AA 284a
Advanced Rocket Propulsion

Lecture 7a
Efficiency of a Rocket

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Efficiency of a Rocket

Objective:

- Derive a closed form expression for the overall efficiency of a rocket vehicle operating in a gravitational field

Assumptions:

- The thrust over the mass of the system is constant
- The gravitational constant times the sine of the trajectory angle is constant
- Drag and lift forces are negligible
- The nozzle is perfectly expanded

General conclusions of this study are independent of the assumptions

Efficiency of a Rocket

Equations of Motion:

$$\frac{dV}{dt} = \frac{T}{M} - g_s = a - g_s$$

$$\frac{dh}{dt} = V$$

$$\frac{dM}{dt} = -\dot{m} = -\frac{T}{I_{sp} g_o} = -\frac{T}{V_e} = -\frac{T}{M}\frac{M}{V_e} = -a\frac{M}{V_e}$$

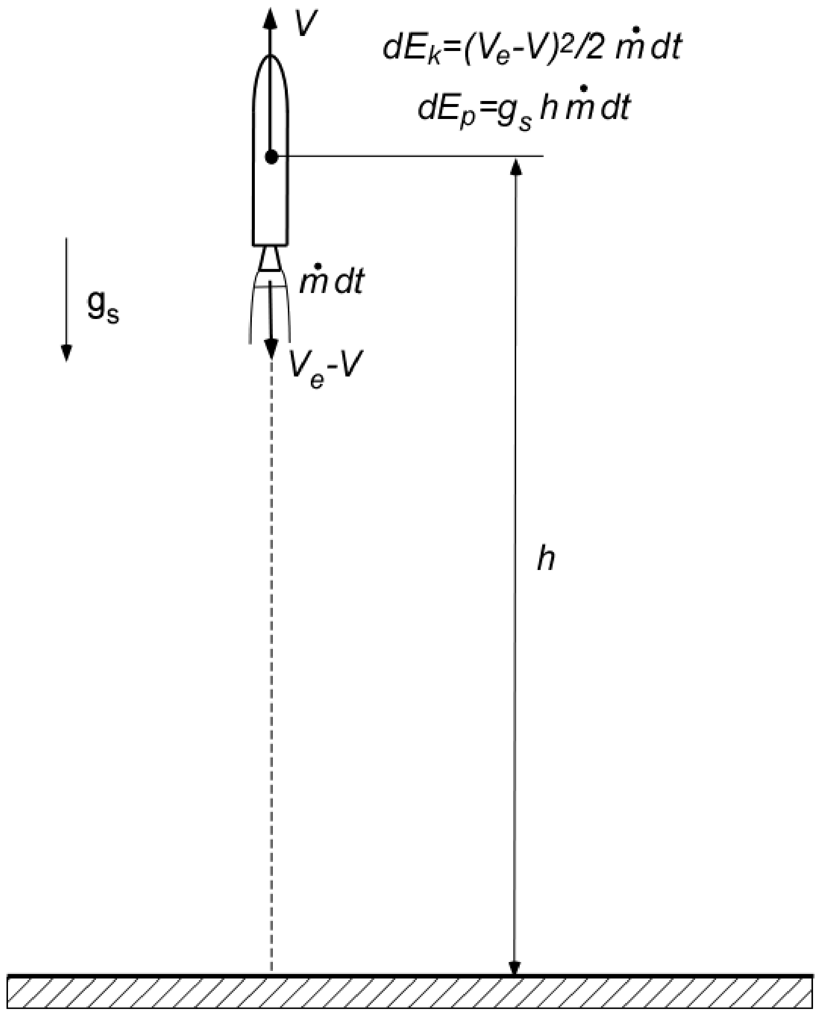
a is the ideal acceleration.

