers moving with the time scale appropriate to the flow. In Cartesian coordinates the equations for particle paths are

$$\frac{dx^i}{dt} = u^i[\boldsymbol{x}, t], \tag{11.125}$$

which, when transformed to similarity variables, become

$$\frac{d\xi^i}{d\tau} = U^i[\boldsymbol{\xi}] - \frac{1}{2}\xi^i, \tag{11.126}$$

where $\xi^i = x^i/(\nu t)^{1/2}$ and $\tau = \ln t$. In the round jet all length scales vary in proportion to $(\nu t)^{1/2}$. For an observer translating according to this function, the appropriate transformation of coordinates is

$$\begin{aligned}
\tilde{x}^j &= x^j + \alpha^j(\nu t)^{1/2}, \\
\tilde{t} &= t, \\
\tilde{u}^i &= u^i + \frac{\alpha^i}{2}\nu^{1/2}t^{-1/2}, \\
\frac{\tilde{p}}{\rho} &= \frac{p}{\rho} + x^k\frac{\alpha^k}{4}\nu^{1/2}t^{-3/2}, \quad \text{sum over } k
\end{aligned} \tag{11.127}$$

where the $\alpha^i$ determine the rate at which the observer moves in each of the three coordinate directions. In terms of similarity variables the transformation of velocities and coordinates, (11.127), becomes

$$\begin{aligned}\tilde{\xi}^j &= \xi^j + \alpha^j,\\ \tilde{U}^i &= U^i + \frac{\alpha^i}{2}.\end{aligned} \tag{11.128}$$

In contrast to the velocity vector field, the vector field of particle paths is invariant. To see this we transform the right-hand side of (11.126):

$$\tilde{U}^i - \frac{1}{2}\tilde{\xi}^i = \left(U^i + \frac{\alpha^i}{2}\right) - \frac{1}{2}(\xi^j + \alpha^j) = U^i - \frac{1}{2}\xi^j. \tag{11.129}$$

Because the $\alpha^i$ cancel all observers, moving or not, would assign the same numerical values to the components $d\xi^i$, $i = 1, 2, 3$, of the particle-path displacement vectors in similarity coordinates. A moving observer would assign these values at points that are uniformly displaced by a fixed amount $(\alpha^i\alpha^i)^{1/2}$ along a ray $\theta$ = constant, but this displacement does not affect the pattern of particle trajectories.

This invariance is extremely important. It means that the location and character of a critical point in similarity coordinates is fixed by the dynamics that govern the flow and not by the incidental choice of speed for a moving observer. In general, when a flow is self-similar in time, the vector field of particle paths is invariant for any observer whose position varies in proportion to the global time scale of the flow. Thus any critical points that may appear in the phase portrait are intrinsic properties of the flow and not a figment of a particular choice of observer. The node of the Landau–Squire jet cannot be changed to a saddle, and similarly the saddle of the unsteady dipole cannot be changed to a node, by merely referring the flow to a new observer. Bifurcations in the phase portrait of particle paths that are produced by changing the Reynolds number cannot be modified by changing frames of reference.

Additional discussion of invariant particle trajectories in the context of a turbulent flow may be found in [11.24], where particle trajectories are used to experimentally identify critical points in the ensemble-averaged flow pattern of a turbulent spot. A photograph of a turbulent spot from these experiments is shown on the front cover of this book. A similar approach is used in [11.25] to collapse data from a turbulent vortex ring. We will look at the turbulent-vortex-ring problem in detail in Chapter 13.

## 11.6 Elliptic Curves and Three-Dimensional Flow Patterns

It is interesting to note the prevalence of rational fractions at the intersections of the invariant trajectories shown in Figure 11.8. Note also the mixture of quadratic and cubic terms in the expressions for the discriminant, (11.98) and (11.99). For constant, nonzero discriminant, these equations belong to a class of functions called *elliptic curves*. We considered this class of functions in Chapter 6, Example 6.10, where we discussed a pair of ODEs that describe the time evolution of the invariants of a cubic equation. Elliptic curves were further discussed in Section 6.7 along with the Diophantine construction used to identify rational roots on elliptic curves. Elliptic curves have the property that there is a unique tangent everywhere on the curve; hence $D = 0$ is excluded, and they are parameterized by elliptic functions. The curve $D = 0$, which has a cusp at the origin, is parameterized by rational functions.

### *11.6.1 Acceleration Field in the Round Jet*

Here some of the theory of elliptic curves and the development in Reference [11.28] will be used to explore the geometry of the forces at the critical points of the round jet. First we need to develop the transport equation for the velocity gradient tensor $a^i_j = \partial u^i/\partial x^j$ by taking the gradient of the Navier–Stokes equations,

$$\frac{\partial}{\partial x^j}\left(\frac{\partial u^i}{\partial t} + u^k\frac{\partial u^i}{\partial x^k} + \frac{1}{\rho}\frac{\partial p}{\partial x^i} - \nu\frac{\partial^2 u^i}{\partial x^k\,\partial x^k}\right) = 0. \tag{11.130}$$

Carrying out the differentiation and applying the continuity equation for incompressible flow, $a^i_i = 0$, leads to

$$\frac{\partial a^i_j}{\partial t} + u^k\frac{\partial a^i_j}{\partial x^k} + a^i_k a^k_j + \frac{1}{\rho}\frac{\partial^2 p}{\partial x^i\,\partial x^j} - \nu\frac{\partial^2 a^i_j}{\partial x^k\,\partial x^k} = 0. \tag{11.131}$$

Now take the trace of (11.131) to generate the Poisson equation for the pressure:

$$\frac{1}{\rho}\frac{\partial^2 P}{\partial x^i\,\partial x^i} = -a^i_k a^k_i. \tag{11.132}$$

Equation (11.132) is subtracted from (11.131) to make the pressure term trace-free. The final result is the transport equation for the velocity gradient tensor,

$$\frac{Da^i_j}{Dt} + a^i_k a^k_j - \frac{1}{3}\left(a^m_n a^n_m\right)\delta^i_j = h^i_j, \tag{11.133}$$

where

$$h^i_j = -\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x^i\,\partial x^j} - \frac{1}{3}\frac{\partial^2 p}{\partial x^k\,\partial x^k}\delta^i_j\right) + \nu\frac{\partial^2 a^i_j}{\partial x^k\,\partial x^k}. \tag{11.134}$$

The tensor $h^i_j$ is the divergence-free ($P_h = 0$) part of the gradient of the acceleration vector field following a fluid element.

When (11.133) is transformed to similarity variables for the round jet, the result is

$$-A^i_j + \left(U_k - \frac{1}{2}\xi_k\right)\frac{\partial A^i_j}{\partial \xi_k} + A^i_k A^k_j - \frac{1}{3}\left(A^m_n A^n_m\right)\delta^i_j = H^i_j, \tag{11.135}$$

where $H$ is the same as (11.134) but expressed in terms of $(U^i, P, \xi^i)$. At a critical point, the convective term in (11.135) is zero, and $A$ and $H$ are algebraically related by

$$-A^i_j + A^i_k A^k_j - \tfrac{1}{3}\left(A^m_n A^n_m\right)\delta^i_j = H^i_j. \tag{11.136}$$

Squaring (11.136) and taking the trace produces

$$Q_H = -\tfrac{1}{3}Q_A^2 + Q_A - 3R_A. \tag{11.137}$$

Cubing (11.136) and taking the trace produces

$$R_H = -R_A^2 - R_A + Q_A R_A - \tfrac{2}{3}Q_A^2 - \tfrac{2}{27}Q_A^3. \tag{11.138}$$

Now switch over, and square (11.138) and cube (11.137) to form the discriminant of the acceleration gradient tensor $H$: The result is

$$Q_H^3 + \tfrac{27}{4}R_H^2 = \left(Q_A^3 + \tfrac{27}{4}R_A^2\right)(I + Q_A - R_A)^2. \tag{11.139}$$

A remarkably simple result! A generalization of this procedure is described in [11.27].

We can express the invariants of $H$ in terms of the invariants of $M$ . The result is

$$\begin{aligned}
Q_H &= 3Q_M - 3R_M - \tfrac{1}{3}Q_M^2 - \tfrac{27}{16},\\
R_H &= -R_M^2 - \tfrac{9}{4}R_M + 2Q_M R_M - \tfrac{2}{27}Q_M^3 - \tfrac{5}{4}Q_M^2 + \tfrac{9}{4}Q_M - \tfrac{27}{32},\\
Q_H^3 + \tfrac{27}{4}R_H^2 &= \left(Q_M^3 + \tfrac{27}{4}R_M^2 + \tfrac{27}{4}R_M\left(\tfrac{1}{2} - Q_M\right) - \tfrac{9}{16}Q_M^2\right)\left(R_M - \tfrac{3}{2}Q_M\right)^2.
\end{aligned} \tag{11.140}$$

Note that the terms of sixth order in $Q_A$ or $Q_M$ that would be expected when the discriminant of $H$ is formed in (11.139) and (11.140) have canceled. At the off-axis critical point in Figure 11.11, where $R_M = 0$ we find,

$$\begin{aligned} Q_H &= 3Q_M - \tfrac{1}{3}Q_M^2 - \tfrac{27}{16}, \\ R_H &= -\tfrac{2}{27}Q_M^3 - \tfrac{5}{4}Q_M^2 + \tfrac{9}{4}Q_M - \tfrac{27}{32}, \\ Q_H^3 + \tfrac{27}{4}R_H^2 &= \tfrac{9}{4}Q_M^4\left(Q_M - \tfrac{9}{16}\right). \end{aligned} \tag{11.141}$$

The trajectory of the critical points of the round jet in the $(R_H, Q_H)$ plane, with the off-axis point parameterized by $Q_M$ as in (11.141), is depicted in Figure 11.12. Four significant points are labeled in these plots:

*Point a.* This corresponds to the zero-Reynolds-number (Stokes flow) limit of the jet, where there is a single stable node on the jet axis. The invariants

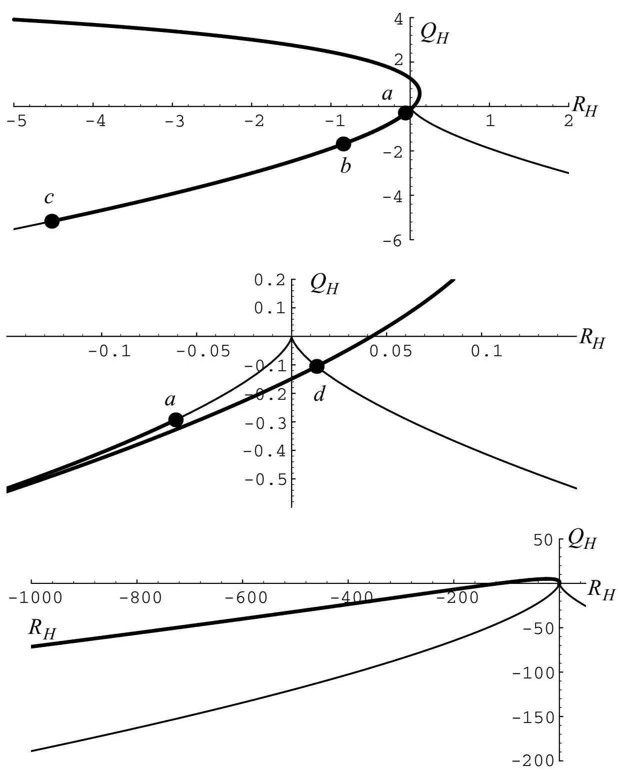


Fig. 11.12. Trajectory of the critical points of the round jet in the $(Q_H, R_H)$ plane at three levels of magnification. Dots indicate several rational roots on the $D_H = 0$ boundary (on-axis critical point) and on the trajectory of the off-axis critical point. The labels $a$, $b$, $c$, and $d$ coincide with the same labels in Figure 11.8.

of this critical point are

$$
\begin{aligned}
(R_A, Q_A) &= \left(\tfrac{1}{32}, -\tfrac{3}{16}\right),\\
(R_M, Q_M) &= \left(\tfrac{1}{16}, \tfrac{9}{16}\right),\\
(R_H, Q_H) &= \left(-\tfrac{125}{2048}, -\tfrac{75}{256}\right).
\end{aligned} \tag{11.142}
$$

*Point b.* Let the Reynolds number increase. At a critical Reynolds number of 5.5 the jet undergoes a bifurcation to a saddle on the jet axis and a stable node off the axis. The invariants at the bifurcation point are

$$
\begin{aligned}
(R_A, Q_A) &= \left(\tfrac{1}{4}, -\tfrac{3}{4}\right),\\
(R_M, Q_M) &= (0, 0),\\
(R_H, Q_H) &= \left(-\tfrac{27}{32}, -\tfrac{27}{16}\right).
\end{aligned} \tag{11.143}
$$

*Point c.* As the jet Reynolds number increases to infinity, the on-axis critical point moves to infinity and the invariants asymptote to the values given at $c$:

$$
\begin{aligned}
(R_A, Q_A) &= \left(\tfrac{27}{32}, -\tfrac{27}{16}\right),\\
(R_M, Q_M) &= \left(\tfrac{1}{8}, -\tfrac{15}{16}\right),\\
(R_H, Q_H) &= \left(-\tfrac{9261}{2048}, -\tfrac{1323}{256}\right).
\end{aligned} \tag{11.144}
$$

*Point d.* Above the first bifurcation Reynolds number, the invariants of the off-axis critical point move upward along a straight line until, at a second critical Reynolds number of 7.545, the off-axis critical point turns into a stable node. Thus a starting vortex from the jet is born. The invariants of the off-axis point at this Reynolds number are

$$
\begin{aligned}
(R_A, Q_A) &= \left(-\tfrac{1}{32}, -\tfrac{3}{16}\right),\\
(R_M, Q_M) &= \left(0, \tfrac{9}{16}\right),\\
(R_H, Q_H) &= \left(\tfrac{27}{2048}, -\tfrac{27}{256}\right).
\end{aligned} \tag{11.145}
$$

We learn quite a bit from this analysis. Virtually every interesting intersection (bifurcation) in the starting jet flow coincides with a rational root in the plane of critical-point invariants. Note that the rational roots on the trajectory of the off-axis critical point in $(Q_H, R_H)$ generated by (11.141) are densely spaced, just as they are on the real line. Moreover they coincide with rational values of the discriminant. This can be exploited to identify at least one rational root on any curve of constant discriminant derived from a rational value of $Q_M$ and/or

$R_M$ and intersected by (11.140). The Diophantene construction can then be used to identify further rational roots.

These results have interesting implications for the limiting behavior of the off-axis critical point, which, eventually closes on the $D_H = 0$, $R_H < 0$ line as $Re \to \infty$. The signs of the discriminant of all three tensors are the same. Thus if $M$ has complex eigenvalues, so have $H$ and $A$. This means that the purely viscous, antisymmetric part of $H^i_j$ remains important but diminishes compared to the symmetric pressure-dominated part as the Reynolds number increases. The viscous contribution to the forces at the critical point is never negligible. Finally, the invariants of the on-axis critical point have finite, rational values as the limit $Re \to \infty$ is taken. Few such infinite-Reynolds-number limits are known in fluid mechanics.

A detailed example of the relationship between elliptic curves, elliptic functions, and fluid mechanics can be found in [11.26].

## 11.7 Classification of Falkner–Skan Boundary Layers

In Chapter 10 we examined the Falkner–Skan class of boundary layers corresponding to a power-law dependence of the free-stream velocity. The general Falkner–Skan ODE cannot be reduced to first order except for the case $\beta = -1$, and therefore it cannot be analyzed on a phase plane in terms of an autonomous system.

However, we can use a rather different approach based on the method used to analyze the impulsively started round jet. Recall that in Section 11.5.2.3 we expressed the steady Landau–Squire solution for the jet in unsteady similarity coordinates. Let's use the same procedure to examine the structure of the Falkner–Skan flow field on a phase plane. The basic idea is to examine particle trajectories of the flow in coordinates that are self-similar in time. The Falkner–Skan variables are

$$\begin{aligned} \alpha &= \left(\frac{M}{2\nu}\right)^{1/2} \frac{y}{(x + x_0)^{(1-\beta)/2}}, \\ F &= \frac{\psi}{(x + x_0)^{(1+\beta)/2}(2\nu M)^{1/2}}. \end{aligned} \tag{11.146}$$

An equivalent form of (11.146) with the variables recast in unsteady self-similar form is

$$\begin{aligned} \alpha &= \frac{\phi}{\theta^{(1-\beta)/2}}, \\ F &= \frac{\psi}{\nu^{1/2} M^{1/(1-\beta)} t^{(1+\beta)/2(1-\beta)}}, \end{aligned} \tag{11.147}$$

where

$$\theta = \frac{x}{(Mt)^{1/(1-\beta)}}, \qquad \phi = \frac{y}{\sqrt{\nu t}}, \tag{11.148}$$

and

$$F[\theta, \phi] = \theta^{(1+\beta)/2} G[\alpha]. \tag{11.149}$$

Note how the normalization of coordinates falls in neatly with the streamwise coordinate normalized by an inertial–convective time scale and the wall-normal coordinate normalized by a viscous–diffusive time scale. This is directly related to the two-parameter dilational invariance of the boundary-layer equations discussed in Chapter 10 and reflects the distinction between the two coordinate directions inherent in the boundary-layer approximation.

Particle paths are determined by the system

$$\begin{aligned} \frac{dx}{dt} &= u[x, y, t] = \frac{\partial \psi}{\partial y}, \\ \frac{dy}{dt} &= v[x, y, t] = -\frac{\partial \psi}{\partial x}. \end{aligned} \tag{11.150}$$

Now we replace $x$ and $y$ in favor of similarity variables (11.148) using the differentials

$$\begin{aligned} dx &= (Mt)^{1/(1-\beta)}\, d\theta + \frac{\theta}{(1-\beta)t}(Mt)^{1/(1-\beta)}\, dt, \\ dy &= (\nu t)^{1/2}\, d\phi + \frac{(\nu t)^{1/2}}{2t}\phi\, dt. \end{aligned} \tag{11.151}$$

The resulting particle-path equations in similarity coordinates are

$$\begin{aligned} \frac{d\theta}{d\tau} &= F_\phi - \frac{\theta}{1-\beta}, \\ \frac{d\phi}{d\tau} &= F_\theta - \frac{\phi}{2}. \end{aligned} \tag{11.152}$$

For the Falkner–Skan *steady* boundary layers, the function $F$ is given by (11.149). Substituting (11.149) into (11.152) produces

$$\begin{aligned} \frac{d\theta}{d\tau} &= \theta^\beta G_\alpha - \frac{\theta}{1-\beta}, \\ \frac{d\phi}{d\tau} &= -\theta^{(\beta-1)/2}\left(\frac{1+\beta}{2}G - \left(\frac{1-\beta}{2}\right)\alpha G_\alpha\right) - \frac{\phi}{2}. \end{aligned} \tag{11.153}$$

In this procedure we have reduced the Falkner–Skan flow to an autonomous system of equations for particle paths whose trajectories can be displayed on a phase plane. Setting the right-hand side of (11.153) equal to zero at a critical point $(\theta_c, \phi_c)$ enables $G$ and $G_\alpha$ to be evaluated at the point:

$$\begin{aligned} G[\alpha_c] &= 0, \\ G_\alpha[\alpha_c] &= \frac{\theta_c^{1-\beta}}{1-\beta}, \end{aligned} \tag{11.154}$$

where $\alpha_c = \phi_c/\theta_c^{(1-\beta)/2}$. Near a critical point (11.153) can be expanded as

$$\begin{aligned} \frac{d\theta}{d\tau} &= a(\theta - \theta_c) + b(\phi - \phi_c), \\ \frac{d\phi}{d\tau} &= c(\theta - \theta_c) + d(\phi - \phi_c), \end{aligned} \tag{11.155}$$

where

$$\begin{aligned} a &= -1 - \left(\frac{1-\beta}{2}\right)\phi_c\theta_c^{(3\beta-3)/2} G_{\alpha\alpha}[\alpha_c], \\ b &= \theta_c^{(3\beta-1)/2} G_{\alpha\alpha}[\alpha_c], \\ c &= \frac{\phi_c}{\theta_c}\left(\frac{3\beta-1}{4}\right) - \left(\frac{1-\beta}{2}\right)^2 \phi_c^2\theta_c^{(3\beta-5)/2} G_{\alpha\alpha}[\alpha_c], \\ d &= -\frac{1+\beta}{2(1-\beta)} + \frac{1-\beta}{2}\left(\phi_c\theta_c^{(3\beta-3)/2} G_{\alpha\alpha}[\alpha_c]\right). \end{aligned} \tag{11.156}$$

Note that, although the individual coefficients in (11.156) depend on $G_{\alpha\alpha}[\alpha_c]$, the invariants $p$ and $q$ do not:

$$\begin{aligned} p &= -(a+d) = \frac{3-\beta}{2(1-\beta)}, \\ q &= ad - bc = \frac{1+\beta}{2(1-\beta)}. \end{aligned} \tag{11.157}$$

Eliminating $\beta$ between the two expressions in (11.157) produces

$$q = p - 1. \tag{11.158}$$

All the various well-known cases are indicated on the plot of critical-point invariants shown in Figure 11.13.

The relevant critical point of (11.153) is at the leading edge of the plate, $(\theta, \phi) = (0, 0)$. As far as the invariants are concerned, the leading edge, where

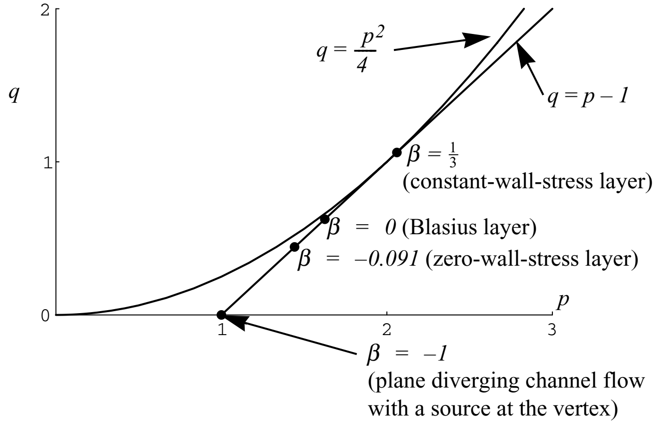


Fig. 11.13. Classification of Falkner–Skan boundary layers in the $(p, q)$ plane.

$G_{\alpha\alpha} \to \infty$, is a removable singularity. For the Blasius layer ($\beta = 0$), the leading-edge critical point is a stable node with invariants $(p, q) = \left(\frac{3}{2}, \frac{1}{2}\right)$.

A constant-wall-stress layer is produced by a free-stream velocity distribution with a mildly favorable pressure gradient:

$$U_e = M(x + x_0)^{1/3}. \tag{11.159}$$

See the velocity profiles in Chapter 10, Figure 10.11. This is a situation where the weak acceleration of the free stream exactly balances diffusion of vorticity from the wall to produce a boundary layer with a wall shear stress that remains constant with $x$. The critical point at the leading edge is a star node with two equal eigenvalues.

The case $\beta = -1$ is the flow generated by a source of area at the plate leading edge and was treated in detail in Chapter 10 and in Section 11.4. It is a case where length scales in both coordinate directions vary like $\sqrt{t}$, the same as for the round jet. This can be seen by setting this value of $\beta$ in (11.148).

## 11.8 Concluding Remarks

In this chapter we have applied symmetry analysis to two problems of incompressible flow governed by the full Navier–Stokes equations. The joining of Lie theory and bifurcation analysis in phase space produces a complete understanding of the Reynolds-number dependence of the space–time structure of the flow. The phase portrait of particle paths in similarity coordinates is invariant under a change of observer. This enabled fundamental questions concerning moving frames of reference and the distinction between streamlines and particle trajectories in unsteady flow to be addressed.

The character of a critical point in the phase portrait is determined by the matrix invariants of the gradient tensor evaluated at the point. A new element was introduced when we analyzed the evolution of flow structure in terms of the trajectory of critical points in the space of matrix invariants. Interestingly, these invariants can often be evaluated without knowing the flow solution. This was exploited in the last example, where we classified the family of Falkner–Skan boundary layers knowing only the similarity form of the solution.

## 11.9 Exercises

11.1 Return to the Jeffery–Hamel problem and explore the invariance of Equation (11.38), $F_{\theta\theta\theta} + 12FF_\theta + 4F_\theta = 0$, under translation in $\theta$. Use the method of differential invariants to reduce the problem to the second-order equation

$$HH_{\phi\phi} + (H_\phi)^2 + 12\phi + 4 = 0. \tag{11.160}$$

The first two terms on the left-hand side can be combined to yield

$$\frac{1}{2}(H^2)_{\phi\phi} + 12\phi + 4 = 0. \tag{11.161}$$

Integrate twice to reproduce the solution (11.41).

11.2 Work out the dilation group that leaves invariant a steady plane laminar jet generated by flow from a narrow slit. The conserved integral is

$$M = \int_{-\infty}^{\infty} u^2\, dy. \tag{11.162}$$

Discuss the problems encountered in searching for a similarity solution of the full Navier–Stokes equations for this problem. Compare your group with that of the boundary-layer equations. Work out a similarity solution using the boundary-layer approximation.

11.3 Use direct substitution to show that the incompressible Navier–Stokes equations,

$$\frac{\partial u_j}{\partial x_j} = 0, \qquad (\text{sum over } j = 1, 2, 3),$$

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} - \nu\frac{\partial^2 u_i}{\partial x_j^2} = 0, \qquad i = 1, 2, 3, \tag{11.163}$$

are invariant under a transformation to a noninertial frame of reference (11.41). Consider two coordinate systems related as follows:

$$\begin{aligned}
\tilde{x}_i &= x_i - f_i[t], \\
\tilde{u}_i &= u_i - \dot{f}_i[t], \\
\tilde{t} &= t, \\
\tilde{p} &= p + x_k \ddot{f}_k[t].
\end{aligned} \tag{11.164}$$

Thus we can always view a flow from the frame of reference of an accelerating observer without changing the physics of the flow. Provide a physical interpretation of the transformation of the pressure.

11.4 A steady, axisymmetric (nonbuoyant) jet is produced by a heated source of momentum. How would you expect the centerline temperature of the jet to vary with distance from the source? The exact solution for the laminar heated jet is presented by Squire [11.21].

11.5 Work out the dependence on distance from the leading edge of the wall-normal velocity component for the Falkner–Skan constant-stress layer $\beta = \frac{1}{3}$.

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# 12

## *Compressible Flow*

The equations that describe compressible flow admit a wide variety of symmetries, far more than we can discuss fully here. For a complete discussion the reader is referred to the seminal work of Ovsiannikov [12.1], the collected results in Ibragimov [12.2], and the recent text by Andreev et al. [12.3]. Rather than try to provide a comprehensive discussion of all the interesting symmetries connected with the equations of compressible flow, three problems will be described in detail that illustrate some of the features that are commonly encountered in the application of point groups. The first has to do with the specification of the relation between pressure and density. The number of group operators increases as the function connecting pressure and density is selected among less and less general forms. The second is the occasional ability of group analysis to generate nontrivial exact solutions without having to solve a differential equation. Finally we will examine a scaling example from aerodynamic theory.

For simplicity, we will focus our attention on the inviscid equations of motion. Applications of group theory to the full viscous, compressible equations are few and far between. Some of the reasons for this are discussed in Reference [12.4] where a weakly compressible viscous flow case is treated. The inviscid equations consist of a coupled system of relatively low order, and provide us with a good opportunity to pick apart the group methodology and to look in detail at the algorithm that leads to the invariance condition for a system with several equations.

## 12.1 Invariance Group of the Compressible Euler Equations

The equations governing inviscid, compressible flow in three dimensions are

$$\begin{aligned}
\Psi^1 &= u_t + uu_x + vu_y + wu_z + \frac{p_x}{\rho} = 0,\\
\Psi^2 &= v_t + uv_x + vv_y + wv_z + \frac{p_y}{\rho} = 0,\\
\Psi^3 &= w_t + uw_x + vw_y + ww_z + \frac{p_z}{\rho} = 0,\\
\Psi^4 &= \rho_t + u\rho_x + v\rho_y + w\rho_z + \rho(u_x + v_y + w_z) = 0,\\
\Psi^5 &= p_t + up_x + vp_y + wp_z + F[p, \rho](u_x + v_y + w_z) = 0.
\end{aligned} \tag{12.1}$$

The first three equations represent conservation of momentum in an inviscid medium, the fourth is the equation of continuity, and the fifth is one form of the energy equation (actually derived from the equation for conservation of entropy). The function $F$ is related to the entropy $S$ of the medium by

$$F[p, \rho] = -\rho \frac{\partial S/\partial \rho}{\partial S/\partial p}. \tag{12.2}$$

For the moment we take $F(p, \rho)$ to be an arbitrary function. Using the correspondence $(x, y, z, t) \to (x^1, x^2, x^3, t)$ and $(u, v, w) \to (u^1, u^2, u^3)$, one expands each equation in the differential system (12.1) in a Lie series in terms of the infinitesimal group

$$\begin{aligned}
\tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x}, t, \boldsymbol{u}, p, \rho],\\
\tilde{t} &= t + s\tau[\boldsymbol{x}, t, \boldsymbol{u}, p, \rho],\\
\tilde{u}^i &= u^i + s\eta^i[\boldsymbol{x}, t, \boldsymbol{u}, p, \rho],\\
\tilde{p} &= p + s\zeta[\boldsymbol{x}, t, \boldsymbol{u}, p, \rho],\\
\tilde{\rho} &= \rho + s\sigma[\boldsymbol{x}, t, \boldsymbol{u}, p, \rho]
\end{aligned} \tag{12.3}$$

and the once extended group operator

$$\begin{aligned}
X_{\{1\}} &= \xi^j \frac{\partial}{\partial x^j} + \tau \frac{\partial}{\partial t} + \eta^i \frac{\partial}{\partial u^i} + \zeta \frac{\partial}{\partial p} + \sigma \frac{\partial}{\partial \rho}\\
&\quad + \eta^i_{\{j\}} \frac{\partial}{\partial u^i_j} + \eta^i_{\{t\}} \frac{\partial}{\partial u^i_t} + \zeta_{\{j\}} \frac{\partial}{\partial p_j} + \zeta_{\{t\}} \frac{\partial}{\partial p_t} + \sigma_{\{j\}} \frac{\partial}{\partial \rho_j} + \sigma_{\{t\}} \frac{\partial}{\partial \rho_t}.
\end{aligned} \tag{12.4}$$

So we have a system with four independent variables, five dependent variables and nine unknown group infinitesimals.

There are five equations in the system, and so five invariance conditions are involved in solving for the groups of (12.1). The invariance conditions written out are listed below:

$$X_{\{1\}}\Psi^1 = X_{\{1\}}\left(u_t + uu_x + vu_y + wu_z + \frac{p_x}{\rho}\right)$$
$$= u_x\eta^1 + u_y\eta^2 + u_z\eta^3 + \left(-\frac{p_x}{\rho^2}\right)\sigma$$
$$+ u\eta^1_{\{1\}} + v\eta^1_{\{2\}} + w\eta^1_{\{3\}} + \eta^1_{\{t\}} + \frac{1}{\rho}\zeta_{\{1\}} = 0, \tag{12.5}$$

$$X_{\{1\}}\Psi^2 = X_{\{1\}}\left(v_t + uv_x + vv_y + wv_z + \frac{p_y}{\rho}\right)$$
$$= v_x\eta^1 + v_y\eta^2 + v_z\eta^3 + \left(-\frac{p_y}{\rho^2}\right)\sigma$$
$$+ u\eta^2_{\{1\}} + v\eta^2_{\{2\}} + w\eta^2_{\{3\}} + \eta^2_{\{t\}} + \frac{1}{\rho}\zeta_{\{2\}} = 0, \tag{12.6}$$

$$X_{\{1\}}\Psi^3 = X_{\{1\}}\left(w_t + uw_x + vw_y + ww_z + \frac{p_z}{\rho}\right)$$
$$= w_x\eta^1 + w_y\eta^2 + w_z\eta^3 + \left(-\frac{p_z}{\rho^2}\right)\sigma$$
$$+ u\eta^3_{\{1\}} + v\eta^3_{\{2\}} + w\eta^3_{\{3\}} + \eta^3_{\{t\}} + \frac{1}{\rho}\zeta_{\{3\}} = 0, \tag{12.7}$$

$$X_{\{1\}}\Psi^4 = X_{\{1\}}(\rho_t + u\rho_x + v\rho_y + w\rho_z + \rho(u_x + v_y + w_z))$$
$$= \eta^1\rho_x + \eta^2\rho_y + \eta^3\rho_z + \sigma(u_x + u_y + u_z) + \rho\left(\eta^1_{\{1\}} + \eta^2_{\{2\}} + \eta^3_{\{3\}}\right)$$
$$+ u\sigma_{\{1\}} + v\sigma_{\{2\}} + w\sigma_{\{3\}} + \sigma_{\{t\}} = 0, \tag{12.8}$$

$$X_{\{1\}}\Psi^5 = X_{\{1\}}(p_t + up_x + vp_y + wp_z + F(p,\rho)(u_x + v_y + w_z))$$
$$= \eta^1 p_x + \eta^2 p_y + \eta^3 p_z$$
$$+ \zeta F_p(u_x + u_y + u_z) + \sigma F_\rho(u_x + u_y + u_z) + F\left(\eta^1_{\{1\}} + \eta^2_{\{2\}} + \eta^3_{\{3\}}\right)$$
$$+ u\zeta_{\{1\}} + v\zeta_{\{2\}} + w\zeta_{\{3\}} + \zeta_{\{t\}} = 0. \tag{12.9}$$

The variables $(u, v, w, p, \rho)$ satisfy the system (12.1), and this condition has to be imposed on (12.5) to (12.9). To accomplish this we use (12.1) to define a set of replacement rules to be inserted in each of the five invariance conditions

(12.5) to (12.9). A reasonable choice would be

$$\begin{aligned}
u_t &\to -\left(uu_x + vu_y + wu_z + \frac{p_x}{\rho}\right),\\
v_t &\to -\left(uv_x + vv_y + wv_z + \frac{p_y}{\rho}\right),\\
w_t &\to -\left(uw_x + vw_y + ww_z + \frac{p_z}{\rho}\right),\\
\rho_t &\to -(u\rho_x + v\rho_y + w\rho_z + \rho(u_x + v_y + w_z)),\\
p_t &\to -(up_x + vp_y + wp_z + F[p, \rho](u_x + v_y + w_z)).
\end{aligned} \tag{12.10}$$

It is important to keep in mind two points when making the replacements:

- All five replacements in (12.10) must be made in *each* of (12.5) to (12.9).
- It is essential to isolate a single term in each of the governing equations, such as the time derivatives in (12.10), in order to make the replacement. Replacing a product such as, say, $uw_x$ is incorrect, because $u$ and $w_x$ do not only appear as that particular product in (12.5) to (12.9). Moreover, $u$ is one of the independent variables of the infinitesimals. To remove it where it might appear explicitly in the invariance conditions would produce an overly restricted system of determining equations. For the same reason, it would not be appropriate to solve for $\rho$ and try to remove it from the invariance conditions. For some complicated nonlinear equations, isolating a single term may be extremely difficult, but such cases are relatively rare.

Running the package **`IntroToSymmetry.m`** reveals that (12.1) is invariant under an 11-parameter group with the following operators:

(1) Invariance under translation in time:

$$X^1 = \frac{\partial}{\partial t}. \tag{12.11}$$

(2) Invariance under translation in $x$:

$$X^2 = \frac{\partial}{\partial x}. \tag{12.12}$$

(3) Invariance under translation in $y$:

$$X^3 = \frac{\partial}{\partial y}. \tag{12.13}$$

(4) Invariance under translation in $z$:

$$X^4 = \frac{\partial}{\partial z}. \tag{12.14}$$

(5) Rotation about the $z$-axis:

$$X^5 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}. \tag{12.15}$$

(6) Rotation about the $x$-axis:

$$X^6 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + w\frac{\partial}{\partial v} - v\frac{\partial}{\partial w}. \tag{12.16}$$

(7) Rotation about the $y$-axis:

$$X^7 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + w\frac{\partial}{\partial u} - u\frac{\partial}{\partial w}. \tag{12.17}$$

(8) Constant-speed translation in the $x$-direction:

$$X^8 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}. \tag{12.18}$$

(9) Constant-speed translation in the $y$-direction:

$$X^9 = t\frac{\partial}{\partial y} + \frac{\partial}{\partial u}. \tag{12.19}$$

(10) Constant-speed translation in the $z$-direction:

$$X^{10} = t\frac{\partial}{\partial z} + \frac{\partial}{\partial u}. \tag{12.20}$$

(11) The fundamental dilation group of the equation:

$$X^{11} = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}. \tag{12.21}$$

Additional groups arise when the function $F[p, \rho]$ is restricted in some way. A few examples are given below.

*Case 1: $F = f[\rho]$.* For this case there is one additional operator:

$$X^{12} = \frac{\partial}{\partial p}. \tag{12.22}$$

*Case 2: $F = f[p]$.* In this case the new symmetry is

$$X^{12} = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w} + 2\rho\frac{\partial}{\partial \rho}. \tag{12.23}$$

*Case 3: $F = A\rho^{\sigma}$, $\sigma \neq 0$.* Two additional symmetries arise:

$$\begin{aligned} X^{12} &= (\sigma-1)t\frac{\partial}{\partial t} - (\sigma-1)u\frac{\partial}{\partial u} - (\sigma-1)v\frac{\partial}{\partial v} - (\sigma-1)w\frac{\partial}{\partial w} \\ &\quad - 2\rho\frac{\partial}{\partial \rho} - 2\sigma p\frac{\partial}{\partial p}, \qquad (12.24) \\ X^{13} &= \frac{\partial}{\partial p}. \end{aligned}$$

*Case 4: $F = Ap$.* This form of $F$ also brings in two additional symmetries:

$$\begin{aligned} X^{12} &= t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w} + 2\rho\frac{\partial}{\partial \rho}, \\ X^{13} &= p\frac{\partial}{\partial p} + \rho\frac{\partial}{\partial \rho}. \end{aligned} \tag{12.25}$$

*Case 5: $F = \frac{5}{3}p$.* This choice of $F$ corresponds to the isentropic flow of a monatomic gas with ratio of specific heats $\gamma = \frac{5}{3}$. In this case three additional group operators arise:

$$\begin{aligned} X^{12} &= t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w} + 2\rho\frac{\partial}{\partial \rho}, \\ X^{13} &= p\frac{\partial}{\partial p} + \rho\frac{\partial}{\partial \rho}, \\ X^{14} &= t^2\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x} + yt\frac{\partial}{\partial y} + zt\frac{\partial}{\partial z} \qquad (12.26) \\ &\quad + (x-ut)\frac{\partial}{\partial u} + (y-vt)\frac{\partial}{\partial v} + (z-wt)\frac{\partial}{\partial w} \\ &\quad - 5pt\frac{\partial}{\partial p} - 3\rho t\frac{\partial}{\partial \rho}. \end{aligned}$$

For more on the compressible flow equations see Ovsiannikov [12.2] as well as the CRC series edited by Ibragimov [12.3].

## 12.2 Isentropic Flow

The Gibbs equation discussed in Chapter 3, Section 3.4 is

$$T\,dS = de + p\,dv. \tag{12.27}$$

For an ideal gas with equation of state $p = \rho R T$, the internal energy and enthalpy depend only on temperature. The heat capacities at constant volume and constant pressure are defined by

$$de = C_v[T]\,dT, \qquad dh = C_p[T]\,dT, \tag{12.28}$$

where $C_v[T]$ and $C_p[T]$ are weakly increasing functions of tempetature. A common approximation is to let $C_v$ and $C_p$ be constant. This permits the Gibbs equation to be integrated explicitly from a reference state to produce

$$\frac{p}{p_r} = e^{(S-S_r)/C_v}\left(\frac{\rho}{\rho_r}\right)^{\gamma}, \tag{12.29}$$

where $\gamma$ is the ratio of specific heats, $\gamma = C_p/C_v$. In the case of homentropic (homogeneously isentropic, $\nabla S = 0$) flow, the inviscid, compressible flow equations take the form

$$\begin{aligned} \frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} + \rho^{\gamma-2}\frac{\partial \rho}{\partial x^i} &= 0, \\ \frac{\partial \rho}{\partial t} + u^j \frac{\partial \rho}{\partial x^j} + \rho \frac{\partial u^j}{\partial x^j} &= 0, \\ i = 1, \ldots, n, \qquad \text{sum over} \quad j &= 1, \ldots, n, \end{aligned} \tag{12.30}$$

where $n$ is the number of space dimensions. The group operator $X^{14}$ in case 5 [Equation (12.26)] appears in a physically subtle and interesting form when the equations (12.30) are examined in one, two, and three dimensions. This is illustrated in the next section.

## 12.3 Sudden Expansion of a Gas Cloud into a Vacuum

Instead of beginning with the problem statement and then searching for an invariant group, let's back into this problem by beginning with the group $X^{14}$ appended to the time translation group $X^1$, and then see what physical problem fits naturally into the chosen symmetries. For a perfect gas in $n$ space dimensions, the group $t_0^2 X^1 + X^{14}$ takes the form

$$\begin{aligned} t_0^2 X^1 + X^{14} = {} & \left(t_0^2 + t^2\right)\frac{\partial}{\partial t} + x^j t \frac{\partial}{\partial x^j} \\ & + (x^j - u^j t)\frac{\partial}{\partial u^j} - (n+2) p t \frac{\partial}{\partial p} - n \rho t \frac{\partial}{\partial \rho}, \end{aligned} \tag{12.31}$$

where $t_0^2$ is an arbitrary constant that will eventually play the role of an effective origin in time. The characteristic equations of (12.31) are

$$\frac{dt}{t_0^2 + t^2} = \frac{dx^j}{x^j t} = \frac{du^j}{x^j - u^j t} = \frac{dp}{(n+2)pt} = \frac{d\rho}{n\rho t}. \tag{12.32}$$

The last two terms generate the invariant

$$\psi = p\rho^{-(n+2)/n}. \tag{12.33}$$

The system (12.30) is invariant under the group (12.31) (more particularly the group $X^{14}$) if and only if

$$\gamma = \frac{n+2}{n}. \tag{12.34}$$

We therefore expect similarity solutions which are invariant under this group for one-, two-, and three-dimensional flow only for $\gamma = 3, 2$, and $\frac{5}{3}$ respectively.

In a way (12.34) is a remarkable result. It is the same one that comes from the kinetic theory of gases, where $n$ is the number of degrees of freedom of an individual gas molecule, yet there is nothing in (12.30) to suggest the corpuscular nature of the medium governed by (12.30). It almost seems that the equations anticipate the existence of monatomic gases with $n = 3$. Essentially, (12.34) expresses the dilation symmetry in the pressure and density common to both theories.

### *12.3.1 The Gasdynamic–Shallow-Water Analogy*

The two-dimensional case with $\gamma = 2$ corresponds to a gas where the molecules are constrained to move in a plane. This is not a physically realizable situation, but it happens that for this value of $\gamma$ the governing equations (12.30) correspond exactly to the equations for the flow of shallow water over a solid wall in the $(x, y)$ plane, where $\rho$ is interpreted as the height of the liquid surface above the wall, as sketched in Figure 12.1. In the shallow-water approximation the flow velocity is assumed constant over the height of the layer.

The one-dimensional case with $\gamma = 3$ corresponds to a gas where the molecules are constrained to move on a line. Neither the one or two dimensional cases are physically realizable however both have been the subject of interesting discrete-particle simulations.

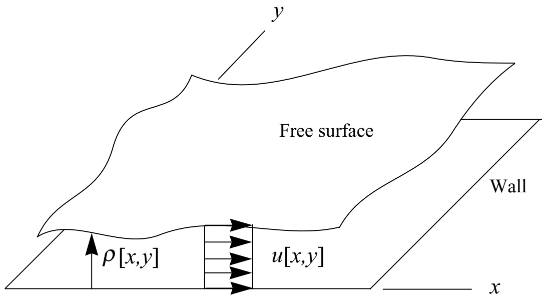


Fig. 12.1. Flow sketch for shallow-water analogy.

### *12.3.2 Solutions*

What kind of solutions come out of this group? Solving the characteristic equations (12.32) leads to the following invariants:

$$\begin{aligned} \alpha^i &= \frac{x^i}{\left(t_0^2 + t^2\right)^{1/2}}, \\ U^i[\alpha] &= u^i\left(t_0^2 + t^2\right)^{1/2} - \frac{x_t^i}{\left(t_0^2 + t^2\right)^{1/2}}, \\ P[\alpha] &= p\left(t_0^2 + t^2\right)^{(n+2)/2}, \\ R[\alpha] &= \rho\left(t_0^2 + t^2\right)^{n/2}. \end{aligned} \tag{12.35}$$

Now substitute (12.35) into (12.30) with $\gamma = (n+2)/n$. The result is

$$\begin{aligned} \alpha^i + U^j \frac{\partial U^i}{\partial \alpha^j} + R^{(2/n)-1} \frac{\partial R}{\partial \alpha^i} &= 0, \\ \frac{\partial (R U^j)}{\partial \alpha^j} &= 0, \end{aligned} \tag{12.36}$$

$$i = 1, \ldots, n, \qquad \text{sum over} \quad j = 1, \ldots, n.$$

The similarity variables (12.35) lead to the expected reduction in the number of independent variables; time is eliminated from the problem.

As was pointed out above, every once in a while group analysis can lead directly to interesting nontrivial solutions of the equations of motion. Let's consider the simplest possible case with $U^i[\alpha] = 0$. The self-similar continuity equation is satisfied and so (12.36) reduces to

$$\alpha^i + R^{(2/n)-1} \frac{\partial R}{\partial \alpha^i} = 0, \qquad i = 1, \ldots, n, \quad \text{sum over} \ \ j = 1, \ldots, n. \tag{12.37}$$

Hence we need only solve for the density function $R[\alpha]$. Let's look at the three relevant cases.

#### 12.3.2.1 Case 1: $n = 1,\ \gamma = 3$

The solution is

$$R = (C^2 - \alpha^2)^{1/2}, \tag{12.38}$$

where $C$ is a constant of integration. The density and velocity profiles in physical coordinates are

$$\begin{aligned} \rho &= \frac{1}{\left(t_0^2 + t^2\right)^{1/2}} \left(C^2 - \frac{x^2}{t_0^2 + t^2}\right)^{1/2}, \\ u &= \frac{xt}{t_0^2 + t^2}. \end{aligned} \tag{12.39}$$

as shown in Figure 12.2.

#### 12.3.2.2 Case 2: $n = 2, \gamma = 2$

The self-similar density is

$$R = \frac{C^2}{2} - \frac{\alpha_1^2 + \alpha_2^2}{2}, \tag{12.40}$$

where $C$ is again a constant of integration. The density and velocity profiles in physical coordinates are

$$\begin{aligned} \rho &= \frac{1}{2\left(t_0^2 + t^2\right)} \left(C^2 - \frac{x^2 + y^2}{t_0^2 + t^2}\right), \\ u &= \frac{xt}{t_0^2 + t^2}, \qquad v = \frac{yt}{t_0^2 + t^2}. \end{aligned} \tag{12.41}$$

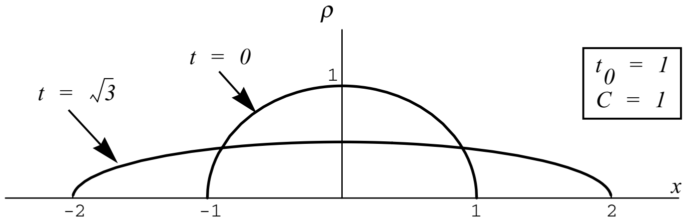


Fig. 12.2. Density profiles for a 1-D gas expanding into vacuum.

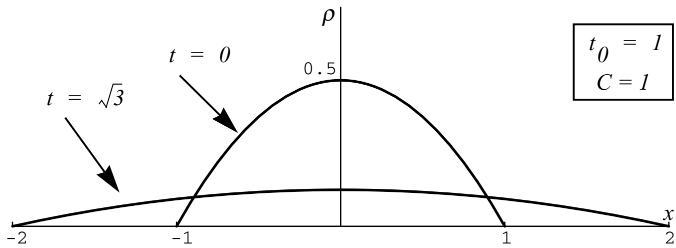


Fig. 12.3. Height profiles of a collapsing parabolic pile of water.

Following Reference [12.5], we can interpret the solution as a parabolic pile of water collapsing under its own weight. The solution is plotted in Figure 12.3.

### *12.3.2.3 Case 3:* $n = 3, \gamma = \frac{5}{3}$

The density solution is

$$R = \left(\frac{C^2}{3} - \frac{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}{3}\right)^{3/2}, \tag{12.42}$$

where $C$ is a constant of integration. The density and velocity profiles for this case are

$$\begin{aligned} \rho &= \frac{1}{3^{3/2}\left(t_0^2 + t^2\right)^{3/2}}\left(C^2 - \frac{x^2 + y^2 + z^2}{t_0^2 + t^2}\right)^{3/2}, \\ u &= \frac{xt}{t_0^2 + t^2}, \qquad v = \frac{yt}{t_0^2 + t^2}, \qquad w = \frac{zt}{t_0^2 + t^2}, \end{aligned} \tag{12.43}$$

as shown in Figure 12.4.

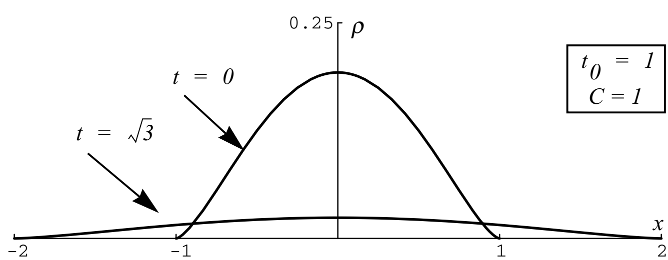


Fig. 12.4. Density profiles for a monatomic gas expanding into vacuum.

This solution represents the expansion into vacuum of a monatomic gas such as helium with an initial density profile given by (12.43). The pressure distribution is generated from $p = \rho^{5/3}$, and the speed of sound comes from $a^2 = \frac{5}{3}(p/\rho)$.

We have been a little cavelier in the use of dimensionless variables. Clearly there are parameters of the initial conditions that can be used to nondimensionalize variables such as the initial radius $r_0$ of the density distribution, and the initial pressure $p_0$ and density $\rho_0$ at the center of the distribution. See Exercise 12.1. The total energy contained in the gas cloud is

$$E = \int_0^{R_s} \left( \rho C_v T + \frac{1}{2}\rho u^2 \right) 4\pi r^2 \, dr = \int_0^{R_s} \left( \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2 \right) 4\pi r^2 \, dr. \tag{12.44}$$

Further discussion of this problem can be found in the classic text by Zel'dovich and Raizer [12.6]. Several of the references cited in [12.6] extend the results above to more complex cases, including the free expansion of an initially ellipsoidal gas cloud. Further applications of group methods to gasdynamics as well as kinetic theory are presented by Meleshko [12.7], [12.8]. This problem serves as a prototype for models of the expanding gas nebula from a supernova.

## 12.4 Propagation of a Strong Spherical Blast Wave

Although the equations of motion described in Section 12.1 do not allow for either viscosity or heat conduction, they can support discontinuous solutions where the effects of diffusion are concentrated in a thin region called a *shock wave*. An outstanding example is the famous blast-wave problem, solved in 1941 by Taylor [12.9], [12.10] and von Neumann [12.11], and also in 1946 by Sedov [12.12], [12.13]. This problem describes the flow behind a spherically expanding shock wave caused by a very strong point explosion in a homogeneous atmosphere. Taylor used the results of this problem, together with a set of photos released by the Army showing the growth of the blast, to estimate the energy released by the first nuclear bomb test at Alamogordo in 1945. His publication sent more than a few shock waves of embarassment through U.S. military circles at the time, since the blast energy was supposed to be a closely guarded secret. The flow situation is depicted in Figure 12.5.

In the blast a huge amount of energy, $E$, is released in a small region, essentially a point, in an ideal gas. The extreme overpressure produced by the blast, which can be thousands of atmospheres, causes a very strong spherical shock wave to propagate away from the source. As the shock wave moves outward,

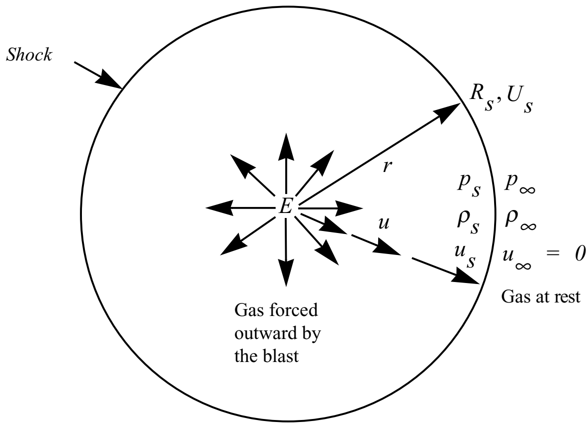


Fig. 12.5. Spherical point explosion.

it weakens, and the pressure just behind the wave decreases. However, if the energy is large enough, the wave will move a considerable distance before the pressure behind the wave begins to approach the ambient pressure $p_\infty$. As a result there is a limited period of time, corresponding to a characteristic radius of propagation $R_0$, over which the shock wave is unaffected by the magnitude of the ambient pressure. One can estimate an upper limit of this radius by calculating the size of a sphere that would encompass a volume of gas at ambient conditions that would contain a total amount of thermal energy equal to the energy released in the explosion. That is,

$$E = \tfrac{4}{3}\pi R_0^3 \rho_\infty C_v T_\infty. \tag{12.45}$$

The characteristic radius is

$$R_0 = \left(\frac{3}{4}\frac{\gamma - 1}{\pi}\frac{E}{p_\infty}\right)^{1/3}, \tag{12.46}$$

where $\gamma$ is the ratio of specific heats, $\gamma = C_p/C_v$.

As long as the radius of propagation of the shock satisfies $R_s \ll R_0$, the following assumptions hold:

(i) The thermal energy per unit volume of the ambient gas can be neglected compared to the energy per unit volume of the gas within the wave.
(ii) The pressure ratio across the shock is large: $p_S/p_\infty \gg 1$.

As a consequence of these two assumptions one can assume that the strong-shock limit can be used to characterize the jump in gas properties across the blast wave. Namely,

$$\begin{aligned}\rho_s &= \frac{\gamma+1}{\gamma-1}\rho_\infty,\\ p_s &= \frac{2}{\gamma+1}\rho_\infty U_s^2,\\ u_s &= \frac{2}{\gamma+1}U_s,\end{aligned} \tag{12.47}$$

where $U_s = dR_s/dt$ is the shock speed and $u_s$ is the gas speed behind the shock.

Notice that the limiting density ratio across the shock is finite. Therefore, in contrast to the ambient pressure, the effect of the ambient density on the shock speed cannot be ignored. Thus the flow pattern generated by the blast wave is completely determined by only two physical parameters, the energy $E$ and the ambient density $\rho_\infty$. Dimensional analysis applied to these parameters, including the time and the shock radius, with dimensions

$$\hat{E} = ML^2/T^2, \qquad \hat{\rho} = M/L^3, \qquad \hat{R}_s = L, \qquad \hat{t} = T, \tag{12.48}$$

leads to

$$\left(\frac{\rho_\infty}{E}\right)^{1/5}\frac{R_s}{t^{2/5}} = \text{constant} = \alpha_s. \tag{12.49}$$

The shock speed is

$$U_s = \frac{dR_s}{dt} = \alpha_s\frac{2}{5}\left(\frac{E}{\rho_\infty}\right)^{1/5}t^{-3/5}. \tag{12.50}$$

The constant $\alpha_s$ remains to be determined. The flow conditions just behind the shock are

$$\begin{aligned}\rho_s &= \frac{\gamma+1}{\gamma-1}\rho_\infty,\\ p_s &= \frac{2}{\gamma+1}\rho_\infty U_s^2 = \alpha_s^2\left(\frac{4}{25}\right)\left(\frac{2}{\gamma+1}\right)(\rho_\infty^{3/5}E^{2/5})t^{-6/5},\\ u_s &= \frac{2}{\gamma+1}U_s = \alpha_s\left(\frac{2}{5}\right)\left(\frac{2}{\gamma+1}\right)\left(\frac{E}{\rho_\infty}\right)^{1/5}t^{-3/5}.\end{aligned} \tag{12.51}$$

In order to determine $\alpha_s$ we turn to the equations of motion and solve for the flow between the shock front and the origin of the explosion, $0 < r < R_s$. The properties of the flow just behind the shock front, (12.47), serve as boundary conditions for the solution. Since there are no shocks in this region, the flow can be assumed to be entropy-conserving. The conservation equations of mass, momentum, and energy for this spherically symmetric flow are

$$\begin{aligned} \rho_t + u\rho_r + \rho\left(u_r + \frac{2u}{r}\right) &= 0, \\ u_t + uu_r + \frac{p_r}{\rho} &= 0, \\ p_t + up_r + \gamma p\left(u_r + \frac{2u}{r}\right) &= 0. \end{aligned} \tag{12.52}$$

The last equation in (12.52) is derived from the equation for conservation of entropy. When combined with the continuity equation it can be written as

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)\ln[p/\rho^\gamma] = 0. \tag{12.53}$$

Noting the formula (12.29) for the entropy of an ideal gas, we can write (12.53) as

$$\frac{DS}{Dt} = 0, \tag{12.54}$$

which states that the entropy $S = \ln[p/\rho^\gamma]$ following a fluid particle inside the blast zone is conserved. There is of course an enormous increase in entropy across the blast wave. This is obviated by the fact that, while the density increase is relatively modest, the pressure and temperature increases across the wave are huge.

If we neglect the internal energy of the ambient fluid being continuously enclosed by the outwardly moving shock, then the total energy of the gas between the origin and the shock front is approximately constant,

$$E = \int_0^{R_s}\left(\rho C_v T + \frac{1}{2}\rho u^2\right)4\pi r^2\,dr = \int^{R_s}\left(\frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2\right)4\pi r^2\,dr. \tag{12.55}$$

See equation (12.44). The spherical symmetry of the problem, together with the result (12.49) from dimensional analysis and the conserved integral (12.55), suggests that the problem may be invariant under a dilation group. Let's try

$$\tilde{r} = e^a r, \qquad \tilde{t} = e^b t, \qquad \tilde{u} = e^c u, \qquad \tilde{p} = e^d p, \qquad \tilde{\rho} = e^f \rho. \tag{12.56}$$

Transforming (12.52) using (12.56) and requiring invariance reduces the group to the following:

$$\tilde{r} = e^{a} r, \qquad \tilde{t} = e^{b} t, \qquad \tilde{u} = e^{a-b} u,$$
$$\tilde{p} = e^{d} p, \qquad \tilde{\rho} = e^{d-2a+2b} \rho. \tag{12.57}$$

The infinitesimal three-parameter group operator corresponding to (12.57) is

$$X = ar\frac{\partial}{\partial r} + bt\frac{\partial}{\partial t} + (a-b)u\frac{\partial}{\partial u} + (d)p\frac{\partial}{\partial p} + (d-2a+2b)\rho\frac{\partial}{\partial \rho}. \tag{12.58}$$

This is a sum of group operators for the basic compressible flow equations, (12.1). In particular, we add the dilation operator $X^{11}$ in (12.21) to the operators $X^{12}$ and $X^{13}$ in (12.25) corresponding to case 4 ($F = A\rho$). That is,

$$X = aX^{11} - (a-b)X^{12} + dX^{13}. \tag{12.59}$$

At this point we have a three-parameter group. However, the conditions of the problem eventually reduce this to a one-parameter group as follows. The position of the shock transforms as

$$\frac{\tilde{R}_s \rho_\infty^{1/5}}{E^{1/5}\tilde{t}^{2/5}} = e^{a-(2/5)b} \frac{R_s \rho_\infty^{1/5}}{E^{1/5} t^{2/5}}, \tag{12.60}$$

or

$$\tilde{\alpha}_s = e^{a-(2/5)b} \alpha_s. \tag{12.61}$$

The constancy of $\alpha_s$ suggested by the results of dimensional analysis (12.49) implies that $b = \frac{5}{2}a$. The group is now simplified to

$$\tilde{r} = e^{a} r, \qquad \tilde{t} = e^{(5/2)a} t, \qquad \tilde{u} = e^{-(3/2)a} u, \qquad \tilde{p} = e^{d} p, \qquad \tilde{\rho} = e^{d+3a} \rho. \tag{12.62}$$

Using (12.62) to transform the energy integral leads to

$$E = \int_0^{R_s} \left( \frac{\tilde{p}}{\gamma - 1} + \frac{1}{2}\tilde{\rho}\tilde{u}^2 \right) 4\pi \tilde{r}^2 \, d\tilde{r} = e^{d+3a} \int_0^{R_s} \left( \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2 \right) 4\pi r^2 \, dr, \tag{12.63}$$

and for invariance we require $d = -3a$. Finally, the one-parameter group that leaves the entire problem invariant is

$$\tilde{r} = e^{a} r, \qquad \tilde{t} = e^{(5/2)a} t, \qquad \tilde{u} = e^{-(3/2)a} u, \qquad \tilde{p} = e^{-3a} p, \qquad \tilde{\rho} = \rho. \tag{12.64}$$

The characteristic equations of (12.64) are

$$\frac{dr}{r} = \frac{2\,dt}{5t} = \frac{2\,du}{-3u} = \frac{dp}{-3p} = \frac{d\rho}{0}. \tag{12.65}$$

Dimensionless similarity variables constructed from the invariants of (12.65) and nondimensionalized using $E$ and $\rho_\infty$ are

$$\alpha = \left(\frac{\rho_\infty}{E}\right)^{1/5} \frac{r}{t^{2/5}}, \quad \alpha U[\alpha] = \frac{5}{2}\left(\frac{\gamma+1}{2}\right)\left(\frac{\rho_\infty}{E}\right)^{1/5} u t^{3/5},$$
$$\alpha^2 P[\alpha] = \frac{25}{4}\left(\frac{\gamma+1}{2}\right)\left(\frac{1}{\rho_\infty^{3/5} E^{2/5}}\right) p t^{6/5}, \quad G[\alpha] = \left(\frac{\gamma-1}{\gamma+1}\right)\frac{\rho}{\rho_\infty}. \tag{12.66}$$

The lack of scaling on the density in (12.64) is a reflection of the fact that, over the time of validity of the solution, the density behind the shock and the density variation to the origin is a time-independent function of $r/R_s$ – the region of density variation simply grows as the shock propagates outward. In contrast, the gas pressure and radial velocity decay fairly rapidly against the backdrop of a frozen density distribution. Only when $R_s$ approaches $R_0$ does the density inside the blast wave begin to fade. The factors involving $\gamma$ are introduced to provide a convenient normalization of the solution.

Now substitute (12.66) into (12.52) and rearrange to put the equations of motion into self-similar form:

$$\left(\left(\frac{2}{\gamma+1}\right)U - 1\right)\left(\frac{2}{\gamma-1}\right)G\frac{dU}{d\ln\alpha} + \left(\frac{2}{\gamma+1}\right)\frac{dP}{d\ln\alpha} - \frac{5}{2}\left(\frac{2}{\gamma-1}\right)GU$$
$$+\left(\frac{4}{\gamma^2-1}\right)GU^2 + \left(\frac{2}{\gamma+1}\right)P = 0,$$
$$\frac{dU}{d\ln\alpha} + \left(V - \frac{\gamma+1}{2}\right)\frac{d\ln G}{d\ln\alpha} + 3U = 0, \tag{12.67}$$
$$\frac{d}{d\ln\alpha}\ln[P/G^\gamma] + \frac{4U - 5(\gamma+1)}{2U-(\gamma+1)} = 0.$$

The governing equations (12.67) were solved exactly by von Neumann [12.11]. The velocity is related to the radial coordinate by

$$\frac{\alpha}{\alpha_s} = U^{-2/5}\left(\frac{\gamma U - \frac{\gamma+1}{2}}{\frac{\gamma-1}{2}}\right)^{\mu_1}\left(\frac{\frac{7-\gamma}{2}}{\frac{5}{2}(\gamma+1)-(3\gamma-1)U}\right)^{\mu_2}, \tag{12.68}$$

where

$$\mu_1 = \frac{\gamma - 1}{2\gamma + 1}, \qquad \mu_2 = \frac{13\gamma^2 - 7\gamma + 12}{5(3\gamma - 1)(2\gamma + 1)}. \tag{12.69}$$

The density is determined as a function of velocity:

$$G = \left( \frac{\gamma U - \frac{\gamma+1}{2}}{\frac{\gamma-1}{2}} \right)^{\mu_3} \left( \frac{\frac{7-\gamma}{2}}{\frac{5}{2}(\gamma + 1) - (3\gamma - 1)U} \right)^{\mu_4} \left( \frac{\frac{\gamma-1}{2}}{\frac{\gamma+1}{2} - U} \right)^{\mu_5}, \tag{12.70}$$

where

$$\mu_3 = \frac{3}{2\gamma + 1}, \qquad \mu_4 = \frac{13\gamma^2 - 7\gamma + 12}{(2 - \gamma)(3\gamma - 1)(2\gamma + 1)}, \qquad \mu_5 = \frac{1}{2 - \gamma}. \tag{12.71}$$

Finally, the pressure is expressed in terms of the density and velocity functions:

$$\frac{P}{G} = \left( \frac{\frac{\gamma+1}{2} - U}{\gamma U - \frac{\gamma+1}{2}} \right) U^2. \tag{12.72}$$

Typical results are shown Figure 12.6.

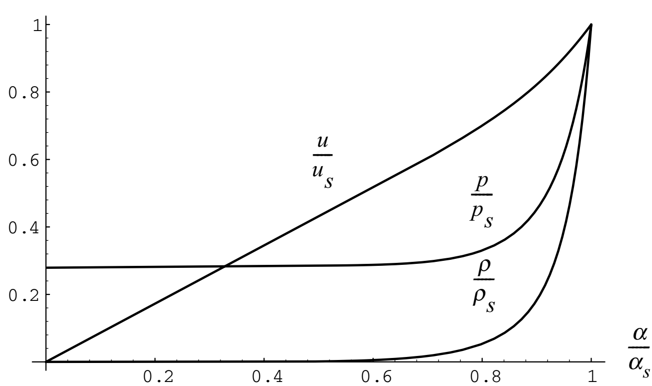


Fig. 12.6. Velocity, density, and pressure for $\gamma = 1.4$.

The ranges of these functions are

$$\begin{aligned} 0 &\le \alpha \le \alpha_s, \\ \frac{\gamma+1}{2\gamma} &\le U[\alpha] \le 1, \\ 0 &\le G[\alpha] \le 1, \\ \infty &\ge P[\alpha] \ge 1. \end{aligned} \tag{12.73}$$

Note that the physical velocity is zero at the origin ($\alpha = 0$) and the physical pressure has a finite limit. This can be seen from the limits,

$$\lim_{\alpha\to 0}\left(\frac{\alpha}{\alpha_s}\right)U = 0,$$

$$\lim_{\alpha\to 0}\left(\frac{\alpha}{\alpha_s}\right)^2 P = \left(\frac{\gamma+1}{2\gamma}\right)^{\frac{6}{5}}\left(\frac{\gamma}{\gamma+1}\right)^{\frac{\gamma-1}{2-\gamma}}\left(\frac{\gamma(7-\gamma)}{(\gamma+1)(2\gamma+1)}\right)^{\frac{(9-2\gamma)(13\gamma^2-7\gamma+12)}{5(2-\gamma)(3\gamma-1)(2\gamma+1)}} \tag{12.74}$$

### *12.4.1 Effect of the Ratio of Specific Heats*

According to kinetic theory, the ratio of specific heats for an ideal gas is

$$\gamma = \frac{n+2}{n}, \tag{12.75}$$

where $n$ is the number of degrees of freedom of the gas molecule (for example, three translational degrees for a monatomic gas such as helium, and additionally two rotational and two vibrational degrees for diatomic molecules plus further degrees arising from chemical reactions and possible dissociation and ionization of the gas as temperatures increase). The gas inside the blast wave is heated to extreme temperatures, and there is a high degree of uncertainty in the appropriate value of $\gamma$. The number of excited degrees of freedom can be quite large, causing $\gamma$ to tend toward one. It is therefore useful to look at the effect of $\gamma$ on the properties of the flow.

First we plot the pressure at the origin as a function of $\gamma$, as shown in Figure 12.7. Note that for values of $\gamma$ close to one, the radial pressure variation becomes very small. The constant $\alpha_s$ is determined from the energy integral,

$$E = \int_0^{R_s}\left(\frac{p}{\gamma-1} + \frac{1}{2}\rho u^2\right) 4\pi r^2\, dr, \tag{12.76}$$

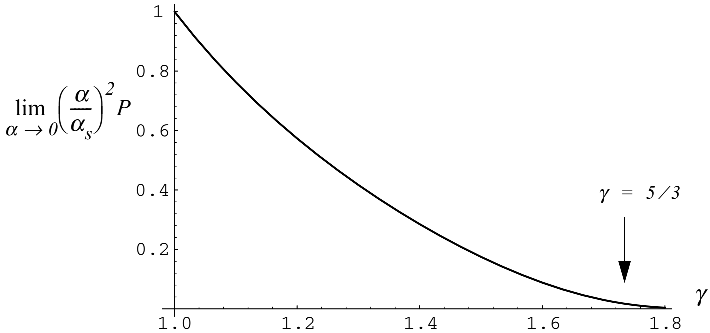


Fig. 12.7. Dependence of the pressure at the origin of the blast on $\gamma$.

which, in nondimensional terms, becomes

$$\frac{32\pi}{25(\gamma^2 - 1)} \int_0^{\alpha_s} (P + GU^2)\alpha^4 \, d\alpha = 1. \tag{12.77}$$

It is actually simpler to carry out the integration in (12.77) by integrating with respect to $U$ (which is monotonic in $\alpha$) and making use of (12.68) to replace $\alpha$. Let $\alpha/\alpha_s = F[U]$. Now

$$\frac{32\pi\alpha_s^5}{25(\gamma^2 - 1)} \int_{(\gamma+1)/2\gamma}^{1} (P + RU^2)(F^4 F_U) \, dU = 1, \tag{12.78}$$

which, using (12.72), becomes

$$\frac{32\pi\alpha_s^5}{25(\gamma^2 - 1)} \int_{(\gamma+1)/2\gamma}^{1} \left(\frac{\gamma - 1}{\gamma U - \frac{\gamma+1}{2}}\right) GU^3(F^4 F_U) \, dU = 1. \tag{12.79}$$

The relation (12.79) is integrated, allowing $\alpha_s$ to be evaluated. This process is carried out for various $\gamma$, and the result is as shown in Figure 12.8. Note that

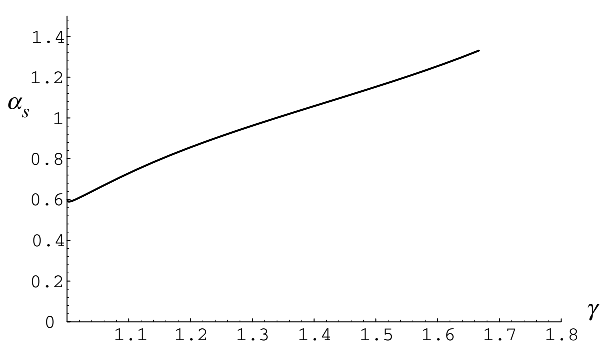


Fig. 12.8. The shock speed parameter as a function of $\gamma$.

$\alpha_s$ varies relatively little for a range of $\gamma$ between 1.1 and 1.4. Since there is considerable uncertainty in the actual value of $\gamma$ inside the blast zone, this is the key feature of the problem that enabled Taylor to use the theory to estimate the energy of the first atomic bomb blast with some reasonable hope of accuracy.

We recall that

$$\left(\frac{\rho_\infty}{E}\right)^{1/5}\frac{R_s}{t^{2/5}} = \text{constant} = \alpha_s, \tag{12.80}$$

or

$$R_s = \alpha_s\left(\frac{E}{\rho_\infty}\right)^{1/5} t^{2/5}, \tag{12.81}$$

or

$$\ln R_s = \ln\left[\alpha_s\left(\frac{E}{\rho_\infty}\right)^{1/5}\right] + \frac{2}{5}\ln t. \tag{12.82}$$

When $\ln R_s$ is plotted versus $\ln t$ with $\alpha_s$ estimated from Figure 12.8 the result is a value for $E$.

## 12.5 Compressible Flow Past a Thin Airfoil

Following World War II there was a greatly increased interest in high speed flight. Lacking analytical tools for handling this complex problem, aerodynamic designers turned to similarity methods to extrapolate low speed wind tunnel data to high Mach numbers. Here we shall examine several of these similarity rules from the point of view of group theory. Figure 12.9 shows the flow past a thin symmetric airfoil at zero angle of attack. The fluid is assumed to be inviscid,

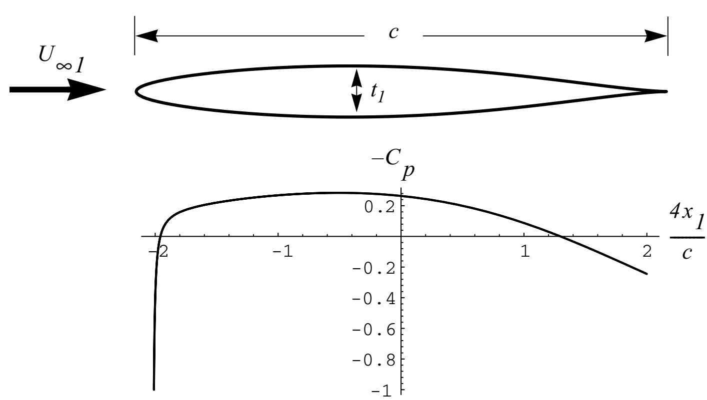


Fig. 12.9. Pressure variation over a thin symmetric airfoil in low-speed flow.

and the flow Mach number $U_\infty/a_\infty$, where $a_\infty^2 = \gamma p_\infty/\rho_\infty$ is the speed of sound, is assumed to be much less than one. The airfoil chord is $c$, and the maximum thickness is $t_1$. The subscript 1 is applied in anticipation of the fact that we will shortly scale the airfoil to a new shape with subscript 2.

