Universal Gravity Turn Trajectories

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One of the simplest trajectory programs for the powered flight of a missile through the atmosphere is the "gravity turn," which results from simply keeping the propulsive thrust always parallel to the vector velocity. However, even for a "point mass" missile, in a uniform gravitational field with constant thrust and no aerodynamic forces, the differential equations for the motion are nonlinear and require numerical integration. To avoid the necessity of doing this computation anew for each missile preliminary design, a method has been found for integrating the equations for the singular case of zero initial velocity. When expressed in terms of appropriate dimensionless variables, the resulting solutions are "universal" in the sense that they constitute a good approximation to any gravity turn with a small, nearly vertical, initial velocity. The solutions depend upon two parameters, the initial thrust to weight ratio η and a parameter k which corresponds to the initial "kick angle" of nonsingular gravity turns.

I. INTRODUCTION

THE powered flight trajectories of long-range missiles, satellites, etc., usually involve an "atmospheric" portion, in which aerodynamic forces and moments are of importance, and a "post atmospheric" section, in which these effects are negligible. During this latter part of the flight, the trajectory to be followed is fairly simple. Holding the thrust attitude angle ψ (relative to the horizontal) constant yields (approximately) maximum range for a surface-surface missile,[1] while varying ψ so that its tangent is a linear function of time, $\tan\psi = a - bt$, results in a satellite orbit of (approximately) maximum altitude.[2] (These programs are the exact solutions of the maximum range or maximum satellite altitude problems, respectively, if the gravitational field is uniform, in magnitude and direction, over the *powered* portion of the flight. They will be good approximations to the optimum programs provided the length of the *powered* flight trajectory is small compared to the earth's radius.)

Neither of these programs is feasible for the atmospheric portion, since both would give rise to large angles of attack. A sensible and (in principle) simple trajectory which can be used for this part is the "gravity turn," sometimes referred to as a "zero angle of attack" or "no-lift" trajectory. It consists simply in keeping the thrust always parallel to the velocity, starting from some nonzero, nonvertical, initial velocity, $\mathbf{v}_0$.

While the equations which describe this trajectory are quite simple, they are nonlinear (even if we neglect aerodynamic forces and thrust variations) and not easy to handle except by numerical methods. Thus, each missile preliminary design involves the integration of these same gravity turn equations on a computing machine. Considerable time and effort could be saved if universal, albeit approximate, solutions of these equations were available.

We describe here a method which has been used to obtain such *universal* solutions for gravity turns which begin very shortly after the launching time. The physical problem is presented in Sec. II, and an analytical solution for the case of constant thrust-weight ratio is given in Sec. III. A method of numerical solution for the case of constant thrust and mass flow rate is discussed in Sec. IV and some illustrative curves are given. A more complete set of such curves has also been prepared.

II. THE GRAVITY TURN EQUATIONS OF MOTION

We treat the missile as a mass point, subject to a thrust $\mathbf{F}$, whose magnitude is a specified function of time and whose direction can be varied at will. The gravitational force field is assumed to be uniform and aerodynamic forces are neglected. The velocity $\mathbf{v}$ then satisfies the equation

$$m d\mathbf{v}/dt = \mathbf{F} - mg\mathbf{k}, \tag{1}$$

where $\mathbf{k}$ is a unit vector in the vertical direction and m is the instantaneous mass.

A "gravity turn" is defined as one of the class of trajectories obtained when the thrust is kept parallel to the velocity,

$$\mathbf{F} = F\boldsymbol{\tau}_1, \quad \text{where} \quad \boldsymbol{\tau}_1 = \mathbf{v}/v. \tag{2}$$

Since

$$\mathbf{v} = v\boldsymbol{\tau}_1,$$

we have

$$d\mathbf{v}/dt = \dot{v}\boldsymbol{\tau}_1 + v\boldsymbol{\sigma}, \tag{3}$$

where $\boldsymbol{\sigma} \equiv d\boldsymbol{\tau}_1/dt$ is normal to $\mathbf{v}$. Resolving (1) along $\boldsymbol{\tau}_1$ and $\boldsymbol{\sigma}$ we have

$$\begin{aligned} \dot{v} &= g(n - \cos\beta), \\ v\dot{\beta} &= g\sin\beta, \end{aligned} \tag{4}$$

where

$$n = F/mg$$

is the thrust to weight ratio and

$$\beta = \cos^{-1}(\boldsymbol{\tau}_1 \cdot \mathbf{k})$$

[1] B. D. Fried and J. M. Richardson, J. Appl. Phys. 27, 955 (1956).
[2] B. D. Fried, "On the powered flight trajectory of an earth satellite," Jet Propulsion (to be published).

is the angle between the velocity vector and the vertical.

