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A New Wall Damping Function for Pipe, Channel and Boundary Layer Flows.

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Recall the UVP mixing length model for the turbulent shear stress

$$\tau^+ = \left(\lambda(y^+)\frac{\mathrm{d}u^+}{\mathrm{d}y^+}\right)^2$$

Prandtl 1934

The first order equation governing pipe/channel flow becomes a quadratic equation in the derivative of the mean velocity

$$\left(\frac{\mathrm{d}u^+}{\mathrm{d}y^+}\right)^2 + \frac{1}{\lambda(y^+)^2}\frac{\mathrm{d}u^+}{\mathrm{d}y^+} - \frac{1}{\lambda(y^+)^2}\left(1 - \frac{y^+}{R_\tau}\right) = 0$$

Take the positive root

$$\frac{\mathrm{d}u^+}{\mathrm{d}y^+} = -\frac{1}{2\lambda(y^+)^2} + \frac{1}{2\lambda(y^+)^2}\left(1 + 4\lambda(y^+)^2\left(1 - \frac{y^+}{R_\tau}\right)\right)^{1/2}$$

Remove the singularity at $\lambda = 0$

$$\frac{du^+}{dy^+} = \frac{2\left(1 - \frac{y^+}{R_\tau}\right)}{1 + \left(1 + 4\lambda\,(y^+)^2\left(1 - \frac{y^+}{R_\tau}\right)\right)^{1/2}}$$

The Universal Velocity Profile (UVP)

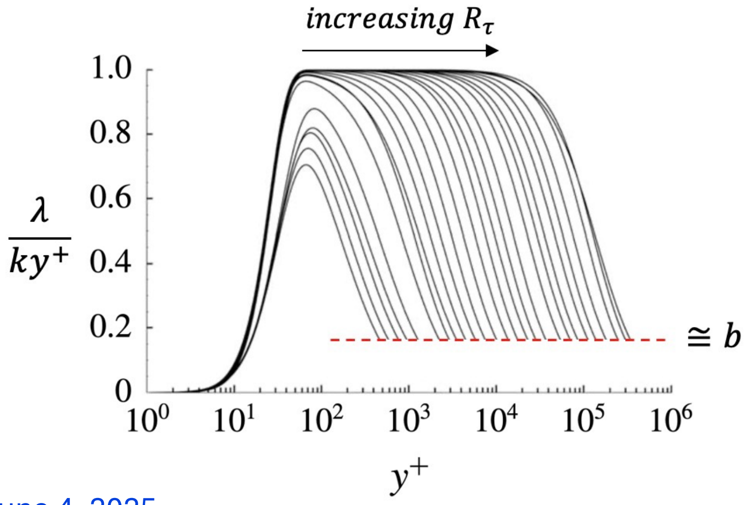
$$\frac{du^+}{dy^+} = \frac{2\left(1 - \frac{y^+}{R_\tau}\right)}{1 + \left(1 + 4\lambda\,(y^+)^2\left(1 - \frac{y^+}{R_\tau}\right)\right)^{1/2}}$$

Integrate the velocity derivative from the wall

$$u^+(k, a, m, b, n, R_\tau, y^+) = \int_0^{y^+} \frac{2\left(1 - \frac{s}{R_\tau}\right)}{1 + \left(1 + 4\lambda^2\left(1 - \frac{s}{R_\tau}\right)\right)^{1/2}} ds$$

Mixing Length function for the UVP

$$\lambda(y^+) = \frac{ky^+\left(1 - e^{-(y^+/a)^m}\right)}{\left(1 + \left(\frac{y^+}{bR_\tau}\right)^n\right)^{1/n}}$$



k - essentially the Karman constant.

a - wall damping length scale similar to the van Driest length.

m - exponent that, along with a, governs the shape and thickness of the near wall profile.

b - length scale proportional to the distance above the wall of the beginning of the outer layer.

n - exponent that, along with b, controls the transition of the profile to the wake function.

We saw that, for boundary layers, b and n can be related through a modified Clauser parameter β_c.

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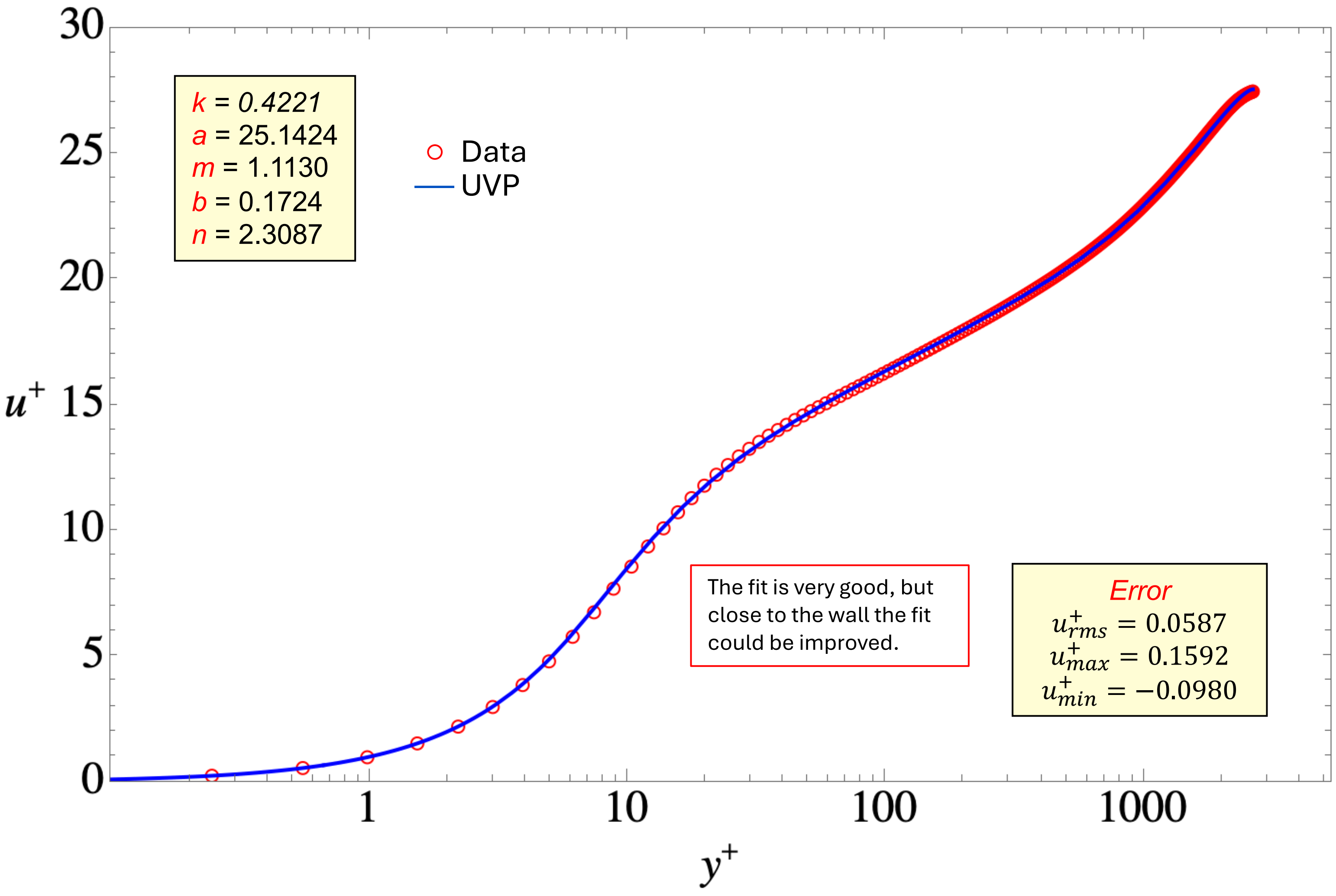
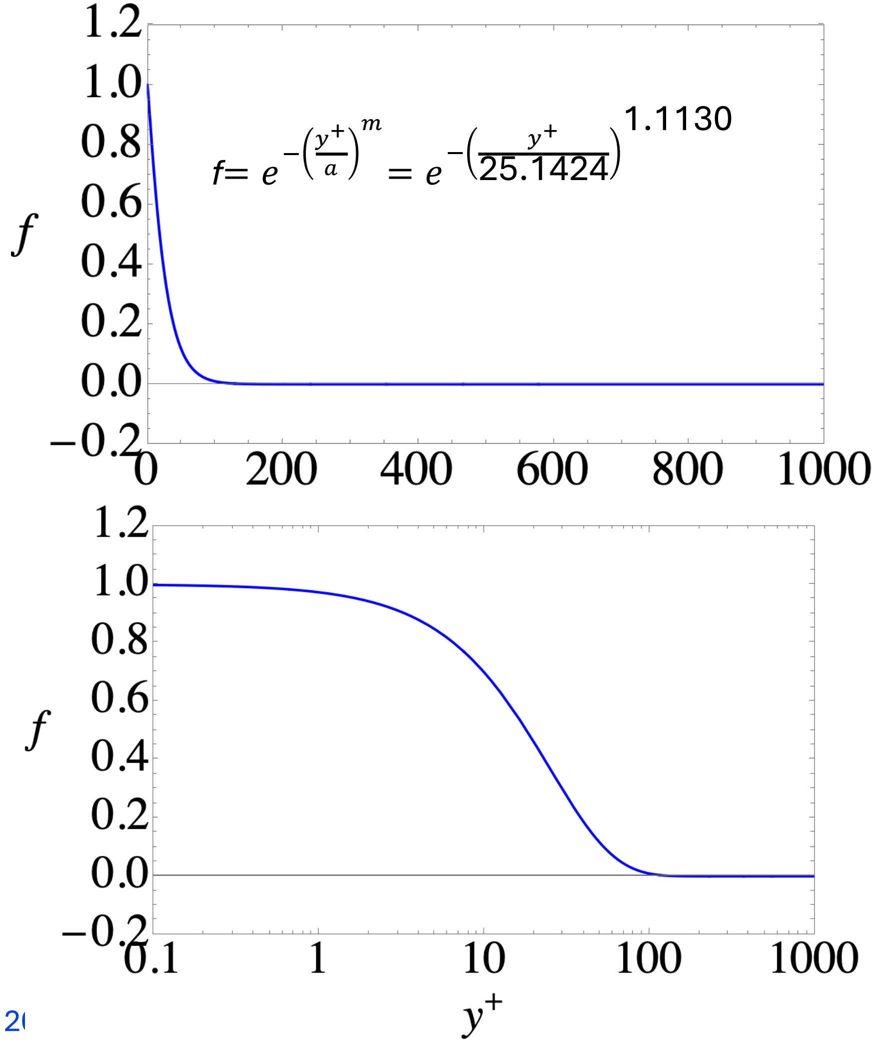
$R_\tau = 2652$ Velocity Profile Comparison

$$\lambda(y^+) = \frac{ky^+(1 - e^{-(y^+/a)^m})}{\left(1 + \left(\frac{y^+}{bR_\tau}\right)^n\right)^{1/n}}$$

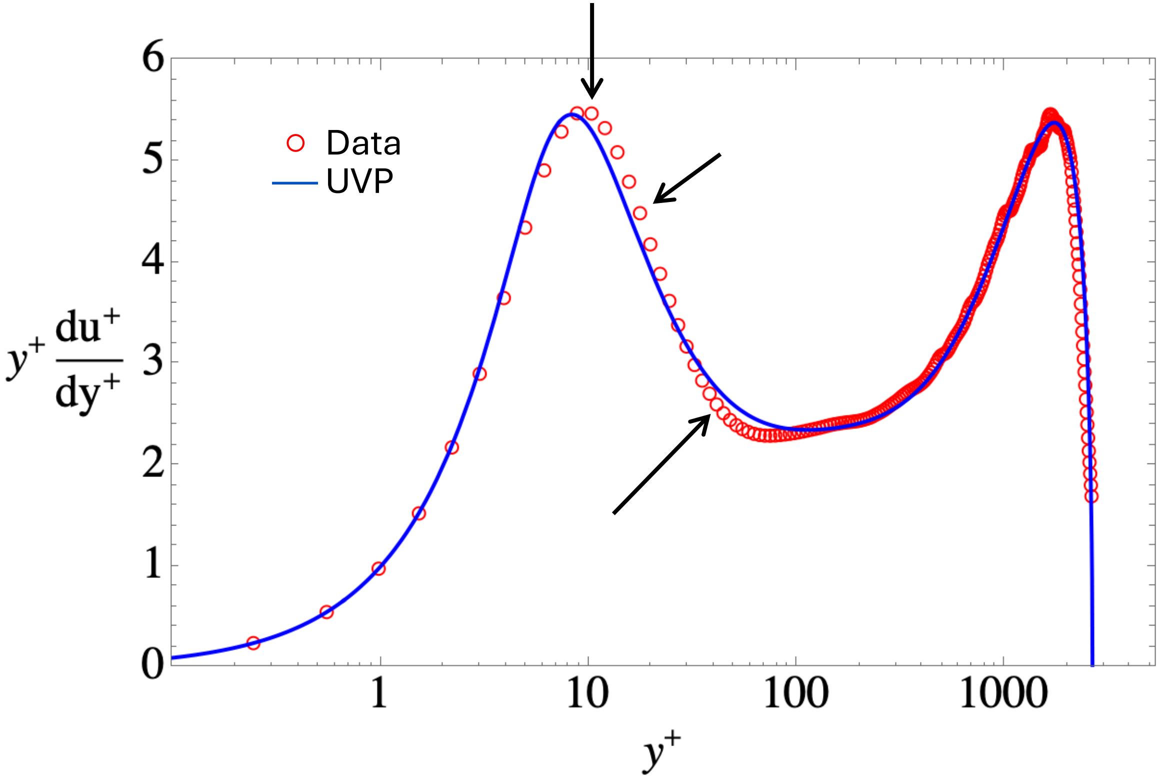


UVP Wall Damping Function

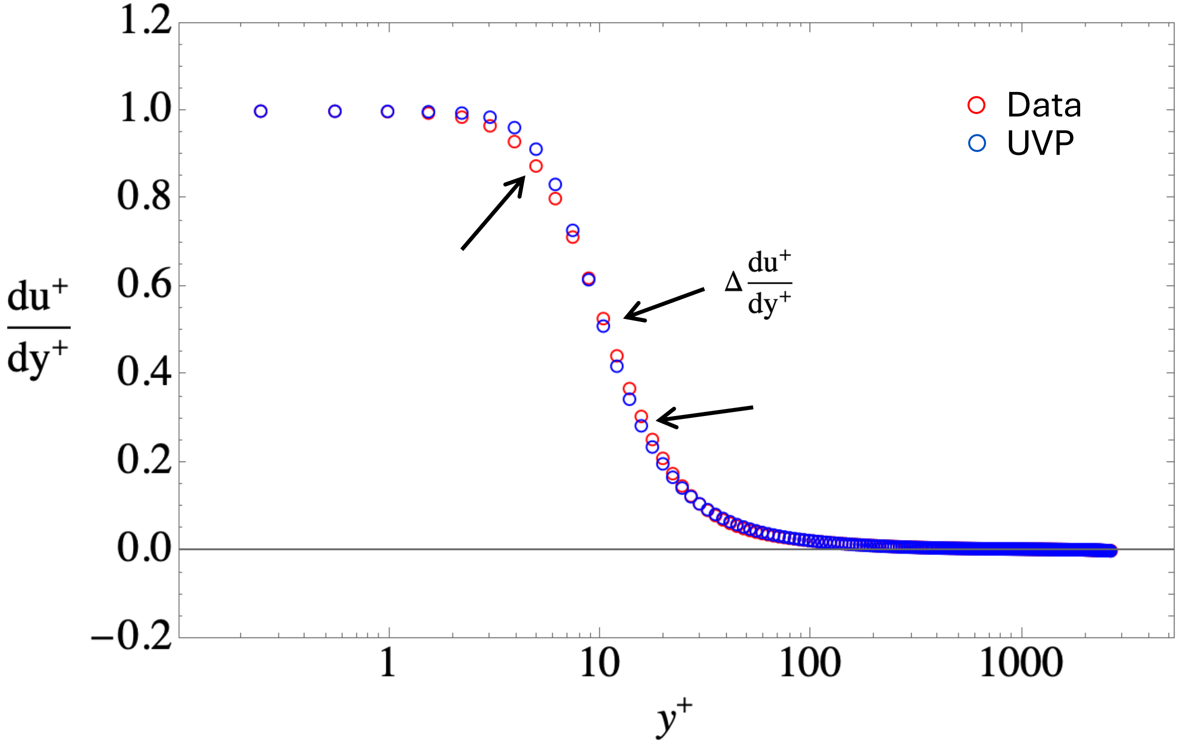


$R_\tau = 2652$ Velocity Derivative Comparison

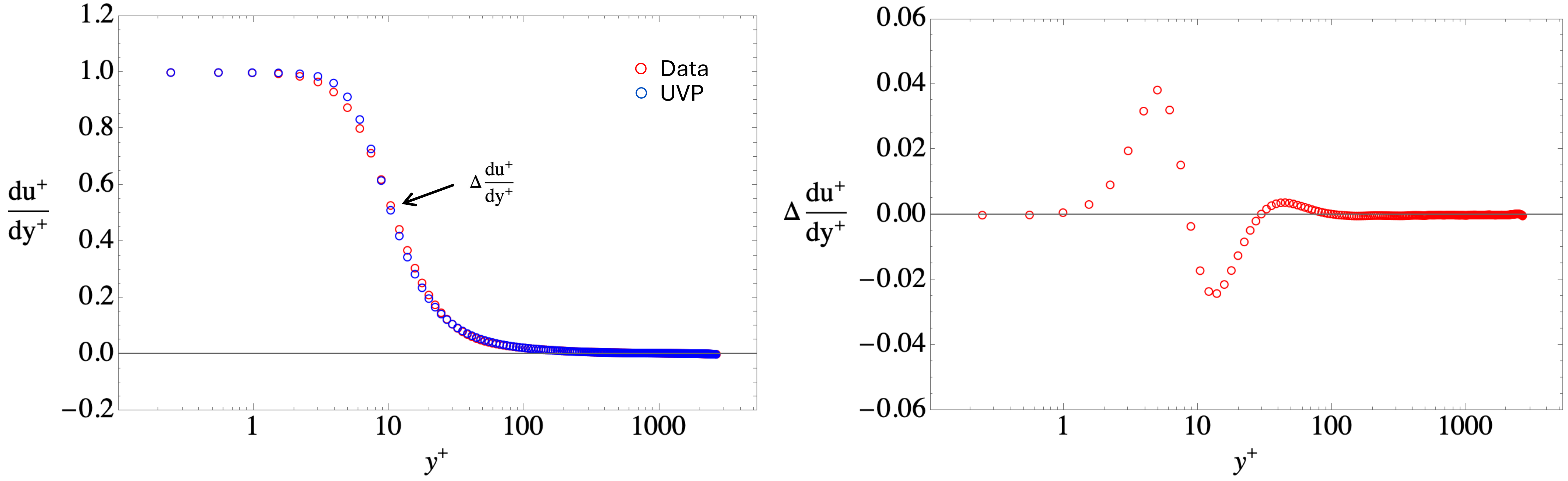
Log Indicator Function y^+dU^+/dy^+



Velocity Derivative dU^+/dy^+



Error in the $R_\tau = 2652$ velocity derivative



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UVP damping function

$$\lambda(y^+) = \frac{k y^+ \left(1 - e^{-(y^+/a)^m}\right)}{\left(1 + \left(\frac{y^+}{bR_\tau}\right)^n\right)^{1/n}}$$

<u>*Ask:*</u>

What damping function would enable the UVP to match the $R_\tau = 2652$ data exactly?