The surface pressure distribution is shown below the wing, expressed in terms of the pressure coefficient

$$C_{p1} = \frac{p_{s1} - p_\infty}{\frac{1}{2}\rho_\infty U_{\infty 1}^2}. \tag{12.83}$$

The pressure and flow speed throughout the flow satisfy the Bernoulli relation. Near the airfoil surface,

$$p_\infty + \tfrac{1}{2}\rho_\infty u_{\infty 1}^2 = p_{s1} + \tfrac{1}{2}\rho_\infty U_{s1}^2. \tag{12.84}$$

The pressure is high at the leading edge where the flow stagnates; then, as the flow accelerates about the body, the pressure falls rapidly at first, then more slowly, reaching a minimum at the point of maximum airfoil thickness. From there the surface velocity decreases and the pressure increases continuously to the trailing edge. The reader can find an instructive set of pictures of the flow being discussed here in the book of flow visualization photographs collected by Van Dyke [12.14]. In the absence of viscosity, the flow is irrotational:

$$\nabla \times \boldsymbol{u}_1 = 0. \tag{12.85}$$

This permits the velocity to be described by a potential function:

$$\boldsymbol{u}_1 = \nabla\phi_1. \tag{12.86}$$

When this is combined with the condition of incompressibility, $\nabla \cdot \boldsymbol{u}_1 = 0$, the result is Laplace's equation,

$$\frac{\partial^2\phi_1}{\partial^2 x_1} + \frac{\partial^2\phi_1}{\partial^2 y_1} = 0. \tag{12.87}$$

Let the shape of the airfoil surface in $(x_1, y_1)$ be given by

$$\frac{y_1}{c} = \tau_1 g[x_1/c], \tag{12.88}$$

where $\tau_1 = t_1/c$ is the thickness-to-chord ratio of the airfoil. The boundary conditions that the velocity potential must satisfy are

$$\left(\frac{\partial\phi_1}{\partial y_1}\right)_{y_1 = c\tau_1 g[x_1/c]} = U_{\infty 1}\left(\frac{dy_1}{dx_1}\right)_{\text{body}} = U_\infty \tau_1 \frac{dg[x_1/c]}{d(x_1/c)},$$
$$\left.\frac{\partial\phi_1}{\partial x_1}\right|_{x_1 \to -\infty} = U_{\infty_1}. \tag{12.89}$$

Any number of methods of solving for the velocity potential are available, including the use of complex variables. In the following we are going to restrict the airfoil to be thin ($\tau_2 \ll 1$). In this context we will take the velocity potential to be a perturbation potential, $\phi'$ so that

$$u_1 = U_{\infty 1} + u_1', \qquad v_1 = v_1', \tag{12.90}$$

or

$$u_1 = U_{\infty 1} + \frac{\partial \phi_1'}{\partial x_1}, \qquad v_1 = \frac{\partial \phi_1'}{\partial y_1}. \tag{12.91}$$

The boundary conditions on the perturbation potential in the thin-airfoil approximation are

$$\begin{aligned} \left(\frac{\partial \phi_1'}{\partial y_1}\right)_{y_1=0} &= U_{\infty 1}\left(\frac{dy_1}{dx_1}\right)_{\text{body}} = U_\infty \tau_1 \frac{dg[x_1/c]}{d(x_1/c)}, \\ \left.\frac{\partial \phi_1'}{\partial x_1}\right|_{x_1\to\infty} &= 0. \end{aligned} \tag{12.92}$$

The surface pressure coefficient in the thin-airfoil approximation is

$$C_{p1} = -\frac{2}{U_{\infty 1}}\left(\frac{\partial \phi_1'}{\partial x_1}\right)_{y_1=0}. \tag{12.93}$$

Note that the boundary condition on the vertical velocity is now applied on the line $y_1 = 0$.

In effect the airfoil has been replaced with a line of volume sources whose strengths are proportional to the local slope of the actual airfoil. This sort of approximation is really unnecessary in the low-Mach-number limit, but it is essential when the Mach number is increased and compressibility effects come into play. Equally, it is essential in this example, where we will map a compressible flow to the incompressible case.

### 12.5.1 Subsonic Flow, $M_\infty < 1$

Now imagine a second flow at a free-stream velocity $U_{\infty 2}$ in a new space $(x_2, y_2)$ over a new airfoil of the same shape (defined by the function $g[x/c]$), but with a new thickness-to-chord ratio $\tau_2 = t_2/c \ll 1$. Part of what we need to do is to determine how $\tau_1$ and $\tau_2$ are related to one another. The boundary conditions that the new perturbation velocity potential must satisfy are

$$\begin{aligned} \left(\frac{\partial \phi_2'}{\partial y_2}\right)_{y_2=0} &= U_{\infty 2}\left(\frac{dy_2}{dx_2}\right)_{\text{body}} = U_{\infty 2} \tau_2 \frac{dg[x_1/c]}{d(x_1/c)}, \\ \left.\frac{\partial \phi_2'}{\partial x_2}\right|_{x_2\to\infty} &= 0. \end{aligned} \tag{12.94}$$

In this second flow the Mach number has been increased to the point where compressibility effects become important: the density begins to vary significantly, and the pressure distribution begins to deviate from the incompressible case. As long as shock waves do not form on the wing, the flow will be nearly isentropic. In this instance the 2-D steady compressible flow equations are

$$\begin{aligned} u_2\frac{\partial u_2}{\partial x_2} + v_2\frac{\partial u_2}{\partial y_2} + \frac{1}{\rho_2}\frac{\partial p_2}{\partial x_2} &= 0, \\ u_2\frac{\partial v_2}{\partial x_2} + v_2\frac{\partial v_2}{\partial y_2} + \frac{1}{\rho_2}\frac{\partial p_2}{\partial y_2} &= 0, \\ u_2\frac{\partial \rho_2}{\partial x_2} + v_2\frac{\partial \rho_2}{\partial y_2} + \rho_2\frac{\partial u_2}{\partial x_2} + \rho_2\frac{\partial v_2}{\partial y_2} &= 0, \\ p_2 &= A\rho_2^{\gamma}. \end{aligned} \tag{12.95}$$

Let

$$u_2 = U_{\infty 2} + u_2', \qquad v_2 = v_2', \qquad \rho_2 = \rho_\infty + \rho_2', \qquad p_2 = p_\infty + \rho_2', \tag{12.96}$$

where the primed quantities are assumed to be small compared to the free-stream conditions. When quadratic terms in the equations of motion are neglected, the equations (12.95) reduce to

$$\left(1 - \frac{\rho_\infty U_{\infty 2}^2}{\gamma p_\infty}\right)\frac{\partial u_2'}{\partial x_2} + \frac{\partial v_2'}{\partial y_2} = 0. \tag{12.97}$$

Introduce the perturbation velocity potential:

$$u_2' = \frac{\partial \phi_2'}{\partial x_2}, \qquad v_2' = \frac{\partial \phi_2'}{\partial y_2}. \tag{12.98}$$

The equation governing the disturbance flow becomes

$$\left(1 - M_{\infty 2}^2\right)\frac{\partial^2 \phi_2'}{\partial x_2^2} + \frac{\partial^2 \phi_2'}{\partial y_2^2} = 0. \tag{12.99}$$

Notice that (12.99) is valid for both sub- and supersonic flow.

Since the flow is isentropic, the pressure and velocity disturbances are related to lowest order by

$$p_2' + \rho_\infty U_{\infty 2} u_2' = 0, \tag{12.100}$$

and the surface pressure coefficient retains the same basic form as in the incompressible case,

$$C_{p2} = -\frac{2}{U_{\infty 2}}\left(\frac{\partial \phi_2'}{\partial x_2}\right)_{y_2=0}. \tag{12.101}$$

This last relation is valid only within the thin-airfoil, small-disturbance approximation and therefore may be expected to be invalid near the leading edge of the airfoil, where the velocity change is of the order of the free-stream velocity. For example, for the "thin" airfoil depicted in Figure 12.9, which is actually not all that thin, the pressure coefficient is within $-0.2 < C_p < 0.2$ except over a very narrow portion of the chord near the leading edge.

Equation (12.99) can be transformed to Laplace's equation (12.87) using the dilation group

$$x_2 = x_1, \qquad y_2 = \frac{1}{\sqrt{1-M_{\infty 2}^2}}y_1, \qquad \phi_2' = \frac{1}{A}\left(\frac{U_{\infty 2}}{U_{\infty 1}}\right)\phi_1', \tag{12.102}$$

where, at the moment, $A$ is an arbitrary constant. The velocity potentials are related by

$$\phi_2'[x_2, y_2] = \frac{1}{A}\left(\frac{U_{\infty 2}}{U_{\infty 1}}\right)\phi_1'[x_1, y_1], \tag{12.103}$$

or

$$\phi_1'[x_1, y_1] = A\left(\frac{U_{\infty 1}}{U_{\infty 2}}\right)\phi_2'\left[x_1, \frac{1}{\sqrt{1-M_{\infty 2}^2}}y_1\right], \tag{12.104}$$

and the boundary conditions transform as

$$\left(\frac{\partial \phi_1'}{\partial y_1}\right)_{y_1=0} = U_{\infty 1}\left(\frac{A\tau_2}{\sqrt{1-M_{\infty 2}^2}}\right)\frac{dg[x_1/c]}{d(x_1/c)},$$

$$\left.\frac{\partial \phi_1'}{\partial x_1}\right|_{x_1\to\infty} = \left.\frac{\partial \phi_2'}{\partial x_2}\right|_{x_2\to\infty} = 0. \tag{12.105}$$

The transformation between flows 1 and 2 is completed by the correspondence

$$\tau_1 = \frac{A\tau_2}{\sqrt{1-M_{\infty 2}^2}}, \tag{12.106}$$

or

$$\frac{t_1}{c} = \frac{A}{\sqrt{1 - M_{\infty 2}^2}} \left( \frac{t_2}{c} \right). \tag{12.107}$$

Finally the transformed pressure coefficient is

$$C_{p1} = AC_{p2}. \tag{12.108}$$

These results may be stated as follows. The solution for incompressible flow over a thin airfoil with shape $g[x_1/c]$ and thickness-to-chord ratio $t_1/c$ at velocity $U_1$ is identical to the subsonic compressible flow at velocity $U_2$ and Mach number $M_2$ over an airfoil with a similar shape but with the thickness-to-chord ratio

$$\frac{t_2}{c} = \frac{\sqrt{1 - M_{\infty 2}^2}}{A} \left( \frac{t_1}{c} \right). \tag{12.109}$$

R. T. Jones [12.15] presents a lucid physical description of the effects of moderate compressibility in terms of the lateral expansion of streamlines compared to the incompressible case. The pressure coefficient for the compressible case is derived by adjusting the incompressible value using $C_{p2} = C_{p1}/A$. This result equates to several different similarity rules that can be found in the aeronautical literature, depending on the choice of the free constant $A$. The one of greatest interest is the so-called Prandtl–Glauert rule that describes the variation of pressure coefficient with Mach number for a body of a given shape and thickness-to-chord ratio. In this case we select

$$A = \sqrt{1 - M_{\infty 2}^2}, \tag{12.110}$$

so that the two bodies being compared in (12.109) have the same shape and the same thickness-to-chord ratio. The pressure coefficient for the compressible flow is

$$C_{p2} = \frac{C_{p1}}{\sqrt{1 - M_{\infty 2}^2}}. \tag{12.111}$$

Several scaled profiles are shown in Figure 12.10. Keep in mind the lack of validity of (12.111) near the leading edge, where the pressure coefficient is scaled to unphysical values greater than one.

### 12.5.2 Supersonic Similarity, $M_\infty > 1$

All the theory developed in the previous subsection can be extended to the supersonic case by simply replacing $1 - M_\infty^2$ with $M_\infty^2 - 1$. In this instance the

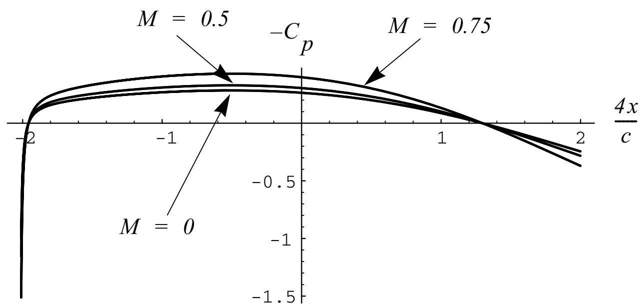


Fig. 12.10. Pressure coefficient over the airfoil in Figure 12.9 at several Mach numbers as estimated using the Prandtl–Glauert rule (12.111).

mapping is between the equation

$$\left(M_{\infty 2}^2 - 1\right)\frac{\partial^2 \phi_2'}{\partial x_2^2} - \frac{\partial^2 \phi_2'}{\partial y_2^2} = 0 \tag{12.112}$$

and the simple wave equation

$$\frac{\partial^2 \phi_1'}{\partial x_1^2} - \frac{\partial^2 \phi_1'}{\partial y_1^2} = 0. \tag{12.113}$$

A generalized form of the pressure coefficient valid for subsonic and supersonic flow is

$$\frac{C_p}{A} = F\left[\frac{\tau}{A\sqrt{\left|1 - M_\infty^2\right|}}\right], \tag{12.114}$$

where $A$ is taken to be a function of $\left|1 - M_\infty^2\right|$.

### 12.5.3 Transonic Similarity, $M_\infty \approx 1$

When the Mach number is close to one, the simple linearization used to obtain (12.99) from (12.95) loses accuracy. In this case the equations (12.95) reduce to the nonlinear equation

$$\left(1 - M_{\infty 1}^2\right)\frac{\partial u_1'}{\partial x_1} + \frac{\partial v_1'}{\partial y_1} - \frac{(\gamma_1 + 1)M_{\infty 1}^2}{U_{\infty 1}} u_1' \frac{\partial u_1'}{\partial x_1} = 0. \tag{12.115}$$

In terms of the perturbation potential,

$$\left(1 - M_{\infty 1}^2\right)\frac{\partial^2 \phi_1'}{\partial x_1^2} + \frac{\partial^2 \phi_1'}{\partial y_1^2} - \frac{(\gamma_1 + 1)M_{\infty 1}^2}{U_{\infty 1}} \frac{\partial \phi_1'}{\partial x_1} \frac{\partial^2 \phi_1'}{\partial x_1^2} = 0. \tag{12.116}$$

This equation is invariant under the scaling

$$x_2 = x_1, \qquad y_2 = \frac{\sqrt{1-M_{\infty 1}^2}}{\sqrt{1-M_{\infty 2}^2}} y_1, \qquad \phi_2' = \frac{1}{A}\left(\frac{U_{\infty 2}}{U_{\infty 1}}\right)\phi_1', \quad (12.117)$$

where

$$A = \left(\frac{1+\gamma_2}{1+\gamma_1}\right)\left(\frac{1-M_{\infty 1}^2}{1-M_{\infty 2}^2}\right)\left(\frac{M_{\infty 2}^2}{M_{\infty 1}^2}\right). \quad (12.118)$$

Notice that, due to the nonlinearity of the transonic equation (12.116), the constant $A$ is no longer arbitrary. The pressure coefficient becomes

$$C_{p1} = \left(\frac{1+\gamma_2}{1+\gamma_1}\right)\left(\frac{1-M_{\infty 1}^2}{1-M_{\infty 2}^2}\right)\left(\frac{M_{\infty 2}^2}{M_{\infty 1}^2}\right) C_{p2}, \quad (12.119)$$

and the thickness-to-chord ratios are related by

$$\frac{t_2}{c} = \left(\frac{1+\gamma_1}{1+\gamma_2}\right)\left(\frac{1-M_{\infty 2}^2}{1-M_{\infty 1}^2}\right)^{3/2}\left(\frac{M_{\infty 1}^2}{M_{\infty 2}^2}\right)\frac{t_1}{c}. \quad (12.120)$$

In the transonic case, it is not possible to compare the same body at different Mach numbers or bodies with different thickness-to-chord ratios at the same Mach number except by selecting gases with different $\gamma$. For a given gas it is only possible to map the pressure distribution for one airfoil to an airfoil with a different thickness-to-chord ratio at a different Mach number. A generalized form of (12.114) valid from subsonic to sonic to supersonic Mach numbers is

$$\frac{C_p\left((\gamma+1)M_\infty^2\right)^{1/3}}{\tau^{2/3}} = F\left[\frac{1-M_\infty^2}{\left(\tau(\gamma+1)M_\infty^2\right)^{2/3}}\right]. \quad (12.121)$$

## 12.6 Concluding Remarks

Prior to the advent of supercomputers capable of solving the equations of high-speed flow, similarity methods and wind-tunnel correlations were the only tools available to the aircraft designer, and these methods played a key role in the early development of transonic and supersonic flight. As a result, similarity solutions and comparisons with experimental data fill the aeronautics literature after World War II. It is only possible to scratch the surface of this vast subject and much more on transonic aerodynamics and the application of similarity rules can be found in References [12.16], [12.17], [12.18], [12.19], [12.20], and [12.21].

## 12.7 Exercises

12.1 Consider the isentropic expansion of a monatomic gas in three dimensions discussed in Section 12.3. Put in the parameters of the problem that characterize the initial conditions, including the initial radius of the cloud, the initial pressure and density at the center of the cloud, and $C_p$ and $C_v$. Discuss the evolution of the cloud front as well as the energy integral (12.44) in terms of the initial conditions. Indicate how you would reach these results using dimensional analysis.

12.2 Use the package **`IntroToSymmetry.m`** to determine the infinitesimal groups of the reduced system of gasdynamic equations (12.36),

$$\alpha^i + U^j \frac{\partial U^i}{\partial \alpha^j} + R^{(2/n)-1} \frac{\partial R}{\partial \alpha^i} = 0,$$
$$\frac{\partial (RU^j)}{\partial \alpha^j} = 0, \tag{12.122}$$
$$i = 1, \ldots, n, \qquad \text{sum over} \quad j = 1, \ldots, n.$$

Solve the system (12.122) for the case $n = 1$. What sort of initial velocity and density distributions are admitted by the solution?

12.3 Solve the blast wave problem in two dimensions. Formulate the problem in cylindrical coordinates, and let the wave be produced by an explosion on a line.

12.4 The equations of homentropic flow (homogeneously isentropic, $\nabla S = 0$) are

$$\frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} + \rho^{\gamma-2} \frac{\partial \rho}{\partial x^i} = 0,$$
$$\frac{\partial \rho}{\partial t} + u^j \frac{\partial \rho}{\partial x^j} + \rho \frac{\partial u^j}{\partial x^j} = 0, \tag{12.123}$$
$$i = 1, \ldots, n, \qquad \text{sum over} \quad j = 1, \ldots, n.$$

Use the packege **`IntroToSymmetry.m`** to work out the infinitesimal groups of (12.123) for general $\gamma$, and compare with those of (12.1).

12.5 Use the package **`IntroToSymmetry.m`** to work out the six-parameter infinitesimal group of the equation

$$u_{yy} - (\gamma + 1) u_x u_{xx} = 0, \tag{12.124}$$

which governs transonic small-disturbance flow over a thin two-dimensional wing. Check your answer by directly transforming the equation. Work out the symmetries of (12.116), and compare with those of (12.124).

12.6 The full viscous equations governing the motion of a compressible gas, including the equation of state, are

$$
\begin{aligned}
\Psi^i &= \frac{\partial \rho u^i}{\partial t} + \frac{\partial}{\partial x^j}\left(\rho u^i u^j + P\delta^i_j\right) \\
&\quad - \frac{\partial}{\partial x^j}\left(\mu\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right) - \frac{2}{3}\mu\delta_{ij}\frac{\partial u^k}{\partial x^k}\right) = 0; \qquad i = 1, 2, 3, \\
\Psi^4 &= \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u^i) = 0, \\
\Psi^5 &= \frac{\partial \rho\left(C_v T + \frac{1}{2}u^k u^k\right)}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho u^i\left(C_p T + \frac{1}{2}u^k u^k\right) - \kappa\frac{\partial T}{\partial x^i}\right) \\
&\quad - u^j\frac{\partial}{\partial x^i}\left(\mu\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right) - \frac{2}{3}\mu\delta_{ij}\left(\frac{\partial u^k}{\partial x^k}\right)\right) = 0, \\
\Psi^6 &= p - \rho RT = 0.
\end{aligned}
\tag{12.125}
$$

Assume for simplicity that the parameters, $\mu$, $C_p$, $C_v$, and $\kappa$ are constant. Use the package **`IntroToSymmetry.m`** to work out the infinitesimal groups of (12.125). Compare with the inviscid case. Note that you will need to fully expand the equations to produce the various derivatives that appear in the twice extended group operator.

12.7 Consider the flow generated by an impulsively started jet in a compressible medium. Carry out the integral momentum balance (similar to that used for the incompressible case in Chapter 11, Section 11.5.1), and determine the fraction of impulse contained in the axial momentum of the fluid and the fraction carried to infinity by the pressure disturbance of the jet. See Reference [12.22], where this problem is dealt with.

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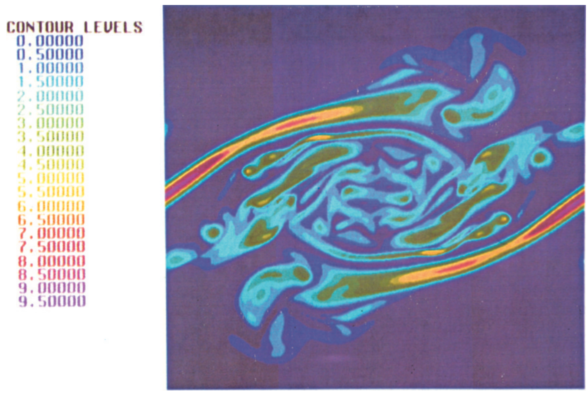
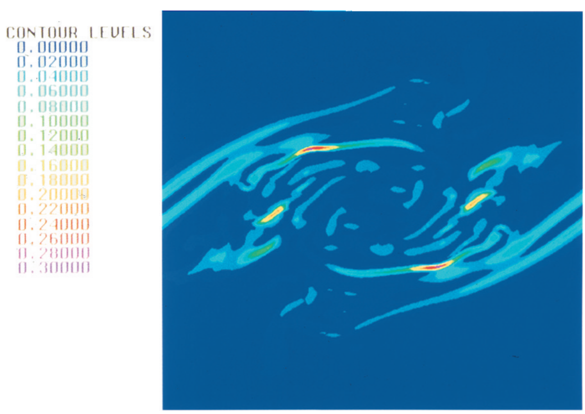
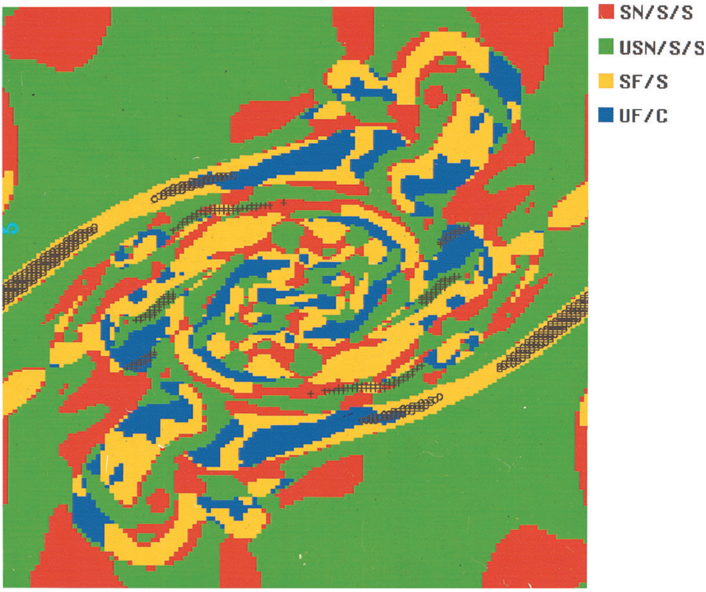



Color plate 1.




Color plate 2.

# 13

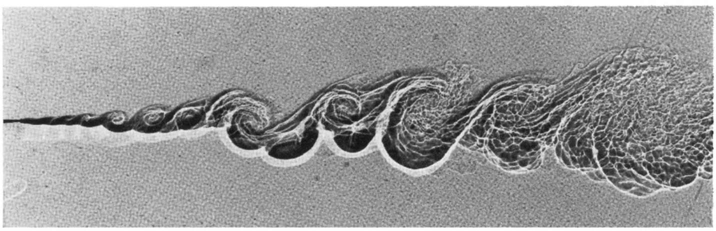
## *Similarity Rules for Turbulent Shear Flows*

Once the Reynolds number of a viscous flow is large enough to produce instability and once the amplitude of the instability is large enough to produce turbulence, then further amplification ceases and the overall behavior of the flow tends to be independent of the viscosity. The purpose of this chapter is to explore some of the ramifications of this ubiquitous property of turbulence in light of the group symmetries of the governing Navier–Stokes equations, the Euler equations, and their variant, the Reynolds-averaged Navier–Stokes equations. First, some of the basic features of Reynolds number invariance are discussed in conventional terms without reference to groups; then the concept of a one-parameter turbulent flow is defined, and Reynolds number invariance of this class of flows is interpreted in terms of the general dilation group of the Euler equations. The result of this process is a set of similarity rules that define the scaling properties of a wide range of geometrically simple flows. The rules are used to design an experiment to measure the structure of turbulent vortex rings at very high Reynolds number. This is then followed by a discussion of two models of the geometry of the fine-scale structure of turbulence. These discussions utilize the tools for analyzing three-dimensional vector fields discussed in Chapter 3 and used in Chapter 11 in the analysis of the impulsively started laminar jet.

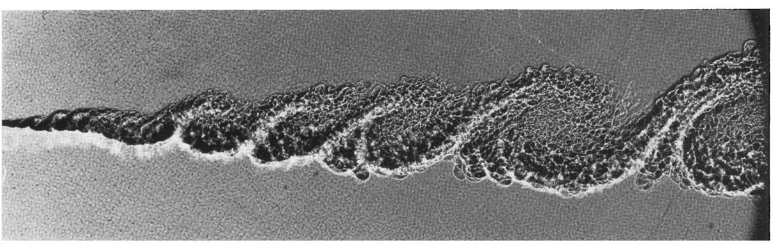
### 13.1 Introduction

The unsteady motion in turbulent shear flows is dominated by large eddies whose size is of the order of the overall thickness of the flow, $\delta$. Moreover the large eddy length scales in the streamwise and cross-stream direction tend to be of the same order. These large-scale motions contain most of the kinetic energy of the flow and, as noted above, are relatively unaffected by changes in the kinematic viscosity. Figure 13.1, from the landmark paper by Brown and

(a)



(b)



(c)

Fig. 13.1. Effects of Reynolds number on a plane mixing layer between helium (upper stream) and nitrogen (lower stream) from Brown and Roshko [13.1] and Roshko[13.2]. The Reynolds number in (a) is approximately $1.3 \times 10^4$ centimeter$^{-1}$. The thickness of the layer at the right side of the picture is approximately 2 cm. The speed of the lower stream is 10 m/s. Test-section pressures in atmospheres are: (a) 2, (b) 4, (c) 8. Dynamic pressures in the upper and lower streams are the same: $\rho_1 u_1^2 = \rho_2 u_2^2$.

Roshko [13.1] depicting a plane mixing layer at three Reynolds numbers, is the best visualization of this behavior that I know of.

The velocities in the upper and lower streams are the same for each picture, while the pressure of the flow increases from top to bottom. Increasing the pressure increases the density, hence decreasing the kinematic viscosity of the fluid and leading to a factor-of-four increase in Reynolds number from top to bottom. Note that the growth rate of the layer is approximately linear ($\delta \propto x$) and the angle of spread of the mixing layer is virtually the same in all three photos. In contrast, the thickness of a *laminar* mixing layer would decrease as the Reynolds number is increased, in proportion to the square root of the kinematic viscosity: $\delta_{\text{laminar}} \propto \sqrt{\nu x}$.

Similar observations hold for jets and wakes: once the Reynolds number is high enough to produce turbulence, the scale of the flow is set by the apparatus that creates the flow and tends to be nearly independent of viscosity. As a result, free shear flows (flows away from walls) are often modeled by neglecting viscous transport of momentum altogether.

The most pronounced dependence on viscosity occurs in the case of turbulent flow along a wall, yet even in this case, the region dominated by viscosity tends to be confined to a very thin layer near the wall, and the wall shear stress is a very slowly decreasing function of the Reynolds number. Over the streamwise distance required for the thickness of a turbulent boundary layer to double, the skin friction may only decrease by a few percent.

## 13.2 Reynolds-Number Invariance

We will use the customary decomposition of the velocity and pressure introduced in 1895 by Osborne Reynolds [13.3],

$$\boldsymbol{u} = \bar{\boldsymbol{u}} + \boldsymbol{u}', \qquad p = \bar{p} + p', \tag{13.1}$$

where $\boldsymbol{u}$ is the vector velocity at an instant, $\bar{\boldsymbol{u}}$ is the ensemble mean over a large number of realizations of the flow, and $\boldsymbol{u}'$ is a velocity fluctuation away from the mean for a given realization.

The notion of an *ensemble* is one of the central statistical tools of turbulence theory and enables the mean flow to be regarded as time-dependent, so that almost any flow can be treated using the Reynolds decomposition. One way to conceive of the ensemble is to imagine repeating a numerical simulation of the flow with some form of randomness in the initial conditions from one realization to another. Each simulation represents a history of the complete three-dimensional flow field. The ensemble mean is formed by averaging over the entire ensemble at each instant in time.

However, there are serious theoretical questions regarding the uniqueness of the mean and its possible dependence on the choice of initial conditions. There is now ample evidence from both numerical simulations and experiment that much of the development of turbulent shear flows at the moderate Reynolds numbers observable in the laboratory does depend on details of the initial conditions, particularly with respect to regular versus randomized initial disturbances [13.4], [13.5], [13.6], [13.7]. The degree of variation tends to be reduced as the Reynolds number is increased, but the jury is still out as to the existence of a uniquely defined mean independent of initial conditions for the extreme high-Reynolds-number limit, which is so extraordinarily hard to observe in the laboratory.

Even if the initial conditions could be suitably randomized and independence of initial conditions achieved, it is still a matter of debate whether the angle of spread of a mixing layer would be truly independent of the Reynolds number or whether there might exist an intrinsic, slow (say logarithmic) dependence on Reynolds number that simply could not be detected over the range of Reynolds numbers available in the laboratory. There is no theory yet that can shed light on this issue. Perhaps symmetry methods will eventually show the way.

In any case, when the Reynolds decomposition is introduced into the Navier–Stokes equations and the equations are averaged, they become the Reynolds-averaged equations,

$$\begin{aligned} \frac{\partial \bar{u}^i}{\partial x^i} &= 0, \\ \frac{\partial \bar{u}^i}{\partial t} + \frac{\partial}{\partial x^j}(\bar{u}^j \bar{u}^i) + \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x^i} - \frac{1}{\rho}\frac{\partial \tau^{ij}}{\partial x^j} - 2\nu \frac{\partial \bar{s}^{ij}}{\partial x^j} &= 0, \end{aligned} \tag{13.2}$$

where $\nu = \mu/\rho$ is the kinematic viscosity, $\bar{p}$ is the mean pressure, and $\bar{s}^{ij}$ is the rate of strain of the mean velocity field,

$$\bar{s}^{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}^i}{\partial x^j} + \frac{\partial \bar{u}^j}{\partial x^i}\right). \tag{13.3}$$

The density $\rho$ is constant. The new term that appears in these equations originates from the quadratic convective velocity term in the Navier–Stokes equations and takes the form of effective stresses arising from the correlation of the velocity fluctuations. These are the so-called *Reynolds stresses*,

$$\frac{\tau^{ij}}{\rho} = -\overline{u'^i u'^j}. \tag{13.4}$$

The linear pressure and viscous diffusion terms contribute to momentum transport only through gradients in the ensemble mean flow. The Reynolds stresses add six new unknowns to the equations of motion, and as they stand the Reynolds equations (13.2) are not closed. In essence the "turbulence problem" boils down to finding additional equations to relate the Reynolds stresses (13.4) to the mean flow and close the equations. This is the domain of *turbulence modeling*. In this chapter we will not delve into the complexities of turbulence modeling, but rather we will concentrate on those properties of turbulent flows that can be deduced from symmetry analysis alone in the absence of a model and with relatively little quantitative knowledge of the flow field. Although we will not address the issue here, symmetry analysis is an extremely useful tool in the construction of rational turbulence models.

Measurements of the fluctuating velocity in a wide variety of turbulent shear flows show that, away from a wall, the Reynolds stresses tend to be much larger than the viscous stresses,

$$-\overline{u'^i u'^j} \gg 2\nu \bar{s}^{ij}. \tag{13.5}$$

As a consequence the last term in (13.2) is often dropped, leading to a simplified form of the Reynolds equations,

$$\begin{aligned} \frac{\partial \bar{u}^j}{\partial x^j} &= 0, \\ \frac{\partial \bar{u}^i}{\partial t} + \frac{\partial}{\partial x^j}(\bar{u}^j \bar{u}^i) + \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x^i} - \frac{1}{\rho}\frac{\partial \tau^{ij}}{\partial x^j} &= 0. \end{aligned} \tag{13.6}$$

Dropping the viscous term has important consequences for the group invariance of the governing equations, as we shall see shortly. First we define

| |
|---|
| $u_0 =$ integral velocity scale characterizing the overall motion,<br>$\delta =$ integral length scale characterizing the overall motion. |

From a wide variety of experiments it is observed that the intensity of turbulence scales with the characteristic integral velocity of the flow. There are several ways to define the turbulence intensity, but the most common method is to use the turbulent kinetic energy. Let

$$u' = \sqrt{\frac{u_1'^2 + u_2'^2 + u_3'^2}{2}}. \tag{13.7}$$

It is found that

$$u' \propto u_0, \tag{13.8}$$

independent of $\nu$. For example, if in the plane mixing layer shown in Figure 13.1 the velocity difference $u_0 = u_1 - u_2$ were to be doubled keeping the velocity ratio $u_1/u_2$ the same, then one could expect the turbulent fluctuations to double.

If the viscosity were decreased keeping everything else the same, the rms turbulent velocity fluctuations would not be expected to change. The spectrum of turbulent fluctuations broadens as the Reynolds number is increased; the range of scales increases, but $u'$ stays about the same and the size of the largest-scale eddies stays about the same.

This change in the range of scales can be clearly seen in Figure 13.1. Perhaps suprising is how wide the range of scales seems to become for only a factor-of-four increase in Reynolds number. The constancy of the angle of spread in these pictures reflects the invariant size of the large eddies; in general, for free shear flows, the scale $\delta$ of the flow is independent of $\nu$.

Using the momentum equation, one can form an equation governing the turbulent kinetic energy. Consideration of the order of magnitude of various terms in this equation reveals that – in contrast to the momentum equation where viscous transport can be neglected – viscous dissipation of turbulent kinetic energy (TKE),

$$\varepsilon = 2\nu\overline{s'^{ij}s'^{ji}}, \tag{13.9}$$

contributes to the energy transport a term that is of the order of the other terms in the equation. The quantity

$$s'^{ij} = \frac{1}{2}\left(\frac{\partial u'^i}{\partial x^j} + \frac{\partial u'^j}{\partial x^i}\right) \tag{13.10}$$

is the fluctuating rate of strain. In a turbulent flow, $s'^{ji} \gg \bar{s}^{ji}$ for reasons that will become clear shortly. The dissipation term cannot be neglected, in spite of the fact that the viscosity may be very small. In general, the dissipation is proportional to the production of TKE:

$$\varepsilon \propto \overline{u'^i u'^j}\frac{\partial \bar{u}^i}{\partial x^j}. \tag{13.11}$$

In the usual notation,

$$\varepsilon \propto \frac{u_0^3}{\delta}. \tag{13.12}$$

The implication of (13.11) is that the fluctuating strain rates must be large and inversely dependent on $\nu$, so that as the Reynolds number (kinematic viscosity) is changed, the fluctuating strain rates change so as to maintain the proportionality indicated in (13.11) and (13.12).

## 13.3 Group Interpretation of Reynolds-Number Invariance

Recall the two-parameter dilation group of the Euler equations discussed in Chapter 11, Section 11.1 [cf. Equation (11.13)],

$$\tilde{x}^i = e^s x^i, \qquad \tilde{t} = e^{s/k} t, \qquad \tilde{\bar{u}}^i = e^{s(1-1/k)} \bar{u}^i,$$
$$\tilde{\tau}^{ij} = e^{s(2-2/k)} \tau^{ij}, \qquad \tilde{\bar{p}} = e^{s(2-2/k)} \bar{p}, \tag{13.13}$$

where $s$ and $k$ are arbitrary group parameters. Note that we have invoked Reynolds-number invariance in writing down the group (13.13). In particular, Equations (13.5) and (13.8) have been used to deduce that $\tau^{ij}$ should be stretched by the square of the factor used to stretch $u^i$. Furthermore, all three coordinate directions are stretched in the same proportion to one another. If we act on the Reynolds equations using this group, the result is

$$\frac{\partial \tilde{\bar{u}}^i}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}^j} \tilde{\bar{u}}^j \tilde{\bar{u}}^i + \frac{\partial \tilde{\bar{p}}}{\partial \tilde{x}^i} - \frac{\partial \tilde{\tau}^{ij}}{\partial \tilde{x}^j}$$
$$= \left( \frac{\partial \bar{u}^i}{\partial t} + \frac{\partial}{\partial x^j} \bar{u}^j \bar{u}^i + \frac{\partial \bar{p}}{\partial x^i} - \frac{\partial \tau^{ij}}{\partial x^j} \right) e^{a(1-2/k)} = 0. \tag{13.14}$$

Recall that the group (13.13) reduces to the dilation group of the Navier–Stokes equations for $k = \frac{1}{2}$. The point of all this is that when we remove the viscous stress term from the Reynolds equations and assume that fluctuating velocities scale with the mean, the result is a system that is invariant under the two-parameter dilation group (13.13) of the Euler equations (see references [13.8], [13.9], [13.10]) rather than just the one-parameter group of the full viscous equations. The additional parameter can be used to define a large class of flows that are self-similar within the assumption of Reynolds-number invariance.

### 13.3.1 One-Parameter Flows

These are turbulent shear flows in open domains governed by a single global parameter with units

$$\hat{M} = L^m T^{-n}. \tag{13.15}$$

Usually $M$ is an integral invariant related to the forces that create the flow. We saw an example of how such an invariant integral is determined in Chapter 11, Section 11.5.1, where the impulse integral was derived. Recall that the analysis in that section is exact, regardless of whether the flow is laminar or turbulent. This is because the volume integral of the momentum can be transformed to a surface integral involving only the exactly known far-field potential flow. The momentum integral can no longer be determined exactly when the control-volume boundary is penetrated by the turbulence as in the momentum balance for a stationary (time-constant ensemble mean) jet. Nevertheless, the dimensions of the conserved quantity remain the same, and the arguements put forth below for determining the self-similar behavior of the flow can still be carried through. Some typical examples are:

*Stationary plane jet.* The integral momentum flux $J/\rho$ is approximately constant at any streamwise position:

$$\frac{J}{\rho} = \int_{-\infty}^{\infty} \tilde{u}^2\, d\tilde{y} = e^{a(3-2/k)} \int_{-\infty}^{\infty} u^2\, dy. \tag{13.16}$$

The integral is invariant under dilation only for $k = \frac{2}{3}$.

*Vortex ring.* The hydrodynamic impulse, $I/\rho$, is the conserved integral for this flow (cf. Chapter 11, Section 11.5.1):

$$\frac{I}{\rho} = \frac{3}{2} \int \tilde{u}\, d\tilde{x}\, d\tilde{y}\, d\tilde{z} = e^{a(4-1/k)} \frac{3}{2} \int u\, dx\, dy\, dz. \tag{13.17}$$

In this case the integral is invariant for $k = \frac{1}{4}$.

Invariance under the group (13.13) implicitly assumes that the flow is created at a point in an infinite domain. It is easy to modify the problem so as to break this symmetry, and any real flow does so. For example, if the force creating the flow (say, a jet tube) is allowed to have a finite size, then the dilation invariance of the problem will be broken. The implication of this is that results based on self-similar behavior really apply to the asymptotic (far field) behavior of the flow. In practice, the self-similar region is not so far from the origin as one might imagine, and similarity behavior is observed in a remarkably broad range of important flows.

Later in this chapter we will look at an experimental investigation of turbulent vortex rings, where all sorts of symmetry-breaking parameters are present but where a substantial region of nicely self-similar flow does occur and is quite accessible experimentally. Nevertheless, one must be aware that the far field of any flow can be affected by length scales that may have been omitted when the near field is collapsed to a point. The results have to be used with some caution, particularly when they are generalized to new geometries.

If we attempted to reincorporate the viscous stress term neglected in (13.6), the symmetry (13.13) would be broken. Since real flows are viscous, one should expect that all turbulent shear flows (except those with $k = \frac{1}{2}$) will include fine-scale dissipating motions that break the symmetry associated with the large eddies, and therefore such flows should all exhibit a weak dependence on the Reynolds number. At the present time there is neither theory nor experiment that sheds much light on how this broken symmetry affects the overall behavior (rate of spread, rate of velocity decay, etc.) of turbulent shear flows. Broadly speaking, there is always a price to be paid when seeking a similarity solution to a physical problem. The price is that the invariance requirements of the applicable group inevitably force the suppression of certain physical parameters of the real problem. And so any claims we might make for the generality of a similarity solution must always be tempered by comparison with experiment.

### 13.3.2 Temporal Similarity Rules

Following Reference [13.9], we can use invariance under the group (13.13) to develop a general set of similarity rules for characterizing the space-time evolution of one-parameter turbulent (and $k = \frac{1}{2}$ laminar) flows. This is accomplished by solving the characteristic equations of (13.13),

$$\frac{dx^i}{x^i} = k\frac{dt}{t} = \left(\frac{k}{k-1}\right)\frac{du^i}{u^i} = \left(\frac{k}{2k-2}\right)\frac{dp}{p} = \left(\frac{k}{2k-2}\right)\frac{d\tau^{ij}}{\tau^{ij}} \qquad (13.18)$$

with integrals

$$\xi^i = \frac{x^i}{\delta[t]}, \qquad U^i = \frac{u^i}{u_0[t]}, \qquad P = \frac{p}{u_0[t]^2}, \qquad T^{ij} = \frac{\tau^{ij}}{u_0[t]^2}. \qquad (13.19)$$

The time-dependent length and velocity scales in (13.19) are

$$\boxed{\delta[t] \propto M^{1/m}(t - t_0)^k, \qquad u_0[t] \propto M^{1/m}(t - t_0)^{k-1},} \qquad (13.20)$$

where $t_0$ is the effective origin in time. The group parameter $k$ is determined by the units of the governing parameter $M$:

$$\boxed{k = n/m.} \qquad (13.21)$$

The general form of the similarity solution that derives from invariance under (13.13) is

$$\frac{\bar{u}^i}{u_0[t]} = U^i\left[\frac{\boldsymbol{x}}{\delta[t]}\right], \qquad \frac{\bar{p}}{u_0[t]^2} = P\left[\frac{\boldsymbol{x}}{\delta[t]}\right], \qquad \frac{\tau^{ij}}{u_0[t]^2} = T^{ij}\left[\frac{\boldsymbol{x}}{\delta[t]}\right]. \qquad (13.22)$$

The functional relationships in (13.22) are consistent with the notion of self-similarity taught by Townsend [13.11]. But there is one important difference: here the characteristic velocity and length scales are functions of time linked by a single governing parameter, whereas traditional approaches usually deal only with stationary flows with scales that depend on streamwise distance. As we shall see shortly, the time-dependent approach includes stationary, spatially developing flows as well.

One of the implications of the similarity form (13.22) derived from the group (13.13) is that, for a flow governed by a single global parameter, the size of the large eddies in the flow scales with the same power of time in all three coordinate directions. This is consistent with the observation that large eddy length scales tend to be of the same order in all three coordinate directions. It also implies that a boundary-layer approximation is not needed to accomplish a simplification of the problem.

### 13.3.3 Frames of Reference

When the similarity variables (13.22) are substituted into the Reynolds equations (13.2), the result is that time drops out of the equations and the number of independent variables is reduced from four to three:

$$\frac{\partial U^j}{\partial \xi^j} = 0,$$

$$(k-1)U^i + (U^j - k\xi^j)\frac{\partial U^i}{\partial \xi^j} + \frac{1}{\rho}\frac{\partial P}{\partial \xi^i} - \frac{1}{\rho}\frac{\partial}{\partial \xi^j}(T^{ij}) = 0. \quad (13.23)$$

This is the same equation encountered in Chapter 11, Section 11.5 [cf. Equation (11.86)] where we analyzed the round jet with $k = \frac{1}{2}$. The equations for particle paths,

$$\frac{dx^i}{dt} = u^i[\boldsymbol{x}, t], \quad (13.24)$$

transform to the autonomous system

$$\frac{d\xi^i}{d\tau} = U^i[\boldsymbol{\xi}] - k\xi^i. \quad (13.25)$$

In these one-parameter flows all lengths scale with the same power of time. If an observer is selected to convect with a particular feature of the flow, then the observer will have to translate nonuniformly according to the power of time appropriate to the flow. Such a transformation is defined by

$$\begin{aligned}\tilde{x}^i &= x^i - \alpha^i M^{1/m}(t-t_0)^k,\\ \tilde{t} &= t,\\ \tilde{\bar{u}}^i &= \bar{u}^i - k\alpha^i M^{1/m}(t-t_0)^{k-1},\\ \tilde{\bar{p}} &= \bar{p} + x^j k(k-1)\alpha^j M^{1/m}(t-t_0)^{k-2},\end{aligned} \quad (13.26)$$

where the $\alpha^i$ determine the relative rates of motion of the observer in the three directions. We already know from the discussion in Chapter 11, Section 11.2, that the Navier–Stokes and Euler equations are invariant under the group (13.26). The Reynolds equations with the viscous term removed, (13.6), are as well. In terms of similarity variables, (13.26) becomes a simple translation,

$$\begin{aligned}\tilde{\xi}^i &= \xi^i - \alpha^i,\\ \tilde{\tau} &= \tau,\\ \tilde{U}^i &= U^i - k\alpha^i,\\ \tilde{P} &= P + \alpha^j \xi^j k(k-1).\end{aligned} \tag{13.27}$$

In similarity coordinates, the equations for particle paths transform as follows:

$$\begin{aligned}\frac{d\tilde{\xi}^i}{d\tilde{\tau}} &= \frac{d\xi^i}{d\tau},\\ \tilde{U}^i[\tilde{\boldsymbol{\xi}}] - k\tilde{\xi}^i &= U^i[\boldsymbol{\xi}] - k\xi^i.\end{aligned} \tag{13.28}$$

The second relation in (13.28) comes directly from subtracting the first and third relations in (13.27) and implies that the vector field of particle paths is independent of the $\alpha^i$. So, whereas the velocity field $\boldsymbol{U}$ changes when the observer is changed, the vector field of particle paths, $\boldsymbol{U} - k\boldsymbol{\xi}$, is the same for all observers who move with the relevant time scale. We saw a beautiful example of this in Chapter 11 when we looked at particle paths in the impulsively started round jet. See Figures 11.11 and 11.12.

### 13.3.4 Spatial Similarity Rules

Using the fact that one-parameter flows evolve in the streamwise direction according to the same law as the evolution in the cross-stream direction, the following conversion between $x$ and $t$ can be used:

$$(x - x_0) \propto M^{1/m}(t - t_0)^k. \tag{13.29}$$

Jetlike flows originate from a point force acting in a surrounding fluid at rest. In this case one uses (13.29) to replace $(t - t_0)$ in (13.20). The result is the following set of spatial similarity rules for jets:

$$\boxed{\delta \propto (x - x_0), \qquad u_0 \propto M^{1/n}(x - x_0)^{1-1/k}.} \tag{13.30}$$

Wakes originate from the drag on a body immersed in a surrounding fluid moving at a velocity $U_\infty$. A typical example is the flow past a sphere studied in Chapter 2, Section 2.3. The drag in this case is

$$\frac{D}{\rho} = C_D\left(\frac{1}{2}U_\infty^2\right)(\pi R^2). \tag{13.31}$$

In the near wake, the drag on the body creating the flow and the free-stream velocity are independent parameters. As a result, wakes are not self-similar in the near field, even with the stipulation that the force is concentrated at a point.

This last statement needs to be clarified. Concentrating the drag force at a point is, of course, not the same as reducing the body to a point. The radius of the body continues to be the appropriate length scale for measuring distance in the flow and for identifying the far field even as the applied force is collapsed to the origin of coordinates.

Far from the origin, an integral of the momentum flux over a control volume that contains the body and traverses the wake reduces to

$$\frac{D}{\rho} \propto \int_A U(U_\infty - U)\,dA, \tag{13.32}$$

where the integral is carried out in a plane perpendicular to the wake and extending to infinity. In the far wake the Oseen approximation $U_\infty - U \ll U_\infty$ becomes valid and the drag integral reduces to

$$\frac{D}{\rho U_\infty} \propto \int_A (U_\infty - U)\,dA. \tag{13.33}$$

In this limit, the drag and free-stream velocity merge into the single governing parameter $D/\rho U_\infty$ with units $L^3/T$ for a three-dimensional wake or $L^2/T$ for a plane wake (where $D$ is drag per unit span). This limiting flow is invariant under the group (13.13) combined with a translation. The transformation between space and time for wakes is

$$(x - x_0) = U_\infty(t - t_0). \tag{13.34}$$

Using (13.34) to replace the time in (13.20) leads to the following spatial similarity rules for wakes:

$$\boxed{\delta \propto M^{1/m}U_\infty^{-k}(x - x_0)^k, \qquad u_0 \propto M^{1/m}U_\infty^{1-k}(x - x_0)^{k-1}.} \tag{13.35}$$

Note that for $k = 1$ jets and wakes have same spatial scaling as is the case for the plane mixing layer depicted in Figure 13.1.

### 13.3.5 Reynolds Number Scaling

The similarity rules (13.20), (13.30), and (13.35) can be used to determine the temporal or spatial evolution of the flow Reynolds number,

$$R_\delta = \frac{U_0\delta}{\nu} \propto \frac{M^{2/m}}{\nu}(t-t_0)^{2k-1}. \tag{13.36}$$

From (13.36) we can see that if $k > \frac{1}{2}$, the Reynolds number increases with time and we would expect the range of scales in the flow to increase as shown in Figure 13.1. If $k < \frac{1}{2}$, then the Reynolds number decreases with time and there is a tendency for the flow to relaminarize. If $k = \frac{1}{2}$, then the Reynolds number is constant.

The case $k = \frac{1}{2}$ is highlighted once again. This case is special in that the full Navier–Stokes equations (including the viscous term) are invariant under the group (13.13). In fact, the few well-known exact solutions of the Navier–Stokes equations that are set in an infinite domain are all cases that correspond to this value of $k$. Included in this group are the Landau–Squire axisymmetric jet and the Jeffrey–Hamel plane flow in a diverging channel, studied in Chapter 11. Several additional cases can be found in Table 13.1, including the Oseen viscous vortex and the round buoyant thermal (with a Boussinesq approximation where the density of the jet is assumed to differ only slightly from the surrounding medium).

Turbulent flows are often characterized by a transport coefficient in the form of an effective "eddy" viscosity $\nu_\tau$. In general $\nu_\tau$ is a function of both space and time. In a thin shear layer the eddy viscosity can be deduced from measurements of the Reynolds stress and mean velocity using the equality

$$-\overline{u'v'} = \nu_\tau \frac{\partial \bar{u}}{\partial y}. \tag{13.37}$$

From the previously discussed scaling relationships we deduce from (13.37)

$$\nu_\tau \propto u_0\,\delta. \tag{13.38}$$

One can define a turbulent Reynolds number as

$$Re_\tau = \frac{u_0\delta}{\nu_\tau} \propto \text{constant} \tag{13.39}$$

where the constant varys from flow to flow and tends to fall between 10 and 50 (Reference behave [13.12]). In a gross sense the overall mean field in a turbulent flow tends to behave somewhat like a very viscous constant Reynolds number flow.

Table 13.1. *Various one-parameter shear flows and the units of the associated governing parameter.*

| Flow | Invariant | $M$ | Units | $k$ |
|---|---|---|---|---|
| | *Jetlike flows* | | | |
| Plane mixing layer | Velocity difference | $U_0$ | $LT^{-1}$ | 1 |
| Plane jet | 2-D momentum flux | $U_0^2\delta$ | $L^3T^{-2}$ | $\frac{2}{3}$ |
| Round jet | 3-D momentum flux | $U_0^2\delta^2$ | $L^4T^{-2}$ | $\frac{1}{2}$ |
| Radial jet | 3-D momentum flux | $U_0^2\delta^2$ | $L^4T^{-2}$ | $\frac{1}{2}$ |
| Vortex pair | 2-D impulse | $U_0\delta^2$ | $L^3T^{-1}$ | $\frac{1}{3}$ |
| Vortex ring | 3-D impulse | $U_0\delta^3$ | $L^4T^{-1}$ | $\frac{1}{4}$ |
| Plane plume | 2-D buoyancy flux | $U_0^3$ | $L^3T^{-3}$ | 1 |
| Round plume | 3-D buoyancy flux | $U_0^3\delta$ | $L^4T^{-3}$ | $\frac{3}{4}$ |
| Plane thermal | 2-D buoyancy | $U_0^2\delta$ | $L^3T^{-2}$ | $\frac{2}{3}$ |
| Round thermal | 3-D buoyancy | $U_0^2\delta^2$ | $L^4T^{-2}$ | $\frac{1}{2}$ |
| Line vortex | Circulation | $U_0\delta$ | $L^2T^{-1}$ | $\frac{1}{2}$ |
| Diverging channel | Area flux | $U_0\delta$ | $L^2T^{-1}$ | $\frac{1}{2}$ |
| Vortex-sheet rollup | Apex $\alpha; n = 1/(2-\alpha/\pi)$ | $U_0^2\delta^{2-n}$ | $L^{3-n}T^{-1}$ | $1/(3-n)$ |
| | *Wakelike flows* | | | |
| Plane wake | (2-D drag)$/U_\infty$ | $U_0\delta$ | $L^2T^{-1}$ | $\frac{1}{2}$ |
| Round wake | (3-D drag)$/U_\infty$ | $U_0\delta^2$ | $L^3T^{-1}$ | $\frac{1}{3}$ |
| Plane jet in cross flow | (2-D mom. flux)$/U_\infty$ | $U_0\delta$ | $L^2T^{-1}$ | $\frac{1}{2}$ |
| Round jet in cross flow | (3-D mom. flux)$/U_\infty$ | $U_0\delta^2$ | $L^3T^{-1}$ | $\frac{1}{3}$ |
| Plane plume in cross flow | (2-D buoy. flux)$/U_\infty$ | $U_0^2$ | $L^2T^{-2}$ | 1 |
| Round plume in cross flow | (3-D buoy. flux)$/U_\infty$ | $U_0^2\delta$ | $L^3T^{-2}$ | $\frac{2}{3}$ |
| Grid turb. initial decay | Saffman invariant | $U_0^2\delta^3$ | $L^5T^{-2}$ | $\frac{2}{5}$ |
| Grid turb. initial decay | Loitsianski invariant | $U_0^2\delta^5$ | $L^7T^{-2}$ | $\frac{2}{7}$ |

## 13.4 Fine-Scale Motions

So far we have used symmetry analysis to derive a great deal of information about the evolution of a turbulent flow at the largest scales of motion. Now let's turn our attention to the fine scales and see what we can learn about the physics of energy dissipation. This necessitates looking closely at fluctuating strain rates, and since in a turbulent flow the strain is closely linked to the vorticity, one is eventually led to a general study of the behavior of the velocity gradient tensor.

Using the scaling relation (13.11) that comes from the turbulent kinetic energy equation, we can write

$$\varepsilon \propto \frac{u_0^3}{\delta}, \tag{13.40}$$

which can be rearranged to read

$$\sqrt{\overline{s'^{ik}s'^{ki}}} \propto \frac{u_0}{\delta}\left(\frac{u_0\delta}{2\nu}\right)^{1/2} \tag{13.41}$$

This affirms the statement made earlier that the instantaneous rates of strain are larger than mean rates of strain by a factor proportional to the square root of the Reynolds number. Given $u_0$ and $\delta$, this result can be used to estimate, the size of the microscale motions that contribute the largest fluctuating strain rates and therefore the bulk of the energy dissipation in a one-parameter flow.

We now define a new length scale, $\lambda$, called the *Taylor microscale*, that, when associated with $u_0$, can account for turbulent kinetic energy dissipation [13.11], [13.13]:

$$\varepsilon \propto \nu\left(\frac{u_0^2}{\lambda^2}\right). \tag{13.42}$$

Combining (13.42) with (13.40) leads to the following estimates for the Taylor microscale:

$$\frac{\lambda}{\delta} \propto \frac{1}{(R_\delta)^{1/2}}, \qquad \lambda \propto (\nu(t - t_0))^{1/2}. \tag{13.43}$$

According to this estimate, there is always some eddying motion in the flow which has a characteristic length that varies like $\sqrt{\nu t}$ and is independent of the governing parameter $M$. In a similar vein note that the velocity gradients of the large-scale motion vary according to

$$\frac{u_0}{\delta} \propto \frac{1}{t - t_0} \qquad (t > t_0), \tag{13.44}$$

which is also independent of $M$. In a sense the large-scale gradients constitute a clock that can be used to date the evolution of the flow just as in the case of the laminar round jet.

Now let's define new length *and* velocity scales that can account for dissipation of TKE. These are the velocity and length scales defined by Kolmogorov [13.14]. See also the discussion of Kolmogorov theory in References [13.15] and [13.16]. The Kolmogorov scales can be regarded as motions that constitute the lower limit for instability – motions with a characteristic Reynolds number of order one. Let

$$\varepsilon \propto \nu\left(\frac{\upsilon^2}{\eta^2}\right), \qquad \frac{\upsilon\eta}{\nu} \approx 1. \tag{13.45}$$

Equation (13.45) can be used in conjunction with (13.12) to generate the following estimates of the Kolmorgorov velocity and length scales:

$$\frac{\eta}{\delta} \propto \frac{1}{(R_\delta)^{3/4}}, \qquad \eta \propto \nu^{3/4} M^{-1/2m} (t - t_0)^{3/4-k/2} \tag{13.46}$$

and

$$\frac{\upsilon}{u_0} \propto \frac{1}{(R_\delta)^{1/4}}, \qquad \upsilon \propto \nu^{1/4} M^{1/2m} (t - t_0)^{k/2-3/4}. \tag{13.47}$$

In a sense, the Taylor and Kolmogorov microscales bracket the range of scales that contribute the bulk of the dissipation of TKE in the flow. At scales larger than the Taylor microscale the turbulent motion is considered to be essentially inviscid. At the smallest scale are the Kolmogorov microscales with a local Reynolds number of order one. The fine-scale gradients over the whole range of dissipating motions vary according to

$$\frac{u_0}{\lambda} \propto \frac{\upsilon}{\eta} \propto \nu^{-1/2} M^{1/m} (t - t_0)^{k-3/2}. \tag{13.48}$$

Toward the end of this chapter we will develop a simple model for the flow geometry of these fine-scale motions.

### *13.4.1 The Inertial Subrange*

In 1941 Kolmogorov [13.14] hypothesized that at high Reynolds number there exists a range of scales, termed the *inertial subrange*, that depends only on the rate of dissipation of TKE imposed by the forces that drive the flow.

Originally the theory was developed in the context of homogeneous and isotropic turbulence, and the question of what parameter governs the large-scale motion was not of primary interest. Several invariants based on volume integrals of moments of the correlation function have been proposed for this seemingly simplest of all flows. Unfortunately, none can be derived from an unassailable first-principles approach, and data from studies of the initial decay of grid turbulence are too scattered to clarify precisely what quantity if any is conserved. In fact, Kolmogorov assumed that the scaling of the inertial subrange is independent of the large-scale motion. Furthermore, there is nothing in the analysis that should necessarily restrict the results to isotropic turbulence. For this reason Kolmogorov's ideas are often used to characterize the high-Reynolds-number behavior of inhomogeneous flows typified by the one-parameter class discussed in Section 13.3.1.

The inertial subrange envisioned by Kolmogorov lies between the $M$-dependent, $\nu$-independent large-scales and the $M$-independent, $\nu$-dependent Taylor microscale that defines the upper size limit of dissipating motions. We can derive one of Kolmogorov's most famous results using purely dimensional reasoning and the similarity rules worked out earlier. Let's accept Kolmogorov's basic tenet and assume that a range of scales exists where the turbulent motion is independent of both $\nu$ and $M$ and is governed only by the local volumetric rate of TKE dissipation. We can think of the eddy motions associated with the inertial subrange as a kind of universal one-parameter flow governed by

$$M = \varepsilon \propto u_0^3/\delta \tag{13.49}$$

with units $\hat{u}_0^3/\hat{\delta} = L^2T^{-3}$ and exponent $k = \frac{3}{2}$. The temporal evolution of the characteristic scales of the inertial subrange should follow the similarity rules in (13.30),

$$\delta \propto \varepsilon^{1/2}(t - t_0)^{3/2}, \qquad u_0 \propto \varepsilon^{1/2}(t - t_0)^{1/2} \tag{13.50}$$

and

$$R_\delta \propto (t - t_0)^2, \qquad \lambda \propto (t - t_0)^{1/2}, \qquad \eta \propto (t - t_0)^0. \tag{13.51}$$

Looking at the examples listed in Table 13.1, it is noteworthy that $k > 1$ never occurs. The value $k = \frac{3}{2}$ implies very strong local forcing of the flow, typically much stronger than the forcing in most common situations. For example, to produce $k = \frac{3}{2}$ at the largest scale of a jet, one would need to apply a force that increased in proportion to the fourth power of the time (see Exercise 13.1).

We can use (13.50) to establish a scaling law for a portion of the TKE spectrum. Assume that a range of scales exists that is characterized by the rules for $\delta$ and $u_0$ given in (13.50). Ask: how is the kinetic energy distributed among the various eddy length scales? Let $\kappa$ be the wave number of an eddy in the inertial subrange,

$$\kappa \propto 1/\delta. \tag{13.52}$$

The kinetic energy per unit wave number at a given wave number can be related to the time as follows:

$$E(\kappa) \propto \frac{u_0^2}{1/\delta} \propto \varepsilon^{3/2}(t - t_0)^{5/2}. \tag{13.53}$$

Solving for the time in (13.50), we have

$$t - t_0 \propto \frac{\delta^{2/3}}{\varepsilon^{1/3}} \propto \frac{\kappa^{-2/3}}{\varepsilon^{1/3}}. \tag{13.54}$$

Substituting (13.54) into (13.53) produces the classical result first postulated by Kolmogorov,

$$E(\kappa) \propto \varepsilon^{2/3}\kappa^{-5/3}. \tag{13.55}$$

The $\kappa^{-5/3}$ rolloff in the TKE spectrum has been confirmed in a wide variety of high-Reynolds-number experiments, and so the arguments of Kolmogorov and the postulated existence of the inertial subrange are generally accepted as correct. The fact that these results can be derived within the framework of the group-theoretical approach used to determine conventional scaling laws of free shear flows adds further support to Kolmogorov's ideas. However, the very strong forcing required to generate the inertial subrange suggests that it can exist only in a very vigorously stirred flow.

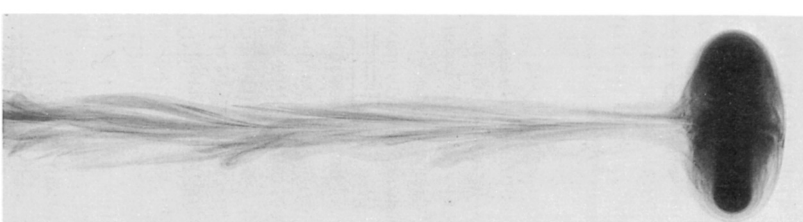
Furthermore, Reynolds-number invariance is still a purely empirically observed property of high-Reynolds-number turbulence. There is no first-principles theory that can draw it out of the equations of motion. Also, we live in a world where the range of Reynolds numbers encountered is very limited. Referring to the drag law for circular cylinders presented in Chapter 2, Figure 2.4, we can see that overall features of turbulent flows tend to change very slowly (logarithmically) with Reynolds number. In fact, unexpected variations tend to occur up to the highest Reynolds numbers tested. As a practical matter there is simply no scale available in the laboratory or in nature on this Earth that can provide a Reynolds number large enough to ensure that truly asymptotic behavior prevails. This is one of the most important stumbling blocks to the development of a theory of turbulence. Taking the flow over a sphere for example, we have no idea what the infinite-Reynolds-number value of the drag coefficient is, nor do we know if the limit is unique for a given set of flow parameters. The same goes for the limiting friction coefficient on a flat plate, and so on.

## 13.5 Application: Experiment to Measure Small Scales in a Turbulent Vortex Ring

To illustrate the practical use of the similarity rules developed in the preceding sections let's design a science experiment with the aim of studying the physics of kinetic energy dissipation in a turbulent vortex ring at very high Reynolds number. The goal is to come up with hard numbers for the apparatus design and choice of working fluid. The dye visualization photos in Figure 13.2, from the paper by Glezer and Coles [13.17], show the flow in question at low and moderately high Reynolds numbers. The upper photo shows a highly turbulent flow, but the experiment we intend to design will be required to reach Reynolds numbers two orders of magnitude higher.

To accomplish our goal it is necessary to construct a laboratory apparatus to contain the flow. This would consist of a large tank full of a transparent fluid, such as air or water, and an impulsively driven pump to produce the forcing needed to generate the flow. In general, we would select the fluid to have as

(a)



(b)

Fig. 13.2. Turbulent and laminar vortex rings produced by an impulsive force, from the paper by Glezer and Coles [13.17]. Initial Reynolds number $\Gamma_0/\nu$ is (a) 27,000, (b) 7,500.

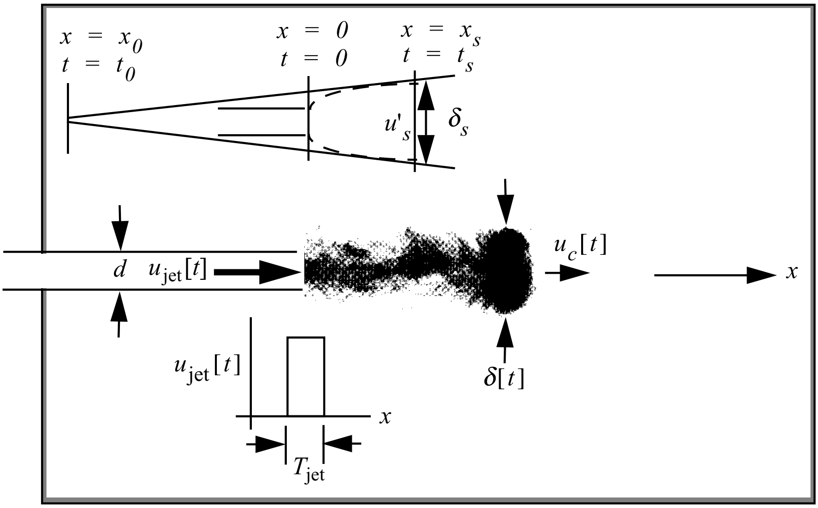



Fig. 13.3. Vortex-ring apparatus with experimental parameters. The sketch in the upper part of the figure defines parameters used to determine the effective origin of the ring.

low a kinematic viscosity as possible. However, this generally implies a high-density fluid and therefore more force required to produce the ring; only through proper analysis of scaling laws can we determine how these effects balance out.

Figure 13.3 shows a typical experimental setup. The vortex rings are produced by turning on and off the flow from an orifice of diameter $d$. The jet exit speed is $u_{\mathrm{jet}}[t]$, and the flow is turned on for a time $T_{\mathrm{jet}}$.

The vortex ring evolves according to its size $\delta[t]$ and convection speed $u_c[t]$. To study kinetic energy dissipation, it is necessary to carry out measurements of the fine scales. The requirements of the experiment are as follows:

(1) The experimental measurements should be capable of reaching values of $R_\lambda = 2000$, where

$$R_\lambda = \frac{u_0 \lambda}{\nu} \tag{13.56}$$

is the Reynolds number based on the Taylor microscale. This is an order of magnitude larger than the Taylor-microscale Reynolds number reached in the experiments of Glezer and Coles and will require an initial jet-tube Reynolds number two orders of magnitude larger than their value of 27,000. Both the characteristic velocity $u_0$ and the Taylor microscale are functions of time. Moreover, the Reynolds number is a decreasing function of time. This means that the measurements need to be made reasonably close to the jet exit, but not so close so as to be within the initial formation region of the ring.

(2) The experimental technique should be capable of spatially resolving the Kolmogorov microscale. The measurements will make use of standard optical diagnostic techniques such as laser Doppler anemometry, in which the measurement volume cannot typically be made smaller than about 50 $\mu$m in diameter. This effectively sets the lower limit for the size of the facility.

(3) The vortex ring must become fully developed within a reasonable distance from the jet exit so that the flow can be accessed by the diagnostic technique at the desired Reynolds number.

For the sake of the estimates to be carried out below, we will assume that the generation mechanism operates with a top-hat exit velocity profile and an ideal on–off characteristic. We will use group methods, the assumption of Reynolds-number invariance, and the data of Glezer and Coles [13.17] to estimate the following quantities:

- The local Reynolds number of the vortex ring needed to satisfy requirement (1).
- The jet exit Reynolds number needed to reach the required local ring Reynolds number at the point where the ring becomes fully developed and begins to follow a similarity law.
- The distance required for the vortex ring to become fully developed.

These estimates, together with the size limitations of the optics, will determine the tank size, orifice diameter, exit velocity, and fluid kinematic viscosity required to generate and study vortex rings at the desired Reynolds number.

### 13.5.1 Similarity Rules for the Turbulent Vortex Ring

Recalling the analysis of a general distribution of forces in an infinite fluid developed in Chapter 11, Section 11.5.1, we know that the hydrodynamic impulse is conserved for an idealized version of this flow. If the force is an impulse function in space and time located at the origin of coordinates, then

$$\frac{3}{2}\int_V u\,dx\,dy\,dz = \int_0^t \int_V \frac{I}{\rho}\delta[x]\delta[y]\delta[z]\delta[t]\,dx\,dy\,dz\,dt = \frac{I}{\rho}, \quad (13.57)$$

where $u$ is the velocity component in the axial direction. The hydrodynamic impulse is the total mechanical impulse generated by the applied force since the onset of the flow.

Experimentally characterizing this integral is somewhat problematic. In principle, the impulse integral requires a control volume that extends to infinity. Later we will invoke invariance of the problem under a dilation group, which carries an implied assumption that there are no length scales that limit the size of the domain or that characterize the generation of the force. In the experiment both of these requirements are violated. The tank is finite, and the force is generated by a finite-diameter jet tube. This greatly limits the size of the region where we should expect self-similar behavior to prevail since we have to be both far from the jet and at the same time far from the end wall of the tank.

Given this situation, it is legitimate to ask whether the impulse is truly constant and, if it is, whether the equality (13.57) holds. We know from the analysis of the impulse integral in Chapter 11, Section 11.5.1 that when the jet is turned on, a roughly cosine-shaped pressure distribution acts on the inside of the tank, with the maximum pressure disturbance occurring on the jet axis. Due to the no-slip condition on the tank surface, this pressure distribution will induce viscous shear layers at the wall. It is possible that these motions slightly modify the impulse, although this effect is probably below the uncertainty of the measurements carried out by Glezer and Coles who found that the impulse is conserved within experimental error.

Thus we should expect various length and velocity scales in the experiment to behave according to the similarity rules developed earlier:

$$\delta[t] \propto (I/\rho)^{1/4}(t-t_0)^{1/4}, \qquad u_0[t] \propto (I/\rho)^{1/4}(t-t_0)^{-3/4}, \quad (13.58)$$

where $t_0$ is a virtual origin in time and $\delta$ and $u_0$ are length and velocity scales that characterize the overall motion. Note that they can be *any* overall length or velocity scale. In Figure 13.3, $\delta$ is used to characterize the width of the ring, but it could just as easily be used for the distance the ring has traveled from the flow origin. The characteristic velocity is generally used to denote a measure of turbulent kinetic energy as in (13.8), but it could equally well denote the convection speed of the ring, $u_c$. Typically, turbulent fluctuation levels are on the order of a third of the mean velocity in a free shear flow, and so we would estimate $u' \propto 0.3u_c$ to hold (roughly).

The basic assumption is that, over some region of the tank, the overall motion is completely determined by just one parameter, $I/\rho$. All the other parameters of the problem – the tank length, the jet tube diameter, turbulent velocity fluctuations in the jet tube, residual motion of the tank fluid from previous ring firings, *and* the kinematic viscosity of the fluid – are ignored. The fact is that all could play a role in determining the ring growth and decay, and the data that exist today are too limited to determine how large this role may be. The transformation (13.29) can be used to convert the temporal similarity rules to spatial rules:

$$\delta \propto (x - x_0), \qquad U_0 \propto \frac{I}{\rho}(x - x_0)^{-3}, \tag{13.59}$$

where $x_0$ is the spatial origin of the flow. Note that angle of growth of the ring is independent of the impulse.

### 13.5.2 Particle Paths in the Turbulent Vortex Ring

Glezer and Coles painstakingly measured the ensemble mean velocity field of a turbulent vortex ring by using laser Doppler anemometry to sample two velocity components at several stations along the axis of the flow. By averaging velocity–time traces from roughly 30 realizations at each of several hundred measurement points they were able to reconstruct the self-similar streamlines and particle paths in a plane through the axis. Their main result is shown in Figure 13.4.

Ignoring all parameters except $I/\rho$, this flow should be invariant under the dilation group,

$$\tilde{x}^i = e^a x^i, \qquad \tilde{t} = e^{4a} t, \qquad \tilde{\bar{u}}^i = e^{-3a} \bar{u}^i,$$
$$\tilde{\tau}^{ij} = e^{-6a} \tau^{ij}, \qquad \tilde{\bar{p}} = e^{-6a} \bar{p}. \tag{13.60}$$

The similarity form derived from (13.60) used to approximate the data is

$$\frac{U^i}{(I/\rho)^{1/4}(t - t_0)^{-3/4}} = G\left[\frac{\boldsymbol{x} - \boldsymbol{x}_0}{(I/\rho)^{1/4}(t - t_0)^{1/4}}\right]. \tag{13.61}$$

The paper contains a good deal of evidence demonstrating that the data collapse well in these variables, confirming that a region does indeed exist where the flow evolves in an approximately self-similar fashion.