We assume that the thrust magnitude and the mass are prescribed functions of time. (Thus we neglect, for example, any variations of thrust with altitude.) Then $n(t)$ is known, and given any nonzero, initial values of v and β we can integrate Eqs. (4) to find $v(t)$ and $\beta(t)$. Zero values for $v(0)$ and $\beta(0)$ are conventionally excluded, (a) on physical grounds, for if $v=0$ the velocity attitude angle is no longer well defined, and (b) on mathematical grounds, for if R is any closed region of the (t,β,v) space, then the system of Eqs. (4) satisfies a Lipschitz condition if and only if R does not contain the origin. We shall return to this question later.

Two particular functions $n(t)$ are of interest. If $n=$constant, then (4) can be integrated explicitly.[3] This is not the situation usually encountered in applications, but the results nevertheless provide useful insight regarding gravity turns. The case where F is constant and m is a linear function of t is a fairly good approximation to many actual situations. With

$$m=m_0-\mu t \quad 0\leq t\leq T$$

and

$$F=g\mu I=\mu c$$

we have

$$n=\mu I(m_0-\mu t)^{-1}=\eta(1-\mu t/m_0)^{-1}, \tag{5}$$

where $\eta=F/m_0g$ is the initial thrust-weight ratio. This case, which requires a numerical solution of the differential equations, will be discussed more fully in Sec. IV.

Before considering either of these special cases, it is convenient to write (4) in terms of dimensionless variables,[4]

$$\begin{aligned} t&=m_0\tau/\mu \\ v&=cu/\eta. \end{aligned} \tag{6}$$

To avoid the manipulation of trigonometric identities, we also introduce

$$z=\tan\beta/2. \tag{7}$$

Then (4) becomes

$$du/d\tau=n(t)-(1-z^2)(1+z^2)^{-1} \tag{8}$$

$$udz/d\tau=z. \tag{9}$$

III. AN EXACT SOLUTION FOR CONSTANT THRUST TO WEIGHT RATIO

We now specialize to the case where $n(t)$ is constant, with $n>1$. The quotient of (8) and (9) gives

$$d(\ln u)/dz=nz^{-1}-(1-z^2)(1+z^2)^{-1}z^{-1} \tag{10}$$

which can be integrated to give

$$u=Az^{n-1}(1+z^2), \tag{11}$$

where A is determined by the initial values, u_0 and z_0,

$$A=u_0[z_0^{n-1}(1+z_0^2)]^{-1}. \tag{12}$$

Then (9) and (11) give

$$\tau=Az^{n-1}[(n-1)^{-1}+z^2(n+1)^{-1}] \tag{13}$$

if we choose our origin of time so that $\tau=0$ when $z=0$.

Examination of (11), (12), and (13) reveals several items of interest.

(a) We can find u as a function of τ by eliminating z between Eqs. (11) and (13). If we write this in the form $u=Af(\tau/A)$, then for a given value of n, the resulting function f will be a universal one, in the sense that it does not depend on the initial conditions u_0 and z_0.

(b) Equations (11) and (13) satisfy the differential equations (8), (9) subject to the initial condition $u=u_0$ when $z=z_0$. (This occurs at a time

$$\tau_0=u_0(1+z_0^2)^{-1}[(n-1)^{-1}+(n+1)^{-1}z_0^2]. \tag{14}$$

However, we notice that they also satisfy the initial condition $u=z=0$ at $\tau=0$, *irrespective* of the value of A. Because of the singularity at $u=0$, this initial condition does not select a unique solution of the differential equations, but rather a one-parameter family of solutions.

This limiting case, $u=z=0$ initially, is of practical interest when we consider gravity turns which begin very shortly after launch. At this time, u_0 and z_0 will have nonzero values and there is no objection in principle to a straightforward integration of the differential equations. However, if u_0, $z_0\ll 1$, the solutions obtained for the limiting case, u_0, $z_0\to 0$, will be a good approximation to the desired ones. In fact, the use of nonzero initial values in computing gravity turns is, in most instances, essentially a computational artifice to avoid dealing with the singularity at $u_0=z_0=0$.