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Consider a generalized Wall Damping Function $\sigma(y^+/a)$. In effect, replace the two parameters, a and m in the function, $e^{-(y^+/a)^m}$, with just a damping length scale. The new damping length scale will still be designated *a*.

$$\lambda(y^+) = \frac{ky^+\left(1-\sigma\left(\frac{y^+}{a}\right)\right)}{\left(1+\left(\frac{y^+}{bR_\tau}\right)^n\right)^{1/n}}$$

What should we use for $\sigma(y^+/a)$?

$$\frac{du^+}{dy^+} = \frac{2\left(1-\frac{y^+}{R_\tau}\right)}{1+\left(1+4\left(\frac{ky^+\left(1-\sigma\left(\frac{y^+}{a}\right)\right)}{\left(1+\left(\frac{y^+}{bR_\tau}\right)^n\right)^{1/n}}\right)^2\left(1-\frac{y^+}{R_\tau}\right)\right)^{1/2}}$$

$$u^+(y^+) = \int_0^{y^+}\left[\frac{2\left(1-\frac{y^+}{R_\tau}\right)}{1+\left(1+4\left(\frac{ky^+\left(1-\sigma\left(\frac{y^+}{a}\right)\right)}{\left(1+\left(\frac{y^+}{bR_\tau}\right)^n\right)^{1/n}}\right)^2\left(1-\frac{y^+}{R_\tau}\right)\right)^{1/2}}\right]ds$$

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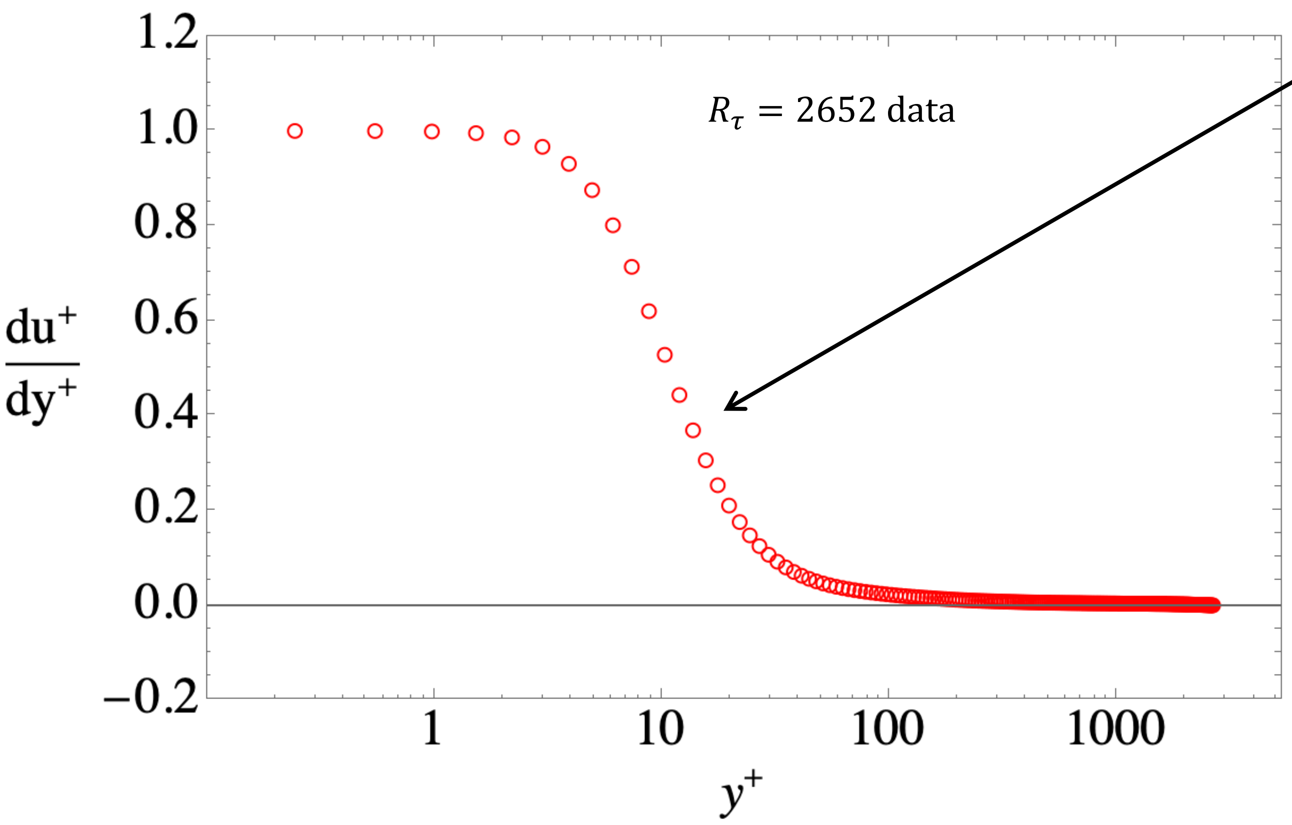
At each y_i^+ solve for the wall damping value, σ_i , in the UVP that exactly matches the derivative given by the data.

What should we use for $\sigma(y^+/a)$?

$k = 0.4221$, $b = 0.1724$, $n = 2.3087$

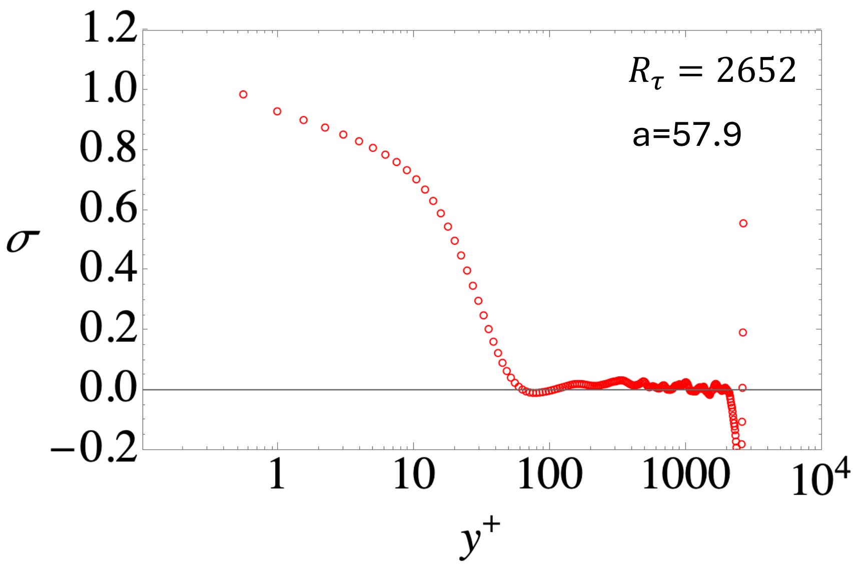
At each y_i^+ solve for σ_i.



$$\left(\frac{du^+}{dy^+}\right)_i = \frac{2\left(1-\frac{y_i^+}{R_\tau}\right)}{1+\left(1+4\left(\frac{k y_i^+(1-\sigma_i)}{\left(1+\left(\frac{y_i^+}{bR_\tau}\right)^n\right)^{1/n}}\right)^2\left(1-\frac{y_i^+}{R_\tau}\right)\right)^{1/2}}$$

From data

The result is an exact match between the UVP and the data



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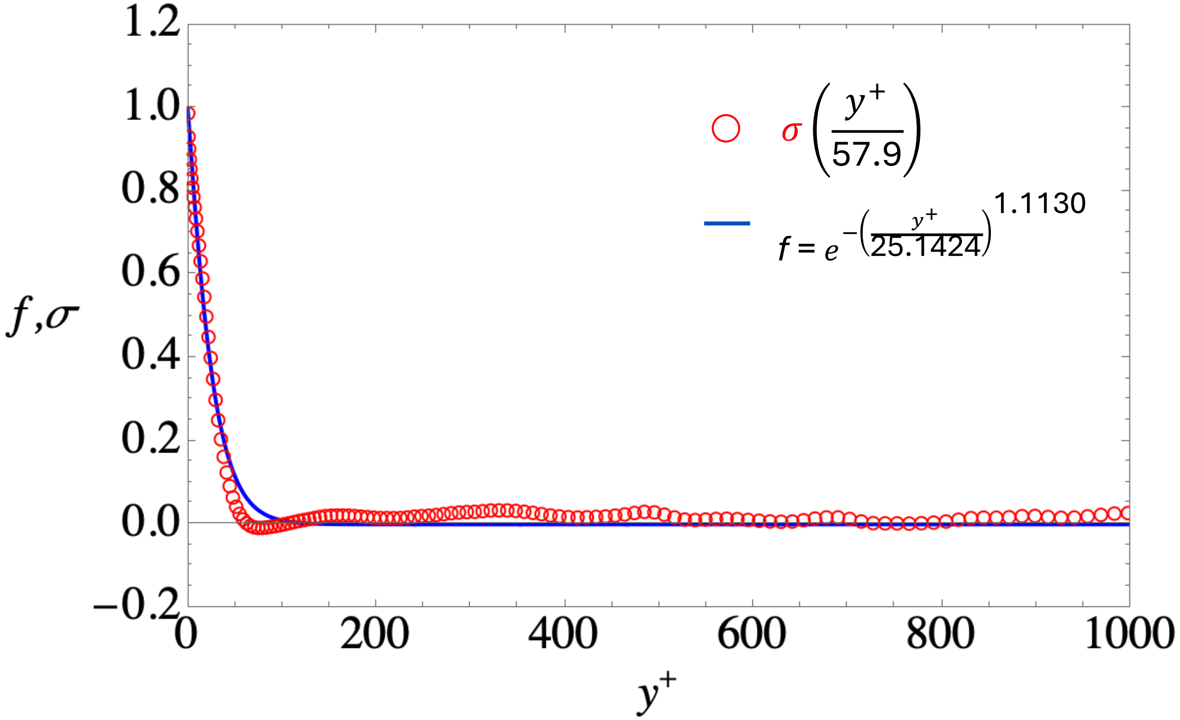
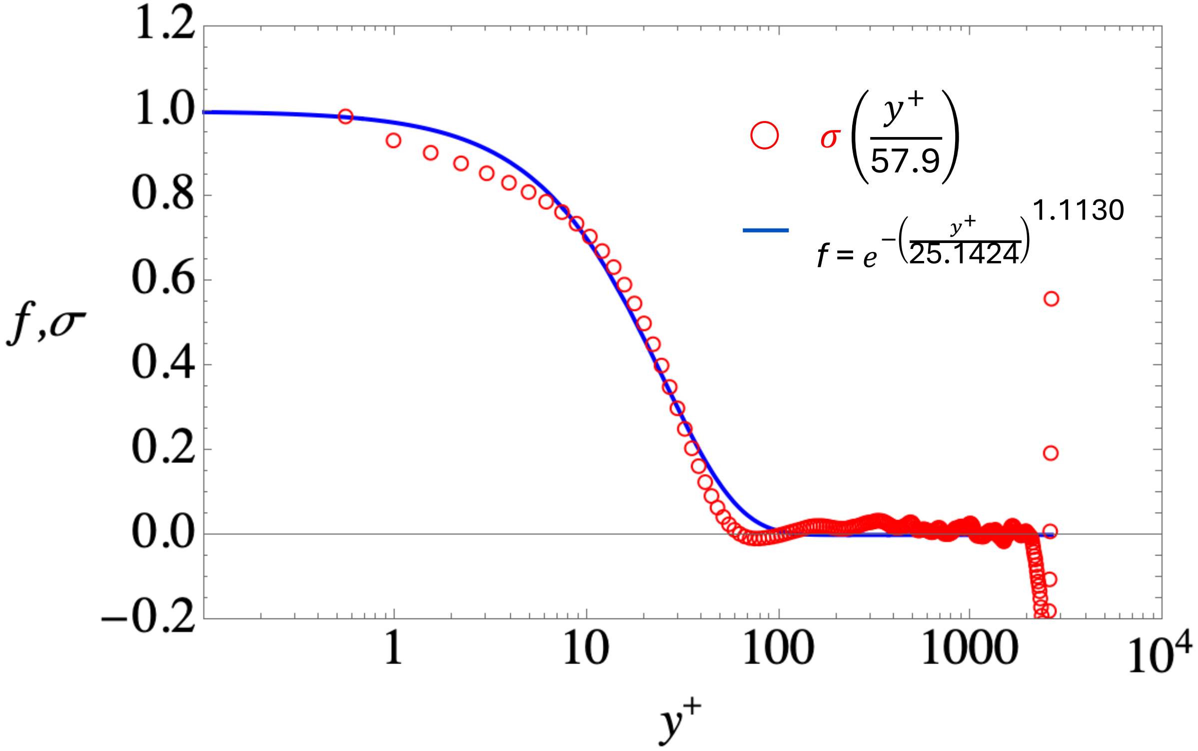
Comparison of old and new wall damping functions





The new damping function decays faster than the van Driest-type exponential and it oscillates about zero several times.

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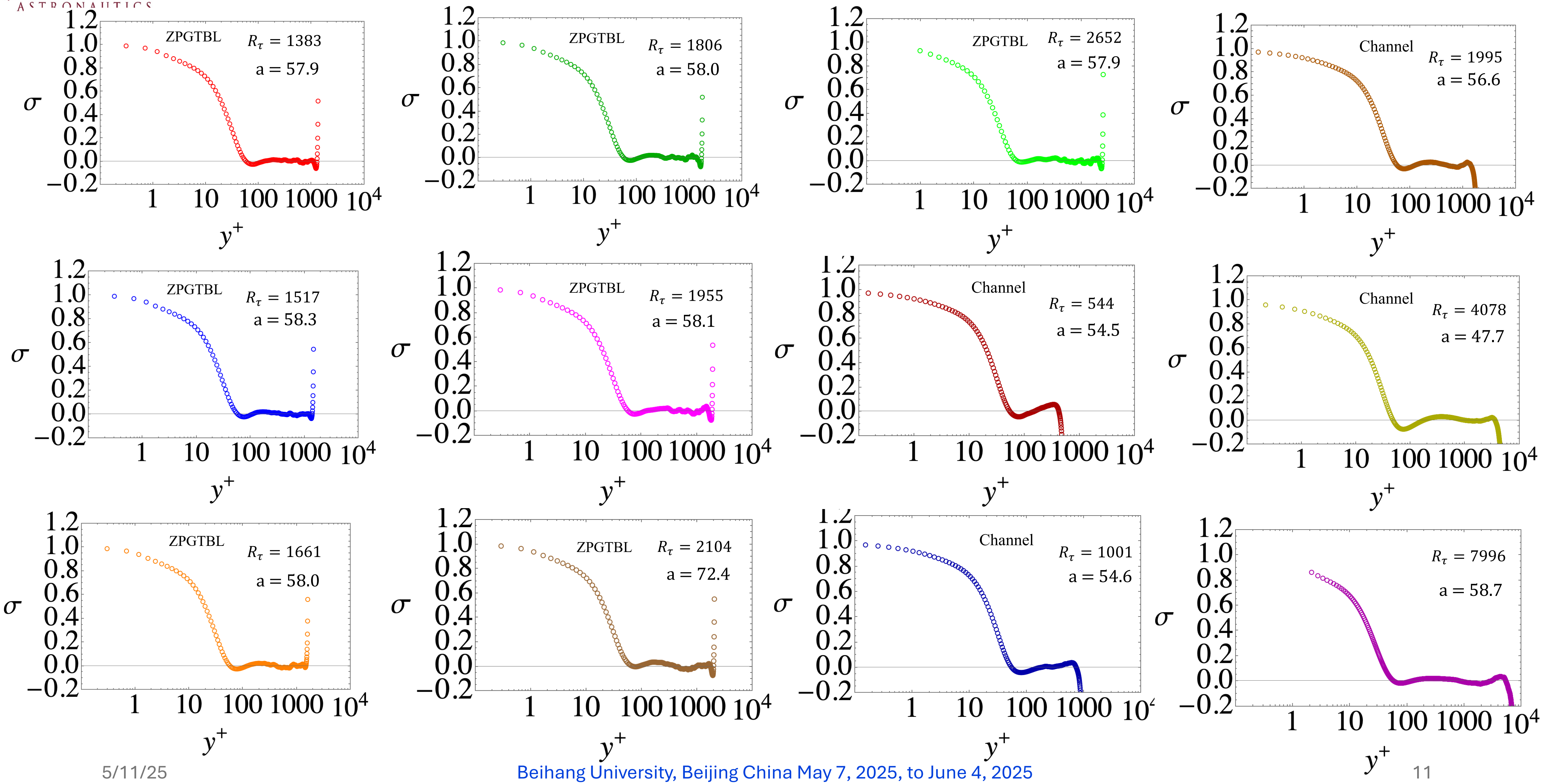
Wall damping functions for 7 ZPGTBL simulations and 5 Channel flow simulations. All show similar behavior



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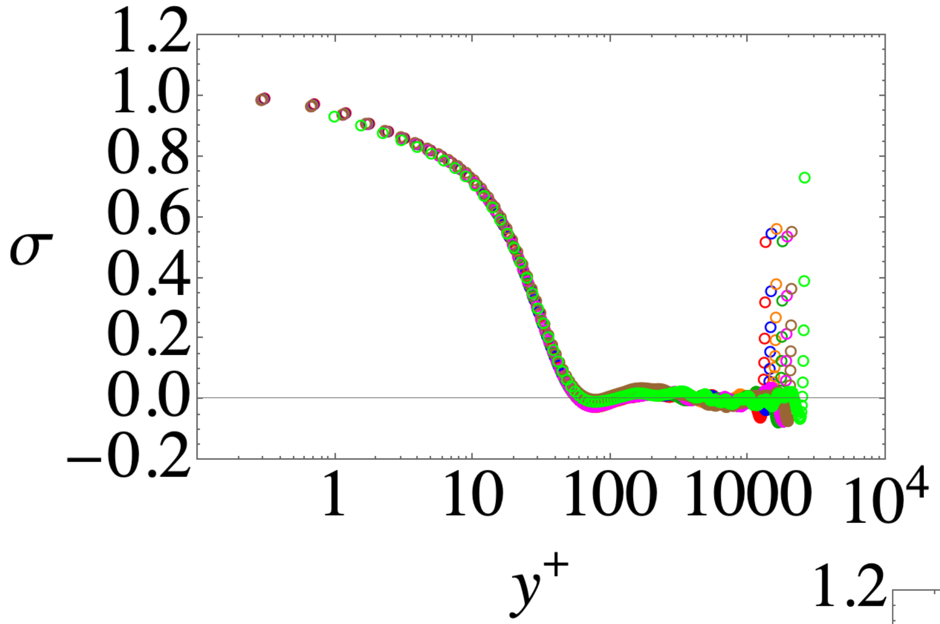
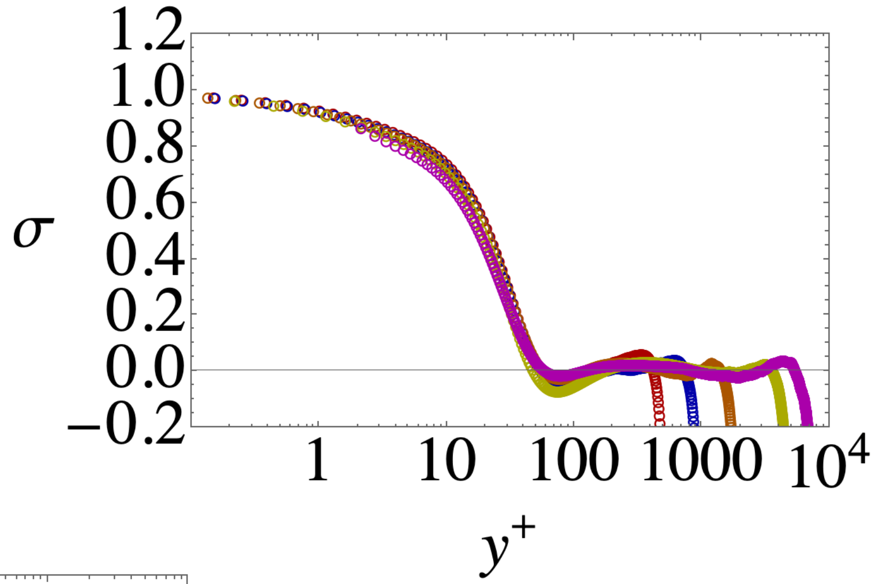
Damping functions superimposed