The coordinates in Figure 13.4 are

$$\xi = \frac{x - x_0}{(I/\rho)^{1/4}(t - t_0)^{1/4}}, \qquad \eta = \frac{y}{(I/\rho)^{1/4}(t - t_0)^{1/4}} \tag{13.62}$$

A perceptive reader will notice a flaw in the way Figure 13.4a is presented. We can understand the problem by first examining Figure 13.4b. This figure shows the phase portrait of particle paths defined by the vector field $\boldsymbol{U} - \boldsymbol{\xi}/4$, which we know from the discussion in Section 13.3.2 is invariant for all observers moving in proportion to $t^{1/4}$. The phase portrait is seen to consist of two on-axis saddles: one at the head of the ring located at $\xi_c = 25.21$, and one at the tail at $\xi_c = 24.85$. In addition, the ring rolls up fluid in an off-axis focus located at $(\xi_c, \eta_c) = (25.03, 0.213)$. This critical point is actually a critical line

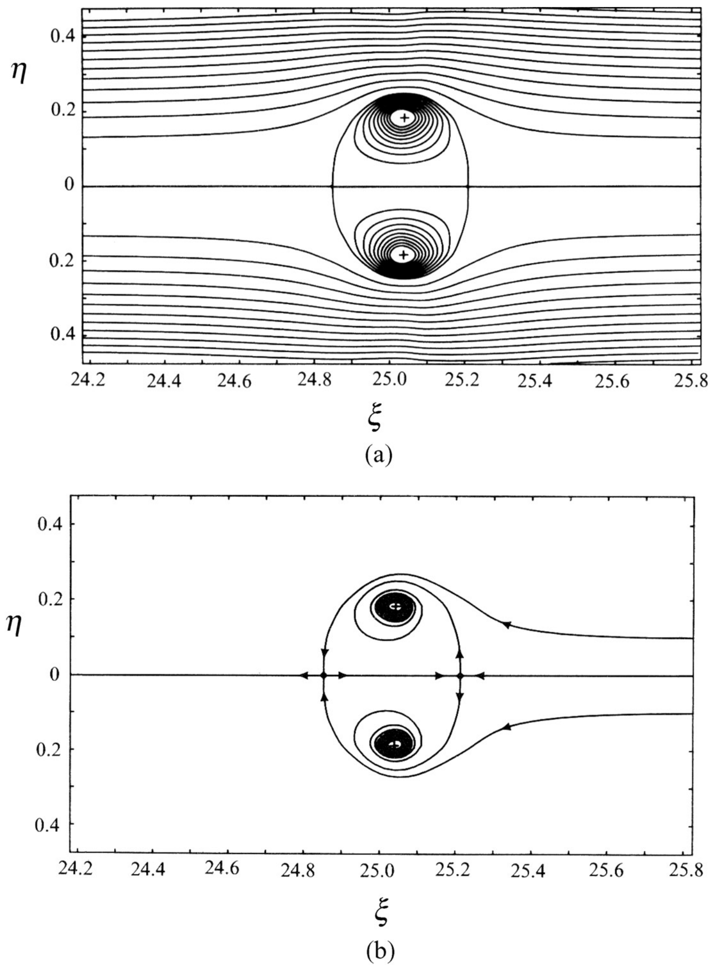



Fig. 13.4. Experimental results from [13.17]: (a) streamline pattern of the ensemble mean velocity field referred to an observer translating to the right with the ring, (b) particle paths of the ensemble mean velocity field.

of foci joined in an azimuthal circle surrounding the axis. The numbers on the horizontal axis in Figure 13.4b are referred to an observer fixed with respect to the laboratory frame, as are the numbers on the horizontal axis in Figure 13.4a. But the streamlines displayed in Figure 13.4a are with respect to an observer that moves with the focal point at $x - x_0 = 25.03(I/\rho)^{1/4}(t - t_0)^{1/4}$. Such an observer will see a closed center in the streamline pattern at $\tilde{\xi}_c = 0$. To be consistent, either the numbers on the horizontal axis in Figure 13.4a should have a zero under the streamline center, or the streamlines should be referred to the laboratory frame, in which case there will be no closed orbits at all, only a slight bump in the region of the vortex ring. This highlights the advantage of the phase portrait of particle paths for identifying flow structure – such ambiguities are completely avoided.

The critical points provide well defined velocity and length scales for analyzing the flow. Using the diameter and speed of the off-axis stable focus to define length and velocity scales, we have from the data in Figure 13.4

$$\begin{aligned} \delta[t] &= 0.426(I/\rho)^{1/4}(t-t_0)^{1/4}, \\ u_0[t] &= \frac{25.03}{4}(I/\rho)^{1/4}(t-t_0)^{-3/4} = 6.27(I/\rho)^{1/4}(t-t_0)^{-3/4}. \end{aligned} \tag{13.63}$$

The time evolution of the turbulent Reynolds number can be estimated as

$$R_\delta = \frac{u'\delta}{\nu} = 0.89\frac{(I/\rho)^{1/2}}{\nu(t-t_0)^{1/2}}, \tag{13.64}$$

where we have used $u' \propto u_0/3$ to estimate the magnitude of the turbulent fluctuations. The spatial evolution of the ring can be estimated in a similar manner. The streamwise position of the ring is $x - x_0 = 25.03(I/\rho)^{1/4}(t-t_0)^{1/4}$, which we can solve for the time in the motion of the ring as

$$t - t_0 = \frac{1}{I/\rho}\left(\frac{x-x_0}{25.03}\right)^4. \tag{13.65}$$

Now substitute (13.65) into (13.63) to get

$$\begin{aligned} \delta[t] &= 0.017(x-x_0), \\ u'[t] &= 3.32 \times 10^4(I/\rho)(x-x_0)^{-3}. \end{aligned} \tag{13.66}$$

The spatial dependence of the Reynolds number is

$$R_\delta = 562\frac{I/\rho}{\nu(x-x_0)^2}. \tag{13.67}$$

### 13.5.3 Estimates of Microscales

In Section 13.4 we used energy considerations to develop a series of relations that can be used to estimate the size of the microscale motions that contribute the bulk of kinetic energy dissipation. We can use results from the classical theory of homogeneous and isotropic turbulence to refine the estimates in (13.43). Using a Taylor microscale based on the curvature of the lateral correlation function [13.11], [13.13], the dissipation of TKE is given by

$$\varepsilon = 15\nu\left(\frac{u'}{\lambda}\right)^2. \tag{13.68}$$

The Reynolds number based on the Taylor microscale is now

$$R_\lambda = \sqrt{15 R_\delta}. \tag{13.69}$$

Our experiment is required to reach values of $R_\lambda = 2000$, which implies a large-scale value of $R_{\delta s} = 267{,}000$. The factor of 15 helps considerably in reaching our experimental goals by reducing the required large-scale Reynolds number $R_\delta$ needed to generate the design value of $R_\lambda$. This optimistic estimate seems to be justified in that recent direct simulations also suggest that the turbulent microscales are not as small as was once assumed.

### 13.5.4 Vortex-Ring Formation

Figure 13.3 shows the geometrical construction used to define the initial formation of a vortex ring. The quantities $t_s$ and $x_s$ are the time and position of the ring when the flow first begins to exhibit self-similar behavior after an initial period of formation. The quantities $\delta_s$ and $u'_s$ are the characteristic width and turbulence level at the same point. In the following, we will use physical reasoning, along with the data of Glezer and Coles [13.17], to estimate these parameters of the formation process.

The impulse generated by forcing a slug of fluid of length $L_{\text{jet}}$ over a period $T_{\text{jet}}$ through a jet orifice as shown in Figure 13.3 is given by

$$\frac{I}{\rho} = \frac{\pi}{4} d^2 u_{\text{jet}}^2 T_{\text{jet}} = \frac{\pi}{4} d^2 u_{\text{jet}} L_{\text{jet}}. \tag{13.70}$$

Substituting (13.70) into (13.64) leads to an expression for the Reynolds number in terms of the jet exit conditions:

$$\frac{u'\delta}{\nu} = 0.89 \left(\frac{\pi}{4}\right)^{1/2} \left(\frac{T_{\text{jet}}}{t - t_0}\right)^{1/2} \left(\frac{u_{\text{jet}} d}{\nu}\right) = 562 \left(\frac{\pi}{4}\right) \left(\frac{L_{\text{jet}} d}{(x - x_0)^2}\right) \left(\frac{u_{\text{jet}} d}{\nu}\right), \tag{13.71}$$

where (13.65) has been used. Equation (13.71) defines a consistency relation between the time and position of vortex-ring formation. This can be written as

$$(0.89)^2 \left(\frac{T_{\text{jet}}}{t - t_0}\right) = (562)^2 \left(\frac{\pi}{4}\right) \frac{(L_{\text{jet}} d)^2}{(x - x_0)^4}. \tag{13.72}$$

The on time of the exit flow, $T_{\text{jet}}$, in Glezer and Coles's experiment was 0.05 s, and the measured virtual origin in time was $t_0 = -0.44$ s. They were able to collapse the mean data beginning approximately 0.7 s after ring initiation. In other words, self-similarity of the mean field was reached at approximately, $t_s - t_0 = 1.14$ s. Using these data, the left-hand side of (13.72) is $(0.89)^2 T_{\text{jet}}/$

$(t_s - t_0) = 0.035$. Their experiments used a piston mechanism to drive a slug of fluid 6.52 cm long through an orifice 1.9 cm in diameter. It was found that the rings first began to follow a similarity law at $x_s = 35$ cm, that is, 5.4 slug lengths downstream of the orifice. The virtual origin of the ring in space was $x_0 = -145$ cm. Using this value, the right-hand side of (13.71) is 0.036. The close correspondence between these two numbers supports the consistency of the choice of virtual origin in space and time used by Glezer and Coles.

The diameter of the vortex ring at the time of formation is determined both by the jet tube diameter and by the length of the slug of fluid used to generate the ring. The ring diameter at the beginning of the self-similar zone is given by (13.66) as $\delta_s = 0.017(x_s - x_0)$. Let's construct a simple model of the formation process using the following assumptions:

- The position of ring formation scales linearly with the slug length with a constant of proportionality that is independent of Reynolds number at high Reynolds number. That is, let

$$x_s = AL_{\text{jet}}. \tag{13.73}$$

  The data of Glezer and Coles [13.17] suggest a value of $A = 5.4$.
- The virtual origin in space scales with the diameter of the jet tube, also with a constant of proportionality that is independent of the Reynolds number at high Reynolds number:

$$x_0 = Bd. \tag{13.74}$$

  The data suggest a value of $B = 77$.

Thus for high-Reynolds-number vortex rings we might expect $x_s - x_0 = 5.4L_{\text{jet}} + 77d$, which implies that the point where similarity behavior begins is characterized by

$$\begin{aligned}
\frac{T_{\text{jet}}}{t_s - t_0} &= 3.9 \times 10^5 \left(\frac{\pi}{4}\right)\left(\frac{L_{\text{jet}}}{d}\right)^2 \frac{1}{\left(5.4\frac{L_{\text{jet}}}{d} + 77\right)^4}, \\
\frac{\delta_s}{d} &= 0.017\left(5.4\frac{L_{\text{jet}}}{d} + 77\right), \\
\frac{u'_s}{u_{\text{jet}}} &= 3.32 \times 10^4 \left(\frac{\pi}{4}\right)\left(\left(\frac{L_{\text{jet}}}{d}\right)\frac{1}{\left(5.4\frac{L_{\text{jet}}}{d} + 77\right)^3}\right).
\end{aligned} \tag{13.75}$$

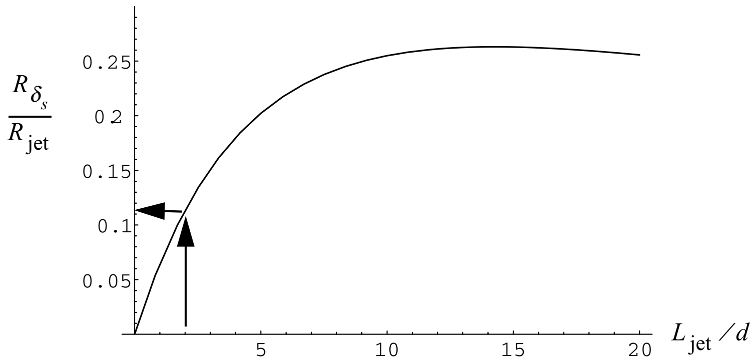



Fig. 13.5. Vortex-ring Reynolds number versus fluid slug length.

The Reynolds number of the ring at this point is

$$\frac{R_{\delta_s}}{R_{jet}} = \frac{u'_s \delta_s}{u_{jet} d} = 562\left(\frac{\pi}{4}\right)\left(\frac{L_{jet}}{d}\right)\frac{1}{\left(5.4\frac{L_{jet}}{d} + 77\right)^2}, \qquad (13.76)$$

where the jet Reynolds number is $R_{jet} = u_{jet} d/v$. Equation (13.76) is plotted in Figure 13.5.

The Reynolds number based on the Taylor microscale at $x_s$ is

$$R_{\delta_s} = R_{\lambda_s}^2/15, \qquad (13.77)$$

and the Kolmogorov microscale is

$$\eta_s/\delta_s = 15^{5/4} R_{\lambda_s}^{-3/2}. \qquad (13.78)$$

All of these relationships highlight the key role played by the ratio of slug length to jet diameter, $L_{jet}/d$.

### 13.5.5 Apparatus Design

The design process begins with the specification of $R_{\lambda_s} = 2000$ and $\eta_s = 50\ \mu$m. The apparatus is then defined through the following steps:

*Step 1*. Use (13.77) and (13.78) to determine $R_{\delta_s}$ and $\delta_s$.

*Step 2*. According to (13.76), a maximum in $R_{\delta_s}/R_{jet}$ occurs for $L_{jet}/d = 14.3$. However, this is likely to be too large for stable ring formation. A very long slug is likely to be unstable and form multiple rings that may collide and scatter in unpredictable directions. In addition, the longer the slug, the longer the tank required to contain the flow. Glezer and Coles used $L_{\mathrm{jet}}/d = 3.4$ and observed relatively large scatter in their vortex-ring trajectories. Axisymmetric jets are known to have a natural Strouhal number, $St = u_{\mathrm{jet}} T_{\mathrm{jet}}/d = 0.3$, corresponding to a slug length per vortex of $L_{\mathrm{jet}} = u_{\mathrm{jet}} T_{\mathrm{jet}} = 3.3d$, and so the observed scatter, is not suprising. In Figure 13.5 we have selected a value of $L_{\mathrm{jet}}/d = 2$ to promote stable ring formation. This generates $R_{\delta_s}/R_{\mathrm{jet}} = 0.11$; the jet exit Reynolds number must be almost an order of magnitude larger than the desired ring Reynolds number at the point where self-similar behavior begins.

*Step 3*. Given $L_{\mathrm{jet}}/d$ and $\delta_s$ from step 1, the required $d$ is determined from the middle relation in (13.75).

*Step 4*. Finally, $x_s/d$ is determined from (13.73). This is used to infer the minimum tank length required to contain the flow. The actual tank size is decided with some discretion with regard to the distance over which self-similar behavior is to be studied.

If the apparatus is designed around the specifications discussed earlier,

$$\begin{aligned} R_{\lambda_s} &= 2000, \\ \eta_s &= 50\ \mu\mathrm{m}, \\ L_{\mathrm{jet}}/d &= 2, \end{aligned} \tag{13.79}$$

then the steps defined above generate the following numbers:

$$\begin{aligned} \delta_s &= 15\ \mathrm{cm}, \\ R_{\mathrm{jet}} &= 2{,}340{,}000, \\ d &= 10\ \mathrm{cm}, \\ x_s &= 107\ \mathrm{cm}. \end{aligned} \tag{13.80}$$

If we assume that the self-similar region begins at the midpoint of the tank, then we require a tank at least 214 cm long.

#### 13.5.5.1 Choice of Working Fluid

Once the jet diameter and Reynolds number have been specified, the ratio $u_{\mathrm{jet}}/\nu$ is determined. A relationship that shows the dependence of this quantity on the specified parameters $R_{\lambda_s}$, $\eta_s$, and $L_{\mathrm{jet}}/d$ can be found by combining (13.76), (13.77), and (13.78):

$$\frac{u_{\mathrm{jet}}\eta_s}{\nu} = 0.76 \times 10^{-4} (R_{\lambda_s})^{1/2} \left( \frac{1}{L_{\mathrm{jet}}/d} \right) \left( 5.4 \frac{L_{\mathrm{jet}}}{d} + 77 \right)^3. \tag{13.81}$$

Recalling that $L_{\rm jet}/d$ is limited by stability considerations to values less than about 3 and noticing the relatively weak dependence on $R_{\lambda_s}$, we recognize that $u_{\rm jet}/\nu$ is essentially inversely proportional to $\eta_s$. The calculations carried out in the previous examples lead to a jet Reynolds number of about two million and jet diameter of 10 cm. The required ratio of exit velocity to kinematic viscosity is $u_{\rm jet}/\nu = 230{,}000\ {\rm cm}^{-1}$. Although a different choice of $R_{\lambda_s}$ or $L_{\rm jet}/d$ would modify this somewhat, it is clear from (13.81) that we do not have a whole lot of flexibility.

At this point we need to consider the working fluid. We will use water at 20°C with $\nu = 0.01\ {\rm cm}^2/{\rm s}$ and $\rho = 1\ {\rm g/cm}^3$. In this case the required jet exit velocity is $u_{\rm jet} = 2340$ cm/s which is approximately a factor of fifteen larger than the value used by Glezer and Coles. The piston stroke time is $T_{\rm jet} = (20/2342)$ sec. The required steady-state piston pressure is

$$J = \rho u_{\rm jet}^2 = 5.5 \times 10^6\ {\rm g/cm\text{-}s}^2 = 80\ {\rm psi}. \tag{13.82}$$

This estimate does not take into account the force required to accelerate the slug of fluid from rest. If we make the assumption that the piston speed must be reached in, say, 10% of the stroke time, then the required acceleration is

$$a = u_{\rm jet}/0.1T_{\rm jet} = 2{,}930{,}000\ {\rm cm/s}^2. \tag{13.83}$$

Neglecting the mass of the piston compared to the mass of the water being set into motion, then the required piston force is

$$F = ma = (\pi/4)\rho d^2 L_{\rm jet} a = 4.6 \times 10^4\ {\rm N} = 10{,}300\ {\rm lbf}, \tag{13.84}$$

corresponding to a piston pressure of 850 psi. Adding the estimated piston mass to these calculations is likely to raise the required pressure to over 1000 psi. Although the pressures involved are large and so is the tank, this design has a reasonable probability of success, although very thick glass or a free surface may be required to withstand the impulsive pressures required. This completes the design of our apparatus. By this time the reader should appreciate the key role of symmetry analysis as well as the need to deal impirically with the virtual origin in space and time where non self-similar behavior occurs.

## 13.6 The Geometry of Dissipating Fine-Scale Motion

In Section 13.2 the turbulence closure problem was briefly discussed. There it was pointed out that the problem of turbulence modeling is that of finding a set of independent equations relating the Reynolds stresses to the ensemble mean flow that can be included with the Reynolds-averaged equations to close the system. To date no universal turbulence model has been developed and the models that are used successfully rarely venture far from the experimental data base of flow cases used to select values for the model parameters. See Wilcox [13.18] for an authoritative treatment of the problem.

Many different kinds of models have been developed over the years and the successful ones all have one thing in common; they do a good job of modeling the transport of kinetic energy, particularly kinetic energy dissipation. For this reason, efforts to develop improved theories of turbulence tend to focus on the physics of turbulent fine scale motions. This has become even more true with the advent in recent years of large eddy simulation techniques (LES) where unsteady large scale motions are computed explicitly while fine scale motions below the spatial resolution of the computational grid must be modeled.

The discussion in Section 13.4 was concerned with estimating the size of dissipating fine-scale motions. We are in the rather odd position of being able to estimate the size of something for which we have no physical picture. It is clear from the discussion of similarity rules that, except for the case $k = \frac{1}{2}$, there are invariably important symmetry-breaking motions in all turbulent shear flows. These motions are characterized by instantaneous vorticity and rates of strain that, according to (13.41), are a factor $\sqrt{R_\delta}$ larger than that associated with the large-scale motion.

To complete the discussion of applications to turbulence, we will turn our attention to a physical picture of the geometry of small-scale motions [13.19], [13.20]. It needs to be stated from the outset that although we now know quite a bit about the local flow patterns associated with turbulent fine-scale motions, there is as yet no rigorous theory for the flow dynamics, and so we shall be necessarily dealing with highly simplified models. The main purpose of this section is to illustrate a further application of the techniques for analyzing 3-D vector fields developed in Chapter 3 and used to investigate the round jet in Chapter 11.

### 13.6.1 Transport Equation for the Velocity Gradient Tensor

In Chapter 11, Section 11.6.1, we developed the transport equation for the velocity gradient tensor from the gradient of the Navier–Stokes equations. The result was

$$\frac{Da^i_j}{Dt} + a^i_k a^k_j - \frac{1}{3}\left(a^m_n a^n_m\right)\delta^i_j = h^i_j, \tag{13.85}$$

where $a^i_j = \partial u^i / \partial x^j$ and

$$h^i_j = -\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x^i\,\partial x^j} - \frac{1}{3}\frac{\partial^2 p}{\partial x^k\,\partial x^k}\delta^i_j\right) + \nu\frac{\partial^2 a^i_j}{\partial x^k\,\partial x^k}. \tag{13.86}$$

If (13.85) is differentiated with respect to time, the result is

$$\frac{D^2 a^i_j}{Dt^2} + \frac{2}{3}Q[t]a^i_j = \frac{Dh^i_j}{Dt} - A^i_k h^k_j - h^i_k A^k_j + \frac{2}{3}\left(a^n_m h^m_n\right)\delta^i_j. \tag{13.87}$$

The first invariant, $P = -a^i_i$, is zero by incompressibility, and the second and third invariants are related to $a^i_j$ by

$$\begin{aligned} Q &= -\tfrac{1}{2}a^i_j a^j_i, \\ R &= -\tfrac{1}{3}a^i_k a^k_j a^j_i. \end{aligned} \tag{13.88}$$

Multiplying (13.85) by $a^p_i$ and then by $a^p_q a^q_i$ and taking the traces of the resulting equations leads to the evolution equations for the invariants (see Reference [13.19]). Thus

$$\frac{dQ}{dt} + 3R = -a^i_k h^k_i \tag{13.89}$$

and

$$\frac{dR}{dt} - \frac{2}{3}Q^2 = -a^i_n a^n_m h^m_i. \tag{13.90}$$

The tensor $h^i_j$ describes the effect of viscous diffusion and pressure forces on the evolution of the velocity gradient tensor; it is essentially the anisotropic part of the acceleration gradient tensor following a fluid particle, $\partial(Du^i/Dt)/\partial x^j$. We looked at the case $h^i_j = 0$ in Chapter 6, Example 6.10.

The character of the eigenvalues of $a^i_j$ is determined by the cubic discriminant,

$$D = Q^3 + \tfrac{27}{4}R^2. \tag{13.91}$$

Also of interest are the nonzero invariants of the rate-of-strain and rate-of-rotation tensors,

$$\begin{aligned} Q_s &= -\tfrac{1}{2}s^i_j s^j_i, \\ R_s &= -\tfrac{1}{3}s^i_k s^k_j s^j_i, \\ Q_w &= -\tfrac{1}{2}w^i_j w^j_i = \omega^i \omega^i, \end{aligned} \tag{13.92}$$

where $a^i_j = s^i_j + w^i_j$. These invariants can be used to investigate the relative importance of strain and vorticity in determining the structure of the flow. They are related to one another through

$$\begin{aligned} Q &= Q_s + Q_w, \\ R &= R_s - w^i_k w^k_j s^j_i. \end{aligned} \tag{13.93}$$

### 13.6.1.1 Simulations of Turbulence

The recognition that the complicated and chaotic fluid motion of turbulence could be simulated on a digital computer originated over forty years ago, and the challenges faced by early pioneering computations are well described in the 1976 review by Reynolds [13.22]. The field has progressed continuously since then, but until the advent of digital computers powerful enough to enable the Navier–Stokes equations to be integrated for moderate values of the Reynolds number, very little was known about the fine-scale motions that exist in a turbulent flow. This can be appreciated by considering the extremely small lengths involved for a typical laboratory-size flow (for example, a few tens of microns in the vortex-ring experiment described in Section 13.5.3). Furthermore, it is necessary to measure all nine components of the velocity gradient field, a challenging task at any scale.

The approach used here to analyze turbulent fine-scale structure is motivated by the example of transition in the round jet described in Chapter 11, Section 11.5, where the formation of a starting vortex was determined by the Reynolds-number dependence of the phase portrait of particle paths. There the onset of a starting vortex was defined by a single critical point that changed from a stable node to a stable focus as the eigenvalues of the local vector field change from real to complex with increasing Reynolds number. The nature of the eigenvalues was determined by the sign of the discriminant $D$: real if $D < 0$, and complex if $D > 0$. The logical generalization of this approach to turbulence is to simply evaluate the gradient tensor everywhere at each instant in time.

Figure 13.6 shows three images of a plane mixing layer. The top image, provided mainly for reference, is a small segment of the high-Reynolds-number, spatially developing flow studied by Brown and Roshko and shown in Figure 13.1. The bottom two images are taken from a direct numerical simulation of a time-developing mixing layer computed by Moser and Rogers [13.7] at a Reynolds number of 3000 based on the vorticity thickness and velocity difference across the layer. See Chen et al. [13.25]. The time-developing case enables the numerics to be simplified by allowing the computation to be carried out in a box with periodic boundary conditions. Crudely speaking, it is intended to

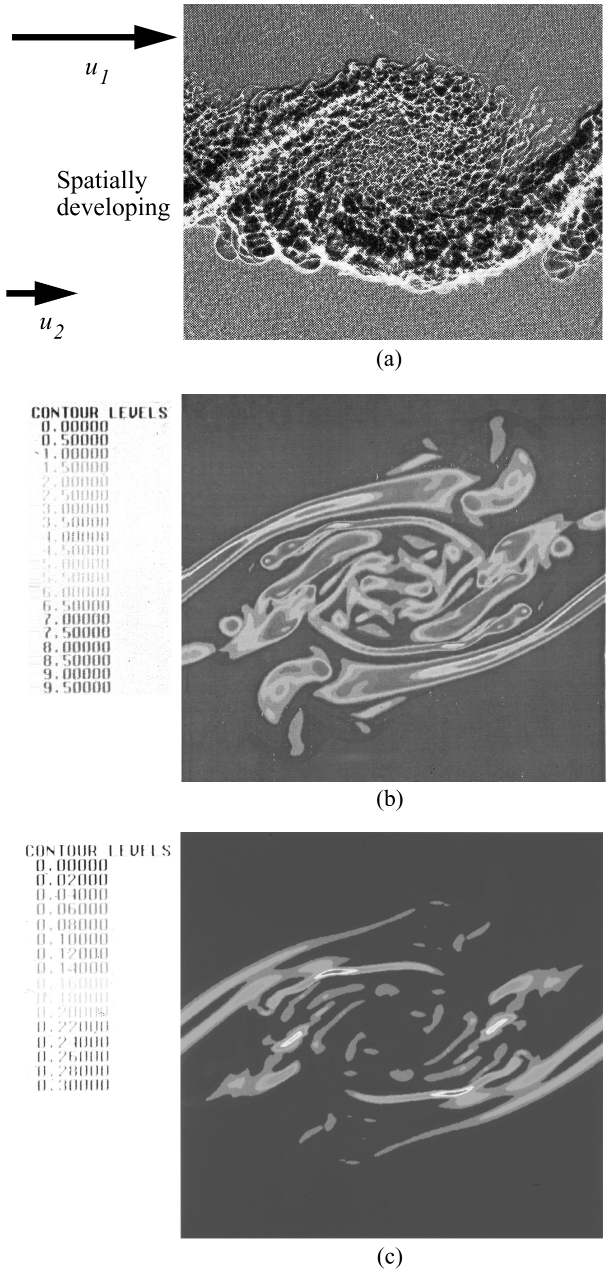



Fig. 13.6. Several views of the plane mixing layer: (a) shadowgraph, (b) vorticity magnitude, (c) dissipation of kinetic energy. (See color plate 1.)

represent the mixing layer as seen by an observer that translates at the average velocity of the upper and lower streams, although the analogy is not exact. The two pictures illustrate the level of detail generated by the simulation. They also illustrate some of the principles we have been discussing above. The middle image shows contours of the vorticity magnitude, while the bottom one shows the dissipation of kinetic energy. In both images these quantities are quite spiky, with large localized peaks separated from one another by a relatively low-level background. These pictures are consistent with the intermittent nature of the fine scales described earlier.

Figure 13.7 from Reference [13.25] is completely different from the others. At every point in the computational domain the invariants have been evaluated; then each point is assigned a color depending on where the $(Q, R)$ coordinates of the point fall in Figure 13.8. The result is a figure that depicts the balance between rotation and strain in the physical space of the flow. The result of this process, applied on a computational grid, is an ensemble that can be analyzed statistically. Typically, the total population of the ensemble is several million samples, depending on the size of the computational grid. The invariants are then cross-plotted in the form of joint probability density functions (pdfs). The

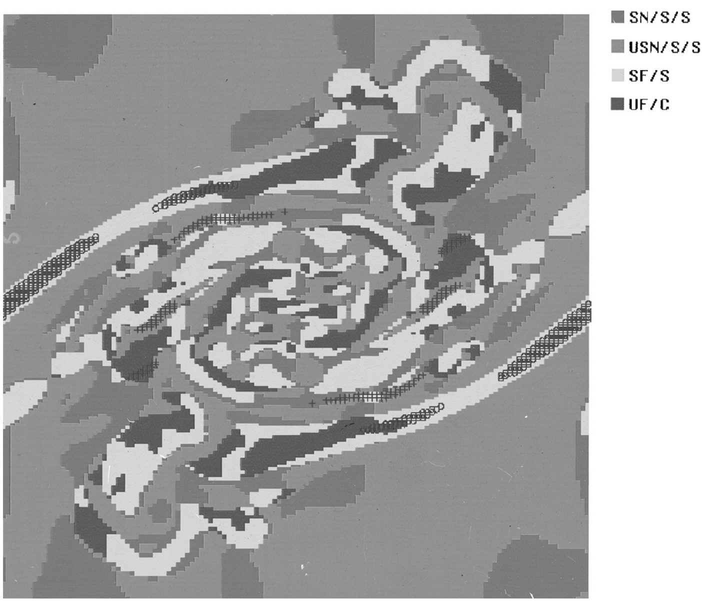



Fig. 13.7. Mixing-layer computation visualized in terms of local flow topology from Reference [13.25]. Colors represent different local flow patterns: red, stable-node–saddle–saddle; green, unstable-node–saddle–saddle; yellow, stable focus, stretching; blue, unstable focus, compressing. See Figure 13.8. (See color plate 2.)

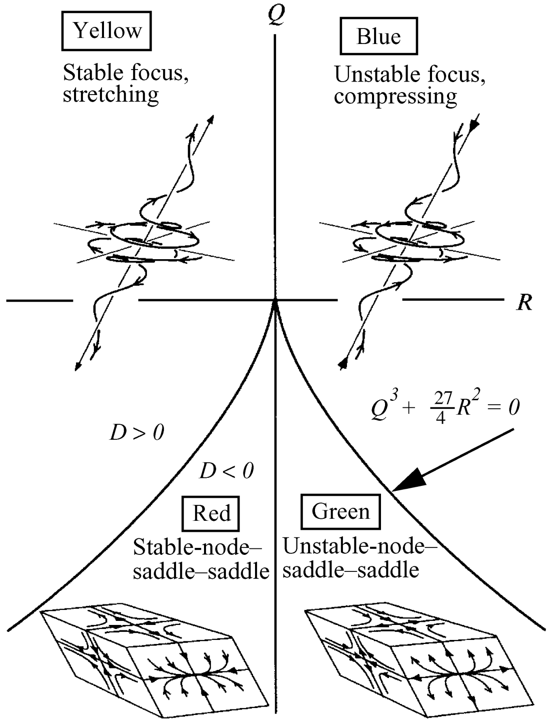



Fig. 13.8. Local flow patterns defined by $Q$ and $R$ with $P = 0$ from [13.21]. Colors refer to Figure 13.7.

velocity gradient field is independent of a nonuniformly translating observer (as are the equations of motion) as long as the observer is not rotating. Therefore this method enjoys the important property that structural features identified in the pdf space of gradient tensor invariants are intrinsic to the flow and not the result of a particular choice of the frame of reference used to view the flow. It should be pointed out that this method is not the only way to look at the geometry of flow structure, and an interesting alternative approach based on a symmetric product of tensors is described in [13.23] and [13.24].

Thus the flow is broadly visualized in physical space and concisely visualized in the space of gradient tensor invariants. The connection between the two views is through the relationship between the invariants and local flow patterns depicted in Figure 13.8, from Soria et al. [13.21].

With this background we can understand the probability density functions (pdfs) in Figure 13.9, from Soria et al. [13.21]. These pdfs were generated from data at one instant in time in the plane-mixing-layer simulation by Moser and Rogers [13.7]. The flow geometry in the simulation is essentially the same as

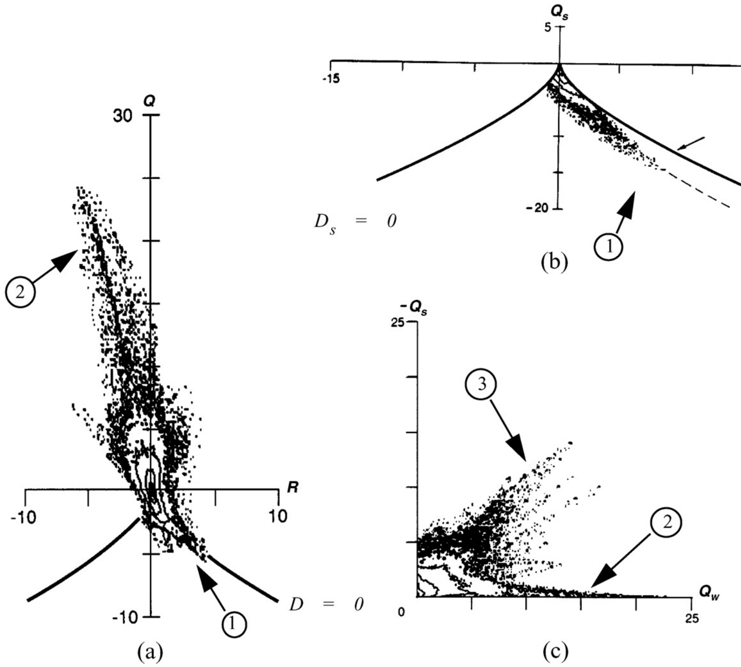



Fig. 13.9. Unnormalized probability density functions of the invariants: (a) velocity gradient tensor $Q$ versus $R$, (b) rate-of-strain tensor $Q_s$ versus $R_s$, (c) rate-of-strain and rate-of-rotation tensors $-Q_s$ versus $Q_w$. Contour levels represent the logarithm of the number of samples.

that depicted in Figure 13.1, except as noted earlier the Reynolds number is somewhat lower and the simulated mixing layer is developing in time rather than space. This permits the flow to be treated as periodic in space, enabling the use of highly accurate spectral methods that require periodic boundary conditions. The pdfs in Figure 13.9 contain a great deal of information concerning the geometry of the fine scales in that simulation. Points near the origin correspond to low gradient values associated with the large-scale motions; points far away characterize the high-gradient fine scales. In the present context we will focus mainly on points distant from the origin. In fact, however, the vast majority of samples pile up near the origin however, and the only way to bring out the fine scales is to plot contours of the log of the sample population.

Several interesting features are identified by circled numbers in Figure 13.6, Figure 13.7, and Figure 13.9:

- There is a general tendency for the $(Q, R)$ pdf to develop a roughly elliptical shape with the major axis of the ellipse aligned with the upper left and lower right quadrants. In fact, the shape is really more like an inclined teardrop with a cusp lying along the $R > 0,\ D = 0$, branch as indicated by the ① in Figure 13.9a. This tendency is seen particularly in the pdf of $(Q_s,\ R_s)$ in Figure 13.9b. When we compare this pdf with Figure 13.8, we can see that the strongest energy-dissipating motions in the flow have a saddle–saddle–unstable-node geometry. This implies that, in a region of high dissipation, the eigenvalues of the rate-of-strain tensor are ordered according to $\alpha > \beta > 0 > \gamma$.

- Occasionally, depending on initial conditions and Reynolds number, a small fraction of the data extend a considerable distance into the upper left quadrant, where the local topology is stable focus stretching. See the ② in Figure 13.9a. For points far from the origin, the local vorticity dominates the rate of strain. Moreover, points far from the origin are associated with very low rates of kinetic energy dissipation, as seen in Figure 13.9c. This suggests that the structure is likely to be quite long-lived. The presence or absence of these kinds of structures is closely related to the regularity of the initial conditions. Such structures are much less prominent in a flow with randomized initial conditions.
- What is the vorticity field like at a point of high dissipation? This is indicated by the ③ in Figure 13.9c. The indication is that points of high dissipation are characterized by high levels of vorticity, $Q_w \propto -Q_s$, although there is a fairly broad distribution about a 45° line in Figure 13.9c. Note that the converse is not so. Points of high vorticity are not necessarily associated with high rates of dissipation.
- At any point one can construct a locally orthogonal system of coordinates from the eigenvectors of the rate-of-strain tensor. When the vorticity vector is located relative to this system, one finds, in a region of high kinetic energy dissipation, a strong tendency for the vorticity to be aligned with the direction of the smaller of the two rate-of-strain eigenvalues (the $\beta$-direction).

#### *13.6.1.2 A Simple Model of the Geometry of Turbulent Fine Scales*

To begin to get an understanding of these results, it is instructive to look at the solution of the restricted Euler version of (13.85) corresponding to the homogeneous case, $h^i_j = 0$. In this model, (13.85) becomes a set of quadratically coupled, nonlinear ODEs for the nine components of the velocity gradient tensor:

$$\frac{da^i_j}{dt} + a^i_k a^k_j - \frac{1}{3}\left(a^m_n a^n_m\right)\delta^i_j = 0. \tag{13.94}$$

Equation (13.94) is a matrix Ricatti equation. The equivalent system, setting $h^i_j = 0$ in (13.87), (13.89), and (13.90), is

$$\begin{aligned} \frac{dQ}{dt} + 3R &= 0, \\ \frac{dR}{dt} - \frac{2}{3}Q^2 &= 0, \\ \frac{d^2 a^i_j}{dt^2} + \frac{2}{3}Q[t]a^i_j &= 0. \end{aligned} \tag{13.95}$$

The equations for the invariants can be combined to give

$$3Q^2\left(\frac{dQ}{dt} + 3R\right) + \frac{27}{2}R\left(\frac{dR}{dt} - \frac{2}{3}Q^2\right) = \frac{d}{dt}\left(Q^3 + \frac{27}{4}R^2\right) = 0. \tag{13.96}$$

The cubic discriminant (13.91) is conserved for all particles in this model. With this integral of the motion known, the time evolution of the two invariants can be determined. Note that all the nonlinearity is in the first two relations in (13.95), which we solved exactly in terms of elliptic functions in Chapter 6, Example 6.10. Once $Q[t]$ is known, the individual components of $a^i_j$ can be determined exactly by solving the linear second-order ODE in (13.95). The full solution is presented in Reference [13.19]. Physically, Equation (13.94) represents the "free" evolution of a fluid particle evolving in the absence of the pressure and viscous stresses normally applied by neighboring fluid particles.

### *13.6.1.3 Asymptotic Behavior*

For any initial condition, the solution of (13.94) evolves to

$$A^i_j = \frac{2^{1/3}}{\tau_{\text{singular}} - \tau} K^i_j \qquad \tau < \tau_{\text{singular}}, \tag{13.97}$$

where the velocity gradient is normalized by the value of the discriminant as determined from the initial values of the invariants ($R_{\text{intial}}$, $Q_{\text{initial}}$):

$$A^i_j = \frac{a^i_j}{D^{1/6}}, \tag{13.98}$$

and where $K^i_j$ satisfies the following matrix equation:

$$K^i_k K^k_j + \frac{1}{2^{1/3}} K^i_j - 2^{1/3}\delta^i_j = 0. \tag{13.99}$$

This is not the first time we have seen a relation like (13.97). See Equation (13.44), which describes the temporal decay of large eddies. The main and crucial difference is that, instead of decaying, (13.97) becomes singular in finite time. Taking the trace of (13.99), then multiplying through by $K_l^j$, and taking the trace again leads to

$$(R_K, Q_K) = \left(1, -3/2^{2/3}\right). \tag{13.100}$$

The invariants plus the relations (13.92) and (13.93) can be used to generate relations for the asymptotic strain and rotation invariants,

$$\begin{aligned} R_{SK} + \frac{1}{2^{1/3}} Q_{SK} + \frac{1}{2} &= 0, \\ Q_{SK} + Q_{WK} &= -\frac{3}{2^{2/3}}, \\ R_{SK} - \frac{1}{2^{1/3}} Q_{WK} &= 1. \end{aligned} \tag{13.101}$$

Note that $Q_{WK} \geq 0$. All initial conditions on $a_j^i$ evolve to a particular strain and rotation state characterized by (13.100) and (13.101).

This simple model has been applied to an ensemble of particles with random initial values of the velocity gradient tensor by Cheng [13.26], and the results are shown in Figure 13.10. The resulting pdfs of the evolved ensemble in Figure 13.10 reproduce virtually all of the significant geometrical features observed in direct numerical simulations and typified by the pdfs in Figure 13.9, including the alignment of the vorticity vector with the principal strain direction of the smaller of the two positive eigenvalues, $\beta$. The only feature that is not reproduced by the model is the occasional occurrence of long-lived stretched streamwise vortices indicated by the label ② in Figure 13.6, Figure 13.7, and Figure 13.9, although vortex stretching is a general feature of (13.94).

Just one small problem: the solution of (13.94) blows up! Luckily it is easy to remove the singularity and to use this model to reproduce the time decay of the gradients of the fine scales derived from energy considerations in (13.48).

Additionally: (13.99) can be rearranged to read

$$\left(\frac{2^{4/3}}{3} K_k^i + \frac{1}{3}\delta_k^i\right)\left(\frac{2^{4/3}}{3} K_k^i + \frac{1}{3}\delta_k^i\right) = \delta_j^i. \tag{13.102}$$

The matrix $(2^{4/3}/3)K_k^i + \frac{1}{3}\delta_k^i$ is its own inverse. Such matrices are called *involutory*. There are an infinite number of such matrices. Indeed, any initial condition on $a_j^i$, when evolved to its asymptotic state through the solution of (13.94), produces an involutory matrix.

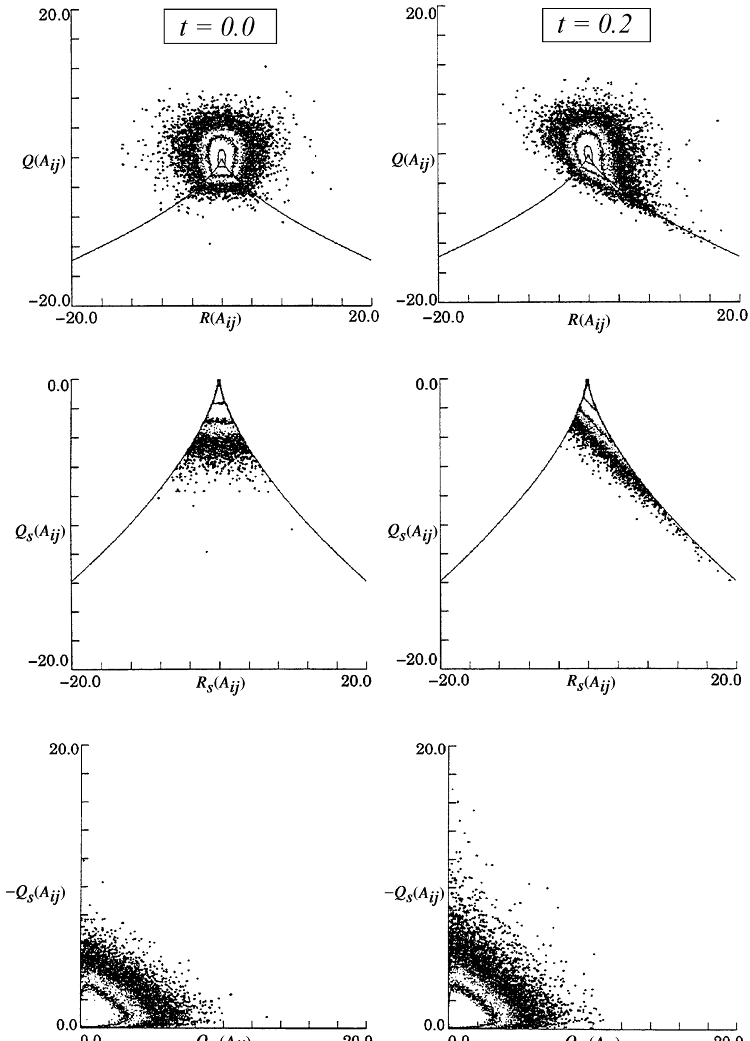



Fig. 13.10. Initially random pdfs of the velocity gradient tensor evolving according to the restricted Euler equation (13.94).

### 13.6.1.4 Nonzero $h^i_j$

Following [13.20], we can get a somewhat different view of the geometry of the fine scales motivated by the form (13.97) of the asymptotic solution of the restricted Euler problem. Now let $h^i_j \neq 0$. We hypothesize the existence of an intermediate state in the evolution of a fluid particle. Assume that some time after a fluid element is set into motion by the flow, during which various components of $a^i_j$ may change at different rates, the particle settles into a state where

$$\frac{da^i_j}{dt} = a^i_j f[t]. \tag{13.103}$$

Under this model, the transport equation for the velocity gradient tensor becomes an algebraic relation between $a^i_j$ and $h^i_j$,

$$f a^i_j + a^i_k a^k_j - \tfrac{1}{3}\left(a^m_n a^n_m\right)\delta^i_j = h^i_j. \tag{13.104}$$

We have seen (13.104) before in Chapter 11, Section 11.6.3, where we considered the acceleration field in the neighborhood of critical points in the round jet. See Equation (11.136). The similarity behavior of the jet gave $f = -1$, and (13.104) was expressed in similarity coordinates. In fact, the model (13.103) presumes that as the flow evolves, fluid particles tend to settle temporarily into critical-point-like regions of the flow. The invariants of $h^i_j$ are formed by squaring and cubing (13.104) and taking traces:

$$\begin{aligned} Q_h &= f^2 Q + 3 f R - \tfrac{1}{3} Q^2, \\ R_h &= f^3 R - f Q R - \tfrac{2}{3} f^2 Q^2 - \tfrac{2}{27} Q^3 - R^2, \end{aligned} \tag{13.105}$$

where

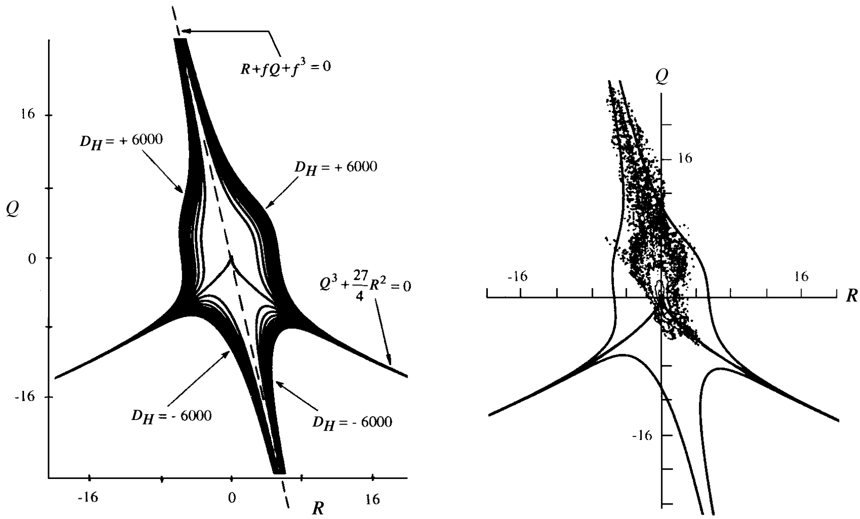
$$Q_h = -\tfrac{1}{2} h^i_j h^j_i, \qquad R_h = -\tfrac{1}{3} h^i_k h^k_j h^j_i. \tag{13.106}$$

The discriminants are related by

$$Q_h^3 + \tfrac{27}{4} R_h^2 = \left(Q^3 + \tfrac{27}{4} R^2\right)(R + f Q + f^3)^2. \tag{13.107}$$

Lines of constant $Q_h^3 + \frac{27}{4} R_h^2$ for $f = -0.2$ are shown in Figure 13.11. This result is the generalization of the relationship between 3-D flow patterns and elliptic curves discussed in Chapter 11 Section 11.6.

We can see that when $h^i_j \neq 0$, a new structure appears in the space of tensor invariants. While in principle $Q$ and $R$ can range over the whole space, very large values of $D_h$ would be required to move outside the extremely steep-sided surface depicted in Figure 13.11a. In effect this surface defines a region of attraction in $(Q, R)$ space. In general, the observations from direct numerical simulations support the conclusion that very large values of $D_h$ probably do not occur. In this model, singular behavior is no longer a necessary property of the solution as in the restricted Euler case.




(a) (b)

Fig. 13.11. Contours of constant $D_h = Q_h^3 + \frac{27}{4} R_h^2$ in ($Q$, $R$) coordinates. (a) See Equation (13.107); note the steepness of the surface near, $D = 0$, (b) Contours superimposed on the data in Figure 13.9a. From [13.20].

Finally, it should be pointed out that these observations of the geometrical properties of turbulent fine-scale structure are not restricted to plane mixing layers. A wide variety of flows have been studied in this fashion [13.7], [13.25], [13.26], including wall-bounded flows [13.27], [13.28]. The general features of the fine scales are essentially the same as those discussed above except very near the wall, where the turbulent fluctuations approach zero [13.28]. This is consistent with Kolmogorov's original assumption and the commonly accepted notion that the behavior of kinetic-energy-dissipating motions is universal and mostly independent of the large-scale structure except for the basic scaling laws (13.46) and (13.47), which depend on the governing parameter $M$.

## 13.7 Concluding Remarks

A great deal of research effort has been spent trying to identify the far-field asymptotic growth and decay rates of elementary free shear flows, including mixing layers, wakes, and jets, and the data tend to have a lot of scatter. Although there is considerable debate concerning whether asymptotic growth rates are unique for a given flow, it is now well recognized that, over the range of scales available in the laboratory, the growth rates of free shear flows are affected by many factors. Mixing-layer growth rates are known to depend on whether the initial splitter-plate boundary layers are laminar or turbulent. Wakes are strongly influenced by initial conditions related to boundary-layer transition, both on the body that creates the wake, and in the separating free shear layers, which feed vorticity of alternating sign into the wake, [13.4]. In addition, low-level free-stream turbulence and acoustic waves can affect the development of a turbulent flow, primarily through the interaction of such disturbances with the initial development region of the flow.

When speaking about growth and decay laws for turbulent shear flows it is important to distinguish between the rate constants, which are subject to all the complexities just described, and the power laws that multiply those constants. In geometrically simple flows, the latter can usually be derived by a two-step procedure. The first step is to integrate the momentum over the volume of the flow, including all forces responsible for its creation. This leads to the identification of a conserved quantity $M$, which governs global conservation of momentum. The second step is to invoke Reynolds-number invariance and make use of the group (13.13). This assumes that the exponent is determined only by the global parameter of the motion, independent of the viscosity. The many parameters that would be required to fully describe the flow are assumed to have their effect only in the rate constants. While this is a reasonable approximation to the available data, it is an open scientific question whether this separation of effects is valid. At the present time we do not have an adequate theoretical understanding of turbulent flow, and so we lack the analytical tools to answer this question. Perhaps Lie theory will eventually show the way.

## 13.8 Exercises

13.1 Show that a jet produced by a force that increases in proportion to the fourth power of the time will produce a large eddy motion with length scales that evolve in proportion to $t^{3/2}$.

13.2 Determine the one-parameter dilation group that leaves invariant a steady plane jet generated by flow from a narrow slit. The conserved integral is

$$M = \int_{-\infty}^{\infty} u^2 \, dy. \tag{13.108}$$

Compare the group invariance of the laminar and turbulent cases. Can either or both be treated using a boundary-layer approximation? Work out the momentum balance of the plane turbulent jet carefully, keeping both turbulent normal stress terms and pressure terms in the integral (13.108). Based on the argument of Reynolds-number invariance, show that the integral remains invariant under the same group. Do the same for the vortex ring with integral invariant (13.17). See Chapter 11, Exercise 11.2.

13.3 Consider the entrainment velocity $v_e$ induced by a plane turbulent jet as shown in Figure 13.12. The area flux of the jet is $Q = \int u\,dy$, and the entrainment velocity is $v_e = dQ/dx$. How does each of these quantities depend on $x$? How would you expect the volume flux and entrainment velocity of an axisymmetric jet to depend on $x$?

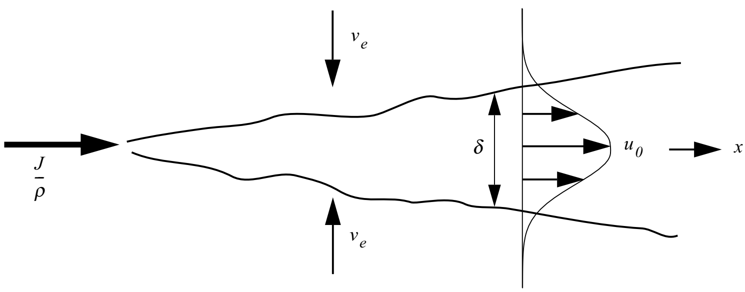



Fig. 13.12.

13.4 Flow past a flat plate of length $L$ is shown in Figure 13.13. Assume an attached laminar Blasius boundary layer over the length of the plate. Show that the drag per unit span of the plate is proportional to $U_\infty^{3/2} L^{1/2}$. How would you expect the turbulence intensity $u'$ to depend on $U_\infty$ and $L$ at a fixed point $x$ in the far wake?

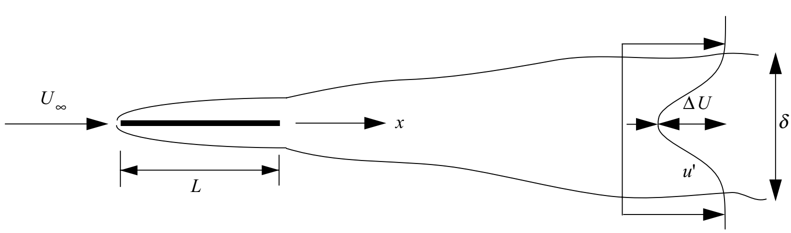



Fig. 13.13.

13.5 Solve the turbulent counterpart of Exercise 13.4. Assume an attached turbulent boundary layer over the length of the plate. The local skin friction coefficient can be taken as $C_f = 0.06(U_\infty x/\nu)^{-1/5}$. How would you expect the turbulence intensity $u'$ to depend on $U_\infty$ and $L$ at a fixed point $x$ in the far wake?

13.6 An axisymmetric buoyant jet is produced by a heated source of momentum. How would you expect the centerline velocity of the jet to vary with distance from the source, in the near field (but away from the source) where the momentum flux dominates the flow, and in the far field where the buoyancy flux dominates?

13.7 Use a control-volume balance to show that the drag and lift of a 3-D wing can be related to appropriate integrals in the downstream wake. Assume the airfoil is in an infinite stream. (See Figure 13.14) suppose a commercial aircraft flies straight and level overhead, leaving behind a downward-drifting, turbulent trailing vortex pair such as that indicated in the Trefftz plane $a$–$a$. The downward momentum of the vortex pair exactly balances the lift on the aircraft. An optical measuring system on the ground is designed to measure the magnitude of turbulent fluctuations in the wake. How would you use such a system to measure the weight of the aircraft? Estimate the downward drift speed $V_d$ of the vortex pair in terms of the aircraft weight, and then develop scaling laws for the behavior of $V_d$ in the far wake. How does the flow Reynolds number vary with distance behind the aircraft? Show that the wake eventually relaminarizes. Neglect all effects associated with the stratification of the atmosphere. Suppose the relaminarization process is to be studied in a long laboratory water channel, using a model that is moved along the channel in a sled. For some reasonable initial wake Reynolds number, say 10,000, how long would a laboratory observer have to wait until the wake decayed to a Reynolds number of 100?

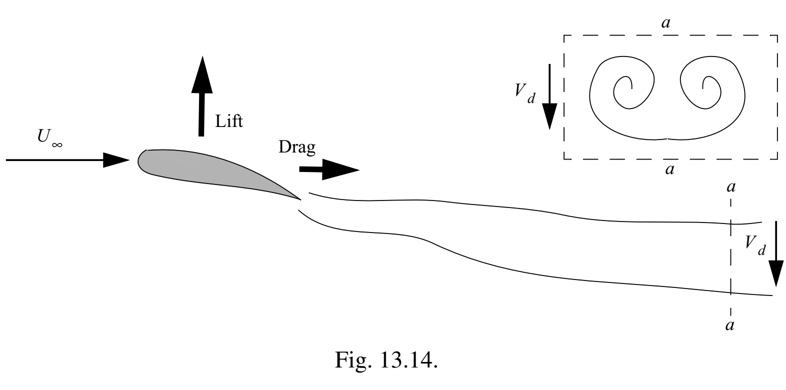



Fig. 13.14.

13.8 Revisit the design of a vortex-ring apparatus. Choose the same parameters as used in Section 13.5.5, but with better instrumentation: let the resolvable Kolmogorov microscale now be reduced by a factor of ten, so that $R_{\lambda_s} = 2000$, $\eta_s = 5\mu$m, and $L_j/d = 2$. Determine the tank size required.

13.9 Use dimensional analysis to estimate the drift speed of a vortex pair and a vortex ring during the late stages of laminar decay. Estimate the drift distance for the asymptotic state. Suppose the vortex-ring apparatus designed in Section 13.5.5 is required to be long enough to permit the ring to be observed all the way to its limiting drift distance. Using the available data, estimate how long the tank should be. See References [13.30], [13.31], [13.32], and [13.33].

13.10 An exact solution for an axisymmetric laminar line vortex is given by

$$u = (4\nu)^{1/2} t^{-1/2} U[\xi, \eta], \qquad v = (4\nu)^{1/2} t^{-1/2} V[\xi, \eta], \tag{13.109}$$

where

$$U = -\left(\frac{\Gamma}{8\pi\nu}\right)\frac{\eta\left(1 - e^{-(\xi^2+\eta^2)}\right)}{\xi^2+\eta^2}, \qquad V = \left(\frac{\Gamma}{8\pi\nu}\right)\frac{\xi\left(1 - e^{-(\xi^2+\eta^2)}\right)}{\xi^2+\eta^2}, \tag{13.110}$$

and the similarity variables are

$$\xi = \frac{x}{(4\nu t)^{1/2}}, \qquad \eta = \frac{y}{(4\nu t)^{1/2}}. \tag{13.111}$$

The constant $\Gamma$ with units $L^2/T$ is the circulation of the vortex, and the combination $\Gamma/(8\pi\nu)$ can be thought of as a Reynolds number. Consider the equations for unsteady particle paths,

$$\frac{dx}{dt} = u[x, y, t], \qquad \frac{dy}{dt} = v[x, y, t]. \tag{13.112}$$

Recast these equations in terms of similarity variables, and show that they reduce to an autonomous system. Identify the critical point at the origin. Sketch the phase portrait of particle paths, paying attention to the flow at large distances from the origin as well as near the critical point. Show that the second invariant at the critical point is related to the Reynolds number by $Q = \Gamma^2/(8\pi\nu)^2 + 1/4$.

13.11 In a footnote to one of his most famous papers, J. M. Burgers [13.29] wrote down an exact solution for a steady stretched vortex. In cylindrical coordinates,

$$u_r = -Ur, \qquad u_\theta = u[r,t], \qquad u_z = 2Uz, \tag{13.113}$$

where $U$ is a constant and $u(r,t)$ satisfies

$$\omega = \frac{A\Gamma}{2\pi\nu} e^{-Ar^2/2\nu} = \frac{1}{r}\frac{\partial(ru)}{\partial r}. \tag{13.114}$$

The pressure is

$$\frac{p}{\rho} = \int \frac{u^2}{r} dr - \frac{1}{2}U^2(r^2 + 4z^2). \tag{13.115}$$

(1) How are $U$ and $A$ related?
(2) Define an appropriate Reynolds number for the flow.
(3) Work out the invariants of both $a^i_j$ and $h^i_j$, and cross-plot the results in the $(Q, R)$ and $(Q_h, R_h)$ planes. Describe how the invariants change as the Reynolds number is varied.
(4) Show that the dissipation of kinetic energy is independent of $\nu$.
(5) Plot the equivalent of Figure 13.11a for this flow. Choose the appropriate value of $F$.

13.12 Prove that any matrix that satisfies (13.99),

$$K^i_k K^k_j + \frac{1}{2^{1/3}} K^i_j - 2^{1/3}\delta^i_j = 0, \tag{13.116}$$

when broken into a symmetric and antisymmetric part,

$$K^i_j = S^i_j + W^i_j, \tag{13.117}$$

has the property that the eigenvalues of $S^i_j$ are ordered so that $\alpha > \beta > 0 > \gamma$ and that the vorticity vector derived from $W^i_j$ is exactly aligned with the principal rate-of-strain direction associated with the smaller of the two positive rate-of-strain eigenvalues, $\beta$.

13.13 Use the package **`IntroToSymmetry.m`** to work out the Lie algebra of the restricted Euler equation,

$$\frac{Da^i_j}{Dt} + a^i_k a^k_j - \frac{1}{3}\left(a^m_n a^n_m\right)\delta^i_j = 0, \tag{13.118}$$

where, $a^i_i = 0$. Compare your result with the symmetries of the equivalent system

$$\frac{dQ}{dt} + 3R = 0,$$
$$\frac{dR}{dt} - \frac{2}{3}Q^2 = 0, \quad (13.119)$$
$$\frac{d^2 a^i_j}{dt^2} + \frac{2}{3}Q[t]a^i_j = 0.$$

13.14 Use the package **`IntroToSymmetry.m`** to work out the Lie algebra of the full velocity-gradient tensor transport equation

$$\frac{Da^i_j}{Dt} + a^i_k a^k_j - \frac{1}{3}\left(a^m_n a^n_m\right)\delta^i_j = h^i_j, \quad (13.120)$$

where $a^i_i = 0$. Work the problem as an unclosed system. What are the symmetries when $h^i_j$ is specified to be a symmetric tensor, as it would be for the conventional Euler equations (i.e., viscous term zero, pressure term nonzero)?

13.15 Consider turbulent parallel flow along a wall as shown in Figure 13.15. All flow properties are independent of $x$. The equations of motion reduce to

$$\frac{d}{dy}\left(\nu \frac{dU}{dy}\right) = 0, \quad (13.121)$$

where $\nu$ is the kinematic viscosity and $U$ is the mean streamwise velocity. If $\nu$ is required to be constant, then the only solution of the equation is the linear profile $U \propto y$. However, if $\nu$ is allowed to be a function of $y$ (i.e., $\nu$ is an effective eddy viscosity), then other profile shapes are possible. Consider the latter case, and show that the equation is invariant under a five-parameter group with three dilations and two translations (Oberlack [13.34]). Which choice of group parameters produces

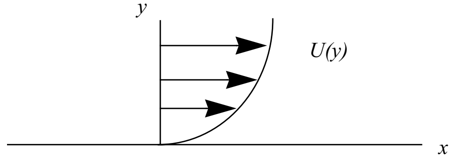



Fig. 13.15.

the logarithmic profile, and what is the corresponding eddy-viscosity profile?

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# 14

## *Lie–Bäcklund Transformations*

Extended Lie groups, together with their transformations of derivatives up to order $p$, are point transformations. They have a very particular hierarchical structure in which the transformation of a point in the space of independent and dependent variables $(\boldsymbol{x}, \boldsymbol{y})$ depends only on the coordinates of the point. Thus at each level of extension, the differential function that carries out the transformation of derivatives depends on derivatives up to and including but not beyond the order of the derivative being transformed:

$$\begin{aligned}(\boldsymbol{x}, \boldsymbol{y}) &\Rightarrow (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}),\\ (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1) &\Rightarrow (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}_1),\\ (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2) &\Rightarrow (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}_1, \tilde{\boldsymbol{y}}_2),\\ &\vdots\\ (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_p) &\Rightarrow (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}_1, \tilde{\boldsymbol{y}}_2, \ldots, \tilde{\boldsymbol{y}}_p).\end{aligned} \tag{14.1}$$

Such transformations are closed diffeomorphisms in the Euclidean space of differential variables $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_p)$ with

$$q = n + m \sum_{k=0}^{p} \frac{(n+k-1)!}{k!(n-1)!} \tag{14.2}$$

dimensions. They clearly enjoy the property that they can be used to transform a differential equation without raising the order of the equation.

It is both possible and fruitful to consider more general transformations, called *Lie–Bäcklund groups* [14.1], [14.2], where the one-parameter mapping of variables can depend on derivatives up to arbitrary order:

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots) \Rightarrow (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}_1, \tilde{\boldsymbol{y}}_2, \ldots). \tag{14.3}$$

In this case, the extension to transformations of derivatives, up to say order $p$, inevitably produces expressions that contain derivatives of arbitrary order greater than $p$. Lie–Bäcklund transformations are not closed in a space of finite dimension, and in Chapter 7 we introduced the space of differential functions, denoted by $\mathcal{A}$, to facilitate the treatment of such groups. In spite of the higher-order derivatives generated by the procedure for constructing the extended group, transformations of this type can be used to transform a differential equation without raising the order of the equation.

I should point out that the name Lie–Bäcklund for these transformations is not universally accepted. Anderson and Ibragimov [14.1] spend a good deal of the early part of their book making a case for the name in the course of reviewing the historical foundations of the field through the work of Lie, Bäcklund and Bianchi. Olver [14.3] rejects "Lie–Bäcklund" altogether, preferring to use the phrase "generalized symmetries," and this is widely used in the literature. Bluman and Kumei [14.4] use the name Lie–Bäcklund but in agreement with olver point out that neither Lie nor Bäcklund ever actually considered such transformations. The first person to use transformations that depend on higher order derivatives was probably Emmy Noether in her investigation of symmetries derived from a variational principle, but the term Noether symmetry is already used to describe a class of symmetries related to her discoveries. In the end I found the arguements in Anderson and Ibragimov sufficiently convincing and decided to use the term Lie–Bäcklund in this book and in my course. In addition it is consistent with the terminology used by Ibragimov in the CRC series [14.5], which I consider an essential desk reference for anyone working in the field.

## 14.1 Lie–Bäcklund Transformations – Infinite-Order Structure

Lie–Bäcklund groups are transformations that preserve infinite order contact but are not necessarily simple extensions of Lie point symmetries [14.1]. To develop the theory of Lie–Bäcklund groups, it is necessary to relax the requirement that the transformation be closed in the space $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p)$. Consider the transformation

$$T^s\colon \left\{ \begin{array}{ll} \tilde{x}^j = F^j[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, s], & j = 1, \ldots, n \\ \tilde{y}^i = G^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, s], & i = 1, \ldots, m \end{array} \right\}. \tag{14.4}$$

The infinite-order extended transformation is

$$T_\infty^s\colon \left\{ \begin{aligned} \tilde{x}^j &= F^j[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, s], \;\; j = 1, \ldots, n \\ \tilde{y}^i &= G^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, s], \;\; i = 1, \ldots, m \\ \tilde{y}^i_j &= G^i_{\{j\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, s] \\ \tilde{y}^i_{j_1 j_2} &= G^i_{\{j_1 j_2\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, s] \\ &\vdots \\ \tilde{y}^i_{j_1 j_2 \ldots j_p} &= G^i_{\{j_1 j_2 \ldots j_p\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, s] \\ &\vdots \end{aligned} \right\}, \tag{14.5}$$

where the dots indicate continuation to arbitrary order. The number of independent variables is *a priori finite or infinite* in the infinite-dimensional space of differential variables $z = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots)$. The transformation (14.4) is called a *Lie–Bäcklund transformation* if it preserves the invariance of the infinite-order system of contact conditions, that is, if the transformation (14.4) is such that

$$\begin{aligned} d\tilde{y}^i - \tilde{y}^i_{j_1}\, d\tilde{x}^{j_1} &= dy^i - y^i_{j_1}\, dx^{j_1} = 0, \\ d\tilde{y}^i_{j_1} - \tilde{y}^i_{j_1 j_2}\, d\tilde{x}^{j_2} &= dy^i_{j_1} - y^i_{j_1 j_2}\, dx^{j_2} = 0, \\ d\tilde{y}^i_{j_1 j_2} - \tilde{y}^i_{j_1 j_2 j_3}\, d\tilde{x}^{j_3} &= dy^i_{j_1 j_2} - y^i_{j_1 j_2 j_3}\, dx^{j_3} = 0, \\ &\vdots \end{aligned} \tag{14.6}$$

A transformation with this property can used to transform a differential equation without raising the order, and therefore might be of great value in constructing methods for solving the equation, since it would allow a given solution to be transformed to a nontrivial, new solution. In some cases the transformation itself could be used to originate a solution.

In Chapter 8, Section 8.1.2, we showed that a once extended point transformation inherits the properties of a group and, by induction, the extended transformation to all orders is a group. This came naturally from the fact that the procedure for generating the extensions used the contact conditions, and so the invariance of the contact conditions was automatically assured. In the case of Lie–Bäcklund transformations, the invariance of the contact conditions (14.6) is actually imposed on the infinitesimal form of the group, and this restricts the kinds of transformations that are permissible. See Appendix 3 for details. Preservation of the infinite-order contact conditions (14.6) ensures that an extended Lie–Bäcklund transformation inherits the properties of a group; and in particular, is a one-to-one invertible map in the space $z$.

The extended group is generated using the total differentiation operator and the contact conditions outlined in Chapter 7. The procedure is exactly the same as that used for point groups in Chapters 8 and 9 and will not be repeated here. For example, the transformation of first derivatives is

$$\tilde{y}^i_j = G^i_{\{j\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, s], \tag{14.7}$$

where

$$G^i_{\{j\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, s] = D_\beta G^i (D_j F^\beta)^{-1}, \tag{14.8}$$

and $(D_j F^\beta)^{-1}$ denotes a matrix inverse. The total differentiation operator in (14.8) reaches to infinite-order, $D_\beta = \partial/\partial x^\beta + \cdots$, i.e., to whatever order may appear in (14.7). The transformation of the $p$th derivative is

$$G^i_{\{j_1 j_2 \ldots j_p\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, s] = D_\beta G^i_{\{j_1 j_2 \cdots j_{p-1}\}} (D_{j_p} F^\beta)^{-1}. \tag{14.9}$$

This expression is identical to that for a point group except that the total differentiation operator acts to whatever order may appear in the transformation functions.

In view of all the high-order derivatives running around in the transformation functions (14.9), it is not at all obvious that it can transform a differential equation without raising the order. Nevertheless, by enforcing higher-order derivatives of the original equation on the invariance condition, such transformations can be realized.

### *14.1.1 Infinitesimal Lie–Bäcklund Transformation*

The infinitesimal form of the group (14.5) is

$$\begin{aligned}
\tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots], \qquad j = 1, \ldots, n,\\
\tilde{y}^i &= y^i + s\eta^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots], \qquad i = 1, \ldots, m,\\
\tilde{y}^i_{j_1} &= y^i_{j_1} + s\eta^i_{\{j_1\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots],\\
&\vdots
\end{aligned} \tag{14.10}$$

The infinitesimals in (14.10) are formed in the usual way by differentiation with respect to the group parameter:

$$\begin{aligned}
\xi^j[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots] &= \frac{\partial}{\partial s} F^j[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, s]|_{s=0},\\
\eta^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots] &= \frac{\partial}{\partial s} G^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, s]|_{s=0}.
\end{aligned} \tag{14.11}$$

The Lie–Bäcklund operator is

$$X = \xi^j \frac{\partial}{\partial x^j} + \eta^i \frac{\partial}{\partial y^i} + \cdots. \tag{14.12}$$

Applying the group operator to the contact conditions leads to the following infinite system of invariance relations for the contact conditions

$$\begin{aligned}
\hat{X}\big(dy^i - y^i_{j_1}\, dx^{j_1}\big) &= d\eta^i - \eta^i_{\{j_1\}}\, dx^{j_1} - y^i_{j_1}\, d\xi^{j_1} = 0,\\
\hat{X}\big(dy^i_{j_1} - y^i_{j_1 j_2}\, dx^{j_2}\big) &= d\eta^i_{j_1} - \eta^i_{\{j_1 j_2\}}\, dx^{j_2} - y^i_{j_1 j_2}\, d\xi^{j_2} = 0,\\
\hat{X}\big(dy^i_{j_1 j_2} - y^i_{j_1 j_2 j_3}\, dx^{j_3}\big) &= d\eta^i_{j_1 j_2} - \eta^i_{\{j_1 j_2 j_3\}}\, dx^{j_3} - y^i_{j_1 j_2 j_3}\, d\xi^{j_3} = 0,\\
&\ \ \vdots
\end{aligned} \tag{14.13}$$

(see Appendix 3 for details and for the definition of the operator $\hat{X}$, which is enlarged to cover the prolonged space including the differentials that appear in the contact conditions).

