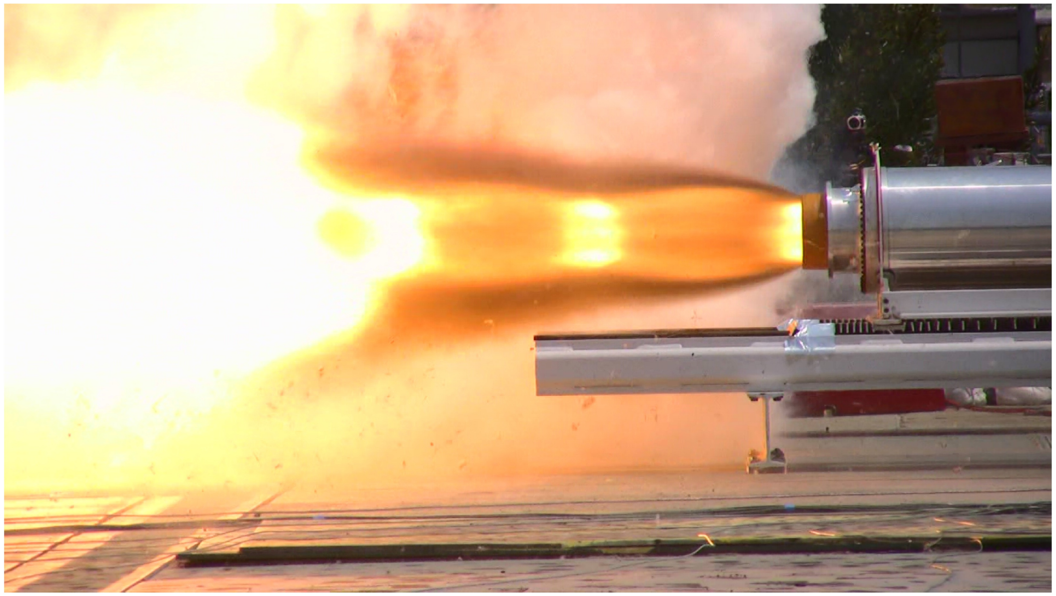
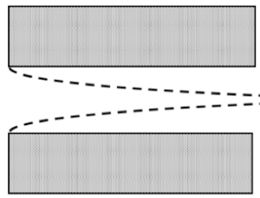
Coupling Between Fuel Mass Transfer and Free-Stream Mass Flow in Hybrid Combustion

Brian Cantwell
Stanford University



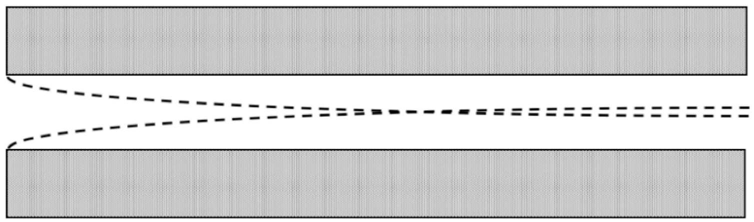
Presentation to the Hybrid Rocket Technical Committee, 50th AIAA/ASME/SAE/ASEE Joint Propulsion Conference & Exhibit, July 29, 2014

The basic hybrid design problem



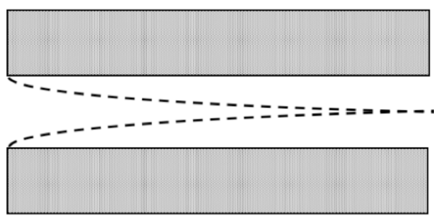
Port too short

Too much unburned oxidizer



Port too long

Too much unburned fuel



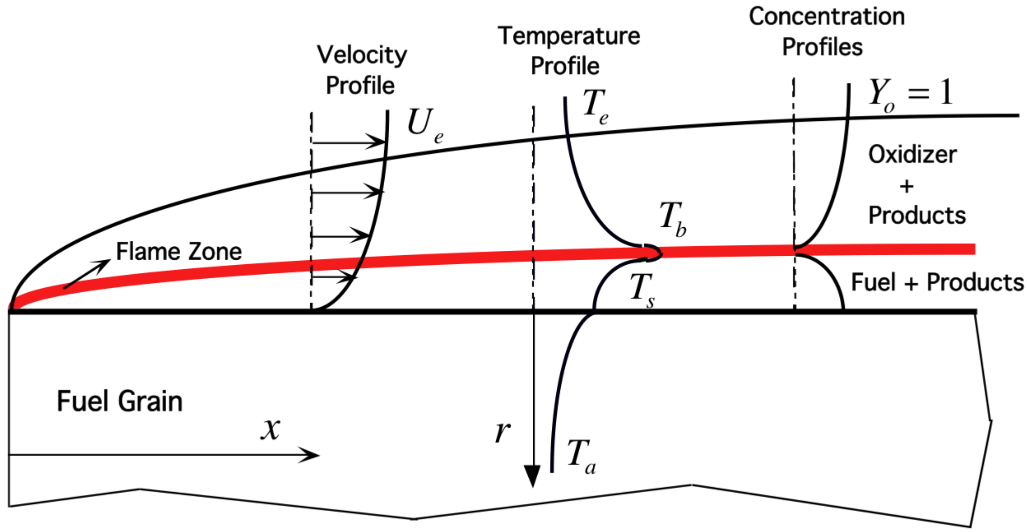
Port length just right

O/F ratio at the end of the port is just right

<u>Problem</u>
The desired O/F ratio varies considerably depending on the choice of oxidizer. But the growth rate of the combustion layer is relatively independent of O/F. The L/D ratio of the port should be about 6 to 10.

<u>The Solution</u>
Mix fuels to produce a regression rate tailored to the desired motor size and O/F.

Port design requires accurate regression rate data and analysis



Burning rate law:

$$\dot{r} = aG^n / x^m$$

where G is the port mass flux $$G = \frac{\left(\dot{m}_{ox} + \dot{m}_f(x,t)\right)}{\pi\ r(x,t)^2}$$

Typically m is small and $0.4 < n < 0.7$. Marxman suggested $m=0.2$, $n=0.8$

Simplified approach

A simplification that is often used is assume the regression rate only depends on oxidizer mass flux

$$\dot{r} = aG_{ox}^{n} \qquad G_{ox} = \frac{\dot{m}_{ox}}{\pi r^2}$$

Substitute and separate variables.

$$\frac{\pi^n}{a} r^{2n} dr = \left(\dot{m}_{ox}(t)\right)^n dt$$

Integrate

$$r(t) = \left(r(0)^{2n+1} + \frac{a(2n+1)}{\pi^n} \int_0^t \left(\dot{m}_{ox}(t')\right)^n dt' \right)^{\frac{1}{2n+1}}$$

In this approximation the radius is constant along the port. This assumption underpredicts the fuel generation rate and is not an accurate predictor of the O/F ratio at the end of the port especially for low O/F ratios.