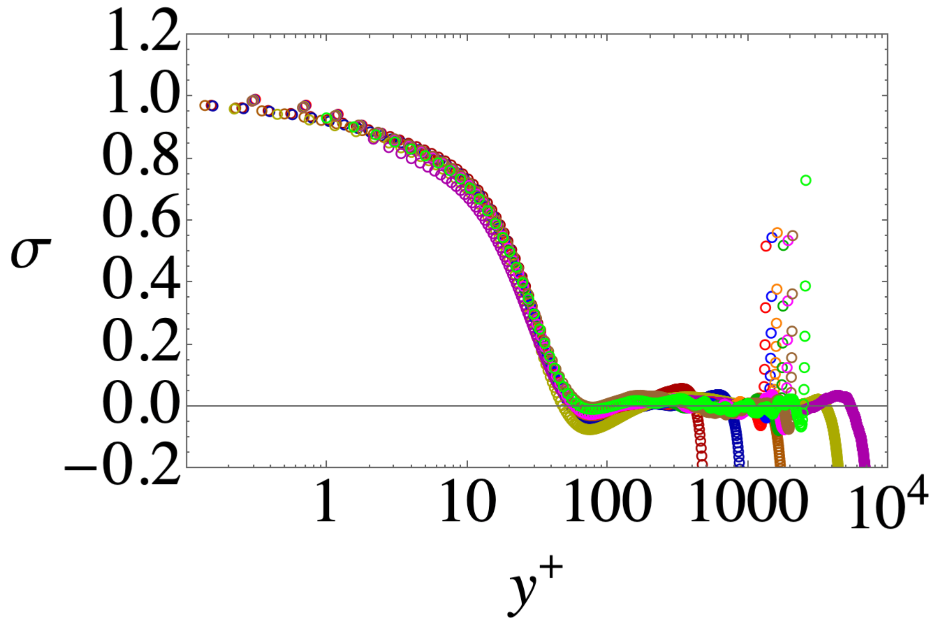
ZPGTBL



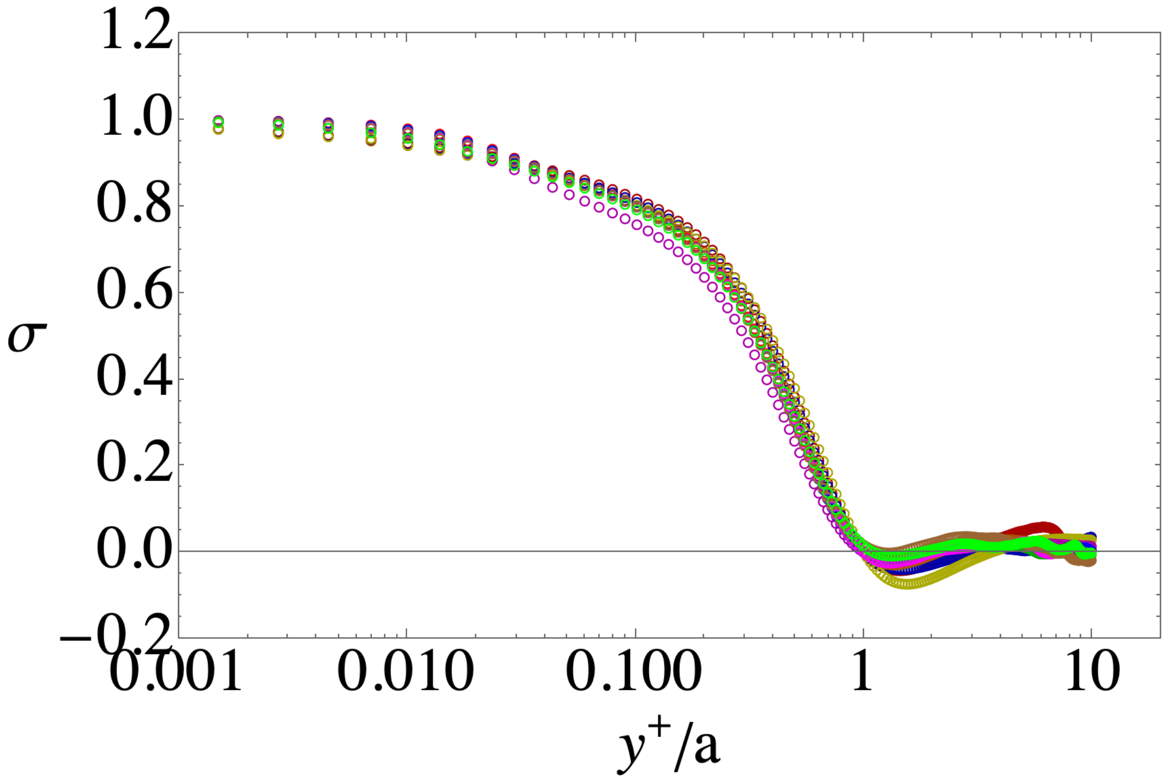
Channel

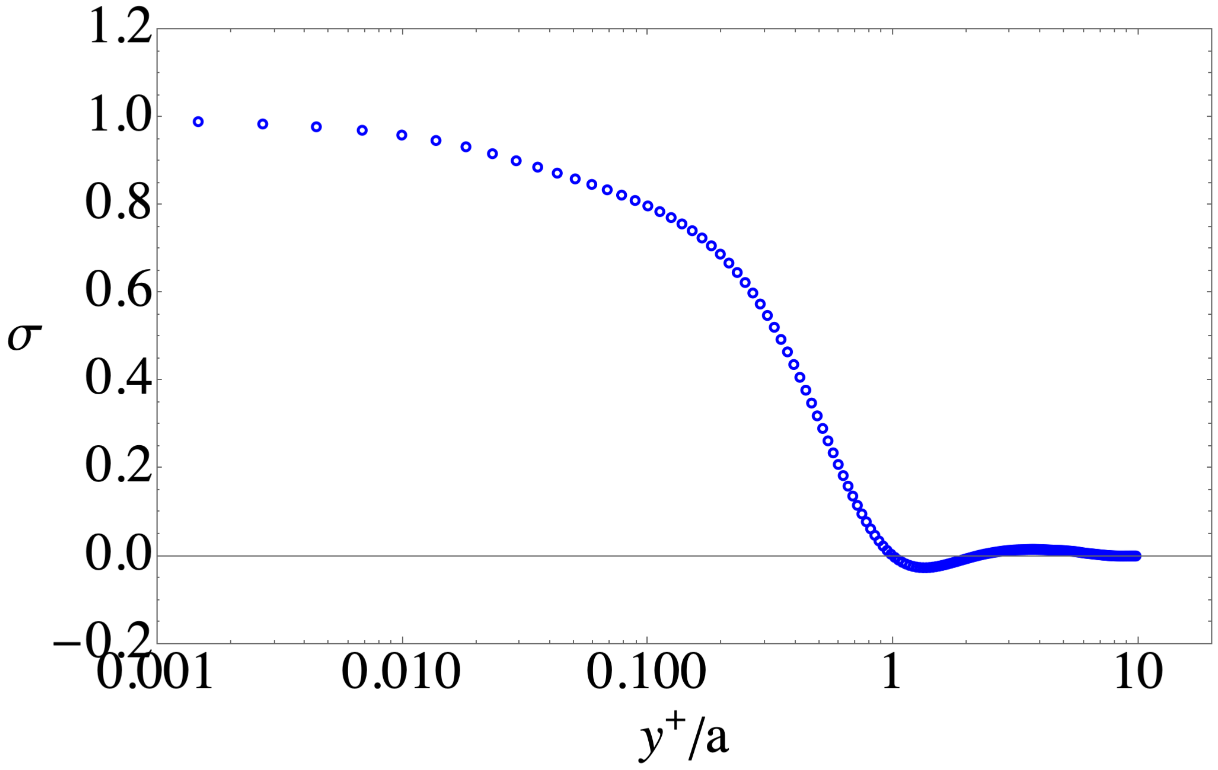


All



Normalize the individual damping functions by a and average over all 11 cases. Assume the average is a *Universal Wall Damping Function*.





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Universal Wall Damping Function $\sigma(y^+/a)$

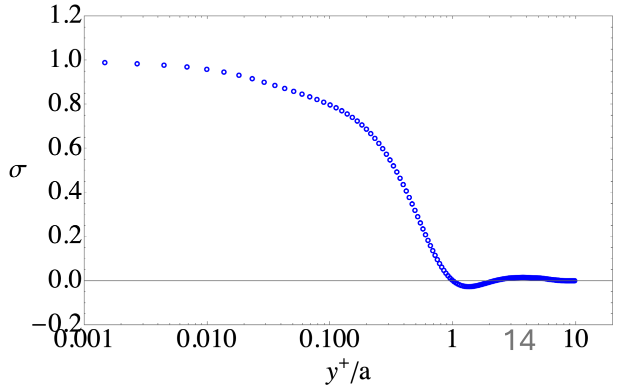
y^+/a	σ
0.	1.
0.0000487964	0.997775
0.000243979	0.997141
0.000683126	0.993751
0.0014638	0.989459
0.00268354	0.984131
0.00443986	0.977742
0.00683022	0.969817
0.00990325	0.958922
0.0136588	0.946515
0.0180964	0.932156
0.023216	0.91647
0.0290172	0.9004
0.0354994	0.885844
0.0426624	0.872147
0.0505055	0.859056
0.0590283	0.846473
0.0682301	0.834213
0.0781104	0.82208
0.0886683	0.809893
0.0999033	0.797445
0.111814	0.784519
0.124401	0.770922
0.137662	0.756473
0.151597	0.741023
0.166204	0.724453
0.181483	0.70668
0.197432	0.687653
0.214051	0.667353
0.231338	0.645791
0.249293	0.623001
0.267913	0.599044
0.287197	0.573998
0.307145	0.547956
0.327755	0.521027
0.349025	0.493333
0.370954	0.465006
0.39354	0.436188
0.416782	0.407037
0.440679	0.377716

y^+/a	σ
0.465228	0.348401
0.490428	0.319276
0.516276	0.29053
0.542773	0.262356
0.569914	0.23495
0.597699	0.208498
0.626126	0.183172
0.655193	0.159123
0.684897	0.136481
0.715237	0.115353
0.74621	0.0958219
0.777815	0.0779395
0.810048	0.0617214
0.842909	0.0471466
0.876394	0.0341665
0.910502	0.0227157
0.945229	0.0127238
0.980574	0.00411646
1.01653	-0.00318513
1.05311	-0.00927209
1.09029	-0.0142528
1.12808	-0.0182423
1.16647	-0.0213453
1.20547	-0.0236521
1.24507	-0.0252479
1.28526	-0.0262252
1.32605	-0.0266688
1.36743	-0.0266552
1.40939	-0.0262386
1.45194	-0.0254644
1.49508	-0.0243813
1.53879	-0.0230561
1.58308	-0.021552
1.62794	-0.0199221
1.67338	-0.0182071
1.71938	-0.0164353
1.76594	-0.0146366
1.81306	-0.0128423
1.86074	-0.0110423
1.90898	-0.00923637

y^+/a	σ
1.95777	-0.00745041
2.0071	-0.00571912
2.05698	-0.00405969
2.10739	-0.00245696
2.15835	-0.000915337
2.20983	0.00055083
2.26185	0.00192454
2.31439	0.00319879
2.36746	0.00437102
2.42104	0.00545728
2.47514	0.00647818
2.52975	0.00742566
2.58487	0.0082987
2.64049	0.00911475
2.69661	0.00989611
2.75323	0.0106404
2.81034	0.0113193
2.86794	0.0119149
2.92602	0.0124338
2.98458	0.0128852
3.04362	0.0132689
3.10313	0.0135865
3.1631	0.0138582
3.22354	0.0140958
3.28444	0.0142857
3.3458	0.014414
3.4076	0.014532
3.46985	0.0146858
3.53255	0.0148693
3.59568	0.0150339
3.65925	0.0151362
3.72324	0.0151699
3.78766	0.0151136
3.8525	0.0149905
3.91775	0.0148529
3.98342	0.0147332
4.04949	0.0146331
4.11597	0.0145032
4.18284	0.0143029
4.2501	0.0140619

y^+/a	σ
4.31775	0.0138424
4.38579	0.0136521
4.4542	0.0134659
4.52299	0.0132918
4.59215	0.0131658
4.66167	0.0130838
4.73155	0.0129952
4.80179	0.0128397
4.87237	0.0126341
4.9433	0.0124007
5.01457	0.0121406
5.08618	0.0118308
5.15811	0.0115039
5.23037	0.0111992
5.30295	0.0109149
5.37585	0.0106297
5.44905	0.0103251
5.52256	0.00996405
5.59637	0.00954495
5.67047	0.00908858
5.74487	0.00860224
5.81954	0.00809378
5.8945	0.00759278
5.96973	0.00713966
6.04523	0.0067288
6.12099	0.0063529
6.19701	0.0059891
6.27328	0.00564721
6.3498	0.00532615
6.42656	0.00502192
6.50356	0.00471528
6.5808	0.00440004
6.65825	0.00408668
6.73593	0.00377557
6.81382	0.00346095
6.89193	0.00315952
6.97023	0.00287094
7.04874	0.00259035
7.12744	0.00230733
7.20633	0.00203514

y^+/a	σ
7.2854	0.00178769
7.36465	0.00155912
7.44407	0.00133184
7.52366	0.0011101
7.60341	0.000904109
7.68331	0.000715307
7.76336	0.000544001
7.84356	0.00039447
7.92389	0.000270937
8.00436	0.00017198
8.08496	0.0000960978
8.16567	0.0000404871
8.24651	2.29574×10^{-6}
8.32745	-0.0000222057
8.4085	-0.0000356204
8.48965	-0.0000410072
8.57089	-0.0000405188
8.65222	-0.0000362685
8.73363	-0.0000301867
8.81512	-0.0000237747
8.89668	-0.0000179403
8.9783	-0.0000130912
9.05998	-9.27361×10^{-6}
9.14172	-6.38029×10^{-6}
9.22351	-4.26843×10^{-6}
9.30534	-2.79322×10^{-6}
9.38721	-1.7995×10^{-6}
9.4691	-1.15047×10^{-6}
9.55103	-7.38603×10^{-7}
9.63297	-4.86361×10^{-7}
9.71493	-3.45919×10^{-7}
9.79689	-3.34863×10^{-7}



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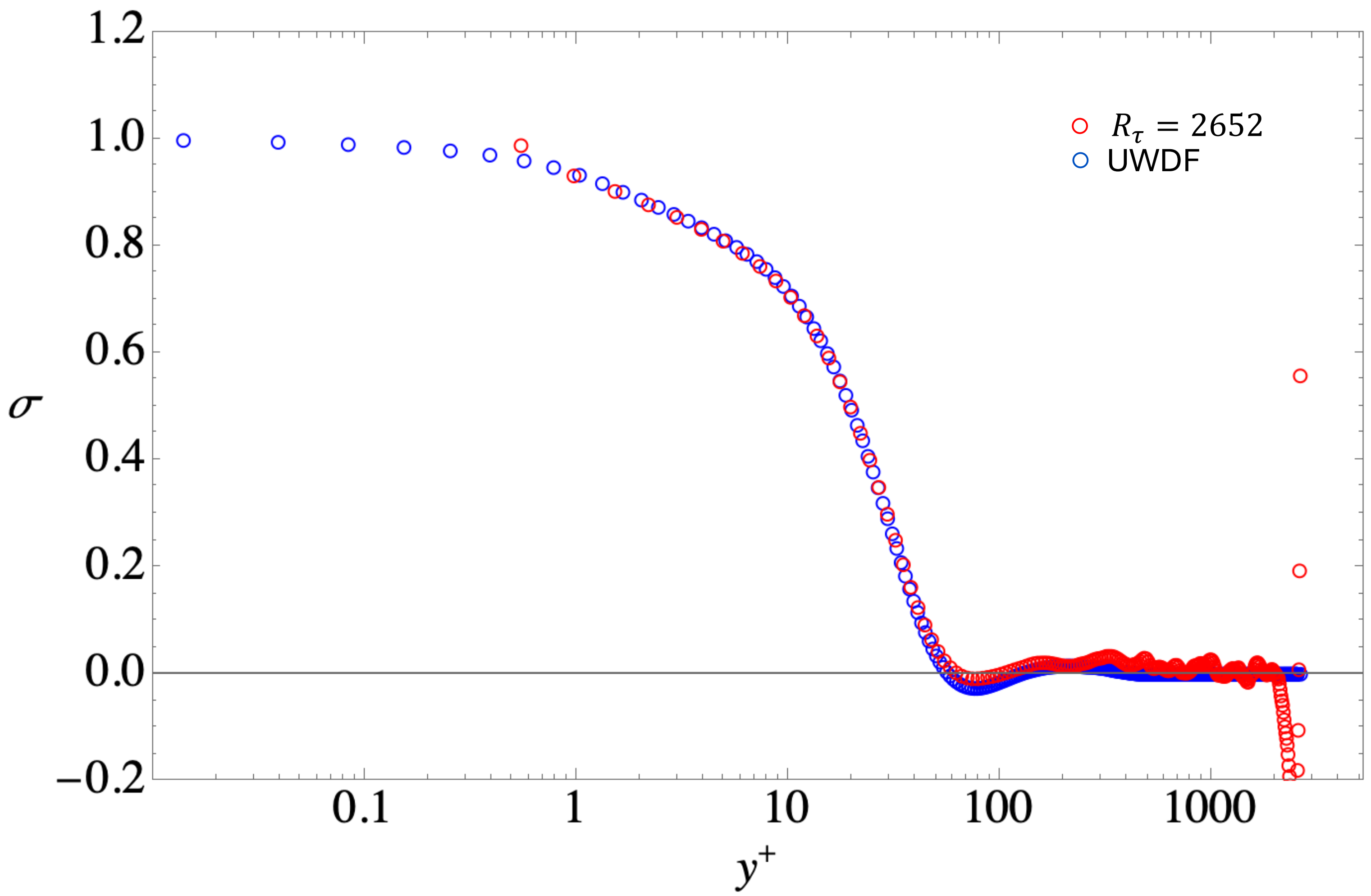
$R_\tau = 2652$ wall damping function compared to the Universal Wall Damping Function.