If we use the contact conditions to replace the differentials $dy^i, dy^i_j, \ldots$ to all orders, the invariance conditions (14.13) become

$$\begin{aligned}
\hat{X}\big(dy^i - y^i_\alpha\, dx^\alpha\big) &= \big(D_{j_1}\eta^i - y^i_\alpha D_{j_1}\xi^\alpha - \eta^i_{\{j_1\}}\big)\, dx^{j_1} = 0,\\
\hat{X}\big(dy^i_{j_1} - y^i_{j_1\alpha}\, dx^\alpha\big) &= \big(D_{j_2}\eta^i_{\{j_1\}} - y^i_{j_1\alpha} D_{j_2}\xi^\alpha - \eta^i_{\{j_1 j_2\}}\big)\, dx^{j_2} = 0,\\
\hat{X}\big(dy^i_{j_1 j_2} - y^i_{j_1 j_2\alpha}\, dx^\alpha\big) &= \big(D_{j_3}\eta^i_{\{j_1 j_2\}} - y^i_{j_1 j_2\alpha} D_{j_3}\xi^\alpha - \eta^i_{\{j_1 j_2 j_3\}}\big)\, dx^{j_3} = 0,\\
&\ \ \vdots
\end{aligned} \tag{14.14}$$

The crucial difference between the infinite- and the finite-order case is that in the infinite-order case, the dependence of the infinitesimals is not restricted to order $p$; rather, the space is expanded naturally as the order of the transformed derivative is increased, just as it was in the case of point groups. The contact condition is satisfied *a priori* to all orders, and so the differentials $dy^i, dy^i_j, \ldots$ are dependent on the $dx^j$ to all orders, i.e., only the $dx^j$ are independent differentials. As a result there are no additional conditions that must be met, which might severely restrict the possible dependence of the infinitesimals on derivatives.

The theory of Lie–Bäcklund transformations is fundamentally a theory of transformations in an infinite-dimensional space [14.2], and the appropriate functional setting is the infinite-dimensional space of differential functions introduced in Chapter 7. Appendix 3 provides full details of this point.

In order for the contact conditions to be invariant under the group, the expressions in parentheses in (14.14) must be zero. In this manner (14.14) produces the conventional expressions for the infinitesimal transformations of partial derivatives by Lie–Bäcklund groups:

$$\begin{aligned} \eta^i_{\{j_1\}} &= D_{j_1}\eta^i - y^i_\alpha D_{j_1}\xi^\alpha, \\ \eta^i_{\{j_1 j_2\}} &= D_{j_2}\eta^i_{\{j_1\}} - y^i_{j_1\alpha} D_{j_2}\xi^\alpha, \\ \eta^i_{\{j_1 j_2 j_3\}} &= D_{j_3}\eta^i_{\{j_1 j_2\}} - y^i_{j_1 j_2\alpha} D_{j_3}\xi^\alpha, \\ &\vdots \end{aligned} \tag{14.15}$$

We now use the characteristic functions

$$\begin{aligned} \mu^i &= \eta^i - y^i_\alpha \xi^\alpha, \\ \mu^i_{\{j_1\}} &= \eta^i_{\{j_1\}} - y^i_{j_1\alpha}\xi^\alpha, \\ \mu^i_{\{j_1 j_2\}} &= \eta^i_{\{j_1 j_2\}} - y^i_{j_1 j_2\alpha}\xi^\alpha, \\ &\vdots \end{aligned} \tag{14.16}$$

introduced in Chapter 9, Section 9.3. The infinitesimals (14.15) become

$$\begin{aligned} \eta^i_{\{j_1\}} &= D_{j_1}\mu^i + y^i_{j_1\alpha}\xi^\alpha, \\ \eta^i_{\{j_1 j_2\}} &= D_{j_2}\mu^i_{\{j_1\}} + y^i_{j_1 j_2\alpha}\xi^\alpha, \\ \eta^i_{\{j_1 j_2 j_3\}} &= D_{j_3}\mu^i_{\{j_1 j_2\}} + y^i_{j_1 j_2 j_3\alpha}\xi^\alpha, \\ &\vdots \end{aligned} \tag{14.17}$$

This is a particularly convenient form for the infinitesimals, since

$$\begin{aligned} \eta^i_{\{j_1\}} &= D_{j_1}\mu^i + y^i_{j_1\alpha}\xi^\alpha, \\ \eta^i_{\{j_1 j_2\}} &= D^2_{j_1 j_2}\mu^i + y^i_{j_1 j_2\alpha}\xi^\alpha, \\ &\vdots \\ \eta^i_{\{j_1 j_2 \dots j_q\}} &= D^q_{j_1 j_2 \dots j_q}\mu^i + y^i_{j_1 j_2 \dots j_q\alpha}\xi^\alpha. \end{aligned} \tag{14.18}$$

In the case where $\xi^\alpha = 0$, (14.18) reduces to the simple form

$$\begin{aligned} \eta^i_{\{j_1\}} &= D_{j_1}\mu^i, \\ \eta^i_{\{j_1 j_2\}} &= D^2_{j_1 j_2}\mu^i, \\ \eta^i_{\{j_1 j_2 j_3\}} &= D^3_{j_1 j_2 j_3}\mu^i, \\ &\vdots \end{aligned} \tag{14.19}$$

In Section 14.3 it will be shown that, without loss of generality, one can always choose $\xi^\alpha = 0$ as long as the transformation is permitted to depend on at least first derivatives.

### 14.1.2 Reconstruction of the Finite Lie–Bäcklund Transformation

Usually all that is known about a Lie–Bäcklund transformation is the infinitesimal form (14.10), and in many cases the finite form cannot be recovered without great effort, if at all. This can be understood by looking at the formal Lie series used to reconstruct the finite transformation,

$$\begin{aligned}
\tilde{x}^j &= x^j + sXx + \frac{s^2}{2!}X(Xx) + \frac{s^3}{3!}X(X(Xx)) + \cdots, \\
\tilde{y}^i &= y^i + sXy + \frac{s^2}{2!}X(Xy) + \frac{s^3}{3!}X(X(Xy)) + \cdots, \\
\tilde{y}^i_j &= y^i_j + sXy^i_j + \frac{s^2}{2!}X\big(Xy^i_j\big) + \frac{s^3}{3!}X\big(X\big(Xy^i_j\big)\big) + \cdots, \\
&\vdots
\end{aligned} \tag{14.20}$$

where $X$ is the infinite-order operator, $X = \xi^j(\partial/\partial x^j) + \eta^i(\partial/\partial y^i) + \cdots$, and the infinitesimals $(\xi^j[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots], \eta^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots])$ depend on derivatives of arbitrary order. In the practical application of Lie–Bäcklund groups, the transformations of $\boldsymbol{x}$ and $\boldsymbol{y}$ are usually assumed to depend on derivatives up to some finite order $r$, more or less arbitrarily chosen by the user. Therefore the dependence of the infinitesimals is simplified to $(\xi^j[\langle z\rangle], \eta^i[\langle z\rangle])$, where $\langle z\rangle$ is any finite subsequence of $z$, namely $\langle z\rangle = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots \boldsymbol{y}_r)$.

In general, each term in the Lie series (14.20) produces an expression involving derivatives of increasingly high-order that may be impossible to sum. Nevertheless, examples will be given shortly where the summation can be carried out quite easily. Summing the Lie series (14.20) is equivalent to integrating the characteristic equations

$$\frac{dx^j}{\xi^j} = \frac{dy^i}{\eta^i} = \frac{dy^i_{j_1}}{\eta^i_{\{j_1\}}} = \frac{dy^i_{j_1 j_2}}{\eta^i_{\{j_1 j_2\}}} = \cdots. \tag{14.21}$$

This is an infinite-order autonomous system of the form

$$\begin{aligned}
\frac{dx^j}{ds} &= \xi^j[\langle z\rangle], & \frac{dy^i}{ds} &= \eta^i[\langle z\rangle], \\
\frac{dy^i_{j_1}}{ds} &= \eta^i_{\{j_1\}}[\langle z\rangle], & \frac{dy^i_{\{j_1 j_2\}}}{ds} &= \eta^i_{\{j_1 j_2\}}[\langle z\rangle], \ldots.
\end{aligned} \tag{14.22}$$

In principle the system (14.22) never closes, since the infinitesimals generate ever higher derivatives at each order. In practice, however, the transformation always arises in the context of a differential equation or system of differential equations. Sometimes the equation and its differential consequences can be used to eliminate all derivatives of order higher than $r$. Thus the system (14.22) can become closed in $\langle z\rangle = \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_r$ on solutions of the equation, and (14.20) becomes a formal power series in the finite vector $\langle z\rangle$. Nevertheless, it is easy to find examples of differential equations where this happy scheme is all but impossible to implement. The differential equation in question has to be such that each of the required derivatives can be isolated to make the required substitution. The integration process is then essentially the same as that for a conventional Lie point group.

## 14.2 Lie Contact Transformations

Before we consider more general Lie–Bäcklund transformations, it is useful to examine Lie contact groups, which are the simplest generalization beyond point groups. Consider an infinitesimal transformation of the following form:

$$T^s\colon \left\{\begin{aligned} \tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x}, y, \boldsymbol{y}_1], \quad j = 1, \ldots, n \\ \tilde{y} &= y + s\eta[\boldsymbol{x}, y, \boldsymbol{y}_1] \\ \tilde{y}_j &= y_j + s\eta_{\{j\}}[\boldsymbol{x}, y, \boldsymbol{y}_1] \end{aligned}\right\}. \tag{14.23}$$

Note that *there is only one dependent variable*. The form of this transformation is suprising at first in view of our experience with extending point groups. The fact that the transformations of $\boldsymbol{x}$, $y$ depend on first derivatives should imply that the transformation of the first derivative depends on second derivatives and so on for higher extensions, but (14.23) does not fit this expectation. By assumption, the transformation of $y_j$ is assumed to depend only on $[\boldsymbol{x}, y, \boldsymbol{y}_1]$.

It is appropriate to ask whether transformation groups of this form can exist. This is addressed by determining what, if any, requirements must be imposed on (14.23) such that the contact conditions

$$\begin{gathered} d\tilde{y} - \tilde{y}_j\, d\tilde{x}^j = dy - y_j\, dx^j = 0 \\ \vdots \end{gathered} \tag{14.24}$$

are preserved to all orders. When we dealt with Lie point transformations in Chapter 5, the preservation of the contact condition was a natural consequence of the differential procedure used to generate the extended group, and the extended transformation automatically inherited the properties of a group. But when dealing with Lie contact or Lie–Bäcklund transformations, the preservation of the contact conditions is a requirement that is *imposed* on the transformation in order to ensure that the transformation inherits the properties of a group – to ensure that it is a one-to-one invertible map.

It turns out that the transformation (14.23) preserves the invariance of the contact condition (14.24) if and only if the infinitesimals of the group are of the following form:

$$\xi^j = -\frac{\partial \omega}{\partial y_j}, \qquad \eta = \omega - y_\sigma \frac{\partial \omega}{\partial y_\sigma}, \qquad \eta_{\{j\}} = \frac{\partial \omega}{\partial x^j} + \frac{\partial \omega}{\partial y} y_j. \quad (14.25)$$

See Appendix 3 for details of the derivation of (14.25). The transformation (14.23) with infinitesimals (14.25) is called a *Lie contact transformation*. Note that all three infinitesimals of the group are determined by a single generating function, $\omega[\boldsymbol{x}, y, \boldsymbol{y}_1]$. Extensions of Lie contact groups to the transformations of higher derivatives are derived using the same algorithm as for point groups. As in the case of point groups, all higher-order contact conditions are invariant under the extended group.

A Lie contact transformation is closed in the space $(\boldsymbol{x}, y, \boldsymbol{y}_1)$. Therefore such a transformation enjoys the same property as a Lie point group, namely, it can be used to transform a differential equation in the source space $(\boldsymbol{x}, y, \boldsymbol{y}_1)$ to a new equation in the target space $(\tilde{\boldsymbol{x}}, \tilde{y}, \tilde{\boldsymbol{y}}_1)$ without raising the order of the equation. This point is discussed in more detail in Appendix 3, where the following theorem is proven.

***Theorem 14.1.*** *There do not exist any transformation groups that preserve pth-order tangency and that are closed in the space* $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p)$*, other than extensions of Lie point transformations for* $m \geq 1$ *and extensions of Lie contact transformations for* $m = 1$ *(one dependent variable).*

True contact symmetries exist only for equations involving one dependent variable. For equations with more than one dependent variable the concept of a contact or tangent transformation is replaced by that of a Lie–Bäcklund transformation.

***Example 14.1 (Reconstructing a finite Lie contact transformation from its generating function).*** Find the finite Lie contact transformation with the generating function

$$\omega = -\left(1 + y_x^2\right)^{1/2}. \quad (14.26)$$

Using (14.25), the infinitesimals are

$$\xi^j = \frac{y_x}{\left(1+y_x^2\right)^{1/2}}, \qquad \eta = -\frac{1}{\left(1+y_x^2\right)^{1/2}}, \qquad \eta_{\{j\}} = 0, \tag{14.27}$$

and the group operator is

$$X = \frac{y_x}{\left(1+y_x^2\right)^{1/2}}\frac{\partial}{\partial x} - \frac{1}{\left(1+y_x^2\right)^{1/2}}\frac{\partial}{\partial y} + (0)\frac{\partial}{\partial y_x}. \tag{14.28}$$

The Lie series (14.20) truncates to the Lie contact group in the form of a translation in $x$ and $y$ depending on $y_x$,

$$\tilde{x} = x + \frac{t y_x}{\left(1+y_x^2\right)^{1/2}}, \qquad \tilde{y} = y - \frac{s}{\left(1+y_x^2\right)^{1/2}}, \qquad \tilde{y}_{\tilde{x}} = y_x, \tag{14.29}$$

where $s$ is the group parameter.

***Example 14.2 (A Lie contact transformation and the corresponding generating function for a second-order PDE).*** Find a Lie contact transformation for the equation

$$U_{xx} + aU_{xy} + U_{yy} = 0. \tag{14.30}$$

Running the package **`IntroToSymmetry.m`** with parameters **`xseon=1`** and **`r=1`** leads to a large number of groups, including all of the point groups of the equation. One of the contact groups found has the infinitesimals

$$\begin{aligned} \xi^1 &= (-a^{215} + b^{114} + b^{120})U_x, \\ \xi^2 &= a^{215}U_y, \\ \eta &= b^{114}U_x^2 + b^{120}U_y^2, \end{aligned} \tag{14.31}$$

where the superscripts label coefficients in the power series used to find the infinitesimals (recall Chapter 9, Section 9.2.8). Now substitute these results into (14.25). The middle relation in (14.25) gives

$$\begin{aligned} \omega &= \eta + U_\sigma \frac{\partial \omega}{\partial U_\sigma} \\ &= b^{114}U_x^2 + b^{120}U_y^2 + (a^{215} - b^{114} - b^{120})U_x^2 - a^{215}U_y^2, \end{aligned} \tag{14.32}$$

and the generating function is

$$\omega = (b^{120} - a^{215})\left(U_y^2 - U_x^2\right). \tag{14.33}$$

Let $b^{120} - a^{215} = \frac{1}{2}$, $a^{215} = -1$, $b^{120} = -\frac{1}{2}$, and $b^{114} = \frac{1}{2}$. The corresponding group operator is

$$X = U_x \frac{\partial}{\partial x} - U_y \frac{\partial}{\partial y} + \frac{1}{2}\left(U_x^2 - U_y^2\right)\frac{\partial}{\partial U} + (0)\frac{\partial}{\partial U_x} + (0)\frac{\partial}{\partial U_y}. \tag{14.34}$$

The finite transformation, generated from a truncated Lie series, is the translation

$$\begin{aligned} \tilde{x} &= x + sU_x, \\ \tilde{y} &= y - sU_y, \\ \tilde{U} = U + \frac{s}{2}\left(U_x^2 - U_y^2\right), \qquad \tilde{U}_{\tilde{x}} &= U_x, \qquad \tilde{U}_{\tilde{y}} = U_y. \end{aligned} \tag{14.35}$$

Any solution $U$ of (14.30) can be mapped to a new solution $\tilde{U}$ using (14.35).

***Example 14.3 (The Legendre transformation).*** The contact transformation

$$\tilde{x}^i = \frac{\partial y}{\partial x^i}, \qquad \tilde{y} = -y + x^i \frac{\partial y}{\partial x^i}, \qquad \frac{\partial \tilde{y}}{\partial \tilde{x}^i} = x^i, \tag{14.36}$$

called the *Legendre transformation*, can be used to map one ODE to another by exchanging variables and derivatives. For example, the mapping

$$\begin{aligned} \tilde{x} &= y_x, \\ \tilde{y} &= -y + xy_x, \\ \tilde{y}_{\tilde{x}} &= x, \\ \tilde{y}_{\tilde{x}\tilde{x}} &= \frac{1}{y_{xx}}, \\ \tilde{y}_{\tilde{x}\tilde{x}\tilde{x}} &= -\frac{y_{xxx}}{y_{xx}^3}, \end{aligned} \tag{14.37}$$

where the higher derivatives are generated using the usual extension formulas, transforms the nonlinear third-order ODE

$$2\tilde{y}_{\tilde{x}}\tilde{y}_{\tilde{x}\tilde{x}\tilde{x}} - 3\tilde{y}_{\tilde{x}\tilde{x}}^2 = 0 \tag{14.38}$$

to the linear third-order equation

$$2xy_{xxx} + 3y_{xx} = 0. \tag{14.39}$$

Letting $x = \theta^2$ transforms (14.39) to $y_{\theta\theta\theta} = 0$, which integrates to the three-parameter family of parabolae,

$$y = a\theta^2 + b\theta + c. \tag{14.40}$$

We encountered the Legendre transformation previously in Chapter 3, Section 3.4, where it was used to change variables in the Gibbs equation of thermodynamics, and then later in Section 3.12, where it was used to transform from the Lagrangian to the Hamiltonian formulation of classical dynamics.

### *14.2.1 Contact Transformations and the Hamilton–Jacobi Equation*

Contact transformations come up commonly in applications in classical dynamics. Consider the time-independent Hamilton–Jacobi equation,

$$H\left[q^1, \ldots, q^n, \frac{\partial S}{\partial q^i}, \ldots, \frac{\partial S}{\partial q^n}\right] = E, \tag{14.41}$$

with constant total energy $E$. We studied the solution of first-order nonlinear PDEs like (14.41) back in Chapter 3, Section 3.8.1. The Lie contact operator with infinitesimals (14.25) applied to (14.41) leads to

$$-\frac{\partial \omega}{\partial S_j}\frac{\partial H}{\partial q^j} + \left(\omega - S\frac{\partial \omega}{\partial S}\right)\frac{\partial H}{\partial S} + \left(\frac{\partial \omega}{\partial q^j} + \frac{\partial \omega}{\partial S}q_j\right)\frac{\partial H}{\partial S_j} = 0. \tag{14.42}$$

If $\omega$ is independent of $S$, (14.42) becomes the Poisson bracket of $\omega$ with $H$,

$$\{H, \omega\} = \frac{\partial \omega}{\partial q^j}\frac{\partial H}{\partial S_j} - \frac{\partial \omega}{\partial S_j}\frac{\partial H}{\partial q^j} = 0. \tag{14.43}$$

Therefore the generating function $\omega$ is an integral of the motion for the Hamiltonian system

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial S_i}, \qquad \frac{dS_i}{dt} = -\frac{\partial H}{\partial q^i}. \tag{14.44}$$

See Chapter 3, Section 3.12.1. Thus searching for all Lie contact symmetries of the system (14.44) is equivalent to searching for all integrals of the motion.

## 14.3 Equivalence Classes of Transformations

The generalization to Lie–Bäcklund transformations permitting the infinitesimals $(\xi^i, \eta^j)$ to depend on derivatives leads to a very useful simplification of the infinitesimal form of a group. Consider the Lie point group

$$T^s\colon \begin{Bmatrix} \tilde{x}^j = x^j + s\xi^j[\boldsymbol{x},\boldsymbol{y}], & j = 1,\dots,n \\ \tilde{y}^i = y^i + s\eta^i[\boldsymbol{x},\boldsymbol{y}], & i = 1,\dots,m \end{Bmatrix} \tag{14.45}$$

applied without loss of generality to functions of the form

$$\Omega^i[\boldsymbol{x},\boldsymbol{y}] = y^i - \Phi^i[\boldsymbol{x}] = 0. \tag{14.46}$$

The action of (14.45) on (14.46) is given by the Lie series

$$\tilde{\Omega}^i[\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}] = \Omega^i[\boldsymbol{x},\boldsymbol{y}] + s(X\Omega^i) + \frac{s^2}{2!}X(X\Omega^i)) + \frac{s^3}{3!}X(X(X\Omega^i)) + \cdots, \tag{14.47}$$

where

$$X = \xi^j\frac{\partial}{\partial x^j} + \eta^i\frac{\partial}{\partial y^i} + \cdots. \tag{14.48}$$

Operating on (14.46) with (14.48) produces

$$X\Omega^i = \eta^i - \frac{\partial \Phi^i}{\partial x_j}\xi^j = \eta^i - y^i_j\xi^j. \tag{14.49}$$

Equation (14.49) shows that the Lie point transformation (14.45) operating on (14.46) is equivalent to the Lie–Bäcklund transformation

$$\begin{aligned} \tilde{x}^j &= x^j, \qquad j = 1,\dots,n, \\ \tilde{y}^i &= y^i + s\big(\eta^i[\boldsymbol{x},\boldsymbol{y}] - y^i_j\xi^j[\boldsymbol{x},\boldsymbol{y}]\big), \qquad i = 1,\dots,m, \end{aligned} \tag{14.50}$$

operating on (14.46). Both transformations generate the same Lie series for $\Omega^i$. Therefore, with full generality and recalling (14.16), we can let the infinitesimal transformation (14.45) have the equivalent form

$$\begin{aligned} \tilde{x}^j &= x^j, \qquad j = 1,\dots,n, \\ \tilde{y}^i &= y^i + s\mu^i[\boldsymbol{x},\boldsymbol{y},\boldsymbol{y}_1], \qquad i = 1,\dots,m, \end{aligned} \tag{14.51}$$

where the Lie–Bäcklund infinitesimal is the characteristic function

$$\mu^i[\boldsymbol{x},\boldsymbol{y},\boldsymbol{y}_1] = \eta^i[\boldsymbol{x},\boldsymbol{y}] - y^i_j\xi^j[\boldsymbol{x},\boldsymbol{y}]. \tag{14.52}$$

In other words, *by generalizing the infinitesimals to allow dependence on at least first derivatives, we need only consider transformations that act on the dependent variables while the independent variables are left unchanged.*

### 14.3.1 Every Lie Point Operator Has an Equivalent Lie–Bäcklund Operator

For example, the operator of the Lie point dilation group is

$$X^{\text{dil}} = x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}. \tag{14.53}$$

This is equivalent to the Lie–Bäcklund operator

$$U^{\text{dil}} = (-2y - xy_x)\frac{\partial}{\partial y}. \tag{14.54}$$

Any decomposition of $\mu^i$ into $\xi^j$ and $\eta^j$ yields an equivalent transformation. For example, the Lie–Bäcklund group

$$U^{\text{dil}} = \frac{2y}{y_x}\frac{\partial}{\partial x} - xy_x\frac{\partial}{\partial y} \tag{14.55}$$

is also equivalent to (14.54). In fact, for every Lie infinitesimal point generator there are an infinite number of Lie–Bäcklund infinitesimal generators that are equivalent to it.

### 14.3.2 Equivalence of Lie–Bäcklund Transformations

Consider the following two Lie–Bäcklund transformations:

$$\begin{aligned} \tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_r], \\ \tilde{y}^i &= y^i + s\eta^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_r], \\ &\vdots \end{aligned} \tag{14.56}$$

and

$$\begin{aligned} \tilde{x}^j &= x^j, \\ \tilde{y}^i &= y^i + s(\eta^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_r] - y^i_\beta \xi^\beta[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_r]), \\ &\vdots \end{aligned} \tag{14.57}$$

The choice of the order of derivative, $r$, that appears in the transformation is arbitrary and is a decision made by the user prior to addressing the question of invariance of a given differential equation. The conventional Lie–Bäcklund operator corresponding to (14.56) is

$$X = \xi^j \frac{\partial}{\partial x^j} + \eta^i \frac{\partial}{\partial y^i} + \left(D_{j_1}\eta^i - y^i_\beta D_{j_1}\xi^\beta\right)\frac{\partial}{\partial y^i_{j_1}} + \left(D_{j_2}\left(D_{j_1}\eta^i - y^i_\beta D_{j_1}\xi^\beta\right) - y^i_{j_1\beta}D_{j_2}\xi^\beta\right)\frac{\partial}{\partial y^i_{j_1 j_2}} + \cdots, \quad (14.58)$$

and the Lie–Bäcklund operator corresponding to (14.57) is

$$U = \eta^i \frac{\partial}{\partial y^i} + \left(D_{j_1}\eta^i - y^i_\beta D_{j_1}\xi^\beta - \xi^\beta y^i_{j_1\beta}\right)\frac{\partial}{\partial y^i_{j_1}} + \left(D_{j_2}\left(D_{j_1}\eta^i - y^i_\beta D_{j_1}\xi^\beta\right) - y^i_{j_1\beta}D_{j_2}\xi^\beta - \xi^\beta y^i_{j_1 j_2\beta}\right)\frac{\partial}{\partial y^i_{j_1 j_2}} + \cdots. \quad (14.59)$$

Generally we will use the symbol $U$ to denote the operator of a Lie–Bäcklund transformation, although when it comes to the application of the theory to a particular problem there is really no pressing need to make such a distinction; one simply has the option of setting the $\xi^j$ to zero if one wishes, as long as at least first derivatives are included in the transformation.

The difference between the two operators (14.58) and (14.59) is a Lie–Bäcklund operator of a particularly simple form:

$$X - U = \xi^\beta \frac{\partial}{\partial x^\beta} + \xi^\beta y^i_{j_1\beta}\frac{\partial}{\partial y^i_{j_1}} + \xi^\beta y^i_{j_1 j_2\beta}\frac{\partial}{\partial y^i_{j_1 j_2}} + \cdots = \xi^\beta D_\beta. \quad (14.60)$$

Any differential function

$$\Psi^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_p] = 0 \quad (14.61)$$

will admit the Lie–Bäcklund operator

$$U^0 = \xi^\beta D_\beta, \quad (14.62)$$

since all derivatives of the equations are zero ($D_\beta \Psi^i = 0$). Thus, for arbitrary functions $\xi^\beta$,

$$U^0\Psi^i = \xi^\beta D_\beta(\Psi^i) = 0. \quad (14.63)$$

The implication of this result is that, if (14.61) is invariant under the operator (14.58), then it must also be invariant under the operator (14.59), and vice versa. The two operators are considered equivalent.

### 14.3.3 Equivalence of Lie–Bäcklund and Lie Contact Operators

The infinitesimal form of a Lie contact transformation is

$$\begin{aligned}
\tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x}, y, \boldsymbol{y}_1] = x^j + s\left(-\frac{\partial \omega}{\partial y_j}\right),\\
\tilde{y} &= y + s\eta[\boldsymbol{x}, y, \boldsymbol{y}_1] = y + s\left(\omega - y_\sigma \frac{\partial \omega}{\partial y_\sigma}\right),\\
\tilde{y}_j &= y_j + s\eta_{\{j\}}[\boldsymbol{x}, y, \boldsymbol{y}_1] = y_j + s\left(\frac{\partial \omega}{\partial x^j} + \frac{\partial \omega}{\partial y} y_j\right),
\end{aligned} \tag{14.64}$$

where we have used (14.25). Note that the infinitesimal transformation of the derivative is

$$\eta_{\{j\}} = D_j\omega + y_{j\alpha}\xi^\alpha = \frac{\partial \omega}{\partial x^j} + \frac{\partial \omega}{\partial y} y_j. \tag{14.65}$$

According to the results of Section 14.3.2, the equivalence form of (14.64) is

$$\begin{aligned}
\tilde{x}^j &= x^j,\\
\tilde{y} &= y + s(\eta - y_\sigma \xi^\sigma) = y + s\omega,\\
\tilde{y}_j &= y_j + sD_j(\eta - y_\sigma \xi^\sigma) = y_j + sD_j\omega,
\end{aligned} \tag{14.66}$$

and so the generating function $\omega$ of a Lie contact transformation is just the characteristic function $\mu$ of an equivalent Lie–Bäcklund transformation.

### 14.3.4 The Extended Infinitesimal Lie–Bäcklund Group

Using the equivalence relations established above, we can, without loss of generality, restrict our attention to Lie–Bäcklund groups of the form

$$\begin{aligned}
\tilde{x}^j &= x^j, \qquad j = 1, \ldots, n,\\
\tilde{y}^i &= y^i + s\mu^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_r], \qquad i = 1, \ldots, m,\\
\tilde{y}^i_{j_1} &= y^i_{j_1} + s\mu^i_{\{j_1\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_r, \boldsymbol{y}_{r+1}],\\
\tilde{y}^i_{j_1 j_2} &= y^i_{j_1 j_2} + s\mu^i_{\{j_1 j_2\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_r, \boldsymbol{y}_{r+1}, \boldsymbol{y}_{r+2}],\\
&\vdots\\
\tilde{y}^i_{j_1 \ldots j_p} &= y^i_{j_1 \ldots j_p} + s\mu^i_{\{j_1 \ldots j_p\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_{r+p}],\\
&\vdots
\end{aligned} \tag{14.67}$$

The extensions are

$$\begin{aligned}
\mu^i_{\{j_1\}} &= D_{j_1}\mu^i, \\
\mu^i_{\{j_1 j_2\}} &= D_{j_2}\mu^i_{\{j_1\}} = D^2_{j_1 j_2}\mu^i, \\
&\vdots \\
\mu^i_{\{j_1\dots j_p\}} &= D^p_{j_1\dots j_p}\mu^i, \\
&\vdots
\end{aligned} \tag{14.68}$$

The relations (14.67) and (14.68) provide the machinery for searching for Lie–Bäcklund symmetries in systems of differential equations.

### 14.3.5 Proper Lie–Bäcklund Transformations

The Lie–Bäcklund operator

$$U = \mu^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1]\frac{\partial}{\partial y^i} \tag{14.69}$$

is equivalent to the operator of a point transformation if

$$\frac{\partial^2 \mu^i}{\partial y^k_j\, \partial y^r_s} = 0 \tag{14.70}$$

for all $i, j, k, r, s$. This follows from the form of the equivalent transformation, $\mu^i = \eta[\boldsymbol{x}, \boldsymbol{y}] - \xi^j[\boldsymbol{x}, \boldsymbol{y}]y^i_j$, which is linear in the $y^i_j$.

***Definition 14.1.*** *A* proper *Lie–Bäcklund transformation is defined as one that is not equivalent to a point transformation.*

Typically quadratic or higher-order products of derivatives of the $y^i$ appear in the infinitesimals $\mu^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1]$ of a proper Lie–Bäcklund transformation.

### 14.3.6 Lie Series Expansion of Differential Functions and the Invariance Condition

The algorithm for finding the unknown infinitesimals $\mu^i$ is similar to that for point transformations. The main complication in working with Lie–Bäcklund groups is that derivatives up to order $r + p$ appear in the group operator. This means that, in order to solve the determining equations, it is necessary to apply to the invariance condition the higher-order differential consequences of the system $\Psi^i = 0$. That is, we require that the system

$$\Psi^i = 0, \qquad D_{j_2}\Psi^i = 0, \qquad D^2_{j_1 j_2}\Psi^i = 0, \qquad D^3_{j_1 j_2 j_3}\Psi^i = 0, \ldots \quad (14.71)$$

up to arbitrary order be invariant under the group. The indices refer to differentiation with respect to any and all of the independent variables.

We can now state the condition for invariance of such a system as follows.

***Theorem 14.2.*** *Let*

$$\Psi^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_p] = 0, \qquad i = 1, \ldots, m, \quad (14.72)$$

*be a system of pth-order differential functions. It may be a system of ODEs or PDEs. The Lie–Bäcklund group assumed to depend on derivatives up to order r is*

$$\begin{aligned} \tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_r], \qquad j = 1, \ldots, n, \\ \tilde{y}^i &= y^i + s\eta^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_r], \qquad i = 1, \ldots, m, \\ &\vdots \end{aligned} \quad (14.73)$$

*The equivalent group is*

$$\begin{aligned} \tilde{x}^j &= x^j, \qquad j = 1, \ldots, n, \\ \tilde{y}^i &= y^i + \mu^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_r]s, \qquad i = 1, \ldots, m, \\ &\vdots \end{aligned} \quad (14.74)$$

*where $\mu^i = \eta^i - y^i_\alpha \xi^\alpha$ and the extensions are given by (14.68).*

*Transform the system (14.72) using (14.74) and expand the result in a Lie series. The series is*

$$\begin{aligned} &\dot{\Psi}^i[\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}_1, \tilde{\boldsymbol{y}}_2, \ldots, \tilde{\boldsymbol{y}}_p] \\ &\quad = \Psi^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_p] + sU\Psi^i + \frac{s^2}{2}U(U\Psi^i) + \cdots, \end{aligned} \quad (14.75)$$

*where U is the Lie–Bäcklund operator*

$$U = \mu^i \frac{\partial}{\partial y_i} + \mu^i_{\{j_1\}} \frac{\partial}{\partial y^i_{j_1}} + \mu^i_{\{j_1 j_2\}} \frac{\partial}{\partial y^i_{j_1 j_2}} + \cdots + \mu^i_{\{j_1 \ldots j_p\}} \frac{\partial}{\partial y^i_{j_1 \ldots j_p}} + \cdots, \quad (14.76)$$

*The system $\Psi^i$ is invariant under the equivalent group with operator $U \dots$, if and only if*

$$\boxed{U\Psi^i = 0, \qquad i = 1, \dots, m.} \tag{14.77}$$

With $\xi^j = 0$ the group extensions simply involve successive total differentiation of $\mu^i$. It follows from (14.77) and the identity

$$D_j(U) - U(D_j) = (D_j\xi^\alpha)D_\alpha \tag{14.78}$$

that the higher-order differential system (14.71) is invariant under the group (14.74), since

$$U\big(D_{j_1}\Psi^i\big) = 0, \qquad U\big(D^2_{j_1 j_2}\Psi^i\big) = 0, \qquad U\big(D^3_{j_1 j_2 j_3}\Psi^i\big) = 0, \dots, \tag{14.79}$$

where the indices take on all possible values. The invariance condition (14.77) is used to generate the determining equations, which are then solved for the unknown infinitesimals $\mu^i$ that leave the system (14.72) invariant. The process is similar to that used to determine the infinitesimals in the case of point groups. But remember that the infinitesimals being sought are assumed to depend on $\boldsymbol{x}$, $\boldsymbol{y}$, *and* derivatives up to order $r$. When the order of derivatives in the transformation is specified, the derivatives are automatically divided into dependent and independent classes. The distinction between dependent and independent derivatives must be carefully adhered to when the invariance condition (14.77) is parsed into the determining equations.

Typically, the search for Lie–Bäcklund symmetries of a given system of equations involves a process of trial and error with different choices of $r$. This is the reason why relatively little is known about the Lie–Bäcklund structure of the various equations of mathematical physics. For a general nonlinear system there is no systematic way of knowing what order will lead to a new symmetry, and the determining equations for the unknown $\mu^i$ derived from (14.77) and (14.71) become extremely complex as the order is increased.

## 14.4 Applications of Lie–Bäcklund Transformations

Several examples will be presented now to give a sense of how these transformations work.

***Example 14.4 (Equivalence transformation – translation group).*** Consider the finite Lie point translation

$$\begin{aligned} \tilde{x} &= x - s, \\ \tilde{y} &= y \end{aligned} \tag{14.80}$$

with infinitesimals $\xi = -1$ and $\eta = 0$. The equivalent extended Lie–Bäcklund transformation with $\mu = \eta - \xi y_x$ is

$$\begin{aligned} \tilde{x} &= x, \\ \tilde{y} &= y + s y_x, \\ \tilde{y}_{\tilde{x}} &= y_x + s y_{xx}, \\ \tilde{y}_{\tilde{x}\tilde{x}} &= y_{xx} + s y_{xxx}, \\ \tilde{y}_{\tilde{x}\tilde{x}\tilde{x}} &= y_{xxx} + s y_{xxxx}, \\ &\vdots \end{aligned} \tag{14.81}$$

The finite transformation of $y$ is determined by summing the Lie series for $y$. The Lie–Bäcklund operator is

$$U = y_x \frac{\partial}{\partial y} + y_{xx} \frac{\partial}{\partial y_x} + y_{xxx} \frac{\partial}{\partial y_{xx}} + \cdots + y_{(p+1)x} \frac{\partial}{\partial y_{px}} + \cdots. \tag{14.82}$$

Various terms in the Lie series for $y$ are

$$Uy = y_x, \qquad U(Uy) = y_{xx}, \qquad U(U(Uy)) = y_{xxx}, \ldots, \tag{14.83}$$

and the series is

$$\tilde{y} = y + \frac{s}{1!} y_x + \frac{s^2}{2!} y_{xx} + \frac{s^3}{3!} y_{xxx} + \cdots = \sum_{k=0}^{\infty} \frac{s^k}{k!} \frac{d^k y}{dx^k}. \tag{14.84}$$

This is simply the Taylor series for a function with a translated arguement. The series is easily summed, and the finite Lie–Bäcklund transformation equivalent to (14.80) is

$$\begin{aligned} \tilde{x} &= x, \\ \tilde{y} &= y[x + s]. \end{aligned} \tag{14.85}$$

The point form (14.80) can be recovered by simply defining $z = x + s$, whereby (14.85) becomes

$$\begin{aligned} \tilde{x} &= z - s, \\ \tilde{y} &= y[z]. \end{aligned} \tag{14.86}$$

This simple example nicely illustrates the interchangeability of point and first-order Lie–Bäcklund groups. It also illustrates the inherently infinite nature of the Lie–Bäcklund operator (14.82). Note that, as it stands, the extended transformation (14.81) is not closed in any finite-dimensional space of differential functions. Yet, there is no difficulty at all in summing the Lie series to recover the finite group. The point is that the ability to sum the Lie–Bäcklund series is not contingent on the transformation existing in a closed space.

***Example 14.5 (Equivalence transformation – dilation group).*** Now consider the finite Lie point dilation group

$$\begin{aligned} \tilde{x} &= e^s x, \\ \tilde{y} &= e^s y. \end{aligned} \tag{14.87}$$

An equivalent finite Lie–Bäcklund group is

$$\begin{aligned} \tilde{x} &= x, \\ \tilde{y} &= e^s y[e^{-s} x]. \end{aligned} \tag{14.88}$$

If we expand (14.88) for small $s$, the result is the infinitesimal Lie–Bäcklund transformation

$$\begin{aligned} \tilde{x} &= x, \\ \tilde{y} &= y[x] + s\left(y[x] - x\frac{dy[x]}{dx}\right). \end{aligned} \tag{14.89}$$

Note that in both Examples 14.4 and 14.5, where there is an equivalent point transformation, the finite Lie–Bäcklund transformation does not depend on derivatives, whereas the infinitesimal form depends linearly on the first derivative.

### *14.4.1 Third-Order ODE Governing a Family of Parabolas*

This is the equation

$$y_{xxx} = 0 \tag{14.90}$$

and is simple enough so that we can work out its complete Lie–Bäcklund structure in detail and examine further how these transformations work. The Lie–Bäcklund series can be summed to see how the finite form of the transformation can be used to generate solutions of the equation.

First let's work out the point group $(\xi[x, y], \eta[x, y])$. We are dealing with a differential function of the form $\psi = \Psi[x, y, y_x, y_{xx}, y_{xxx}]$, and the invariance condition in this case reduces to a single term,

$$X\Psi = \eta_{\{3\}} = 0, \tag{14.91}$$

which fully written out is

$$\begin{aligned}\eta_{\{3\}} = \eta_{xxx} &+ (3\eta_{xxy} - \xi_{xxx})y_x + (3\eta_{xyy} - 3\xi_{xxy})y_x^2 \\ &+ (\eta_{yyy} - 3\xi_{xyy})y_x^3 - \xi_{yyy}y_x^4 + (3\eta_{xy} - 3\xi_{xx})y_{xx} \\ &+ (3\eta_{yy} - 9\xi_{xy})y_x y_{xx} - (6\xi_{yy})y_x^2 y_{xx} - (3\xi_y)y_{xx}^2 \\ &+ (\eta_y - 3\xi_x)y_{xxx} - (4\xi_y)y_x y_{xxx} = 0. \end{aligned} \tag{14.92}$$

Using $y_{xxx} = 0$ in the invariance condition and equating the remaining coefficients to zero leads to the determining equations of the group,

$$\begin{gathered} \eta_{xxx} = 0, \qquad 3\eta_{xxy} - \xi_{xxx} = 0, \qquad 3\eta_{xyy} - 3\xi_{xxy} = 0, \\ \eta_{yyy} - 3\xi_{xyy} = 0, \qquad \xi_{yyy} = 0, \qquad 3\eta_{xy} - 3\xi_{xx} = 0, \\ 3\eta_{yy} - 9\xi_{xy} = 0, \qquad 6\xi_{yy} = 0, \qquad 3\eta_y = 0 \end{gathered} \tag{14.93}$$

with the solution

$$\begin{aligned} \xi &= a^5 + a^6 x + a^7 (x)^2, \\ \eta &= a^1 + a^2 x + a^3 (x)^2 + a^4 y + a^7 (2xy). \end{aligned} \tag{14.94}$$

Equation (14.90) is invariant under a seven-parameter projective point group.

Note that the group itself actually generates the solution of the equation as follows. If we select $a^5 = a^6 = a^7 = 0$ and $a^4 = -1$, the group operator becomes

$$X = (a^1 + a^2 x + a^3 (x)^2 - y)\frac{\partial}{\partial y}. \tag{14.95}$$

A function $y(x)$ will automatically satisfy the invariance condition $Xy = 0$ if

$$y = a^1 + a^2 x + a^3 (x)^2. \tag{14.96}$$

It is obvious that substituting this function into the equation puts no restriction on the parameters, and so the function as it stands is a solution; in fact, it is the general solution of the equation.

Now let's work out the first-order Lie–Bäcklund transformation that leaves $y_{xxx} = 0$ invariant. The invariance condition is

$$\mu_{\{3\}} = 0. \tag{14.97}$$

Let the Lie–Bäcklund transformation depend on the first derivative; i.e., choose $r = 1$:

$$\begin{aligned} \tilde{x} &= x, \\ \tilde{y} &= y + s\mu[x, y, y_x]. \end{aligned} \tag{14.98}$$

The first extension is

$$\mu_{\{1\}} = D\mu = \mu_x + y_x\mu_y + y_{xx}\mu_{y_x}. \tag{14.99}$$

The second extension is

$$\begin{aligned}\mu_{\{2\}} = D\mu_{\{1\}} &= \left(\mu_{xx} + 2y_x\mu_{xy} + y_x^2\mu_{yy} + 2y_x\eta_{yy_x}\right)\\ &\quad + y_{xx}\left(2\mu_{xy_x} + \mu_y\right) + y_{xx}^2\left(\mu_{y_xy_x}\right) + y_{xxx}\left(\mu_{y_x}\right),\end{aligned} \tag{14.100}$$

and the third extension is

$$\begin{aligned}\mu_{\{3\}} = D\mu_{\{2\}} &= \left(\mu_{xxx} + 3y_x\mu_{xxy} + 3y_x^2\mu_{xyy} + y_x^3\mu_{yyy}\right)\\ &\quad + y_{xx}\left(3\mu_{xxy_x} + 6y_x\mu_{xyy_x} + 3\mu_{xy} + 3y_x\mu_{yy}\right)\\ &\quad + y_{xx}^2\left(3\mu_{xy_xy_x} + 3\mu_{yy_x} + 3y_x\mu_{yy_xy_x}\right) + y_{xx}^3\left(\mu_{y_xy_xy_x}\right)\\ &\quad + y_{xxx}\left(3y_x\mu_{yy_x} + \mu_y\right) + y_{xx}y_{xxx}\left(3\mu_{y_xy_x}\right) + y_{xxxx}(\mu_{y_x}).\end{aligned} \tag{14.101}$$

The equation and its differential consequences

$$y_{xxx} = 0, \qquad y_{xxxx} = 0 \tag{14.102}$$

are applied to the invariance condition (14.91), and the remaining terms are gathered together. The result is the determining equations of the Lie–Bäcklund group,

$$\begin{aligned}\mu_{xxx} + 3y_x\mu_{xxy} + 3y_x^2\mu_{xyy} + y_x^3\mu_{yyy} &= 0,\\ 3\mu_{xxy_x} + 6y_x\mu_{xyy_x} + 3\mu_{xy} + 3y_x\mu_{yy} &= 0,\\ 3\mu_{xy_xy_x} + 3\mu_{yy_x} + 3y_x\mu_{yy_xy_x} &= 0,\\ \mu_{y_xy_xy_x} &= 0.\end{aligned} \tag{14.103}$$

The unknown infinitesimal turns out to be

$$\begin{aligned}\mu[x, y, y_x] &= a^1 + a^2x + a^3(x)^2 + a^4y + a^5(y_x) + a^6(xy_x)\\ &\quad + a^7\left(2xy - x^2y_x\right) + a^8\left(4y^2 - 4xyy_x + x^2y_x^2\right)\\ &\quad + a^9\left(xy_x^2 - 2yy_x\right) + a^{10}\left(y_x^2\right).\end{aligned} \tag{14.104}$$

Several points should be noted:

- In the process of generating the Lie–Bäcklund group, the point groups determined in (14.94) are recovered as equivalent transformations. These correspond to the parameters $a^1$ through $a^7$. The parameters $a^8$ through $a^{10}$ represent new proper Lie–Bäcklund symmetries, as evidenced by the dependence of these symmetries on $y_x^2$.

- Occasionally the equation and its higher-order differential consequences can be used to eliminate higher-order derivatives that appear in the extended transformation and thus achieve closure of the transformation on solutions of the equation. The Lie–Bäcklund transformation in this example, in conjunction with the equation, is closed in the space $(x, y, y_x, y_{xx}, y_{xxx})$. Since $\mu_{\{3\}} = 0$, the extended transformation is

$$\begin{aligned}
\tilde{x} &= x, \\
\tilde{y} &= y + s\mu[x, y, y_x], \\
\tilde{y}_{\tilde{x}} &= y_x + s\mu_{\{1\}}[x, y, y_x, y_{xx}], \\
\tilde{y}_{\tilde{x}\tilde{x}} &= y_{xx} + s\mu_{\{2\}}[x, y, y_x, y_{xx}, y_{xxx}], \\
\tilde{y}_{\tilde{x}\tilde{x}\tilde{x}} &= y_{xxx}, \\
\tilde{y}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} &= y_{xxxx}, \\
&\vdots
\end{aligned} \tag{14.105}$$

- We could continue to generate Lie–Bäcklund transformations dependent on higher-order derivatives. For a general equation, however, there is no way to predict the outcome of this process. More often than not, no new symmetries are found beyond the point and first-order symmetries. In the case of a linear equation and certain nonlinear equations a systematic procedure for generating higher-order transformations can be developed using recursion operators. These will be discussed in the next section.

What good is a Lie–Bäcklund transformation? For one thing, it can be used to generate new solutions of the equation in two somewhat distinct ways. This can be seen as follows. Let $y[x]$ be a function that satisfies the invariance condition (14.91). Such a function can be constructed from the infinitesimal itself. Consider the group corresponding to $a^9$. The invariance condition applied to $y$, $X^9 y = 0$, leads to a first-order ODE for $y[x]$,

$$\left(xy_x^2 - 2yy_x\right)\frac{\partial}{\partial y}(y) = xy_x^2 - 2yy_x = 0, \tag{14.106}$$

with the solution

$$\psi = y/x^2. \tag{14.107}$$

This is a special case of the general solution (14.96). The groups $a^7$ and $a^8$ generate the same solution, while the groups $a^1$, $a^2$, $a^3$, $a^4$ reduce to the general solution described earlier, (14.96).

Secondly, if the Lie series can be summed, the finite Lie–Bäcklund transformation can be generated from the infinitesimal to produce $\tilde{y} = G[x, y, y_x]$. In this case, quite complex solutions can be generated from simple ones.

This is a particularly simple example where the Lie series can be summed quite easily. Consider the Lie–Bäcklund operator with parameter $a^7$. The extended transformation, with $y_{xxx} = 0$ imposed, is

$$\begin{aligned} \tilde{x} &= x, \\ \tilde{y} &= y + s(2xy - x^2 y_x), \\ \tilde{y}_{\tilde{x}} &= y_x + s(2y - x^2 y_{xx}), \\ \tilde{y}_{\tilde{x}\tilde{x}} &= y_{xx} + s(2y_x - 2xy_{xx}), \\ \tilde{y}_{\tilde{x}\tilde{x}\tilde{x}} &= y_{xxx}, \\ &\vdots \end{aligned} \tag{14.108}$$

and is clearly closed in the space $(x, y, y_x, y_{xx}, y_{xxx})$. Because all higher-order extensions are zero, the Lie–Bäcklund operator truncates in this particular case to

$$U = (2xy - x^2 y_x)\frac{\partial}{\partial y} + (2y - x^2 y_{xx})\frac{\partial}{\partial y_x} + (2y_x - 2xy_{xx})\frac{\partial}{\partial y_{xx}}. \tag{14.109}$$

The various terms in the Lie series for $y$ are as follows:

$$\begin{aligned} Uy &= 2xy - x^2 y_x, \\ U(Uy) &= 2x^2 y - 2x^3 y_x + x^4 y_{xx}, \\ U(U(Uy)) &= 0. \end{aligned} \tag{14.110}$$

The series truncates to the finite one-parameter transformation

$$\tilde{y} = y + s(2xy - x^2 y_x) + \frac{s^2}{2!}(2x^2 y - 2x^3 y_x + x^4 y_{xx}), \tag{14.111}$$

where $s$ is not necessarily small and can be regarded as an arbitrary constant. The corresponding transformations of the first and second derivatives are as follows:

$$\begin{aligned} \tilde{y}_{\tilde{x}} &= y_x + s(2y - x^2 y_{xx}) + \frac{s^2}{2!}(4xy - 4x^2 y_x + 2x^3 y_{xx}), \\ \tilde{y}_{\tilde{x}\tilde{x}} &= y_{xx} + s(2y_x - 2xy_{xx}) + \frac{s^2}{2!}(4y - 4xy_x + 2x^2 y_{xx}). \end{aligned} \tag{14.112}$$

The transformations (14.111) and (14.112) completely define (along with $\tilde{x} = x$) the finite Lie–Bäcklund invariance of the equation $y_{xxx} = 0$ under the group $a^7$. Note that, in contrast to Examples 14.4 and 14.5, this symmetry is not equivalent to a point symmetry, and that the finite form of the transformation depends explicitly on derivatives.

The transformation (14.111) takes a given solution $y$ and transforms it to a new solution $\tilde{y}$. Let's rearrange the transformation as follows:

$$\tilde{y} = (1 + 2sx + s^2x^2)y - (sx^2 + s^2x^3)y_x + \left(\tfrac{1}{2}s^2x^4\right)y_{xx}. \quad (14.113)$$

If we apply (14.113) successively to a seed solution $y =$ constant, we get the following sequence of solutions:

$$\begin{aligned} y &= C, \\ \tilde{y} &= C(1 + sx)^2, \\ \tilde{\tilde{y}} &= C(1 + 2sx)^2, \\ \tilde{\tilde{\tilde{y}}} &= C(1 + 3sx)^2, \\ &\vdots \end{aligned} \quad (14.114)$$

This is an especially simple example where all aspects of the theory can be carried out in a transparent way for a particular group. The solutions (14.114) are fairly trivial and less general than the exact solution (14.96); however, they illustrate the point that transformations can generate solutions.

### 14.4.2 The Blasius Equation $y_{xxx} + yy_{xx} = 0$

The invariance condition for the point group of this equation is

$$\eta y_{xx} + \eta_{\{2\}} y + \eta_{\{3\}} = 0. \quad (14.115)$$

The point group of the Blasius equation was worked out in Chapter 8. Letting

$$\begin{aligned} \tilde{x} &= x + s\xi[x, y], \\ \tilde{y} &= y + s\eta[x, y] \end{aligned} \quad (14.116)$$

and solving the invariance condition leads to the infinitesimals

$$\xi = a + bx, \qquad \eta = -by. \quad (14.117)$$

Now let's work out the first-order Lie–Bäcklund group. Since the equation does not depend explicitly on $x$, the invariance condition has the same form as (14.115):

$$\mu y_{xx} + \mu_{\{2\}} y + \mu_{\{3\}} = 0. \tag{14.118}$$

Let the transformation depend on the first derivative:

$$\begin{aligned} \tilde{x} &= x, \\ \tilde{y} &= y + s\mu[x, y, y_x]. \end{aligned} \tag{14.119}$$

One expands the invariance condition and uses the differential consequences of the equation to replace higher-order derivatives:

$$y_{xxx} = -y y_{xx}, \qquad y_{xxxx} = -y_x y_{xx} + y^2 y_{xx}. \tag{14.120}$$

Solving the determining equations leads to

$$\xi = 0, \qquad \eta = a y_x + b(y + x y_x). \tag{14.121}$$

This result is just the two-parameter group of Lie–Bäcklund transformations, which is equivalent to the point group of the Blasius equation. In this case no new symmetries are found. Nevertheless, this form of the point group can be used to generate an invariant solution of the Blasius equation as follows:

$$\eta \frac{\partial}{\partial y}(y) = 0 \quad \Rightarrow \quad y + x y_x = 0 \quad \Rightarrow \quad \psi = (x + C)y. \tag{14.122}$$

Where the group (14.121) is used with $a = 0$. If we substitute (14.122) into the Blasius equation to evaluate $\psi$, the result is the exact solution

$$y = \frac{3}{x + C}. \tag{14.123}$$

We have seen this solution before in Chapter 10, Section 10.3.2, where the phase portrait of the Blasius equation was discussed. The solution (14.123) corresponds to the invariant solution at the spiral-node critical point in Figure 10.2 at position $(\gamma, H) = (-\frac{1}{3}, -\frac{2}{3})$.

Let's repeat the process, and this time allow the infinitesimal to depend on the second derivative:

$$\begin{aligned} \tilde{x} &= x, \\ \tilde{y} &= y + s\mu[x, y, y_x, y_{xx}]. \end{aligned} \tag{14.124}$$

Now the invariance condition is

$$\begin{aligned} y_{xx}\mu[x, y, y_x, y_{xx}] + y\mu_{\{2\}}[x, y, y_x, y_{xx}, y_{xxx}, y_{xxxx}] \\ + \mu_{\{3\}}[x, y, y_x, y_{xx}, y_{xxx}, y_{xxxx}, y_{xxxxx}] = 0, \end{aligned} \quad (14.125)$$

where the dependences on derivatives up to order $r + p = 5$ are shown. The Blasius equation and its differential consequences are as follows:

$$\begin{aligned} y_{xxx} &= -yy_{xx}, \\ y_{xxxx} &= -y_x y_{xx} + y^2 y_{xx}, \\ y_{xxxxx} &= -y_{xx}^2 + 3yy_x y_{xx} - y^3 y_{xx}. \end{aligned} \quad (14.126)$$

Using (14.126), no $y$-derivative of greater than second order will appear in the invariance condition, and the extended group is closed in the space $(x, y, y_x, y_{xx})$:

$$\begin{aligned} \tilde{x} &= x, \\ \tilde{y} &= y + s\mu[x, y, y_x, y_{xx}], \\ \tilde{y}_{\tilde{x}} &= y_x + s\mu_{\{1\}}[x, y, y_x, y_{xx}, -yy_{xx}], \\ \tilde{y}_{\tilde{x}\tilde{x}} &= y_{xx} + s\mu_{\{2\}}[x, y, y_x, y_{xx}, -yy_{xx}, -y_x y_{xx} + y^2 y_{xx}]. \end{aligned} \quad (14.127)$$

In principle the infinitesimal transformation (14.127) could be used in a Lie series to generate the finite form of the transformation:

$$\begin{aligned} \tilde{x} &= x, \\ \tilde{y} &= G[x, y, y_x, y_{xx}, s]. \end{aligned} \quad (14.128)$$

Such a transformation, if it could be found, would be extremely interesting, because it could be used to generate nontrivial solutions of the Blasius equation from a given solution merely by differentiation and substitution in the transformation. However, when the invariance condition (14.125) is worked out and the determining equations of the group are solved, the result is

$$\xi = 0, \qquad \mu = ay_x + b(y + xy_x) \quad (14.129)$$

The first-order Lie–Bäcklund transformation is retrieved again. Nothing new is generated at the second order.

This example highlights the fundamental conundrum of Lie–Bäcklund transformations. There is usually no way of knowing at what order, if any, new nontrivial results will be obtained. It is therefore critical that the process of finding

Lie–Bäcklund transformations be automated to allow one to search the various possible orders. Now let the transformation depend on third derivatives,

$$\begin{aligned}\tilde{x} &= x,\\ \tilde{y} &= y + s\mu[x, y, y_x, y_{xx}, y_{xxx}].\end{aligned} \tag{14.130}$$

When higher-order consequences of the Blasius equation (up to sixth derivatives) are generated and the invariance condition is solved, the result is the following:

$$\begin{aligned}\mu = a(y_x) + b(y + xy_x) + c(y_x)(y_{xxx} + yy_{xx})\\ + d(y_x)(y_{xxx} + yy_{xx})^2 + e(y + xy_x)(y_{xxx} + yy_{xx}).\end{aligned} \tag{14.131}$$

At this order, three new proper Lie–Bäcklund symmetries are found, but they are rather trivial in that they are formed from products of the two basic symmetries and the equation itself. Since the variable satisfies the equation, these three symmetries are identically zero. If we proceed to include the fourth derivative in the infinitesimal, we get back the same result (14.131).

### *14.4.3 A Particle Moving under the Influence of a Spherically Symmetric Inverse-Square Body Force*

Let's return to the Kepler problem discussed in Chapter 2, Section 2.2 and in Chapter 3, Example 3.11. A particle moving in three dimensions under the influence of a spherically symmetric body force satisfies the following system of three ODEs:

$$\begin{aligned}\Psi^x[t, x, y, z, x_t, y_t, z_t, x_{tt}, y_{tt}, z_{tt}] &= mx_{tt} + \frac{\gamma x}{r^3} = 0,\\ \Psi^y[t, x, y, z, x_t, y_t, z_t, x_{tt}, y_{tt}, z_{tt}] &= my_{tt} + \frac{\gamma y}{r^3} = 0,\\ \Psi^z[t, x, y, z, x_t, y_t, z_t, x_{tt}, y_{tt}, z_{tt}] &= mz_{tt} + \frac{\gamma z}{r^3} = 0,\end{aligned} \tag{14.132}$$

where

$$r = (x^2 + y^2 + z^2)^{1/2}. \tag{14.133}$$

If $\gamma > 0$ the force is attracting, and if $\gamma < 0$ the force is repelling.

First, let's find the point groups that leave the system (14.132) invariant. Assume an infinitesimal transformation of the following form:

$$\begin{aligned}\tilde{t} &= t + s\xi[t, x, y, z],\\ \tilde{x} &= x + s\eta^{x}[t, x, y, z],\\ \tilde{y} &= y + s\eta^{y}[t, x, y, z],\\ \tilde{z} &= z + s\eta^{z}[t, x, y, z].\end{aligned} \tag{14.134}$$

Expanding each equation in a Lie series and requiring invariance under the group (14.134) leads to the following three invariance conditions:

$$\begin{aligned}X_{\{2\}}\Psi^{x}[t, x, y, z, x_t, y_t, z_t, x_{tt}, y_{tt}, z_{tt}] &= 0,\\ X_{\{2\}}\Psi^{y}[t, x, y, z, x_t, y_t, z_t, x_{tt}, y_{tt}, z_{tt}] &= 0,\\ X_{\{2\}}\Psi^{z}[t, x, y, z, x_t, y_t, z_t, x_{tt}, y_{tt}, z_{tt}] &= 0,\end{aligned} \tag{14.135}$$

where the group operator is of the form

$$\begin{aligned}X_{\{2\}} = \xi\frac{\partial}{\partial t} + \eta^{x}\frac{\partial}{\partial x} + \eta^{y}\frac{\partial}{\partial y} + \eta^{z}\frac{\partial}{\partial z} + \eta^{x}_{\{t\}}\frac{\partial}{\partial x_t} + \eta^{y}_{\{t\}}\frac{\partial}{\partial y_t} + \eta^{z}_{\{t\}}\frac{\partial}{\partial z_t}\\ + \eta^{x}_{\{tt\}}\frac{\partial}{\partial x_{tt}} + \eta^{y}_{\{tt\}}\frac{\partial}{\partial y_{tt}} + \eta^{z}_{\{tt\}}\frac{\partial}{\partial z_{tt}}.\end{aligned} \tag{14.136}$$

Carrying through the differentiation leads to the following three invariance conditions:

$$\begin{aligned}m\eta^{x}_{\{2\}} - \eta^{x}\left(-\frac{\gamma}{r^3} + 3\gamma\frac{x^2}{r^5}\right) - \eta^{y}\left(3\gamma\frac{xy}{r^5}\right) - \eta^{z}\left(3\gamma\frac{xz}{r^5}\right) &= 0,\\ m\eta^{y}_{\{2\}} - \eta^{x}\left(3\gamma\frac{yx}{r^5}\right) - \eta^{y}\left(-\frac{\gamma}{r^3} + 3\gamma\frac{y^2}{r^5}\right) - \eta^{z}\left(3\gamma\frac{yz}{r^5}\right) &= 0,\\ m\eta^{z}_{\{2\}} - \eta^{x}\left(3\gamma\frac{zx}{r^5}\right) - \eta^{y}\left(3\gamma\frac{zy}{r^5}\right) - \eta^{z}\left(-\frac{\gamma}{r^3} + 3\gamma\frac{z^2}{r^5}\right) &= 0.\end{aligned} \tag{14.137}$$

Substituting the expressions for the extensions, gathering terms, and solving the resulting set of 48 determining equations for the unknowns $(\xi, \eta^{x}, \eta^{y}, \eta^{z})$ using the package **IntroToSymmetry.m** leads to the following infinitesimals:

$$\begin{aligned}
\xi &= a^1 + a^5\left(\tfrac{3}{2}t\right),\\
\eta^x &= a^5 x - a^4 y - a^3 z,\\
\eta^y &= a^4 x + a^5 y - a^2 z,\\
\eta^z &= a^3 x + a^2 y + a^5 z.
\end{aligned} \tag{14.138}$$

The system (14.132) is invariant under a five-parameter group of time translation, three rotations, and one dilation:

$$\begin{gathered}
X^1 = \frac{\partial}{\partial t},\\
X^2 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \qquad X^3 = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}, \qquad X^4 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},\\
X^5 = \frac{2}{3}x\frac{\partial}{\partial x} + \frac{2}{3}y\frac{\partial}{\partial y} + \frac{2}{3}z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}.
\end{gathered} \tag{14.139}$$

Now let's look for first order Lie–Bäcklund groups. We seek a transformation of the following form that leaves the system (14.132) invariant:

$$\begin{aligned}
\tilde{t} &= t,\\
\tilde{x} &= x + s\eta^x[t, x, y, z, x_t, y_t, z_t],\\
\tilde{y} &= y + s\eta^y[t, x, y, z, x_t, y_t, z_t],\\
\tilde{z} &= z + s\eta^z[t, x, y, z, x_t, y_t, z_t].
\end{aligned} \tag{14.140}$$

Here we have used the symbol $\eta$ for the unknown infinitesimal instead of $\mu$, in recognition of the fact that there is no fundamental need to distinguish symbolically between the two types of transformations. This is consistent with the software package provided with the text, which only uses the symbol $\eta$. Thus the invariance condition (14.135) remains symbolically the same, and the twice extended operator (14.136) and differentiated operator condition (14.137) retain the same form except that the detailed expressions for the infinitesimals $\eta_{\{1\}}$ and $\eta_{\{2\}}$ are considerably longer and the group operator symbol $X_{\{2\}}$ is replaced by $U$ to indicate that the independent variable is left untransformed.

The invariance conditions are

$$\begin{aligned}
U\Psi^x[t, x, y, z, x_t, y_t, z_t, x_{tt}, y_{tt}, z_{tt}] = 0,\\
U\Psi^y[t, x, y, z, x_t, y_t, z_t, x_{tt}, y_{tt}, z_{tt}] = 0,\\
U\Psi^z[t, x, y, z, x_t, y_t, z_t, x_{tt}, y_{tt}, z_{tt}] = 0,
\end{aligned} \tag{14.141}$$

where

$$U = \eta^x \frac{\partial}{\partial x} + \eta^y \frac{\partial}{\partial y} + \eta^z \frac{\partial}{\partial z} + \eta^x_{\{t\}} \frac{\partial}{\partial x_t} + \eta^y_{\{t\}} \frac{\partial}{\partial y_t} + \eta^z_{\{t\}} \frac{\partial}{\partial z_t} + \eta^x_{\{tt\}} \frac{\partial}{\partial x_{tt}} + \eta^y_{\{tt\}} \frac{\partial}{\partial y_{tt}} + \eta^z_{\{tt\}} \frac{\partial}{\partial z_{tt}} + \cdots . \quad (14.142)$$

Expanding the invariance conditions (14.141) and collecting terms leads to (only) three determining equations for the unknown infinitesimals of the group. The reason for the dramatic reduction in the number of determining equations from 48 to 3 is that now the first derivatives are regarded as independent variables and not free coefficients in the invariance condition. As a result, the invariance condition parses into a much smaller set of grouped terms.

Here is the result. First, here are the Lie–Bäcklund groups, which are equivalent to the five original point groups:

$$U^1 = x_t \frac{\partial}{\partial x} + y_t \frac{\partial}{\partial y} + z_t \frac{\partial}{\partial z},$$

$$U^2 = -z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}, \qquad U^3 = -z \frac{\partial}{\partial x} + x \frac{\partial}{\partial z}, \qquad U^4 = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y},$$

$$U^5 = \left(-\frac{2}{3}x + t x_t\right) \frac{\partial}{\partial x} + \left(-\frac{2}{3}y + t y_t\right) \frac{\partial}{\partial y} + \left(-\frac{2}{3}z + t z_t\right) \frac{\partial}{\partial z}. \quad (14.143)$$

In addition to these five, three new symmetries are identified

$$U^6 = \left(-\frac{1}{2}y y_t - \frac{1}{2}z z_t\right) \frac{\partial}{\partial x} + \left(-\frac{1}{2}y x_t + x y_t\right) \frac{\partial}{\partial y} + \left(-\frac{1}{2}z x_t + x z_t\right) \frac{\partial}{\partial z},$$

$$U^7 = \left(y x_t - \frac{1}{2}x y_t\right) \frac{\partial}{\partial x} + \left(-\frac{1}{2}x x_t - \frac{1}{2}z z_t\right) \frac{\partial}{\partial y} + \left(-\frac{1}{2}z y_t + y z_t\right) \frac{\partial}{\partial z},$$

$$U^8 = \left(z x_t - \frac{1}{2}x z_t\right) \frac{\partial}{\partial x} + \left(z y_t - \frac{1}{2}y z_t\right) \frac{\partial}{\partial y} + \left(-\frac{1}{2}x x_t - \frac{1}{2}y y_t\right) \frac{\partial}{\partial z}. \quad (14.144)$$

The operators (14.144) can be written more concisely as

$$U^j = \left(x^j x^i_t - \frac{1}{2}x^i x^j_t - \frac{1}{2}(x^k x^k_t)\delta^j_i\right) \frac{\partial}{\partial x^i}, \qquad j = 1, 2, 3. \quad (14.145)$$

In (14.145) the index $j$ refers to each coordinate direction in turn and should not be confused with that on the first three operators in (14.143). We can use these groups to generate particular solutions of the system of equations. For example, the operator $U^5$ generates the system of first-order equations

$$\begin{aligned} U^5 x &= -\frac{2}{3}x + t x_t = 0, \\ U^5 y &= -\frac{2}{3}y + t y_t = 0, \\ U^5 z &= -\frac{2}{3}z + t z_t = 0, \end{aligned} \tag{14.146}$$

The system (14.146) generates the solution for a particle beginning at the origin with a singular velocity and moving outward along a radius against an attractive force,

$$\begin{aligned} x &= \frac{3^{1/6}}{2^{1/3}}\left(\frac{\gamma}{m}\right)^{1/3} t^{2/3}, \\ y &= \frac{3^{1/6}}{2^{1/3}}\left(\frac{\gamma}{m}\right)^{1/3} t^{2/3}, \\ z &= \frac{3^{1/6}}{2^{1/3}}\left(\frac{\gamma}{m}\right)^{1/3} t^{2/3}, \end{aligned} \tag{14.147}$$

where $\gamma > 0$. The finite point transformation equivalent to $U^5$ is just the dilation group of the Kepler equations,

$$\tilde{x} = e^{2a}x, \qquad \tilde{y} = e^{2a}y, \qquad \tilde{z} = e^{2a}z, \qquad \tilde{t} = e^{3a}t, \qquad \tilde{m} = m, \tag{14.148}$$

used in Chapter 4, Example 4.4 [see Equation (4.112)] to define Kepler's third law. In Chapter 15 we will use these groups to generate invariants of the motion for the system (14.132).

## 14.5 Recursion Operators

In some cases, higher-order Lie–Bäcklund symmetries can be generated directly from lower-order symmetries without having to solve the determining equations generated by an increasingly complicated invariance condition at every level. Instead one can use what are called *recursion operators*. Given some differential equation, the basic idea is to search for nontrivial differential operators that commute with the Lie–Bäcklund operator derived from the invariance condition. The basic ideas behind recursion operators will be described primarily through several selected examples. An excellent treatment can be found in Olver [14.3] and in Chapter 5 of Bluman and Kumei [14.4].

### 14.5.1 Linear Equations

For linear equations, recursion operators can be generated directly from the equivalence form of the Lie–Bäcklund operator corresponding to the point groups of the equation.

#### 14.5.1.1 The Heat Equation

Consider the linear heat equation $y_{xx} - y_t = 0$. The corresponding invariance condition is $\eta_{\{xx\}} - \eta_{\{t\}} = 0$. If $\eta$ is the infinitesimal of a Lie–Bäcklund transformation with the $\xi_j$ equal to zero, then using (14.68) the invariance condition (14.77) can be written in the operator form

$$L\eta = 0, \tag{14.149}$$

where $L$ is the differential operator

$$L = D_x^2 - D_t \tag{14.150}$$

and $\eta$ in general depends on $(x, t, y, y_x, y_t, y_{xx}, y_{xt}, y_{tt}, \ldots)$. Clearly the invariance condition is satisfied by $\eta = y$, simply by the assumption that $y$ is a solution of the heat equation. In other words,

$$Ly = D_x^2 y - D_t y = y_{xx} - y_t = 0. \tag{14.151}$$

Similarly, (14.149) is satisfied by $\eta = f[x, t]$ where $f$ is any function that satisfies the heat equation. Such solutions of the invariance condition are called *trivial* groups.

In general, for a linear equation, $\eta$ has a special structure in that it depends linearly on the dependent variables and their derivatives. For example, the first-order Lie–Bäcklund infinitesimal of the heat equation is

$$\begin{aligned}\eta &= ay_x + b(xy + 2ty_x) + c(2ty_t + xy_x)\\ &\quad + d\left(\left(\frac{t}{2} + \frac{x^2}{4}\right)y + t^2 y_t + xty_x\right) + ey_t + fy.\end{aligned} \tag{14.152}$$

Notice that every term in (14.152) contains $y$ or some derivative of $y$. This is always the case when the groups are equivalent to point groups, and generally the case for nontrivial Lie–Bäcklund groups that leave linear equations invariant. We can use this fact to write (14.152) as

$$\eta = Ry, \tag{14.153}$$

where $R$ is the operator

$$\begin{aligned} R = {} & aD_x + b(x + 2tD_x) + c(2tD_t + xD_x) \\ & + d\left(\left(\frac{t}{2} + \frac{x^2}{4}\right) + t^2 D_t + xtD_x\right) + eD_t + f. \end{aligned} \tag{14.154}$$

The invariance condition (14.149) now becomes

$$LRy = 0. \tag{14.155}$$

So if $y$ satisfies $Ly = 0$, then $\eta = Ry$ satisfies $L\eta = 0$. Similarly, $\tilde{\eta} = R\eta = R^2 y$ satisfies $L\tilde{\eta} = 0$, and so forth, up to any order.

The operator $R$ in (14.154) is a recursion operator for the heat equation and can be used to generate Lie–Bäcklund symmetries of any order we wish. For example, the group with parameter $c$ and recursion operator $R^c = 2tD_t + xD_x$ generates

$$\begin{aligned} R^c y &= 2ty_t + xy_x, \\ R^c(R^c y) &= 4ty_t + 4t^2 y_{tt} + 4xty_{xt} + xy_x + x^2 y_{xx}, \\ &\vdots \end{aligned} \tag{14.156}$$

as the first few symmetries.

Notice that, when one begins with a Lie–Bäcklund infinitesimal equivalent to a point group, at every level the infinitesimal is linear in $y$ and the various derivatives of $y$. In general, a system of linear equations has Lie–Bäcklund infinitesimals of the form

$$\eta^i = f_k^i[x]y^k + f_k^{ij}[x]y_j^k + f_k^{ij_1j_2}[x]y_{j_1j_2}^k + \cdots \tag{14.157}$$

with the matrix of operators

$$R_k^i = f_k^i[x] + f_k^{ij}[x]D_j + f_k^{ij_1j_2}[x]yD_{j_1}D_{j_2} + \cdots, \tag{14.158}$$

enabling one to state the following theorem.

***Theorem 14.3.*** *Let the vector y be the solution of a system of linear differential equations* $\psi^i = \Psi^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots]$ *with invariance condition operators* $L^i$

*and infinitesimals* $\eta^i$ *and with recursion operators of the form (14.158). Given that*

$$L^i y^i = 0 \quad (no\ sum) \tag{14.159}$$

*and*

$$L^i \eta^i = 0 \quad (no\ sum), \tag{14.160}$$

*where* $\eta^i = R^i_k y^k$. *Then* $\tilde{\eta}^i = R^i_k \eta^k$ *satisfies* $L\tilde{\eta}^i = 0$, $\tilde{\tilde{\eta}}^i = R^i_k \tilde{\eta}^k$, *satisfies* $L\tilde{\tilde{\eta}}^i = 0$, *and so forth to any order.*

The whole process of constructing the recursion operator for linear equations relies on the fact that one immediately has a solution of the invariance condition in the form of a solution of the original equation.

### 14.5.2 Nonlinear Equations

Recursion operators can often be constructed for nonlinear equations by solving a system of determining equations for the coefficients in the operator. The basic idea is to search for a differential operator that commutes with the Lie–Bäcklund operator of the invariance condition. The procedure is illustrated in the following examples.