STANFORD
AERONAUTICS & ASTRONAUTICS

Solve the nonlinear coupled mass-flow-regression-rate problem

The regression rate equation is

$$\frac{\partial r(x,t)}{\partial t} = a\frac{\dot{m}^n_{port}}{\pi^n x^m r^{2n}} \qquad \dot{m}_{port} = \dot{m}_{ox} + \dot{m}_f$$

The mass flow rate increase along the port is determined by the rate at which mass is swept up from the fuel surface.

$$\frac{\partial \dot{m}_{port}(x,t)}{\partial x} = 2\pi^{1-n} a \rho_f \frac{\dot{m}^n_{port}}{x^m r^{2n-1}}$$

These first order PDEs need to be solved simultaneously for the local mass flow rate and port radius.

STANFORD
AERONAUTICS & ASTRONAUTICS

The case *n=1/2*

The coupled problem can be solved exactly for the case $n = 1/2$.

$$\dot{m}_{port}(x,t) = \left(\dot{m}_{ox}(t)^{1/2} + \frac{\pi^{1/2} a \rho_f x^{1-m}}{1-m} \right)^2$$

$$r(x,t) = \left(r(0,t)^2 + \frac{2\pi^{-1/2} a}{x^m} \left(\int_0^t \dot{m}_{ox}(t')^{1/2} dt' + \frac{\pi^{1/2} a \rho_f x^{1-m} t}{1-m} \right) \right)^{1/2}$$

The increase in port surface area exactly balances the decrease in mass flux and so the mass flow rate is constant for constant oxidizer mass flow rate. The O/F at the end of the port is constant.

Numerical solution of the coupled problem

Karabeyoglu, M.A., Cantwell, B.J. and Zilliac, G., Development of Scalable Space-Time Averaged Regression Rate Expressions For Hybrid Rockets, 41st AIAA/ASME/SAE/ASEE Joint Propulsion Conference & Exhibit AIAA 2005-3544 10 - 13 July 2005, Tucson, Arizona

Table 1: Summary of motor test data used to evaluate the length exponent for the paraffin/GOX system.

Test	4L-04	4L-05	4L-08	4P-01
Oxidizer Mass Flow Rate, kg/sec	4.44	4.43	4.42	4.43
Burn Time, sec	8.30	8.25	8.15	8.40
Average O/F	2.66	2.72	2.64	2.69
Initial Port Diameter, cm	8.93	10.01	10.30	11.38
Grain Length, m	1.149	1.149	1.148	1.148
Final Port Diameter (Fore), cm	15.53	15.88	16.17	16.67
Final Port Diameter (Aft), cm	16.23	16.63	16.64	17.54
% Change in the Port Diameter	4.34	4.54	2.84	4.99
Flux Exponent, *n*	0.62	0.62	0.62	0.62
Length Exponent, *m*	-0.018	-0.009	-0.033	0.000
Regression Rate Coefficient, a^b	$9.36\ 10^{-2}$	$9.24\ 10^{-2}$	$9.10\ 10^{-2}$	$9.36\ 10^{-2}$

[b]: Note that the units of the regression rate coefficient are based on mm/sec for the regression rate, meters for the length and kg/m^2-sec for the mass flux.

Initial and final port geometry data from four tests were used to estimate *a, n* and *m* in the full space-time coupled problem.

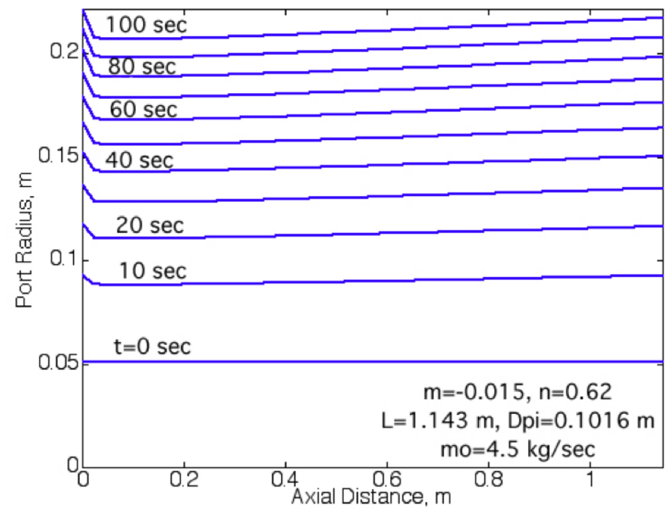


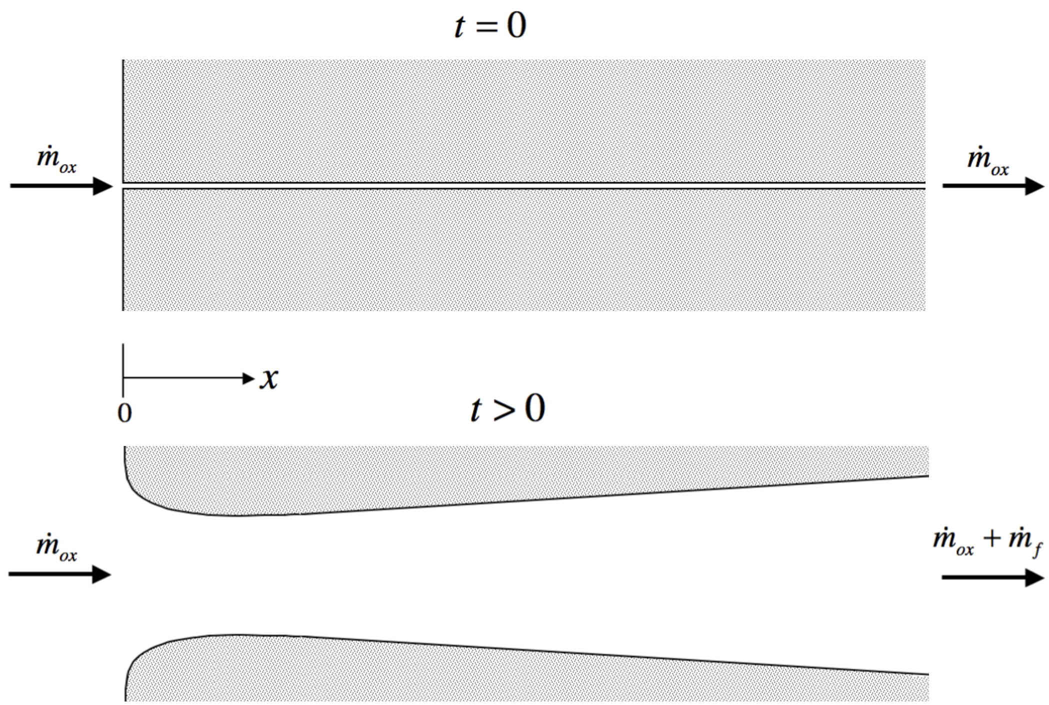
Figure 11: Port diameters contours calculated at various times for *m= -0.015*, *n=0.62* and $a = 0.0927$. Oxidizer mass flow rate is constant at 4.5 kg/sec, grain initial diameter is 0.106 m and grain length is 1.143 m.