Solution:

$$V = (a - g_s)t \qquad h = \frac{(a - g_s)}{2}t^2 \qquad a\,t = V_e \ln\left(\frac{M_i}{M}\right)$$

Efficiency of a Rocket

Energy Loss Estimation:



Efficiency of a Rocket

1) Kinetic energy of the expelled propellant loss:

$$E_k = \int_o^{t_b} \frac{(V - Ve)^2}{2} \dot{m}\, dt$$

$$E_k = -\frac{1}{2} \int_{M_i}^{M_f} \left[(a - g_s)t - V_e\right]^2 dM$$

$$E_k = -\frac{V_e^2}{2} \int_{M_i}^{M_f} \left[\left(1 - \frac{g_s}{a}\right) \ln\left(\frac{M_i}{M}\right) - 1\right]^2 dM$$

$$\frac{E_k}{1/2 V_e^2 M_f} = \left[1 - 2\frac{g_s}{a} + 2\left(\frac{g_s}{a}\right)^2\right]\left(\frac{M_i}{M_f} - 1\right) + 2\frac{g_s}{a}\left(1 - \frac{g_s}{a}\right)\ln\left(\frac{M_i}{M_f}\right) - \left(1 - \frac{g_s}{a}\right)^2 \ln^2\left(\frac{M_i}{M_f}\right)$$

In the following limit, this loss does NOT disappear

$$g_s / a \to 0$$

Efficiency of a Rocket

1) Potential energy loss due to expelled propellant:

$$E_p = \int_o^{t_b} g_s h \dot{m}\, dt$$

$$E_p = -\int_{M_i}^{M_f} g_s (a - g_s) \frac{t^2}{2} dM$$

$$E_p = -\frac{V_e}{2} \int_{M_i}^{M_f} \frac{g_s}{a}\left(1 - \frac{g_s}{a}\right) \ln^2\left(\frac{M_i}{M}\right) dM$$

$$\frac{E_p}{1/2 V_e^2 M_f} = \frac{g_s}{a}\left(1 - \frac{g_s}{a}\right)\left\{2\left(\frac{M_i}{M_f} - 1\right) + 2\ln\left(\frac{M_i}{M_f}\right) - \ln^2\left(\frac{M_i}{M_f}\right)\right\}$$

In the following limits, this loss disappears

$$g_s/a \to 0 \qquad g_s/a \to 1$$

Efficiency of a Rocket

Combined loss is

$$\frac{E_k + E_p}{1/2 V_e^2 M_f} = \left(\frac{M_i}{M_f} - 1\right) - \left(1 - \frac{g_s}{a}\right)\ln^2\left(\frac{M_i}{M_f}\right)$$

Useful Energy:

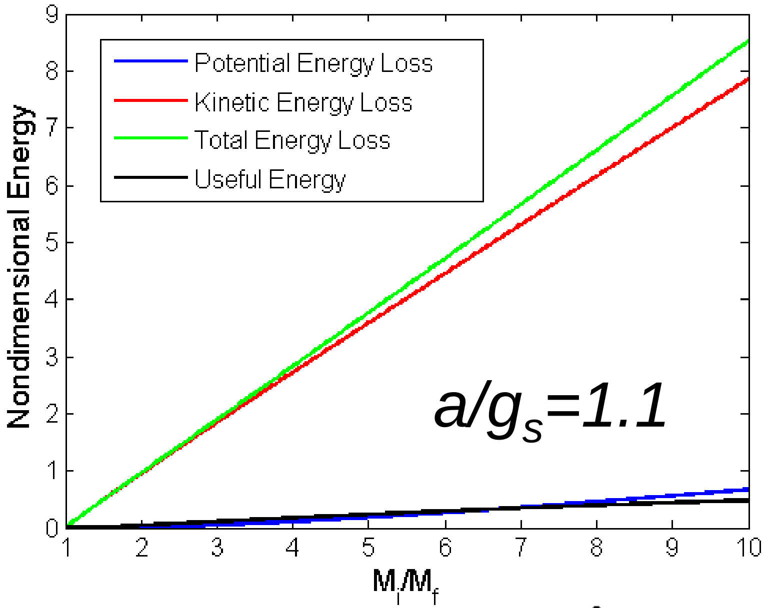
$$E_u = \left(\frac{V_f^2}{2} + g_s h_f\right) M_f$$