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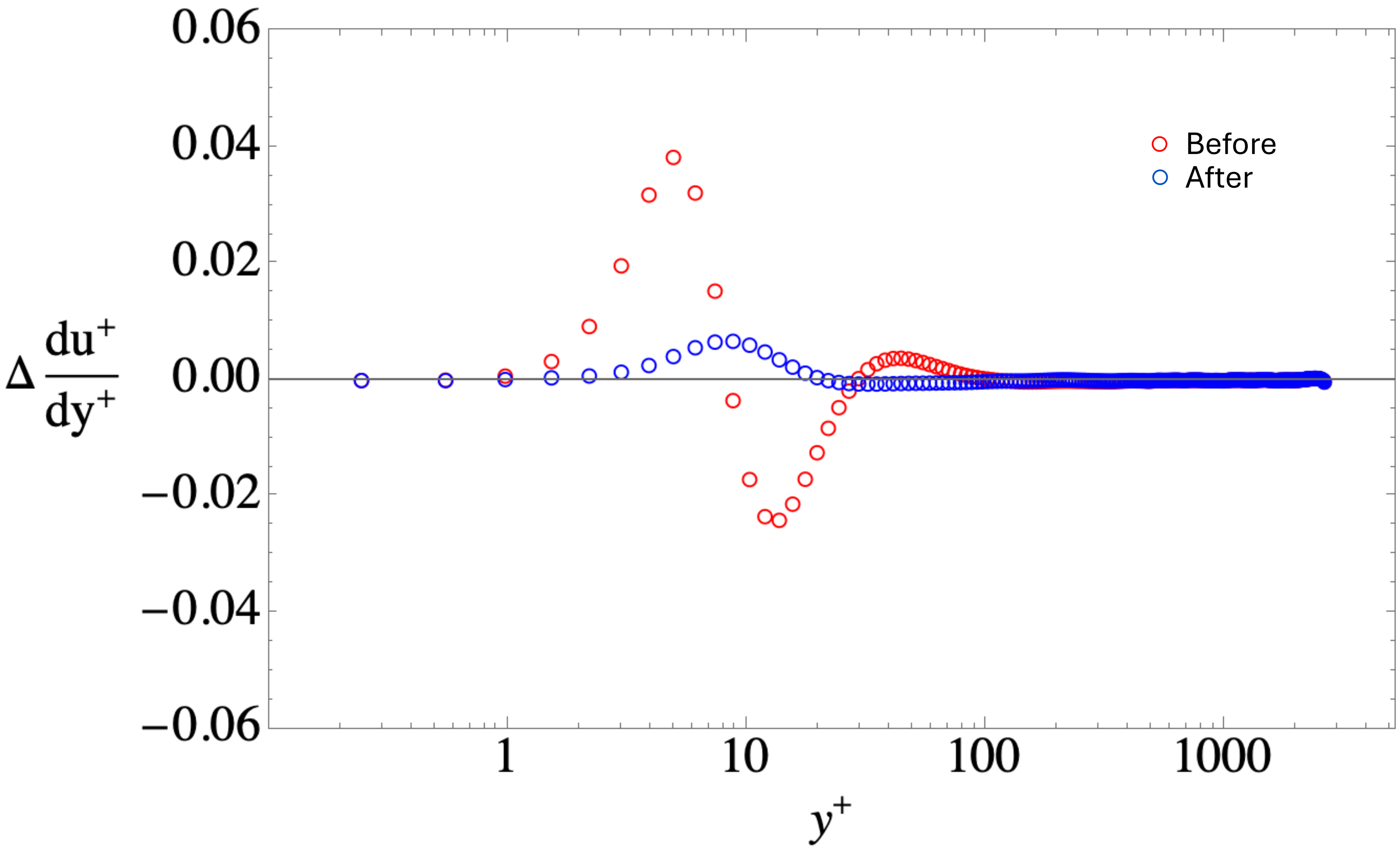
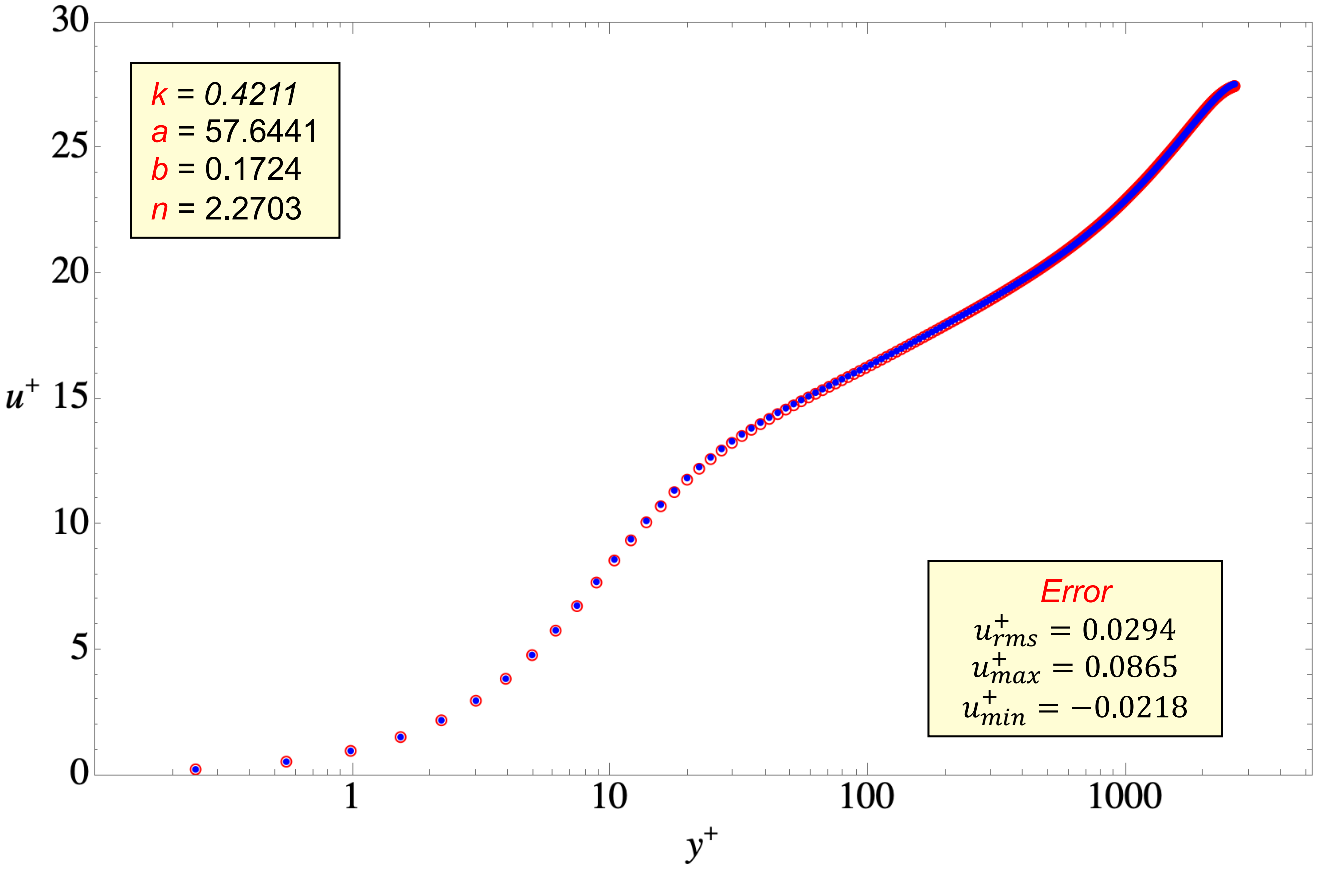
$R_\tau = 2652$ reduction in the velocity derivative error using $\sigma(y^+/a)$.



$R_\tau = 2652$ reduction in the velocity error using $\sigma(y^+/a)$.



Optimum values of k, b and n remain about the same.

$k = 0.4221$
$a = 25.1424$
$m = 1.1130$
$b = 0.1724$
$n = 2.3087$

Previous error using the exponential damping function

Error
$u^+_{rms} = 0.0587$
$u^+_{max} = 0.1592$
$u^+_{min} = -0.0980$

$$u^+(y^+) = \int_0^{y^+} \left[\frac{2\left(1 - \frac{y^+}{R_\tau}\right)}{1 + \left(1 + 4\left(\frac{ky^+\left(1-\sigma\left(\frac{y^+}{a}\right)\right)}{\left(1+\left(\frac{y^+}{bR_\tau}\right)^n\right)^{1/n}}\right)^2\left(1 - \frac{y^+}{R_\tau}\right)\right)^{1/2}} \right] ds$$

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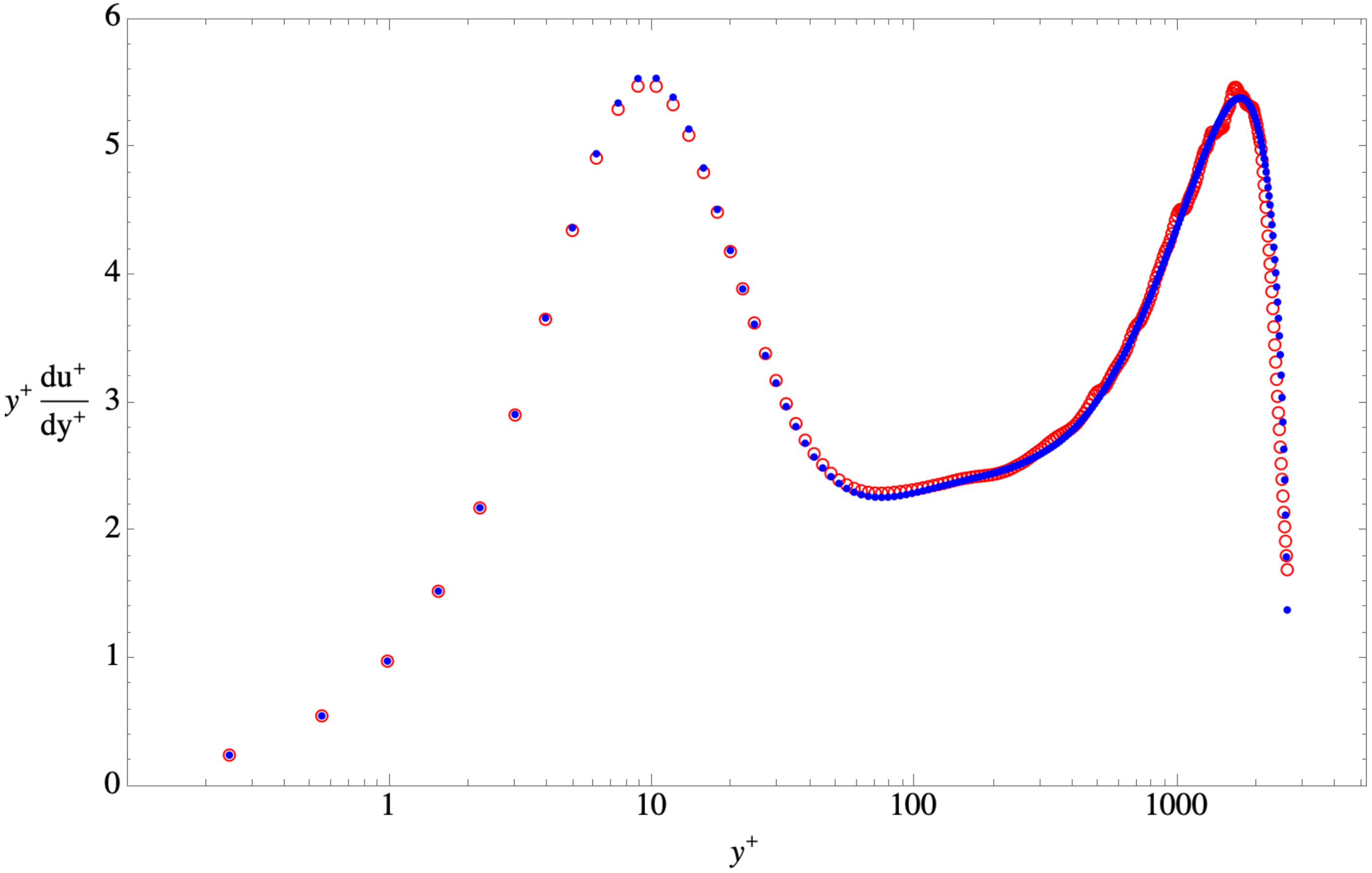
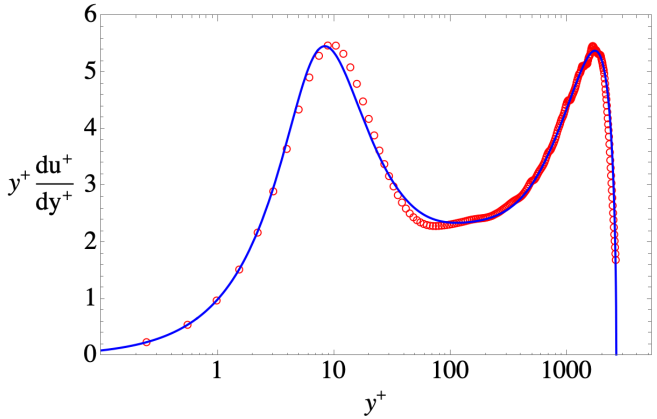
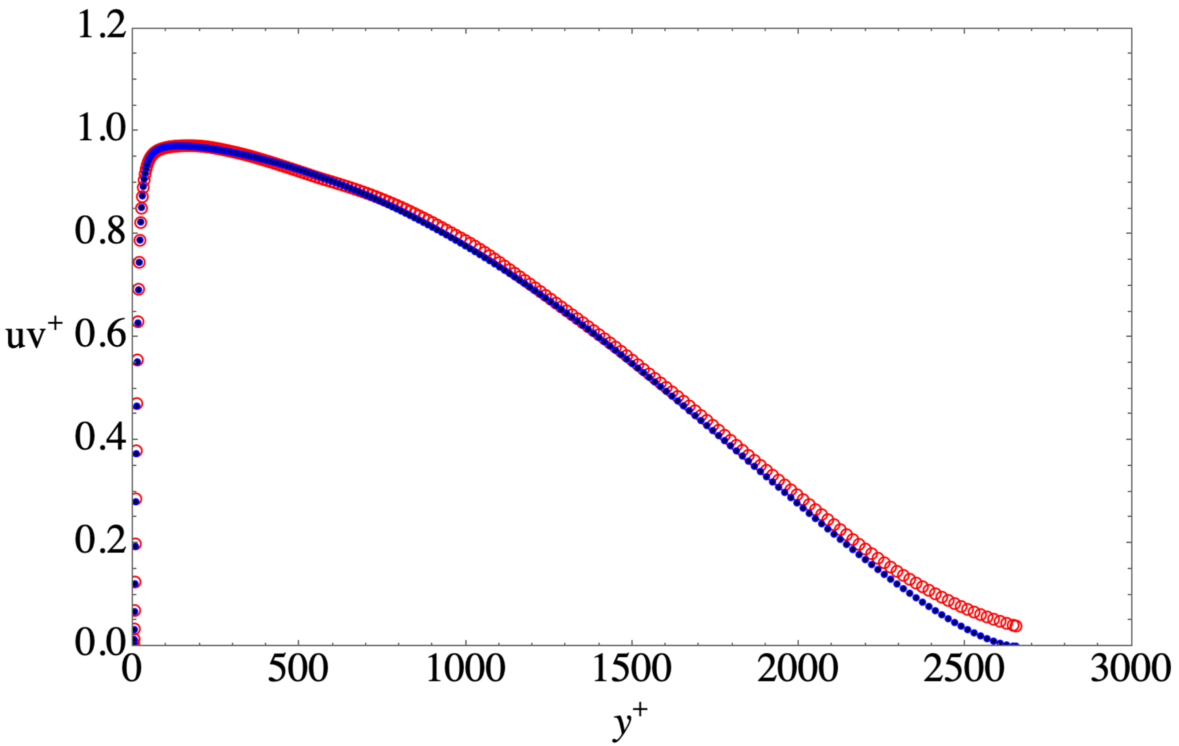
$R_\tau = 2652,$ reduced error in the log indicator function.

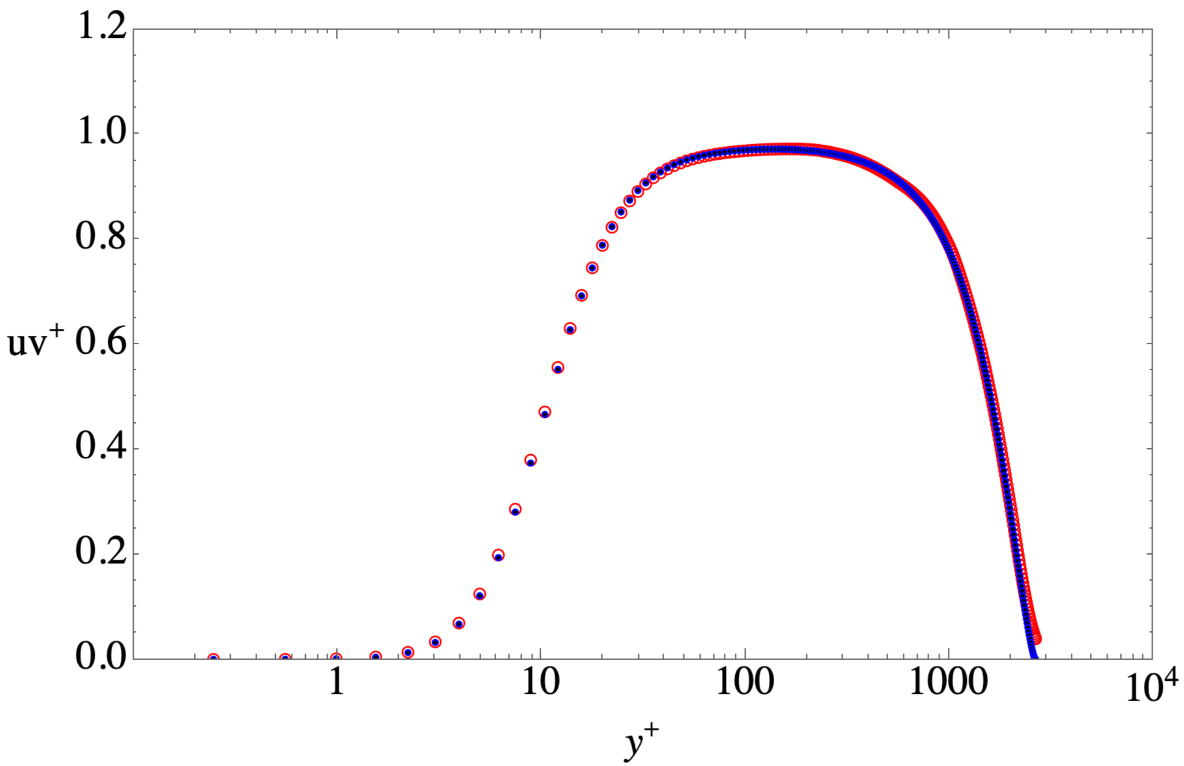


With exponential damping



$R_\tau = 2652,$ UVP Reynolds shear stress closely matches DNS

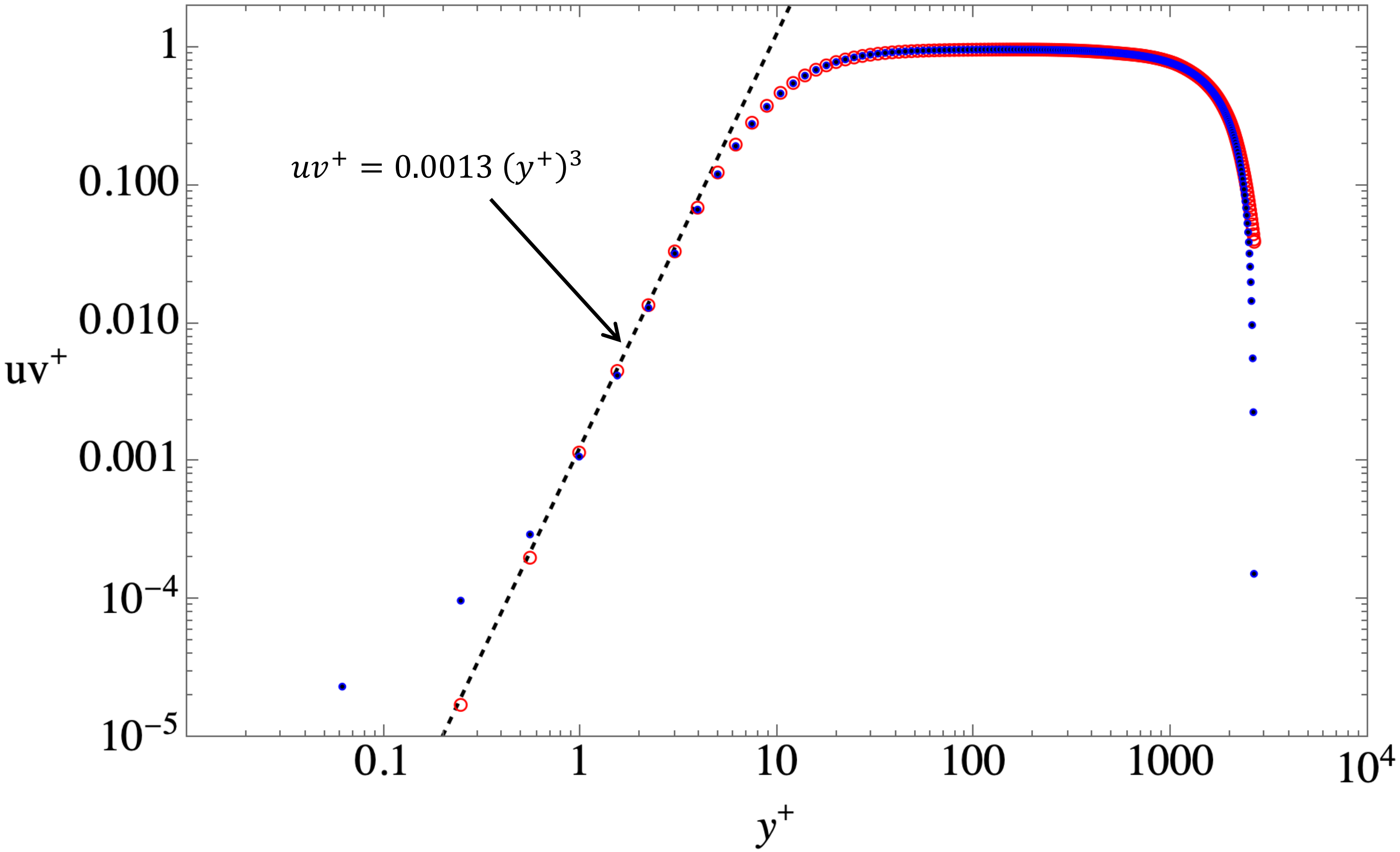




Generation of the Reynolds shear stress for boundary layers requires the boundary layer equation to be integrated.