Numerical solution captures the "coning" effect due to the mass flow increase in the port as well as the port minimum due to a nonzero *m* exponent on *x*.

Similarity solution

For constant oxidizer mass flow rate the coupled equations admit a similarity solution. Importantly this solution is relevant to the design of a working system.

Let the fuel be a semi-infinite block filling the right half plane in 3-D. At $t = 0$ a finite oxidizer mass flow rate is initiated along the x-axis. At first the mass flux is infinite but as the port opens up the mass flux drops to reasonable values in the range of interest to a designer.



Cantwell, B. J. , Similarity solution of fuel mass transfer, port mass flux coupling in hybrid propulsion, J. Engr. Math. Vol 84 issue 1, Feb 2014. Special issue in rememberance of Milton Van Dyke.

STANFORD AERONAUTICS & ASTRONAUTICS

Governing parameters

Nondimensionalize variables. Scales can be defined in terms of the basic parameters of the problem.

$$\left[2\pi\rho_f\right] = Mass\,/\,Length^3$$

$$\left[\dot{m}_{ox}\right] = Mass\,/\,Time$$

$$\left[\pi^{-n}a\right] = Mass^{-n}Length^{2n+m+1}Time^{n-1}$$

Solve for characteristic length, time and mass.

$$T_{ch} = \left[\frac{\left(\dot{m}_{ox}\right)^{\frac{1-n+m}{2-2n-m}}}{\left(\pi^{-n}a\right)^{\frac{3}{2-2n-m}}\left(2\pi\rho_f\right)^{\frac{1+2n+m}{2-2n-m}}}\right]$$

$$L_{ch} = \left[\frac{\left(\dot{m}_{ox}\right)^{\frac{1-n}{2-2n-m}}}{\left(\pi^{-n}a\right)^{\frac{1}{2-2n-m}}\left(2\pi\rho_f\right)^{\frac{1}{2-2n-m}}}\right]$$

$$M_{ch} = \left[\frac{\left(\dot{m}_{ox}\right)^{\frac{3-3n}{2-2n-m}}}{\left(\pi^{-n}a\right)^{\frac{3}{2-2n-m}}\left(2\pi\rho_f\right)^{\frac{1+2n+m}{2-2n-m}}}\right]$$

Dimensionless variables and equations

Define

$$\chi = \frac{x}{L_{ch}} \quad \tau = \frac{t}{T_{ch}} \quad R = \frac{r}{L_{ch}} \quad J = \frac{\dot{m}_{port}}{\dot{m}_{ox}}$$

Note

$$O/F = \frac{\dot{m}_{ox}}{\dot{m}_f} = \frac{1}{J-1}$$

Dimensionless governing equations

$$\frac{\partial R(\chi,\tau)}{\partial \tau} = \frac{J^n}{\chi^m R^{2n}}$$

$$\frac{\partial J(\chi,\tau)}{\partial \chi} = \frac{J^n}{\chi^m R^{2n-1}}$$

The boundary conditions of the semi-infinite problem defined earlier are

$$R(\chi,0) = 0 \quad J(0,\tau) = 1$$

Group operators

These equations admit a three dimensional Lie algebra

$$X^a = \frac{\partial}{\partial \tau}$$

$$X^b = \left(\frac{1-n}{1-m}\right)\chi\frac{\partial}{\partial \chi} - \left(\frac{n-m}{1-m}\right)\tau\frac{\partial}{\partial \tau} + J\frac{\partial}{\partial J}$$

$$X^c = \chi\frac{\partial}{\partial \chi} + \left(\frac{2n-2m+1}{2n-1}\right)\tau\frac{\partial}{\partial \tau} + \left(\frac{1-m}{2n-1}\right)R\frac{\partial}{\partial R}$$

The group

$$X^c = \chi\frac{\partial}{\partial \chi} + \left(\frac{2n-2m+1}{2n-1}\right)\tau\frac{\partial}{\partial \tau} + \left(\frac{1-m}{2n-1}\right)R\frac{\partial}{\partial R}$$

holds invariant the constant oxidizer mass flow rate in the port

Similarity variables

The group X^c is used to construct similarity variables.

$$\frac{d\chi}{\chi} = \frac{d\tau}{\left(\frac{2n-2m+1}{2n-1}\right)\tau} = \frac{dR}{\left(\frac{1-m}{2n-1}\right)R} = \frac{dJ}{0}$$

$$\theta = \frac{\chi}{\tau^{\left(\frac{2n-1}{2n-2m+1}\right)}} \qquad K(\theta) = \frac{R}{\chi^{\left(\frac{1-m}{2n-1}\right)}} \qquad J(\theta) = J$$

The governing equations reduce to a pair of coupled ODEs

$$\frac{dK}{d\theta} = -\left(\frac{2n-2m+1}{2n-1}\right)\frac{J^n}{\theta^{\left(\frac{4n-2m}{2n-1}\right)}K^{2n}}$$

$$\frac{dJ}{d\theta} = \frac{J^n}{\theta K^{2n-1}}$$

$$K(\infty) = 0 \qquad J(0) = 1$$

Further reduction to an autonomous pair of ODEs

This system of ODEs admits a dilation group.

$$\tilde{\theta} = e^{a}\theta$$

$$\tilde{K} = e^{-(1-n)\left(\frac{2n-2m+1}{2n-1}\right)a}K$$

$$\tilde{J} = e^{(2n-2m+1)a}J$$

Construct new similarity variables.

$$\frac{d\theta}{\theta} = \frac{dK}{-(1-n)\left(\dfrac{2n-2m+1}{2n-1}\right)K} = \frac{dJ}{(2n-2m+1)J}$$

$$u(\theta) = \frac{1}{\theta^{(1-n)\left(\frac{2n-2m+1}{2n-1}\right)}K(\theta)} \qquad v(\theta) = \frac{\theta^{(2n-2m+1)}}{J(\theta)}$$

Finally the problem reduces to an autonomous system

$$\frac{du}{d\alpha} = \left(\frac{1-n}{2n-1}\right)\left(\frac{1}{(1-n)}u^{2n+2}v^{-n} - u\right)$$

$$\frac{dv}{d\alpha} = \left(v - \frac{1}{(2n-2m+1)}v^{2-n}u^{2n-1}\right)$$

$$\alpha = (2n-2m+1)Log(\theta)$$

Critical points

The system has two critical points at

$$(u_c, v_c) = (0,0)$$

and

$$(u_c, v_c) = \left((2n-2m+1)^n (1-n)^{1-n}, \frac{(2n-2m+1)^{2n+1}}{(1-n)^{(2n-1)}} \right)$$