#### 14.5.2.1 Burgers Potential Equation

Let's construct recursion operators for the Burgers potential equation,

$$\phi_t + \frac{1}{2}(\phi_x)^2 - \phi_{xx} = 0. \tag{14.161}$$

First analyze the point symmetries of (14.161). Let the variables be transformed as

$$\begin{aligned} \tilde{x} &= x + s\xi[x, t, \phi], \\ \tilde{t} &= t + s\tau[x, t, \phi], \\ \tilde{\phi} &= \phi + s\eta[x, t, \phi]. \end{aligned} \tag{14.162}$$

The invariance condition is

$$\phi_x \eta_{\{x\}} + \eta_{\{t\}} - \eta_{\{xx\}} = 0. \tag{14.163}$$

The various extensions in (14.163) are substituted, and the resulting expression is parsed into the determining equations of the group. Using the package

**IntroToSymmetry.m** to solve the resultant set of determining equations leads to the basic point symmetries of the Burgers potential equation, which constitute a six-parameter Lie algebra with the following infinitesimals:

$$\begin{aligned} \xi &= a^1 + a^2 t + a^3 x + a^4(2xt), \\ \tau &= a^5 + a^3(2t) + a^4(2t^2), \\ \eta &= a^6 + a^2 x + a^4(2t + x^2). \end{aligned} \tag{14.164}$$

The corresponding group operators are

$$\begin{aligned} X^1 &= \frac{\partial}{\partial x}, \qquad X^2 = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial \phi}, \qquad X^3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \\ X^4 &= (2xt)\frac{\partial}{\partial x} + (2t^2)\frac{\partial}{\partial t} + (2t + x^2)\frac{\partial}{\partial \phi}, \qquad X^5 = \frac{\partial}{\partial t}, \qquad X^6 = \frac{\partial}{\partial \phi}. \end{aligned} \tag{14.165}$$

In addition, the invariance condition (14.163) is satisfied by the infinite-dimensional group

$$X^7 = f[x,t]e^{\phi/2}\frac{\partial}{\partial \phi}, \tag{14.166}$$

where $f[x,t]$ is a solution of the heat equation $f_{xx} = f_t$.

Formulating the equation in terms of a potential leads to two additional symmetries over the five-parameter point group of the Burgers equation: invariance under translation in $\phi$, and the infinite-dimensional group (14.166). The latter will be discussed again in Chapter 16, where it will be seen to be related to a nonlocal group of the Burgers equation.

Next we need to work out the first-order Lie–Bäcklund transformation of (14.161). Let the variables be transformed as

$$\begin{aligned} \tilde{x} &= x, \\ \tilde{t} &= t, \\ \tilde{\phi} &= \phi + s\eta[x,t,\phi,\phi_x,\phi_t]. \end{aligned} \tag{14.167}$$

Since Equation (14.161) does not depend explicitly on the independent variables, the invariance condition retains the same basic form as (14.163):

$$\begin{aligned} &\phi_x \eta_{\{x\}}[x,t,\phi,\phi_x,\phi_t,\phi_{xx},\phi_{xt},\phi_{tt}] + \eta_{\{t\}}[x,t,\phi,\phi_x,\phi_t,\phi_{xx},\phi_{xt},\phi_{tt}] \\ &\quad - \eta_{\{xx\}}[x,t,\phi,\phi_x,\phi_t,\phi_{xx},\phi_{xt},\phi_{tt},\phi_{xxx},\phi_{xxt},\phi_{xtt},\phi_{ttt}] = 0. \end{aligned} \tag{14.168}$$

The extensions are worked out, and the determining equations are solved for the infinitesimal yielding

$$\eta = a^1\phi_x + a^2(x - t\phi_x) + a^3(-x\phi_x - 2t\phi_t) + a^4\left(x^2 + 2t - 2xt\phi_x - 2t^2\phi_t\right) + a^5\phi_t + a^6. \tag{14.169}$$

In addition there is the Lie–Bäcklund symmetry identical to (14.166),

$$U = f[x, t]e^{\phi/2}\frac{\partial}{\partial\phi}. \tag{14.170}$$

At the first order, we only find the seven Lie–Bäcklund symmetries equivalent to the point groups (14.165). No new symmetries are revealed. If we let $\eta$ depend on second derivatives, the same result prevails. But if we go to third order, four additional symmetries are found. To develop these higher-order symmetries we could return to the invariance condition (14.168), allow the unknown infinitesimal $\eta$ to depend on third derivatives, and then sort and solve the determining equations. An alternative is to use recursion operators.

Recursion operators for the Burgers potential equation can be determined as follows. Let $\eta[x, t, \phi, \phi_x, \phi_t]$ be a solution of the invariance condition (14.168),

$$\eta_{xx} - \phi_x\eta_x - \eta_t = 0, \tag{14.171}$$

which we write as follows

$$L\eta[x, t, \phi, \phi_x, \phi_t] = 0, \tag{14.172}$$

where the differential operator is

$$L = D_{xx} - \phi_x D_x - D_t. \tag{14.173}$$

The equation (14.172) is a linear PDE for the unknown $\eta$. The idea now is to seek an operator that will commute with (14.173). By analogy with the recursion operator for a linear equation, (14.158), let's adopt the ansatz that a recursion operator exists of the form

$$R = f[x, t, \phi, \phi_x, \phi_t] + g[x, t, \phi, \phi_x, \phi_t]D_x \tag{14.174}$$

such that

$$L(Rv) = 0, \tag{14.175}$$

where $v(x, t)$ is a solution of the linear PDE

$$v_{xx} - \phi_x v_x - v_t = 0. \tag{14.176}$$

Now search for the conditions on $f$ and $g$ such that

$$(D_{xx} - \phi_x D_x - D_t)(f + gD_x)v = 0. \tag{14.177}$$

In (14.177) $\phi[x, t]$ is a solution of the nonlinear equation (14.161), and $v[x, t]$ is a solution of the linear PDE (14.176). Expanding (14.177) leads to

$$D_{xx}(fv + gv_x) - \phi_x D_x(fv + gv_x) - D_t(fv + gv_x) = 0. \tag{14.178}$$

The next step is to fully expand the invariance condition (14.178) and gather together various factors of $v[x, t]$ and its derivatives:

$$\begin{aligned} &v(D_{xx}f - \phi_x D_x f - D_t f) + v_x(2D_x f + D_{xx}g - \phi_x D_x g - D_t g - f\phi_x) \\ &\quad + v_{xx}(2D_x g + f - g\phi_x) + v_{xxx}(g) + v_{xt}(-g) + v_t(-f) = 0. \end{aligned} \tag{14.179}$$

The function $v[x, t]$ is a solution of (14.176), leading to the following replacements:

$$\begin{aligned} v_{xxx} &= \phi_{xx}v_x + \phi_x v_{xx} + v_{xt}, \\ v_t &= v_{xx} - \phi_x v_x. \end{aligned} \tag{14.180}$$

These reduce (14.179) to

$$\begin{aligned} &v(D_{xx}f - \phi_x D_x f - D_t f) + v_x(2D_x f + D_{xx}g - \phi_x D_x g - D_t g + \phi_{xx}g) \\ &\quad + v_{xx}(2D_x g) = 0. \end{aligned} \tag{14.181}$$

Since $v$, $v_x$, and $v_{xx}$ are arbitrary, the result is three determining equations for the two unknowns $f$ and $g$:

$$\begin{aligned} D_{xx}f - \phi_x D_x f - D_t f &= 0, \\ 2D_x f + D_{xx}g - \phi_x D_x g - D_t g + \phi_{xx}g &= 0, \\ 2D_x g &= 0. \end{aligned} \tag{14.182}$$

Solving the determining equations (14.182) leads to expressions for the unknown functions $f = 1, -\frac{1}{2}\phi_x, \frac{1}{2}(-t\phi_x - x)$ and $g = 0, 1, t$ and to the following three recursion operators for the Burgers potential equation:

$$R^1 = 1, \qquad R^2 = -\tfrac{1}{2}\phi_x + D_x, \qquad R^3 = \tfrac{1}{2}(-t\phi_x - x) + tD_x. \tag{14.183}$$

Let's check $R^2$ in (14.175). Let $\eta[x, t, \phi, \phi_x, \phi_t]$ be a solution of

$$D_{xx}\eta - \phi_x D_x\eta - D_t\eta = 0. \tag{14.184}$$

The invariance condition (14.175) is

$$(D_{xx} - \phi_x D_x - D_t)\big(-\tfrac{1}{2}\phi_x + D_x\big)\eta = 0, \tag{14.185}$$

or

$$D_{xx}\big(-\tfrac{1}{2}\phi_x\eta + D_x\eta\big) - \phi_x D_x\big(-\tfrac{1}{2}\phi_x\eta + D_x\eta\big) - D_t\big(-\tfrac{1}{2}\phi_x\eta + D_x\eta\big) = 0. \tag{14.186}$$

Expand the first term

$$D_{xx}\big(-\tfrac{1}{2}\phi_x\eta + D_x\eta\big) = -\tfrac{1}{2}\phi_{xxx}\eta - \phi_{xx}D_x\eta - \tfrac{1}{2}\phi_x D_{xx}\eta + D_{xxx}\eta, \tag{14.187}$$

the second term

$$\phi_x D_x\big(-\tfrac{1}{2}\phi_x\eta + D_x\eta\big) = -\tfrac{1}{2}\phi_x\phi_{xx}\eta - \tfrac{1}{2}\phi_x^2 D_x\eta + \phi_x D_{xx}\eta, \tag{14.188}$$

and the third term

$$D_t\big(-\tfrac{1}{2}\phi_x\eta + D_x\eta\big) = -\tfrac{1}{2}\phi_{xt}\eta - \tfrac{1}{2}\phi_x D_t\eta + D_{xt}\eta. \tag{14.189}$$

Fully expanded, (14.186) can be organized as follows:

$$\begin{aligned}\tfrac{1}{2}\eta D_x\big(\phi_t + \tfrac{1}{2}(\phi_x)^2 - \phi_{xx}\big) \;-\; \tfrac{1}{2}\phi_x(D_{xx}\eta - \phi_x D_x\eta - D_t\eta)\\ + D_x(D_{xx}\eta - \phi_x D_x\eta - D_t\eta) = 0.\end{aligned} \tag{14.190}$$

The first term in (14.190) is zero by (14.161), and so we have the expected result that the recursion operator $R^2$ commutes with the Lie–Bäcklund operator (14.173): $L(R\eta) = R(L\eta)$, or

$$\begin{aligned}&(D_{xx} - \phi_x D_x - D_t)\big(\big(-\tfrac{1}{2}\phi_x - D_x\big)\eta\big)\\ &\quad = \big(-\tfrac{1}{2}\phi_x - D_x\big)((D_{xx} - \phi_x D_x - D_t)\eta) = 0.\end{aligned} \tag{14.191}$$

The same sort of result can be worked out for $R^3$.

Since the recursion operator produces a function that satisfies the invariance condition, it can be used repeatedly to produce a symmetry of any order. Beginning with the Lie–Bäcklund transformation corresponding to the point translation group in $x$, the first few symmetries generated by $R^2$ are

$$\begin{aligned}
\eta &= \phi_x, \\
R^2\eta &= \left(-\tfrac{1}{2}\phi_x + D_x\right)\phi_x = -\tfrac{1}{2}\phi_x^2 + \phi_{xx} = \phi_t, \\
R^2(R^2\eta) &= \left(-\tfrac{1}{2}\phi_x + D_x\right)(\phi_t) = -\tfrac{1}{2}\phi_x\phi_t + \phi_{xt}, \\
R^2(R^2(R^2\eta)) &= \left(-\tfrac{1}{2}\phi_x + D_x\right)\left(-\tfrac{1}{2}\phi_x\phi_t + \phi_{xt}\right) \\
&= \tfrac{1}{4}\phi_x^2\phi_t - \phi_x\phi_{xt} - \tfrac{1}{2}\phi_{xx}\phi_t + \phi_{xxt}.
\end{aligned} \tag{14.192}$$

In (14.192), operating once generates the group equivalent to a translation in time. Operating a second or third time produces proper Lie–Bäcklund symmetries (symmetries that are not equivalent to point symmetries).

#### *14.5.2.2 Burgers Equation and Integro-differential Operators*

If we differentiate (14.161) with respect to $x$, the result is

$$\phi_{xt} + \phi_x\phi_{xx} - \phi_{xxx} = 0. \tag{14.193}$$

Thus $u = \phi_x$ is a solution of the Burgers equation

$$u_t + uu_x - uu_{xx} = 0, \tag{14.194}$$

and $\phi$ is a potential function. First let's work out the point groups of (14.194). Let the variables be transformed as

$$\begin{aligned}
\tilde{x} &= x + s\xi[x, t, u], \\
\tilde{t} &= t + s\tau[x, t, u], \\
\tilde{u} &= u + s\eta[x, t, u].
\end{aligned} \tag{14.195}$$

The invariance condition is

$$u_x\eta + u\eta_{\{x\}} + \eta_{\{t\}} - \eta_{\{xx\}} = 0. \tag{14.196}$$

Working out the various extensions and solving the determining equations leads to the following infinitesimals of the five-parameter point group that leaves the Burgers equation invariant:

$$\begin{aligned}
\xi &= a^1 + a^2 t + a^3 x + a^4 xt, \\
\tau &= a^5 + 2a^3 t + a^4(t^2), \\
\eta &= a^2 - a^3 u + a^4(x - tu),
\end{aligned} \tag{14.197}$$

with corresponding operators

$$X^1 = \frac{\partial}{\partial x}, \qquad X^2 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \qquad X^3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u},$$
$$X^4 = x\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + (x - tu)\frac{\partial}{\partial u}, \qquad X^5 = \frac{\partial}{\partial t}. \tag{14.198}$$

Now let's work out the first-order Lie–Bäcklund group of (14.194). Let the variables be transformed as

$$\begin{aligned} \tilde{x} &= x, \\ \tilde{t} &= t, \\ \tilde{u} &= u + s\eta[x, t, u, u_x, u_t]. \end{aligned} \tag{14.199}$$

Since the equation does not depend explicitly on the independent variables, the invariance condition (14.196) retains the same basic form:

$$\begin{aligned} u_x\eta[x, t, u, u_x, u_t] &+ u\eta_{\{x\}}[x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}] \\ &+ \eta_{\{t\}}[x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}] \\ - \eta_{\{xx\}}[x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt}] &= 0. \end{aligned} \tag{14.200}$$

Using the package **`IntroToSymmetry.m`** the extensions are worked out, and the determining equations are solved for the infinitesimal

$$\begin{aligned} \eta = a^1u_x &+ a^2(1 - tu_x) + a^3(-u - xu_x - 2tu_t) \\ &+ a^4(x - ut - xtu_x - t^2u_t) + a^5u_t. \end{aligned} \tag{14.201}$$

At the first order we only find the five symmetries equivalent to the point groups. No new symmetries are revealed. If we let the infinitesimal depend on second derivatives, the same result (14.201) prevails. But if we go to third order, four additional symmetries are found:

$$\begin{aligned} U^6 &= \left(4u_{xxx} - 6uu_{xx} - 6u_x^2 + 3u^2u_x\right)\frac{\partial}{\partial u}, \\ U^7 &= \left(4tu_{xxx} + (2x - 6tu)u_{xx} - 6tu_x^2 + (3tu^2 - 2xu)u_x - u^2\right)\frac{\partial}{\partial u}, \\ U^8 &= \left(4t^2u_{xxx} + (4tx - 6t^2u)u_{xx} - 6t^2u_x^2 \right. \\ &\quad \left. + (3t^2u^2 - 4txu + x^2)u_x - 2tu^2 + 2xu + 6\right)\frac{\partial}{\partial u}, \end{aligned} \tag{14.202}$$

$$\begin{aligned}U^9 = \big(&4t^3u_{xxx} + (6t^2x - 6t^3u)u_{xx} - 6t^3u_x^2\\ &+ (3t^3u^2 - 6t^2xu + 3tx^2 + 12t^2)u_x - 3t^2u^2\\ &+ 6xtu - 3x^2 - 6t\big)\frac{\partial}{\partial u}.\end{aligned}$$

Would we find more symmetries at higher-order? The effort required, even computationally, is very substantial, so let's seek an alternative approach.

Additional symmetries of (14.194) can be generated using recursion operators. Let's identify a couple derived from recursion operators of the Burgers potential equation. The potential $\phi$ can be expressed in terms of $u$ using the integral operator

$$\phi = D_x^{-1}u. \tag{14.203}$$

A first-order Lie–Bäcklund transformation of $\phi$ is written as follows:

$$\begin{aligned}\tilde{x} &= x,\\ \tilde{t} &= t,\\ \tilde{\phi} &= \phi + s\eta[x, t, \phi, \phi_x, \phi_t].\end{aligned} \tag{14.204}$$

The equivalent transformation of $u$ is

$$\begin{aligned}\tilde{x} &= x,\\ \tilde{t} &= t,\\ D_x^{-1}\tilde{u} &= D_x^{-1}u + s\eta\big[x, t, D_x^{-1}u, D_xD_x^{-1}u, D_tD_x^{-1}u\big],\end{aligned} \tag{14.205}$$

or

$$\begin{aligned}\tilde{x} &= x,\\ \tilde{t} &= t,\\ \tilde{u} &= u + s\eta',\end{aligned} \tag{14.206}$$

where $\eta' = D_x\eta[x, t, D_x^{-1}u, u, D_tD_x^{-1}u]$ depends on integrals of the dependent variable. For example, the dilation group of the Burgers potential equation,

$$\begin{aligned}\tilde{x} &= x,\\ \tilde{t} &= t,\\ \tilde{\phi} &= \phi + s\big(-t\phi_t - \tfrac{1}{2}x\phi_x\big),\end{aligned} \tag{14.207}$$

generates

$$\begin{aligned}\tilde{x} &= x,\\ \tilde{t} &= t,\\ \tilde{u} &= u + sD_x\big(-tD_tD_x^{-1}u - \tfrac{1}{2}xu\big),\end{aligned} \tag{14.208}$$

which can be written as

$$\begin{aligned}\tilde{x} &= x,\\ \tilde{t} &= t,\\ \tilde{u} &= u + sD_x\big(-tD_tD_x^{-1}u - \tfrac{1}{2}xu\big) = u + s\big(-\tfrac{1}{2}u - tu_t - \tfrac{1}{2}xu_x\big),\end{aligned} \tag{14.209}$$

which is equivalent to the point dilation group of the Burgers equation.

Now let's look at the recursion operators of the Burgers potential equation. If

$$\begin{aligned}\tilde{x} &= x,\\ \tilde{t} &= t,\\ \tilde{\phi} &= \phi + sR\eta[x, t, \phi, \phi_x, \phi_t],\end{aligned} \tag{14.210}$$

then

$$\begin{aligned}\tilde{x} &= x,\\ \tilde{t} &= t,\\ \tilde{u} &= u + sD_xR\big(D_x^{-1}\eta'\big).\end{aligned} \tag{14.211}$$

The recursion operators of the Burgers equation corresponding to $R^2$ and $R^3$ for the Burgers potential equation are

$$\begin{aligned}R^2_{\text{Burgers}} &= D_x\big(-\tfrac{1}{2}u + D_x\big)D_x^{-1} = D_x\big(-\tfrac{1}{2}uD_x^{-1} + 1\big)\\ &= -\tfrac{1}{2}u_xD_x^{-1} - \tfrac{1}{2}u + D_x,\\ R^3_{\text{Burgers}} &= D_x\big(\tfrac{1}{2}(-tu - x) + tD_x\big)D_x^{-1}\\ &= \tfrac{1}{2}(-tu_x - 1)D_x^{-1} + \tfrac{1}{2}(-tu - x) + tD_x.\end{aligned} \tag{14.212}$$

These are both integro differential recursion operators (see Bluman and Kumei [14.4, Chapter 5]). Let's apply $R^2_{\text{Burgers}}$ to the point dilation group of the Burgers equation expressed as an equivalent Lie–Bäcklund operator:

$$\begin{aligned}&\left(-\tfrac{1}{2}u_x D_x^{-1} - \tfrac{1}{2}u + D_x\right)\left(-\tfrac{1}{2}u - tu_t - \tfrac{1}{2}xu_x\right)\\ &\quad= \frac{1}{4}u_x\left(D_x^{-1}u\right) + -\frac{3}{4}tu^2u_x + \frac{t}{2}u_x^2 + \frac{x}{2}uu_x + \frac{x^2}{8}u_x^2\\ &\qquad + \frac{1}{4}u^2 + \frac{3}{2}tuu_{xx} - \frac{1}{2}u_x + tu_xu_{xx} - tu_{xxx} - \frac{1}{2}u_x - \frac{1}{2}xu_{xx}.\end{aligned} \tag{14.213}$$

The result (14.213) defines a proper Lie–Bäcklund symmetry of the Burgers equation. Moreover, the symmetry is nonlocal in that it depends on the integral of $u$ through the term $\frac{1}{4}u_x(D_x^{-1}u)$. We shall have more to say about nonlocal symmetries in Chapter 16.

#### *14.5.2.3 The Classical Recursion Operator for the Korteweg–de Vries Equation*

Let's work out an integrodifferential recursion operator for the Korteweg–de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0. \tag{14.214}$$

The invariance condition is

$$L\eta = 0, \tag{14.215}$$

where the operator $L$ is

$$L = D_{xxx} + u_x + uD_x + D_t. \tag{14.216}$$

Let's assume that an integrodifferential recursion operator exists of the form

$$\begin{aligned}R = {}& f_1[x,t,u,u_x,u_t]D_x^{-1} + f_2[x,t,u,u_x,u_t]\\ &+ f_3[x,t,u,u_x,u_t]D_x + f_4[x,t,u,u_x,u_t]D_{xx}\end{aligned} \tag{14.217}$$

such that

$$LRv = 0 \tag{14.218}$$

when $v[x,t]$ is solution of the linear PDE

$$v_{xxx} + u_xv + uv_x + v_t = 0 \tag{14.219}$$

and $u[x,t]$ is a solution of the nonlinear PDE (14.214). Equation (14.217) is of course one of those guesses that is aided hugely by the fact that the answer is already known. That's the problem with nonlinear equations – a brilliant ansatz is needed to get anywhere.

Now expand (14.218):

$$(D_{xxx} + u_x + uD_x + D_t)\big(f_1 D_x^{-1} + f_2 + f_3 D_x + f_4 D_{xx}\big)v = 0. \quad (14.220)$$

The first term fully expanded is

$$\begin{aligned} &D_{xxx}\big(f_1 D_x^{-1}v + f_2 v + f_3 v_x + f_4 v_{xx}\big) \\ &\quad = (D_{xxx} f_1)D_x^{-1}v + 3(D_{xx} f_1)v + 3(D_x f_1)v_x + f_1 v_{xx} \\ &\qquad + (D_{xxx} f_2)v + 3(D_{xx} f_2)v_x + 3(D_x f_2)v_{xx} + f_2 v_{xxx} \\ &\qquad + (D_{xxx} f_3)v_x + 3(D_{xx} f_3)v_{xx} + 3(D_x f_3)v_{xxx} + f_3 v_{xxxx} \\ &\qquad + (D_{xxx} f_4)v_{xx} + 3(D_{xx} f_4)v_{xxx} + 3(D_x f_4)v_{xxxx} + f_4 v_{xxxxx}. \end{aligned} \quad (14.221)$$

The second term is

$$\begin{aligned} &u_x\big(f_1 D_x^{-1}v + f_2 v + f_3 v_x + f_4 v_{xx}\big) \\ &\quad = (u_x f_1)D_x^{-1}v + (u_x f_2)v + (u_x f_3)v_x + (u_x f_4)v_{xx}. \end{aligned} \quad (14.222)$$

The third term is

$$\begin{aligned} &uD_x\big(f_1 D_x^{-1}v + f_2 v + f_3 v_x + f_4 v_{xx}\big) \\ &\quad = (uD_x f_1)D_x^{-1}v + (uf_1)v + (uD_x f_2)v + (uf_2)v_x \\ &\qquad + (uD_x f_3)v_x + (uf_3)v_{xx} + (uD_x f_4)v_{xx} + (uf_4)v_{xxx}, \end{aligned} \quad (14.223)$$

and the fourth term is

$$\begin{aligned} &D_t\big(f_1 D_x^{-1}v + f_2 v + f_3 v_x + f_4 v_{xx}\big) \\ &\quad = (D_t f_1)D_x^{-1}v + (f_1)D_x^{-1}v_t + (D_t f_2)v + (f_2)v_t \\ &\qquad + (D_t f_3)v_x + (f_3)v_{xt} + (D_t f_4)v_{xx} + f_4 v_{xxt}. \end{aligned} \quad (14.224)$$

The invariance condition (14.220) can now be rearranged to read as follows:

$$\begin{aligned} &(D_{xxx} + u_x + uD_x + D_t)\big(f_1 D_x^{-1} + f_2 + f_3 D_x + f_4 D_{xx}\big)v \\ &\quad = f_2(v_{xxx} + u_x v + uv_x + v_t) \\ &\qquad + D_x^{-1}v(D_{xxx} f_1 + u_x f_1 + uD_x f_1 + D_t f_1) \\ &\qquad + v(3D_{xx} f_1 + D_{xxx} f_2 + uf_1 + uD_x f_2 + D_1 f_2) \\ &\qquad + v_x(3D_x f_1 + 3D_{xx} f_2 + D_{xxx} f_3 + u_x f_3 + uD_x f_3 + D_t f_3) \\ &\qquad + v_{xx}(f_1 + 3D_x f_2 + 3D_{xx} f_3 + D_{xxx} f_4 + u_x f_4 + uf_3 + uD_x f_4 + D_t f_4) \\ &\qquad + v_{xxx}(3D_x f_3 + 3D_{xx} f_4 + uf_4) + v_{xxxx}(f_3 + 3D_x f_4) \\ &\qquad + v_{xxxxx}(f_4) + v_{xt}(f_3) + v_{xxt}(f_4) + D_x^{-1}v_t(f_1). \end{aligned} \quad (14.225)$$

We now substitute the following rules derived from (14.219), $v_{xxx} + u_x v + u v_x + v_t = 0$:

$$\begin{aligned} D_x^{-1} v_t &= -v_{xx} - uv, \\ v_{xt} &= -v_{xxxx} - u_{xx} v - 2u_x v_x - u v_{xx}, \\ v_{xxt} &= -v_{xxxxx} - u_{xxx} v - 3u_{xx} v_x - 3u_x v_{xx} - u v_{xxx}. \end{aligned} \tag{14.226}$$

With these substitutions (14.225) becomes the following:

$$\begin{aligned} &(D_{xxx} + u_x + uD_x + D_t)\big(f_1 D_x^{-1} + f_2 + f_3 D_x + f_4 D_{xx}\big)v \\ &\quad = f_2(v_{xxx} + u_x v + u v_x + v_t) \\ &\qquad + D_x^{-1} v(D_{xxx} f_1 + u_x f_1 + uD_x f_1 + D_t f_1) \\ &\qquad + v(3D_{xx} f_1 + D_{xxx} f_2 + uf_1 + uD_x f_2 \\ &\qquad + D_t f_2 - uf_1 - u_{xxx} f_4 - u_{xx} f_3) \\ &\qquad + v_x(3D_x f_1 + 3D_{xx} f_2 + D_{xxx} f_3 + u_x f_3 + uD_x f_3 \\ &\qquad + D_t f_3 - 3u_{xx} f_4 - 2u_x f_3) \\ &\qquad + v_{xx}(f_1 + 3D_x f_2 + 3D_{xx} f_3 + D_{xxx} f_4 + u_x f_4 \\ &\qquad + uf_3 + uD_x f_4 + D_t f_4 - f_1 - 3u_x f_4 - uf_3) \\ &\qquad + v_{xxx}(3D_x f_3 + 3D_{xx} f_4 + uf_4 - uf_4) \\ &\qquad + v_{xxxx}(f_3 + 3D_x f_4 - f_3) + v_{xxxxx}(f_4 - f_4). \end{aligned} \tag{14.227}$$

The first term on the right-hand side of (14.227) is zero, and we finally we have the following six determining equations for the four unknowns, $f_1$, $f_2$, $f_3$, and $f_4$:

$$\begin{aligned} D_{xxx} f_1 + u_x f_1 + uD_x f_1 + D_t f_1 &= 0, \\ 3D_{xx} f_1 + D_{xxx} f_2 + uf_1 + uD_x f_2 + D_t f_2 - uf_1 - u_{xxx} f_4 - u_{xx} f_3 &= 0, \\ 3D_x f_1 + 3D_{xx} f_2 + D_{xxx} f_3 + u_x f_3 + uD_x f_3 + D_t f_3 - 3u_{xx} f_4 - 2u_x f_3 &= 0, \\ 3D_x f_2 + 3D_{xx} f_3 + D_{xxx} f_4 + u_x f_4 + uf_3 + uD_x f_4 + D_t f_4 - 3u_x f_4 - uf_3 &= 0, \\ 3D_x f_3 + 3D_{xx} f_4 &= 0, \\ 3D_x f_4 &= 0. \end{aligned} \tag{14.228}$$

We can simplify (14.228) to the following:

$$\begin{aligned} D_{xxx} f_1 + u_x f_1 + uD_x f_1 + D_t f_1 &= 0, \\ 3D_{xx} f_1 + D_{xxx} f_2 + uD_x f_2 + D_t f_2 - u_{xxx} f_4 - u_{xx} f_3 &= 0, \\ 3D_x f_1 + 3D_{xx} f_2 + D_{xxx} f_3 + uD_x f_3 + D_t f_3 - 3u_{xx} f_4 - u_x f_3 &= 0, \end{aligned}$$

$$\begin{aligned}3D_x f_2 + 3D_{xx} f_3 + D_{xxx} f_4 + uD_x f_4 + D_t f_4 - 2u_x f_4 &= 0,\\ 3D_x f_3 + 3D_{xx} f_4 &= 0,\\ 3D_x f_4 &= 0.\end{aligned} \tag{14.229}$$

Using the last two determining equations, (14.229) becomes

$$\begin{aligned}D_{xxx} f_1 + u_x f_1 + uD_x f_1 + D_t f_1 &= 0,\\ 3D_{xx} f_1 + D_{xxx} f_2 + uD_x f_2 + D_t f_2 - u_{xxx} f_4 - u_{xx} f_3 &= 0,\\ 3D_x f_1 + 3D_{xx} f_2 + D_t f_3 - 3u_{xx} f_4 - u_x f_3 &= 0,\\ 3D_x f_2 + D_t f_4 - 2u_x f_4 &= 0,\\ D_x f_3 &= 0,\\ D_x f_4 &= 0.\end{aligned} \tag{14.230}$$

This final set of determining equations can be solved, yielding

$$f_1 = \tfrac{1}{3}u_x, \qquad f_2 = \tfrac{2}{3}u, \qquad f_3 = 0, \qquad f_4 = 1. \tag{14.231}$$

The final result is the classical recursion operator of the Korteweg de Vries equation:

$$R = \tfrac{1}{3}u_x D_x^{-1} + \tfrac{2}{3}u + D_{xx}. \tag{14.232}$$

The four-parameter point group of the KdV equation is as follows:

$$\begin{aligned}X^a = \frac{\partial}{\partial x}, \qquad X^b = \frac{\partial}{\partial t}, \qquad X^c = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u},\\ X^d = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u},\end{aligned} \tag{14.233}$$

with equivalent Lie–Bäcklund operators

$$\begin{aligned}U^a = u_x\frac{\partial}{\partial u}, \qquad U^b = u_t\frac{\partial}{\partial u}, \qquad U^c = (1 - tu_x)\frac{\partial}{\partial u},\\ U^d = (2u + xu_x + 3tu_t)\frac{\partial}{\partial u}.\end{aligned} \tag{14.234}$$

Now use the recursion operator (14.232) to study the symmetries that can be generated from the translational group $U^a$. Note that this symmetry is in the form of a total differential and so the inverse operator is well defined. The first two symmetries generated by (14.232) are the following:

$$\begin{aligned}\eta^0 &= D_x u = u_x,\\ \eta^1 &= \left(\tfrac{1}{3}u_x D_x^{-1} + \tfrac{2}{3}u + D_{xx}\right)D_x u = uu_x + u_{xxx} = -u_t.\end{aligned} \tag{14.235}$$

Note that $\eta^1$ is simply the infinitesimal corresponding to a point translation in time. Using the KdV equation, it can be written as a total derivative,

$$\eta^1 = D_x\big(\tfrac{1}{2}u^2 + u_{xx}\big). \tag{14.236}$$

The next symmetry is

$$\begin{aligned}\eta^2 &= \big(\tfrac{1}{3}u_x D_x^{-1} + \tfrac{2}{3}u + D_{xx}\big)(uu_x + u_{xxx}) \\ &= \tfrac{5}{6}u^2u_x + \tfrac{10}{3}u_x u_{xx} + \tfrac{5}{3}uu_{xxx} + u_{xxxxx},\end{aligned} \tag{14.237}$$

which is a proper Lie–Bäcklund symmetry. This expression can also be written as a total derivative,

$$\eta^2 = D_x\big(\tfrac{5}{18}u^3 + \tfrac{5}{6}u_x^2 + \tfrac{5}{3}uu_{xx} + u_{xxxx}\big). \tag{14.238}$$

The next symmetry is

$$\eta^3 = \big(\tfrac{1}{3}u_x D_x^{-1} + \tfrac{2}{3}u + D_{xx}\big)\big(\tfrac{5}{6}u^2u_x + \tfrac{10}{3}u_x u_{xx} + \tfrac{5}{3}uu_{xxx} + u_{xxxxx}\big), \tag{14.239}$$

which becomes

$$\begin{aligned}\eta^3 &= \tfrac{35}{54}u^3u_x + \tfrac{35}{18}u_x^3 + \tfrac{70}{9}uu_x u_{xx} + \tfrac{35}{18}u^2u_{xxx} \\ &\quad + \tfrac{35}{3}u_{xx}u_{xxx} + \tfrac{7}{3}uu_{xxxx} + u_{xxxxxxx} + \tfrac{20}{3}u_x u_{xxxx} + \tfrac{1}{3}u_x u_{xxx}.\end{aligned} \tag{14.240}$$

Once again this expression can be put in the form of a total derivative,

$$\begin{aligned}\eta^3 = D_x\Big(&\tfrac{1}{3}u\big(\tfrac{5}{18}u^3 + \tfrac{5}{6}u_x^2 + \tfrac{5}{3}uu_{xx} + u_{xxxx}\big) \\ &+ \tfrac{1}{3}\big(\tfrac{5}{24}u^4 + \tfrac{5}{3}u^2u_{xx} + \tfrac{1}{2}u_{xx}^2 + uu_{xxxx} - u_x u_{xxx}\big) \\ &+ D_{xx}\big(\tfrac{5}{18}u^3 + \tfrac{5}{6}u_x^2 + \tfrac{5}{3}uu_{xx} + u_{xxxx}\big)\Big).\end{aligned} \tag{14.241}$$

It looks like there is a pattern here! Each of the generated symmetries can be written as a total derivative. In general one can write

$$\eta^{k+1} = D_x\big(\tfrac{1}{3}u\big(D_x^{-1}\eta^k\big) + \tfrac{1}{3}D_x^{-1}(u\eta^k) + D_x\eta^k\big). \tag{14.242}$$

Carrying through the differentiation to check gives

$$\eta^{k+1} = \tfrac{1}{3}u_x\big(D_x^{-1}\eta^k\big) + \tfrac{2}{3}u\eta^k + D_{xx}\eta^k, \tag{14.243}$$

which is the correct form of the recursion operator acting on the current infinitesimal.

Both integrals appearing inside the outer parentheses in (14.242) can be removed as follows. The middle term $u\eta^k$ generates the infinite sequence

$$
\begin{aligned}
u\eta^0 &= uD_x u = D_x\left(\tfrac{1}{2}u^2\right),\\
u\eta^1 &= uD_x\left(\tfrac{1}{2}u^2 + u_{xx}\right) = D_x\left(\tfrac{1}{3}u^3 + uu_{xx} - \tfrac{1}{2}u^2\right),\\
u\eta^2 &= uD_x\left(\tfrac{5}{18}u^3 + \tfrac{5}{6}u_x^2 + \tfrac{5}{3}uu_{xx} + u_{xxxx}\right)\\
&= D_x\left(\tfrac{5}{24}u^4 + \tfrac{5}{3}u^2u_{xx} + \tfrac{1}{2}u_{xx}^2 + uu_{xxxx} - u_xu_{xxx}\right),\\
&\vdots
\end{aligned}
\tag{14.244}
$$

In other words, $u\eta^k = D_x\alpha^k$, where $\alpha^k$ is a differential function, the first few terms of which are in (14.244) and so

$$
\eta^{k+1} = D_x\left(\tfrac{1}{3}u\left(D_x^{-1}\eta^k\right) + \tfrac{1}{3}\alpha^k + D_x\eta^k\right). \tag{14.245}
$$

The term $D_x^{-1}\eta^k$ can be expressed recursively in terms of $\eta^k, \ldots, \eta^0$.

In summary, the recursion operator (14.232) acting on the translational symmetry, of the KdV equation generates an infinite sequence of Lie–Bäcklund symmetries, the first few of which are

$$
\begin{aligned}
\eta^0 &= D_x u,\\
\eta^1 &= D_x\left(\tfrac{1}{2}u^2 + u_{xx}\right),\\
\eta^2 &= D_x\left(\tfrac{5}{18}u^3 + \tfrac{5}{6}u_x^2 + \tfrac{5}{3}uu_{xx} + u_{xxxx}\right),\\
\eta^3 &= D_x\Big(\tfrac{1}{3}u\left(\tfrac{5}{18}u^3 + \tfrac{5}{6}u_x^2 + \tfrac{5}{3}uu_{xx} + u_{xxxx}\right)\\
&\qquad + \frac{1}{3}\left(\tfrac{5}{24}u^4 + \tfrac{5}{3}u^2u_{xx} + \tfrac{1}{2}u_{xx}^2 + uu_{xxxx} - u_xu_{xxx}\right)\\
&\qquad + D_{xx}\left(\tfrac{5}{18}u^3 + \tfrac{5}{6}u_x^2 + \tfrac{5}{3}uu_{xx} + u_{xxxx}\right)\Big),\\
&\vdots
\end{aligned}
\tag{14.246}
$$

The corresponding sequence of Lie–Bäcklund operators is $U^k = \eta^k\partial/\partial u$ and since $u$ is a solution of the KdV equation, $U^k u = 0$. The result is an infinite sequence of conservation laws for the KdV equation, $\eta^k = D_x(\,) = 0$.

A close look at how this all comes about reveals that the key feature of the KdV equation that enables the succession of infinitesimals to be put in conservation form is the fact that the highest and second highest derivatives in the equation are separated by two orders. The same analysis would not work on the Burgers equation, for example.

## 14.6 Concluding Remarks

The discussion of Lie–Bäcklund groups is nearly done. Our work won't quite be complete, however, until we discuss the use of Lie–Bäcklund transformations to establish the invariance properties of integrals in Chapter 15. A generalization to many independent variables of the theory of classical dynamics for one independent variable, covered in Chapter 4, will be used to derive a generalized system of Euler–Lagrange equations. In the process we will establish Noether's theorem connecting symmetries and conservation laws. This whole subject is an area of active research, and much more about these fascinating symmetries can be found in References [14.1] to [14.6].

It was pointed out in Section 14.4.3, where the Kepler problem was described, and it bears repeating here, that the software package **`IntroToSymmetry.m`** uses the symbol $\eta$ for the unknown infinitesimal instead of $\mu$ in recognition of the fact that there is no fundamental need to distinguish symbolically between the two quantities. As far as the software is concerned, the difference between a Lie point and a Lie–Bäcklund transformation is a matter of choosing the order of derivative that the unknown infinitesimals are assumed to depend on (the program parameter **`r`**) and deciding whether the transformations of independent variables should be turned off or not (the program parameter **`xseon`**).

## 14.7 Exercises

14.1 Derive the expressions for the infinitesimals (14.18) from (14.16).

14.2 Show by hand that (14.30) is invariant under the contact transformation (14.35).

14.3 Show that the equation for a harmonic oscillator, $y_{xx} + y = 0$, admits the contact symmetries

$$\begin{aligned}(\xi, \eta) &= (yy_x, -y^3),\\ (\xi, \eta) &= (y_x \sin x, -y^2 \sin x), \qquad (14.247)\\ (\xi, \eta) &= (y_x \cos x, -y^2 \cos x).\end{aligned}$$

14.4 Use the software package **`IntroToSymmetry.m`** to work out all the point and first-order Lie–Bäcklund symmetries of the equation

$$U_{xx} + aU_{xy} + U_{yy} = 0. \qquad (14.248)$$

Which, if any, are proper Lie–Bäcklund symmetries? Develop recursion operators for each of the point groups of the equation. Use these to generate the first three Lie–Bäcklund symmetries from each point group.

14.5 Use the package to work out the first order Lie–Bäcklund symmetries (14.169), (14.201) and (14.202)

14.6 Show that

$$R^3 = \tfrac{1}{2}(-t\phi_x - x) + t D_x \tag{14.249}$$

is a valid recursion operator for the Burgers potential equations $\phi_t + \frac{1}{2}(\phi_x)^2 - \phi_{xx} = 0$. Use it to generate the first three Lie–Bäcklund symmetries arising from translational invariance in space, $\eta = \phi_x$. Compare with the symmetries generated by $R^2$.

14.7 Use the package **`IntroToSymmetry.m`** to work out all point and first-order Lie–Bäcklund symmetries of the equation

$$\psi_{tt} + \psi_{xxxx} + 2\psi_{xxyy} + \psi_{yyyy} = 0. \tag{14.250}$$

Note that the linearity of the equation means that an arbitrary solution of (14.250) will appear in $\eta$, the infinitesimal transformation of $\psi$. This will be evidenced by the package as successively higher-order terms in the solution for $\eta$. These will need to be collected together in order to identify the remaining groups.

## References


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# 15

## *Variational Symmetries and Conservation Laws*

### 15.1 Introduction

In Chapter 4 the Euler–Lagrange equations of classical dynamics were developed from a variational integral for the action $S$. In one of the seminal contributions to modern physics in the early twentieth century, Amalie Emmy Noether, in her famous 1918 paper [15.1], showed that the conservation laws of classical physics are directly related to Lie symmetries of the corresponding system of Euler–Lagrange equations. Noether (1882–1935) grew up in Erlangen, where Felix Klein a decade before her birth had established his "*Erlangen programm*" designed to unify group theory and geometry. She joined Kline and David Hilbert at the University of Göttingen in 1915, eventually gaining recognition as one of the foremost algebraic theorists of her time. She remained there until 1933, when she and many other Jewish professors were dismissed when the Nazis came to power. She emigrated to the United States and became a visiting professor of mathematics at Bryn Mawr College and lecturer at the Princeton Institute for Advanced Study. Noether's ideas connecting symmetries to conservation laws have had a profound effect on the development of modern field theories impacting a vast range of disciplines in mechanics, quantum mechanics and relativity.

One of the main themes of this book has been to present in some detail the procedure for transforming a differential function under a group. In each case the end result is an invariance condition for the differential function being transformed. This chapter is the logical extension of this theme to the transformation of a volume integral of a differential function. The procedure is fundamentally the same although the details are a little more complicated. In this case the invariance condition yields a generalized system of Euler–Lagrange equations and a formula by which conserved vectors can be constructed from the invariant groups.

### 15.1.1 Transformation of Integrals by Lie–Bäcklund Groups

In Chapter 4 we followed a conventional approach and considered Lagrangians of the form $L[\boldsymbol{q}, d\boldsymbol{q}/dt, t]$. This covers many of the most common applications in Physics. However, Noether recognized that much more general Lagrangians could be defined, and she extended the theory to the vector of independent variables, $\boldsymbol{x} = (x^1, \ldots, x^n)$, and to Lagrangians dependent on higher-order derivatives. Here we let the Lagrangian be a differential function of the form

$$L = L[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p]. \tag{15.1}$$

Our goal is to establish the conditions under which the action integral

$$S = \int_V L[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p]\, dx_1\, dx_2 \cdots dx_n \tag{15.2}$$

is invariant under the extended infinitesimal Lie–Bäcklund group with group parameter $s$:

$$\begin{aligned}
\tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_r],\\
\tilde{y}^i &= y^i + s\eta^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_r],\\
\tilde{y}^i_j &= y^i_j + s\eta^i_{\{j\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_r, \boldsymbol{y}_{r+1}],\\
\tilde{y}^i_{j_1 j_2} &= y^i_{j_1 j_2} + s\eta^i_{\{j_1 j_2\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_r, \boldsymbol{y}_{r+1}, \boldsymbol{y}_{r+2}],\\
&\vdots\\
\tilde{y}^i_{j_1 j_2 \cdots j_p} &= y^i_{j_1 j_2 \cdots j_p} + s\eta^i_{\{j_1 j_2 \cdots j_p\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_r, \ldots, \boldsymbol{y}_{r+p}],
\end{aligned} \tag{15.3}$$

where for any order $p$,

$$\eta^i_{\{j_1 j_2 \cdots j_p\}} = D_{j_p}\eta^i_{\{j_1 j_2 \cdots j_{p-1}\}} - y^i_{j_1 j_2 \cdots j_{p-1}\alpha} D_{j_p}\xi^\alpha. \tag{15.4}$$

### 15.1.2 Transformation of the Differential Volume

First we need to develop the transformation law for the product of differentials

$$dV = dx^1\, dx^2 \cdots dx^n. \tag{15.5}$$

The infinitesimal transformation (15.3) gives the prolongation,

$$d\tilde{x}^j = dx^j + s\left(\frac{\partial \xi^j}{\partial x_\alpha} + \frac{\partial \xi^j}{\partial y_\beta}\frac{dy_\beta}{dx_\alpha} + \frac{\partial \xi^j}{\partial y^\gamma_\beta}\frac{\partial y^\gamma_\beta}{\partial x^\alpha} + \cdots\right) dx^\alpha. \tag{15.6}$$

The only term that contributes a quantity of order $s$ to the sum over $\alpha$ in (15.6) is the term with $\alpha = j$. Therefore, to lowest order in $s$, Equation (15.6) becomes

$$d\tilde{x}^j = dx^j + s\left(\frac{\partial \xi^j}{\partial x^j} + \frac{\partial \xi^j}{\partial y_\beta}\frac{dy_\beta}{dx^j} + \frac{\partial \xi^j}{\partial y^\gamma_\beta}\frac{\partial y^\gamma_\beta}{\partial x^j} + \cdots\right) dx^j \qquad \text{(no sum over } j\text{)}, \tag{15.7}$$

or

$$d\tilde{x}^j = (1 + sD_j\xi^j)\, dx^j \qquad \text{(no sum over } j\text{)}. \tag{15.8}$$

The product of differentials is now

$$d\tilde{x}^1\, d\tilde{x}^2 \cdots d\tilde{x}^n = (1 + sD_1\xi^1)(1 + sD_2\xi^2)\cdots(1 + sD_n\xi^n)dx^1\, dx^2 \cdots dx^n \tag{15.9}$$

Retaining only terms of order $s$, we have

$$d\tilde{V} = dV + sD_j\xi^j\, dV. \qquad \text{sum over } j = 1, 2, \ldots, n \tag{15.10}$$

The differential function $L[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p]$ can be expanded in a Lie series

$$L[\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}_1, \ldots, \tilde{\boldsymbol{y}}_p] = L[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p] + sX_{\{p\}}L + O(s^2) + \cdots, \tag{15.11}$$

where $X_{\{p\}}$ is the $p$th-order Lie–Bäcklund operator,

$$X_{\{p\}} = \xi^j \frac{\partial}{\partial x^j} + \eta^i \frac{\partial}{\partial y^i} + \eta^i_{\{j_1\}} \frac{\partial}{\partial y^i_{j_1}} + \cdots + \eta^i_{\{j_1 j_2 \cdots j_p\}} \frac{\partial}{\partial y^i_{j_1 j_2 \cdots j_p}}. \tag{15.12}$$

Now expand the integral (15.2). For small $s$,

$$\begin{aligned} S &= \int L[\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}_1, \ldots, \tilde{\boldsymbol{y}}_p]\, d\tilde{x}^1\, d\tilde{x}^2 \cdots d\tilde{x}^n \\ &\approx \int (L[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p] + sX_{\{p\}}L)(1 + sD_j\xi^j)\, dV. \end{aligned} \tag{15.13}$$

### *15.1.3 Invariance Condition for Integrals*

Retaining only the lowest-order terms in (15.13), the transformation of the integral (15.2) becomes

$$\tilde{S} = S + s\int (X_{\{p\}}L + L(D_j\xi^j))\, dV + D(s^2) + \cdots \tag{15.14}$$

The integral (15.2) is invariant under the group (15.3) if and only if

$$X_{\{p\}}L + L(D_j\xi^j) = 0. \tag{15.15}$$

Notice that in deriving the result (15.15) there was no consideration of the surface terms that normally come up in the context of variational calculus. In a sense the condition (15.15) is overly restrictive. A slightly weaker, and therefore more broadly applicable, condition is to require that the kernel of the integral in (15.14) be equal to the divergence of a vector:

$$X_{\{p\}}L + L(D_j\xi^j) = D_j\beta^j \tag{15.16}$$

(cf. Equation 4.20 and the related discussion in Chapter 4). The integral over $V$ can be converted to an integral over the surface $A$ using the divergence theorem:

$$\tilde{S} = S + s\int_A \beta^j \, dA_j, \tag{15.17}$$

where $dA_j$ is a component of the differential outward normal vector on the surface. The integral (15.2) is invariant if and only if

$$\int_A \beta^j \, dA_j = 0. \tag{15.18}$$

Here it is convenient to introduce the characteristic function discussed in Chapter 14, Section 14.1.1:

$$\mu^i = \eta^i - y^i_\alpha \xi^\alpha. \tag{15.19}$$

The functions for the infinitesimal transformations of derivatives take on the following form:

$$\begin{aligned}
\eta^i_{\{j_1\}} &= D_{j_1}\mu^i + y^i_{j_1\alpha}\xi^\alpha, \\
\eta^i_{\{j_1 j_2\}} &= D_{j_1 j_2}\mu^i + y^i_{j_1 j_2\alpha}\xi^\alpha, \\
&\vdots \\
\eta^i_{\{j_1 j_2 \cdots j_p\}} &= D_{j_1 j_2 \cdots j_p}\mu^i + y^i_{j_1 j_2 \cdots j_p\alpha}\xi^\alpha,
\end{aligned} \tag{15.20}$$

and the left-hand side of Equation (15.16) becomes

$$\begin{aligned}
&X_{\{p\}}L + L(D_j\xi^j) \\
&\quad = L(D_j\xi^j) + \xi^j \frac{\partial L}{\partial x^j} + \eta^i \frac{\partial L}{\partial y^i} \\
&\qquad + D_{j_1}\mu^i \frac{\partial L}{\partial y^i_{j_1}} + D_{j_1 j_2}\mu^i \frac{\partial L}{\partial y^i_{j_1 j_2}} + \cdots + D_{j_1 j_2 \cdots j_p}\mu^i \frac{\partial L}{\partial y^i_{j_1 j_2 \cdots j_p}} \\
&\qquad + \frac{\partial L}{\partial y^i_{j_1}} y^i_{j_1 j}\xi^j + \frac{\partial L}{\partial y^i_{j_1 j_2}} y^i_{j_1 j_2 j}\xi^j + \cdots + \frac{\partial L}{\partial y^i_{j_1 j_2 \cdots j_p}} y^i_{j_1 j_2 \cdots j_p j}\xi^j.
\end{aligned} \tag{15.21}$$

Combining terms and replacing $\eta^i$ with $\mu^i + y^i_j \xi^j$ in (15.21), yields

$$X_{\{p\}}L + L(D_j\xi^j) = D_j(L\xi^j) + \mu^i \frac{\partial L}{\partial y^i} + D_{j_1}\mu^i \frac{\partial L}{\partial y^i_{j_1}} + D_{j_1 j_2}\mu^i \frac{\partial L}{\partial y^i_{j_1 j_2}} + \cdots + D_{j_1 j_2 \cdots j_q}\mu^i \frac{\partial L}{\partial y^i_{j_1 j_2 \cdots j_p}}. \quad (15.22)$$

Introduce the Euler operator

$$E_i(\;) = \frac{\partial(\;)}{\partial y^i} - D_{j_1}\left(\frac{\partial(\;)}{\partial y^i_{j_1}}\right) + D_{j_1 j_2}\left(\frac{\partial(\;)}{\partial y^i_{j_1 j_2}}\right) - \cdots + \cdots + (-1)^p D_{j_1 j_2 \cdots j_p}\left(\frac{\partial(\;)}{\partial y^i_{j_1 j_2 \cdots j_p}}\right), \quad (15.23)$$

and use (15.23) to replace $\partial L/\partial y^i$ in (15.22):

$$\begin{aligned} X_{\{p\}}L &+ L(D_j\xi^j) \\ &= D_j(L\xi^j) + \mu^i(E_i L) \\ &\quad + \mu^i D_{j_1}\left(\frac{\partial L}{\partial y^i_{j_1}}\right) - \mu^i D_{j_1 j_2}\left(\frac{\partial L}{\partial y^i_{j_1 j_2}}\right) + \cdots (-1)^{p-1}\mu^i D_{j_1 j_2 \cdots j_p}\left(\frac{\partial L}{\partial y^i_{j_1 j_2 \cdots j_p}}\right) \\ &\quad + D_{j_1}\mu^i\left(\frac{\partial L}{\partial y^i_{j_1}}\right) + D_{j_1 j_2}\mu^i\left(\frac{\partial L}{\partial y^i_{j_1 j_2}}\right) + \cdots + D_{j_1 j_2 \cdots j_p}\mu^i\left(\frac{\partial L}{\partial y^i_{j_1 j_2 \cdots j_p}}\right). \end{aligned} \quad (15.24)$$

Integrating by parts repeatedly, (15.24) can be rearranged to read as follows:

$$X_{\{p\}}L + L(D_j\xi^j) = D_j(L\xi^j) + \mu^i(E_i L) + D_{j_1}\theta^{j_1}, \quad (15.25)$$

where

$$\begin{aligned} \theta^{j_1} = \Bigg[ & \mu^i \left\{ \frac{\partial L}{\partial y^i_{j_1}} - D_{j_2}\frac{\partial L}{\partial y^i_{j_1 j_2}} + D_{j_2 j_3}\frac{\partial L}{\partial y^i_{j_1 j_2 j_3}} - \cdots + (-1)^{p-1} D_{j_2 \cdots j_p}\frac{\partial L}{\partial y^i_{j_1 \cdots j_p}} \right\} \\ & + D_{j_2}\mu^i \left\{ \frac{\partial L}{\partial y^i_{j_1 j_2}} - D_{j_3}\frac{\partial L}{\partial y^i_{j_1 j_2 j_3}} + D_{j_3 j_4}\frac{\partial L}{\partial y^i_{j_1 \cdots j_4}} \cdots \right. \\ & \left. (-1)^{p-2} D_{j_3 \cdots j_p}\frac{\partial L}{\partial y^i_{j_1 \cdots j_p}} \right\} \end{aligned}$$

$$+ D_{j_2 j_3}\mu^i \left\{ \frac{\partial L}{\partial y^i_{j_1 j_2 j_3}} - D_{j_4}\frac{\partial L}{\partial y^i_{j_1\cdots j_4}} + \cdots (-1)^{p-3} D_{j_4\cdots j_p}\frac{\partial L}{\partial y^i_{j_1\cdots j_p}} \right\}$$
$$+ \cdots$$
$$+ D_{j_2\cdots j_{p-1}}\mu^i \left\{ \frac{\partial L}{\partial y^i_{j_1 j_2\cdots j_{p-1}}} - D_{j_p}\frac{\partial L}{\partial y^i_{j_1\cdots j_p}} \right\}$$
$$+ D_{j_2\cdots j_p}\mu^i \left\{ \frac{\partial L}{\partial y^i_{j_1 j_2\cdots j_p}} \right\} \Bigg]. \tag{15.26}$$

The transformation of the integral (15.14) finally becomes

$$\tilde{S} = S + s\int (\mu^i(E_i L) + D_{j_1}(L\xi^{j_1} + \theta^{j_1}))\, dV. \tag{15.27}$$

Since the volume of integration is arbitrary, invariance of the integral (15.2) can hold only if the integrand in (15.27) is zero. We can now state Noether's theorem in the following form.

***Theorem 15.1.*** *Let $L[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p]$ be a differential function. The action integral*

$$S = \int L[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p]\, dx^1\, dx^2 \cdots dx^n \tag{15.28}$$

*is invariant under the Lie–Bäcklund group (15.3) with infinitesimals $(\xi^j, \eta^i)$ $(j = 1, \ldots, n, i = 1, \ldots, m)$ if and only if*

$$\mu^i(E_i L) + D_{j_1}(L\xi^{j_1} + \theta^{j_1}) = D_{j_1}\beta^{j_1}, \tag{15.29}$$

*where $\mu^i = \eta^i - y^i_\alpha \xi^\alpha$ and $\int \beta^j\, dA_j = 0$. The vector $\theta^{j_1}$ is*

$$\theta^{j_1} = \Bigg[ \mu^i \left\{ \frac{\partial L}{\partial y^i_{j_1}} + \sum_{k=2}^{p} (-1)^{k-1} D_{j_2\cdots j_k}\frac{\partial L}{\partial y^i_{j_l\cdots j_k}} \right\}$$
$$+ \sum_{\lambda=2}^{p-1} \left[ D_{j_2\cdots j_\lambda}\mu^i \left\{ \frac{\partial L}{\partial y^i_{j_1\cdots j_\lambda}} + \sum_{k=\lambda}^{p-1} (-1)^{k-\lambda+1} D_{j_{\lambda+1}\cdots j_{k+1}} \frac{\partial L}{\partial y^i_{j_l\cdots j_{\lambda+1}\cdots j_{k+1}}} \right\} \right]$$
$$+ D_{j_2\cdots j_p}\mu^i \left\{ \frac{\partial L}{\partial y^i_{j_1 j_2\cdots j_p}} \right\} \Bigg]. \qquad j_1 = i, \ldots, n \tag{15.30}$$

*The condition (15.29) is met if* $\mathbf{y}$ *is a solution of the generalized Euler–Lagrange system*

$$E_i L = \frac{\partial L}{\partial y^i} + \sum_{k=1}^{p} (-1)^k D_{j_1 j_2 \cdots j_k} \frac{\partial L}{\partial y^i_{j_1 j_2 \cdots j_k}} = 0 \qquad (15.31)$$

*and if*

$$D_j(L\xi^j + \theta^j - \beta^j) = 0 \qquad (15.32)$$

*holds on solutions of (15.31). The combination*

$$\Gamma^j = L\xi^j + \theta^j - \beta^j \qquad (15.33)$$

*is a conserved vector for the system (15.31), and (15.32) is a conservation law.*

The infinitesimals that leave the integral (15.28) invariant are found by investigating the invariance properties of the generalized Euler–Lagrange system $E_i L = 0$. The operator $X = \xi^j(\partial/\partial x^j) + \eta^i(\partial/\partial y^i)$ that generates a conserved vector (15.33) is called a *variational symmetry*. For a given equation, derivable from a Lagrangian, the variational symmetries are a subset of the set of Lie point and Lie–Bäcklund symmetries. Stephani [15.2] distinguishes between variational symmetries, where $\int_A \beta^j dA_j = 0$, and Noether symmetries, where $\int_A \beta^j dA_j = \text{constant}$, causing the action in (15.17) to be shifted by a constant.

Noether's theorem is the justification for the discussion in Chapter 1, where it was stated that the symmetries are as fundamental as the conservation laws themselves. It is one of the major advances in modern physics for it highlights the key role of symmetries in the analysis of physical phenomena.

## 15.2 Examples

***Example 15.1 (The free motion of a mass in the absence of body forces).*** The Lagrangian for a mass moving in the absence of any body forces is given by its kinetic energy

$$L = \tfrac{1}{2} m\left(\left(x_t^1\right)^2 + \left(x_t^2\right)^2 + \left(x_t^3\right)^2\right). \qquad (15.34)$$

The Euler equations corresponding to (15.34) are as follows

$$E_i L = \frac{\partial L}{\partial x^i} - D_t\left(\frac{\partial L}{\partial x_t^i}\right) = -D_t(m x_t^i) = -m x_{tt}^i = 0, \qquad (15.35)$$

which states that the acceleration of the particle is zero. The Euler system (15.35) is invariant under the four-parameter group of space–time translations,

$$X^1 = \frac{\partial}{\partial x^1}, \qquad X^2 = \frac{\partial}{\partial x^2}, \qquad X^3 = \frac{\partial}{\partial x^3}, \qquad X^4 = \frac{\partial}{\partial t}, \tag{15.36}$$

and the three-parameter group of rotations SO(3),

$$X^5 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \qquad X^6 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \qquad X^7 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}. \tag{15.37}$$

Actually the system (15.35) is invariant under a much more general projective group, of which the operators (15.36) and (15.37) are a subgroup. See Chapter 8, Section 8.6.1. Using (15.26) and (15.33), conserved vectors corresponding to each of the groups can be constructed. In this case, where time is the only independent variable, the conserved vectors have only one component.

Conserved vectors corresponding to each of the independent space translations are the three components of the momentum,

$$\begin{aligned} P^1 &= L\xi^j + \left(\eta^i - y^i_\alpha \xi^\alpha\right)\frac{\partial L}{\partial y^i_j} = \frac{\partial L}{\partial x^1_t} = m x^1_t, \\ P^2 &= L\xi^j + \left(\eta^i - y^i_\alpha \xi^\alpha\right)\frac{\partial L}{\partial y^i_j} = \frac{\partial L}{\partial x^2_t} = m x^2_t, \\ P^3 &= L\xi^j + \left(\eta^i - y^i_\alpha \xi^\alpha\right)\frac{\partial L}{\partial y^i_j} = \frac{\partial L}{\partial x^3_t} = m x^3_t. \end{aligned} \tag{15.38}$$

The vector momentum is usually denoted $\boldsymbol{P}$, and so the conserved quantity is

$$\boldsymbol{P} = m\boldsymbol{v}. \tag{15.39}$$

The conserved "vector" corresponding to invariance under time translation is the kinetic energy of the particle,

$$\begin{aligned} -E &= L\xi^j + (\eta^i - y^i_\alpha \xi^\alpha)\frac{\partial L}{\partial y^i_j} = L(1) + (-x^i_t)\frac{\partial L}{\partial x^i_t} \\ &= \tfrac{1}{2}m\left(\left(x^1_t\right)^2 + \left(x^2_t\right)^2 + \left(x^3_t\right)^2\right) - m\left(\left(x^1_t\right)^2 + \left(x^2_t\right)^2 + \left(x^3_t\right)^2\right) \\ &= -\tfrac{1}{2}m\left(\left(x^1_t\right)^2 + \left(x^2_t\right)^2 + \left(x^3_t\right)^2\right), \end{aligned} \tag{15.40}$$

or

$$E = \tfrac{1}{2}mv^2. \tag{15.41}$$

Invariance under rotation produces the three components of angular momentum,

$$M^1 = L\xi^j + \left(\eta^i - y^i_\alpha \xi^\alpha\right)\frac{\partial L}{\partial y^i_j} = x^3\frac{\partial L}{\partial x^2_t} - x^2\frac{\partial L}{\partial x^3_t} = m\left(x^2 x^3_t - x^3 x^2_t\right),$$
$$M^2 = L\xi^j + \left(\eta^i - y^i_\alpha \xi^\alpha\right)\frac{\partial L}{\partial y^i_j} = x^1\frac{\partial L}{\partial x^3_t} - x^3\frac{\partial L}{\partial x^1_t} = m\left(x^3 x^1_t - x^1 x^3_t\right),$$
$$M^3 = L\xi^j + \left(\eta^i - y^i_\alpha \xi^\alpha\right)\frac{\partial L}{\partial y^i_j} = x^2\frac{\partial L}{\partial x^1_t} - x^1\frac{\partial L}{\partial x^2_t} = m\left(x^1 x^2_t - x^2 x^1_t\right), \tag{15.42}$$

or

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{P}. \tag{15.43}$$

In Chapter 1 it was stated that the fundamental symmetries of free space are homogeneity and isotropy. Time is also homogeneous and isotropic. The equations of motion remain the same when the sign of time is changed. Thus the homogeneity of space leads to conservation of momentum, the isotropy of space leads to conservation of angular momentum, and the homogeneity of time leads to conservation of energy.

***Example 15.2 (A particle moving under the influence of a spherically symmetric inverse-square body force).*** The Lagrangian for such a particle is

$$L = \frac{1}{2}m\left(\left(x^1_t\right)^2 + \left(x^2_t\right)^2 + \left(x^3_t\right)^2\right) + \frac{\gamma}{((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2}}. \tag{15.44}$$

If $\gamma < 0$, the force is repelling from the origin. If $\gamma > 0$, the force is attracting to the origin. The corresponding Euler-Lagrange equations are

$$\frac{\partial L}{\partial x^i} - D_t\left(\frac{\partial L}{\partial x^i_t}\right) = -\left(\frac{\gamma x^i}{r^3} + m x^i_{tt}\right) = 0, \qquad i = 1, 2, 3, \tag{15.45}$$

where

$$r = ((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2}. \tag{15.46}$$

From Chapter 14 Section 14.4.3 we know that the system (15.45) is invariant under a five-parameter group of time translation, three rotations, and one dilation,

$$X^1 = \frac{\partial}{\partial t},$$
$$X^2 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \qquad X^3 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \qquad X^4 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \tag{15.47}$$
$$X^5 = 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z} + 3t\frac{\partial}{\partial t}.$$

Invariance under time translation leads to the expression for the total energy of the system,

$$\begin{aligned}
-E &= L\xi^j + \left(\eta^i - y^i_\alpha \xi^\alpha\right)\frac{\partial L}{\partial y^i_j} = L(1) + \left(-x^i_t\right)\frac{\partial L}{\partial x^i_t} \\
&= \frac{1}{2}m\left(\left(x^1_t\right)^2 + \left(x^2_t\right)^2 + \left(x^3_t\right)^2\right) + \frac{\gamma}{((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2}} \\
&\quad - m\left(\left(x^1_t\right)^2 + \left(x^2_t\right)^2 + \left(x^3_t\right)^2\right) \\
&= -\frac{1}{2}m\left(\left(x^1_t\right)^2 + \left(x^2_t\right)^2 + \left(x^3_t\right)^2\right) + \frac{\gamma}{((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2}},
\end{aligned} \tag{15.48}$$

or

$$E = \frac{1}{2}mv^2 - \frac{\gamma}{r}, \tag{15.49}$$

as a conserved quantity.

Invariance under rotation leads to conservation of the three components of angular momentum,

$$\begin{aligned}
M^1 &= L\xi^j + \left(\eta^i - y^i_\alpha \xi^\alpha\right)\frac{\partial L}{\partial y^i_j} = x^3\frac{\partial L}{\partial x^2_t} - x^2\frac{\partial L}{\partial x^3_t} = m\left(x^2 x^3_t - x^3 x^2_t\right), \\
M^2 &= L\xi^j + \left(\eta^i - y^i_\alpha \xi^\alpha\right)\frac{\partial L}{\partial y^i_j} = x^1\frac{\partial L}{\partial x^3_t} - x^3\frac{\partial L}{\partial x^1_t} = m\left(x^3 x^1_t - x^1 x^3_t\right), \\
M^3 &= L\xi^j + \left(\eta^i - y^i_\alpha \xi^\alpha\right)\frac{\partial L}{\partial y^i_j} = x^2\frac{\partial L}{\partial x^1_t} - x^1\frac{\partial L}{\partial x^2_t} = m\left(x^1 x^2_t - x^2 x^1_t\right),
\end{aligned} \tag{15.50}$$

or

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{P}. \tag{15.51}$$

Now, if we try to follow Noether's theorem and use invariance under the dilation group $X^5$ to produce a conservation law, the result is

$$
\begin{aligned}
R &= L\xi^j + \left(\eta^i - y^i_\alpha \xi^\alpha\right)\frac{\partial L}{\partial y^i_j} \\
&= \frac{3}{4}mt\left(\left(x^1_1\right)^2 + \left(x^2_t\right)^2 + \left(x^3_t\right)^2\right) - \frac{3}{2}\frac{\gamma t}{((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2}} \\
&\quad + \left(x^1 - \frac{3}{2}tx^1_t\right)mx^1_t + \left(x^2 - \frac{3}{2}tx^2_t\right)mx^2_t + \left(x^3 - \frac{3}{2}tx^3_t\right)mx^3_t \\
&= -\left(\frac{3}{4}\right)mt\left(\left(x^1_t\right)^2 + \left(x^2_t\right)^2 + \left(x^3_t\right)^2\right) - \frac{3}{2}\frac{\gamma t}{\left((x^1)^2 + (x^2)^2 + (x^3)^2\right)^{1/2}} \\
&\quad + m\left(x^1 x^1_t + x^2 x^2_t + x^3 x^3_t\right),
\end{aligned}
\tag{15.52}
$$

or

$$
R = -\tfrac{3}{2}Et + \boldsymbol{P}\cdot\boldsymbol{x}. \tag{15.53}
$$

If we take the divergence of (15.53), the result using (15.45) is

$$
D_t\left(-\tfrac{3}{2}\boldsymbol{E}\boldsymbol{t} + \boldsymbol{P}\cdot\boldsymbol{x}\right) = \tfrac{1}{2}mv^2, \tag{15.54}
$$

which is not a conserved quantity. The dilation group $X^5$ is not a variational symmetry of the Kepler system.

The system (15.45) is also invariant under the three-parameter Lie–Bäcklund transformation

$$
\begin{aligned}
X^6 &= \left(2x^1 x^i_t - x^i x^1_t - \left(x^k x^k_t\right)\delta^i_1\right)\frac{\partial}{\partial x^i}, \\
X^7 &= \left(2x^2 x^i_t - x^i x^2_t - \left(x^k x^k_t\right)\delta^i_2\right)\frac{\partial}{\partial x^i}, \\
X^8 &= \left(2x^3 x^i_t - x^i x^3_t - \left(x^k x^k_t\right)\delta^i_3\right)\frac{\partial}{\partial x^i}.
\end{aligned}
\tag{15.55}
$$

Let's construct the one-component conserved vectors corresponding to each of these groups. Let $x = x^1$, $y = x^2$, and $z = x^3$. Thus the once extended group corresponding to $X^6$ is

$$
\begin{aligned}
X^6_{\{1\}} &= (-yy_t - zz_t)\frac{\partial}{\partial x} + (2xy_t - yx_t)\frac{\partial}{\partial y} + (2xz_t - zx_t)\frac{\partial}{\partial z} \\
&\quad + \left(-y_t^2 - yy_{tt} - z_t^2 - zz_{tt}\right)\frac{\partial}{\partial x_t} \\
&\quad + (x_t y_t + 2xy_{tt} - yx_{tt})\frac{\partial}{\partial y_t} \\
&\quad + (x_t z_t + 2xz_{tt} - zx_{tt})\frac{\partial}{\partial z_t}.
\end{aligned}
\tag{15.56}
$$

The action of this operator on the Lagrangian is

$$X^6_{\{1\}}L = \frac{2\gamma}{r^3}\left(xyy_t - y^2x_t + xzz_t - z^2x_t\right), \tag{15.57}$$

where (15.45) has been used to eliminate the second-derivative terms. The result (15.57) can be written as

$$X^6_{\{1\}}L = -D_t\left(2\gamma\frac{x}{r}\right), \tag{15.58}$$

where

$$r = (x^2 + y^2 + z^2)^{1/2}. \tag{15.59}$$

The expression $2\gamma x/r$ is the vector $B^j$ that appears in (15.16). Note that $\xi^j = 0$ for the groups (15.55). This process can be repeated for the remaining operators in (15.55) to give

$$\begin{aligned} X^7_{\{1\}}L &= -D_t\left(2\gamma\frac{y}{r}\right), \\ X^8_{\{1\}}L &= -D_t\left(2\gamma\frac{z}{r}\right). \end{aligned} \tag{15.60}$$

Now the conserved vector (15.33) can be constructed as follows:

$$\begin{aligned} -2Q^1 &= L\xi^j + \left(\eta^i - y^i_\alpha\xi^\alpha\right)\frac{\partial L}{\partial y^i_j} - B^j = \eta^i\frac{\partial L}{\partial y^i_j} + 2\gamma\frac{x^1}{r} \\ &= -mx^i_t\eta^i + 2\gamma\frac{x^1}{r} \\ &= \left(2x^1x^i_t - x^ix^1_t - \left(x^kx^k_t\right)\delta^i_1\right)\left(-mx^i_t\right) + 2\gamma\frac{x^1}{r} \\ &= m\left(x^kx^k_tx^1_t + x^ix^i_tx^1_t - 2x^1x^i_tx^i_t\right) + 2\gamma\frac{x^1}{r}. \end{aligned} \tag{15.61}$$

Similarly,

$$\begin{aligned} -2Q^2 &= m\left(x^kx^k_tx^2_t + x^ix^i_tx^2_t - 2x^2x^i_tx^i_t\right) + 2\gamma\frac{x^2}{r}, \\ -2Q^3 &= m\left(x^kx^k_tx^3_t + x^ix^i_tx^3_t - 2x^3x^i_tx^i_t\right) + 2\gamma\frac{x^3}{r}. \end{aligned} \tag{15.62}$$

In vector notation,

$$\boldsymbol{Q} = m((\boldsymbol{u}\cdot\boldsymbol{u})\boldsymbol{x} - (\boldsymbol{x}\cdot\boldsymbol{u})\boldsymbol{u}) - \gamma\frac{x}{r}, \tag{15.63}$$

where $\boldsymbol{u} = \boldsymbol{x}_t$. One can show that $D_t(\boldsymbol{Q}) = 0$.