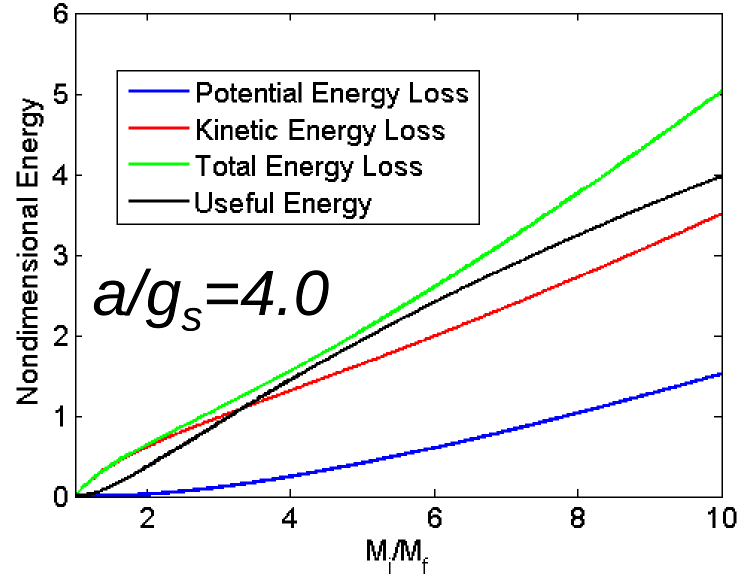
$$\frac{E_u}{1/2 V_e^2 M_f} = \left(1 - \frac{g_s}{a}\right)\ln^2\left(\frac{M_i}{M_f}\right)$$

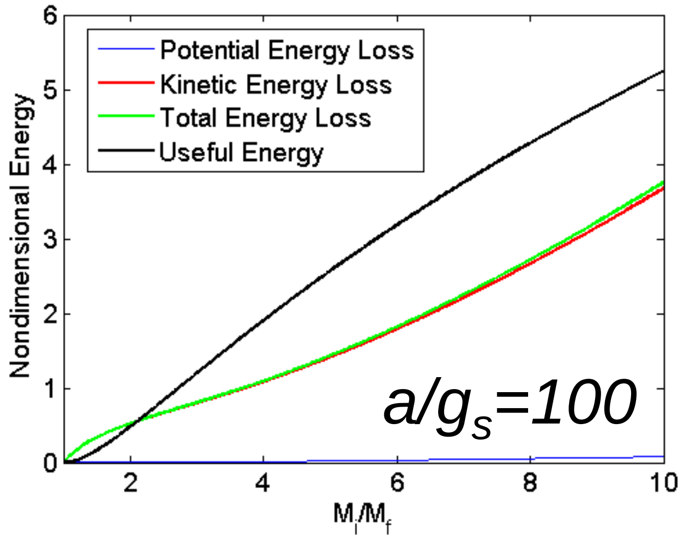
Efficiency which governs the conversion of the kinetic energy of the plume to the useful energy of the rocket vehicle

$$\frac{E_u}{E_p + E_k + E_u} = \left(1 - \frac{g_s}{a}\right)\frac{\ln^2\left(M_i/M_f\right)}{M_i/M_f - 1}$$

Efficiency of a Rocket







Efficiency of a Rocket

Overall efficiency of a Rocket System:

- The overall efficiency of a chemical rocket can be defined as the ratio of the useful energy over the chemical energy stored in the propellant

$$\eta_R \equiv \frac{Useful\ Energy}{Stored\ Chemical\ Energy} = \frac{E_u}{\left(M_i - M_f\right) Q_{prop}}$$

– Qprop is the heat of combustion per unit mass of the propellant

$$\eta_R = \left(1 - \frac{g_s}{a}\right) \frac{\ln^2\left(M_i/M_f\right)}{M_i/M_f - 1} \frac{1/2 V_e^2}{Q_{prop}}$$

$$\eta_R = \left(1 - \frac{g_s}{a}\right) \frac{\ln^2\left(M_i/M_f\right)}{M_i/M_f - 1} \frac{1/2 V_e^2}{h_{t2}} \frac{h_{t2}}{Q_{prop}}$$