Normalize by the coordinates of the nonzero critical point

$$U = \frac{u}{u_c} \qquad V = \frac{v}{v_c}$$

Finally

$$\frac{dU}{d\alpha} = \left(\frac{1-n}{2n-1}\right) U \left(\frac{U^{2n+1}}{V^n} - 1\right)$$

$$\frac{dV}{d\alpha} = V\left(1 - \left(\frac{V}{U^2}\right)\frac{U^{2n+1}}{V^n}\right)$$

Notice the absence of m

Connection back to R and J

$$\theta = \frac{\chi}{\tau^{\left(\frac{2n-1}{2n-2m+1}\right)}}$$

$$R = \frac{\chi^{\left(\frac{1-m}{2n-1}\right)}}{(2n-2m+1)^{n}(1-n)^{1-n}\,\theta^{(1-n)\left(\frac{2n-2m+1}{2n-1}\right)}U(\theta)}$$

$$J = \frac{(1-n)^{(2n-1)}\,\theta^{(2n-2m+1)}}{(2n-2m+1)^{2n+1}V(\theta)}$$

The preservation of $J(0,\tau)=1$ is accomplished through the asymptotic behavior of $V(\theta)$

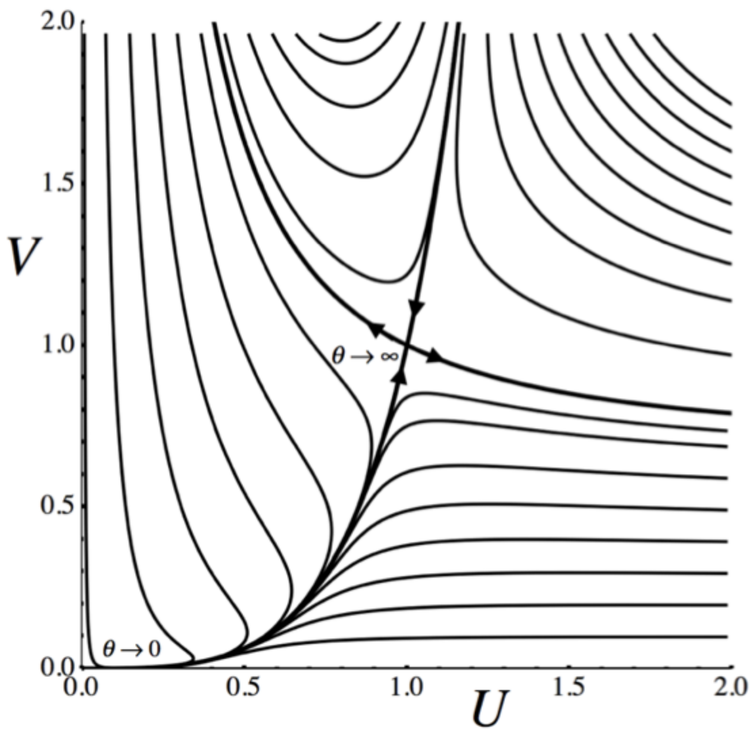
STANFORD
AERONAUTICS & ASTRONAUTICS

The problem boils down to the solution of a single first order ODE

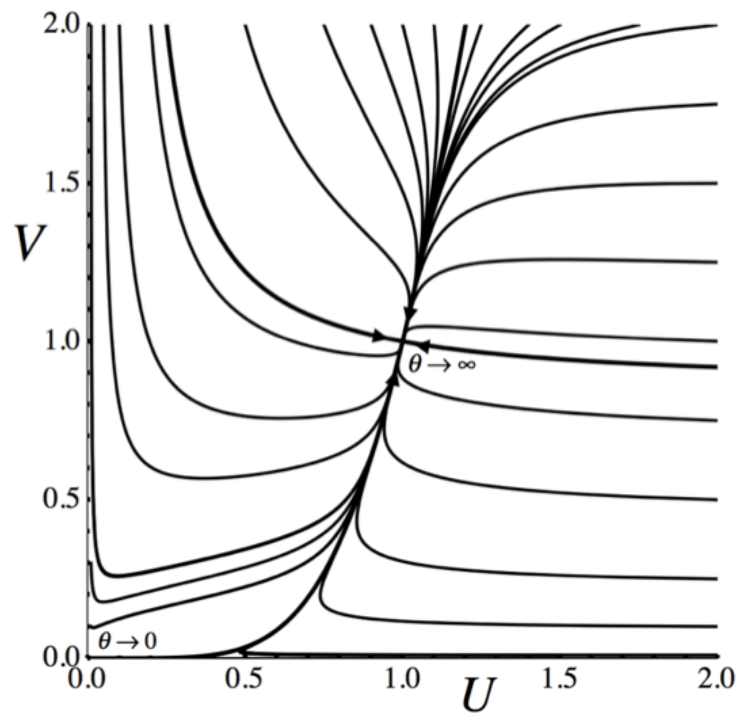
$$\frac{dV}{dU}=\left(\frac{2n-1}{1-n}\right)\frac{V}{U}\frac{\left(1-\left(\frac{V}{U^2}\right)\frac{U^{2n+1}}{V^n}\right)}{\left(\frac{U^{2n+1}}{V^n}-1\right)}$$