In the following section we shall show how to obtain such a one-parameter family of solutions of the differential equations, for the case where n is given by (5), which satisfy the initial conditions $u=z=0$.

IV. CONSTANT THRUST AND MASS FLOW RATE

For a thrust to weight ratio which varies with time, e.g., as given by (5), the differential equations must be solved numerically. Because of the singularity at $u=z=0$, a straightforward numerical integration starting from this initial condition is impossible. Instead, it is convenient to convert (8) and (9) to a single integral equation, with the desired boundary conditions included. With

$$z=e^{\varphi} \tag{15}$$

(8) and (9) become

$$du/d\tau=\eta(1-\tau)^{-1}+\tanh\varphi, \tag{16}$$

$$u(d\varphi/d\tau)=1, \tag{17}$$

[3] Kooy and Uytenbogaart, *Ballastics of the Future* (Technical Publishing Company, H. Stam, Haarlem, Netherlands, 1946), pp. 223 ff.

[4] Note that the final value of τ (corresponding to $t=T$) is just $\nu=\mu T/m_0=(r-1)/r$, where r (mass ratio) and ν (propellant weight/gross weight) are parameters commonly used to characterize one stage of a missile. Thus, $0\leq\tau\leq 1$. Since a gravity turn is usually ended before $\beta=\pi/2$, we have also $0\leq z\leq 1$.

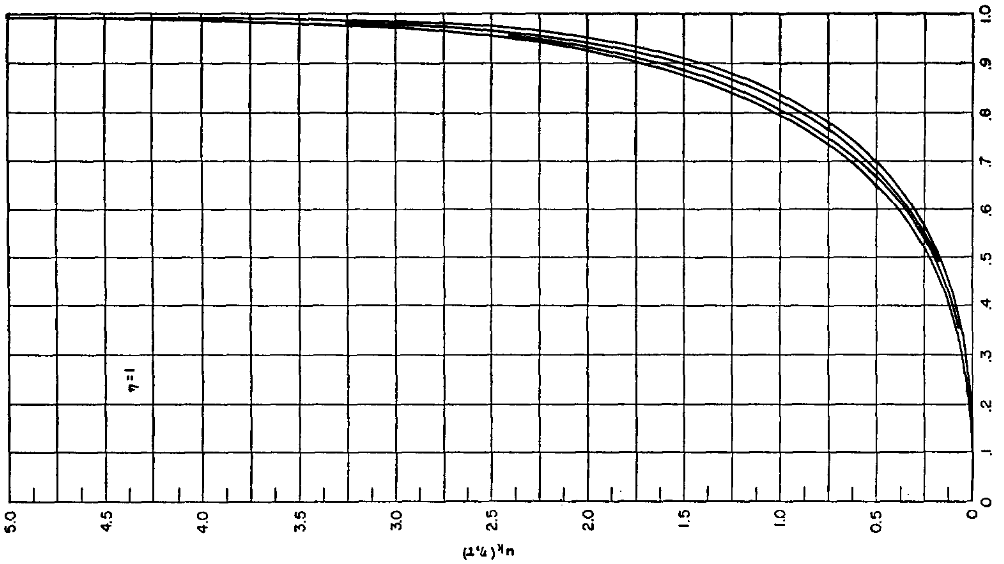

FIG. 1. Total speed u as a function of time for $\eta=1$. The curves from top to bottom, in monotonic order, correspond to values of $k=0$; 0.223; 0.7; 1.5, respectively. The near coincidence of the curves, which is found also for other values of η, is in contrast to the plots of z *vs* time for the same range of k values (see Fig. 2).

whence

$$u(\tau)=-\eta\log(1-\tau)+\int_0^\tau \tanh\varphi(s)ds \qquad (18)$$

a form which already takes care of the initial condition $u(0)=0$. From (17) we have

$$\varphi(\tau)=\int_1^\tau \frac{ds}{u(s)}-k$$

$$=\int_\tau^1 \frac{ds}{\eta\log(1-s)-\int_0^s \tanh\varphi(s')ds'}-k. \qquad (19)$$