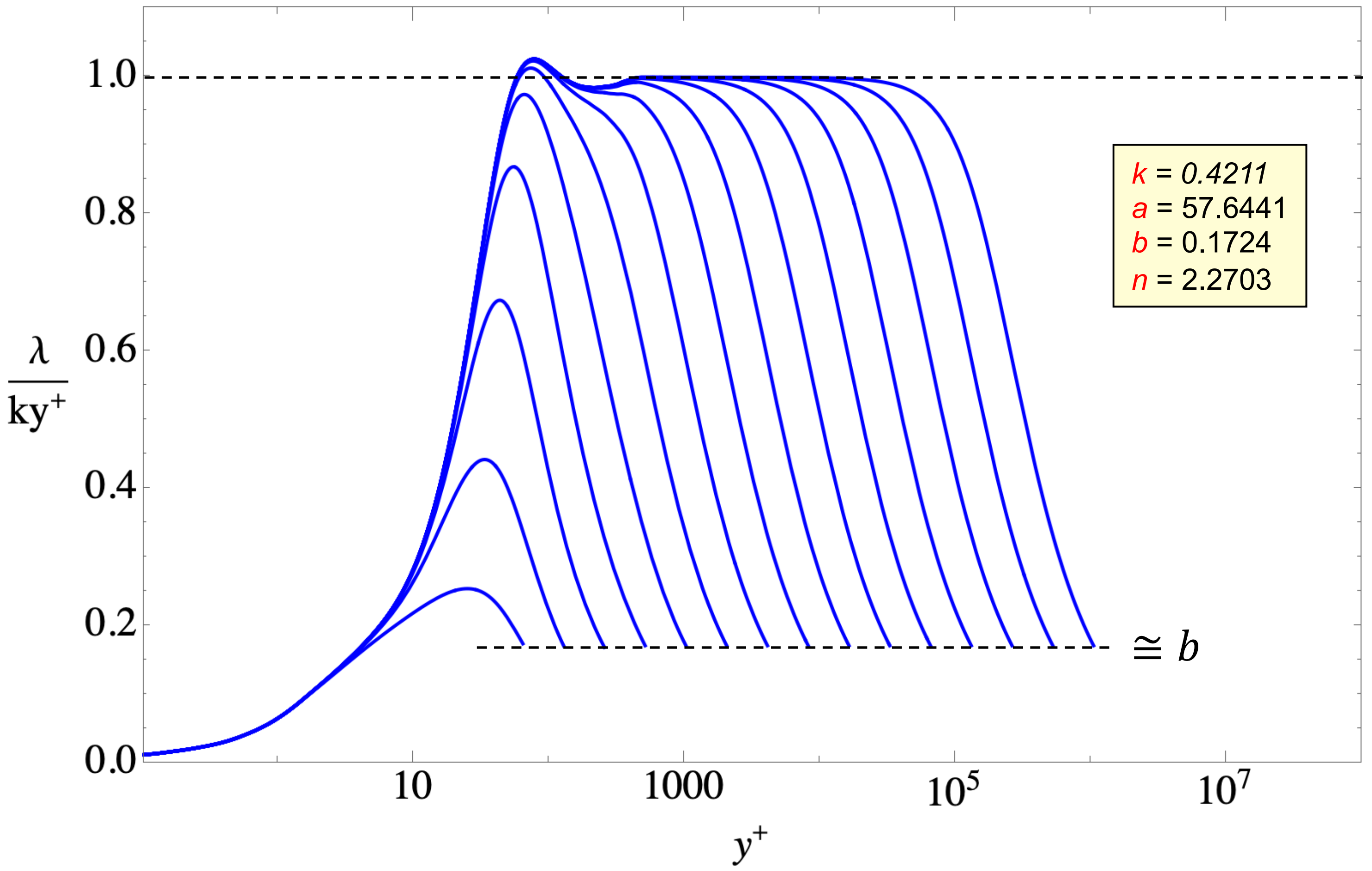
$$\mathbf{uv}^+ = 1 - \frac{\partial u^+}{\partial y^+} - \left(\frac{u^+ \frac{\partial}{\partial R_\tau} \int_0^{y^+} u^+ \mathrm{d}y^+ - \frac{\partial}{\partial R_\tau} \int_0^{y^+} u^{+2} \mathrm{d}y^+ - \left(\frac{u_e}{u_\tau}\right) \frac{\mathrm{d}}{\mathrm{d}R_\tau} \left(\frac{u_\tau}{u_e}\right) \int_0^{y^+} u^{+2} \mathrm{d}y^+}{\left(\frac{u_e}{u_\tau}\right) \frac{\mathrm{d}}{\mathrm{d}R_\tau} \int_0^{R_\tau} u^+ \mathrm{d}y^+ - \frac{\mathrm{d}}{\mathrm{d}R_\tau} \int_0^{R_\tau} u^{+2} \mathrm{d}y^+ - \left(\frac{u_e}{u_\tau}\right) \frac{\mathrm{d}}{\mathrm{d}R_\tau} \left(\frac{u_\tau}{u_e}\right) \int_0^{R_\tau} u^{+2} \mathrm{d}y^+} \right)$$

$R_\tau = 2652,$ UVP Reynolds shear stress is proportional to $(y^+)^3$ near the wall.

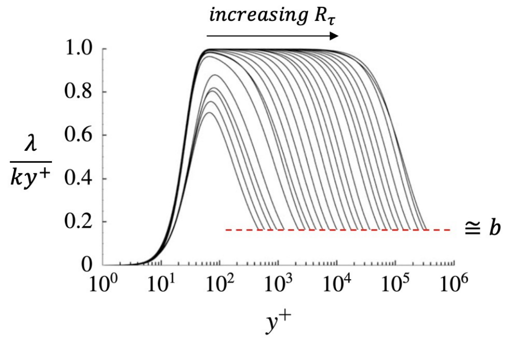


New mixing length function, λ/ky^+ versus R_τ using parameters from $R_\tau = 2652$ data

The new λ/ku^+ has a peak of 1.027 at $y^+ = 77.54$



With exponential damping



$R_\tau =$ 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768, 65536, 131072, 262144, 524288, 1,048576

Determination of best fit model parameters

Minimize G with respect to k, a, b, n

$$G = \sum_{i=1}^{N} \left(u^+(k, a, m, b, n, y_i^+) - u_i^+(y_i^+)\right)^2$$

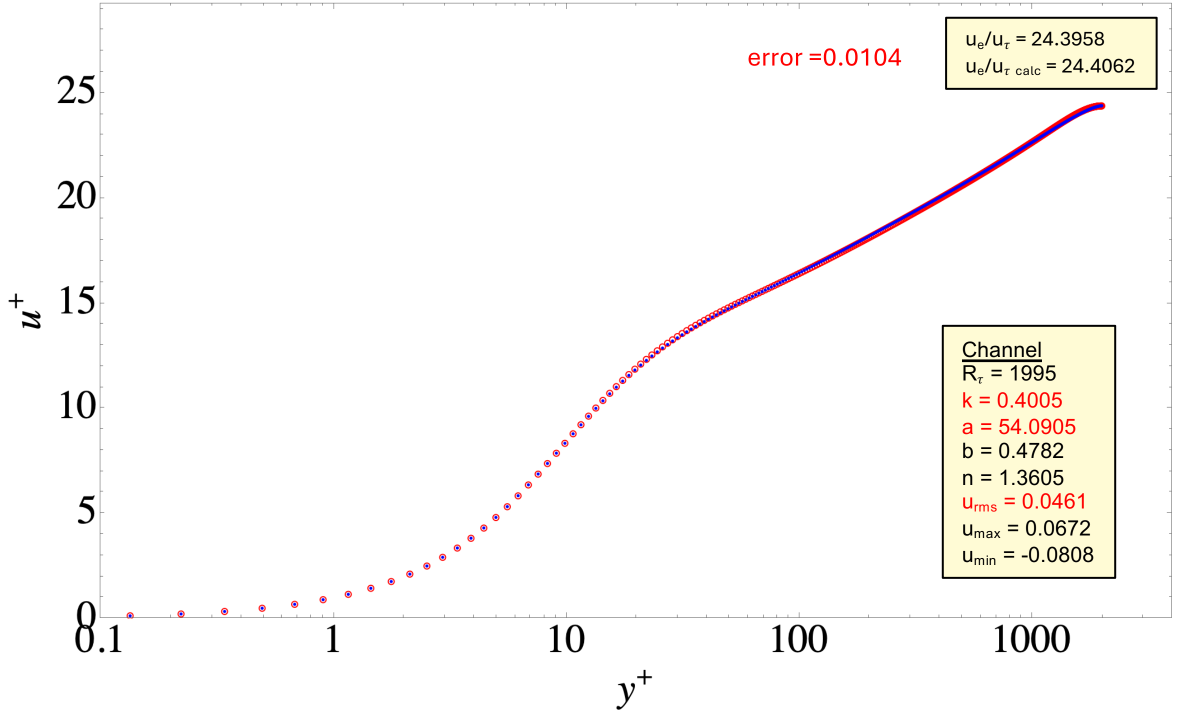
UVP with new damping function

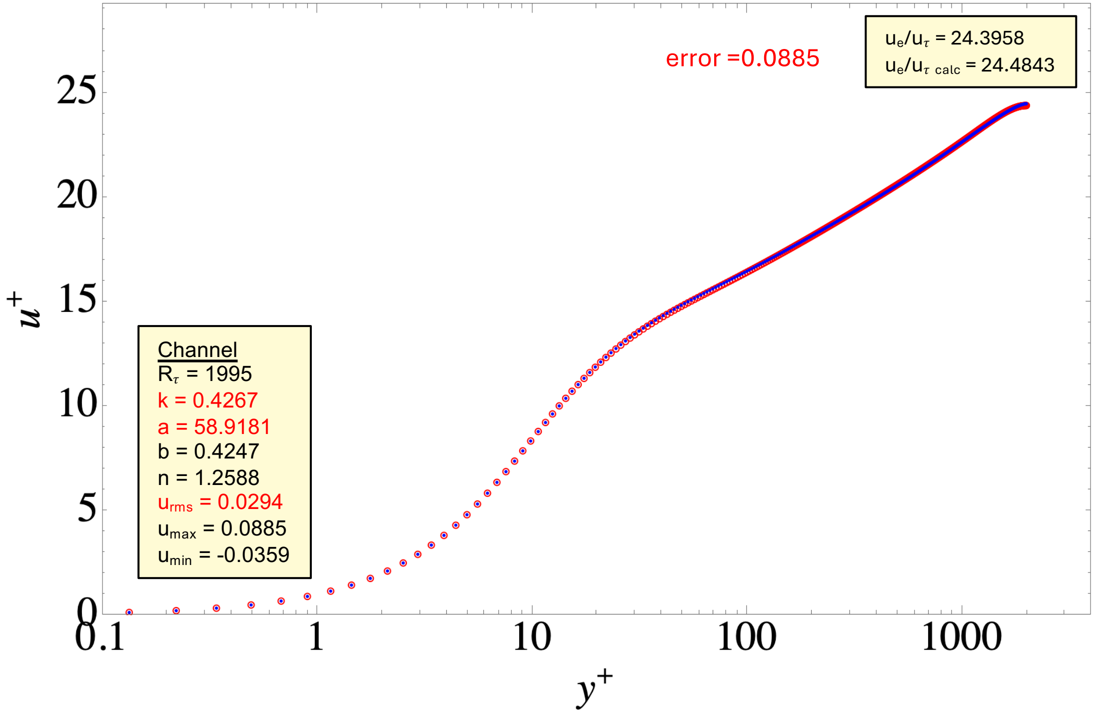
DNS data

Finding the minimum is much easier!

At low Reynolds number, when the error is small, further reducing the error to reach a minimum might not produce the most accurate value of the friction.

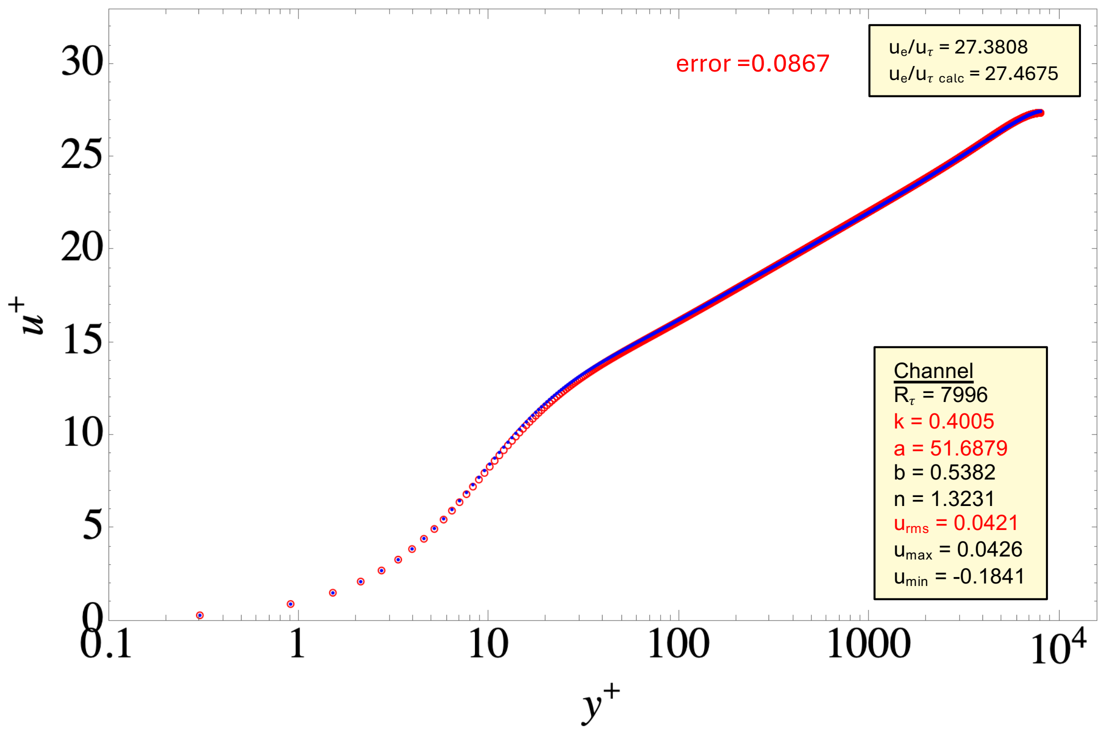
At low Reynolds number the optimal values of *k* and *a* tend to track each other.

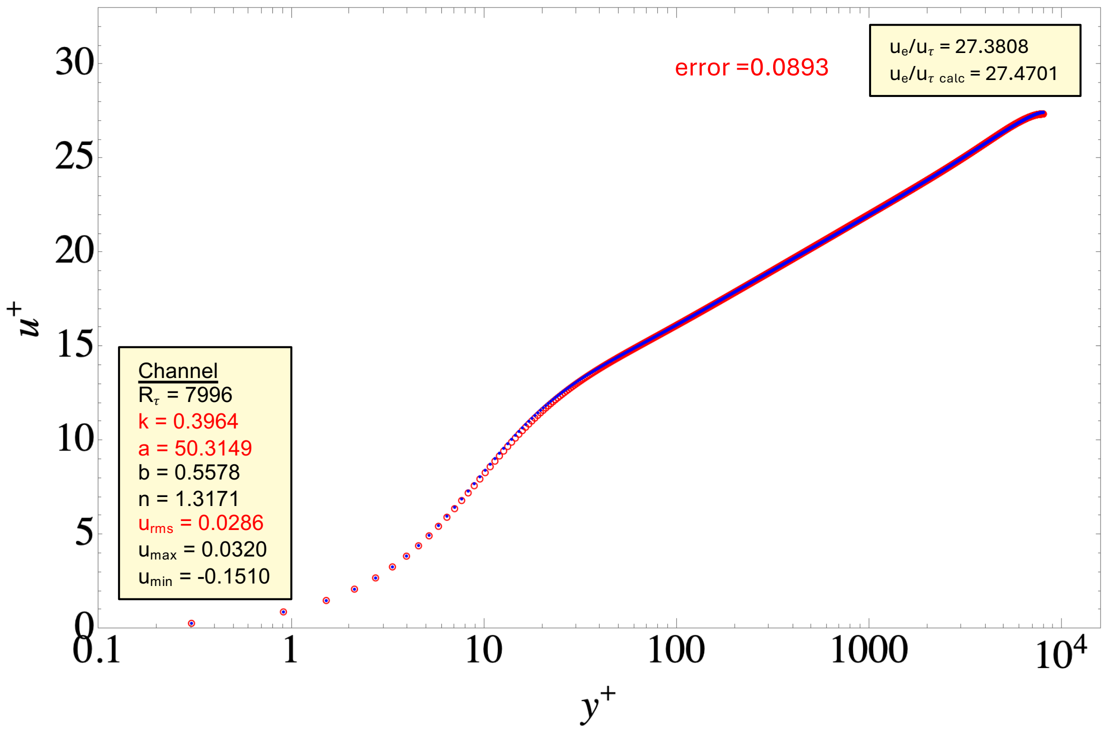




At moderate Reynolds numbers profiles with somewhat different (but still very small) levels of accuracy may give almost identical values of the friction.

At moderate Reynolds number the optimal values of *k* and *a* tend to track each other.





At low and moderate Reynolds number, by not requiring the most extreme level of accuracy, a 'soft constraint' can be applied to *k* and *a* when minimizing the error. This may not be true at high Reynolds number, R_τ > 20,000. Experience suggests that for large R_τ the minima in parameter space may lie very close to one another.

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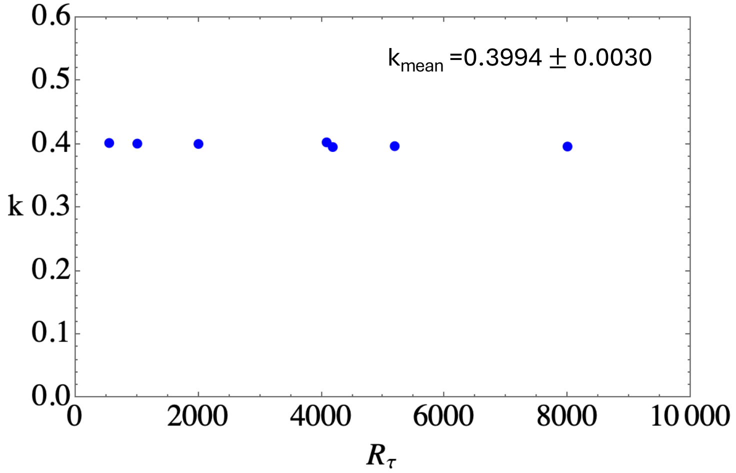
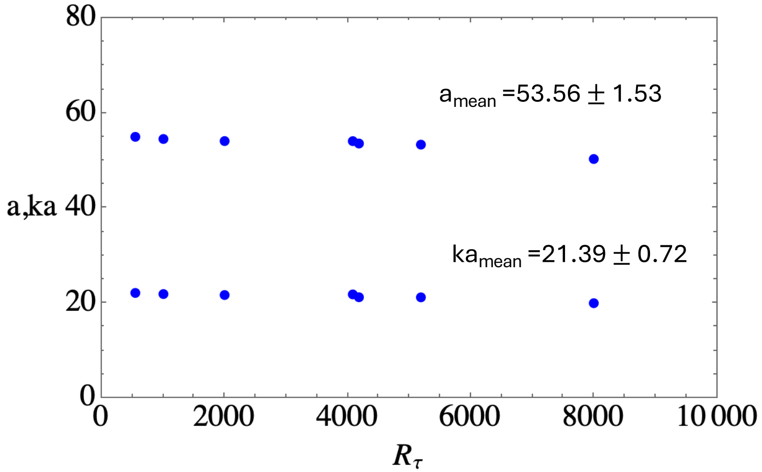
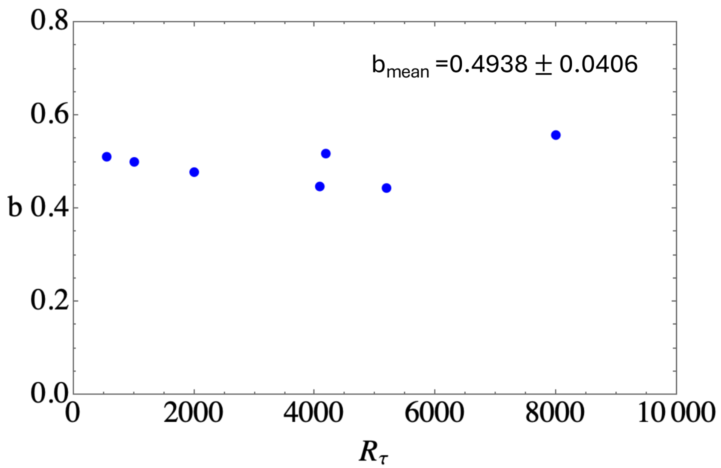
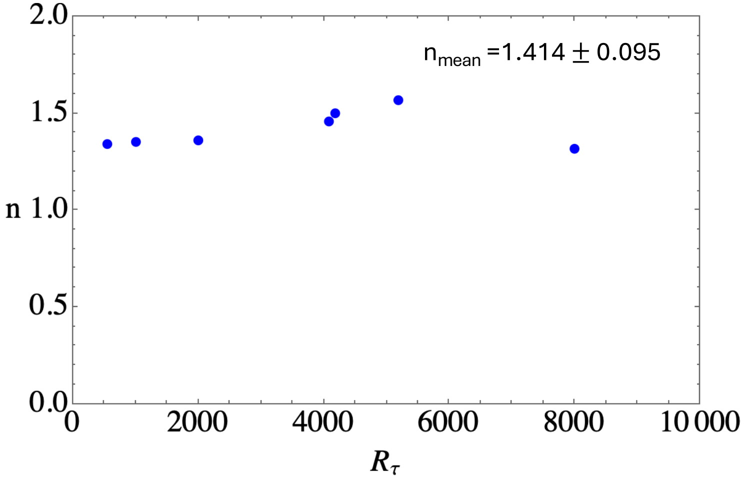
R_τ	$\frac{u_e}{u_{\tau\, data}}$	$\frac{u_e}{u_{\tau\, uvp}}$	k	a	b	n	u^+_{rms}	u^+_{max}	u^+_{min}
543.500	21.0003	21.0081	0.4020	54.9869	0.5112	1.3415	0.0554	0.0760	-0.1141
1000.51	22.5929	22.5875	0.4010	54.5329	0.5002	1.3525	0.0583	0.0922	-0.0797
1995.00	24.3958	24.4062	0.4005	54.0905	0.4782	1.3605	0.0461	0.0672	-0.0808
4078.86	25.9545	26.0442	0.4030	54.0763	0.4474	1.4582	0.0429	0.0897	-0.0549
4178.88	25.9565	25.9685	0.3956	53.5765	0.5180	1.5004	0.0419	0.0980	-0.0738
5185.90	26.5753	26.6711	0.3970	53.3417	0.4441	1.5680	0.0333	0.0958	-0.0380
7996.01	27.3808	27.4701	0.3964	50.3149	0.5578	1.3171	0.0286	0.0320	-0.1510