Phase portrait

$n > 1/2$



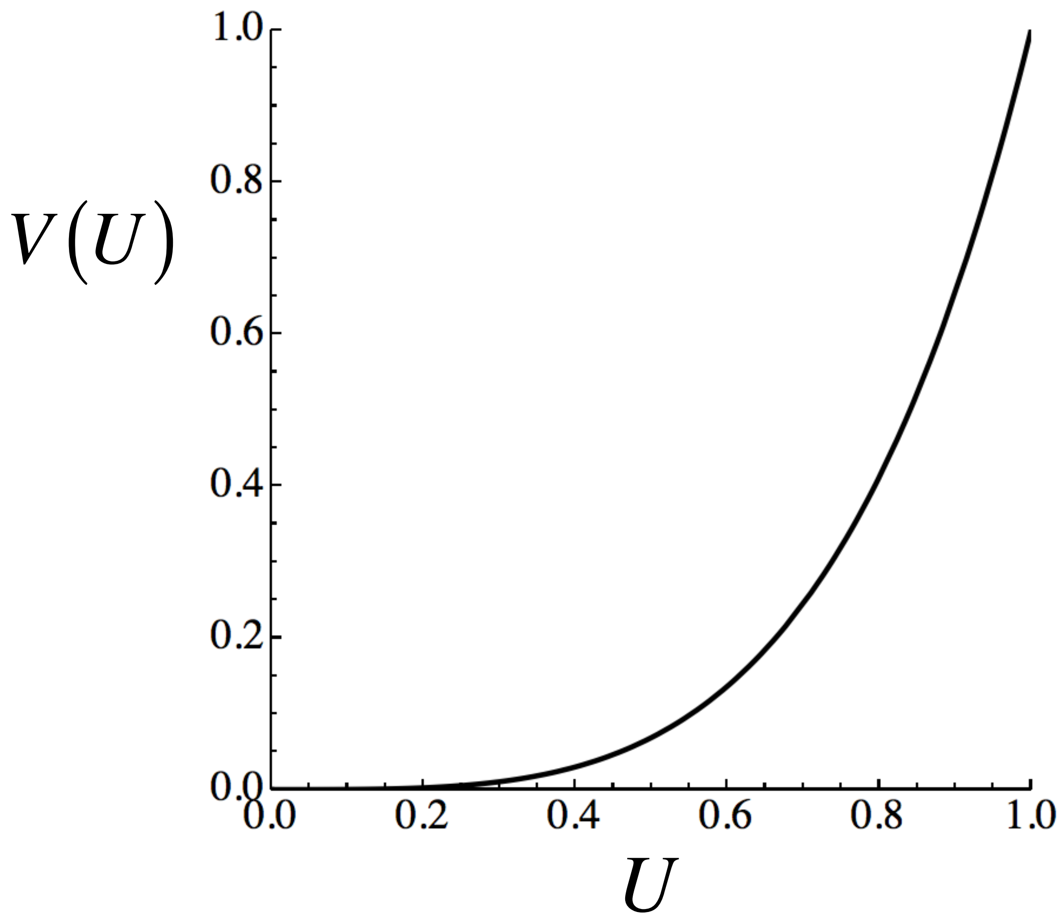
$n < 1/2$



One trajectory in the phase portrait preserves the boundary condition $J(0,\tau)=1$.

Along this trajectory $\lim\limits_{\theta \to 0} V(\theta) = (1-n)^{2n-1}\,\theta^{2n-2m+1} / (2n-2m+1)^{2n+1}$.