Using the vector identity $\boldsymbol{u} \times (\boldsymbol{x} \times \boldsymbol{u}) = \boldsymbol{x}(\boldsymbol{u} \cdot \boldsymbol{u}) - \boldsymbol{u}(\boldsymbol{u} \cdot \boldsymbol{x})$, the conserved vector (15.63) can be written as

$$\boldsymbol{Q} = \boldsymbol{u} \times \boldsymbol{M} - \gamma \frac{\boldsymbol{x}}{r}. \tag{15.64}$$

This vector originates in a Lie–Bäcklund symmetry of the Kepler equations and is called in the literature Laplace's vector or the Runge–Lenz vector. The vector $\boldsymbol{Q}$ lies in the plane of the orbit and points along the major axis from the origin (which in the reduced-mass problem is a focus of the orbit) toward the perihelion (the point of closest approach to the origin). The magnitude of $\boldsymbol{Q}$ is proportional to the eccentricity of the orbit:

$$|\boldsymbol{Q}| = \gamma e = \gamma \left(1 + \frac{2H\Gamma^2}{m\alpha^2}\right)^{1/2}. \tag{15.65}$$

In summary, there are a total of seven conserved quantities for this problem:

$$\begin{aligned} E &= \frac{1}{2} m v^2 - \frac{\gamma}{r}, \\ \boldsymbol{M} &= \boldsymbol{r} \times \boldsymbol{P}, \\ \boldsymbol{Q} &= \boldsymbol{u} \times \boldsymbol{M} - \gamma \frac{\boldsymbol{x}}{r}. \end{aligned} \tag{15.66}$$

In addition to these constants of the motion, there is the scaling symmetry $X^5$ that gives Kepler's third law. Although $X^5$ does not produce a conservation law for the Kepler system, such a possibility is not precluded simply because the symmetry is a dilation. See Exercise 15.3.

***Example 15.3 (Conservation law for a higher-order PDE).*** The fourth-order PDE

$$\psi_{tt} + \psi_{xxxx} + 2\psi_{xxyy} + \psi_{yyyy} = 0 \tag{15.67}$$

can be derived from the Lagrangian

$$L = \tfrac{1}{2}\psi_t^2 - (\psi_{xx} + \psi_{yy})^2. \tag{15.68}$$

Equation (15.67) is invariant under the translation in time, $X = \partial/\partial t$. The conserved vector generated from this symmetry using (15.30) is

$$\Gamma^{j_1} = L\xi^{j_1} + \theta^{j_1}, \tag{15.69}$$

where

$$\theta^{j_1} = \mu^i \left\{ \frac{\partial L}{\partial y^i_{j_1}} - D_{j_2} \frac{\partial L}{\partial y^i_{j_1 j_2}} \right\} + D_{j_2}\mu^i \left\{ \frac{\partial L}{\partial y^i_{j_1 j_2}} \right\} \tag{15.70}$$

For the present problem, noting that $\eta = 0$, $\xi^t = 1$, $\xi^x = 0$, and $\xi^y = 0$, these relations become, $\Gamma^t = L + \theta^t$, $\Gamma^x = \theta^x$, and $\Gamma^y = \theta^y$, and, taking the indicated derivatives, the components of $\theta^{j_1}$ are

$$\begin{aligned} \theta^t = &-\psi_t \left\{ \frac{\partial L}{\partial \psi_t} - D_t \frac{\partial L}{\partial \psi_{tt}} - D_x \frac{\partial L}{\partial \psi_{tx}} - D_y \frac{\partial L}{\partial \psi_{ty}} \right\} \\ &+ D_t(-\psi_t)\left\{ \frac{\partial L}{\partial \psi_{tt}} \right\} + D_x(-\psi_t)\left\{ \frac{\partial L}{\partial \psi_{tx}} \right\} + D_y(-\psi_t)\left\{ \frac{\partial L}{\partial \psi_{ty}} \right\} = -\psi_t^2, \end{aligned} \tag{15.71}$$

$$\begin{aligned} \theta^x &= -\psi_t \left\{ \frac{\partial L}{\partial \psi_x} - D_t \frac{\partial L}{\partial \psi_{xt}} - D_x \frac{\partial L}{\partial \psi_{xx}} - D_y \frac{\partial L}{\partial \psi_{xy}} \right\} \\ &\quad + D_t(-\psi_t)\left\{ \frac{\partial L}{\partial \psi_{xt}} \right\} + D_x(-\psi_t)\left\{ \frac{\partial L}{\partial \psi_{xx}} \right\} + D_y(-\psi_t)\left\{ \frac{\partial L}{\partial \psi_{xy}} \right\} \\ &= 2D_x\psi_t(\psi_{xx} + \psi_{yy}) - 2\psi_t D_x(\psi_{xx} + \psi_{yy}), \end{aligned} \tag{15.72}$$

and

$$\begin{aligned} \theta^y &= (-\psi_t)\left\{ \frac{\partial L}{\partial \psi_y} - D_t \frac{\partial L}{\partial \psi_{yt}} - D_x \frac{\partial L}{\partial \psi_{yx}} - D_y \frac{\partial L}{\partial \psi_{yy}} \right\} \\ &\quad + D_t(-\psi_t)\left\{ \frac{\partial L}{\partial \psi_{yt}} \right\} + D_x(-\psi_t)\left\{ \frac{\partial L}{\partial \psi_{yx}} \right\} + D_y(-\psi_t)\left\{ \frac{\partial L}{\partial \psi_{yy}} \right\} \\ &= 2D_y\psi_t(\psi_{xx} + \psi_{yy}) - 2\psi_t D_y(\psi_{xx} + \psi_{yy}), \end{aligned} \tag{15.73}$$

giving the conservation law for (15.67), namely

$$\begin{aligned} &D_t\left\{ \tfrac{1}{2}\psi_t^2 + (\psi_{xx} + \psi_{yy})^2 \right\} \\ &\quad + D_x\{2\psi_t D_x(\psi_{xx} + \psi_{yy}) - 2(\psi_{xx} + \psi_{yy})D_x\psi_t\} \\ &\quad + D_y\left\{2\psi_t D_y(\psi_{xx} + \psi_{yy}) - 2(\psi_{xx} + \psi_{yy})D_y\psi_t\right\} = 0. \end{aligned} \tag{15.74}$$

## 15.3 Concluding Remarks

This concludes our discussion of the invariance condition for an integral of a differential function. The derivation of Noether's theorem provides a fitting end to our theoretical development of symmetry analysis. The main results (15.31) and (15.32) unite two vast disciplines: Lagrangian–Hamiltonian Dynamics and Symmetry theory that by themselves unify a wide diversity of physical phenomena. The brief introduction presented here is mainly intended to bring closure to the material on dynamics in Chapter 4 by revisiting it in the context of Lie–Bäcklund symmetries presented in Chapter 14. The result is a mere glimpse into an important and growing field. In the final chapter we will discuss a class of generalized symmetries called Bäcklund transformations. These are normally not defined as parametric mappings, and at first sight there would seem to be no direct connection to either Lie point or Lie–Bäcklund groups. However, as we shall see, such a connection can often be made, and when it is, nonlocal groups are often involved.

## 15.4 Exercises

15.1 Take the Kepler system,

$$\left(\frac{\gamma x^i}{r^3} + m x^i_{tt}\right) = 0, \qquad i = 1, 2, 3, \tag{15.75}$$

and put it into canonical form as a set of six first-order ODEs. Set up the characteristic equations, and write down the first-order PDE that governs integrals of the system. Use the software package **IntroToSymmetry.m** to work out all the point and first-order Lie–Bäcklund symmetries of the resulting system. Compare with the results in Example 15.2. Work out the integrals of this system, and compare with the results in (15.66). Are all seven conserved quantities in (15.66) independent?

15.2 Show that $D_t \boldsymbol{Q} = 0$, where $\boldsymbol{Q}$ is the Runge–Lenz vector (15.64).

15.3 In Chapter 4, Example 4.4 we solved the two-body problem in a central force field with a general potential function $V[r]$. The equations of motion are

$$m x^i_{tt} + \frac{x^i}{r}\left(\frac{\partial V}{\partial r}\right) = 0. \tag{15.76}$$

The solution for the radius is expressed implicitly in terms of the time

$$t = \int_{r_0}^{r} \frac{dr}{\left(\frac{2}{m}(H - V[r]) - \frac{\Gamma^2}{m^2 r^2}\right)^{1/2}}, \tag{15.77}$$

and the angle is determined from conservation of angular momentum:

$$\theta - \theta_0 = \int_{r_0}^{r} \frac{\Gamma dr}{r^2\big(2m(H - V[r]) - \frac{\Gamma^2}{r^2}\big)^{1/2}}. \tag{15.78}$$

For a general $V[r]$ the particle is constrained to move in an annular disk between two radii, $r_{\min}$ and $r_{\max}$. Eventually the particle motion fills the region between the two radii. Only when $V = -\gamma/r$ does the trajectory execute a closed path, and this is the situation for the Kepler problem. The quantity $\Gamma^2/(m^2r^2)$ is called the centrifugal energy [15.3] and becomes infinite as the radius of the orbit goes to zero, preventing the particle from falling in to the origin for any initial condition with a finite angular momentum. The term $\left(\frac{2}{m}\right)V[r] + \Gamma^2/(m^2r^2)$ in the integrands in (15.77) and (15.78) is called the effective potential energy, and the only way the particle can fall to the origin is if this term becomes negative or zero as the radius goes to zero. In this case the attractive force at the origin is strong enough to overcome the centrifugal energy. Study the symmetries and conservation laws for the system (15.76) for $V = -\gamma/r^n$. Show that for the case $n = 2$ the dilation group generates a variational symmetry. Compare your result with Example 15.2, and interpret it physically. What is special about $n = 2$?

15.4 Show that the fourth-order PDE treated in Example 15.3,

$$\psi_{tt} + \psi_{xxxx} + 2\psi_{xxyy} + \psi_{yyyy}, = 0 \tag{15.79}$$

can be derived from the Lagrangian

$$L = \tfrac{1}{2}\psi_t^2 - (\psi_{xx} + \psi_{yy})^2. \tag{15.80}$$

Use the package **`IntroToSymmetry.m`** to work out all point symmetries of the equation (15.79). Note that the linearity of the equation means that an arbitrary solution of (15.79) will appear in $\eta$, the infinitesimal transformation of $\psi$. This will be evidenced by the package as successively higher-order terms in the solution for $\eta$ that will need to be collected together in order to identify the remaining groups. Use Noether's theorem to generate a conservation law from each of the point symmetries of the equation.

## References

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[15.2] Stephani, H. 1989. *Differential Equations: Their Solution Using Symmetries*. Cambridge University Press.
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# 16

## *Bäcklund Transformations and Nonlocal Groups*

In the discussion of one-parameter Lie groups we made quite a fuss over the parametric nature of the transformations and the fact that the group definition ensures that they are single-valued, one-to-one invertible maps. The analytic nature of Lie groups is the main attribute that makes them so useful and that enabled Lie to develop the infinitesimal theory.

Lie contact and Lie Bäcklund transformations were given the same underpinning through the introduction of the space of differential functions and the enforcement of the contact conditions. These higher-order parametric transformations enjoy the same features as conventional groups. In particular, they can be used to transform a differential equation either to itself or to another equation without raising the order – all in all, a remarkable property, given the dependence of the basic (nonextended) transformation on first and possibly higher derivatives.

There exists a large class of nonparametric, many-valued transformations that also have the property that they can be used to transform an equation without raising the order. These are generically called Bäcklund transformations and arise in many different forms and contexts. A general discussion of Bäcklund transformations and their categorization can be found in Forsyth [16.1].

Most (but not all) applications involve two independent variables and one dependent variable $u(x, y)$. A typical Bäcklund transformation is of the form

$$\begin{aligned} \tilde{x} &= x, \\ \tilde{y} &= y, \\ \tilde{u}_{\tilde{x}} &= G_1[x, y, u, u_x, u_y], \\ \tilde{u}_{\tilde{y}} &= G_2[x, y, u, u_x, u_y]. \end{aligned} \tag{16.1}$$

Recalling the discussion of equivalence transformations in Chapter 14, we recognize that there is no real loss of generality in leaving the independent variables untransformed.

In contrast to a one-parameter Lie group, the Bäcklund transformation (16.1) has no standing in and of itself. It must be accompanied by the integrability condition,

$$\frac{\partial^2 \tilde{u}}{\partial x\,\partial y} = \frac{\partial G_1}{\partial y} = \frac{\partial G_2}{\partial x}. \tag{16.2}$$

This condition is generally satisfied only if the functions $u(x, y)$ and $\tilde{u}(\tilde{x}, \tilde{y})$ are solutions of some PDE or pair of PDEs. If both solve the same PDE, the transformation is termed an *auto-Bäcklund* transformation. It is in this sense that the transformation (16.1) must be viewed as many-valued. If we supply specific numbers for $(x, y, u, u_x, u_y)$ to the right-hand side of (16.1), then specific values for the first derivatives on the left are determined, but the function $\tilde{u}$ is not assigned a value. It can range over the entire subset of solutions corresponding to the particular values assigned to the first derivatives.

Bäcklund transformations can be extremely useful for generating solutions of nonlinear equations. Great progress has been made, especially in the past four decades, and a large number of Bäcklund transformations are now known. A modern exposition of this topic can be found in the text by Rogers and Ames [16.2]. Bäcklund transformations tend to fall generally into two categories: those that transform a nonlinear equation to a linear one, and auto-Bäcklund transformations (mentioned above) that map a nonlinear equation to itself. In the first case, linear methods can be used to solve the equation completely for a broad class of boundary and initial conditions. In the second case the transformation can be used recursively to generate a sequence of solutions, sometimes leading to a nonlinear superposition principle for the equation. The great value of these transformations is in their ability to generate large classes of nontrivial exact solutions for nonlinear partial differential equations and the deep insight into the fundamental nature of nonlinearity that results.

At first sight, Bäcklund transformations seem unrelated to Lie groups. Although arbitrary constants often appear in the transformations, they generally do not define a one-to-one invertible map as in one-parameter Lie groups. Nevertheless, a Bäcklund transformation is an expression of a fundamental symmetry of an equation, and so some connection to a Lie group might be expected. In fact such a connection can be drawn, and in the several examples described in this chapter, Bäcklund transformations are shown to arise directly from a one-parameter point or nonlocal group symmetry of a potential equation one or even two integrations removed from the original differential equation.

## 16.1 Two Classical Examples

A couple of examples will illustrate some of the basic ideas. Both examples are generalizations of the famous Klein–Gordon equation

$$\theta_{tt} - \theta_{xx} + \theta = 0, \tag{16.3}$$

which admits periodic solutions of the form

$$\theta = A\cos(\kappa x - \omega t), \tag{16.4}$$

where the frequency $\omega$ is a real function of the wave number $\kappa$. Solutions of the Klein–Gordon equation are dispersive in that the phase speed of a wave depends on the wave number, $c(\kappa) = \omega(\kappa)/\kappa$. In a superposition of waves of different wavelength, unless $\omega(\kappa) = c_0\kappa$ (i.e., $\omega_{\kappa\kappa} \neq 0$), the various modes moving at different speeds will tend to separate, and an initially narrow solution will tend to broaden as it evolves. The Klein–Gordon equation is the protypical example of a linear equation with a nontrivial dispersion relation, $\omega^2 = \kappa^2 + 1$, $c = \kappa + 1/\kappa$.

The simplest nonlinear generalization of the Klein–Gordon equation is

$$\theta_{tt} - \theta_{xx} + f(\theta) = 0. \tag{16.5}$$

Two of the most widely studied choices of $f(\theta)$ will be discussed below.

### *16.1.1 The Liouville Equation*

This is

$$\theta_{\bar{y}\bar{y}} - \theta_{\bar{x}\bar{x}} + e^{-\theta} = 0. \tag{16.6}$$

Using the new variables

$$x = \frac{\bar{x} - \bar{y}}{2}, \qquad y = \frac{\bar{x} + \bar{y}}{2}, \tag{16.7}$$

equation (16.6) can be transformed to the familiar cross-derivative form of the Liouville equation,

$$\theta_{xy} = e^{-\theta}. \tag{16.8}$$

This equation admits the Bäcklund transformation

$$\begin{aligned} \frac{\tilde{\theta}_x - \theta_x}{2} &= ae^{-(\tilde{\theta}+\theta)/2}, \\ \frac{\tilde{\theta}_y + \theta_y}{2} &= be^{-(\tilde{\theta}-\theta)/2}. \end{aligned} \tag{16.9}$$

To see the relationship between (16.8) and (16.9) we have to generate the integrability condition (16.2) by differentiating the first transformation with respect to $y$ and the second with respect to $x$:

$$\begin{aligned} \frac{\tilde{\theta}_{xy} - \theta_{xy}}{2} &= -ae^{-(\tilde{\theta}+\theta)/2}\,\frac{\tilde{\theta}_y + \theta_y}{2} = -abe^{-\tilde{\theta}}, \\ \frac{\tilde{\theta}_{xy} + \theta_{xy}}{2} &= -be^{-(\tilde{\theta}-\theta)/2}\,\frac{\tilde{\theta}_x - \theta_x}{2} = -abe^{-\tilde{\theta}}. \end{aligned} \tag{16.10}$$

Now add and subtract the equations in (16.10). The functions $\theta$ and $\tilde{\theta}$ must satisfy the PDEs

$$\begin{aligned} \tilde{\theta}_{xy} &= -2abe^{-\tilde{\theta}}, \\ \theta_{xy} &= 0. \end{aligned} \tag{16.11}$$

If we choose $b = -1/(2a)$, then

$$\tilde{\theta}_{xy} = e^{-\tilde{\theta}} \tag{16.12}$$

and

$$\theta_{xy} = 0. \tag{16.13}$$

The general solution of (16.13) is

$$\theta = f[x] + g[y], \tag{16.14}$$

where $f$ and $g$ are arbitrary functions. The Bäcklund transformation (16.9) provides the solution of (16.12) through the quadrature $d\tilde{\theta} = \tilde{\theta}_{\tilde{x}}\, d\tilde{x} + \tilde{\theta}_{\tilde{y}}\, d\tilde{y}$:

$$d\tilde{\theta} = \left(2ae^{-(\tilde{\theta}+f+g)/2} + f_x\right) dx + \left(-\frac{1}{a}e^{-(\tilde{\theta}-f-g)/2} - g_y\right) dy. \tag{16.15}$$

For example, if we seed the process with $f = g = 0$, the corresponding exact solution of the Liouville equation found by integrating (16.15) is

$$\tilde{\theta} = \ln\left[ax - \frac{1}{2a}y + C\right]^2, \tag{16.16}$$

which exhibits a singularity propagating along the line $y = 2a^2x + 2aC$ as shown in Figure 16.1. Further solutions can be generated by selecting other seed functions.

Now let's consider an example of an auto-Bäcklund transformation.

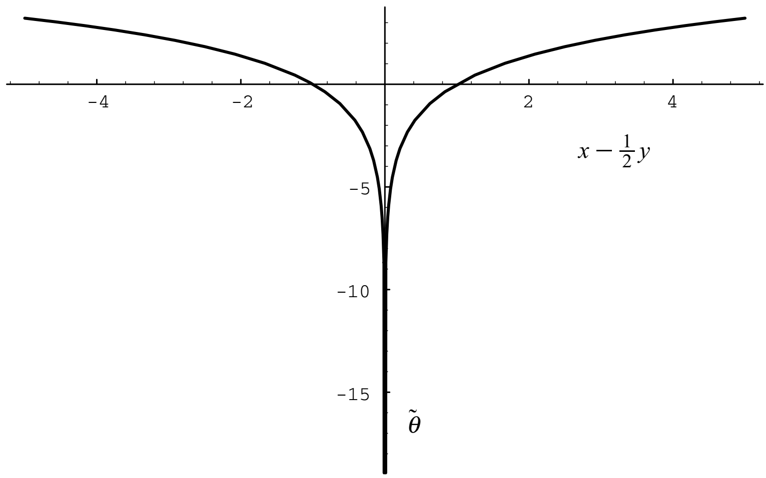


Fig. 16.1. Basic solution (16.16) of the Liouville equation with $a = 1$, $C = 0$.

### *16.1.2 The Sine–Gordon Equation*

Now choose $f[\theta] = \sin[\theta]$. Transforming to new variables as in the above example, the resulting equation is

$$\theta_{xy} = \sin[\theta]. \tag{16.17}$$

This equation admits the auto-Bäcklund transformation

$$\begin{aligned} \frac{\tilde{\theta}_x - \theta_x}{2} &= a \sin\left[\frac{\tilde{\theta} + \theta}{2}\right], \\ \frac{\tilde{\theta}_y + \theta_y}{2} &= \frac{1}{a} \sin\left[\frac{\tilde{\theta} - \theta}{2}\right]. \end{aligned} \tag{16.18}$$

Now generate the integrability condition

$$\begin{aligned} \frac{\tilde{\theta}_{xy} - \theta_{xy}}{2} &= a \cos\left[\frac{\tilde{\theta} + \theta}{2}\right] \frac{\tilde{\theta}_y + \theta_y}{2} = \cos\left[\frac{\tilde{\theta} + \theta}{2}\right] \sin\left[\frac{\tilde{\theta} - \theta}{2}\right], \\ \frac{\tilde{\theta}_{xy} + \theta_{xy}}{2} &= \frac{1}{a} \cos\left[\frac{\tilde{\theta} - \theta}{2}\right] \frac{\tilde{\theta}_x - \theta_x}{2} = \cos\left[\frac{\tilde{\theta} - \theta}{2}\right] \sin\left[\frac{\tilde{\theta} + \theta}{2}\right]. \end{aligned} \tag{16.19}$$

First add the two equations above yielding

$$\tilde{\theta}_{xy} = \sin[\tilde{\theta}]. \tag{16.20}$$

Then subtract to give

$$\theta_{xy} = \sin[\theta]. \tag{16.21}$$

We can generate solutions from the Bäcklund transformation just as in the previous example. Let's seed the process with the solution $\theta = 0$. Then (16.18) gives

$$\tilde{\theta}_x = 2a \sin\left[\frac{\tilde{\theta}}{2}\right],$$
$$\tilde{\theta}_y = \frac{2}{a} \sin\left[\frac{\tilde{\theta}}{2}\right]. \tag{16.22}$$

The solution is generated from the quadrature

$$d\tilde{\theta} = \tilde{\theta}_x\, dx + \tilde{\theta}_y\, dy = 2a \sin\left[\frac{\tilde{\theta}}{2}\right] dx + \frac{2}{a} \sin\left[\frac{\tilde{\theta}}{2}\right] dy. \tag{16.23}$$

Integrating (16.23),

$$\int \frac{d\tilde{\theta}}{\sin\,[\tilde{\theta}/2]} = 2ax + \frac{2}{a} y + 2C \tag{16.24}$$

produces the exact solution

$$\tilde{\theta} = 4 \tan^{-1}\left[e^{ax+(1/a)y+C}\right], \tag{16.25}$$

where $C$ is a constant of integration. Equation (16.25), shown in Figure 16.2, depicts a smooth step located at the origin.

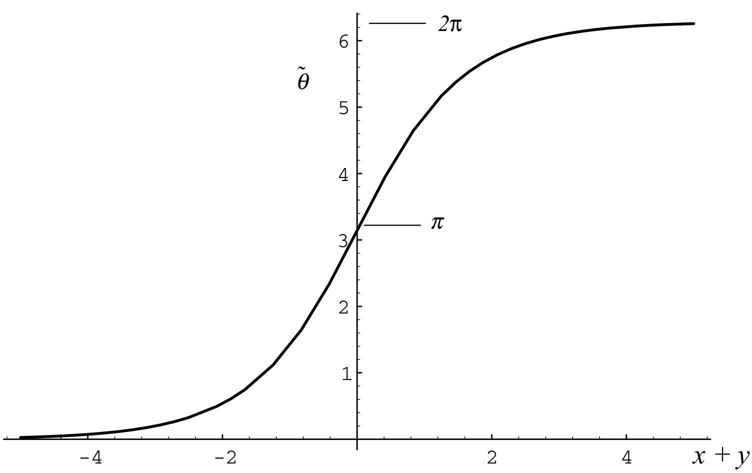


Fig. 16.2. Fundamental solution of (16.17) for $a = 1$, $C = 0$.

The sine–Gordon equation was studied in relation to the theory of surfaces of negative curvature by Darboux [16.3] and Bianchi [16.4], [16.5]. Bianchi showed that a theorem of permutability (a law of nonlinear superposition) exists for the sine–Gordon equation. If $\theta_1$, $\theta_2$, $\theta_3$ are solutions of (16.17), then a fourth solution can be determined from

$$\tan\left[\frac{\theta_4-\theta_1}{4}\right]=\frac{a_2+a_3}{a_2-a_3}\tan\left[\frac{\theta_2-\theta_3}{4}\right]. \tag{16.26}$$

In recent years the sine–Gordon equation has been shown to govern a wide variety of wave phenomena where the coordinate $y$ is timelike. The solution (16.25) can be regarded as a "front" that propagates undeformed in the minus $x$-direction. It illustrates, for this equation, a balance that can occur between the effects of dispersion, which tend to spread the front, and those of nonlinearity, which tend to steepen the front. This is a prototypical example of a solitary wave.

## 16.2 Symmetries Derived from a Potential Equation; Nonlocal Symmetries

In the previous examples, the Bäcklund transformations (16.9) and (16.18) were simply stated *a priori*, and the implications for finding solutions of the associated nonlinear PDE were then discussed. But the nagging question is: Where do these transformations come from? In particular, are they related to a Lie symmetry of the equation, and if they are, how can this symmetry be found? The general answer to this question is not known. However, in the remainder of this chapter several important examples will be described where the required transformation can be generated from a Lie symmetry of a related potential equation. The equivalent symmetry of the original equation is generally nonlocal. That is, it involves an integral of the dependent variable. The nonlocality of these symmetries is both encouraging and troublesome. The upside is that when such a symmetry can be found, it can often be used in quite remarkable ways to generate exact solutions of nonlinear problems. The downside is that when we solve the determining equations of the group for any given system of PDEs, we must be prepared to search for solutions that may involve integrals of the dependent variables. This is not an easy task, since there are no general methods for finding such solutions. For this reason they are often termed *hidden* symmetries.

Perhaps the clearest example of this is the use of a nonlocal symmetry to achieve the complete integration of the Burgers equation.

### 16.2.1 Solution of the Burgers Equation

In the late 1930s, J. M. Burgers [16.6], [16.7] at the University of Delft looked at a variety of model equations that were intended to reproduce several key aspects of the physical behavior of turbulent flow. He was particularly interested in modeling the well-known kinetic energy cascade where, at high Reynolds number, turbulence comprises a wide range of eddy length scales. Most of the kinetic energy is contained in the viscosity-independent, low-velocity-gradient, large scales and most of the dissipation of kinetic energy occurs in the viscosity-dependent, high-velocity-gradient, small scales. We discussed this ubiquitous feature of turbulence at some length in Chapter 13.

Burgers's idea was to seek evolution equations that were simplifications of the Navier–Stokes equations but contained the essential features of nonlinear convection and linear diffusion, in the hope of reproducing some of the essential physics. In his famous 1939 paper he looked at several candidates and finally concentrated his attention on the nonlinear convective heat equation,

$$u_t + uu_x - \nu u_{xx} = 0, \tag{16.27}$$

which has been known as the Burgers equation ever since. In Burgers' notation $u$ is the flow velocity in the $x$-direction and $t$ is time. The equation can be integrated in space to yield

$$\frac{d}{dt}\int_{-\infty}^{\infty} u\,dx + \left(\frac{1}{2}u^2 - \nu\frac{\partial u}{\partial x}\right)_{-\infty}^{\infty} = 0. \tag{16.28}$$

As long as the velocity and velocity derivative are zero at infinity, the integral of $u$ is conserved.

Let

$$A = \int_{-\infty}^{\infty} u\,dx, \tag{16.29}$$

which we can interpret as the total one-dimensional impulse (cf. Chapter 11, Section 11.5.1). One more quantity is needed to fully specify the problem. Let $u_0$ be the amplitude of the initial distribution of velocity:

$$u[x,0] = u_0 g[x], \tag{16.30}$$

where $g$ is dimensionless. Define new variables as follows:

$$U = \frac{u}{u_0}, \qquad \chi = \frac{xu_0}{\nu}, \qquad \tau = \frac{u_0^2 t}{\nu}. \tag{16.31}$$

Equations (16.27) and (16.29) become

$$U_\tau + UU_\chi - U_{\chi\chi} = 0 \tag{16.32}$$

and

$$Re = \int_{-\infty}^{\infty} U[\chi]\, d\chi, \tag{16.33}$$

where $Re = A/\nu$ can be thought of as an effective Reynolds number for the problem.

#### *16.2.1.1 Burgers Potential Equation Revisited*

We want to solve (16.32) and (16.33) for general initial conditions (16.30). To this end we first consider the potential equation that governs the incompletely integrated impulse. Let

$$\phi[\chi] = \int_{-\infty}^{\chi} U[\hat{\chi}]\, d\hat{\chi}. \tag{16.34}$$

The new variable $\phi$ is a potential function for $U$:

$$U = \phi_\chi. \tag{16.35}$$

Substitute (16.35) into (16.32) to get

$$\frac{D}{D\chi}\left(\phi_\tau + \frac{1}{2}(\phi_\chi)^2 - \phi_{\chi\chi}\right) = 0. \tag{16.36}$$

The Burgers potential equation

$$\phi_\tau + \frac{1}{2}(\phi_\chi)^2 - \phi_{\chi\chi} = 0 \tag{16.37}$$

is invariant under the infinitesimal group

$$\begin{aligned}
\tilde{\chi} &= \chi + s\xi[\chi, \tau, \phi],\\
\tilde{\tau} &= \tau + s\zeta[\chi, \tau, \phi],\\
\tilde{\phi} &= \phi + s\eta[\chi, \tau, \phi]
\end{aligned} \tag{16.38}$$

with the invariance condition

$$\eta_{\{\tau\}} + \phi_\chi \eta_{\{\chi\}} - \eta_{\{\chi\chi\}} = 0. \tag{16.39}$$

We analyzed (16.37) and the invariance condition (16.39) in Chapter 14, Section 14.5.2.1, where we used the package **IntroToSymmetry.m** to solve for the basic point symmetries of the equation. The equation is invariant under a six-parameter Lie algebra with the following operators:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial \chi}, \qquad X_2 = \frac{\partial}{\partial \tau}, \qquad X_3 = \frac{\partial}{\partial \phi}, \\ X_4 &= \tau \frac{\partial}{\partial \chi} + \chi \frac{\partial}{\partial \phi}, \qquad X_5 = \frac{1}{2}\chi \frac{\partial}{\partial \chi} + \tau \frac{\partial}{\partial \tau}, \\ X_6 &= 2\chi\tau \frac{\partial}{\partial \chi} + 2\tau^2 \frac{\partial}{\partial \tau} + (\chi^2 + 2\tau)\frac{\partial}{\partial \phi}. \end{aligned} \tag{16.40}$$

These comprise all of the conventional point groups where the polynomial form of the infinitesimals truncates, but, as we noted in Chapter 14, an additional symmetry can also be found that solves (16.39), and it is worthwhile showing some of the details. Let the transformation (16.38) be of the form

$$\begin{aligned} \tilde{\chi} &= \chi, \\ \tilde{\tau} &= \tau, \\ \tilde{\phi} &= \phi + s\eta[\chi, \tau, \phi]. \end{aligned} \tag{16.41}$$

The invariance condition (16.39) becomes

$$D_\tau \eta + \phi_\chi D_\chi \eta - D_{\chi\chi}\eta = 0. \tag{16.42}$$

Equation (16.42) is satisfied by the infinite-dimensional point group

$$\eta = f[\chi, \tau] e^{\phi/2}, \tag{16.43}$$

where $f$ is any solution of the heat equation, $f_\tau = f_{\chi\chi}$. Let's check (16.43)

$$\begin{aligned} D_\tau \eta &= \left( f_\tau + \frac{f}{2}\phi_\tau \right) e^{\phi/2}, \\ \phi_\chi D_\chi \eta &= \left( \phi_\chi f_\chi + \frac{f}{2}\phi_\chi^2 \right) e^{\phi/2}, \\ D_{\chi\chi}\eta &= \left( f_{\chi\chi} + f_\chi \phi_\chi + \frac{f}{4}\phi_\chi^2 + \frac{f}{2}\phi_{\chi\chi} \right) e^{\phi/2}, \end{aligned} \tag{16.44}$$

and the invariance condition (16.42) is

$$D_\tau \eta + \phi_\chi D_\chi \eta - D_{\chi\chi}\eta = (f_\tau - f_{\chi\chi}) + e^{\phi/2}\frac{f}{2}\left( \phi_\tau + \frac{1}{2}\phi_\chi^2 - \phi_{\chi\chi} \right). \tag{16.45}$$

It works! The invariance condition is satisfied if $f$ is a solution of the heat equation and $\phi$ is a solution of (16.37). Now the question is: what finite transformation corresponds to (16.43)? To answer this we must sum the Lie series,

$$\tilde{\phi} = \phi + sX\phi + \frac{s^2}{2!}X(X\phi) + \frac{s^3}{3!}X(XX(\phi)) + \cdots, \tag{16.46}$$

where the group operator is

$$X = f[\chi, \tau]e^{\phi/2}\frac{\partial}{\partial\phi}. \tag{16.47}$$

The first few terms are as follows:

$$\begin{aligned}
X\phi &= 2\left(\frac{f}{2}e^{\phi/2}\right), \\
X^2\phi &= 2(1)\left(\frac{f}{2}e^{\phi/2}\right)^2, \\
X^3\phi &= 2(1\times 2)\left(\frac{f}{2}e^{\phi/2}\right)^3, \\
X^4\phi &= 2(1\times 2\times 3)\left(\frac{f}{2}e^{\phi/2}\right)^4, \\
X^5\phi &= 2(1\times 2\times 3\times 4)\left(\frac{f}{2}e^{\phi/2}\right)^5, \\
&\vdots
\end{aligned} \tag{16.48}$$

Let

$$a = s\frac{f}{2}e^{\phi/2}. \tag{16.49}$$

The Lie series (16.46) becomes

$$\tilde{\phi} = \phi + 2\left\{a + \frac{a^2}{2} + \frac{a^3}{3} + \frac{a^4}{4} + \frac{a^5}{5} + \cdots\right\}. \tag{16.50}$$

Consider the series

$$\begin{aligned}
g(a) &= a + \frac{a^2}{2} + \frac{a^3}{3} + \frac{a^4}{4} + \frac{a^5}{5} + \cdots, \\
\frac{dg}{da} &= a + a^2 + a^3 + a^4 + a^5 + \cdots = \frac{1}{1-a}.
\end{aligned} \tag{16.51}$$

Integrating the second equation in (16.51) leads to

$$g(a) = -\ln[1-a]. \tag{16.52}$$

Using this result in (16.50) generates the finite transformation corresponding to the infinitesimal operator (16.47):

$$\boxed{\tilde{\phi} = \phi - 2\ln\left[1 - \frac{s}{2} f[\chi, \tau] e^{\phi/2}\right], \qquad \tilde{\chi} = \chi, \qquad \tilde{\tau} = \tau.} \tag{16.53}$$

Once the process is complete, the group parameter $s$ can obviously be merged with $f[\chi, \tau]$, but for the moment we will retain it as a reminder of the origin of (16.53) in a Lie series expansion of the group operator (16.47).

What is the implication of this result? Well, if, in (16.53), $\phi$ is a solution of (16.37), then for any $f[\chi, \tau]$ that is a solution of the heat equation the new function $\tilde{\phi}$ is a new solution of (16.37). In effect, (16.53) is a machine for churning out exact solutions of the Burgers potential equation (16.37).

To satisfy ourselves that the finite transformation (16.53) is correct, let's work out the transformations of derivatives to check that it leaves (16.37) invariant. The transformation formulas for derivatives greatly simplify in this case, because the independent variables are not transformed. First derivatives transform as follows:

$$\tilde{\phi}_{\tilde{\tau}} = D_\tau\left(\phi - 2\ln\left[1 - \frac{s}{2} f[\chi, \tau] e^{\phi/2}\right]\right) = \phi_\tau + \frac{\left(f_\tau + \frac{f}{2}\phi_\tau\right) s e^{\phi/2}}{1 - \frac{s}{2} f e^{\phi/2}} \tag{16.54}$$

and

$$\tilde{\phi}_{\tilde{\chi}} = D_x\left(\phi - 2\ln\left[1 - \frac{s}{2} f[\chi, \tau] e^{\phi/2}\right]\right) = \phi_\chi + \frac{\left(f_\chi + \frac{f}{2}\phi_\chi\right) s e^{\phi/2}}{1 - \frac{s}{2} f e^{\phi/2}}. \tag{16.55}$$

Squaring the first derivative produces

$$(\tilde{\phi}_{\tilde{\chi}})^2 = (\phi_\chi)^2 + \frac{\left(2\phi_\chi f_\chi + f\phi_\chi^2\right) s e^{\phi/2}}{1 - \frac{s}{2} f e^{\phi/2}} + \frac{\left(f_\chi + \frac{f}{2}\phi_\chi\right)^2 s^2 e^{\phi}}{\left(1 - \frac{s}{2} f e^{\phi/2}\right)^2}. \tag{16.56}$$

The second derivative transforms as

$$\tilde{\phi}_{\tilde{\chi}\tilde{\chi}} = D_{\chi\chi}\left(\phi - 2\ln\left[1 - \frac{s}{2} f[\chi, \tau] e^{\phi/2}\right]\right), \tag{16.57}$$

or

$$\tilde{\phi}_{\tilde{\chi}\tilde{\chi}} = \phi_{\chi\chi} + \frac{\left(f_{\chi\chi} + f_\chi \phi_\chi + \frac{f}{2}\phi_{\chi\chi} + \frac{f}{4}\phi_\chi^2\right) s e^{\phi/2}}{1 - \frac{s}{2} f e^{\phi/2}} + \frac{\left(f_\chi + \frac{f}{2}\phi_\chi\right)^2 \frac{s^2}{2} e^{\phi}}{\left(1 - \frac{s}{2} f e^{\phi/2}\right)^2}. \tag{16.58}$$

Summing (16.54), (16.56), and (16.58) and canceling terms leads to the finite transformation of the Burgers potential equation

$$\tilde{\phi}_{\tilde{\tau}} + \frac{1}{2}(\tilde{\phi}_{\tilde{\chi}})^2 - \tilde{\phi}_{\tilde{\chi}\tilde{\chi}} = \left(\phi_\tau + \frac{1}{2}(\phi_\chi)^2 - \phi_{\chi\chi}\right) + \frac{(f_\tau - f_{\chi\chi}) + \frac{f}{2}\left(\phi_\tau + \frac{1}{2}(\phi_\chi)^2 - \phi_{\chi\chi}\right)}{\left(1 - \frac{s}{2} f e^{\phi/2}\right)} s e^{\phi/2} \tag{16.59}$$

The right-hand side of (16.59) is proportional to Burgers potential equation, if and only, if $f[\chi, \tau]$ is a solution of the heat equation: $f_\tau - f_{\chi\chi} = 0$.

#### *16.2.1.2 The Nonlocal Group of the Burgers Equation*

The finite transformation of the Burgers potential equation (16.53) will now be used to generate solutions of the Burgers equation. Recall that $U = \phi_\chi$. Replace $\phi$ in (16.53) with

$$\phi = \int_{-\infty}^{\chi} U \, d\hat{\chi} = D_\chi^{-1} U. \tag{16.60}$$

The result is the finite one-parameter group corresponding to (16.53),

$$\begin{aligned} \tilde{\chi} &= \chi, \\ \tilde{\tau} &= \tau, \\ \tilde{U} &= U + D_\chi \ln\left[1 - \frac{s}{2} f[\chi, \tau] e^{D_\chi^{-1} U/2}\right]^{-2} \\ &= U + \frac{s\left(f_\chi + \frac{f}{2} U\right) e^{D_\chi^{-1} U/2}}{1 - \frac{s}{2} f e^{D_\chi^{-1} U/2}}, \end{aligned} \tag{16.61}$$

which leaves the Burgers equation invariant. The transformation (16.61) is of the general form

$$\begin{aligned}\tilde{\chi} &= F_1[\chi, \tau, U, s],\\ \tilde{\tau} &= F_2[\chi, \tau, U, s],\\ \tilde{U} &= G[\chi, \tau, U, D_x^{-1}U, s]\end{aligned} \tag{16.62}$$

and represents an example of a *nonlocal* group – one that depends on all values of the dependent variable over some interval of the independent variable.

Although the transformation (16.62) is rather nonstandard in its appearance, it fits quite nicely within Lie theory in that it simply represents another type of solution of the invariance condition, albeit one that is often not easy to find. The fact that such a transformation can lead to a general solution of a nonlinear PDE is a powerful reminder that there may be important symmetries out there that are not simple point symmetries or conventional Lie–Bäcklund symmetries. We need to recognize that solutions of the invariance condition can include integrals – even multiple integrals. Thus, the search for symmetries of an equation is an open-ended process until all possible solutions of the invariance condition have been nailed down.

Expand (16.61) for small values of $s$ to generate the infinitesimal group corresponding to (16.61),

$$\begin{aligned}\tilde{\chi} &= \chi,\\ \tilde{\tau} &= \tau,\\ \tilde{U} &= U + s\left(f_\chi + \frac{f}{2}U\right)e^{D_\chi^{-1}U/2}.\end{aligned} \tag{16.63}$$

The infinitesimal

$$\eta = \left(f_\chi + \frac{f}{2}U\right)e^{D_\chi^{-1}U/2} \tag{16.64}$$

is a nonlocal solution of the invariance condition for Burgers equation, namely

$$\eta_{\{\tau\}} + \eta U_\chi + U\eta_{\{\chi\}} - \eta_{\{\chi\chi\}} = 0. \tag{16.65}$$

Since the independent variables are not transformed, the invariance condition becomes

$$D_\tau\eta + \eta U_\chi + UD_x\eta - D_{\chi\chi}\eta = 0. \tag{16.66}$$

When the various terms appearing in (16.66) are formed and summed, the result is

$$\left(f_\chi + \frac{f}{2}U\right)e^{D_\chi^{-1}U/2}\left(\frac{D_x^{-1}U_\tau}{2} + \frac{U^2}{4} - \frac{U_\chi}{2}\right)$$
$$+\,e^{D_\chi^{-1}U/2}\left((f_\tau - f_{\chi\chi})_\chi + (f_\tau - f_{\chi\chi})\frac{U}{2} + \frac{f}{2}(U_\tau + UU_\chi - U_{\chi\chi})\right) = 0. \tag{16.67}$$

Note that the coefficient of the first term is the integral of the Burgers equation,

$$\frac{D_x^{-1}U_\tau}{2} + \frac{U^2}{4} - \frac{U_\chi}{2} = D_\chi^{-1}(U_\tau + UU_\chi - U_{\chi\chi}). \tag{16.68}$$

Thus the invariance condition (16.66) is satisfied as long as $f_\tau - f_{\chi\chi} = 0$ and $U$ is a solution of $U_\tau + UU_\chi - U_{\chi\chi} = 0$.

The remarkable transformation (16.61) can be used to generate solutions of the Burgers equation by succesive transformation beginning with an initial seed solution and with an arbitrary choice of $f[\chi, \tau]$ at each step. Note that the transformation (16.61) preserves the conserved integral of the Burgers equation. Substitute (16.61) into (16.29):

$$\int_{-\infty}^{\infty} \tilde{U} d\tilde{x} = \int_{-\infty}^{\infty} U dx - 2\ln\left[1 - \frac{s}{2}f[\chi, \tau]e^{D_\chi^{-1}U/2}\right]_{-\infty}^{\infty}. \tag{16.69}$$

For $f \to 0$ at plus and minus infinity the integral is invariant:

$$\int_{-\infty}^{\infty} \tilde{U}\, d\tilde{x} = \int_{-\infty}^{\infty} U\, dx. \tag{16.70}$$

#### *16.2.1.3 The Cole–Hopf Transformation*

Suppose we take $U = 0$ as a first solution. Then (16.61) becomes

$$\tilde{U} = \frac{sf_\chi}{1 - \frac{s}{2}f}. \tag{16.71}$$

If the function $f$ is a solution of the heat equation, then so is

$$\theta[\chi, \tau] = 1 - \frac{s}{2}f[\chi, \tau]. \tag{16.72}$$

The transformation (16.71) becomes

$$\tilde{U} = -2\frac{\theta_\chi}{\theta}. \tag{16.73}$$

This is the usual form given for the famous transformation of the Burgers equation, discovered independently in the early 1950s by Hopf [16.8] and Cole [16.9], that bears their names.

Although the Cole–Hopf transformation is often regarded as an infinitely many-valued transformation and therefore outside the class of one-parameter Lie groups, we can see from the development above that (16.73) actually arises from a point group, (16.43), of the Burgers potential equation that leads, in turn, to a nonlocal group, (16.61), of the Burgers equation.

The general transformation (16.61) can be rearranged in the form (16.73), and so at first sight (16.61) would seem to be no more general than (16.73). However, (16.61) has one attribute that is not shared by (16.73), which we will now investigate. The Cole–Hopf transformation can be expressed as a Bäcklund transformation,

$$\begin{aligned} \theta_\chi &= -\tfrac{1}{2}U\theta, \\ \theta_\tau &= -\tfrac{1}{2}(U\theta)_\chi, \end{aligned} \tag{16.74}$$

where the second relation is generated by differentiating the first with respect to $x$ and making the replacement $\theta_{\chi\chi} \to \theta_\tau$. Equating cross derivatives produces the integrability condition

$$U(\theta_\tau - \theta_{\chi\chi}) + \theta(U_\tau + UU_\chi - U_{\chi\chi}) = 0 \tag{16.75}$$

that must be satisfied by (16.74) in order for it to have any meaning. Are the transformations (16.74) and (16.61) the same? In one sense they are: they generate the same solutions of the Burgers equation. But in one important way they are not the same: the Bäcklund form (16.74) has no standing by itself; it must be accompanied by the integrability condition $\theta_{\chi\tau} = \theta_{\tau\chi}$ and only makes sense when $U$ is a Burgers-equation solution and $\theta$ is a heat-equation solution. In contrast, (16.61) is a one-to-one invertible map; $U$ and all its derivatives are defined by the transformation. It can be used to transform any integrable $U$ given any $f$. In this respect, the Burgers-equation and heat-equation solutions constitute only a subset of the general group defined by (16.61).

#### *16.2.1.4 Burgers Equation Related to Turbulence*

What about Burgers's original program to search for solutions with behavior that resembles turbulence? Let's look at the Burgers-equation solution corresponding to the following solution of the heat equation involving the complementary error function $\mathrm{erfc}[x] = (2/\sqrt{\pi}) \int_x^\infty e^{-\zeta^2} d\zeta$:

$$\theta = 1 + \frac{e^{Re/2} - 1}{\sqrt{\pi}} \int_{\chi/\sqrt{4\tau}}^{\infty} e^{-\zeta^2} d\zeta. \tag{16.76}$$

The integral (16.33) is satisfied identically, and the corresponding Burgers-equation solution is

$$U = \frac{\left(e^{Re/2} - 1\right) e^{-\chi^2/4\tau}}{\sqrt{\tau}\left(\sqrt{\pi} + \left(e^{Re/2} - 1\right) \int_{\chi/\sqrt{4\tau}}^{\infty} e^{-\zeta^2} d\zeta\right)}. \tag{16.77}$$

At small Reynolds numbers the solution (16.77) becomes

$$U = \frac{1}{\sqrt{\pi\tau}} \left(\frac{Re}{2}\right) e^{-\chi^2/4\tau} \tag{16.78}$$

In dimensioned variables (16.78) is the elementary-source solution of the heat equation,

$$u = \frac{A}{\sqrt{4\pi\nu t}} e^{-x^2/4\nu t}. \tag{16.79}$$

To study the solution at high Reynolds number, we follow Whitham [16.10] and rearrange (16.77) to read as follows:

$$V = \frac{U\sqrt{\pi\tau}}{\sqrt{Re}} = \frac{\left(e^{Re/2} - 1\right) e^{-\lambda^2 Re}}{\sqrt{Re}\left(1 + \frac{e^{Re/2} - 1}{\sqrt{\pi}} \int_{\lambda\sqrt{Re}}^{\infty} e^{-\zeta^2} d\zeta\right)}. \tag{16.80}$$

The single-hump velocity distribution (16.80) is shown in Figure 16.3 for several Reynolds numbers. If $\lambda < 0$ as $Re \to \infty$ the complimentary error function in the denominator of (16.80) goes to one and the solution simplifies to

$$V \approx \frac{1}{\sqrt{Re}} e^{-\lambda^2 Re}. \tag{16.81}$$

For $\lambda > 0$ the integral can be approximated asymptotically by

$$\frac{1}{\sqrt{\pi}} \int_{\lambda\sqrt{Re}}^{\infty} e^{-\zeta} d\zeta \simeq \frac{e^{-\lambda^2 Re}}{2\sqrt{\pi}(\lambda\sqrt{Re})}. \tag{16.82}$$

If $0 < \lambda^2 < \frac{1}{2}$ and $Re \to \infty$,

$$V\sqrt{\tau} \to 2\lambda. \tag{16.83}$$

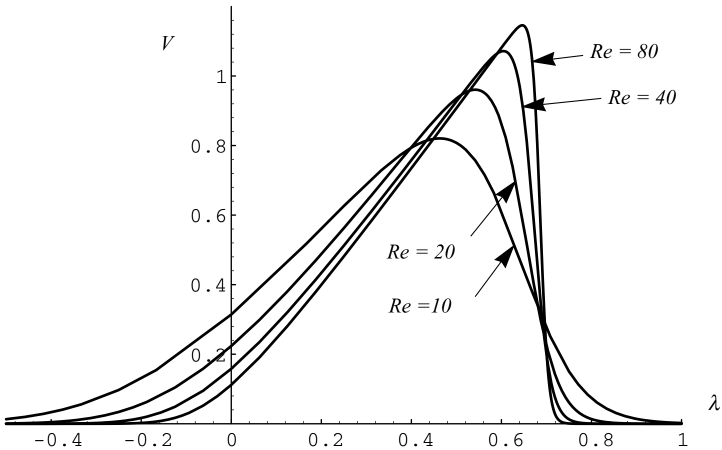


Fig. 16.3. The single-hump solution of the Burgers equation at several Reynolds numbers.

If $\lambda > 1/\sqrt{2}$ and $Re \to \infty$,

$$V\sqrt{\tau} \to 0, \tag{16.84}$$

where $\chi/\sqrt{4\tau} = \lambda\sqrt{Re}$. Thus at high Reynolds number there is a sudden transition in $V$ located near $\lambda = 1/\sqrt{2}$. In the transition layer we can make the approximation $\lambda^2 \approx \sqrt{2}(\lambda - 1/\sqrt{2})$, and the velocity distribution becomes

$$V\sqrt{\tau} \simeq \frac{\sqrt{2}}{1 + (\sqrt{2\pi\, Re})e^{\sqrt{2}\, Re\,(\lambda - 1/\sqrt{2})}}. \tag{16.85}$$

The solution (16.80) has precisely the behavior sought by Burgers. As the Reynolds number is increased, two transition layers develop at the upstream and downstream ends of a hump of rightward-moving fluid. In the upstream layer, confined to negative $\chi$ (or $\lambda$), the moderately large velocity gradient, using (16.81), is approximately

$$\frac{\partial U}{\partial \chi} \simeq -\frac{4\chi}{\sqrt{4\pi\tau}} e^{-\chi^2/4\tau} \tag{16.86}$$

and is independent of the Reynolds number – a property shared by the inertial subrange of turbulence. In the neighborhood of the origin there is also a transition in the profile shape from (16.81) to (16.83). In the downstream layer, near $\lambda = 1/\sqrt{2}$, the very large velocity gradient is, in the limit of high Reynolds number,

$$\frac{\partial U}{\partial \chi} \simeq -\sqrt{\frac{8Re}{\pi}}. \tag{16.87}$$

In this layer the velocity gradient increases in proportion to the square root of the Reynolds number, precisely the same Reynolds-number dependence observed in the fine-scale (large-gradient) motions in three-dimensional turbulence. Burgers's great achievement was to demonstrate the nature of nonlinear steepening in a convective–diffusive environment. What makes the one-dimensional Burgers model interesting in the context of turbulence is that the steepening of the velocity gradient occurs in the absence of any mechanism akin to the vortex stretching that occurs in three-dimensional turbulence.

### 16.2.2 Solitary-Wave Solutions of the Korteweg–de Vries Equation

The Burgers equation provides the canonical example illustrating the balance that can occur, in more general phenomena, between nonlinear convection and diffusion. In a study of the motion of long waves in a rectangular channel, Korteweg and de Vries in 1895 [16.15] developed the following model equation,

$$u_t + 6uu_x + u_{xxx} = 0 \tag{16.88}$$

which has since become the canonical example of the combined effects of nonlinear convection and dispersion. When water is shallow (but not so shallow so as to bring into play surface tension), waves of small amplitude become slightly dispersive. In this case a localized disturbance on the surface of a channel will tend to spread as it propagates. However, if the amplitude is not small, the tendency to spread due to dispersion may be balanced by the proportionality between wave speed and water depth, $c \propto \sqrt{h}$, which causes a wave of sufficiently large amplitude to tend to steepen and form a bore. The result is a localized bump in the surface that propagates but does not spread at all.

This is an outstanding example of a solitary wave, or soliton, as introduced in Section 16.1 in the context of the sine–Gordon equation. The first observation of a solitary wave of this type was on a canal near Edinburgh, Scotland in 1834 by an engineer named Scott Russell. Russell later wrote of his experience following the wave on horseback for over a kilometer, describing a "large solitary elevation . . . which continued its course along the channel apparently without change of form." The speed of a solitary wave increases with the height of the wave, so that if a high-amplitude wave is formed behind a low-amplitude one, it will catch up. It turns out that when it does so, it passes through the low-amplitude wave and reemerges ahead with its shape unchanged but shifted forward in position.

It is now known that many of the nonlinear wave equations that arise in diverse branches of physics admit large-amplitude solitary-wave solutions, and it is recognized that the suprising ability of solitons to survive collisions with other solitons is one of the universal traits of nonlinear wave behavior. Whitham [16.10] gives a very complete exposition of the sine–Gordon equation along with its relation to other types of nonlinear wave equations. The role of the sine–Gordon equation in various physical phenomena, including the self-focusing of light beams in nonlinear optics is described by Barone et al. [16.11]. Segev and Stegeman [16.12] describe experimental observations of spatial solitons in nonlinear optical media governed by the related cubic Schrödinger equation. An exposition of the wide variety of physical phenomena involving solitary waves, as well as a history of their discovery, can be found in Remoissenet [16.13].

We worked out the point groups and the classical recursion operator for (16.88) in Chapter 14, Section 14.5.2.3. The variable $u$ is the height of the free surface, and, as we shall see later, the dispersion relation is

$$\omega = \kappa^3, \tag{16.89}$$

where $\kappa$ is the wave number. For the coming analysis it is convenient to generalize equation [16.88] slightly to

$$u_t + 3\beta u u_x + \frac{\beta}{2} u_{xxx} = 0, \tag{16.90}$$

where $\beta$ is an arbitrary constant.

### *16.2.2.1 Nonlocal Group of the KdV Potential Equation*

Our search for a Bäcklund transformation of this equation begins with a *potential* equation two derivatives removed from the KdV equation. We consider

$$\theta_t + \frac{\beta}{2}\theta_x^3 - \beta\theta_{xxx} = 0, \tag{16.91}$$

where $\beta$ is arbitrary. The relationship between this equation and the KdV equation is described by Lamb [16.16] and will be elucidated in the following. The invariance condition for (16.91) is

$$\eta_{\{t\}} + \frac{3\beta}{2}\theta_x^2 \eta_{\{x\}} - \beta\eta_{\{xxx\}} = 0. \tag{16.92}$$

The elementary four-parameter point group of (16.91) has the operators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial t}, \qquad X_3 = \frac{\partial}{\partial \theta}, \qquad X_4 = \frac{x}{3}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}. \quad (16.93)$$

The last operator corresponds to invariance under the one-parameter dilation group,

$$\begin{aligned} \tilde{x} &= e^s x, \\ \tilde{t} &= e^{3s} t, \\ \tilde{\theta} &= \theta. \end{aligned} \quad (16.94)$$

which leaves invariant the dispersion relation (16.89).

But these are not all the possible solutions of the invariance condition (16.92); there is also a nonlocal solution. We search for an invariant group of (16.91) with the following infinitesimal form:

$$\begin{aligned} \tilde{x} &= x, \\ \tilde{t} &= t, \\ \tilde{\theta} &= \theta + s\eta[x, t, \theta]. \end{aligned} \quad (16.95)$$

The independent variables are not transformed, and so the invariance condition for (16.91) takes on the simplified form

$$D_t\eta + \frac{3\beta}{2}\theta_x^2 D_x\eta - \beta D_{xxx}\eta = 0. \quad (16.96)$$

Equation (16.96) is satisfied by the nonlocal group

$$\eta = \int e^{-\theta}\, dx = D_x^{-1}(e^{-\theta}). \quad (16.97)$$

Let's verify (16.97). We have

$$\begin{aligned} D_t\eta &= -D_x^{-1}(e^{-\theta}\theta_t), \\ D_x\eta &= e^{-\theta}, \\ D_{xx}\eta &= -e^{-\theta}\theta_x, \\ D_{xxx}\eta &= e^{-\theta}\theta_x^2 - e^{-\theta}\theta_{xx}. \end{aligned} \quad (16.98)$$

Substituting (16.98) into (16.96) leads to

$$
\begin{aligned}
-D_x^{-1}\big(e^{-\theta}\theta_t\big) &+ \frac{\beta}{2}\theta_x^2 e^{-\theta} + \beta e^{-\theta}\theta_{xx} \\
&= D_x^{-1}\left(-e^{-\theta}\left(\theta_t + \frac{\beta}{2}\theta_x^3 - \beta\theta_{xxx}\right)\right) = 0,
\end{aligned}
\tag{16.99}
$$

which confirms that (16.97) satisfies (16.96) on solutions of (16.91). Thus (16.97) is a symmetry of (16.91). A second nonlocal solution of (16.96) is

$$
\eta = \int e^{\theta}\, dx = D_x^{-1}(e^{\theta}). \tag{16.100}
$$

The linearity in $\eta$ of the invariance condition (16.96) enables any combination of exponentials to be used to construct nonlocal symmetries, including

$$
\begin{aligned}
\eta &= D_x^{-1}(\sinh\theta),\\
\eta &= D_x^{-1}(\cosh\theta).
\end{aligned}
\tag{16.101}
$$

Notice that the transformation (16.97) also leaves invariant the spatial terms in (16.91). If we consider just the equation

$$
\tfrac{1}{2}\theta_x^3 - \theta_{xxx} = 0, \tag{16.102}
$$

the invariance condition becomes

$$
\begin{aligned}
\tfrac{3}{2}\theta_x^2 D_x\eta &- D_{xxx}\eta \\
&= \tfrac{1}{2}\theta_x^2 e^{-\theta} + e^{-\theta}\theta_{xx} \\
&= D_x^{-1}\big(-e^{-\theta}\big(\tfrac{1}{2}\theta_x^3 - \theta_{xxx}\big)\big) = 0.
\end{aligned}
\tag{16.103}
$$

The integrated form of (16.91) is

$$
-D_x^{-1}(e^{-\theta}\theta_t) + \frac{\beta}{2}\theta_x^2 e^{-\theta} + \beta e^{-\theta}\theta_{xx} = 0, \tag{16.104}
$$

for which the corresponding steady equation is

$$
\tfrac{1}{2}\theta_x^2 + \theta_{xx} = 2e^{-\theta/2}\big(e^{\theta/2}\big)_{xx} = 0 \tag{16.105}
$$

In summary, the nonlocal group (16.97) is a symmetry of all three equations, (16.91), (16.102), and (16.105), or for that matter any linear combination of these equations.

Equation (16.105) can be generalized to

$$\tfrac{1}{2}\theta_x^2 + \varepsilon\theta_{xx} = 2e^{-\varepsilon\theta/2}\big(e^{\varepsilon\theta/2}\big)_{xx} = 0, \tag{16.106}$$

where $\varepsilon = \pm\, 1$. The invariance condition (16.96) is satisfied by the nonlocal group generalized from (16.97) and (16.100):

$$\eta = D_x^{-1}\big(e^{-\varepsilon\theta}\big). \tag{16.107}$$

The group (16.107) satisfies the invariance condition for (16.106) identically as follows:

$$\theta_x D_x\eta + \varepsilon(D_{xx}\eta) = \theta_x e^{-\varepsilon\theta} + \varepsilon(-\varepsilon e^{-\varepsilon\theta}\theta_x) = 0. \tag{16.108}$$

Notice that a function satisfying (16.105) or (16.102) can have an arbitrary dependence on time.

Now let's generate the finite transformation corresponding to the group (16.97) by summing the Lie series

$$\tilde{\theta} = \theta + sX\theta + \frac{s^2}{2!}X^2\theta + \frac{s^3}{3!}X^3\theta + \frac{s^4}{4!}X^4\theta + \cdots, \tag{16.109}$$

where the group operator is

$$X = D_x^{-1}(e^{-\theta})\frac{\partial}{\partial\theta}. \tag{16.110}$$

Various terms in the series are

$$\begin{aligned} X\theta &= D_x^{-1}(e^{-\theta}), \\ X^2\theta &= -D_x^{-1}(e^{-\theta})^2, \\ X^3\theta &= 2D_x^{-1}(e^{-\theta})^3, \\ X^4\theta &= -(2\times 3)D_x^{-1}(e^{-\theta})^4. \\ &\vdots \end{aligned} \tag{16.111}$$

The Lie series becomes

$$\tilde{\theta} = \theta - \left(\alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{3} + \frac{\alpha^4}{4} + \cdots\right), \tag{16.112}$$

where

$$\alpha = -sD_x^{-1}(e^{-\theta}). \tag{16.113}$$

Consider the series

$$f(\alpha) = \alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{3} + \frac{\alpha^4}{4} + \cdots \quad (16.114)$$

with

$$\frac{df}{d\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \cdots = \frac{1}{1-\alpha}. \quad (16.115)$$

Integrating (16.115) leads to

$$f(\alpha) = \ln[1-\alpha] + \ln\gamma, \quad (16.116)$$

where $\gamma$ is a constant of integration. The finite transformation (16.112) becomes the following:

$$\tilde{\theta} = \theta - \lambda \ln\left[\gamma + \gamma s D_x^{-1}(e^{-\theta})\right] + \lambda \ln\gamma. \quad (16.117)$$

The added constant makes the identity element of the transformation $s = 0$. Notice that the constant $\beta$ that appears in equation (16.91) does not appear in (16.117).

There is a scaling ambiguity in the finite transformation (16.117). If we differentiate (16.117) with respect to the group parameter $s$, the result is

$$\eta = \frac{d}{ds}\left(\theta - \lambda \ln\left[\gamma + \gamma s D_x^{-1}(e^{-\theta})\right]\right)_{s=0} = -\lambda D_x^{-1}(e^{-\theta}), \quad (16.118)$$

which matches the infinitesimal (16.97) up to the factor $-\lambda$. Note that the invariance condition (16.96) is satisfied for any value of $\lambda$ because of its linearity.

To develop the final form of the transformation, (16.117) is substituted into (16.91) to evaluate $\lambda$. The final result is the non local transformation

$$\boxed{\tilde{\theta} = \theta + 2\ln\left[1 - sD_x^{-1}(e^{-\theta})\right], \quad \tilde{x} = x, \quad \tilde{t} = t} \quad (16.119)$$

where, without loss of generality, the quantity $s/\gamma$ has, for convenience, been replaced by the equivalent group parameter $-s$. The parameter $s$ is an arbitrary constant in the finite, nonlocal transformation (16.119). Substitution of (16.119) into the KdV potential equation (16.91) confirms that $\tilde{\theta}$ is a solution of (16.91) as long as $\theta$ is a solution.

We can use (16.119) to generate a Bäcklund transformation for the KdV potential equation (16.91). This is accomplished by differentiating (16.119) with respect to $x$ and then $t$ and then reusing (16.119) to remove the argument of the logarithim in the result:

$$\tilde{\theta}_x - \theta_x = \frac{-2se^{-\theta}}{1 - sD_x^{-1}(e^{-\theta})} = \frac{-2se^{-\theta}}{e^{(\tilde{\theta}-\theta)/2}} = -2se^{-(\tilde{\theta}+\theta)/2}, \quad (16.120)$$

or

$$\frac{\tilde{\theta}_x - \theta_x}{2} = -se^{-(\tilde{\theta}+\theta)/2} \quad (16.121)$$

Now differentiate with respect to time

$$\begin{aligned}\tilde{\theta}_t - \theta_t &= \frac{2sD_x^{-1}(e^{-\theta}\theta_t)}{1-sD_x^{-1}(e^{-\theta})} = \frac{2sD_x^{-1}(e^{-\theta}\theta_t)}{e^{(\tilde{\theta}-\theta)/2}} \\ &= \frac{2s\big((\beta/2)\theta_x^2 e^{-\theta} + \beta e^{-\theta}\theta_{xx}\big)}{e^{(\tilde{\theta}-\theta)/2}}\end{aligned} \quad (16.122)$$

Finally, a Bäcklund transformation for (16.91) derived from the nonlocal group (16.119) is

$$\begin{aligned}\frac{\tilde{\theta}_x - \theta_x}{2} &= -se^{-(\tilde{\theta}+\theta)/2}, \\ \frac{\tilde{\theta}_t - \theta_t}{2} &= s\left(\frac{\beta}{2}\theta_x^2 + \beta\theta_{xx}\right)e^{-(\tilde{\theta}+\theta)/2}.\end{aligned} \quad (16.123)$$

If we use (16.123) to generate an exact solution of (16.91) beginning with the vacuum seed solution $\theta = 0$, we end up with the quadrature

$$d\tilde{\theta} = -(2s)e^{-\tilde{\theta}/2}\,dx + (0)\,dt, \quad (16.124)$$

leading to the solution

$$\tilde{\theta} = \ln[1 - s(x - x_a)]^2, \quad (16.125)$$

where $x_a$ is an effective origin in space. The solution (16.125) corresponds to inserting $\theta = 0$ into (16.119). If we iterate on this solution, the result is a trivial sequence of solutions identical to (16.125) with varying values of $s$.

The simplest time-dependent solution of (16.91) is the linear phase front,

$$\theta = \kappa(x - x_a) - \frac{\beta}{2}\kappa^3(t - t_a) + c \quad (16.126)$$

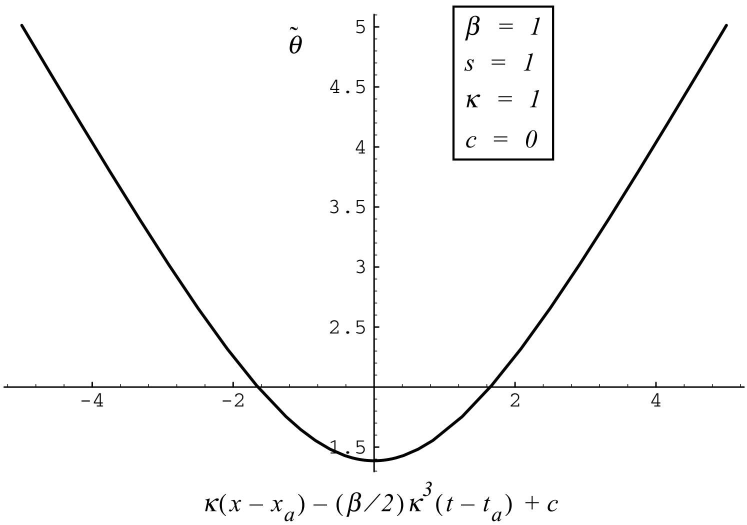


Fig. 16.4. Elementary solution of the potential KdV equation.

(see Figure 16.4). The factors $\kappa$ and $\kappa^3$ reflect the dispersion relation (16.89) that we expect for the KdV equation and are directly linked to the invariance of the KdV potential equation (16.91) under the dilation group (16.94). The nontrivial solution generated from (16.119) using (16.126) is

$$\begin{aligned}\tilde{\theta} &= \kappa(x - x_a) - \frac{\beta}{2}\kappa^3(t - t_a) + c \\ &\quad + \ln\left[1 + \frac{s}{\kappa}e^{-(\kappa(x-x_a)-(\beta/2)\kappa^3(t-t_a)+c)}\right]^2. \end{aligned} \tag{16.127}$$

plotted in Figure 16.4.

Note that we have seen the first of the transformations in (16.123) in the context of the Bäcklund transformation for the Liouville equation. If, for the moment, we imagine that $\theta$ is a function of both $x$ and $y$, and we write the transformation twice with a change in sign and a new parameter, then

$$\begin{aligned}\frac{\tilde{\theta}_x - \theta_x}{2} &= -se^{-(\tilde{\theta}+\theta)/2}, \\ \frac{\tilde{\theta}_y + \theta_y}{2} &= -\hat{s}e^{-(\tilde{\theta}-\theta)/2}. \end{aligned} \tag{16.128}$$

If we choose $\hat{s} = -1/(2s)$, the result is the Bäcklund transformation (16.10) for the Liouville equation (16.8).

### 16.2.2.2 Exact Solutions of the KdV Equation

Differentiate (16.91) with respect to $x$, and let

$$h = \tfrac{1}{2}\theta_x. \tag{16.129}$$

The function $h$ satisfies the modified KdV equation

$$h_t + (6\beta)h^2 h_x - (\beta)h_{xxx} = 0. \tag{16.130}$$

Returning to (16.105), let's look at the equation that would govern a variable constructed from the combination $(\beta/2)\theta_x^2 + \varepsilon\beta\theta_{xx}$, which is invariant under (16.107). Let

$$w = \left(\frac{\beta}{4}\right)\theta_x^2 + \varepsilon\left(\frac{\beta}{2}\right)\theta_{xx} = \beta h^2 + \varepsilon\beta h_x, \tag{16.131}$$

where $\varepsilon$ can take on the values $+1$ or $-1$. The relation (16.131) is recognized to be the well-known Miura transformation [16.17] connecting $h$ and $w$. Using (16.130) and (16.131), we can derive the relation

$$h_t = \varepsilon w_{xx} - 2(hw)_x. \tag{16.132}$$

Now bring together, (16.131) and (16.132)

$$\begin{aligned} h_x &= \varepsilon\left(\frac{1}{\beta}w - h^2\right), \\ h_t &= \varepsilon w_{xx} - 2(hw)_x, \end{aligned} \tag{16.133}$$

where $h$ satisfies (16.130) and $w$ satisfies

$$w_t + 6ww_x - \beta w_{xxx} = 0. \tag{16.134}$$

The transformation (16.133) constitutes the classical form of the Bäcklund transformation of the KdV equation (see [16.18]). The transformation is directly connected to the one-parameter nonlocal transformation (16.119) that leaves the KdV potential equation (16.91) invariant.

The mapping accomplished by the Miura transformation (16.131) using either sign of $\varepsilon$ produces a solution of the KdV equation (16.90). That is,

$$\begin{aligned} &w_t + 6ww_x - \beta w_{xxx} \\ &\quad = \left(2\beta h + \varepsilon\beta\frac{\partial}{\partial x}\right)(h_t + (6\beta)h^2 h_x - (\beta)h_{xxx}) = 0, \end{aligned} \tag{16.135}$$

where $\beta$ can be selected to match whatever form of the KdV equation is being considered. Let

$$\begin{aligned} w^{+} &= \left(\frac{\beta}{4}\right)\theta_x^2 + \left(\frac{\beta}{2}\right)\theta_{xx} = \beta h^2 + \beta h_x, \\ w^{-} &= \left(\frac{\beta}{4}\right)\theta_x^2 - \left(\frac{\beta}{2}\right)\theta_{xx} = \beta h^2 - \beta h_x. \end{aligned} \tag{16.136}$$

The superscripts in (16.136) correspond to letting $\varepsilon = \pm 1$ in (16.131). Substituting (16.119) into the $\varepsilon = +1$ relation in (16.136) results in

$$\tilde{w}_1^{+} = \left(\frac{\beta}{4}\right)\tilde{\theta}_x^2 + \left(\frac{\beta}{2}\right)\tilde{\theta}_{xx} = \left(\frac{\beta}{4}\right)\theta_x^2 + \left(\frac{\beta}{2}\right)\theta_{xx}, \tag{16.137}$$

which is already implied by the result (16.108). In other words, the $\varepsilon = +1$ transformation reaches all the way back to the original solution $\theta$, no matter how many recursions of the transformation (16.119) may have been performed. The $\varepsilon = -1$ transformation in (16.136) produces the following KdV solution:

$$\begin{aligned} \tilde{w}_1^{-} &= \left(\frac{\beta}{4}\right)\tilde{\theta}_x^2 - \left(\frac{\beta}{2}\right)\tilde{\theta}_{xx} = \left(\frac{\beta}{4}\right)\theta_x^2 - \left(\frac{\beta}{2}\right)\theta_{xx} \\ &\quad + \frac{2\beta s^2 - 2\beta s e^{\theta}\theta_x\left(1 - sD_x^{-1}(e^{-\theta})\right)}{e^{2\theta}\left(1 - sD_x^{-1}(e^{-\theta})\right)^2}. \end{aligned} \tag{16.138}$$

The transformation (16.138) in conjuction with (16.119) provides a procedure for generating solutions of the KdV equation.

Now let's look at a particular case, beginning with the simplest propagating solution of the KdV potential equation (16.91):

$$\theta_0 = \kappa(x - x_a) - \frac{\beta}{2}\kappa^3(t - t_a) + c, \tag{16.139}$$

where $x_a$ and $t_a$ are the origins in time and space, and where $\kappa$ is a constant that determines both the speed of propagation and the amplitude of the solution. The solution of the KdV potential equation (16.91) generated by the transformation (16.119) – repeated here for convenience:

$$\tilde{\theta} = \theta + 2\ln\left[1 - sD_x^{-1}(e^{-\theta})\right]^2 \tag{16.140}$$

– is

$$\theta_1 = \theta_0 + 2\ln\left[1 + \frac{s}{\kappa}e^{-\theta_0}\right]^2. \tag{16.141}$$

The corresponding solutions of the KdV equation

$$\tilde{w}_{\tilde{t}} + 6\tilde{w}\tilde{w}_{\tilde{x}} - \beta\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}} = 0 \tag{16.142}$$

generated from (16.137) and (16.138) are

$$\begin{aligned} \tilde{w}^{+} &= \frac{\beta}{4}\kappa^2, \\ \tilde{w}^{-} &= \frac{\beta}{4}\kappa^2 - \frac{2(\beta s)\kappa e^{\theta_0}}{\left(e^{\theta_0} + \frac{s}{\kappa}\right)^2}. \end{aligned} \tag{16.143}$$

Both regular and singular cases of solution $\tilde{w}^-$ the are plotted in Figure 16.5 for several choices of $\beta$ and $s$.