This satisfies the initial condition $z(0)=0$ (i.e., $\varphi(0)=-\infty$), irrespective of the value of the constant of integration k, for with any a, $0<a<1$, we have $(\eta-1)s\leq u(s)\leq bs$ for $0\leq s\leq a$, where

$$b\equiv 1+\eta-\eta\log(1-a)>0.$$

Then

$$\frac{\ln a/\tau}{b}\leq\int_1^a \frac{ds}{u(s)}-(\varphi+k)\leq\frac{\ln a/\tau}{\eta-1}$$

and since $\int_1^a ds/u(s)$ is finite it follows that for any (finite) value of k,

$$\varphi(0)\equiv\lim_{\tau\to 0}\varphi(\tau)=-\infty.$$

Because the range of φ is not finite, it is convenient to introduce

$$\psi=\tanh\varphi. \qquad (20)$$

Then ψ satisfies the integral equation

$$\psi(\tau)=\tanh\left[\int_\tau^1 \frac{ds}{\eta\log(1-s)-\int_0^s \psi(s')ds'}-k\right] \qquad (21)$$

and has the correct boundary condition, $\psi(0)=-1$. This equation is well adapted for iteration, starting, e.g., with the initial choice $\psi(s)\equiv -1$.

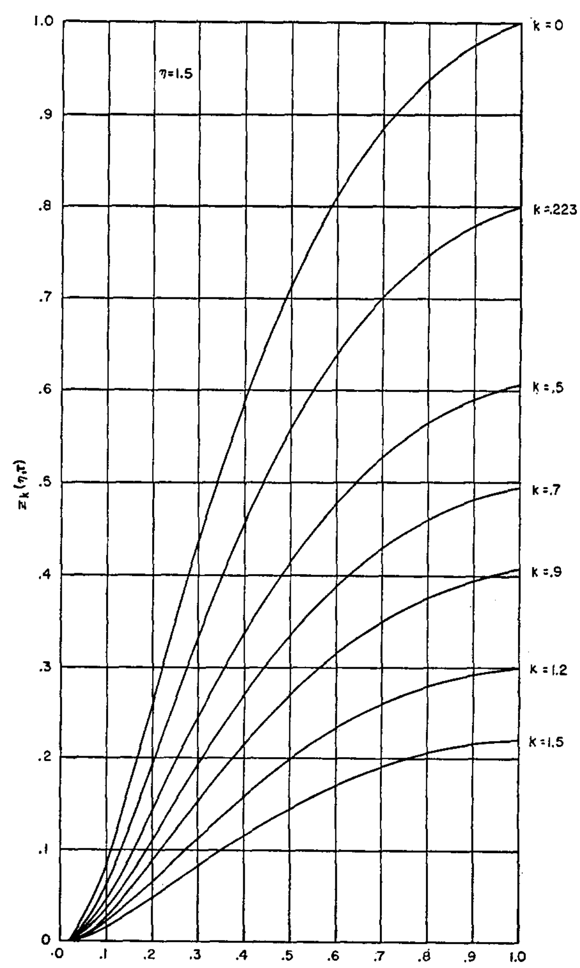

FIG. 2. Attitude variable, $z=\tan\beta/2$, as a function of time for $\eta=1.5$ and several values of k.

The parameter k, which plays a role analogous to that of A in the previous section, labels the various curves of the one-parameter family.[5] From (7), (15), and (19) we see that

$$k=-\varphi(1)=-\ln z(1)=-\ln\tan\frac{\beta(1)}{2} \tag{22}$$

$\beta(1)$ being the attitude angle at "time" $\tau=1$, i.e., that time at which the mass becomes zero. In practice, the final value of τ, i.e., ν, is frequently in the range between 0.7 and 1.0 so that k is approximately equal to $-\ln z(\nu)$. (Since $u\rightarrow\infty$ at $\tau=1$, and $\dot{z}\equiv z/u$, the slope of the z *vs* τ curve will be small in this region.)