Solution V versus U for $n=0.62$



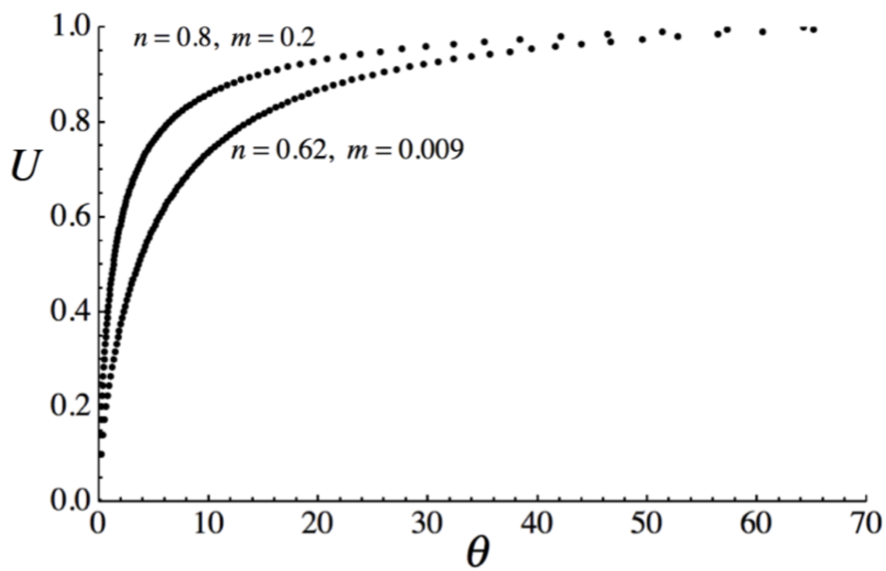
Recall

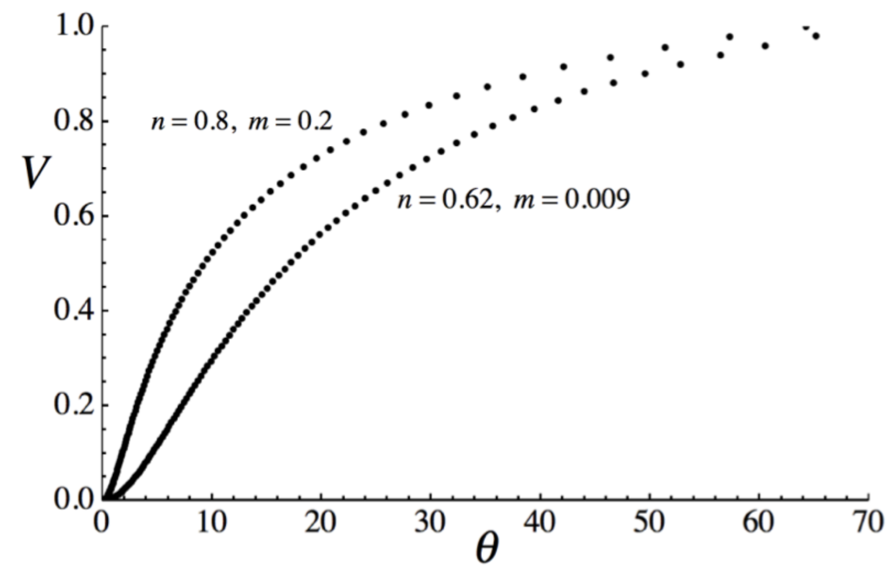
$$V \sim \frac{1}{R}$$

$$U \sim \frac{1}{J}$$

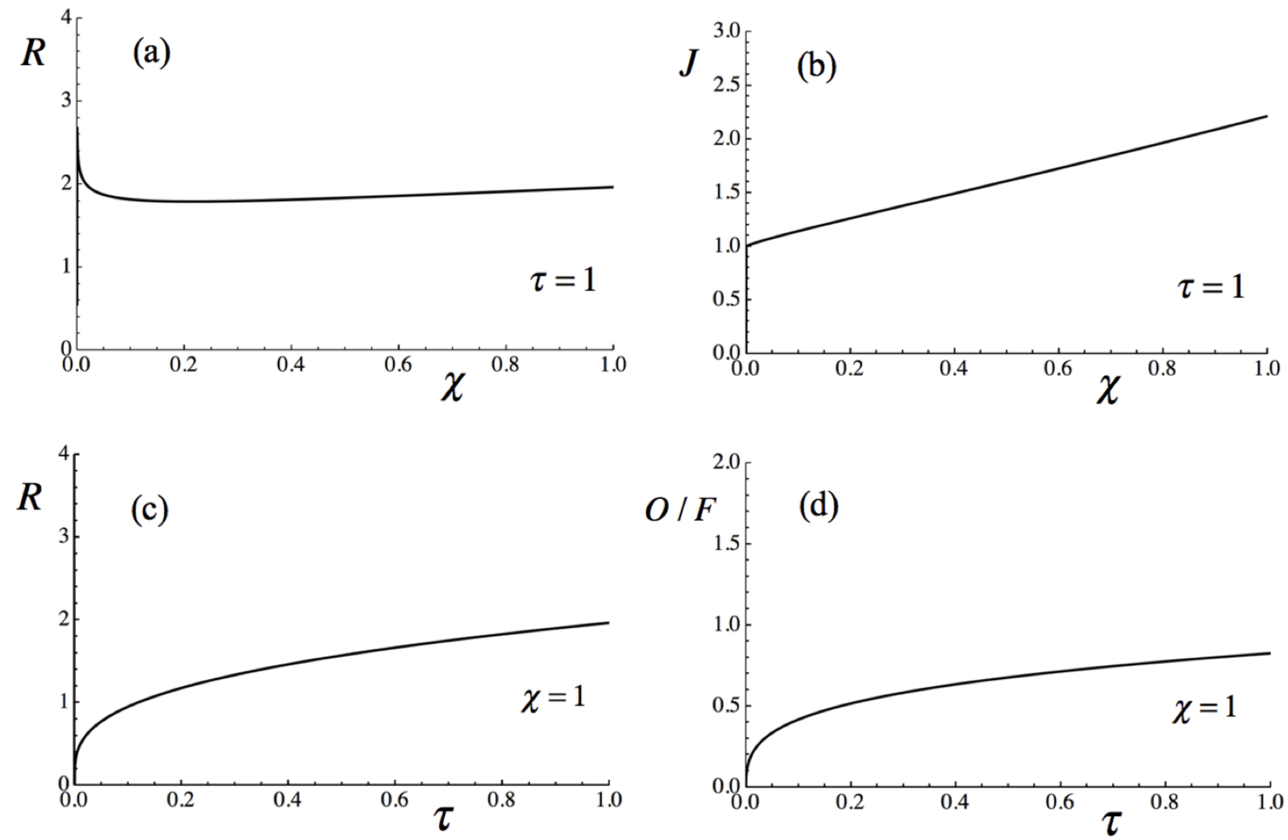
Solution in terms of θ

Numerical approximation to $U(\theta)$ and $V(\theta)$

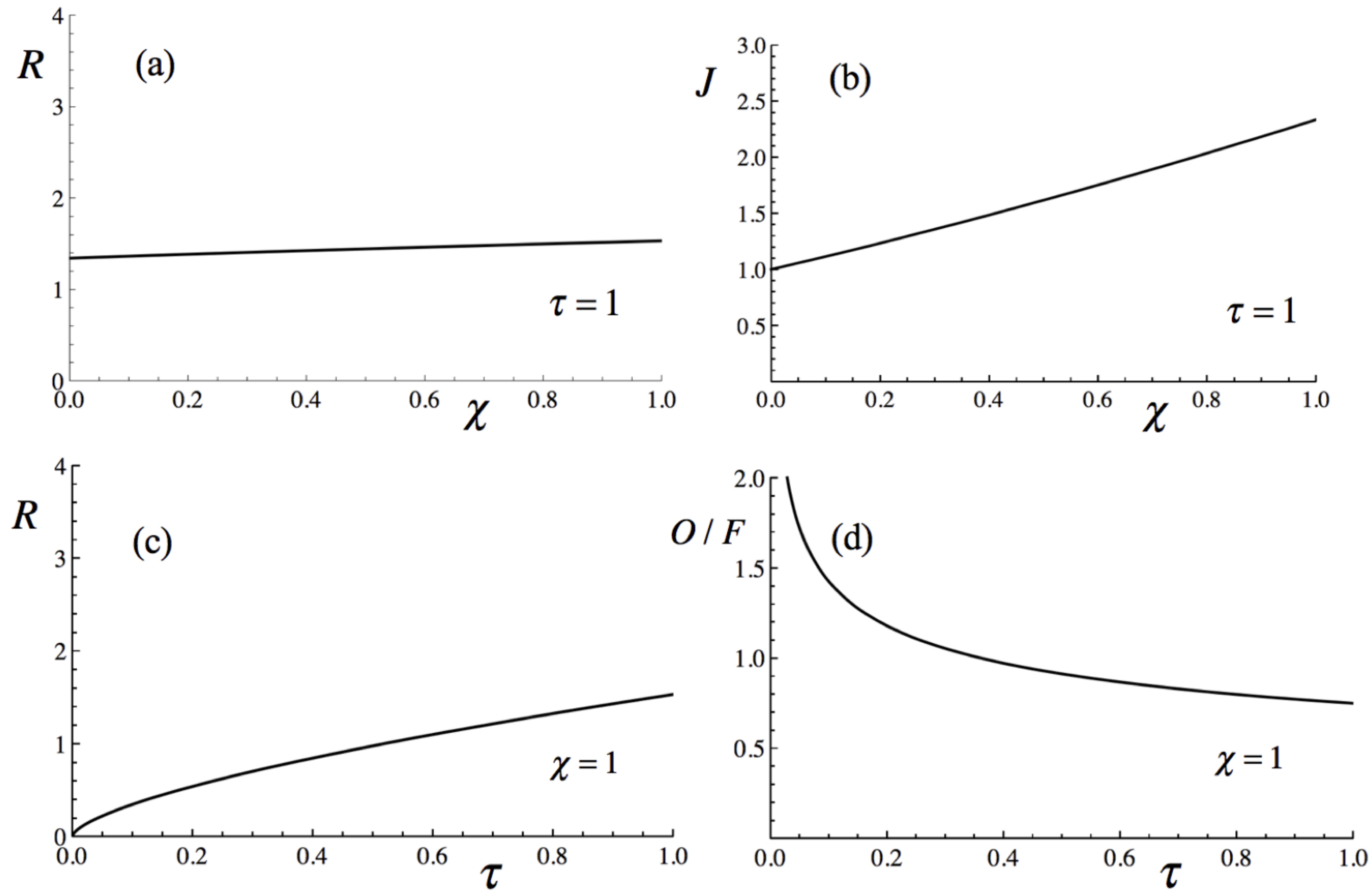




Dimensionless radius and mass flow solutions. $n > 1/2$



Dimensionless radius and mass flow solutions. $n < 1/2$



Parameters of some typical oxidizer/fuel combinations

Fuel/Oxidizer	Paraffin/O_2	HDPE/O_2	HDPE/N_2O
$\rho_f \times 10^{-3}$	0.920	0.941	0.941
n	0.62	0.498	0.331
m	0.009	0.0	0.0
$a \times 10^5$	9.240	4.193	11.573
$T_{ch} / \left(\dot{m}_{ox}\right)^{\frac{1-n+m}{2-2n-m}} \times 10^{-6}$	1.206	2.095	0.03237
$L_{ch} / \left(\dot{m}_{ox}\right)^{\frac{1-n}{2-2n-m}}$	5.931	7.077	1.763
$M_{ch} / \left(\dot{m}_{ox}\right)^{\frac{3-3n}{2-2n-m}} \times 10^{-6}$	1.206	2.095	0.03237