The solutions generated thus far are displaced forms of the single-hump solitary-wave solution that is the distinguishing feature of the KdV equation. The undisplaced solution can be easily constructed from the two solutions in (16.143). Let

$$u = \frac{1}{\beta}\left(\frac{\beta}{4}\kappa^2 - \tilde{w}^-\right) = 2s\kappa\frac{e^{\theta_0}}{\left(e^{\theta_0} + \frac{s}{\kappa}\right)^2}. \tag{16.144}$$

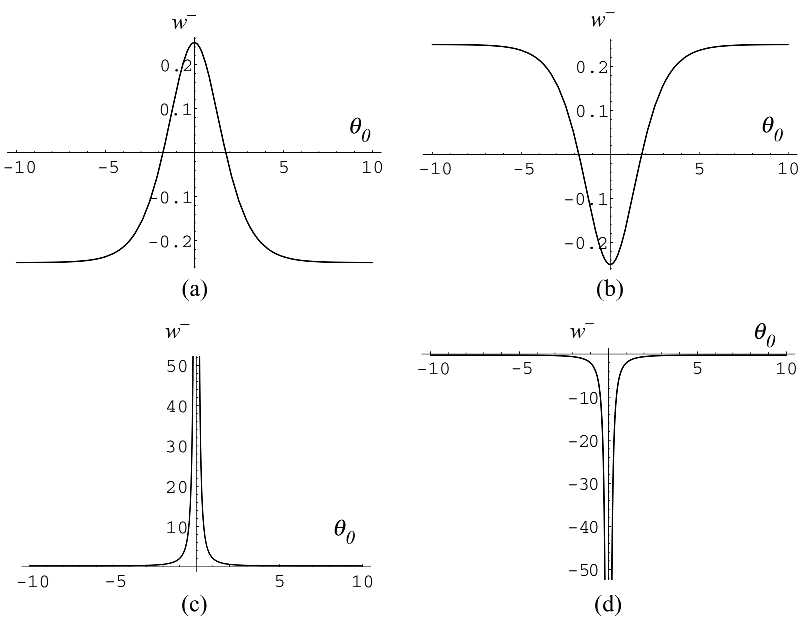


Fig. 16.5. Solutions of the KdV equation (16.142) corresponding to $w^-$ in (16.143). Parameter values are: $\kappa = 1$, and (a) $\beta = -1$, $s = 1$; (b) $\beta = 1$, $s = 1$; (c) $\beta = 1$, $s = -1$; (d) $\beta = -1$, $s = -1$.

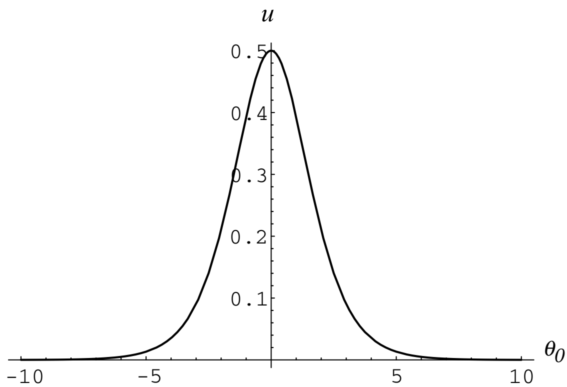


Fig. 16.6. Classical single-hump solution of the KdV equation presented in Equation (16.144). Parameter values are $k = 1, s = 1$.

The function $u$ satisfies

$$u_t + 3\beta u u_x + \frac{\beta}{2} u_{xxx} = 0. \tag{16.145}$$

The choice $\beta = 2$ in the KdV potential equation (16.91) produces the usual form of the KdV equation in (16.145). The solution (16.144) with $\beta = 2$ corresponds to a localized disturbance propagating along

$$\theta_0 = \kappa(x - x_a) - \kappa^3(t - t_a) + c. \tag{16.146}$$

This is the classical undisplaced single-hump solution of the KdV equation shown in Figure 16.6.

We can get to this same solution in a much more direct way. Begin with the group

$$\theta_1 = \theta_0 + 2\ln\left[1 + \frac{s}{\kappa} e^{-\theta_0}\right]^2, \tag{16.147}$$

where $\theta_0$ is given by (16.139). If we differentiate (16.147) twice with respect to $x$, we immediately get (16.144):

$$u = \theta_{1xx} = 2s\kappa \frac{e^{\theta_0}}{\left(e^{\theta_0} + \frac{s}{\kappa}\right)^2}. \tag{16.148}$$

This result follows directly from the transformations (16.137) and (16.138) as follows:

$$u = \frac{1}{\beta}\left(\frac{\beta}{4}\kappa^2 - \frac{\beta}{4}\theta_{1x}^2 + \frac{\beta}{2}\theta_{1xx}\right)$$
$$= \frac{1}{\beta}\left(\frac{\beta}{4}\kappa^2 - \left(\frac{\beta}{4}\kappa^2 - \frac{\beta}{2}\theta_{1xx}\right) + \frac{\beta}{2}\theta_{1xx}\right) = \theta_{1xx}. \quad (16.149)$$

### *16.2.2.3 Colliding Solitons*

Multiple interacting soliton solutions of the KdV equation can be generated following the procedure described by Whitham [16.10]. Let

$$u = p_x. \quad (16.150)$$

The KdV equation (16.145) can be integrated once to give

$$u_t + 3\beta u u_x + \frac{\beta}{2}u_{xxx} = \frac{\partial}{\partial x}\left(p_t + \frac{3\beta}{2}p_x^2 + \frac{\beta}{2}p_{xxx}\right). \quad (16.151)$$

The basic solution of

$$p_t + \frac{3\beta}{2}p_x^2 + \frac{\beta}{2}p_{xxx} = 0 \quad (16.152)$$

is, using (16.144) in (16.150),

$$p = \theta_x - \kappa = 2(\ln F)_x \quad (16.153)$$

where $F$ is

$$F(x,t) = 1 + \frac{s}{\kappa}e^{-\theta} \quad (16.154)$$

and

$$\theta = \kappa(x - x_a) - \frac{\beta}{2}\kappa^3(t - t_a) + c. \quad (16.155)$$

Substituting (16.153) into (16.152) leads to the equation for $F(x,t)$:

$$\left\{\frac{F_{xt}}{F} - \frac{F_t F_x}{F^2}\right\} - \frac{3\beta s}{\kappa}\left\{\frac{F_{xx}^2}{F^2} - 2\frac{F_x^2 F_{xx}}{F^3} + \frac{F_x^4}{F^4}\right\}$$
$$+ \frac{\beta}{2}\left\{\frac{F_{xxxx}}{F} - 3\frac{F_{xx}^2}{F^2} - 4\frac{F_x F_{xxx}}{F^2} + 12\frac{F_x^2 F_{xx}}{F^3} - 6\frac{F_x^4}{F^4}\right\} = 0. \quad (16.156)$$

To make further progress, we must choose the group parameter to be $s = -\kappa$. Equation (16.156) becomes

$$F\left\{F_t + \frac{\beta}{2}F_{xxx}\right\}_x - F_x\left\{F_t + \frac{\beta}{2}F_{xxx}\right\} + \frac{3\beta}{2}\left\{F_{xx}^2 - F_x F_{xxx}\right\} = 0. \tag{16.157}$$

Consider two solutions:

$$F_0 = 1 - e^{-\theta_0}, \tag{16.158}$$

where

$$\theta_0 = \kappa_0(x - x_{a0}) - \frac{\beta}{2}\kappa_0^3(t - t_{a0}) + c_0, \tag{16.159}$$

and

$$F_1 = 1 - e^{-\theta_1}, \tag{16.160}$$

where

$$\theta_1 = \kappa_1(x - x_{a1}) - \frac{\beta}{2}\kappa_1^3(t - t_{a1}) + c_1. \tag{16.161}$$

Equation (16.157) is known to admit the exact solution

$$F = 1 + e^{-\theta_0} + e^{-\theta_1} + \left(\frac{\kappa_1 - \kappa_0}{\kappa_1 + \kappa_0}\right)^2 e^{-(\theta_1+\theta_0)}. \tag{16.162}$$

The corresponding KdV solution is generated using (16.150) and (16.153). After differentiating (16.162) twice, we have

$$u = 2\left\{\frac{\kappa_0^2 e^{-\theta_0} + \kappa_1^2 e^{-\theta_1} + (\kappa_1 - \kappa_0)^2 e^{-(\theta_1+\theta_0)}}{1 + e^{-\theta_0} + e^{-\theta_1} + \left(\frac{\kappa_1-\kappa_0}{\kappa_1+\kappa_0}\right)^2 e^{-(\theta_1+\theta_0)}}\right\} + 2\left\{\frac{\left(\kappa_0 e^{-\theta_0} + \kappa_1 e^{-\theta_1} + \frac{(\kappa_1-\kappa_0)^2}{\kappa_1+\kappa_0} e^{-(\theta_1+\theta_0)}\right)^2}{\left(1 + e^{-\theta_0} + e^{-\theta_1} + \left(\frac{\kappa_1-\kappa_0}{\kappa_1+\kappa_0}\right)^2 e^{-(\theta_1+\theta_0)}\right)^2}\right\}. \tag{16.163}$$

The solution (16.163) is plotted in Figure 16.7 at various times.

The solution in Figure 16.7 depicts a collision between two solitons of different amplitude. The larger-amplitude "particle" overtakes the smaller amplitude one and, after an interaction, emerges with its shape unchanged. The residual

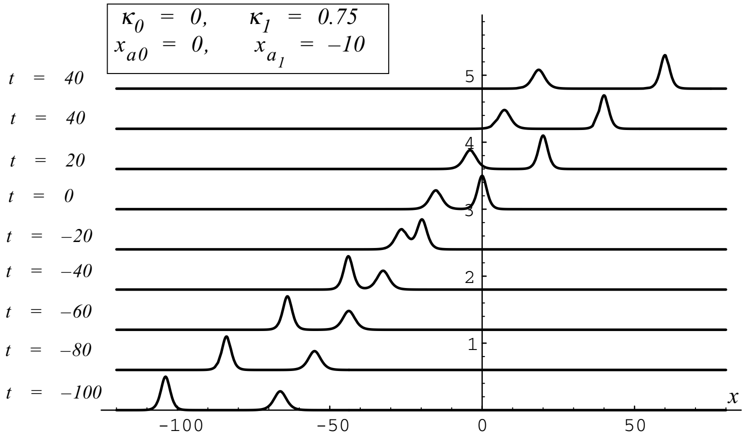


Fig. 16.7. Exact solution of the KdV equation depicting the collision of two solitons.

effect of the collision is to induce a phase shift causing the overtaking particle to be suddenly "flung forward" to a position ahead of where it would have been in the absence of the collision. This example and the others treated in this chapter involve one space dimension and time but one should be aware that solitary wave behavior in two space dimensions is well known. See References [16.19] and [16.20].

## 16.3 Concluding Remarks

The discovery of solitons and the associated methods of integration of nonlinear wave equations can fairly be described as one of the most important advances in mathematical physics in the 20th century. The solutions provide insight into the nature of nonlinearity and into the nonlinear interactions that can occur between colliding "particles" of all sorts. Most importantly, approximately solitonic behavior to a high degree of accuracy is readily observed in the laboratory. The numerous examples described by Remoissenet [16.13] vividly illustrate the breadth of concrete applications.

The Bäcklund transformations described in this chapter possess a clear link to an underlying one parameter group leaving invariant a potential equation. Moreover the group can be used to generate classical soliton solutions of the related wave equation. It is quite satisfying to think that Bäcklund transformations can fit into the general framework of Lie theory. Perhaps the most salutary lesson from all this is to remind us of the open-ended nature of the invariance condition $X_{\{p\}}\Psi^i = 0$ and of the continuing pursuit of solutions that is the ongoing task of symmetry analysis.

## 16.4 Exercises

16.1 Derive the Bianchi permutation relation for the sine–Gordon equation. Show that if $\theta_1, \theta_2, \theta_3$ are three solutions of the sine–Gordon equation, then a fourth can be obtained from the permutation relation

$$\tan\left(\frac{\theta_4 - \theta_1}{4}\right) = \frac{a_2 + a_3}{a_2 - a_3}\tan\left(\frac{\theta_2 - \theta_3}{4}\right), \tag{16.164}$$

where $a_2$ and $a_3$ are arbitrary constants. See Anderson and Ibragimov [16.21].

16.2 Show that the infinitesimal

$$\eta = \left(f_\chi + \frac{f}{2}U\right)e^{D_\chi^{-1}U/2} \tag{16.165}$$

is a nonlocal solution of the invariance condition for the Burgers equation,

$$D_\tau\eta + \eta U_\chi + U D_\chi\eta - D_{\chi\chi}\eta = 0, \tag{16.166}$$

as long as $f_\tau - f_{\chi\chi} = 0$ and $U$ is a solution of $U_\tau + UU_\chi - U_{\chi\chi} = 0$.

16.3 Confirm the nonlocal solution of the Burgers equation

$$\tilde{U} = U + \frac{(f_\chi + \frac{f}{2}U)se^{D_\chi^{-1}U/2}}{1 - \frac{s}{2}fe^{D_\chi^{-1}U/2}}, \tag{16.167}$$

by direct substitution. That is, show that if $U$ is a solution of the Burgers equation and $f$ is a solution of the heat equation, then $\tilde{U}$ is a solution of the Burgers equation. In the process you will need to use the higher-order differential consequences of the equations

$$\begin{aligned} f_{\chi\tau} - f_{\chi\chi\chi} &= 0, \\ U_{\chi\tau} + U_\chi^2 + UU_{\chi\chi} - U_{\chi\chi\chi} &= 0. \end{aligned} \tag{16.168}$$

16.4 Derive the equation governing the Burgers kinetic energy,

$$\frac{\partial k}{\partial t} + u\frac{\partial k}{\partial x} - \nu\frac{\partial^2 k}{\partial x^2} + \nu\left(\frac{\partial u}{\partial x}\right)^2 = 0, \tag{16.169}$$

where $k = u^2/2$. Using the single-hump solution of the Burgers equation, discuss the Reynolds-number dependence of the energy dissipation integral

$$\Phi = \int_{-\infty}^{\infty} \nu\left(\frac{\partial u}{\partial x}\right)^2 dx. \tag{16.170}$$

What part of the solution contributes most to the integral (16.170)? What part contributes most to the total kinetic energy

$$E = \int_{-\infty}^{\infty} \frac{u^2}{2}\, dx \, ? \tag{16.171}$$

16.5 The point-source solution of the heat equation is

$$\phi = \left(\frac{2}{\pi}\right)^{1/2} A(2\nu t)^{-1/2} e^{-\xi^2/2} + B, \qquad \xi = \frac{x}{(2\nu t)^{1/2}}. \tag{16.172}$$

(i) Use this to generate the classical $N$-wave solution of the Burgers equation. Discuss the nature of this solution at small and large time. This solution has been used to model the shock structure associated with the sonic boom.

(ii) Generate a periodic solution of the Burgers equation by taking for $\phi$ a periodic distribution of heat sources spaced a distance $\lambda$ apart. Show that for small time this corresponds to a periodic set of shocks, and for large time to a decaying sine wave. See Whitham [16.10] for the solution.

16.6 Show by direct substitution that

$$\tilde{\theta} = \theta + 2\ln\left(1 - sD_x^{-1}(e^{-\theta})\right) \tag{16.173}$$

is a solution of the KdV potential equation

$$\theta_t + \frac{\beta}{2}\theta_x^3 - \beta\theta_{xxx} = 0 \tag{16.174}$$

as long as $\theta$ is a solution.

16.7 Use (16.136) to construct an auto-Bäcklund transformation of the KdV equation (16.134). Recall that $\theta_x = 2h$, and consider the change of variable $w = p_x$.

16.8 Show that the 1 – D compressible flow equations

$$u_t + uu_x + (1/\rho)\, p_x = 0, \qquad \rho_t + u\rho_x + \rho u_x = 0,$$
$$p_t + up_x + F[\rho]u_x = 0$$

Are invariant under the group with operator

$$X = \frac{t^2}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial u} - R\frac{\partial}{\partial p}$$

where $R$ is a non-local variable that satisfies the equations $R_x = \rho$, $R_t = -\rho u$. Note that the integrability condition $R_{xt} = R_{tx}$ is satisfied according to the continuity equation. Show that the corresponding finite transformation is

$$\tilde{t} = t, \quad \tilde{x} = x + at^2/2, \quad \tilde{u} = u + at, \tilde{\rho} = \rho, \tilde{p} = p - a\int \rho d_x$$

This is a transformation to a coordinate system accelerating at the rate $a$ and is an example of a hidden symmetry. See Reference [16.21].

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# Appendix 1

## *Review of Calculus and the Theory of Contact*

It is useful to review certain of the tools of calculus that play a central role in the development of Lie theory. A thorough understanding of the chain rule and the concept of total differentiation of implicit functions is crucial to the manipulation of transformations that act on variables and their derivatives.

First, the concept of a differential is reviewed along with the chain rule for partial differentiation. This discussion leads to the definition of the total differentiation operator, which is required to overcome certain notational ambiguities that arise when we take partial derivatives of a function of a function. This operator plays a central role in the development of Lie theory and is of immense help in maintaining concise, unambiguous notation. Finally the notion of contact between a curve and a surface is reviewed.

### A1.1 Differentials and the Chain Rule

Consider the function

$$u = H[f]. \tag{A1.1}$$

Assume that $H$ is a differentiable function of $f$. Then two symbols, $du$ and $df$, can be introduced that allow the derivative to be manipulated as a fraction. Let $df$ denote a real number, and define $du = d(H(f))$ to be a function of the two independent variables $f$ and $df$ prescribed by the equation

$$du \equiv H_f[f]\,df. \tag{A1.2}$$

The quantity $H_f[f]$ is the slope of the function $H[f]$ at the point $f$ and is constructed geometrically by the usual limiting process. The differential notation just described is convenient for the treatment of explicit, implicit, inverse, and parametrically defined functions.

If $u$ is a function of several variables, say

$$u = H[f^1, f^2, \ldots, f^q], \tag{A1.3}$$

then the total differential of $u$ is

$$du = \frac{\partial H}{\partial f^1} df^1 + \frac{\partial H}{\partial f^2} df^2 + \cdots + \frac{\partial H}{\partial f^q} df^q. \tag{A1.4}$$

The variable $du$ is a function of $2q$ independent variables: the coordinates $f^1$, $f^2, \ldots, f^q$, and the differentials $df^1, \ldots, df^q$.

### A1.1.1 A Problem with Notation

Suppose that the $f^1, f^2, \ldots, f^q$ are assumed to depend on new variables $x^1, x^2, \ldots, x^n$. The functional dependence of $u$ is

$$u = H[f^1[x^1, x^2 \ldots, x^n], f^2[x^1, x^2 \ldots, x^n], \ldots, f^q[x^1, x^2 \ldots, x^n]]. \tag{A1.5}$$

According to the chain rule, the partial derivative of $u$ with respect to $x^j$ is

$$\frac{\partial u}{\partial x^j} = \frac{\partial H}{\partial f^1}\frac{\partial f^1}{\partial x^j} + \frac{\partial H}{\partial f^2}\frac{\partial f^2}{\partial x^j} + \cdots + \frac{\partial H}{\partial f^q}\frac{\partial f^q}{\partial x^j}, \qquad j = 1, \ldots, n. \tag{A1.6}$$

In the derivative notation adopted in Chapter 7, Equation (A1.6) would be written

$$\frac{\partial u}{\partial x^j} = H_{f^1} f^1_j + H_{f^2} f^2_j + \cdots + H_{f^q} f^q_j, \qquad j = 1, \ldots, n. \tag{A1.7}$$

The notation of (A1.6) or (A1.7) is perfectly adequate in most circumstances. However a common ambiguity can arise. Consider the function

$$u = H[x^1, x^2 \ldots, x^n, f^1, \ldots, f^q], \tag{A1.8}$$

where $x^1, x^2 \ldots, x^n, f^1, \ldots, f^q$ are initially assumed to be independent variables. The partial derivative of $u$ with respect to $x^j$ is

$$\frac{\partial u}{\partial x^j} = \frac{\partial H}{\partial x^j}. \tag{A1.9}$$

However, if $f^1, \ldots, f^q$ depend on $x^1, x^2 \ldots, x^n$, the functional dependence of $u$ is

$$u = H[x^1, x^2 \ldots, x^n, f^1[x^1, x^2 \ldots, x^n], \ldots, f^q[x^1, x^2 \ldots, x^n]], \tag{A1.10}$$

and the partial derivative of $u$ with respect to $x^j$ is now written

$$\frac{\partial u}{\partial x^j} = \frac{\partial H}{\partial x^j} + \frac{\partial H}{\partial f^1}\frac{\partial f^1}{\partial x^j} + \cdots + \frac{\partial H}{\partial f^q}\frac{\partial f^q}{\partial x^j}. \tag{A1.11}$$

Here we have a problem. The symbols $\partial u/\partial x^j$ and $\partial H/\partial x^j$ have two entirely different relationships to one another – one given by (A1.9), the other given by (A1.11). Our partial derivative notation has failed to unambiguously define exactly what operation is intended in the second case. As it happens, the functional forms that we commonly deal with in group theory present this ambiguity so often that it is necessary to define a special derivative symbol.

### *A1.1.2 The Total Differentiation Operator*

Symmetry analysis is concerned with quantities that depend on independent variables $\boldsymbol{x} = [x^1, x^2, \ldots, x^n]$ and dependent variables $\boldsymbol{y} = [y^1, y^2, \ldots, y^m]$ via functions of the form

$$\psi = \Psi[\boldsymbol{x}, \mathbf{y}[\mathbf{x}], f^1[\boldsymbol{x}], f^2[\boldsymbol{x}], \ldots, f^q[\boldsymbol{x}]], \qquad i = 1, \ldots, m. \tag{A1.12}$$

The derivative of (A1.12) with respect to $x^j$ is

$$\frac{\partial \psi}{\partial x^j} = \frac{D\Psi}{Dx^j}, \tag{A1.13}$$

where the operator $D/Dx^j$ is defined as

$$\begin{aligned}\frac{D(\,)}{Dx^j} &= D_j(\,)\\ &= \frac{\partial(\,)}{\partial x^j} + \frac{\partial(\,)}{\partial y^i}\frac{\partial y^i}{\partial x^j} + \frac{\partial(\,)}{\partial f^1}\frac{\partial f^1}{\partial x^j} + \frac{\partial(\,)}{\partial f^2}\frac{\partial f^2}{\partial x^j} + \cdots + \frac{\partial(\,)}{\partial f^q}\frac{\partial f^q}{\partial x^j},\end{aligned} \tag{A1.14}$$

and the Einstein summation condition on repeated indices is used.

The operator (A1.14) is called the *total differentiation operator with respect to the $j$th independent variable*. Throughout the literature on group theory, there is a tendency to shorten the name and call it merely the total derivative, thus causing some confusion with the concept of a total differential. This is unfortunate given the fact that, for more than one independent variable, (A1.14) well and truly defines a partial derivative. One could perhaps come up with a more appropriate name and call (A1.14), say, the complete partial derivative operator, but this sounds like an oxymoron. Since current usage is so pervasive, there is probably no way to change it without causing added confusion, and so we will adopt the traditional nomenclature. To the usual partial derivative notation, $\partial(\ )/\partial(\ )$ we will assign the meaning of (A1.9) where *the derivative is with respect to the explicit dependence of* $\Psi$ *on* $x^j$. Note that in fluid mechanics, $D(\ )/Dt$ is called the *substantial derivative* and has the interpretation of describing the change with time of some property of a fluid particle.

### *A1.1.3 The Inverse Total Differentiation Operator*

Chapters 14 and 16 describe the application of nonlocal group operators involving integrals of the dependent variables. Here it is necessary to consider the inverse of the operation of total differentiation. Define the integral operator

$$D_j^{-1}(\ ) = \int_{-\infty}^{x^j} (\ )\, dx^j. \tag{A1.15}$$

This operator satisfies

$$D_{x^j} D_{x^j}^{-1} g[\boldsymbol{x}] = g[\boldsymbol{x}], \qquad D_{x^j}^{-1} D_{x^j} g[\boldsymbol{x}] = g[\boldsymbol{x}], \tag{A1.16}$$

where $g[\boldsymbol{x}]$ is any differentiable function of the vector $\boldsymbol{x}$ with compact support such that (A1.15) converges.

## A1.2 The Theory of Contact

The extended transformations discussed in Chapters 8 and 9 are required to preserve all tangency conditions (contact conditions) up to order $p$. It turns out that transformations that satisfy this requirement actually preserve tangency up to infinite order, and this is utilized in the theory of Lie–Bäcklund transformations described in Chapter 14. This property, together with the parametric representation of a Lie group, ensures that the transformation, extended to whatever order, constitutes a single-valued, invertible map.

To help introduce the notion of contact conditions, the theory that describes the degree of contact between a curve and a surface is reviewed here.

### *A1.2.1 Finite-Order Contact between a Curve and a Surface*

Let's examine the degree to which a curve $C$ can make contact with a surface $\psi$. Assume that both $C$ and $\psi$ are continuous and differentiable to order at least $q$. Let $C$ be a space curve in $n$ dimensions described parametrically by

$$x^j = F^j[s], \qquad j = 1, \ldots, n, \tag{A1.17}$$

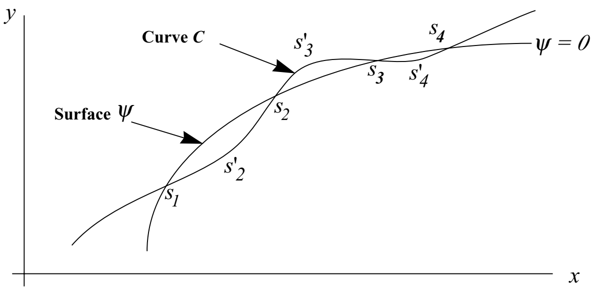


Fig. A1.1. Four intersection points between curve $C$ and surface $S$.

where the variable $s$ is a parameter along the curve. The surface $\psi$ in $n$ dimensions is given by

$$\Psi[\boldsymbol{x}] = 0. \tag{A1.18}$$

Now suppose that $C$ and $\psi$ intersect at a point $x_1^j = F^j[s_1]$ and also at $q-1$ other points $x_2^j = F^j[s_2]$, $x_3^j = F^j[s_3], \ldots, x_q^j = F^j[s_q]$ in some neighborhood of $\boldsymbol{x}_1$. Figure A1.1 depicts the situation where $q = 4$.

Now consider the function

$$g[s] = \Psi[\boldsymbol{x}[s]]. \tag{A1.19}$$

At the successive intersections we have

$$\begin{aligned} g[s_1] &= \Psi[\boldsymbol{x}[s_1]] = 0, \\ g[s_2] &= \Psi[\boldsymbol{x}[s_2]] = 0, \\ g[s_3] &= \Psi[\boldsymbol{x}[s_3]] = 0, \\ &\vdots \\ g[s_q] &= \Psi[\boldsymbol{x}[s_q]] = 0. \end{aligned} \tag{A1.20}$$

It is clear from Figure A1.1 that there are intermediate points $s'_2, s'_3, \ldots, s'_q$ where

$$s_1 \leq s'_2 \leq s_2, \qquad s_2 \leq s'_3 \leq s_3, \ldots, \qquad s_{q-1} \leq s'_q \leq s_q \tag{A1.21}$$

such that the first derivatives of $g[s]$ are zero:

$$\frac{dg[s'_2]}{ds} = \frac{dg[s'_3]}{ds} = \frac{dg[s'_4]}{ds} = \cdots = \frac{dg[s'_q]}{ds} = 0. \tag{A1.22}$$

This follows from the definition of the derivative and the fact that between two intersection points of $C$ where the function $\Psi$ is zero, the derivative of $g[s]$ must change sign. Therefore, between the two points there must be a point where the derivative of $g[s]$ is zero. Similarly there must exist points $s_3'', s_4'', \ldots, s_q''$ where

$$s_2' \le s_3'' \le s_3', \qquad s_3' \le s_4'' \le s_4', \ldots, \qquad s_{q-1}' \le s_q'' \le s_q' \tag{A1.23}$$

such that the second derivatives of $g(s)$ are zero:

$$\frac{d^2 g[s_3'']}{ds^2} = \frac{d^2 g[s_4'']}{ds^2} = \cdots = \frac{d^2 g[s_q'']}{ds^2} = 0. \tag{A1.24}$$

Continuing in this manner, one finds that there exist $s_1, s_2', s_3'', s_4''', \ldots, s_q^{(q-1)}$, all in a neighborhood of $s_1$, such that

$$g[s_1] = \frac{dg[s_2']}{ds} = \frac{d^2 g[s_3'']}{ds^2} = \frac{d^3 g[s_4''']}{ds^3} = \cdots = \frac{d^{q-1} g\left[s_q^{(q-1)}\right]}{ds^{q-1}} = 0. \tag{A1.25}$$

Now consider the limit as $s_2', s_3'', s_4''', \ldots, s_q^{(q-1)}$ approach $s_1$. That is, we imagine that the curve $C$ is modified and distorted while remaining smooth as the points of intersection with the surface $\psi$ are collapsed toward $s_1$. In the limit we find that

$$g[s_1] = \frac{dg[s_1]}{ds} = \frac{d^2 g[s_1]}{ds^2} = \frac{d^3 g[s_1]}{ds^3} = \cdots = \frac{d^{q-1} g[s_1]}{ds^{q-1}} = 0. \tag{A1.26}$$

To picture this, imagine that the surface is a piece of cloth and the curve is a length of thread stitched $q$ times through the cloth with a needle. Let the thread be pulled tight, and then let the length of the stitches be reduced to zero.

The curve $C$ is then said to have $q$-order contact with the surface $\psi$. This leads to the following definition.

***Definition A1.1.*** *A curve $x^j = F^j[s]$, $j = 1, \ldots, n$, is said to have $q$-order contact with a surface $\Psi[\boldsymbol{x}] = 0$ at the point $\boldsymbol{x}$ corresponding to $s_1$ if and only if the function $g[s] = \Psi[\boldsymbol{x}[s]]$ satisfies (A1.26) but has the $q$th derivative at $s_1$ not equal to zero, i.e., $d^q g[s_1]/ds^q \neq 0$.*

# Appendix 2

## *Invariance of the Contact Conditions under Lie Point Transformation Groups*

### A2.1 Preservation of Contact Conditions – One Dependent and One Independent Variable

#### *A2.1.1 Invariance of the First-Order Contact Condition*

The once extended group ($\xi$, $\eta$, $\eta_{\{1\}}$) leaves invariant the contact condition

$$d\tilde{y} - \tilde{y}_{\tilde{x}}\, d\tilde{x} = 0. \tag{A2.1}$$

To show this, we prolong the infinitesimal form of the once extended group to include the transformation of differentials,

$$\begin{aligned}
\tilde{x} &= x + s\xi[x, y],\\
\tilde{y} &= y + s\eta[x, y],\\
\tilde{y}_{\tilde{x}} &= y_x + s\eta_{\{1\}}[x, y, y_x],\\
d\tilde{x} &= dx + s(\xi_x\, dx + \xi_y\, dy) = dx + s(d\xi),\\
d\tilde{y} &= dy + s(\eta_x\, dx + \eta_y\, dy) = dy + s(d\eta).
\end{aligned} \tag{A2.2}$$

The differentials $dx$ and $dy$ are independent variables. Therefore the differentials $d\xi$ and $d\eta$ are functions of four independent variables $x$, $y$, $dx$, and $dy$ [ignoring, for the moment, the contact condition (A2.1)]. In this sense the underlying space has been prolonged from two to four variables. The group operator corresponding to (A2.2) is

$$\hat{X}_{\{1\}} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \eta_{\{1\}}\frac{\partial}{\partial y_x} + d\xi\,\frac{\partial}{\partial(dx)} + d\eta\,\frac{\partial}{\partial(dy)}, \tag{A2.3}$$

where $\hat{X}$ (with the ˆ) is used to distinguish a group operator that includes differentials as variables.

The contact condition (A2.1) is invariant under the prolonged group (A2.2). To demonstrate this we apply the operator (A2.3) to (A2.1). The result is

$$\hat{X}_{\{1\}}(dy - y_x\, dx) = d\eta - \eta_{\{1\}}\, dx - y_x\, d\xi. \tag{A2.4}$$

Writing out the differentials in (A2.4) in full, we have

$$\begin{aligned}
&\hat{X}_{\{1\}}(dy - y_x\, dx)\\
&\quad = \eta_x\, dx + \eta_y\, dy - (D_x\eta - y_x D_x\xi)\, dx - y_x(\xi_x\, dx + \xi_y\, dy)\\
&\quad = \eta_x\, dx + \eta_y\, dy - (\eta_x + \eta_y y_x - (\xi_x + \xi_y y_x)y_x)\, dx\\
&\qquad - y_x(\xi_x\, dx + \xi_y\, dy)\\
&\quad = \eta_y\, dy - (\eta_y y_x - \xi_y y_x y_x)\, dx - y_x\xi_y\, dy\\
&\quad = (\eta_y - \xi_y y_x)(dy - y_x\, dx) = 0.
\end{aligned} \tag{A2.5}$$

Thus the contact condition (A2.1) is invariant under the extended group (A2.2), and we can write

$$d\tilde{y} - \tilde{y}_{\tilde{x}}\, d\tilde{x} = dy - y_x\, dx. \tag{A2.6}$$

### *A2.1.2 Invariance of the Second-Order Contact Condition*

The transformation $(\xi, \eta)$ leaves invariant the second-order contact condition

$$dy_x - y_{xx}\, dx = 0. \tag{A2.7}$$

To show this, we now consider the infinitesimal form of the twice extended group including the transformation of differentials:

$$\begin{aligned}
\tilde{x} &= x + s\xi[x, y],\\
\tilde{y} &= y + s\eta[x, y],\\
\tilde{y}_{\tilde{x}} &= y_x + s\eta_{\{1\}}[x, y, y_x],\\
\tilde{y}_{\tilde{x}\tilde{x}} &= y_{xx} + s\eta_{\{2\}}[x, y, y_x, y_{xx}],\\
d\tilde{x} &= dx + s(\xi_x\, dx + \xi_y\, dy) = dx + s(d\xi),\\
d\tilde{y} &= dy + s(\eta_x\, dx + \eta_y\, dy) = dy + s(d\eta),\\
d\tilde{y}_{\tilde{x}} &= dy_x + s\big(\eta_{\{1\}x}\, dx + \eta_{\{1\}y}\, dy + \eta_{\{1\}y_x}\, dy_x\big) = dy + s\big(d\eta_{\{1\}}\big),
\end{aligned} \tag{A2.8}$$

where $dx$, $dy$, and $dy_x$ are independent variables. The group operator corresponding to (A2.8) is

$$\hat{X}_{\{2\}} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \eta_{\{1\}}\frac{\partial}{\partial y_x} + \eta_{\{2\}}\frac{\partial}{\partial y_{xx}} + d\xi\,\frac{\partial}{\partial(dx)} + d\eta\,\frac{\partial}{\partial(dy)} + d\eta_{\{1\}}\,\frac{\partial}{\partial(dy_x)}. \tag{A2.9}$$

If we apply this operator to the contact condition (A2.7), the result is

$$\hat{X}_{\{2\}}(dy_x - y_{xx}\,dx) = d\eta_{\{1\}} - \eta_{\{2\}}\,dx - y_{xx}\,d\xi. \tag{A2.10}$$

Writing out the differentials in (A2.10) in full, we have

$$\begin{aligned}
&\hat{X}_{\{2\}}(dy_x - y_{xx}\,dx)\\
&\quad= \eta_{\{1\}x}\,dx + \eta_{\{1\}y}\,dy + \eta_{\{1\}y_x}\,dy_x\\
&\qquad - \left(D_x\eta_{\{1\}} - y_{xx}D_x\xi\right)dx - y_{xx}(\xi_x\,dx + \xi_y\,dy)\\
&\quad= \eta_{\{1\}x}\,dx + \eta_{\{1\}y}\,dy + \eta_{\{1\}y_x}\,dy_x\\
&\qquad - \left(\eta_{\{1\}x} + \eta_{\{1\}y}y_x + \eta_{\{1\}y_x}y_{xx} - (\xi_x + \xi_y y_x)y_{xx}\right)dx\\
&\qquad - y_{xx}(\xi_x\,dx + \xi_y\,dy)\\
&\quad= \eta_{\{1\}y}\,dy + \eta_{\{1\}y_x}\,dy_x - \left(\eta_{\{1\}y}y_x + \eta_{\{1\}y_x}y_{xx} - (\xi_y y_x)y_{xx}\right)dx\\
&\qquad - y_{xx}\xi_y\,dy\\
&\quad= \left(\eta_{\{1\}y} - \xi_y y_{xx}\right)(dy - y_x\,dx) + \eta_{\{1\}y_x}(dy_x - y_{xx}\,dx) = 0,
\end{aligned} \tag{A2.11}$$

and so the second-order contact condition is preserved:

$$d\tilde{y}_{\tilde{x}} - \tilde{y}_{\tilde{x}\tilde{x}}\,d\tilde{x} = dy_x - y_{xx}\,dx. \tag{A2.12}$$

### *A2.1.3 Invariance of Higher-Order Contact Conditions*

Now we consider the infinitesimal form of the $p$th extended group, including its prolongation to include transformations of differentials up to order $p-1$:

$$\begin{aligned}
\tilde{x} &= x + s\xi[x, y],\\
\tilde{y} &= y + s\eta[x, y],\\
\tilde{y}_{\tilde{x}} &= y_x + s\eta_{\{1\}}[x, y, y_x],\\
\tilde{y}_{\tilde{x}\tilde{x}} &= y_{xx} + s\eta_{\{2\}}[x, y, y_x, y_{xx}],\\
&\vdots
\end{aligned}$$

$$
\begin{aligned}
\tilde{y}_{p\tilde{x}} &= y_{px} + s\eta_{\{p\}}[x, y, y_x, y_{xx}, \ldots, y_{px}],\\
d\tilde{x} &= dx + s(\xi_x\, dx + \xi_y\, dy)s = dx + s(d\xi),\\
d\tilde{y} &= dy + s(\eta_x\, dx + \eta_y\, dy)s = dy + s(d\eta),\\
d\tilde{y}_{\tilde{x}} &= dy_x + s\big(\eta_{\{1\}x}\, dx + \eta_{\{1\}y}\, dy + \eta_{\{1\}y_x}\, dy_x\big)\\
&= dy + s\big(d\eta_{\{1\}}\big)\\
&\vdots\\
d\tilde{y}_{(p-1)\tilde{x}} &= dy_{(p-1)x} + s\big(\eta_{\{p-1\}_x}\, dx + \cdots + \eta_{\{p-1\}y_{(p-1)x}}\, dy_{(p-1)x}\big)\\
&= dy_{(p-1)x} + s\big(d\eta_{\{p-1\}}\big),
\end{aligned}
\tag{A2.13}
$$

where $dx$, $dy$, and $dy_x, \ldots, dy_{(p-1)x}$ are regarded as independent variables. The group operator corresponding to (A2.8) is

$$
\begin{aligned}
\hat{X}_{\{p\}} &= \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \cdots + \eta_{\{p\}}\frac{\partial}{\partial y_{px}}\\
&\quad + (d\xi)\frac{\partial}{\partial(dx)} + \cdots + \big(d\eta_{\{p-1\}}\big)\frac{\partial}{\partial\big(dy_{(p-1)x}\big)}.
\end{aligned}
\tag{A2.14}
$$

If we apply this operator to the $p$th-order contact condition, the result is

$$
\hat{X}_{\{p\}}\big(d\big(y_{(p-1)x}\big) - y_{px}\, dx\big) = d\eta_{\{p-1\}} - \eta_{\{p\}}\, dx - y_{px}\, d\xi. \tag{A2.15}
$$

Writing out the differentials in (A2.15) in full, we have

$$
\begin{aligned}
&\hat{X}_{\{p\}}\big(d\big(y_{(p-1)x}\big) - y_{px}\, dx\big)\\
&\quad = \eta_{\{p-1\}x} dx + \eta_{\{p-1\}y}\, dy + \cdots + \eta_{\{p-1\}y_{(p-1)x}}\, dy_{(p-1)x}\\
&\qquad - \big(D_x\big(\eta_{\{p-1\}}\big) - y_{px} D_x\xi\big)\, dx - y_{px}(\xi_x\, dx + \xi_y\, dy)\\
&\quad = \eta_{\{p-1\}x}\, dx + \eta_{\{p-1\}y}\, dy + \cdots + \eta_{\{p-1\}y_{(p-1)x}}\, dy_{(p-1)x}\\
&\qquad - \big(\eta_{\{p-1\}x} + \eta_{\{p-1\}y} y_x + \cdots\\
&\qquad + \eta_{\{p-1\}y_{(p-1)x}} y_{px} - (\xi_x + \xi_y y_x) y_{px}\big)\, dx\\
&\qquad - y_{px}(\xi_x\, dx + \xi_y\, dy)\\
&\quad = \eta_{\{p-1\}y}\, dy + \eta_{\{p-1\}y_x}\, dy_x + \cdots + \eta_{\{p-1\}y_{(p-1)x}}\, dy_{(p-1)x}\\
&\qquad - \big(\eta_{\{p-1\}y} y_x + \eta_{\{p-1\}y_x} y_{xx} + \cdots + \eta_{\{p-1\}y_{(p-1)x}} y_{px} - \xi_y y_x y_{px}\big)\, dx\\
&\qquad - y_{px}\xi_y\, dy\\
&\quad = \big(\eta_{\{p-1\}y} - \xi_y y_{px}\big)(dy - y_x\, dx) + \eta_{\{p-1\}y_x}(dy_x - y_{xx}\, dx)\\
&\qquad + \cdots + \eta_{\{p-1\}y_{(p-1)x}}\big(dy_{(p-1)x} - y_{px}\, dx\big) = 0.
\end{aligned}
\tag{A2.16}
$$

By induction the contact condition (A2.17) is preserved to all orders, and we can state the following theorem.

***Theorem A2.1.*** *The pth-order contact condition*

$$d\big(y_{(p-1)x}\big) - y_{px}\,dx = 0 \tag{A2.17}$$

*is preserved by the group* $(\xi, \eta)$. *The transformation* $(\xi, \eta)$ *preserves the contact conditions*

$$\begin{aligned} d\tilde{y} - \tilde{y}_{\tilde{x}}\,d\tilde{x} &= dy - y_x\,dx, \\ d\tilde{y}_{\tilde{x}} - \tilde{y}_{\tilde{x}\tilde{x}}\,d\tilde{x} &= dy_x - y_{xx}\,dx, \\ &\vdots \\ d\big(\tilde{y}_{(p-1)\tilde{x}}\big) - \tilde{y}_{p\tilde{x}}\,d\tilde{x} &= d\big(y_{(p-1)x}\big) - y_{px}\,dx, \end{aligned} \tag{A2.18}$$

*and so forth to all orders.*

## A2.2 Preservation of the Contact Conditions – Several Dependent and Independent Variables

### *A2.2.1 Invariance of the First-Order Contact Condition*

The first extension leaves invariant the first-order contact condition

$$dy^i - y^i_j\,dx^j = 0. \tag{A2.19}$$

To prove this we use the once extended infinitesimal transformation including its prolongation to include the transformation of differentials:

$$\begin{aligned} \tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x}, \boldsymbol{y}], \qquad j = 1, \ldots, n, \\ \tilde{y}^i &= y^i + s\eta^i[\boldsymbol{x}, \boldsymbol{y}]; \qquad i = 1, \ldots, m, \\ \tilde{y}^i_j &= y^i_j + s\eta^i_{\{j\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1], \\ d\tilde{x}^j &= dx^j + s\left(\frac{\partial \xi^j}{\partial x^\alpha}\,dx^\alpha + \frac{\partial \xi^j}{\partial y^\beta}\,dy^\beta\right) = dx^j + s(d\xi^j), \\ d\tilde{y}^i &= dy^i + s\left(\frac{\partial \eta^i}{\partial x^\alpha}\,dx^\alpha + \frac{\partial \eta^i}{\partial y^\beta}\,dy^\beta\right) = dy^i + s(d\eta^i). \end{aligned} \tag{A2.20}$$

The differentials $dx^\alpha$ and $dy^\beta$ are independent variables.

The prolonged group operator corresponding to (A2.20) is

$$\hat{X}_{\{1\}} = \xi^j \frac{\partial}{\partial x^j} + \eta^i \frac{\partial}{\partial y^i} + \eta^i_{\{j\}} \frac{\partial}{\partial \left(y^i_j\right)} + d\xi^j \frac{\partial}{\partial (dx^j)} + d\eta^i \frac{\partial}{\partial (dy^i)}. \quad \text{(A2.21)}$$

If we apply this operator to the contact condition, the result is

$$\hat{X}_{\{1\}}\left(dy^\alpha - y^\alpha_\beta\, dx^\beta\right) = (d\eta^i)\delta^\alpha_i - y^\alpha_\beta (d\xi^j)\delta^\beta_j - \eta^i_{\{j\}}\delta^\alpha_i \delta^j_\beta (dx^\beta), \quad \text{(A2.22)}$$

or

$$\hat{X}_{\{1\}}\left(dy^\alpha - y^\alpha_\beta\, dx^\beta\right) = d\eta^\alpha - y^\alpha_\beta\, d\xi^\beta - \eta^\alpha_{\{\beta\}}\, dx^\beta. \quad \text{(A2.23)}$$

Replacing indices, we have

$$\hat{X}_{\{1\}}(dy^i - y^i_j\, dx^j) = d\eta^i - y^i_j\, d\xi^j - \eta^i_{\{j\}}\, dx^j, \quad \text{(A2.24)}$$

or

$$\hat{X}_{\{1\}}\left(dy^i - y^i_j\, dx^j\right) = \left(\frac{\partial \eta^i}{\partial x^\alpha} dx^\alpha + \frac{\partial \eta^i}{\partial y^\beta} dy^\beta\right) - y^i_j \left(\frac{\partial \xi^j}{\partial x^\alpha} dx^\alpha + \frac{\partial \xi^j}{\partial y^\beta} dy^\beta\right) - \left(D_j \eta^i - y^i_\beta D_j \xi^\beta\right) dx^j. \quad \text{(A2.25)}$$

The total differentiation indicated in (A2.25), can be written out as

$$D_j \eta^i - y^i_\beta D_j \xi^\beta = \frac{\partial \eta^i}{\partial x^j} + y^\alpha_j \frac{\partial \eta^i}{\partial y^\alpha} - y^i_\beta \left(\frac{\partial \xi^\beta}{\partial x^j} + y^\alpha_j \frac{\partial \xi^\beta}{\partial y^\alpha}\right). \quad \text{(A2.26)}$$

Now (A2.25) becomes

$$\hat{X}_{\{1\}}\left(dy^i - y^i_j\, dx^j\right) = \left(\frac{\partial \eta^i}{\partial x^\alpha} dx^\alpha + \frac{\partial \eta^i}{\partial y^\beta} dy^\beta\right) - y^i_j \left(\frac{\partial \xi^j}{\partial x^\alpha} dx^\alpha + \frac{\partial \xi^j}{\partial y^\beta} dy^\beta\right) - \left(\frac{\partial \eta^i}{\partial x^j} + y^\alpha_j \frac{\partial \eta^i}{\partial y^\alpha} - y^i_\beta \left(\frac{\partial \xi^\beta}{\partial x^j} + y^\alpha_j \frac{\partial \xi^\beta}{\partial y^\alpha}\right)\right) dx^j. \quad \text{(A2.27)}$$

Equation (A2.27) can be rearranged to read, after canceling terms,

$$\hat{X}_{\{1\}}\left(dy^i - y^i_j\, dx^j\right) = \left(\frac{\partial \eta^i}{\partial y^\beta} - y^i_j \frac{\partial \xi^j}{\partial y^\beta}\right) dy^\beta - \left(\frac{\partial \eta^i}{\partial y^\alpha} - y^i_\beta \frac{\partial \xi^\beta}{\partial y^\alpha}\right) y^\alpha_j\, dx^j. \quad \text{(A2.28)}$$

Upon exchanging indices again,

$$\hat{X}_{\{1\}}\big(dy^i - y^i_j\,dx^j\big) = \left(\frac{\partial\eta^i}{\partial y^\alpha} - y^i_\beta\,\frac{\partial\xi^\beta}{\partial y^\alpha}\right)\big(dy^\alpha - y^\alpha_j\,dx^j\big) = 0, \quad \text{(A2.29)}$$

which proves that the contact condition (A2.1) is invariant under the group (A2.20). In other words, if (A2.19) were to be expanded in a Lie series in the operator $\hat{X}_{\{1\}}$, the series would truncate to

$$d\tilde{y}^i - \tilde{y}^i_j\,d\tilde{x}^j = dy^i - y^i_j\,dx^j. \quad \text{(A2.30)}$$

### *A2.2.2 Invariance of the Second-Order Contact Conditions*

The twice extended group leaves invariant the second-order contact condition

$$d\tilde{y}^i_{j_1} - \tilde{y}^i_{j_1 j_2}\,d\tilde{x}^{j_2} = 0. \quad \text{(A2.31)}$$

To show this, we consider the prolongation of the twice extended group to include differentials:

$$\begin{aligned}
\tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x},\boldsymbol{y}], \qquad j = 1,\ldots,n,\\
\tilde{y}^i &= y^i + s\eta^i[\boldsymbol{x},\boldsymbol{y}], \qquad i = 1,\ldots,m,\\
\tilde{y}^i_{j_1} &= y^i_{j_1} + s\eta^i_{\{j_1\}}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{y}_1],\\
\tilde{y}^i_{j_1 j_2} &= y^i_{j_1 j_2} + s\eta^i_{\{j_1 j_2\}}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{y}_1,\boldsymbol{y}_2], \qquad \text{(A2.32)}\\
d\tilde{x}^j &= dx^j + s\left(\frac{\partial\xi^j}{\partial x^\alpha}\,dx^\alpha + \frac{\partial\xi^j}{\partial y^\beta}\,dy^\beta\right) = dx^j + s(d\xi^j),\\
d\tilde{y}^i &= dy^i + s\left(\frac{\partial\eta^i}{\partial x^\alpha}\,dx^\alpha + \frac{\partial\eta^i}{\partial y^\beta}\,dy^\beta\right) = dy^i + s(d\eta^i),\\
d\tilde{y}^i_{j_1} &= dy^i_{j_1} + s\left(\frac{\partial\eta^i_{\{j_1\}}}{\partial x^\alpha}\,dx^\alpha + \frac{\partial\eta^i_{\{j_1\}}}{\partial y^\beta}\,dy^\beta + \frac{\partial\eta^i_{\{j_1\}}}{\partial y^\beta_\alpha}dy^\beta_\alpha\right)\\
&= dy^i + s\big(d\eta^i_{\{j_1\}}\big).
\end{aligned}$$

Again, the differentials $dx^j$, $dy^i$, and $dy^i_j$ are independent variables. The group operator corresponding to (A2.32) is

$$\begin{aligned}
\hat{X}_{\{2\}} = \xi^j\frac{\partial}{\partial x^j} + \eta^i\frac{\partial}{\partial y^i} + \eta^i_{\{j_1\}}\frac{\partial}{\partial\big(y^i_{j_1}\big)} + \eta^i_{\{j_1 j_2\}}\frac{\partial}{\partial\big(y^i_{j_1 j_2}\big)}\\
+ (d\xi^{j_1})\frac{\partial}{\partial(dx^{j_1})} + (d\eta^i)\frac{\partial}{\partial(dy^i)} + \big(d\eta^i_{\{j_1\}}\big)\frac{\partial}{\partial\big(dy^i_{j_1}\big)}. \quad \text{(A2.33)}
\end{aligned}$$

Operating with (A2.33) on the contact condition (A2.31) gives

$$\hat{X}_{\{2\}}\big(dy^{\alpha}_{\beta_1} - y^{\alpha}_{\beta_1\beta_2}\,dx^{\beta_2}\big)$$
$$= d\eta^{i}_{\{j_1\}}\delta^{\alpha}_{i}\delta^{j_1}_{\beta_1} - \big(\eta^{i}_{\{j_1 j_2\}}\delta^{\alpha}_{i}\delta^{j_1}_{\beta_1}\delta^{j_2}_{\beta_2}\big)\,dx^{\beta_2} - y^{\alpha}_{\beta_1\beta_2}\,d\xi^{j_1}\,\delta^{\beta_2}_{j_1}. \quad \text{(A2.34)}$$

After taking the indicated sums, (A2.34) becomes

$$\hat{X}_{\{2\}}\big(dy^{\alpha}_{\beta_1} - y^{\alpha}_{\beta_1\beta_2}\,dx^{\beta_2}\big) = d\eta^{\alpha}_{\{\beta_1\}} - \eta^{\alpha}_{\{\beta_1\beta_2\}}\,dx^{\beta_2} - y^{\alpha}_{\beta_1\beta_2}\,d\xi^{\beta_2}. \quad \text{(A2.35)}$$

Now we supply the various expressions on the right-hand side of (A2.35):

$$d\eta^{\alpha}_{\{\beta_1\}} - \eta^{\alpha}_{\{\beta_1\beta_2\}}\,d\tilde{x}^{\beta_2} - \tilde{y}^{\alpha}_{\{\beta_1\beta_2\}}\,d\xi^{\beta_2}$$
$$= \frac{\partial\eta^{\alpha}_{\{\beta_1\}}}{\partial x^{\gamma}}\,dx^{\gamma} + \frac{\partial\eta^{\alpha}_{\{\beta_1\}}}{\partial y^{\varepsilon}}\,dy^{\varepsilon} + \frac{\partial\eta^{\alpha}_{\{\beta_1\}}}{\partial y^{\varepsilon}_{\gamma}}\,dy^{\varepsilon}_{\gamma}$$
$$- \tilde{y}^{\alpha}_{\beta_1\beta_2}\left(\frac{\partial\xi^{\beta_2}}{\partial x^{\gamma}}\,dx^{\gamma} + \frac{\partial\xi^{\beta_2}}{\partial y^{\varepsilon}}\,dy^{\varepsilon}\right)$$
$$- \left(D_{\beta_2}\eta^{\alpha}_{\{\beta_1\}} - \frac{\partial^2 y^{\alpha}}{\partial x^{\beta_1}\partial x^{\gamma}}\,D_{\beta_2}\xi^{\gamma}\right)d\tilde{x}^{\beta_2}. \quad \text{(A2.36)}$$

Inserting the total differentiation operator, Equation (A2.36) can be rearranged to read

$$d\eta^{\alpha}_{\{\beta_1\}} - \eta^{\alpha}_{\{\beta_1\beta_2\}}\,d\tilde{x}^{\beta_2} - \tilde{y}^{\alpha}_{\beta_1\beta_2}\,d\xi^{\beta_2}$$
$$= \frac{\partial\eta^{\alpha}_{\{\beta_1\}}}{\partial x^{\gamma}}\,dx^{\gamma} + \frac{\partial\eta^{\alpha}_{\{\beta_1\}}}{\partial y^{\varepsilon}}\,dy^{\varepsilon} + \frac{\partial\eta^{\alpha}_{\{\beta_1\}}}{\partial y^{\varepsilon}_{\gamma}}\,dy^{\varepsilon}_{\gamma}$$
$$- \tilde{y}^{\alpha}_{\beta_1\beta_2}\left(\frac{\partial\xi^{\beta_2}}{\partial x^{\gamma}}\,dx^{\gamma} + \frac{\partial\xi^{\beta_2}}{\partial y^{\varepsilon}}\,dy^{\varepsilon}\right)$$
$$- \left(\frac{\partial\eta^{\alpha}_{\{\beta_1\}}}{\partial x^{\beta_2}} + \frac{\partial y^{\theta}}{\partial x^{\beta_2}}\frac{\partial\eta^{\alpha}_{\{\beta_1\}}}{\partial y^{\theta}} + \frac{\partial y^{\theta}_{\varepsilon}}{\partial x^{\beta_2}}\frac{\partial\eta^{\alpha}_{\{\beta_1\}}}{\partial y^{\theta}_{\{\varepsilon\}}}\right)d\tilde{x}^{\beta_2}$$
$$+ y^{\alpha}_{\beta_1\gamma}\left(\frac{\partial\xi^{\gamma}}{\partial x^{\beta_2}} + \frac{\partial y^{\theta}}{\partial x^{\beta_2}}\frac{\partial\xi^{\gamma}}{\partial y^{\theta}}\right)d\tilde{x}^{\beta_2}. \quad \text{(A2.37)}$$

Finally,

$$\hat{X}_{\{2\}}\big(dy^{\alpha}_{\beta_1} - y^{\alpha}_{\beta_1\beta_2}\,dx^{\beta_2}\big)$$
$$= d\eta^{\alpha}_{\{\beta_1\}} - \eta^{\alpha}_{\{\beta_1\beta_2\}}\,dx^{\beta_2} - y^{\alpha}_{\beta_1\beta_2}\,d\xi^{\beta_2}$$
$$= \left(\frac{\partial\eta^{\alpha}_{\{\beta_1\}}}{\partial y^{\varepsilon}} - \tilde{y}^{\alpha}_{\beta_1\beta_2}\frac{\partial\xi^{\beta_2}}{\partial y^{\varepsilon}}\right)\big(dy^{\varepsilon} - y^{\varepsilon}_{\gamma}\,dx^{\gamma}\big)$$
$$+ \frac{\partial\eta^{\alpha}_{\{\beta_1\}}}{\partial y^{\varepsilon}_{\gamma}}\big(dy^{\varepsilon}_{\gamma} - y^{\varepsilon}_{\gamma\beta_2}\,dx^{\beta_2}\big) = 0. \quad \text{(A2.38)}$$

The result (A2.38) demonstrates the invariance of the second-order contact condition. As in the case of two variables, operating on the contact condition with $\hat{X}$ produces two terms on the left-hand side, each of which is zero.

### *A2.2.3 Invariance of Higher-Order Contact Conditions*

The $p$th-order extended group leaves invariant the $p$th-order contact condition

$$d\tilde{y}^i_{j_1 j_2 \cdots j_{p-1}} - \tilde{y}^i_{j_1 j_2 \ldots j_p}\, d\tilde{x}^{j_p} = 0. \tag{A2.39}$$

To prove this we now consider the prolongation of the $p$th extended group,

$$\begin{aligned}
\tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x}, \boldsymbol{y}], \qquad j = 1, \ldots, n,\\
\tilde{y}^i &= y^i + s\eta^i[\boldsymbol{x}, \boldsymbol{y}], \qquad i = 1, \ldots, m,\\
\tilde{y}^i_{j_1} &= y^i_{j_1} + s\eta^i_{\{j_1\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1],\\
\tilde{y}^i_{j_1 j_2} &= y^i_{j_1 j_2} + s\eta^i_{\{j_1 j_2\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2],\\
&\vdots\\
\tilde{y}^i_{j_1 j_2 \ldots j_p} &= y^i_{j_1 j_2 \ldots j_p} + s\eta^i_{\{j_1 j_2 \ldots j_p\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_p],\\
d\tilde{x}^j &= dx^j + s\left(\frac{\partial \xi^j}{\partial x^\alpha}\, dx^\alpha + \frac{\partial \xi^j}{\partial y^\beta} dy^\beta\right) = dx^j + s(d\xi^j),\\
d\tilde{y}^i &= dy^i + s\left(\frac{\partial \eta^i}{\partial x^\alpha}\, dx^\alpha + \frac{\partial \eta^i}{\partial y^\beta} dy^\beta\right) = dy^i + s(d\eta^i),\\
d\tilde{y}^i_{j_1} &= dy^i_{j_1} + s\left(\frac{\partial \eta^i_{\{j_1\}}}{\partial x^\alpha}\, dx^\alpha + \frac{\partial \eta^i_{\{j_1\}}}{\partial y^\beta}\, dy^\beta + \frac{\partial \eta^i_{\{j_1\}}}{\partial y^\beta_\alpha}\, dy^\beta_\alpha\right)\\
&= dy^i + s\big(d\eta^i_{\{j_1\}}\big),\\
&\vdots\\
d\tilde{y}^i_{j_1 j_2 \cdots j_{p-1}} &= dy^i_{j_1 j_2 \cdots j_{p-1}}\\
&\quad + s\left(\frac{\partial \eta^i_{\{j_1 j_2 \cdots j_{p-1}\}}}{\partial x^\alpha}\, dx^\alpha + \frac{\partial \eta^i_{\{j_1 j_2 \cdots j_{p-1}\}}}{\partial y^\beta}\, dy^\beta\right.\\
&\quad \left. + \frac{\partial \eta^i_{\{j_1 j_2 \cdots j_{p-1}\}}}{\partial y^\beta_\alpha}\, dy^\beta_\alpha + \cdots + \frac{\partial \eta^i_{\{j_1 j_2 \cdots j_{p-1}\}}}{\partial y^\beta_{\alpha_1 \alpha_2 \ldots \alpha_{p-1}}}\, dy^\beta_{\alpha_1 \alpha_2 \ldots \alpha_{p-1}}\right)\\
&= dy^i_{j_1 j_2 \cdots j_{p-1}} + s\big(d\eta^i_{\{j_1 j_2 \cdots j_{p-1}\}}\big).
\end{aligned} \tag{A2.40}$$

The differentials $dx^j$, $dy^i$, and $dy^i_{j_1}, \ldots, dy^i_{j_1 j_2 \ldots j_{p-1}}$ are independent variables. Notice that differentials only up to order $p-1$ are required. The group operator corresponding to (A2.40) is

$$\begin{aligned}\hat{X}_{\{p\}} &= \xi^j \frac{\partial}{\partial x^j} + \eta^i \frac{\partial}{\partial y^i} + \eta^i_{\{j_1\}} \frac{\partial}{\partial y^i_{j_1}} + \cdots \\ &\quad + \eta^i_{\{j_1 j_2 \ldots j_p\}} \frac{\partial}{\partial y^i_{j_1 j_2 \ldots j_p}} \\ &\quad + d\xi^{j_1} \frac{\partial}{\partial (dx^{j_1})} + d\eta^i \frac{\partial}{\partial (dy^i)} + d\eta^i_{\{j_1\}} \frac{\partial}{\partial \left(dy^i_{j_1}\right)} + \cdots \\ &\quad + d\eta^i_{\{j_1 \ldots j_{p-1}\}} \frac{\partial}{\partial \left(dy^i_{j_1 \ldots j_{p-1}}\right)}. \qquad \text{(A2.41)}\end{aligned}$$

Operating on the contact condition (A2.39) gives

$$\begin{aligned}&\hat{X}_{\{p\}}\left(dy^\alpha_{\beta_1\beta_2\ldots\beta_{p-1}} - y^\alpha_{\beta_1\beta_2\ldots\beta_p} dx^{\beta_p}\right) \\ &\quad = \left(d\eta^i_{\{j_1 j_2 \ldots j_{p-1}\}}\right)\delta^\alpha_i \delta^{j_1}_{\beta_1} \ldots \delta^{j_1}_{\beta_{p-1}} - \left(\eta^i_{\{j_1 j_2 \ldots j_p\}}\right)\delta^\alpha_i \delta^{j_1}_{\beta_1}\delta^{j_2}_{\beta_2} \ldots \delta^{j_2}_{\beta_p} dx^{\beta_p} \\ &\qquad - y^\alpha_{\beta_1\beta_2\ldots\beta_p} d\xi^{j_1} \delta^{\beta_p}_{j_1}. \qquad \text{(A2.42)}\end{aligned}$$

After taking the indicated sums, (A2.42) becomes

$$\begin{aligned}&\hat{X}_{\{p\}}\left(dy^\alpha_{\beta_1\beta_2\ldots\beta_{p-1}} - y^\alpha_{\beta_1\beta_2\ldots\beta_p} dx^{\beta_p}\right) \\ &\qquad = d\eta^\alpha_{\{\beta_1\beta_2\ldots\beta_{p-1}\}} - \eta^\alpha_{\{\beta_1\beta_2\ldots\beta_p\}} dx^{\beta_p} - \tilde{y}^\alpha_{\beta_1\beta_2\ldots\beta_p} d\xi^{\beta_p}. \qquad \text{(A2.43)}\end{aligned}$$

When we supply the various expressions on the right-hand side of (A2.43), the result is

$$\begin{aligned}&d\eta^\alpha_{\{\beta_1\beta_2\ldots\beta_{p-1}\}} - \eta^\alpha_{\{\beta_1\beta_2\ldots\beta_p\}} dx^{\beta_p} - y^\alpha_{\beta_1\beta_2\ldots\beta_p} d\xi^{\beta_p} \\ &\quad = \frac{\partial \eta^i_{\{j_1 j_2 \ldots j_{p-1}\}}}{\partial x^\alpha} dx^\alpha + \frac{\partial \eta^i_{\{j_1 j_2 \ldots j_{p-1}\}}}{\partial y^\beta} dy^\beta \\ &\qquad + \frac{\partial \eta^i_{\{j_1 j_2 \ldots j_{p-1}\}}}{\partial y^\beta_\alpha} dy^\beta_\alpha + \cdots + \frac{\partial \eta^i_{\{j_1 j_2 \ldots j_{p-1}\}}}{\partial y^\beta_{\alpha_1\alpha_2\ldots\alpha_{p-1}}} dy^\beta_{\alpha_1\alpha_2\ldots\alpha_{p-1}} \\ &\qquad - y^\alpha_{\beta_1\beta_2\ldots\beta_p}\left(\frac{\partial \xi^{\beta_p}}{\partial x^\gamma} dx^\gamma + \frac{\partial \xi^{\beta_p}}{\partial y^\varepsilon} dy^\varepsilon\right) \\ &\qquad - \left(D_{\beta_p}\left(\eta^\alpha_{\{\beta_1\beta_2\ldots\beta_{p-1}\}}\right) - \frac{\partial^p y^\alpha}{\partial x^{\beta_1}\,\partial x^{\beta_2}\cdots\partial x^{\beta_{p-1}}\,\partial x^\alpha} D_{\beta_p}\xi^\alpha\right) dx^{\beta_p}. \\ &\qquad \text{(A2.44)}\end{aligned}$$

Working out the total derivatives, Equation (A2.44) can be rearranged as follows:

$$\begin{aligned}
& d\eta^{\alpha}_{\{\beta_1\beta_2\ldots\beta_{p-1}\}} - \eta^{\alpha}_{\{\beta_1\beta_2\ldots\beta_p\}}\, dx^{\beta_p} - y^{\alpha}_{\beta_1\beta_2\ldots\beta_p}\, d\xi^{\beta_p} \\
&\quad = \frac{\partial \eta^{\alpha}_{\{\beta_1\beta_2\ldots\beta_{p-1}\}}}{\partial x^{\alpha}} dx^{\alpha} + \frac{\partial \eta^{\alpha}_{\{\beta_1\beta_2\ldots\beta_{p-1}\}}}{\partial y^{\beta}}\, dy^{\beta} + \frac{\partial \eta^{\alpha}_{\{\beta_1\beta_2\ldots\beta_{p-1}\}}}{\partial y^{\beta}_{\alpha}}\, dy^{\beta}_{\alpha} + \cdots \\
&\qquad + \frac{\partial \eta^{\alpha}_{\{\beta_1\beta_2\ldots\beta_{p-1}\}}}{\partial y^{\beta}_{\alpha_1\alpha_2\ldots\alpha_{p-1}}}\, dy^{\beta}_{\alpha_1\alpha_2\ldots\alpha_{p-1}} - y^{\alpha}_{\beta_1\beta_2\ldots\beta_p}\left(\frac{\partial \xi^{\beta_p}}{\partial x^{\gamma}}\, dx^{\gamma} + \frac{\partial \xi^{\beta_p}}{\partial y^{\varepsilon}} dy^{\varepsilon}\right) \\
&\qquad - \left(\frac{\partial \eta^{\alpha}_{\{\beta_1\beta_2\ldots\beta_{p-1}\}}}{\partial x^{\beta_p}} + \frac{\partial y^{\theta}}{\partial x^{\beta_p}} \frac{\partial \eta^{\alpha}_{\{\beta_1\beta_2\ldots\beta_{p-1}\}}}{\partial y^{\theta}}\right. \\
&\qquad \left. + \frac{\partial y^{\theta}_{\varepsilon}}{\partial x^{\beta_p}} \frac{\partial \eta^{\alpha}_{\{\beta_1\beta_2\ldots\beta_{p-1}\}}}{\partial y^{\theta}_{\varepsilon}} + \cdots + \frac{\partial y^{\theta}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{p-1}}}{\partial x^{\beta_p}} \frac{\partial \eta^{\alpha}_{\{\beta_1\beta_2\ldots\beta_{p-1}\}}}{\partial y^{\theta}_{\varepsilon_1\varepsilon_2\ldots\,\varepsilon_{p-1}}}\right) dx^{\beta_p} \\
&\qquad + y^{\alpha}_{\beta_1\beta_2\ldots\beta_{p-1}\gamma}\left(\frac{\partial \xi^{\gamma}}{\partial x^{\beta_p}} + \frac{\partial y^{\theta}}{\partial x^{\beta_p}} \frac{\partial \xi^{\gamma}}{\partial y^{\theta}}\right) dx^{\beta_p}. \qquad \text{(A2.45)}
\end{aligned}$$

Finally, the prolonged group operator acting on the $p$th-order contact condition gives

$$\begin{aligned}
& \hat{X}_{\{p\}}\left(dy^{\alpha}_{\beta_1\beta_2\ldots\beta_{p-1}} - y^{\alpha}_{\beta_1\beta_2\ldots\beta_p} dx^{\beta_p}\right) \\
&\quad = d\eta^{\alpha}_{\{\beta_1\beta_2\ldots\beta_{p-1}\}} - \eta^{\alpha}_{\{\beta_1\beta_2\ldots\beta_p\}}\, dx^{\beta_p} - y^{\alpha}_{\beta_1\beta_2\ldots\beta_p}\, d\xi^{\beta_p} \\
&\quad = \left(\frac{\partial \eta^{\alpha}_{\{\beta_1\beta_2\ldots\beta_{p-1}\}}}{\partial y^{\varepsilon}} - y^{\alpha}_{\beta_1\beta_2\ldots\beta_p} \frac{\partial \xi^{\beta_p}}{\partial y^{\varepsilon}}\right)\left(dy^{\varepsilon} - y^{\varepsilon}_{\gamma}\, dx^{\gamma}\right) \\
&\qquad + \frac{\partial \eta^{\alpha}_{\{\beta_1\beta_2\ldots\beta_{p-1}\}}}{\partial y^{\varepsilon}_{\sigma}}\left(dy^{\varepsilon}_{\sigma} - y^{\varepsilon}_{\sigma\gamma}\, dx^{\gamma}\right) + \ldots \\
&\qquad + \frac{\partial \eta^{\alpha}_{\{\beta_1\beta_2\ldots\beta_{p-1}\}}}{\partial y^{\varepsilon}_{\sigma_1\sigma_2\ldots\sigma_{p-1}}}\left(dy^{\varepsilon}_{\sigma_1\sigma_2\ldots\sigma_{p-1}} - y^{\varepsilon}_{\sigma_1\sigma_2\ldots\sigma_{p-1}\gamma}\, dx^{\gamma}\right) = 0, \qquad \text{(A2.46)}
\end{aligned}$$

and the $p$th-order contact condition is invariant under the $p$th extended group. By induction the contact conditions (A2.39) are invariant to all orders under the group $(\xi^j, \eta^i)$.

# Appendix 3

## *Infinite-Order Structure of Lie–Bäcklund Transformations*

The proper functional setting of the theory of Lie–Bäcklund groups is the space $\mathcal{A}$ of differential functions of arbitrary derivative order [A3.1]. The reason can be found in the requirement that such transformations preserve the contact conditions discussed in the context of Lie point groups in Appendix 2. The purpose of this appendix is to explain this concept. The discussion outlines the proof presented in Anderson and Ibragimov [A3.1], and the reader should look there for further details.

### A3.1 Lie Point Groups

Classical Lie groups comprise point transformations together with their extensions to include transformations of derivatives up to order $p$. At each level of extension the differential function that carries out the transformation of derivatives depends on derivatives up to and including but not beyond the order of the derivative being transformed:

$$\begin{aligned}
(\boldsymbol{x}, \boldsymbol{y}) &\Rightarrow (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}),\\
(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1) &\Rightarrow (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}_1),\\
(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2) &\Rightarrow (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}_1, \tilde{\boldsymbol{y}}_2),\\
&\vdots\\
(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_p) &\Rightarrow (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}_1, \tilde{\boldsymbol{y}}_2, \ldots, \tilde{\boldsymbol{y}}_p).
\end{aligned} \tag{A3.1}$$

Such transformations are closed in the Euclidean space of differential variables $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_p)$ with $q$ dimensions, where

$$q = n + m \sum_{k=0}^{p} \frac{(n+k-1)!}{k!(n-1)!} \tag{A3.2}$$

### A3.2 Lie–Bäcklund Groups

Lie–Bäcklund groups are more general invertible mappings where the transformation of a point can depend on derivatives of the dependent variables up to arbitrary order:

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots) \Rightarrow (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}_1, \tilde{\boldsymbol{y}}_2, \ldots). \tag{A3.3}$$

In this case the extension to transformations of derivatives, up to say order $p$, inevitably produces expressions that contain derivatives of arbitrary order greater than $p$. Anderson and Ibragimov [A3.1] show that, except in the case of Lie contact transformations (to be discussed shortly), such transformations, when extended to include derivatives, cannot be closed in a space of finite dimension.

### A3.3 Lie Contact Transformations

To begin, consider the case where the transformation of coordinates can depend on the first derivative. Consider the one-parameter transformation

$$T^s : \left\{ \begin{aligned} \tilde{x}^j &= F^j[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, s],\ j = 1, \ldots, n \\ \tilde{y}^i &= G^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, s],\ i = 1, \ldots, m \\ \tilde{y}^i_j &= G^i_{\{j\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, s] \end{aligned} \right\} \tag{A3.4}$$

with infinitesimal form

$$T^s : \left\{ \begin{aligned} \tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, s],\quad j = 1, \ldots, n \\ \tilde{y}^i &= y^i + s\eta^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, s],\quad i = 1, \ldots, m \\ \tilde{y}^i_j &= y^i_j + s\eta^i_{\{j\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, s] \end{aligned} \right\}, \tag{A3.5}$$

where

$$\xi^j = \left.\frac{\partial F^j}{\partial s}\right|_{s=0}, \qquad \eta^i = \left.\frac{\partial G^i}{\partial s}\right|_{s=0}, \qquad \eta^i_{\{j\}} = \left.\frac{\partial G^i_{\{j\}}}{\partial s}\right|_{s=0}. \tag{A3.6}$$

Here we have used the convention adopted in Chapter 7 that subscripts without braces denote partial differentiation with respect to the $j$th independent variable, while subscripts encased in braces represent labels of the function that transforms the $j$th derivative. Such a transformation, if it could be found, would be of considerable interest because it could be used to transform an equation in the source space $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1)$ to an equation in the target space $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}_1)$ without raising the order of the equation.