$$\eta_R = \eta_g \eta_p \eta_T \eta_n \eta_c$$

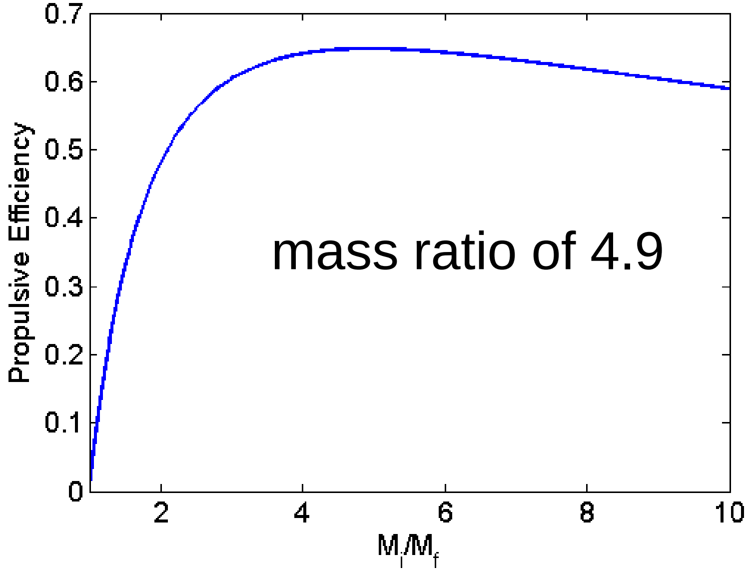
Efficiency of a Rocket

Gravity Loss Efficiency:

$$\eta_g = \left(1 - \frac{g_s}{a}\right)$$

Propulsive Efficiency:

$$\eta_p \equiv \frac{\ln^2\left(M_i / M_f\right)}{M_i / M_f - 1}$$



$$\eta_p \equiv \frac{\left(\Delta V / V_e\right)^2}{e^{\Delta V / V_e} - 1}$$

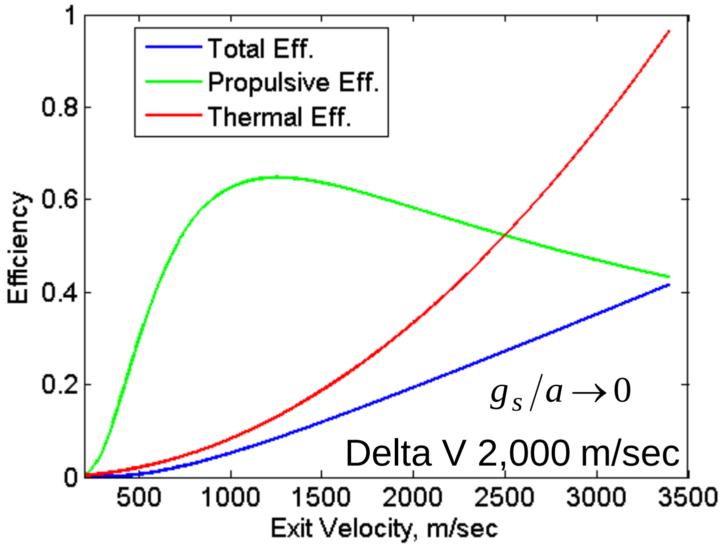
Efficiency of a Rocket

Thermal Efficiency:

$$\eta_T = \frac{1/2 V_{ei}^2}{h_{t2}} = 1 - \frac{T_e}{T_{t2}}$$

Note that for larger nozzle expansion ratios for which the temperature ratio is large, the thermal efficiency is higher. For a calorically perfect gas, one can express the thermal efficiency in terms of the Mach number at the nozzle exit as

$$\eta_T = \frac{M_e^2}{2/(\gamma - 1) + M_e^2}$$



Even though the propulsive efficiency is maximized at a certain exhaust velocity, the product of the propulsive and thermal efficiencies increases monotonically with increasing exit velocity (i.e. specific impulse).

Efficiency of a Rocket

Nozzle Efficiency: This term includes all the losses that take place in the nozzle (heat transfer losses, viscous losses, 3D losses, two phase losses, etc)

$$\eta_n = \frac{V_e^2}{V_{ei}^2}$$

- Nozzle efficiency is typically in the range of 0.97-0.99.

Combustion Efficiency: This governs the efficiency of converting the chemical bond energy to the thermal energy.

$$\eta_c = \frac{h_{t2}}{Q_{prop}}$$

- Combustion efficiency is typically in the range of 0.90-0.98.