STANFORD AERONAUTICS & ASTRONAUTICS

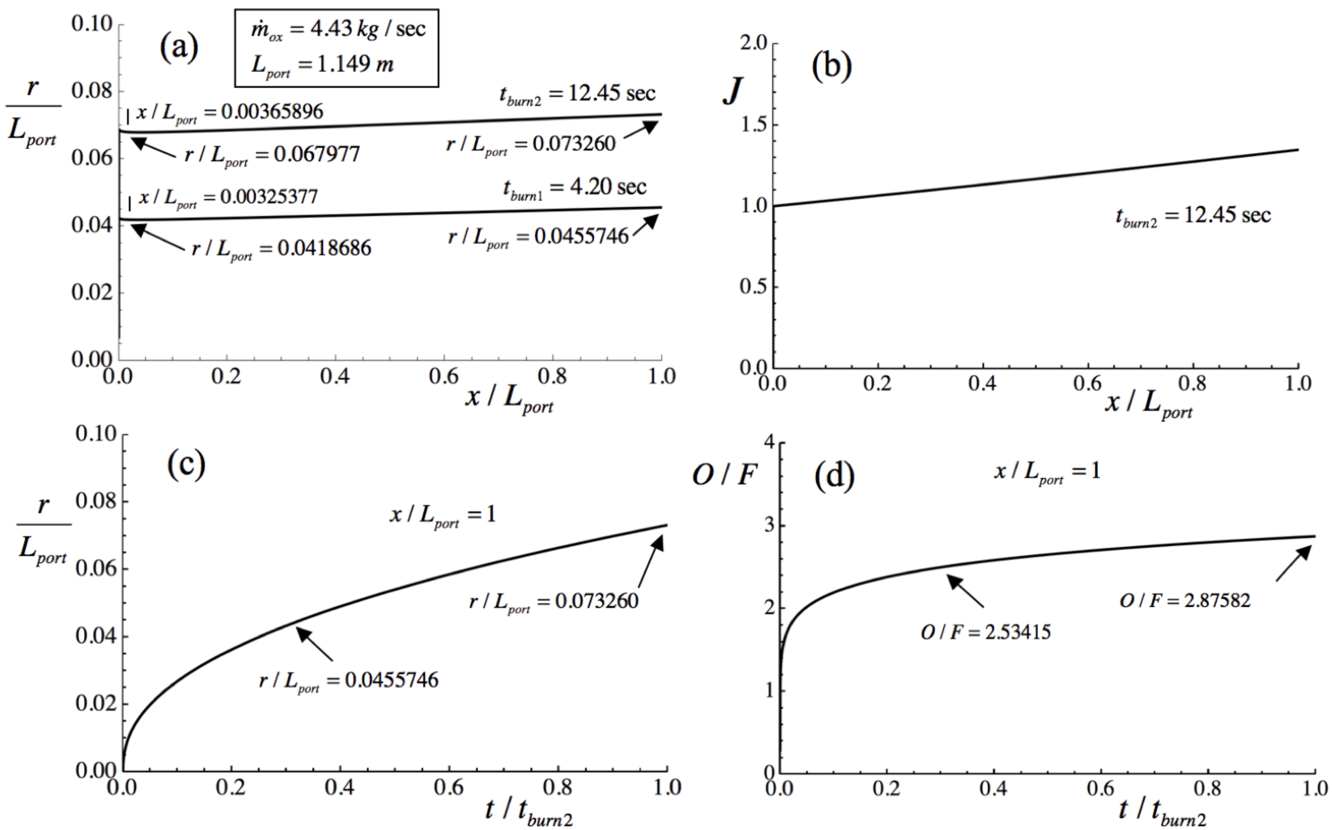
Comparison with NASA Ames test 4L-05

Case 1 – Paraffin burning with oxygen $(n,m)=(0.62,0.009)$.

Set the oxidizer mass flow rate to $\dot{m}_{ox}=4.43\ kg/\sec$. Characteristic scales for this case are

$$T_{ch}=2.607\times10^{6}\ seconds \quad L_{ch}=12.594\ meters \quad M_{ch}=1.155\times10^{7}\ kilograms \qquad (73)$$

Take the port length to be $L_{port}=1.149\ m$ meters. The initial and final burn times are $t_{burn1}=4.20$ seconds and $t_{burn2}=12.45$ seconds. For these burn times, $\tau_{\max 1}=1.611\times10^{-6}$ and $\tau_{\max 2}=4.776\times10^{-6}$.



