AA 284a
Advanced Rocket Propulsion

Lecture 7a
Efficiency of a Rocket

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Objective:
- Derive a closed form expression for the overall efficiency of a rocket vehicle operating in a gravitational field

Assumptions:
- The thrust over the mass of the system is constant
- The gravitational constant times the sine of the trajectory angle is constant
- Drag and lift forces are negligible
- The nozzle is perfectly expanded

General conclusions of this study are independent of the assumptions
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Equations of Motion:

\[ \frac{dV}{dt} = \frac{T}{M} - g_s = a - g_s \]

\[ \frac{dh}{dt} = V \]

\[ \frac{dM}{dt} = -\dot{m} = -\frac{T}{I_{sp} g_o} = -\frac{T}{V_e} = -\frac{T}{M} \frac{M}{V_e} = -a \frac{M}{V_e} \]

\( a \) is the ideal acceleration.

Solution:

\[ V = (a - g_s)t \]

\[ h = \frac{(a - g_s)}{2} t^2 \]

\[ at = V_e \ln \left( \frac{M_i}{M} \right) \]
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Energy Loss Estimation:

\[ dE_k = (V_e - V)^2 / 2 \dot{m} \, dt \]

\[ dE_p = g_s \, h \dot{m} \, dt \]
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1) Kinetic energy of the expelled propellant loss:

\[ E_k = \int_{0}^{t_b} \frac{(V - Ve)^2}{2} m dt \]

\[ E_k = -\frac{1}{2} \int_{M_i}^{M_f} [(a - g_s)t - Ve]^2 dM \]

\[ E_k = -\frac{Ve^2}{2} \int_{M_i}^{M_f} \left[ \left(1 - \frac{g_s}{a}\right) \ln \left(\frac{M_i}{M}\right) - 1 \right]^2 dM \]

\[ \frac{E_k}{1/2 Ve^2 M_f} = \left[ 1 - 2 \frac{g_s}{a} + 2 \left(\frac{g_s}{a}\right)^2 \right] \left(\frac{M_i}{M_f} - 1\right) + 2 \frac{g_s}{a} \left(1 - \frac{g_s}{a}\right) \ln \left(\frac{M_i}{M_f}\right) - \left(1 - \frac{g_s}{a}\right)^2 \ln^2 \left(\frac{M_i}{M_f}\right) \]

In the following limit, this loss does NOT disappear

\[ g_s / a \rightarrow 0 \]
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1) Potential energy loss due to expelled propellant:

\[ E_p = \int_{0}^{t_b} g_s h \dot{m} \, dt \]

\[ E_p = -\int_{M_i}^{M_f} g_s (a - g_s) \frac{t^2}{2} \, dM \]

\[ E_p = -\frac{V_e}{2} \int_{M_i}^{M_f} \frac{g_s}{a} \left(1 - \frac{g_s}{a}\right) \ln^2 \left(\frac{M_i}{M} \right) \, dM \]

\[ \frac{E_p}{1/2V_e^2M_f} = \frac{g_s}{a} \left(1 - \frac{g_s}{a}\right) \left\{ 2 \left(\frac{M_i}{M_f} - 1\right) + 2 \ln \left(\frac{M_i}{M_f} \right) - \ln^2 \left(\frac{M_i}{M_f} \right) \right\} \]

In the following limits, this loss disappears

\[ g_s/a \to 0 \quad g_s/a \to 1 \]
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Combined loss is

\[
\frac{E_k + E_p}{1/2V_e^2M_f} = \left(\frac{M_i}{M_f} - 1\right) - \left(1 - \frac{g_s}{a}\right) \ln^2\left(\frac{M_i}{M_f}\right)
\]

Useful Energy:

\[
E_u = \left(\frac{V_f^2}{2} + g_s h_f\right)M_f
\]

\[
\frac{E_u}{1/2V_e^2M_f} = \left(1 - \frac{g_s}{a}\right) \ln^2\left(\frac{M_i}{M_f}\right)
\]

Efficiency which governs the conversion of the kinetic energy of the plume to the useful energy of the rocket vehicle

\[
\frac{E_u}{E_p + E_k + E_u} = \left(1 - \frac{g_s}{a}\right) \ln^2\left(\frac{M_i}{M_f}\right) \frac{M_i}{M_i / M_f - 1}
\]
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\[ a/g_s = 1.1 \]

\[ a/g_s = 4.0 \]

\[ a/g_s = 100 \]
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Overall efficiency of a Rocket System:

- The overall efficiency of a chemical rocket can be defined as the ratio of the useful energy over the chemical energy stored in the propellant

\[
\eta_R = \frac{\text{Useful Energy}}{\text{Stored Chemical Energy}} = \frac{E_u}{(M_i - M_f)Q_{prop}}
\]

- \( Q_{prop} \) is the heat of combustion per unit mass of the propellant

\[
\eta_R = \left(1 - \frac{g_s}{a}\right) \frac{\ln^2 M_i/M_f}{M_i/M_f - 1} \frac{1/2 V_e^2}{Q_{prop}}
\]

\[
\eta_R = \left(1 - \frac{g_s}{a}\right) \frac{\ln^2 M_i/M_f}{M_i/M_f - 1} \frac{1/2 V_e^2}{h_{t2}} \frac{h_{t2}}{Q_{prop}}
\]

\[
\eta_R = \eta_g \eta_p \eta_T \eta_n \eta_c
\]
Gravity Loss Efficiency:

$$\eta_g = \left(1 - \frac{g_s}{a}\right)$$

Propulsive Efficiency:

$$\eta_p \equiv \frac{\ln^2\left(M_i/M_f\right)}{M_i/M_f - 1}$$

mass ratio of 4.9
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Thermal Efficiency:

\[ \eta_T = \frac{1/2 V_{ei}^2}{h_{t2}} = 1 - \frac{T_e}{T_{t2}} \]

Note that for larger nozzle expansion ratios for which the temperature ratio is large, the thermal efficiency is higher. For a calorically perfect gas, one can express the thermal efficiency in terms of the Mach number at the nozzle exit as

\[ \eta_T = \frac{M_e^2}{2(\gamma - 1) + M_e^2} \]

Even though the propulsive efficiency is maximized at a certain exhaust velocity, the product of the propulsive and thermal efficiencies increases monotonically with increasing exit velocity (i.e. specific impulse).
Nozzle Efficiency: This term includes all the losses that take place in the nozzle (heat transfer losses, viscous losses, 3D losses, two phase losses, etc)

\[ \eta_n = \frac{V_e^2}{V_{ei}} \]

- Nozzle efficiency is typically in the range of 0.97-0.99.

Combustion Efficiency: This governs the efficiency of converting the chemical bond energy to the thermal energy.

\[ \eta_c = \frac{h_{t2}}{Q_{prop}} \]

- Combustion efficiency is typically in the range of 0.90-0.98.