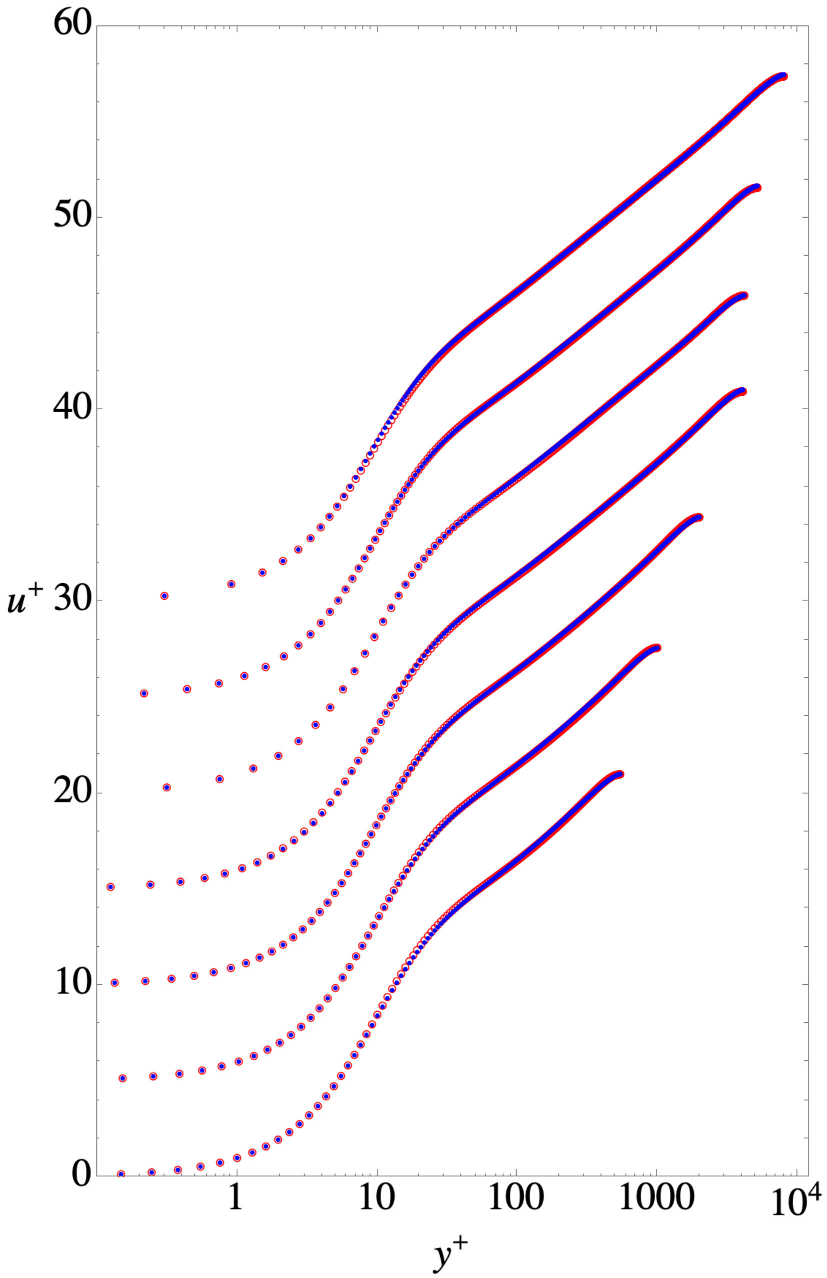
Channel



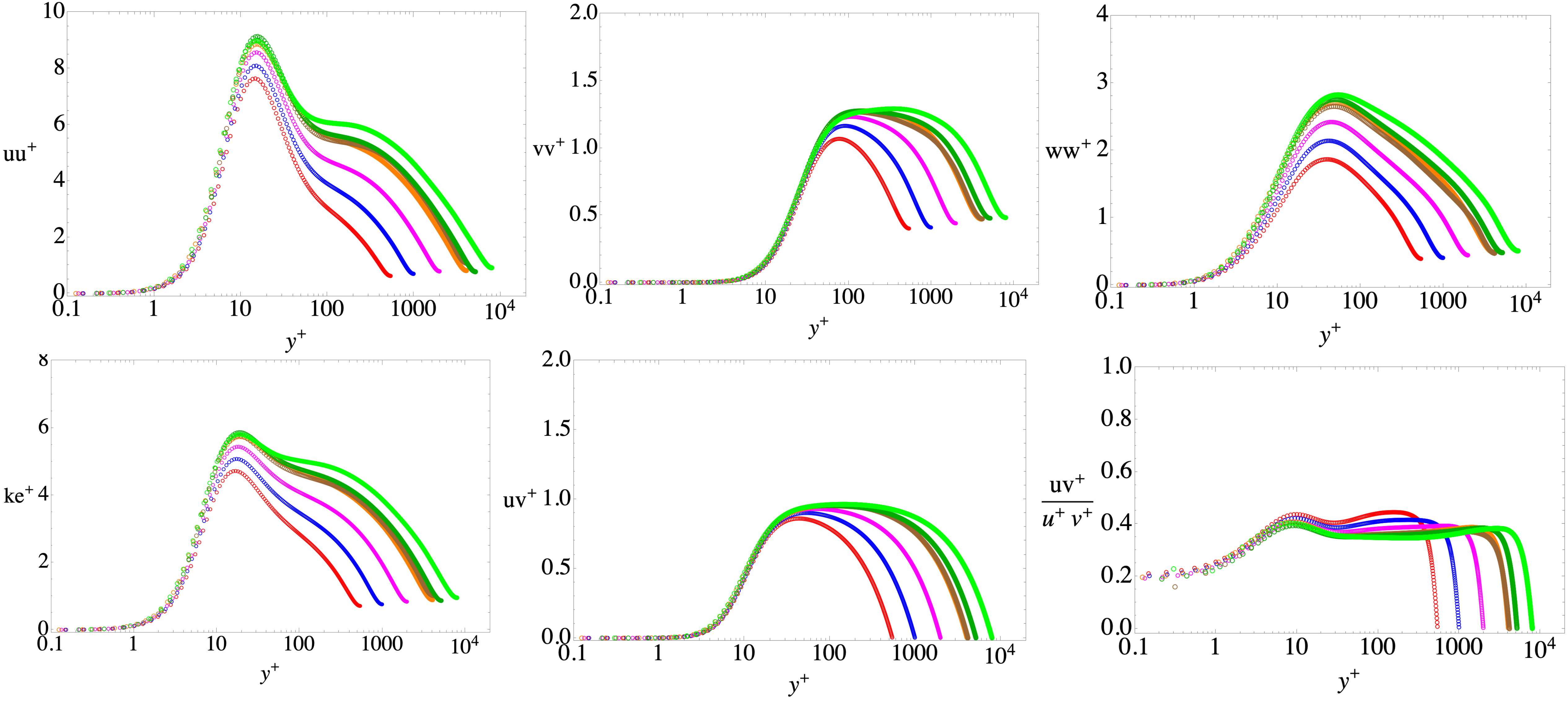








Channel turbulence from simulations



$R_\tau =$ 544, 1001, 1995, 4079, 4179, 5186, 7996

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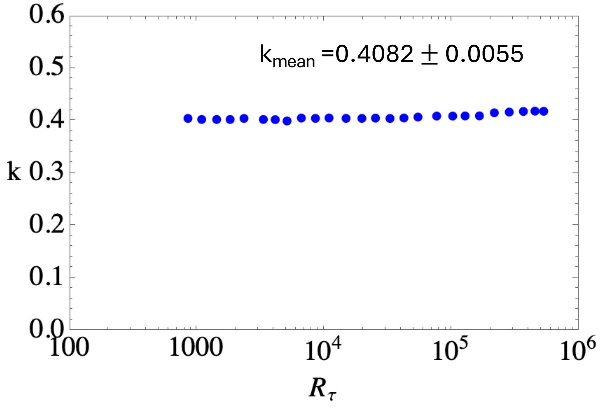
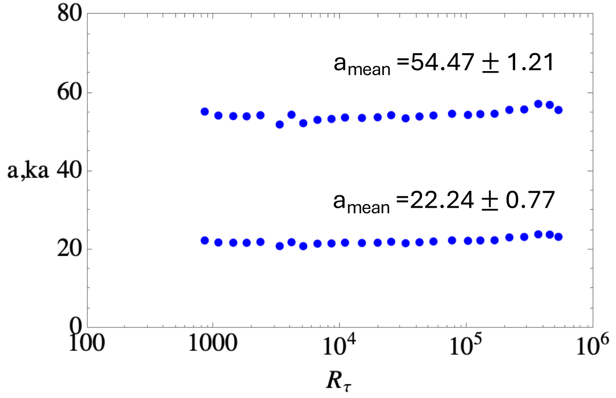
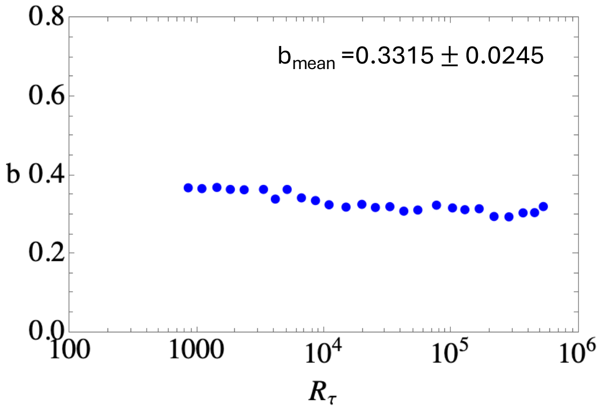
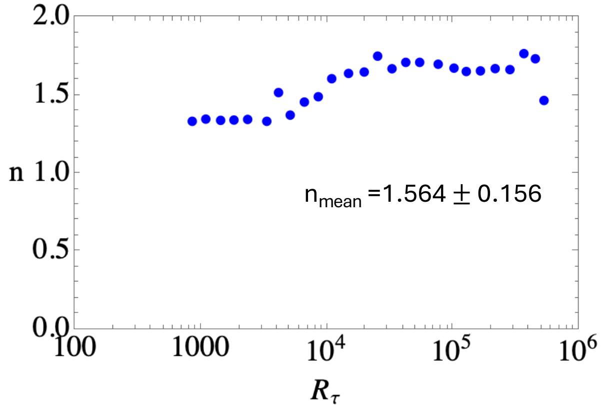
$PSP\#$	R_τ	$\frac{u_e}{u_{\tau\,data}}$	$\frac{u_e}{u_{\tau\,uvp}}$	k	a	b	n	$u^+{}_{rms}$	$u^+{}_{max}$	$u^+{}_{min}$
1	850.947	23.0788	23.0187	0.4054	55.3277	0.3692	1.3354	0.1402	0.3014	-0.3515
2	1090.56	23.6738	23.6043	0.4034	54.3234	0.3672	1.3494	0.1646	0.2090	-0.3306
3	1430.26	24.3436	24.2704	0.4034	54.1727	0.3702	1.3414	0.1672	0.2907	-0.3662
4	1824.72	24.9410	24.9194	0.4034	54.1146	0.3652	1.3434	0.0858	0.2004	-0.1615
5	2344.74	25.5345	25.4860	0.4054	54.3996	0.3642	1.3474	0.1488	0.2040	-0.3920
6	3327.37	26.1918	26.2473	0.4034	52.0356	0.3652	1.3354	0.1041	0.3163	-0.2044
7	4124.89	26.8018	26.8141	0.4031	54.5318	0.3405	1.5196	0.1070	0.1782	-0.2879
8	5108.56	27.2840	27.3753	0.4005	52.3625	0.3651	1.3751	0.0810	0.3327	-0.1265
9	6617.44	27.6954	27.8247	0.4061	53.2176	0.3436	1.4587	0.0644	0.2168	-0.1345
10	8536.62	28.3537	28.4962	0.4055	53.4499	0.3367	1.4930	0.0644	0.2306	-0.1264
11	10914.38	28.8432	28.9788	0.4062	53.8116	0.3260	1.6086	0.0484	0.1355	-0.1115
12	14848.87	29.6175	29.7748	0.4055	53.7105	0.3201	1.6423	0.0570	0.1574	-0.1430
13	19778.30	30.2906	30.4074	0.4055	53.8858	0.3271	1.6513	0.0441	0.1167	-0.1035
14	25278.07	30.8868	30.9618	0.4060	54.4156	0.3191	1.7531	0.0417	0.0750	-0.1466
15	32869.08	31.5881	31.6753	0.4053	53.6086	0.3212	1.6723	0.0538	0.0872	-0.1675
16	42293.51	32.2268	32.3582	0.4062	54.0781	0.3098	1.7132	0.0647	0.1427	-0.1257
17	54530.62	32.7430	32.8613	0.4080	54.3395	0.3127	1.7132	0.0714	0.2227	-0.1163
18	76479.83	33.4563	33.4893	0.4099	54.7671	0.3250	1.7014	0.0860	0.2524	-0.2771
19	102200.19	34.2462	34.2870	0.4099	54.4778	0.3180	1.6763	0.0774	0.2345	-0.1674
20	127913.55	34.8458	34.9315	0.4101	54.6843	0.3135	1.6546	0.0755	0.2213	-0.1830
21	165704.41	35.5102	35.5322	0.4102	54.7826	0.3159	1.6586	0.0597	0.2218	-0.1154
22	216978.54	36.0106	36.0936	0.4161	55.7691	0.2962	1.6731	0.0707	0.2942	-0.0967
23	284254.01	36.5586	36.6758	0.4175	55.8720	0.2953	1.6668	0.0596	0.1996	-0.1084
24	366972.50	36.8963	37.0508	0.4187	57.2946	0.3054	1.7693	0.1015	0.1574	-0.2655
25	452379.62	37.3313	37.5257	0.4193	57.0466	0.3060	1.7364	0.0740	0.1944	-0.1704
26	530023.42	37.9002	38.1287	0.4190	55.7425	0.3215	1.4689	0.0974	0.2794	-0.1186

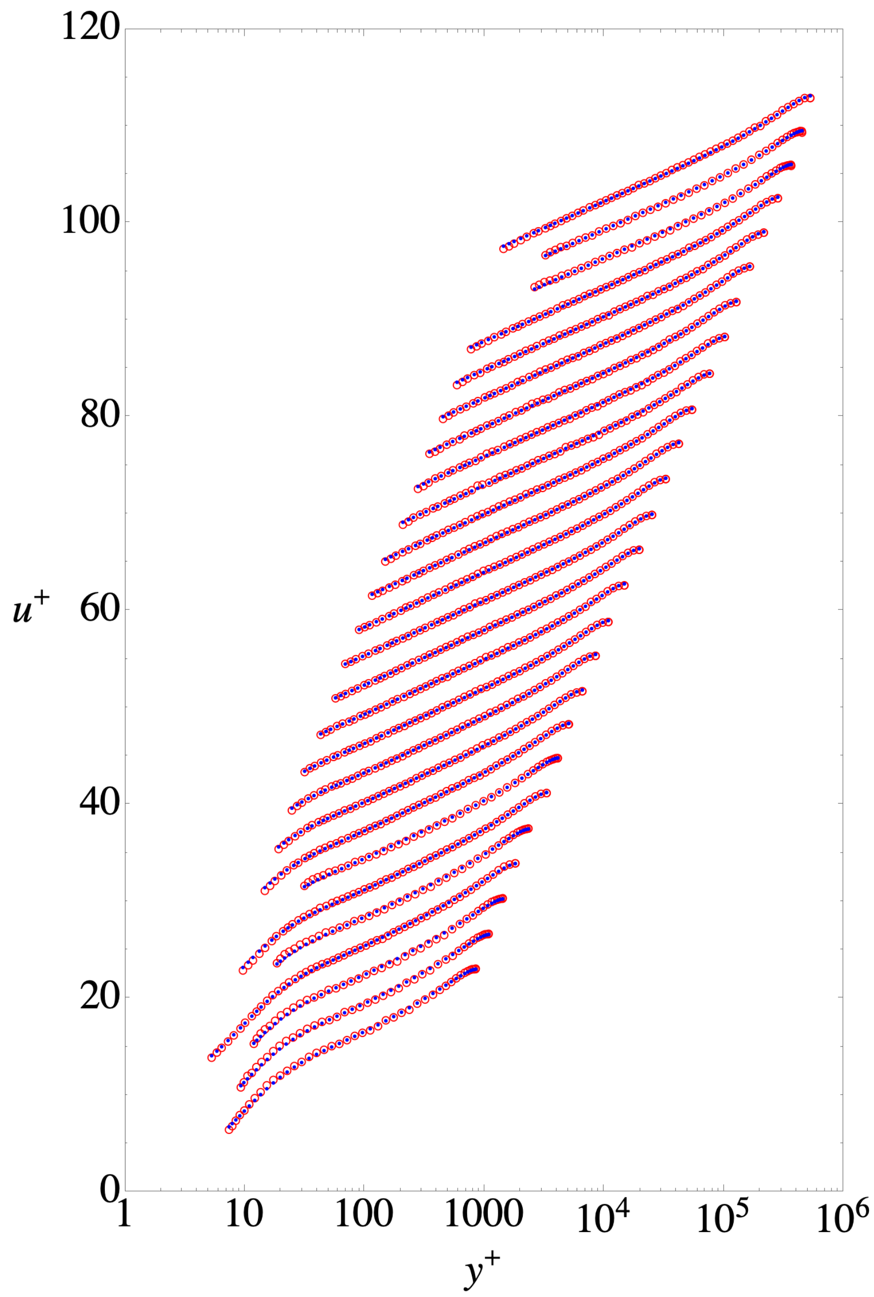
PSP Pipe











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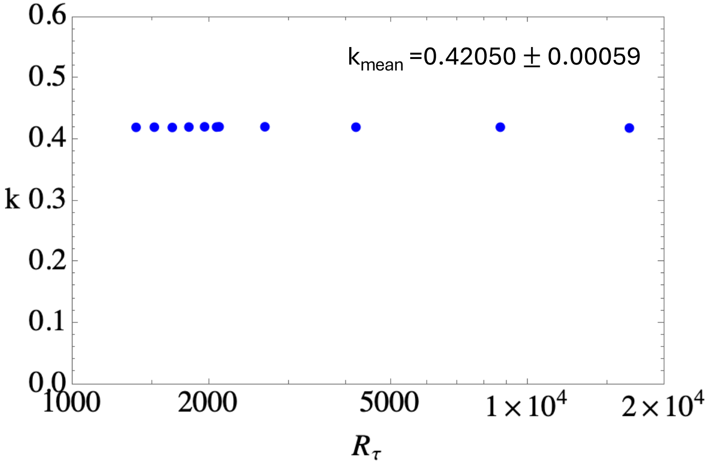
R_τ	$\frac{u_e}{u_\tau}_{data}$	$\frac{u_e}{u_\tau}_{uvp}$	k	a	b	n	${u^+}_{rms}$	${u^+}_{max}$	${u^+}_{min}$	$\frac{u_e}{U}$
1383.36	25.5602	25.6151	0.4202	58.6033	0.1828	2.3718	0.0294	0.0655	-0.0381	0.995
1516.99	25.9827	25.9999	0.4205	58.9419	0.1787	2.3842	0.0327	0.0635	-0.0420	0.995
1660.91	26.3251	26.3489	0.4200	58.7051	0.1764	2.3328	0.0237	0.0410	-0.0339	0.995
1806.16	26.6193	26.6304	0.4207	58.7189	0.1757	2.2392	0.0197	0.0402	-0.0337	0.995
1955.32	26.8496	26.8544	0.4211	58.8018	0.1747	2.2373	0.0184	0.0356	-0.0385	0.995
2104.15	27.0464	27.0652	0.4211	58.8125	0.1727	2.2933	0.0150	0.0383	-0.0167	0.995
2651.82	27.4729	27.5508	0.4211	57.6441	0.1724	2.2703	0.0294	0.0218	-0.0865	0.995
2077.23	26.7834	26.7838	0.4207	57.4787	0.1775	2.2416	0.0494	0.0782	-0.1488	0.995
4199.96	28.8297	28.8929	0.4205	57.3455	0.1685	2.1976	0.0850	0.1971	-0.1379	0.995
8708.43	30.5176	30.6446	0.4205	57.7037	0.1715	2.1045	0.1476	0.3032	-0.2258	0.995
16711.52	32.0348	32.2518	0.4191	57.0626	0.1856	1.7177	0.1780	0.2853	-0.2620	0.995