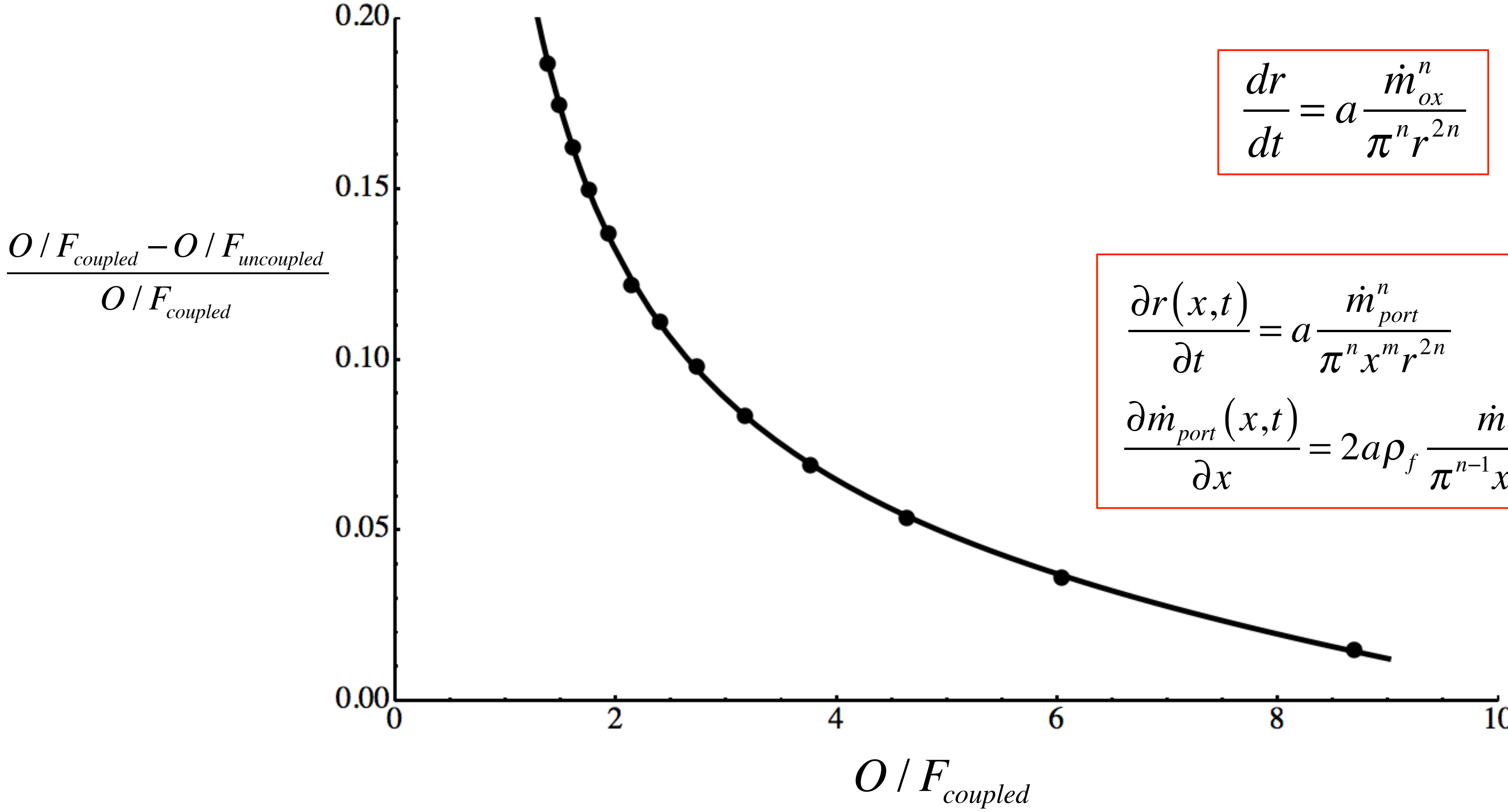
O/F at the end of the port looks very good over the entire burn

Comparison with NASA Ames test 4L-05

Measured variable	Test 4L-05	Similarity solution (Fig 15)
Oxidizer mass flow rate, kg/sec	4.43	4.43
Burn time, sec	8.25	8.25
Grain length, m	1.149	1.149
Initial port radius, m	0.05005	0.05005 (averaged along port)
Final port radius (fore end), m	0.0794	0.0783
Final port radius (aft end), m	0.0832	0.0842
Percent increase in radius along the port	4.79	7.54
Overall O/F ratio	2.72	2.73

Excellent agreement with port geometry and O/F

O/F comparison between coupled and uncoupled regression rate formulations



$$\frac{dr}{dt} = a\frac{\dot{m}_{ox}^{n}}{\pi^{n} r^{2n}}$$

$$\frac{\partial r(x,t)}{\partial t} = a\frac{\dot{m}_{port}^{n}}{\pi^{n} x^{m} r^{2n}}$$

$$\frac{\partial \dot{m}_{port}(x,t)}{\partial x} = 2a\rho_f \frac{\dot{m}_{port}^{n}}{\pi^{n-1} x^{m} r^{2n-1}}$$

STANFORD
AERONAUTICS & ASTRONAUTICS

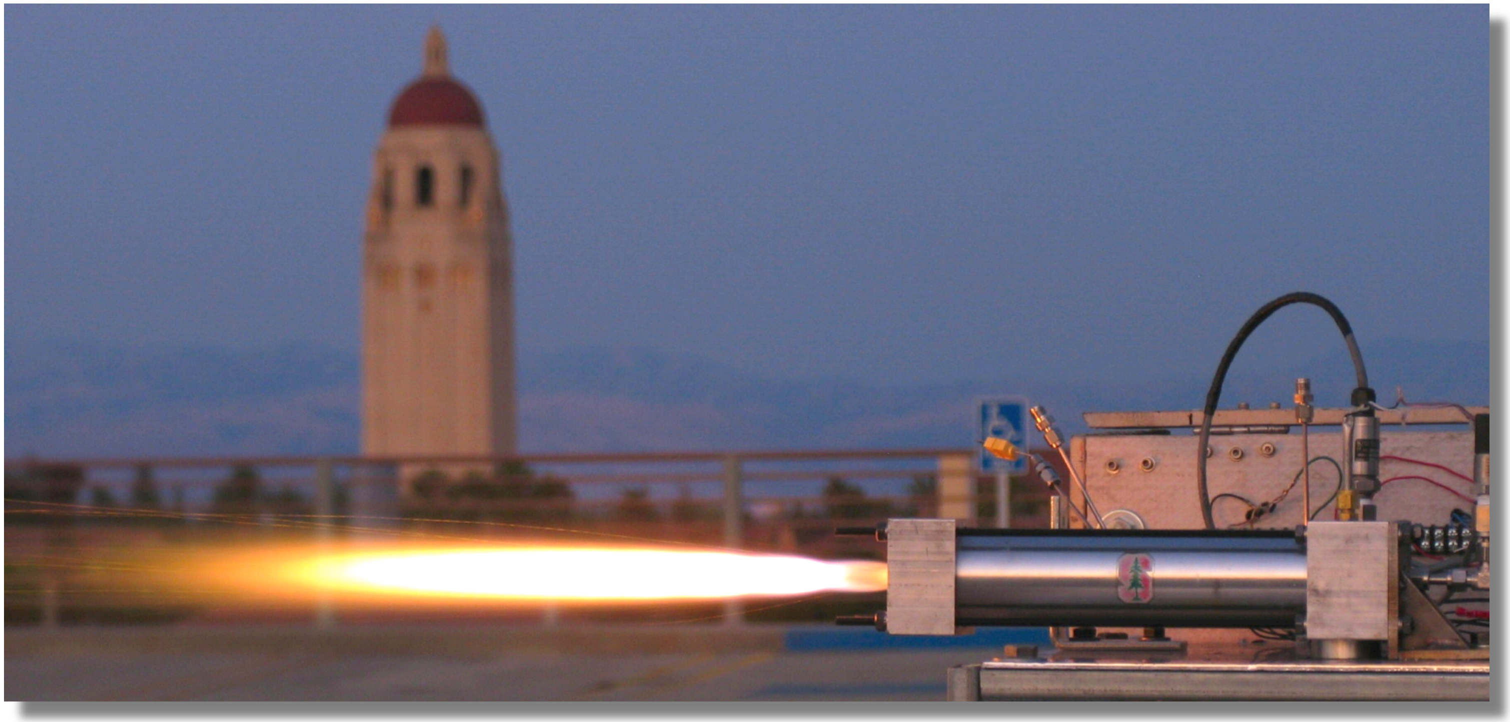


Photo - Kevin Lohner