Once (21) has been iterated to give a sufficiently accurate $\psi(\tau)$, we can find $u(z)$ and $z(\tau)$,

$$u(\tau)=-\eta\log(1-\tau)+\int_0^\tau\psi(s)ds \tag{23}$$

$$z=\left(\frac{1+\psi}{1-\psi}\right)^{\frac{1}{2}}. \tag{24}$$

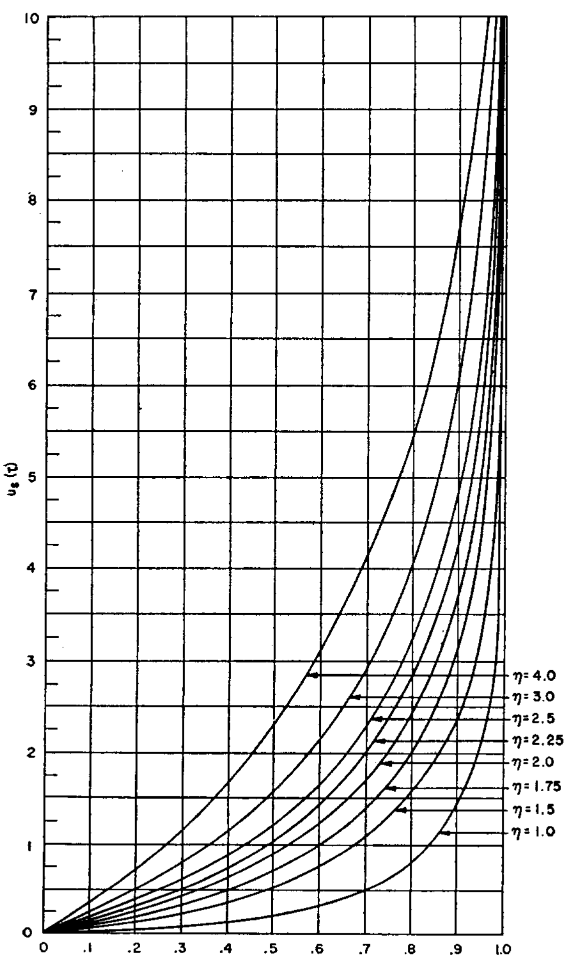

FIG. 3. Sounding speed u_s as a function of time for several values of η.

[5] It can be shown that Eq. (21) has exactly one solution for each positive value of k.

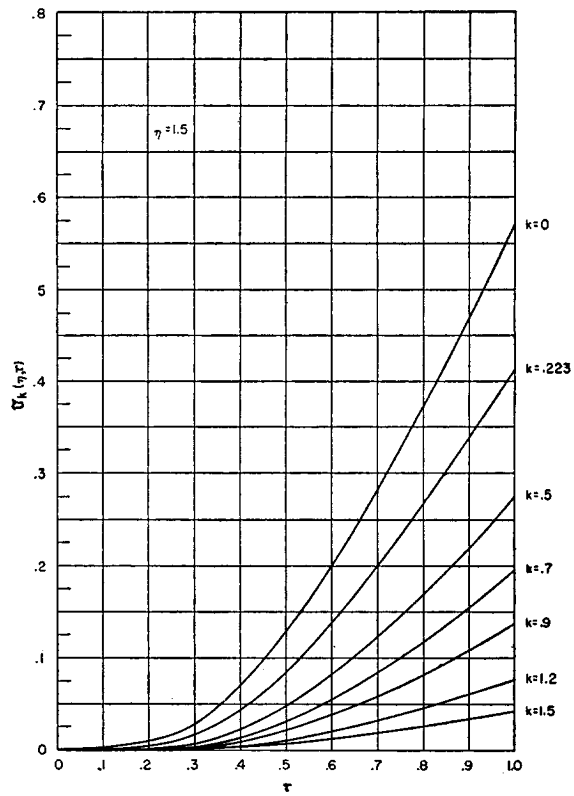

FIG. 4. Incremental speed ũ as a function of time for $\eta=1.5$ and several values of k.

We find that the curves of u *vs* τ for given η depend so little on the value of k that it is difficult to make a readable plot showing more than a few curves of the family (see Fig. 1). It is therefore convenient to split $u(\tau)$ into two parts,

$$u(\tau)=u_s(\tau)+\tilde{u}(\tau), \tag{25}$$

where

$$u_s(\tau)\equiv-\eta\log(1-\tau)-\tau \tag{26}$$

is the velocity corresponding to a vertical "sounding" flight ($z\equiv0$) and is thus independent of k, while

$$\tilde{u}(\tau)\equiv\int_0^\tau[\psi(s)+1]ds \tag{27}$$

is a function of k as well as of η. Since $|\tilde{u}(\tau)|\ll|u_s(\tau)|$ in almost all of the cases which have been calculated, $u_s(\tau)$ provides a good first approximation, which can be corrected, if necessary, by adding $\tilde{u}(\tau)$.