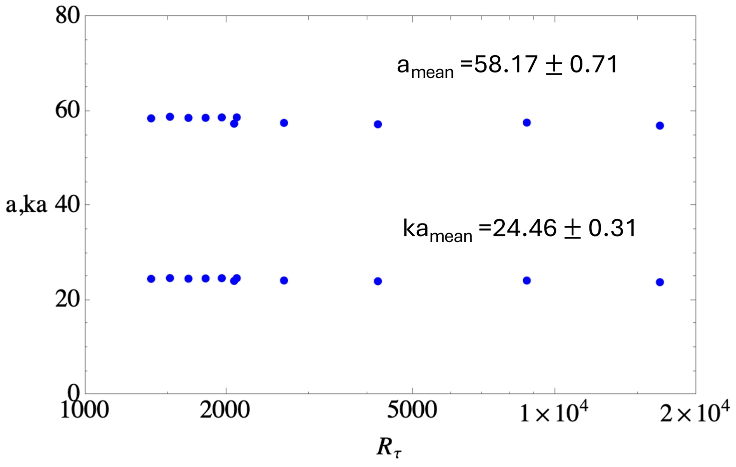
ZPGTBL

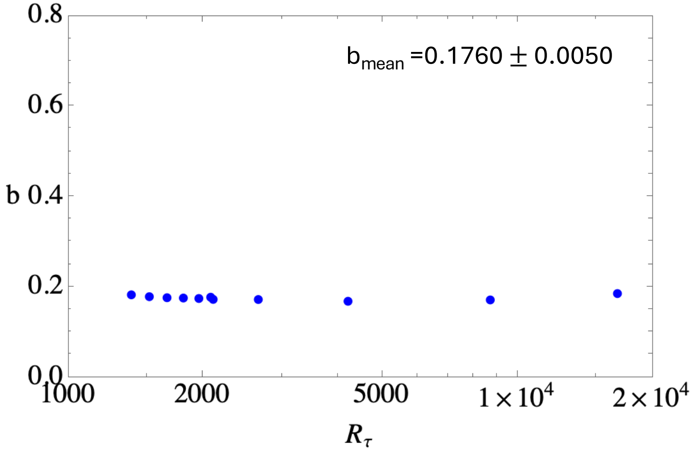
For **all** boundary layer cases

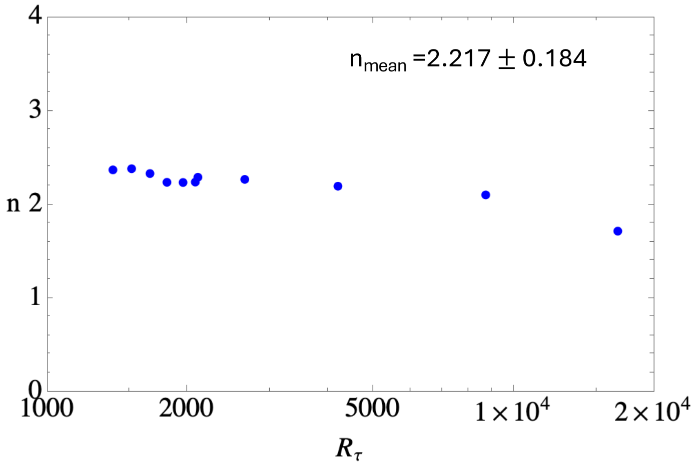
$$\delta_h = \delta_{0.995}$$

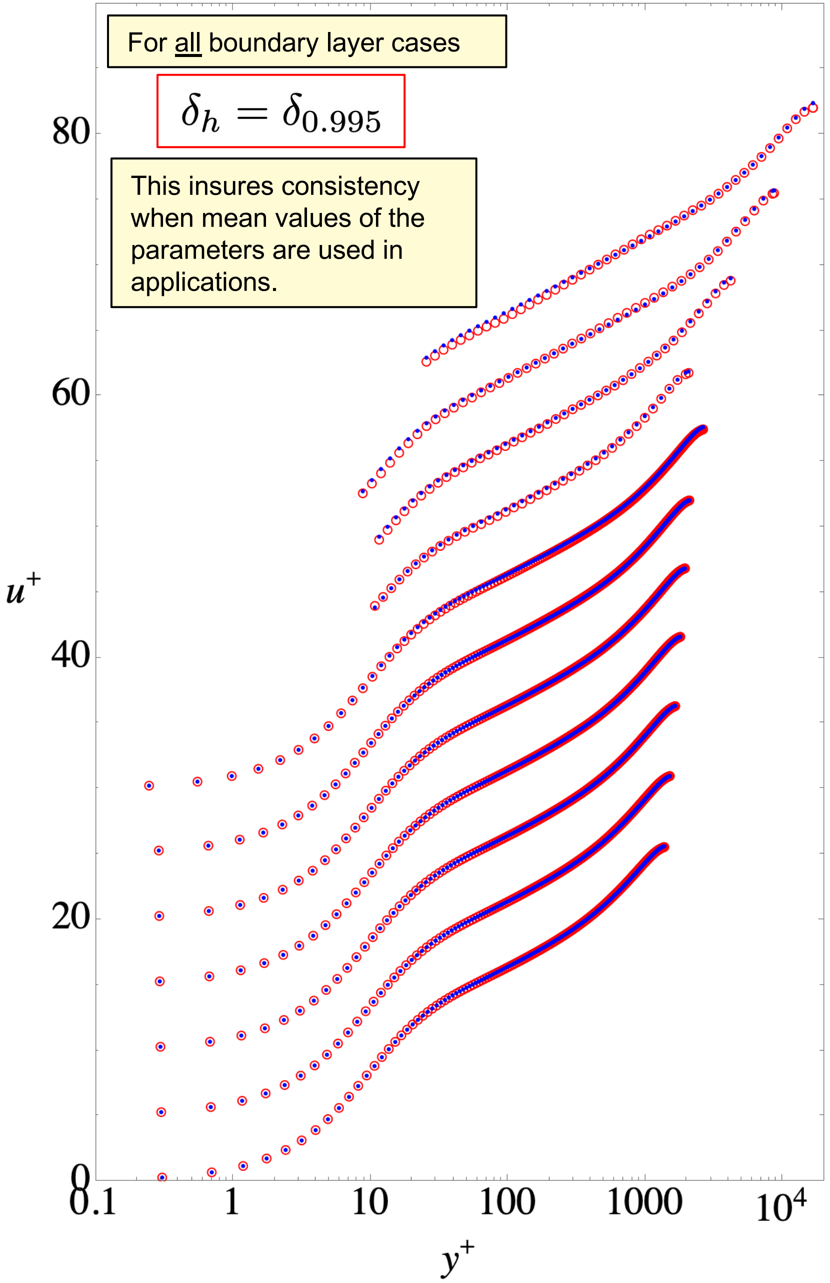
This insures consistency when mean values of the parameters are used in applications.











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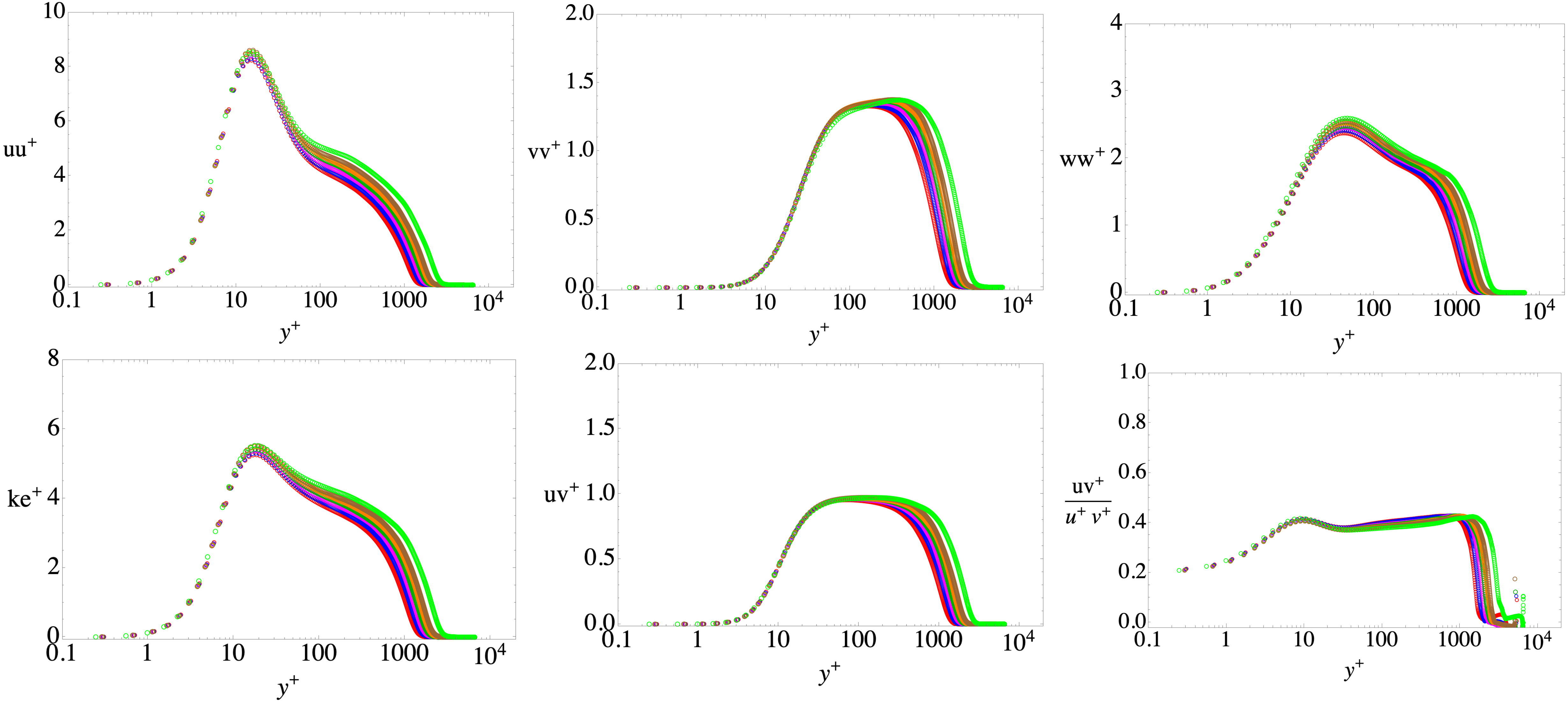
ZPGTBL turbulence from simulations



$R_\tau =$ 1383, 1517, 1661, 1806, 1955, 2104, 2652

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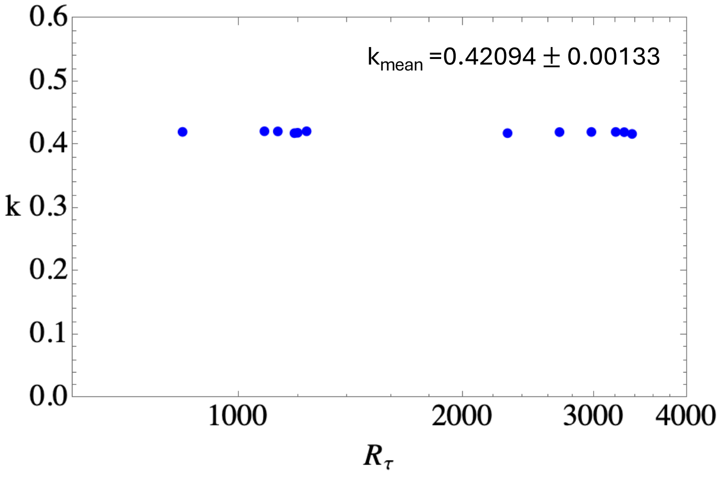
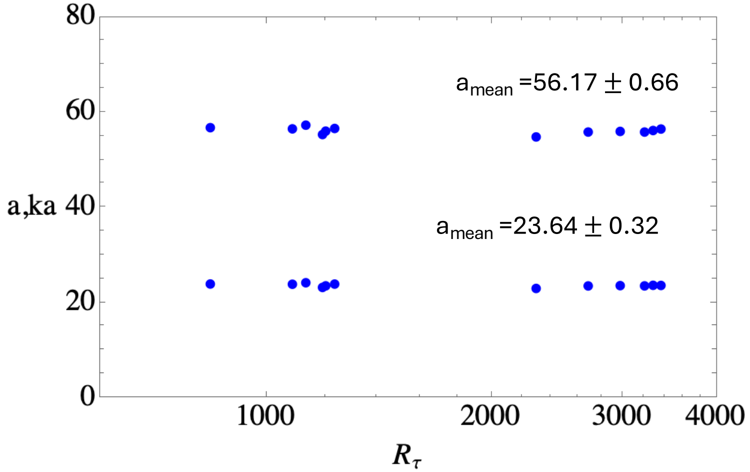
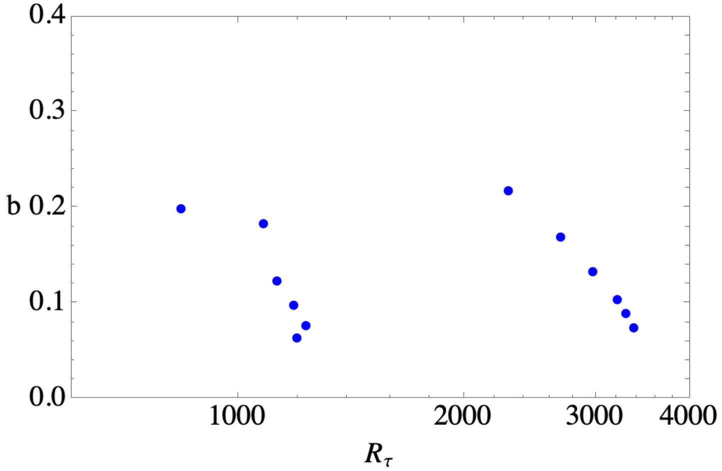
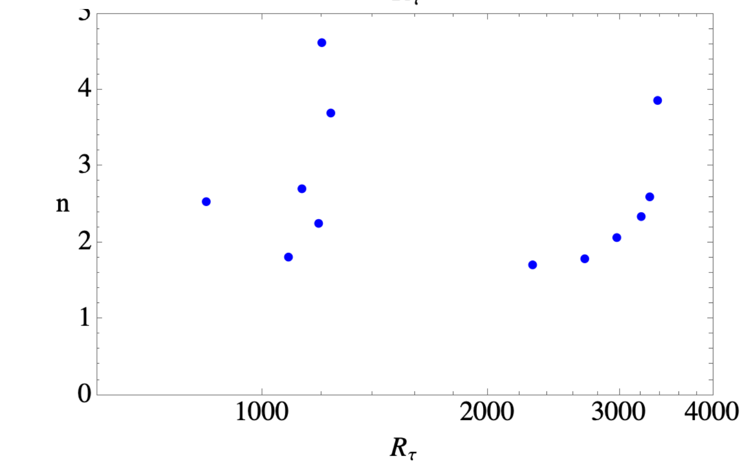
R_τ	$\frac{u_e}{u_\tau}_{data}$	$\frac{u_e}{u_\tau}_{uvp}$	k	a	b	n	u^+_{rms}	u^+_{max}	u^+_{min}	$\frac{u_e}{U}$
840.788	23.4405	23.5363	0.4215	56.7993	0.1992	2.5400	0.0796	0.1111	-0.1701	0.995
1082.73	25.2970	25.3327	0.4225	56.5478	0.1836	1.8138	0.1024	0.1403	-0.2467	0.995
1128.48	27.9189	27.9205	0.4223	57.3206	0.1237	2.7091	0.0964	0.2086	-0.2439	0.995
1188.00	31.3791	31.0872	0.4195	55.3516	0.0983	2.2553	0.1771	0.2975	-0.2918	0.995
1233.35	34.3316	34.1980	0.4225	56.6264	0.0770	3.7015	0.1667	0.2862	-0.3337	0.995
1199.70	38.0131	37.5798	0.4201	56.0683	0.0641	4.6239	0.2884	0.5872	-0.4433	0.995
2294.49	26.2558	26.2637	0.4195	54.8484	0.2180	1.7118	0.0819	0.1196	-0.1950	0.995
2693.75	28.0896	28.1486	0.4212	55.8802	0.1696	1.7915	0.0384	0.0741	-0.1007	0.995
2972.75	29.9626	30.0105	0.4214	56.0065	0.1334	2.0693	0.0516	0.0636	-0.1271	0.995
3204.70	32.7188	32.6572	0.4214	55.8777	0.1042	2.3448	0.0805	0.1439	-0.1591	0.995
3289.93	34.9745	34.6631	0.4211	56.2013	0.0897	2.6032	0.1590	0.2544	-0.3208	0.995
3370.72	37.8787	37.6014	0.4183	56.5134	0.0747	3.8665	0.1707	0.2846	-0.4330	0.995