By the same token, the form of (A3.4) is somewhat unexpected. Recalling the procedure used to generate transformations of derivatives in the case of point groups, one would, because of the dependence of $F^j$ and $G^i$ on $\boldsymbol{y}_1$, expect the function $G^i_{\{j\}}$ to depend on $\boldsymbol{y}_2$. But we have assumed that it doesn't. So it is legitimate to ask whether groups (with all that is implied by the definition of a group) of the form (A3.4) can exist. In other words, what, if any, restrictions must be placed on $F^j$, $G^i$, and $G^i_{\{j\}}$ to ensure that the transformation (A3.4) preserves the contact condition

$$d\tilde{y}^i - \tilde{y}^i_j \, d\tilde{x}^j = dy^i - y^i_j \, dx^j \tag{A3.7}$$

To examine this question, we make use of the prolongation of (A3.5) to include the differentials of $x$ and $y$:

$$\begin{aligned}
\tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, s], \qquad j = 1, \ldots, n,\\
\tilde{y}^i &= y^i + s\eta^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, s], \qquad i = 1, \ldots, m,\\
\tilde{y}^i_j &= y^i_j + s\eta^i_{\{j\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, s],\\
d\tilde{x}^j &= dx^j + s\left(\frac{\partial \xi^j}{\partial x^\alpha} dx^\alpha + \frac{\partial \xi^j}{\partial y^\beta} dy^\beta + \frac{\partial \xi^j}{\partial y^\beta_\gamma} dy^\beta_\gamma\right) = dx^j + s(d\xi^j),\\
d\tilde{y}^i &= dy^i + s\left(\frac{\partial \eta^i}{\partial x^\alpha} dx^\alpha + \frac{\partial \eta^i}{\partial y^\beta} dy^\beta + \frac{\partial \eta^i}{\partial y^\beta_\gamma} dy^\beta_\gamma\right) = dy^i + s(d\eta^i),
\end{aligned} \tag{A3.8}$$

where the differentials $dx^j$, $dy^i$, and $dy^i_j$ are, prior to the application of the contact condition (A3.7), assumed to be independent. The space is closed in that the transformations involve at most the first derivative, and the corresponding contact condition only involves the first derivative.

The prolonged infinitesimal operator corresponding to (A3.8) is

$$\hat{X}_{\{1\}} = \xi^j \frac{\partial}{\partial x^j} + \eta^i \frac{\partial}{\partial y^i} + \eta^i_{\{j\}} \frac{\partial}{\partial y^i_j} + (d\xi^j) \frac{\partial}{\partial (dx^j)} + (d\eta^i) \frac{\partial}{\partial (dy^i)}. \tag{A3.9}$$

The contact condition (A3.7) is invariant under the prolonged group (A3.8) if and only if

$$\hat{X}_{\{1\}}\left(dy^i - y^i_j \, dx^j\right) = 0. \tag{A3.10}$$

Following the same procedure used in Appendix 2, we now apply (A3.9) to (A3.7). The result is

$$\hat{X}_{\{1\}}\left(dy^i - y^i_j \, dx^j\right) = d\eta^i - \eta^i_{\{j\}} \, dx^j - y^i_j \, d\xi^j = 0. \tag{A3.11}$$

Written out in full, (A3.11) is

$$\hat{X}_{\{1\}}\left(dy^i - y^i_j\,dx^j\right) = \left(\frac{\partial\eta^i}{\partial x^\alpha}\,dx^\alpha + \frac{\partial\eta^i}{\partial y^\beta}\,dy^\beta + \frac{\partial\eta^i}{\partial y^\beta_\gamma}\,dy^\beta_\gamma\right) - \eta^i_{\{j\}}\,dx^j$$
$$- y^i_j\left(\frac{\partial\xi^j}{\partial x^\alpha}\,dx^\alpha + \frac{\partial\xi^j}{\partial y^\beta}\,dy^\beta + \frac{\partial\xi^j}{\partial y^\beta_\gamma}\,dy^\beta_\gamma\right) = 0. \tag{A3.12}$$

Using $dy^i = y^i_j\,dx^j$ to replace $dy^i$ in (A3.12), the invariance condition becomes

$$\hat{X}_{\{1\}}\left(dy^i - y^i_j\,dx^j\right) = \left(\frac{\partial\eta^i}{\partial x^j} + \frac{\partial\eta^i}{\partial y^\beta}\,y^\beta_j - y^i_\sigma\left(\frac{\partial\xi^\sigma}{\partial x^j} + \frac{\partial\xi^\sigma}{\partial y^\beta}y^\beta_j\right) - \eta^i_{\{j\}}\right)dx^j$$
$$- \left(\frac{\partial\eta^i}{\partial y^\beta_j} - y^i_\sigma\,\frac{\partial\xi^\sigma}{\partial y^\beta_j}\right)dy^\beta_j = 0. \tag{A3.13}$$

Now here is the key point. Notice that in forming the transformation of differentials we have, because of the dependence of $\xi^j$ and $\eta^i$ on $\boldsymbol{y}_1$, generated the differential $dy^\beta_j$ in (A3.13). However, because the space is assumed to be closed in $\boldsymbol{x}, \boldsymbol{y}$, and $\boldsymbol{y}_1$ [cf. Equation (A3.4)], there is nothing to restrict the possible values of this differential. Therefore, in order for (A3.13) to be satisfied, *both* terms in parentheses in (A3.13) must be individually equal to zero. Thus the infinitesimals of the group (A3.4) must satisfy

$$\eta^i_{\{j\}} = \frac{\partial\eta^i}{\partial x^j} + \frac{\partial\eta^i}{\partial y^\beta}y^\beta_j - y^i_\sigma\left(\frac{\partial\xi^\sigma}{\partial x^j} + \frac{\partial\xi^\sigma}{\partial y^\beta}y^\beta_j\right) \tag{A3.14}$$

*and*

$$\frac{\partial\eta^i}{\partial y^\beta_j} - y^i_\sigma\,\frac{\partial\xi^\sigma}{\partial y^\beta_j} = 0. \tag{A3.15}$$

Here it is convenient to use the characteristic function

$$\mu^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1] = \eta^i - y^i_\sigma\xi^\sigma \tag{A3.16}$$

used in Chapter 14. In terms of $\mu^i$ the conditions (A3.14) and (A3.15) become

$$\begin{aligned}\eta^i_{\{j\}} &= \frac{\partial\mu^i}{\partial x^j} + \frac{\partial\mu^i}{\partial y^\beta}y^\beta_j,\\ 0 &= \frac{\partial\mu^i}{\partial y^\beta_j} + \delta^i_\beta\xi^j.\end{aligned} \tag{A3.17}$$

### A3.3.1 The Case m>1

For $m > 1$, the second relation in (A3.17) is

$$
\begin{aligned}
0 &= \frac{\partial \mu^i}{\partial y_j^\beta}, \qquad i \neq \beta, \\
-\xi^j &= \frac{\partial \mu^1}{\partial y_j^1} = \frac{\partial \mu^2}{\partial y_j^2} = \cdots = \frac{\partial \mu^m}{\partial y_j^m}, \qquad i = \beta.
\end{aligned}
\tag{A3.18}
$$

Note that the $\xi^j$ are highly constrained by having to satisfy each of the multiple equalities in (A3.18). The only solution of (A3.18) is

$$
\mu^i = \eta^i[\boldsymbol{x}, \boldsymbol{y}] - y^i_\sigma \xi^\sigma[\boldsymbol{x}, \boldsymbol{y}], \tag{A3.19}
$$

which is the Lie–Bäcklund transformation equivalent to a point group, i.e., $\eta^i$ and $\xi^j$ cannot depend on $\boldsymbol{y}_1$.

So for $m > 1$ the transformation (A3.4) can preserve the contact condition (A3.7) only if it is an extension of a Lie point group. Moreover, we know from our earlier discussion that point groups preserve tangency up to infinite order. Notice that the first relation in (A3.17) is the conventional point-group formula for the infinitesimal transformation of the first partial derivative.

### A3.3.2 The Case m = 1

Now consider the case of one dependent variable. In this case (A3.17) reduces to

$$
\begin{aligned}
\eta_{\{j\}} &= \frac{\partial \mu}{\partial x^j} + \frac{\partial \mu}{\partial y} y_j, \\
0 &= \frac{\partial \mu}{\partial y_j} + \xi^j.
\end{aligned}
\tag{A3.20}
$$

In (A3.20) each $\xi^j$ is specified by only a single equality. The infinitesimals of the group for the case $m = 1$ are derived from (A3.20):

$$
\xi^j = -\frac{\partial \mu}{\partial y_j}, \qquad \eta = \mu - y_\sigma \frac{\partial \mu}{\partial y_\sigma}, \qquad \eta_{\{j\}} = \frac{\partial \mu}{\partial x^j} + \frac{\partial \mu}{\partial y} y_j. \tag{A3.21}
$$

In this instance $\mu$ can depend on $\boldsymbol{x}$, $y$, and $\boldsymbol{y}_1$. The conclusion is that first-order contact transformations of the form (A3.4) can preserve the contact condition (A3.7) only when there is just one dependent variable. Otherwise (A3.4) must be a simple first extension of a point group. The transformation (A3.5) with infinitesimals (A3.21) is called a *Lie contact group*. Note that all three infinitesimals of the group are defined by a single function, $\mu[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1]$.

Extensions of the group to higher derivatives are generated using the same algorithm based on the contact conditions used for point groups. Also, as in the case of point groups, the contact conditions are invariant under the extended group to all orders.

## A3.4 Higher-Order Tangent Transformation Groups

Now the question is whether there are any transformations that are not simple extensions of Lie contact groups or Lie point groups and for which higher-order tangency is an invariant condition. To examine this question, we now consider a generalization of (A3.4) to a transformation that can depend on derivatives of order higher than one, say of order $p$. Consider the transformation

$$T^s : \left\{ \begin{aligned} \tilde{x}^j &= F^j[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p, s], \quad j = 1, \ldots, n \\ \tilde{y}^i &= G^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p, s], \quad i = 1, \ldots, m \\ \tilde{y}^i_{j_1} &= G^i_{\{j_1\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p, s] \\ &\ \vdots \\ \tilde{y}^i_{j_1 \ldots j_p} &= G^i_{\{j_1 \ldots j_p\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p, s] \end{aligned} \right\} \tag{A3.22}$$

with infinitesimal form

$$T^s : \left\{ \begin{aligned} \tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p, s], \quad j = 1, \ldots, n \\ \tilde{y}^i &= y^i + s\eta^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p, s], \quad i = 1, \ldots, m \\ \tilde{y}^i_{j_1} &= y^i_{j_1} + s\eta^i_{\{j_1\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p, s] \\ &\ \vdots \\ \tilde{y}^i_{j_1 \ldots j_p} &= y^i_{j_1 \ldots j_p} + s\eta^i_{\{j_1 \ldots j_p\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p, s] \end{aligned} \right\}, \tag{A3.23}$$

where

$$\begin{gathered} \xi^j = \left.\frac{\partial F^j}{\partial s}\right|_{s=0}, \qquad \eta^i = \left.\frac{\partial G^i}{\partial s}\right|_{s=0}, \qquad \eta^i_{\{j\}} = \left.\frac{\partial G^i_{\{j\}}}{\partial s}\right|_{s=0}, \ldots, \\ \eta^i_{\{j_1 \ldots j_p\}} = \left.\frac{\partial G^i_{\{j_1 \ldots j_p\}}}{\partial s}\right|_{s=0}. \end{gathered} \tag{A3.24}$$

Here again, as in the case of Lie contact transformations, the form of (A3.22) and (A3.23) is somewhat unexpected in that a straightforward procedure for generating the extensions of the group should raise by one the order of derivatives appearing in the functional dependence of each derivative being transformed. So again, we have to ask what restrictions must be placed on $F^j$, $G^i$, and $G^i_{\{j\}}, \dots, G^i_{\{j_1\dots j_p\}}$ to ensure that (A3.22) preserves the invariance of the contact conditions

$$\begin{aligned} dy^i - y^i_{j_1} dx^{j_1} &= 0, \\ dy^i_{j_1} - y^i_{j_1 j_2} dx^{j_2} &= 0, \\ &\vdots \\ dy^i_{j_1\dots j_{p-1}} - y^i_{j_1\dots j_p} dx^{j_p} &= 0. \end{aligned} \tag{A3.25}$$

Initially we assume:

(1) The transformation (A3.23) is closed in the space $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \dots, \boldsymbol{y}_p$ (as are $p$th extended point groups and first-order contact groups).
(2) Extensions of the transformation are generated using the contact conditions (A3.25). This is required if the transformation (A3.23) is to preserve $p$th-order tangency and to inherit the properties of a group.

So, as in the case treated in Section A3.3, the question of the existence of such transformations boils down to the identification of the conditions that must be met in order to preserve tangency in the closed space of the transformation.

Now prolong (A3.23) to include differentials $dx^j$, $dy^i$ up to $dy^i_{j_1}, \dots, dy^i_{j_1\dots j_{p-1}}$:

$$\begin{aligned} \tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \dots, \boldsymbol{y}_p, s], \qquad j = 1, \dots, n, \\ \tilde{y}^i &= y^i + s\eta^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \dots, \boldsymbol{y}_p, s], \qquad i = 1, \dots, m, \\ \tilde{y}^i_{j_1} &= y^i_{j_1} + s\eta^i_{\{j_1\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \dots, \boldsymbol{y}_p, s], \\ &\vdots \\ \tilde{y}^i_{j_1\dots j_p} &= y^i_{j_1\dots j_p} + s\eta^i_{\{j_1\dots j_p\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \dots, \boldsymbol{y}_p, s], \\ d\tilde{x}^j &= dx^j + s\left(\frac{\partial \xi^j}{\partial x^\alpha} dx^\alpha + \frac{\partial \xi^j}{\partial y^\beta} dy^\beta + \frac{\partial \xi^j}{\partial y^\beta_{\gamma_1}} dy^\beta_{\gamma_1} + \dots + \frac{\partial \xi^j}{\partial y^\beta_{\gamma_1\dots\gamma_p}} dy^\beta_{\gamma_1\dots\gamma_p}\right) \\ &= dx^j + s(d\xi^j), \\ d\tilde{y}^i &= dy^i + s\left(\frac{\partial \eta^i}{\partial x^\alpha} dx^\alpha + \frac{\partial \eta^i}{\partial y^\beta} dy^\beta + \frac{\partial \eta^i}{\partial y^\beta_{\gamma_1}} dy^\beta_{\gamma_1} + \dots + \frac{\partial \eta^i}{\partial y^\beta_{\gamma_1\dots\gamma_p}} dy^\beta_{\gamma_1\dots\gamma_p}\right) \end{aligned}$$

$$= dy^i + s(d\eta^i),$$
$$\vdots$$
$$\begin{aligned} d\tilde{y}^i_{j_1\dots j_{p-1}} &= dy^i_{j_1\dots j_{p-1}} \\ &\quad + s\left(\frac{\partial \eta^i_{\{j_1\dots j_{p-1}\}}}{\partial x^\alpha}\,dx^\alpha + \dots + \frac{\partial \eta^i\{j_1\dots j_{p-1}\}}{\partial y^\beta_{\gamma_1\dots\gamma_p}}\,dy^\beta_{\gamma_1\dots\gamma_p}\right) \\ &= dy^i_{j_1\dots j_{p-1}} + s\big(d\eta^i_{\{j_1\dots j_{p-1}\}}\big), \end{aligned} \tag{A3.26}$$

where the differentials $dx^j$, $dy^i$, and $dy^i_j, \dots, dy^i_{j_1\dots j_{p-1}}$ are initially assumed to be independent. The prolonged infinitesimal operator corresponding to (A3.26) is

$$\begin{aligned} \hat{X}_{\{p\}} &= \xi^j\frac{\partial}{\partial x^j} + \eta^i\frac{\partial}{\partial y^i} + \eta^i_{\{j_1\}}\frac{\partial}{\partial y^i_{j_1}} + \dots + \eta^i_{\{j_1\dots j_p\}}\frac{\partial}{\partial y^i_{j_1\dots j_p}} \\ &\quad + d\xi^j\frac{\partial}{\partial(dx^j)} + d\eta^i\frac{\partial}{\partial(dy^i)} + d\eta^i_{\{j\}}\frac{\partial}{\partial\big(dy^i_j\big)} + \cdots \\ &\quad + d\eta^i_{\{j_1\dots j_{p-1}\}}\frac{\partial}{\partial\big(dy^i_{j_1\dots j_{p-1}}\big)}. \end{aligned} \tag{A3.27}$$

Apply the prolonged operator (A3.27) to the contact conditions (A3.26):

$$\begin{aligned} \hat{X}_{\{p\}}\big(dy^i - y^i_{j_1}\,dx^{j_1}\big) &= d\eta^i - \eta^i_{\{j_1\}}\,dx^{j_1} - y^i_{j_1}\,d\xi^{j_1} = 0, \\ \hat{X}_{\{p\}}\big(dy^i_{j_1} - y^i_{j_1 j_2}\,dx^{j_2}\big) &= d\eta^i_{\{j\}} - \eta^i_{\{j_1 j_2\}}\,dx^{j_2} - y^i_{j_1 j_2}\,d\xi^{j_2} = 0, \\ &\vdots \\ \hat{X}_{\{p\}}\big(dy^i_{j_1\dots j_{p-1}} - y^i_{j_1\dots j_p}\,dx^{j_p}\big) &= d\eta^i_{\{j_1\dots j_{p-1}\}} - \eta^i_{\{j_1\dots j_p\}}\,dx^{j_p} - y^i_{j_1\dots j_p}\,d\xi^{j_p} = 0, \end{aligned} \tag{A3.28}$$

Expand the first relation in (A3.28):

$$\begin{aligned} &\hat{X}_{\{p\}}\big(dy^i - y^i_{j_1}\,dx^{j_1}\big) \\ &\quad = \frac{\partial\eta^i}{\partial x^\alpha}\,dx^\alpha + \frac{\partial\eta^i}{\partial y^\beta}\,dy^\beta + \frac{\partial\eta^i}{\partial y^\beta_{\gamma_1}}\,dy^\beta_{\gamma_1} + \dots + \frac{\partial\eta^i}{\partial y^\beta_{\gamma_1\dots\gamma_p}}\,dy^\beta_{\gamma_1\dots\gamma_p} \\ &\quad - \eta^i_{\{j_1\}}\,dx^{j_1} \\ &\quad - y^i_{j_1}\left(\frac{\partial\xi^{j_1}}{\partial x^\alpha}\,dx^\alpha + \frac{\partial\xi^{j_1}}{\partial y^\beta}\,dy^\beta + \frac{\partial\xi^{j_1}}{\partial y^\beta_{\gamma_1}}\,dy^\beta_{\gamma_1} + \dots + \frac{\partial\xi^{j_1}}{\partial y^\beta_{\gamma_1\dots\gamma_p}}\,dy^\beta_{\gamma_1\dots\gamma_p}\right). \end{aligned} \tag{A3.29}$$

Now use the contact conditions up to order $p$ to replace $dy^\beta$ by $y^\beta_\alpha\,dx^\alpha$, $dy^\beta_{\gamma_1}$ by $y^\beta_{\gamma_1\alpha}\,dx^\alpha$, and so on up to replacing $dy^\beta_{\gamma_1\ldots\gamma_{p-1}}$ by $y^\beta_{\gamma_1\ldots\gamma_{p-1}\alpha}\,dx^\alpha$.

Here again is the key point. Note that, because we have assumed that the transformation is closed in the space $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p)$, the differential $dy^\beta_{\gamma_1\ldots\gamma_p}$, that appears in the prolonged transformation (A3.26) must remain an independent variable, since the corresponding contact condition would require the inclusion of $y^\beta_{\gamma_1\ldots\gamma_{p+1}}$ in the variable space, which, through the differential $dy^\beta_{\gamma_1\ldots\gamma_{p+1}}$, would in turn require the inclusion of $y^\beta_{\gamma_1\ldots\gamma_{p+2}}$, and so on. Equation (A3.29) now becomes

$$\hat{X}_{\{p\}}\big(dy^i - y^i_{j_1}\,dx^{j_1}\big) = \big(D_{j_1}(\eta^i)\big|_{p-1} - y^i_\alpha D_{j_1}(\xi^\alpha)\big|_{p-1} - \eta^i_{\{j_1\}}\big)\,dx^{j_1} + \left(\frac{\partial\eta^i}{\partial y^\beta_{\gamma_1\ldots\gamma_p}} - y^i_{j_1}\frac{\partial\xi^{j_1}}{\partial y^\beta_{\gamma_1\ldots\gamma_p}}\right)dy^\beta_{\gamma_1\ldots\gamma_p} = 0, \qquad \text{(A3.30)}$$

where $\alpha$ has been replaced by $j$ and we have used a modified total differentiation operator that involves differentiation only up to order $p-1$ of functions that, so far, could depend on derivatives up to order $p$. For example,

$$D_\alpha(\eta^i)|_{p-1} = \frac{\partial\eta^i}{\partial x^\alpha} + y^\beta\frac{\partial\eta^i}{\partial y^\beta} + y^\beta_{\gamma_1}\frac{\partial\eta^i}{\partial y^\beta_{\gamma_1}} + \cdots + y^\beta_{\gamma_1\ldots\gamma_{p-1}}\frac{\partial\eta^i}{\partial y^\beta_{\gamma_1\ldots\gamma_{p-1}}}. \qquad \text{(A3.31)}$$

Expanding the second relation in (A3.28) produces

$$\begin{aligned}\hat{X}_{\{p\}}&\big(dy^i_{j_1} - y^i_{j_1 j_2}\,dx^{j_2}\big)\\ &= \big(D_{j_2}\big(\eta^i_{\{j_1\}}\big)\big|_{p-1} - y^i_{j_1\alpha}D_{j_2}(\xi^\alpha)\big|_{p-1} - \eta^i_{\{j_1 j_2\}}\big)\,dx^{j_2}\\ &\quad + \left(\frac{\partial\eta^i_{\{j\}}}{\partial y^\beta_{\gamma_1\ldots\gamma_p}} - y^i_{j_1 j_2}\frac{\partial\xi^{j_2}}{\partial y^\beta_{\gamma_1\ldots\gamma_p}}\right)dy^\beta_{\gamma_1\ldots\gamma_p} = 0,\end{aligned} \qquad \text{(A3.32)}$$

and so on up to order $p-1$:

$$\begin{aligned}\hat{X}_{\{p\}}&\big(dy^i_{j_1\ldots j_{p-1}} - y^i_{j_1\ldots j_p}\,dx^{j_p}\big)\\ &= \big(D_{j_p}\big(\eta^i_{\{j_1\ldots j_{p-1}\}}\big)\big|_{p-1}\\ &\quad - y^i_{j_1\ldots j_{p-1}\alpha}D_{j_p}(\xi^\alpha)\big|_{p-1} - \eta^i_{\{j_1\ldots j_p\}}\big)\,dx^{j_p}\\ &\quad + \left(\frac{\partial\eta^i_{\{j_1\ldots j_{p-1}\}}}{\partial y^\beta_{\gamma_1\ldots\gamma_p}} - y^i_{j_1\ldots j_p}\frac{\partial\xi^{j_p}}{\partial y^\beta_{\gamma_1\ldots\gamma_p}}\right)dy^\beta_{\gamma_1\ldots\gamma_p} = 0.\end{aligned} \qquad \text{(A3.33)}$$

The differentials $dx^j$ and $dy^\beta_{\gamma_1\ldots\gamma_p}$ in (A3.33) are independent, and so the invariance of the contact conditions (A3.25) leads to the following relations for the infinitesimals of the group from (A3.30), (A3.32), and (A3.33):

$$\begin{aligned}
\eta^i_{\{j_1\}} &= D_{j_1}(\eta^i)|_{p-1} - y^i_\alpha D_{j_1}(\xi^\alpha)|_{p-1},\\
\eta^i_{\{j_1 j_2\}} &= D_{j_2}\big(\eta^i_{\{j_1\}}\big)\big|_{p-1} - y^i_{j_1\alpha} D_{j_2}(\xi^\alpha)|_{p-1},\\
&\;\vdots\\
\eta^i_{\{j_1\ldots j_p\}} &= D_{j_p}\big(\eta^i_{\{j_1\ldots j_{p-1}\}}\big)\big|_{p-1} - y^i_{j_1\ldots j_{p-1}\alpha} D_{j_p}(\xi^\alpha)|_{p-1},
\end{aligned} \tag{A3.34}$$

together with the extra conditions from (A3.30), (A3.32), and (A3.33),

$$\begin{aligned}
\frac{\partial \eta^i}{\partial y^\beta_{\gamma_1\ldots\gamma_p}} - y^i_{j_1}\frac{\partial \xi^{j_1}}{\partial y^\beta_{\gamma_1\ldots\gamma_p}} &= 0,\\
\frac{\partial \eta^i_{\{j\}}}{\partial y^\beta_{\gamma_1\ldots\gamma_p}} - y^i_{j_1 j_2}\frac{\partial \xi^{j_1}}{\partial y^\beta_{\gamma_1\ldots\gamma_p}} &= 0,\\
&\;\vdots\\
\frac{\partial \eta^i_{\{j_1\ldots j_{p-1}\}}}{\partial y^\beta_{\gamma_1\ldots\gamma_p}} - y^i_{j_1\ldots j_p}\frac{\partial \xi^{j_p}}{\partial y^\beta_{\gamma_1\ldots\gamma_p}} &= 0,
\end{aligned} \tag{A3.35}$$

that must be satisfied in order for (A3.30), (A3.32), and (A3.33) to be satisfied. Now use the characteristic functions introduced in Chapter 14:

$$\begin{aligned}
\mu^i &= \eta^i - y^i_{j_1}\xi^{j_1},\\
\mu^i_{\{j_1\}} &= \eta^i_{\{j_1\}} - y^i_{j_1 j_2}\xi^{j_2},\\
&\;\vdots\\
\mu^i_{\{j_1\ldots j_{p-1}\}} &= \eta^i_{\{j_1\ldots j_{p-1}\}} - y^i_{j_1\ldots j_{p-1} j_p}\xi^{j_p}.
\end{aligned} \tag{A3.36}$$

Equations (A3.35) become

$$\begin{aligned}
\frac{\partial \mu^i}{\partial y^\beta_{\gamma_1\ldots\gamma_p}} &= 0,\\
\frac{\partial \mu^i_{\{j\}}}{\partial y^\beta_{\gamma_1\ldots\gamma_p}} &= 0,\\
&\;\vdots\\
\frac{\partial \mu^i_{\{j_1\ldots j_{p-2}\}}}{\partial y^\beta_{\gamma_1\ldots\gamma_p}} &= 0,\\
\frac{\partial \mu^i_{\{j_1\ldots j_{p-1}\}}}{\partial y^\beta_{\gamma_1\ldots\gamma_p}} + \delta^i_\beta\delta^{j_1}_{\gamma_1}\ldots\delta^{j_p}_{\gamma_p}\xi^{\gamma_p} &= 0.
\end{aligned} \tag{A3.37}$$

Compare (A3.37) with (A3.18). Note that in the case $m > 1$, the $\xi^{\gamma_p}$ are required to satisfy multiple equalities by the last equation in (A3.37). That is,

$$\begin{aligned} &\frac{\partial \mu^i_{\{j_1 \ldots j_{p-1}\}}}{\partial y^\beta_{\gamma_1 \ldots \gamma_p}} = 0, \qquad i \neq \beta, \\ &\frac{\partial \mu^i_{\{j_1 \ldots j_{p-1}\}}}{\partial y^i_{\gamma_1 \ldots \gamma_p}} = 0 \qquad \text{(lower indices not all equal)}, \\ &\frac{\partial \mu^i_{\{j_1 \ldots j_{p-1}\}}}{\partial y^i_{j_1 \ldots j_p}} = -\xi^{j_p} \qquad \text{(no sum)}. \end{aligned} \tag{A3.38}$$

Thus, as in the case $p = 1$ treated in Section A3.3 (Lie contact transformations), the $\xi^j$ cannot depend on $\boldsymbol{y}_p$, and the general solution of (A3.38) is

$$\begin{aligned} \mu^i_{\{j_1 \ldots j_{p-1}\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{p-1}] = {} & \eta^i_{\{j_1 \ldots j_{p-1}\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{p-1}] \\ & - y^i_{j_1 \ldots j_{p-1} j_p} \xi^{j_p}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{p-1}], \end{aligned} \tag{A3.39}$$

i.e., the infinitesimals $\xi^j$ and $\eta^i_{\{j_1 \ldots j_{p-1}\}}$ cannot depend on $\boldsymbol{y}_p$. Similarly, from the remaining relations in (A3.37) we can deduce that all of the infinitesimals $\eta^i, \eta^i_{\{j_1\}}, \ldots, \eta^i_{\{j_1 \ldots j_{p-2}\}}$ are independent of $\boldsymbol{y}_p$. Repeating this process $p - 1$ times shows by induction that the infinitesimals $(\xi^j, \eta^i)$ can only depend on $\boldsymbol{x}$ and $\boldsymbol{y}$ for $m > 1$. Thus the group (A3.23) must be the $p$th-order extension of a Lie point group.

For $m = 1$ the relations (A3.38) become

$$\begin{aligned} &\frac{\partial \mu_{\{j_1 \ldots j_{p-1}\}}}{\partial y_{\gamma_1 \ldots \gamma_p}} = 0 \qquad \text{(lower indices not all equal)}, \\ &\frac{\partial \mu_{\{j_1 \ldots j_{p-1}\}}}{\partial y_{j_1 \ldots j_p}} = -\xi^{j_p} \qquad \text{(no sum)}. \end{aligned} \tag{A3.40}$$

Once again the $\xi^j$ cannot depend on $\boldsymbol{y}_{\{p\}}$, and the general solution of (A3.40) is

$$\begin{aligned} \mu_{\{j_1 \ldots j_{p-1}\}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{p-1}) = {} & \eta_{\{j_1 \ldots j_{p-1}\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1 \ldots, \boldsymbol{y}_{p-1}] \\ & - y_{j_1 \ldots j_{p-1} j_p} \xi^{j_p}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{p-1}], \end{aligned} \tag{A3.41}$$

i.e., the infinitesimals $\xi^j$ and $\eta_{\{j_1 \ldots j_{p-1}\}}$ cannot depend on $\boldsymbol{y}_p$. Similarly, from the remaining relations in (A3.37) we can deduce that all of the infinitesimals $\eta^i, \eta^i_{\{j_1\}}, \ldots, \eta^i_{\{j_1 \ldots j_{p-2}\}}$ are independent of $\boldsymbol{y}_p$. Again, by induction, the infinitesimals can only depend on $\boldsymbol{x}$, $y$, and $\boldsymbol{y}_1$ for $m = 1$. Therefore, if $m = 1$

the group (A3.23) must be the $p$th-order extension of a Lie point or Lie contact transformation.

## A3.5 One Dependent Variable and One Independent Variable

For the case $m = n = 1$ the relations (A3.36) are

$$\begin{aligned} \mu &= \eta - y_x \xi, \\ \mu_{\{1\}} &= \eta_{\{1\}} - y_{xx}\xi, \\ &\vdots \\ \mu_{\{p-2\}} &= \eta_{\{p-2\}} - y_{(p-1)x}\xi, \\ \mu_{\{p-1\}} &= \eta_{\{p-1\}} - y_{px}\xi, \end{aligned} \tag{A3.42}$$

where

$$\begin{aligned} \eta_{\{1\}} &= D(\eta)|_{p-1} - y_x D(\xi)|_{p-1}, \\ \eta_{\{2\}} &= D\big(\eta_{\{1\}}\big)|_{p-1} - y_{xx} D(\xi)|_{p-1}, \\ &\vdots \\ \eta_{\{p-1\}} &= D\big(\eta_{\{p-2\}}\big)\Big|_{p-1} - y_{(p-1)x} D(\xi)\Big|_{p-1}, \\ \eta_{\{p\}} &= D\big(\eta_{\{p-1\}}\big)|_{p-1} - y_{px} D(\xi)|_{p-1}, \end{aligned} \tag{A3.43}$$

and (A3.40) reduces to

$$\begin{aligned} \frac{\partial \mu}{\partial y_{px}} &= 0, \\ \frac{\partial \mu_{\{p-2\}}}{\partial y_{px}} &= 0, \\ \frac{\partial \mu_{\{p-1\}}}{\partial y_{px}} &= -\xi. \end{aligned} \tag{A3.44}$$

The last of (A3.44) combined with (A3.42) shows that $\xi, \eta, \eta_1, \ldots, \eta_{p-1}$ cannot depend on $y_{px}$. Nor, by induction, can they depend on any derivatives beyond the first. So the case $m = 1,\ n = 1$ reduces to the extension of a Lie contact transformation.

***Theorem A3.1.*** *There do not exist any transformation groups that preserve pth-order tangency and that are closed in the space* $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p$ *other than extensions of Lie point transformations for* $m > 1$ *and extensions of Lie contact transformations for* $m = 1$.

## A3.6 Infinite Order Structure

Lie–Bäcklund groups are transformations that preserve infinite-order contact and are not simple extensions of Lie point or Lie tangent transformations. They are interesting transformations, which can be used to transform systems of differential equations without raising the order. To develop the theory of Lie–Bäcklund groups, it is necessary to relax the requirement that the transformation be closed in the space $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p$. Consider a one-parameter transformation of the form

$$T^s : \left\{ \begin{aligned} \tilde{x}^j &= F^j[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, s], \quad j = 1, \ldots, n \\ \tilde{y}^i &= G^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, s], \quad i = 1, \ldots, m \\ \tilde{y}^i_{j_1} &= G^i_{\{j_1\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, s] \\ &\vdots \end{aligned} \right\}, \tag{A3.45}$$

where the dots indicate continuation to infinite order.

The number of independent variables in $F^j$, $G^i$, $G^i_{\{j_1\}}, \ldots$ is *a priori* finite or infinite in the infinite-dimensional space $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, s)$. The group is prolonged to include the transformations of the differentials

$$\begin{aligned} \tilde{dx}^j &= \frac{\partial F^j}{\partial x^\alpha}\, dx^\alpha + \frac{\partial F^j}{\partial y^\beta}\, dy^\beta + \frac{\partial F^j}{\partial y^\beta_\gamma}\, dy^\beta_\gamma + \cdots, \\ \tilde{dy}^i &= \frac{\partial G^i}{\partial x^\alpha}\, dx^\alpha + \frac{\partial G^i}{\partial y^\beta}\, dy^\beta + \frac{\partial G^i}{\partial y^\beta_\gamma}\, dy^\beta_\gamma + \cdots, \\ d\tilde{y}^i_{j_1} &= \frac{\partial G^i_{\{j_1\}}}{\partial x^\alpha}\, dx^\alpha + \frac{\partial G^i_{\{j_1\}}}{\partial y^\beta}\, dy^\beta + \frac{\partial G^i_{\{j_1\}}}{\partial y^\beta_\gamma}\, dy^\beta_\gamma + \cdots, \\ &\vdots \end{aligned} \tag{A3.46}$$

***Definition A3.1.*** *The transformation (A3.45) is called a Lie-Bäcklund transformation if the infinite-order system of contact conditions*

$$\begin{aligned} dy^i - y^i_{j_1}\, dx^{j_1} &= 0, \\ dy^i_{j_1} - y^i_{j_1 j_2}\, dx^{j_2} &= 0, \\ dy^i_{j_1 j_2} - y^i_{j_1 j_2 j_3}\, dx^{j_3} &= 0, \\ &\vdots \end{aligned} \tag{A3.47}$$

*is invariant with respect to the action of the prolonged group (A3.45) and (A3.46).*

### A3.6.1 Infinitesimal Transformation

The infinitesimal form of the extended group (A3.45) is

$$T^s : \left\{ \begin{aligned} \tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots], \quad j = 1, \ldots, n \\ \tilde{y}^i &= y^i + s\eta^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots], \quad i = 1, \ldots, m \\ \tilde{y}^i_{j_1} &= y^i_{j_1} + s\eta^i_{\{j_1\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots] \\ &\vdots \end{aligned} \right\}, \tag{A3.48}$$

and the infinitesimal form of the group prolonged to include the transformations of differentials is (A3.48) plus

$$\begin{aligned} d\tilde{x}^j &= dx^j + s\left( \frac{\partial \xi^j}{\partial x^\alpha} dx^\alpha + \frac{\partial \xi^j}{\partial y^\beta} dy^\beta + \frac{\partial \xi^j}{\partial y^\beta_{\gamma_1}} dy^\beta_{\gamma_1} + \cdots \right) = dx^j + s(d\xi^j), \\ d\tilde{y}^i &= dy^i + s\left( \frac{\partial \eta^i}{\partial x^\alpha} dx^\alpha + \frac{\partial \eta^i}{\partial y^\beta} dy^\beta + \frac{\partial \eta^i}{\partial y^\beta_{\gamma_1}} dy^\beta_{\gamma_1} + \cdots \right) = dy^i + s(d\eta^i), \\ d\tilde{y}^i_{j_1} &= dy^i_{j_i} + s\left( \frac{\partial \eta^i_{\{j_1\}}}{\partial x^\alpha} dx^\alpha + \frac{\partial \eta^i_{\{j_1\}}}{\partial y^\beta} dy^\beta + \frac{\partial \eta^i_{\{j_1\}}}{\partial y^\beta_{\gamma_1}} dy^\beta_{\gamma_1} + \cdots \right) \\ &= dy^i_{j_1} + s\big(d\eta^i_{\{j_1\}}\big), \\ &\vdots \end{aligned} \tag{A3.49}$$

Prior to the application of the contact conditions, the differentials $dx^j$ and $dy^i, d\eta^i_{\{j\}}, \ldots$ are independent. The infinitesimal operator corresponding to the extended group (A3.48) and (A3.49) is

$$\begin{aligned} \hat{X} &= \xi^j \frac{\partial}{\partial x^j} + \eta^i \frac{\partial}{\partial y^i} + \eta^i_{\{j_1\}} \frac{\partial}{\partial y^i_{j_1}} + \cdots \\ &\quad + (d\xi^j) \frac{\partial}{\partial (dx^j)} + (d\eta^i) \frac{\partial}{\partial (dy^i)} + \big(d\eta^i_{\{j_1\}}\big) \frac{\partial}{\partial (dy^i_{j_1})} + \cdots. \end{aligned} \tag{A3.50}$$

Applying the extended operator to the contact conditions leads to the following infinite system of invariance conditions:

$$\begin{aligned} \hat{X}\big(dy^i - y^i_{j_1} dx^{j_1}\big) &= d\eta^i - \eta^i_{\{j_1\}} dx^{j_1} - y^i_{j_1} d\xi^{j_1} = 0, \\ \hat{X}\big(dy^i_{j_1} - y^i_{j_1 j_2} dx^{j_2}\big) &= d\eta^i_{\{j_1\}} - \eta^i_{\{j_1 j_2\}} dx^{j_2} - y^i_{j_1 j_2} d\xi^{j_2} = 0, \\ \hat{X}\big(dy^i_{j_1 j_2} - y^i_{j_1 j_2 j_3} dx^{j_3}\big) &= d\eta^i_{\{j_1 j_2\}} - \eta^i_{\{j_1 j_2 j_3\}} dx^{j_3} - y^i_{j_1 j_2 j_3} d\xi^{j_3} = 0, \\ &\vdots \end{aligned} \tag{A3.51}$$

If we use the infinite-order contact conditions to replace the differentials $dy^i, dy^i_j, \ldots$ to all orders, the invariance conditions (A3.51) become

$$\begin{aligned}
\hat{X}\big(dy^i - y^i_\alpha\, dx^\alpha\big) &= \big(D_{j_1}\eta^i - y^i_\alpha D_{j_1}\xi^\alpha - \eta^i_{\{j_1\}}\big)\, dx^{j_1} = 0,\\
\hat{X}\big(dy^i_{j_1} - y^i_{j_1\alpha}\, dx^\alpha\big) &= \big(D_{j_2}\eta^i_{\{j_1\}} - y^i_{j_1\alpha} D_{j_2}\xi^\alpha - \eta^i_{\{j_1 j_2\}}\big)\, dx^{j_2} = 0,\\
\hat{X}\big(dy^i_{j_1 j_2} - y^i_{j_1 j_2\alpha}\, dx^\alpha\big) &= \big(D_{j_3}\eta^i_{\{j_1 j_2\}} - y^i_{j_1 j_2\alpha} D_{j_3}\xi^\alpha - \eta^i_{\{j_1 j_2 j_3\}}\big)\, dx^{j_3} = 0,\\
&\;\;\vdots
\end{aligned} \tag{A3.52}$$

The crucial difference between the infinite- and the finite-order case is that, in the infinite-order case, the dependence of the infinitesimals is not restricted to order $p$; rather, the space is expanded naturally as the order of the transformed derivative is increased, just as it was in the case of point groups. The contact condition is satisfied *a priori* to all orders, and so the differentials $dy^i, dy^i_j, \ldots$ are dependent on the $dx^j$ to all orders, i.e., only the $dx^j$ are independent differentials. As a result, there are no extra conditions such as (A3.35) that must be met and that might severely restrict the possible dependence of the infinitesimals on derivatives. Therefore the theory of Lie–Bäcklund transformations is fundamentally a theory of transformations in an infinite-dimensional space, and the appropriate functional setting is the infinite-dimensional space $\mathcal{A}$ of differential functions. Lie point and tangent groups are regarded as special cases of Lie–Bäcklund transformations.

## REFERENCE

[A3.1] Anderson, R. L. and Ibragimov, N. H. 1979. *Lie–Bäcklund Transformations in Applications*. SIAM Studies in Applied Mathematics.

# Appendix 4

## *Symmetry Analysis Software*

Throughout this chapter **`Courier boldface`** will be used to denote *Mathematica*® format. The application *Mathematica*® has been used to create a software package that eases the labor of finding the invariant groups of a differential equation. Without any question, the bulk of the effort in group analysis comes in the generation of the formulae for the determining equations of the group. The software provided with this text automates this process. The package is called **`IntroToSymmetry.m.`** Details for how to load and use the package on various computer systems is provided in the Readme file on the CD enclosed with this book. The CD also contains the source file for the package, an auto-save file and a large number of sample notebooks organized by chapter numbers corresponding to those in the text.

There are a number of functions provided with the package but the two core ones are called **`FindDeterminingEquations`** and **`SolveDeterminingEquations`**.

The function **`FindDeterminingEquations`** takes a set of ODEs or PDEs provided by the user and executes a purely algorithmic process to produce a list of linear PDEs called the determining equations of the group. This is where the vast bulk of hand calculation was needed in the past to implement the Lie algorithm. This function has proven to be fast, reliable and requires only a modest amount of computer memory. There are a number of variables and tables that the user can access to follow the process at any desired level of detail.

The function **`SolveDeterminingEquations`** attempts to solve the system of determining equations using a multivariate polynomial approach. This function is useful for finding groups of algebraic type but is not useful for finding groups that depend on special or transcendental functions nor does it have an option for simplifying the system of determining equations. This can usually be accomplished using *Mathematica*® built-in functions.

After entering relevant data, the user begins by calling the function **FindDeterminingEquations**. The software first generates the long sum of terms in the invariance condition $X_{\{p\}}\Psi = 0$. Then the original equation or system of equations and perhaps their differential consequences are used to create a table of rules for the elimination of dependent derivative terms in the invariance condition. Finally, derivatives of the unknown infinitesimals are separated from products of $y$-derivatives (derivatives of the dependent variables) and collected together to form the determining equations of the group. The resulting (usually overdetermined) system of linear PDE's for the unknown infinitesimals is presented as a table of string expressions. The series of steps leading to the determining equations is purely algorithmic and the package can be relied on to produce the correct result. Although, in the case of Lie–Bäcklund transformations, the user may wish to apply replacement rules that differ from the standard set produced internally by the program. The software is set up to allow user defined rules to be applied to the invariance condition. The software is very robust and will produce the determining equations for virtually any mathematical expression or set of expressions including systems of under determined equations.

Also included in the package, is a function called **SolveDeterminingEquations** which enables the user to attempt a first pass at solving the determining equations for the unknown infinitesimals. The approach is to let each of the unknown infinitesimal functions be approximated by a multivariable polynomial expansion up to some order selected by the user. These are substituted into the determining equations. Terms multiplying the same products of expansion variables are collected together and the coefficients are set to zero enabling various polynomial coefficients to be evaluated in terms of a small subset using the *Mathematica*® function **Solve**. The remaining subset constitutes the infinitesimal parameters of the symmetry group of the system of equations being analyzed. The rationale for this approach is that the determining equations are always linear and are usually highly overdetermined. It is therefore reasonable to expect solutions in the form of truncated power series. Quite often the infinitesimals contain arbitrary functions and **SolveDeterminingEquations** can be used to pick these up by searching over a range of orders of the trial polynomial looking for terms in the infinitesimals that fail to truncate.

However, there are many examples where the infinitesimals contain transcendental functions including logarithms and exponentials as well as periodic functions, etc. These cannot be found through a simple polynomial expansion and must be treated using more sophisticated approaches (See References [4.1], [4.2], and the review of software by Hereman [4.3]). Nevertheless, solving for the infinitesimal functions that do truncate is extremely useful and a major first step toward the complete solution of the determining equations. Furthermore, *Mathematica*® provides a variety of built-in functions that can be used to manipulate and simplify the system of determining equations. A number of examples are included with the sample notebooks on the CD.

## A4.1 Summary of the Theory

The system, $\Psi^i$, of $m$, $p$th order partial differential equations,

$$\Psi^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_p] = 0; \qquad i = 1, \ldots, m \tag{A4.1}$$

is transformed under the extended infinitesimal Lie–Bäcklund group

$$\begin{aligned}
\tilde{x} &= x + s\xi^j[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_r]; \qquad j = 1, \ldots, n \\
\tilde{y} &= y + s\eta_i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_r]; \qquad i = 1, \ldots, m \\
\tilde{y}^i_{j_1} &= y^i_{j_1} s\eta^i_{\{j_1\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_{r+1}] \\
\tilde{y}^i_{j_1 j_2} &= y^i_{j_1 j_2} + s\eta^i_{\{j_1 j_2\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_{r+2}] \\
&\cdots\cdots\cdots \\
\tilde{y}^i_{j_1 j_2 \cdots j_p} &= y^i_{j_1 j_2 \cdots j_p} + s\eta^i_{\{j_1 j_2 \cdots j_p\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_{r+p}]
\end{aligned} \tag{A4.2}$$

The transformed equation is expanded in the form of a Lie series

$$\begin{aligned}
&\Psi^i[\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}_1, \tilde{\boldsymbol{y}}_2, \ldots, \tilde{\boldsymbol{y}}_p] = \Psi^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_p] \\
&+ sX_{\{p\}}\Psi^i + \frac{s^2}{2!}X_{\{p\}}\left(X_{\{p\}}\Psi^i\right) + \frac{s^3}{3!}X_{\{p\}}\left(X_{\{p\}}\left(X_{\{p\}}\Psi^i\right)\right) + \cdots\cdots\cdots
\end{aligned} \tag{A4.3}$$

where $X_{\{p\}}$ is the p-times extended group operator

$$X_{\{p\}} = \xi^j \frac{\partial}{\partial x^j} + \eta^i \frac{\partial}{\partial y^i} + \eta^i_{\{j_1\}} \frac{\partial}{\partial y^i_{j_1}} + \eta^i_{\{j_1 j_2\}} \frac{\partial}{\partial y^i_{j_1 j_2}} + \cdots + \eta^i_{\{j_1 j_2 \cdots j_p\}} \frac{\partial}{\partial y^i_{j_1 j_2 \cdots j_p}} \tag{A4.4}$$

The system (A4.1) is invariant under the group (A4.2) if and only if

$$X_{\{p\}}\Psi^i = 0; \qquad i = 1, \ldots, m \tag{A4.5}$$

Generally the system of equations, (A4.1), is given and the infinitesimal transformation functions ($\xi^j$, $\eta^i$) which leave the equation invariant are unknowns that need to be determined. The infinitesimals of the extended transformation appearing in (A4.2) and (A4.4) are given by the following formulae.

$$
\begin{aligned}
\eta^i_{\{j_1\}} &= D_{j_1}\eta^i - y^i_\alpha D_{j_1}\xi^\alpha \\
\eta^i_{\{j_1 j_2\}} &= D_{j_2}\eta^i_{\{j_1\}} - y^i_{j_1\alpha} D_{j_2}\xi^\alpha = D_{j_2}D_{j_1}\eta^i - y^i_\alpha D_{j_2}D_{j_1}\xi^\alpha \\
&\qquad - y^i_{j_1\alpha} D_{j_2}\xi^\alpha - y^i_{j_2\alpha} D_{j_1}\xi^\alpha \\
&\cdots\cdots\cdots\cdots \\
\eta^i_{\{j_1 j_2\cdots j_p\}} &= D_{j_p}\eta^i_{\{j_1 j_2\cdots j_{p-1}\}} - y^i_{j_1 j_2\cdots j_{p-1}\alpha} D_{j_p}\xi^\alpha
\end{aligned}
\tag{A4.6}
$$

where the operator of total differentiation with respect to the variable $x_j$ is,

$$
\begin{aligned}
D_j = \frac{\partial}{\partial x_j} + y^i_j \frac{\partial}{\partial y^i} + y^i_{j_1 j}\frac{\partial}{\partial y^i_{j_1}} + \cdots\cdots + y^i_{j_1 j_2\cdots j_p j}\frac{\partial}{\partial y^i_{j_1 j_2\cdots j_p}} \\
+ y^i_{j_1 j_2\cdots j_{p+1} j}\frac{\partial}{\partial y^i_{j_1 j_2\cdots j_{p+1}}} + \cdots\cdots + y^i_{j_1 j_2\cdots j_{p+r-1} j}\frac{\partial}{\partial y^i_{j_1 j_2\cdots j_{p+r-1}}}
\end{aligned}
\tag{A4.7}
$$

The invariance condition (A4.5) is typically a very long sum. Each term in the sum is a product of two kinds of factors; the first is a product of derivatives of the dependent variables, $y^i$, and the second is an isolated derivative of one of the unknown infinitesimal functions, $(\xi^j, \eta^i)$. The highest $y$-derivative appearing in (A4.6) is of order $p + r$. The highest derivative of an unknown infinitesimal is of order $p$. The y-derivatives are constrained by the requirement that the solution, $y^i$ solves the original system (A4.1) *and* its differential consequences up to order $r$,

$$
D_j\Psi^i = 0; \quad D_{j_1 j_2}\Psi^i = 0; \quad D_{j_1 j_2 j_3}\Psi^i = 0; \ldots\ldots; \quad D_{j_1 j_2\cdots j_r}\Psi^i = 0
\tag{A4.8}
$$

where the indices denote all possible combinations of derivatives. In practice, these constraints are applied by substituting (A4.1) and (A4.8) into the invariance condition (A4.5). Common $y$-derivative terms are collected together and, since these remaining $y$-derivatives are free to take on any value, the only way the invariance condition can be satisfied is to set the coefficients of these terms to zero. The coefficients form the set of *determining equations* for the infinitesimal functions of the group $(\xi^j, \eta^i)$. Because of the linearity of the extensions, (A4.6), in the $\xi$'s and $\eta$'s and their derivatives the determining equations of the group are always linear (except in the non-classical approach, several examples of which are included with the CD). Furthermore, for equations of second order and higher the determining equations generally form an overdetermined set. For first order ODE's and simple functions of **x** and **y** there is only a single determining equation for the two unknown infinitesimals (i.e., the invariance condition itself) and in this case, the invariant groups are undetermined.

Note that when seeking the infinitesimals of a system of differential equations each equation in the system generates its own set of determining equations subject to the constraints implied by the *entire* system. The full set of determining equations is generated by concatenating those found for each equation.

Some care is required when substituting (A4.1) and (A4.8) into the invariance condition. This has to do with the variables upon which the infinitesimals are assumed to depend (independent variables, dependent variables and possibly derivatives up to order $r$). If none of these variables appear in the input equation (A4.1) then it makes no difference which term in the input equation is replaced in the invariance condition. However if any of the these variables do appear then care is needed to insure that one does not create an infinitely recursive loop when the replacement is made. Generally one replaces the highest derivative in the equation and its differential consequences. However if the equation involves complicated expressions given as general functions this may not be possible. In any case the rule of thumb is to isolate a $y$-derivative term in the equation *which the infinitesimals do not depend on* and use that term and its derivatives to make the substitution. In some cases the replacement process may be tailored to enable one to search for some particular restricted type of Lie–Bäcklund transformation.

## A4.2 The Program

The program generates the determining equations of the infinitesimals of the Lie point, Lie contact or Lie–Bäcklund group which leaves a given input equation invariant. The package can take as input simply a general function, an ODE, a PDE, a system of ODEs or a system of PDEs. Moreover the system need not be closed. For example one can analyze the invariance of a system of two equations in three unknowns. Such systems commonly arise in the development of engineering models such as in turbulence modeling. If necessary, the determining equations can be manipulated using *Mathematica*® built-in functions prior to attempting to solve them. The infinitesimals are designated `xsej` (for $\xi^j$) and `etai` (for $\eta^i$). About the spelling of the greek letter $\xi$: it is usual to use '`xi`'. However the software involves a lot of string manipulations with combinations such as '`xj`' appearing in reference to the independent variables. To avoid any possible conflict I decided to coin my own spelling. The spelling of $\eta$ is the conventional one.

### A4.2.1 Getting Started

Open a new *Mathematica*® notebook and read the package into memory using

```
Needs["SymmetryAnalysis'IntroToSymmetry'"]
```

or

```
<<IntroToSymmetry.m.
```

The equation is entered as a string which I usually call **inputequation** with the derivatives written as *Mathematica*® input. Please note that for the purposes of this appendix, I will use the names that I customarily employ in the sample notebooks but the choice of what to call any variable input to the package is entirely up to the user. The Blasius equation would be entered as

```
inputequation="D[y[x],x,x,x]+y[x]*D[y[x],x,x]".
```

The heat equation would be entered as

```
inputequation="D[u[x,t],t]-k*D[u[x,t],x,x]"
```

and so forth. Note that the symbols, **==0**, are not entered.

The next piece of data required by the package is the table of rules that need to be applied to the invariance condition. The rule appropriate to the heat equation would be entered as

```
rulesarray={"D[u[x,t],x,x]->(1/k)*D[u[x,t],t]".
```

Alternatively, the rule could be entered as

```
rulesarray={"D[u[x,t],t]->k*D[u[x,t],x,x]".
```

Either form is suitable however if the intention is to solve for, say, the first order Lie–Bäcklund transformation where the infinitesimals depend on first derivatives, only the first rule may be used. If the second rule is used inadvertently *Mathematica*® will try to replace $u_t$ everywhere in the determining equations when in fact it needs to be treated as an independent variable. Generally, when searching for a point group, **rulesarray** will contain only the elements gotten by rearranging the original system of equations. If one is analyzing, say, the incompressible Navier–Stokes equations then **rulesarray** will contain four items corresponding to the constraints implied by the three momentum and one continuity equations.

The user can express the equation and rules in terms of any symbols permitted by *Mathematica*®. The variable names must be entered as strings. For example, for the heat equation the user would enter **independentvariables={"x","t"}** and **dependentvariables={"u"}**. Internally, the package works entirely in terms of generic variables **xj** and **yi** and one of the first things to occur is that the equation and rules are converted to these variables. So the heat equation would be converted internally to read **D[y1[x1,x2],x2]-k*D[y1[x1,x2],x1,x1]**. The user needs to know this because if there are constant or function names that might be corrupted by the conversion process then they must be protected. For example if the function **Exp** appears in the equation then it will be converted to **Ex1p** causing an error. To prevent this the user must enter a table of names that are to be preserved. In the case of the heat equation one would enter **frozennames={"k"}** although in this case no corruption would occur. Even if there are no names that need to be protected a null table must still be entered, i.e., **frozennames={""}**.

In addition to this data, there are several other input parameters.

**p** – the order of the *highest* derivative in the system of equations being analyzed.

**r** – the order of the highest derivative which the infinitesimals **xsej** and **etai** are assumed to depend on. In the case of Lie point groups the dependence is only on the **xj**'s and **yi**'s and **r** would be set to zero. However for Lie–Bäcklund groups the infinitesimals can depend on derivatives up to whatever order, **r**, is selected by the user.

**xseon** – Setting this parameter to one or any number greater than zero causes the **xsej**'s to be included in the transformation of variables. This is essential for determining Lie point groups (**r=0**) and an option if one is considering Lie–Bäcklund groups (**r=1** or more). In the case of Lie–Bäcklund transformations, one can, without loss of generality, transform only the dependent variables. This point is addressed in the discussion of equivalence classes of transformations in Chapter 14. In this case, **xseon** may be set to zero so that the **xsej**'s are not included in the transformation of variables. The length and complexity of the extended transformations and the invariance condition is greatly reduced in this case.

**internalrules** – In the case of Lie–Bäcklund transformations, the original equation must be supplemented with additional replacement rules corresponding to the differential consequences of the original equation. When **internalrules=1**, a standard set of replacement rules is generated internally by constructing all the derivatives up to order **r** of the original set of input equations. However additional rules can be included at the discretion of the user. This may be useful when looking for certain restricted types of Lie–Bäcklund transformations. The internally generated rules are concatenated to those contained in **`rulesarray`** to produce a new table of rules called **`rulesarrayexpanded`**. If **`internalrules=0`**, this is not done and only the rules in **`rulesarray`** are applied. This way the user has complete flexibility to put in whatever replacement rules may be appropriate to a given problem.

### *A4.2.2 Using the Program*

Once the data for **`independentvariables,dependentvariables, frozennames,p,r,xseon,inputequation,rulesarray`** and **`internalrules`** have been entered, the user calls the function

**`FindDeterminingEquations[independentvariables, dependentvariables,frozennames,p,r,xseon,inputequation,rulesarray,internalrules]`**.

To save space the information can be entered directly into the various slots in the function call. When the process is complete the function returns a message to that effect. The output determining equations of the group are contained in the table of strings

**`determiningequations.`**

This is the primary result of the function **`FindDeterminingEquations`**. However the user may be interested in following the progress of the calculation and so several other arrays are also available to see how the the procedure fared. The y-derivative factors which multiply the determining equations are given in the table of expressions

**`yderivfactortable.`**

The invariance condition in the form of a table of terms is contained in

**`termsoftheinvarianceconditionrulesapplied.`**

In addition, the invariance condition is available as a sum called

**`invarianceconditiontablerulesappliedsum.`**

The invariance condition prior to the application of **`rulesarray`** is also available as the table

**`termsoftheinvarianceconditionnorules.`**

The various partial derivatives of the input equation which appear in (A4.5) are available as the table

**`invarconditiontable`**.

The string expressions contained in the table **`determiningequations`** can be somewhat difficult to read in the usual *Mathematica*® output form with the derivatives and dependencies of each infinitesimal indicated by somewhat lengthy strings of variable names. To simplify the look of the output, each independent variable in the infinitesimals (that is, the **`xj`**'s, **`yi`**'s and derivatives of the **`yi`**'s up to order **`r`**) is replaced by a simpler form; **`x1->z1,x2->z2,y1->z3,y2->z4`**, and so forth. The determining equations expressed in terms of **`z`**-variables are contained in the table of strings

**`zdeterminingequations`**.

The correspondence between conventional variables and **`z`**-variables is contained in

**`ztableofrules`**.

These last two tables are the primary output of the program which generates the determining equations. When analyzing a system of equations the user calls the function **`FindDeterminingEquations`** once for each equation. Before attempting to solve for the unknown infinitesimals, the determining equation tables must be joined into a single long table of string expressions using the *Mathematica*® built-in function **`Join`**.

Finally, I would like to say a word about the variable names that I used in the package. *Mathematica*® places no restriction on the length of a name and in the interest of clarity, sometimes very lengthy variable names are used which contain as much information about the entity being named as possible. The reason for this is simple. In the future, when it might become necessary to modify or upgrade the program or to correct a bug it is essential for me to be able to exactly reconstruct what I was doing; what the motivation was for a particular step and what line of reasoning led to a particular approach. The long, but descriptive, variable names will make this much easier for me. So, for example, **`termsoftheinvarianceconditionrulesapplied`** denotes a table containing the terms of the invariance condition after the application of the rules contained in **`rulesarray`** or **`rulesarrayexpanded`**. For the same reason every single line in the package is commented.

Generally speaking, the function **FindDeterminingEquations** works extremely well. Even rather lengthy systems of equations with fairly high order derivatives can be analyzed on a machine with fairly modest memory capacity very quickly. In fact the input equation does not have to be a differential equation; it can be merely a function of independent variables **(p=0)** and the routine will generate the appropriate (not very useful) invariance condition.

### *A4.2.3 Solving the Determining Equations and Viewing the Results*

A solution for the infinitesimals in the form of power series can be attempted by calling the function
**SolveDeterminingEquations[independentvariables,dependentvariables,r,xseon,zdeterminingequations,order].**

When the process is complete the function returns a message that it has finished executing. The parameter **order** is used to select the order of the multivariable polynomial used in the expansion. The polynomials are of the form

```
xsej[z1,z2,z3,...,zq]=
        aj0+aj1*z1+·· ajq*zq+aj(q+1)*z1^2+···
etai[z1,z2,z3,...,zq]=
        bi0+bi1*z1+···biq * zq+bi(q+1)*z1^2+····.
```

Recall that the variable count for a given $n$, $m$ and $r$ is

$$q(n,m,r) = n + m\sum_{k=0}^{r}\frac{(n+k-1)!}{k!(n-1)!}. \tag{A4.9}$$

One approach is to call **SolveDeterminingEquations** successively with increasing orders. The procedure is stopped when the infinitesimals stop changing. If after a reasonable number of steps, any of the expressions continues to grow longer, one should consider whether the infinitesimal in question might admit an arbitrary function. The results of the solver are available in two forms. The tables

**xsefunctions**

and

**etafunctions**

contain lists of strings for the infinitesimals in the form of polynomial expressions with the group parameters as coefficients. In addition the table

**`infinitesimalgroups`**

contains a list of the individual groups with the group parameters stripped away. The expressions in these three tables are given in terms of **`zvariables`** and are ordered according to the usual ordering of independent and dependent variables. Usually the user will want to look at the list of groups by converting the **`zvariables`** to ordinary variables using the command

```
infinitesimalgroupsxy = infinitesimalgroups/.{z1->
x,z2->t,...}
```

The best way to view the results is to present the list **`infinitesimalgroupsxy`** as a column using the *Mathematica*® built-in function **`ColumnForm`**.

The package function

```
MakeCommutatorTable[independentvariables,dependent
variables,infinitesimalgroupsxy]
```

generates the commutator table of the group from the list **`infinitesimalgroupsxy`**. The table is contained in **`commutatortable`** and is best displayed as a matrix using **`MatrixForm[commutatortable].`**

## A4.3 Timing, Memory and Saving Intermediate Data

The function **`SolveDeterminingEquations`** does not always find the complete solution of the determining equations for the reasons described above. Furthermore for a complicated equation or system of equations it tends to be slow and hog a lot of memory. The problem is with the *Mathematica*® built-in function **`Solve`** which, at a certain point in the program, uses Gaussian elimination to symbolically solve what may be a very large algebraic system for the coefficients of the infinitesimals, most of which are zero. For an expansion of order three or more the number of coefficients begins to get quite large and, even though the equations for the coefficients are linear and quite simple there is a fairly large number of them and **`Solve`** tends to bog down rather badly. To help the user address this, the *Mathematica* built-in functions **`Timing`** and **`MaxMemoryUsed`** are used to inform the user of the time and memory requirements of each sample notebook.

When the function **`FindDeterminingEquations[...]`** is called I usually imbed it in the timing function. Thus the usual command is **`Timing[FindDeterminingEquations[...]]`** so that at the end of execution, the time in seconds required to execute the function is output to the notebook.

For most simple input equations the time required to find the determining equations is usually only a few seconds on a modern desktop machine. For a complex equation that may be one of a system of equations such as one component of the 3-D Navier Stokes equations the time may be several minutes.

Searching for Lie–Bäcklund symmetries of a system of equations is another matter. It is not difficult to define a problem where **`FindDetermining-Equations`** can take a very long time and use enormous amounts of memory. The infinitesimals of Lie–Bäcklund symmetries depend on derivatives of the dependent variables and if the number of dependent variables is large then the number of variables on which the infinitesimals depend can be quite large. For example, the four infinitesimals of the first order Lie–Bäcklund symmetry of the incompressible, unsteady, 3-D Navier–Stokes equations depend on 4 independent variables, 4 dependent variables and 16 first derivatives; twenty four in all! Generating the determining equations can take 4–5 hours per equation.

Therefore it is important to realize that the user need not generate all the results in a single run of the notebook. Results can be written to the hard disc by the *Mathematica*® built-in function **`Save`**. For example, the command

```
Save["zdeterminingequations1file", zdeterminingeq-
uations]
```

takes the table of determining equations called **`zdeterminingequations`** and saves it on the hard disc in a file called **`zdeterminingequations1-file`**. The file is located in the *Mathematica*® folder. Intermediate results that take a long time to generate can be saved and the notebook can be closed. At a later date when the notebook is reopened the table **`zdetermininge-quations`** can be called back into memory using the command **`<<zdeter-miningequations1file.`** The notebook can then pick up where it left off without having to re-execute the function **`FindDeterminingEquations`** that generated **`zdeterminingequations`** previously. This way the results of a long calculation can be generated and saved in several sessions.

Similarly, when the function **`SolveDeterminingEquations[...]`** is called I generally imbed it in the timing function. The call is **`Timing[Solve DeterminingEquations[...]]`** so that at the end of execution the time in seconds needed to solve the determining equations is output to the notebook. The package function **`SolveDeterminingEquations`** substitutes a multivariable polynomial expansion of the infinitesimals into the determining equations and various products of the dependent and independent variables are then gathered together. The factors multiplying each product are set to zero forming a linear system of equations for the polynomial coefficients. The function **`SolveDeterminingEquations`** then calls the *Mathematica*®

built-in function **Solve** which uses Gaussian elimination to symbolically determine the polynomial coefficients (most of which are usually zero). In the end, the nonzero coefficients are expressed in terms of a small subset of the original polynomial coefficients and these remaining coefficients become the set of group parameters of the infinitesimal transformation. If the number of linear equations for the polynomial coefficients is large then the process of symbolic solution via Gaussian elimination can take a long time. To help the user estimate the time required, **SolveDeterminingEquations** outputs to the notebook the number of unknowns being solved for and the (generally larger) number of equations being solved before **Solve** begins executing. If it looks like the time is going to be excessive the user can interrupt the calculation. The time required to execute can be roughly estimated from

$$\frac{T}{T_{ref}} = \left(\frac{\textit{Number of equations}}{\textit{Number of equations-ref}}\right)^n \tag{A4.10}$$

where the exponent $n$ is between 2.4 and 2.7. Running **SolveDetermining-Equations** for low order polynomials, say 1 and 2, provides good reference data for the execution time, $T_{ref}$, for the number of equations being solved *Number of equations-ref*, and for estimating the exponent. This way the user can estimate whether the time required to run **SolveDeterminingEquations** is measured in minutes, hours or days. If the time is excessive then the best strategy is to use built-in *Mathematica*® functions to manually reduce the number of determining equations by using some of the simpler, typically one-term, equations as rules applied to the rest. A few iterations of this process can usually reduce the set of determining equations to a manageable size.

Memory requirements also grow rapidly as the polynomial order increases. The user should appreciate that it is quite easy to pose a problem for **SolveDeterminingEquations** that will bring even the largest, fastest computer to its knees. This is particularly true when one is trying to find Lie–Backlund transformations that depend on derivatives of order two or more. As noted above, the number of variables that the infinitesimals are assumed to depend on grows rapidly with the derivative order. In addition, rather high order polynomials are required to capture high order Lie–Backlund groups. For example, searching for 3rd derivative order Lie–Backlund transformations of the Blasius equation (a modest third-order ODE with two unknown infinitesimals) expressed as 5th order polynomials requires the solution of 730 equations. Searching for third order Lie–Backlund transformations of the Burgers potential equation, a PDE with three unknown infinitesimals expressed as 5th order polynomials, requires the solution of 3850 equations.

The package is fully capable of finding high order Lie–Bäcklund symmetries of a system of equations but this requires very substantial supercomputing resources to execute.

### *A4.3.1 Why Give the Output in the Form of Strings?*

Any of the tables of strings generated by the program can be immediately converted to expressions by using the built-in *Mathematica*® function **ToExpression**. The reason for presenting the determining equations and infinitesimal functions as strings is to provide the user with an essentially immutable form of the results. Any expressions created from these strings become active and *Mathematica*® will immediately evaluate **xsej** and **etai** wherever they appear. For the same reason the **xsej** and **etai** symbols are cleared at the beginning of the functions **FindDeterminingEquations** and **SolveDeterminingEquations**. This avoids any name conflicts which might occur if the user decides to call these functions more than once.

### *A4.3.2 Summary of Program Functions*

**GenerateVariableTable[numberindependentvars,number dependentvars,porderoftheequation]**
This function generates a table of all the various variables in $[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_p]$. The output is a table of strings contained in **variablestringtable.** The indices of the various y-derivatives which appear in the table are also available to the user as a table called **derivativeindextable.**

**SecondTerm[numberindependentvars,numberdependentvars,yindex,plocal,rorderofinfinitesimals, j1,j2,j3, j4,j5,j6,j7,j8,j9,j10,j11,j12,j13,j14]**
This function generates the second term in the *p*th order infinitesimal transformation function (See equation (A4.6)). The parameters **j1** to **j14** are the indices of the partial derivative being transformed. Derivatives up to 14th order are permitted by the program. The function is defined so that the indices **j2** to **j14** are optional arguements. Indices which are not explicitely defined are automatically set to zero. The limitation to 14th order is due to the fact that *Mathematica*® apparently will not allow more than 13 optional arguements. The fact is, that any circa 2000 desktop computer will be quickly overwhelmed for derivatives above 5th or 6th order.

**PthInfinitesimal[numberindependentvars,numberdependentvars,yindex,porderoftheequation,rorderofinfinitesimals,xseon, j1,j2,j3,j4,j5,j6,j7,j8,j9,j10,j11, j12,j13,j14]**

This nested function generates the *p*th order extended infinitesimal transformation expression (See (A4.6). In the process it repeatedly calls the function **SecondTerm** however many times is required to reach the required order. For high order derivatives and/or many variables the result can be extremely long. This is the most time consuming step in the function **FindDetermining-Equations** although it will typically run in a few seconds on a reasonably fast desktop machine. It is, of course, precisely the calculus we want to avoid having to carry out by hand.

**GenerateInfinitesimalTable[numberindependentvars, numberdependentvars,porderoftheequation,rorderofinf-initesimals,xseon]**

This function generates a table of strings with the same indices as **generate-variabletable**. Each element in the table is a call to **PthInfinitesi-mal** which will be evaluated once it is determined which extended infinitesimals will be needed based on the structure of the **inputequation.**

**InvarianceConditionNoRules[numberindependentvars, numberdependentvars,porderoftheequation,rorderofinf-initesimals,xseon,inputequation]**

This function generates a table of expressions for the terms in the invariance condition before any transformation rules have been applied. The result is contained in the table **termsoftheinvarianceconditionnorules.**

**InvarianceConditionRulesApplied[numberindependent-vars,numberdependentvars,porderoftheequation,rorder-ofinfinitesimals,xseon, inputequation,rulesarray,in-ternalrules]**

This function generates a table of expressions for the terms in the invariance condition after the transformation rules have been applied. The result is contained in **termsoftheinvarianceconditionrulesapplied.** In addition, the invariance condition is available in the form of a sum called **invari-anceconditiontablerulesappliedsum.**

**MakeRulesArray[numberindependentvars,numberdepend-entvars, mequindexsep98,rorderofinfinitesimals, rul-esarray]**

This function generates the supplementary array of rules which are applied to the invariance condition in the case of a Lie–Bäcklund transformation. The rules are standard in the sense that they are simply all the possible derivatives of the input equation up to order **r**. The final set of rules, concatenated with those input in **rulesarray**, are contained in the table **rulesarrayexpanded.** If **internal=1** then **rulesarrayexpanded** is applied to **finalinvari-anceconditionsumnorules.** Whereas if **internal=0** then only the rules in **rulesarray** are applied.

**FindDeterminingEquations[independentvariables,dependentvariables,frozenstrings,porderoftheequation,rorderofinfinitesimals,xseon, inputequation,rulesarray,internalrules]**
This is the main function of the *Mathematica*® package **IntroToSymmetry.m.** The results are contained in the table of strings **determiningequations** which contains the determining equations of the group. For convenience, the variables upon which the infinitesimals depend are converted to a set of dummy variables,i.e., **x1->z1, x2->z2, y1->z3,** etc. This result is contained in **zdeterminingequations** and the correspondence between variables is contained in **ztableofrules**. These tables are the main output of the package. The derivatives of the input equation which appear in the invariance condition (See (A4.4)and (A4.5)) are also available as a table **invarconditiontable.** All these various outputs allow the user to follow the progress of the calculation in considerable detail.

**SolveDeterminingEquations[independentvariables,dependentvariables,rorderofinfinitesimals,xseon,zdeterminingequationstable,order]**

This function is used to attempt a solution of the determining equations in the form of a multivariable polynomial. The parameter **order** is used to select the order of the polynomial used in the expansion. The results of the solver are contained in the tables **xsefunctions** and **etafunctions**. The strings in these two tables are given in terms of **zvariables** and ordered according to the usual ordering of independent and dependent variables.

**MakeCommutatorTable[independentvariables,dependentvariables,infinitesimalgroupsxy]**

This function is used to generate the commutator table of the Lie algebra of the groups contained in **infinitesimalgroupsxy.** The symbols used for **independentvariables** and **dependentvariables** must correspond to the symbols in **infinitesimalgroupsxy.** The output is contained in the table **commutatortable**. This is the last function in the package.

All function names in the package are protected using the *Mathematica*® **Protect** function.

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