In Fig. 2, we show z *vs* τ for $\eta=1.5$ and several values of k. Figure 3 shows the function $u_s(\tau)$, defined by Eq. (26), for several values of η, while Fig. 4 gives the velocity correction $\tilde{u}$ which must be added to u_s for the case $\eta=1.5$ and various values of k. Finally, Fig. 5 shows u plotted against z for the same values of η and k. Lines of constant τ are also given. The small slope of these is another expression of the already noted

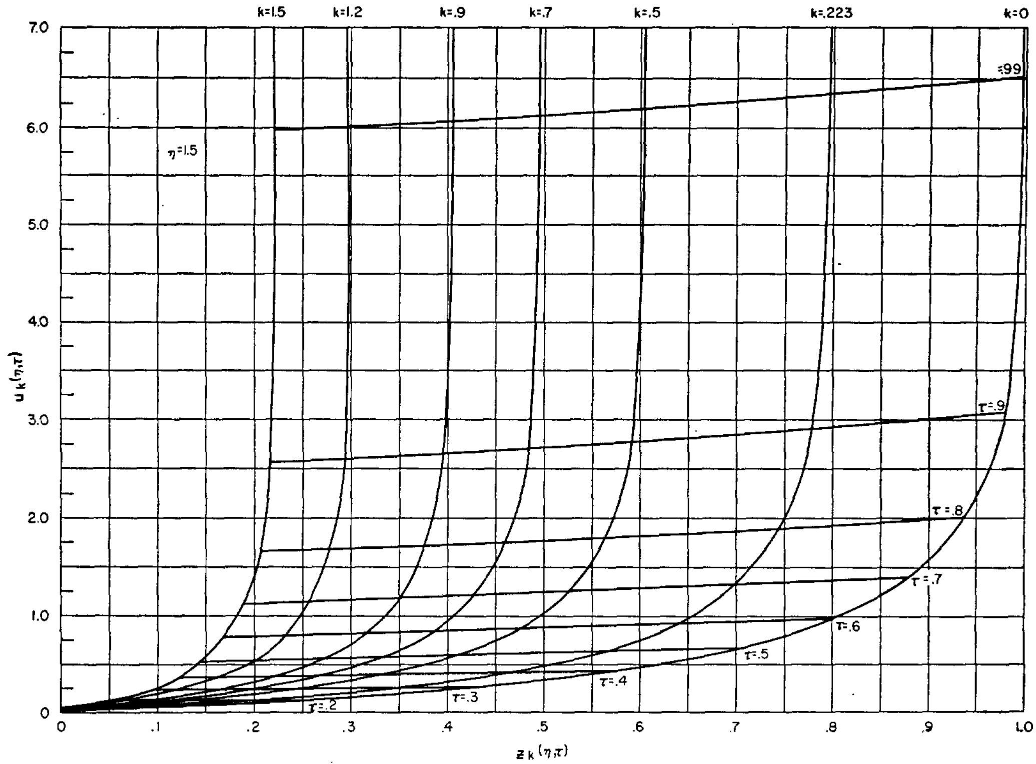

FIG. 5. Total speed u as a function of z for several values of k, with $\eta=1.5$. Lines of constant time are also shown.

circumstance that u is a much weaker function of k, for given η, than is z.

We have made many curves, similar to Figs. 2, 4, and 5, with values of η ranging from 1.0 to 4.0.

V. CONCLUSIONS

A technique has been described for obtaining "universal" solutions to the equations describing the gravity turn of a point missile with constant thrust and mass flow rate. Although numerical integration is required, the number of parameters is diminished by using appropriate dimensionless variables and by working with the limiting case obtained by allowing the initial velocity magnitude to approach zero, while its direction approaches the vertical. The singularity in the differential equations associated with this limiting initial condition is avoided by using the equivalent integral equation. For each choice of initial thrust-to-weight ratio, a one-parameter family of trajectories results. The parameter k which distinguishes individual members of the family is analogous to the "initial kick angle" of nonsingular gravity turns. The results obtained by the present method may be considered as a good approximation to those obtained by numerical integration of the differential equations (16) and (17), starting with a small, nearly vertical initial velocity.