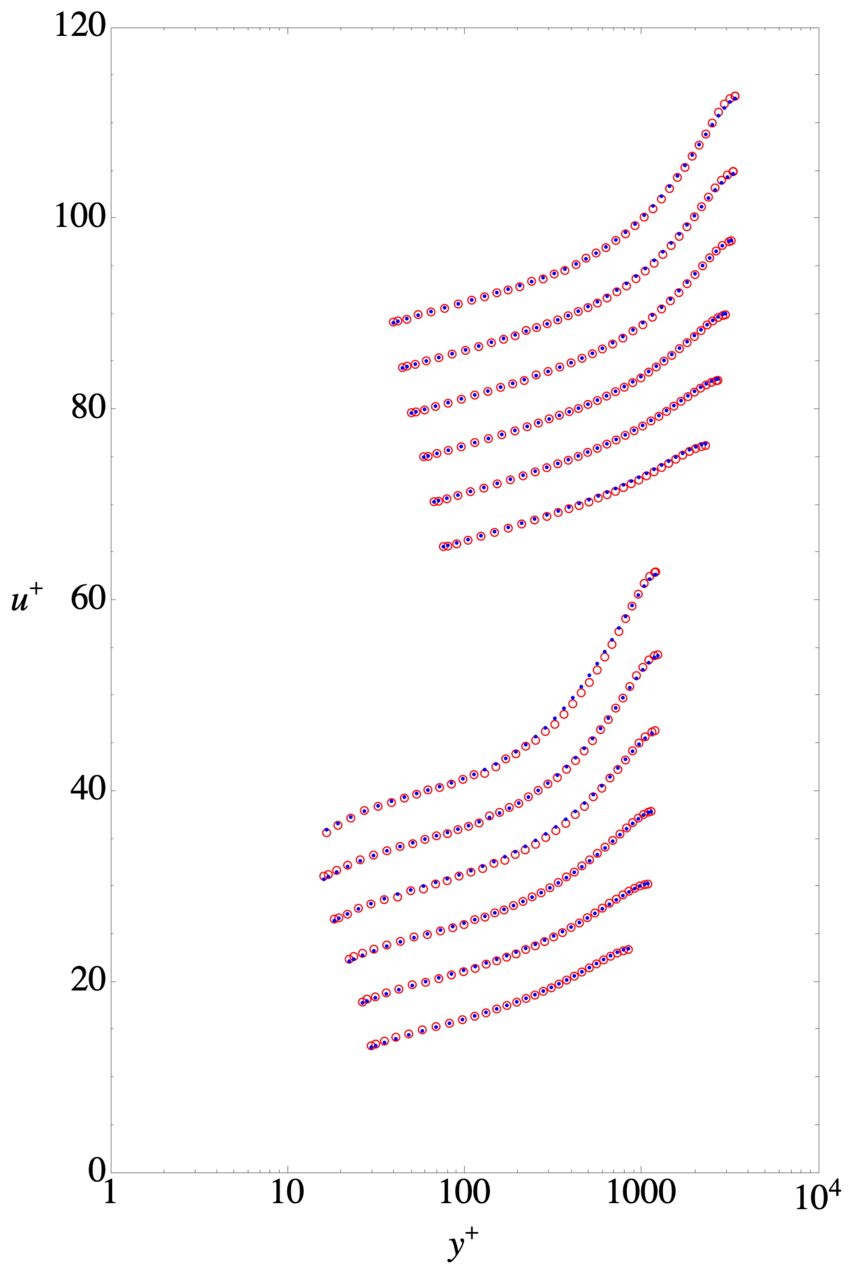
AdvPGTBL











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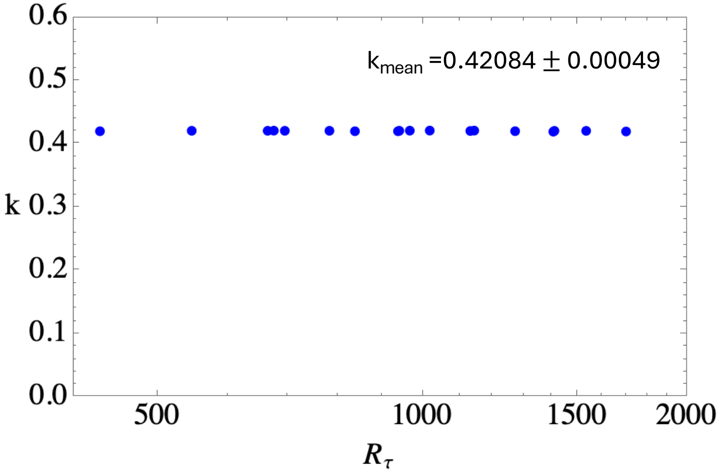
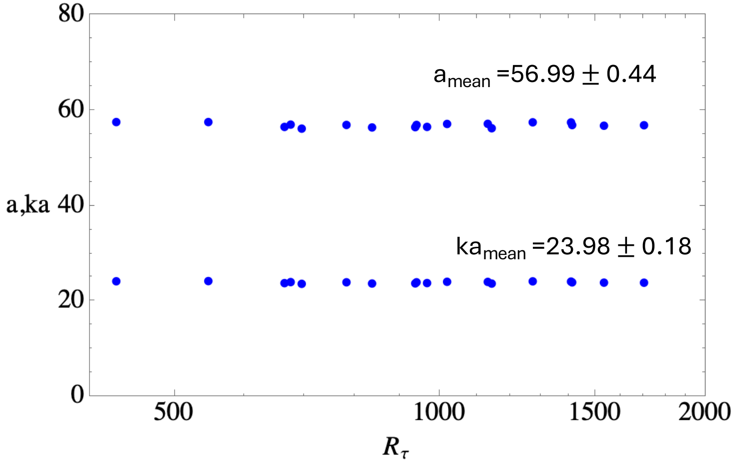
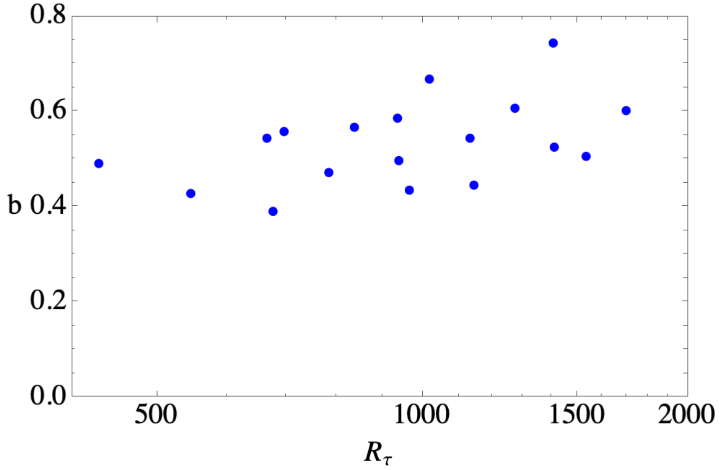
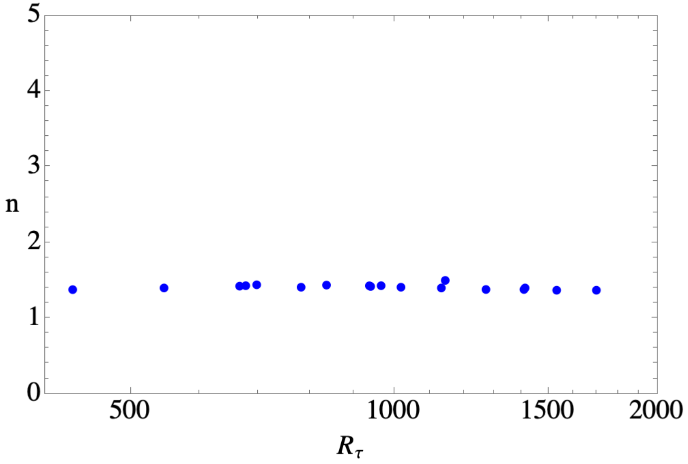
R_τ	$\frac{u_e}{u_\tau}_{data}$	$\frac{u_e}{u_\tau}_{uvp}$	k	a	b	n	u^+_{rms}	u^+_{max}	u^+_{min}	$\frac{u_e}{U}$
429.39	20.1597	20.0852	0.4204	57.6329	0.4920	1.3805	0.0861	0.1577	-0.1210	0.995
666.20	20.7287	20.7431	0.4211	56.6151	0.5450	1.4265	0.0746	0.2021	-0.1149	0.995
696.78	20.8750	20.7386	0.4211	56.2556	0.5590	1.4445	0.0664	0.0913	-0.1364	0.995
837.26	21.1776	21.2092	0.4204	56.5069	0.5680	1.4405	0.0504	0.1764	-0.0847	0.995
937.18	21.3760	21.4448	0.4204	56.5698	0.5870	1.4315	0.0539	0.0900	-0.0906	0.995
1018.42	21.3920	21.4892	0.4214	57.2364	0.669	1.4125	0.0676	0.1057	-0.1350	0.995
546.03	21.1474	20.9520	0.4211	57.6327	0.4290	1.4025	0.0847	0.1585	-0.1954	0.995
783.15	21.6813	21.5114	0.4211	57.0318	0.4730	1.4125	0.0813	0.1599	-0.1699	0.995
940.33	21.8366	21.7718	0.4211	57.0327	0.4980	1.4225	0.0406	0.1033	-0.0648	0.995
1132.57	22.0424	22.1489	0.4208	57.2613	0.5450	1.4025	0.0499	0.1327	-0.0883	0.995
1273.59	22.2230	22.3037	0.4205	57.5810	0.6079	1.3825	0.0460	0.0982	-0.1190	0.995
1407.51	22.1511	22.2514	0.4197	57.5650	0.7447	1.3825	0.0500	0.1003	-0.0957	0.995
676.97	21.7204	21.6131	0.4213	57.0988	0.3915	1.4325	0.0880	0.2387	-0.1108	0.995
966.54	22.2435	22.1385	0.4213	56.6218	0.4360	1.4325	0.0667	0.1558	-0.1050	0.995
1144.22	22.4216	22.3211	0.4214	56.3373	0.4465	1.5025	0.0656	0.2158	-0.1005	0.995
1411.65	22.6737	22.7223	0.4210	57.0052	0.5263	1.4025	0.0426	0.0729	-0.0647	0.995
1533.96	22.9257	23.0538	0.4210	56.8663	0.5069	1.3725	0.0872	0.1769	-0.1126	0.995
1702.99	22.8822	23.0006	0.4200	56.9691	0.6029	1.3725	0.0921	0.1781	-0.1237	0.995

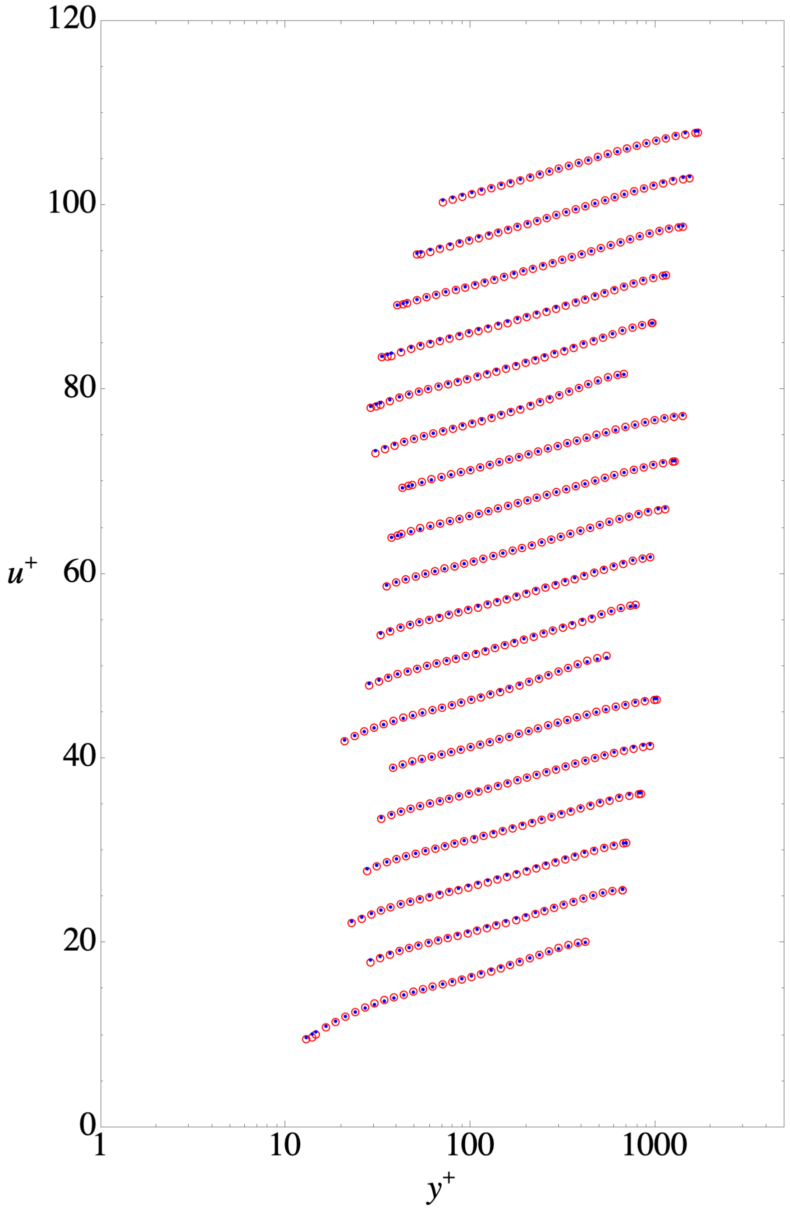
FavPGTBL



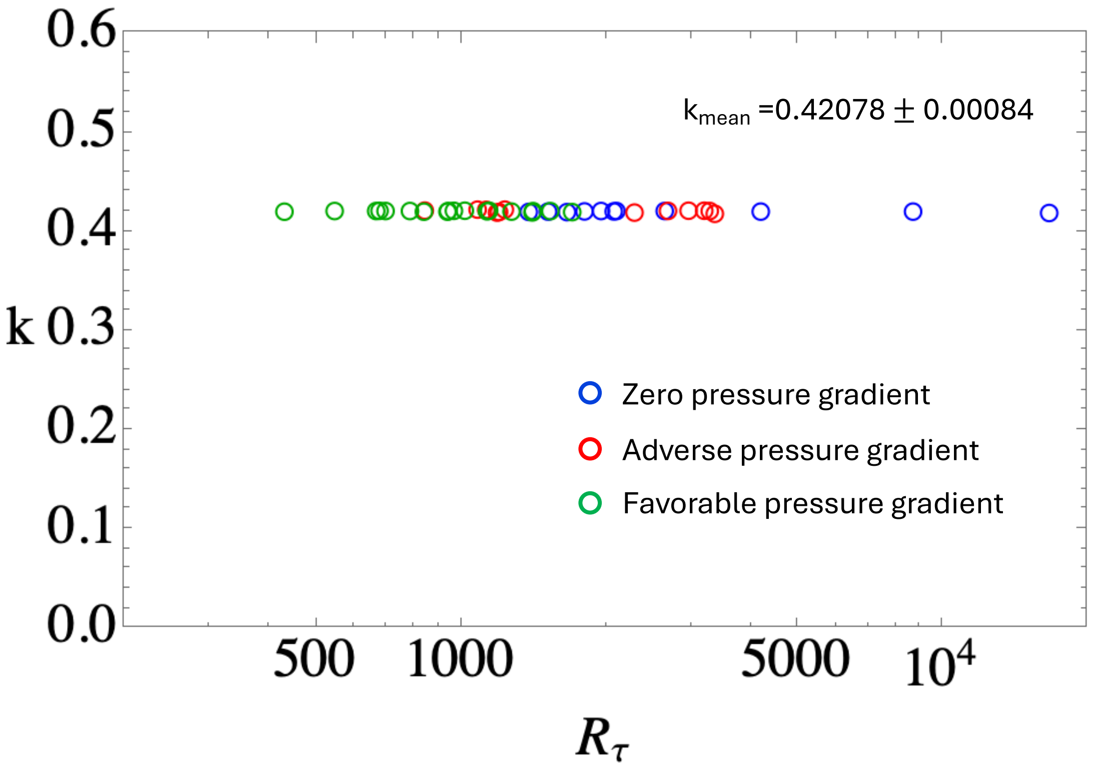


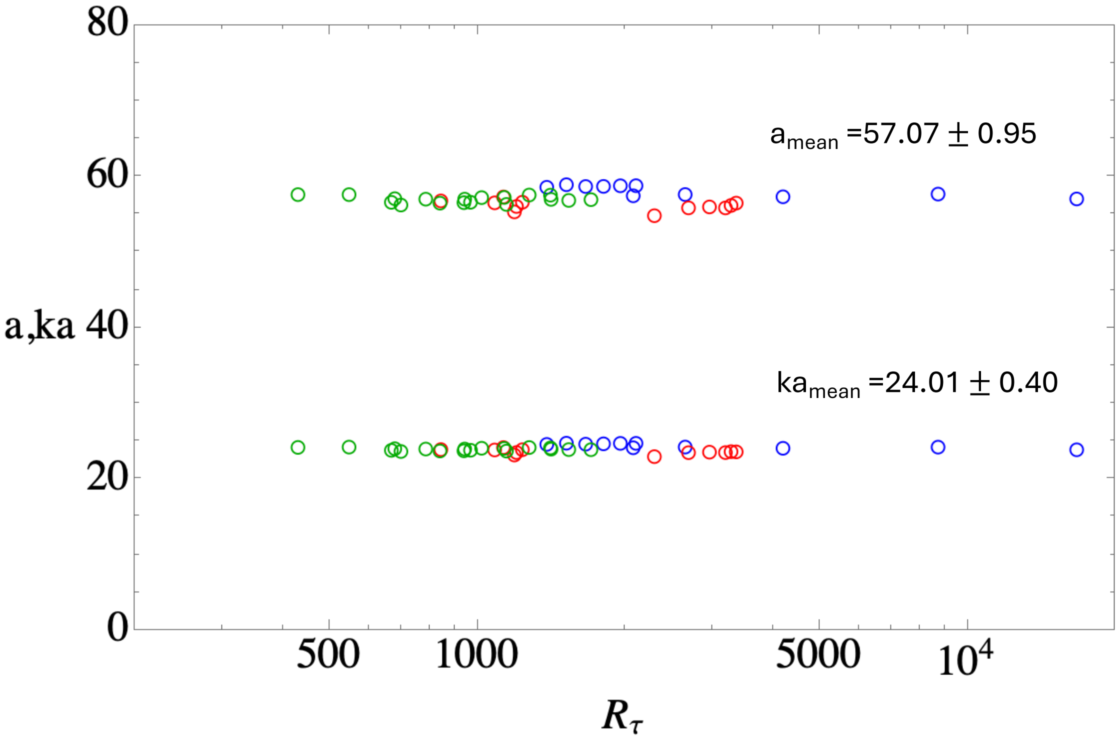




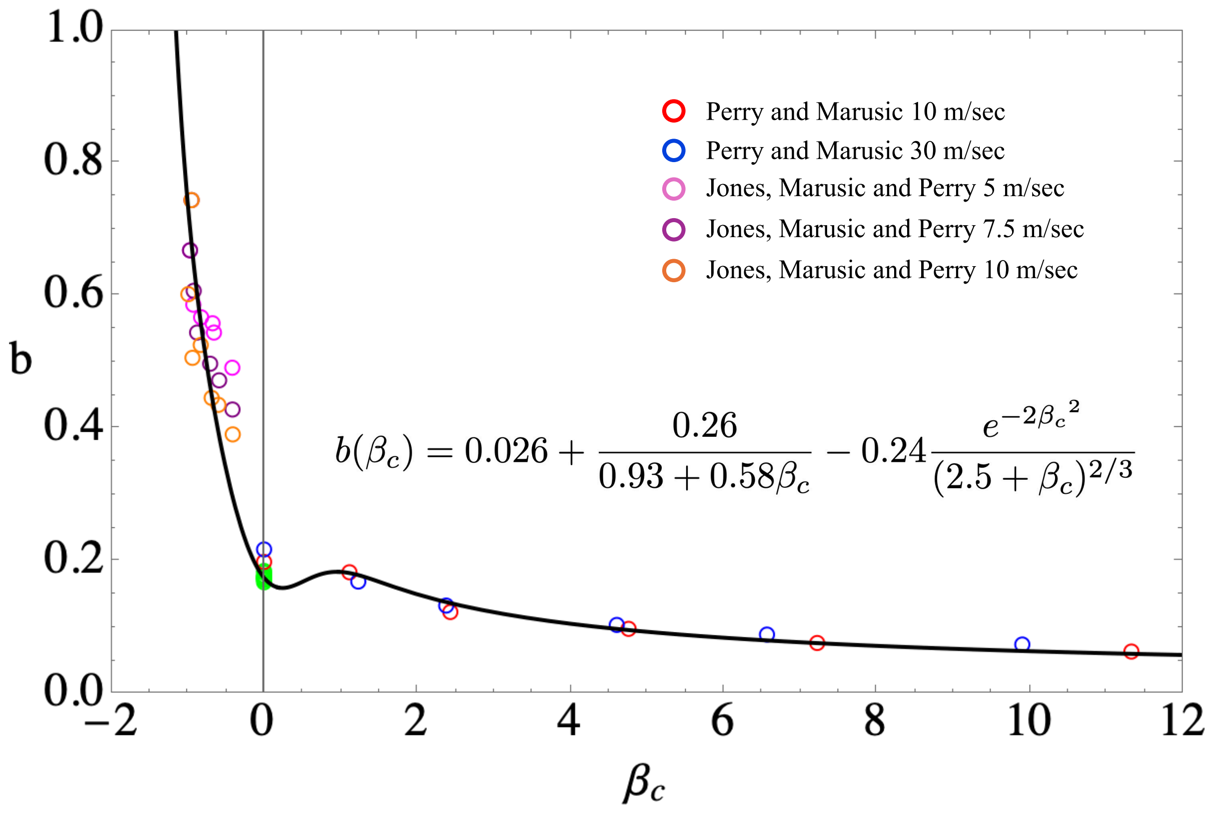


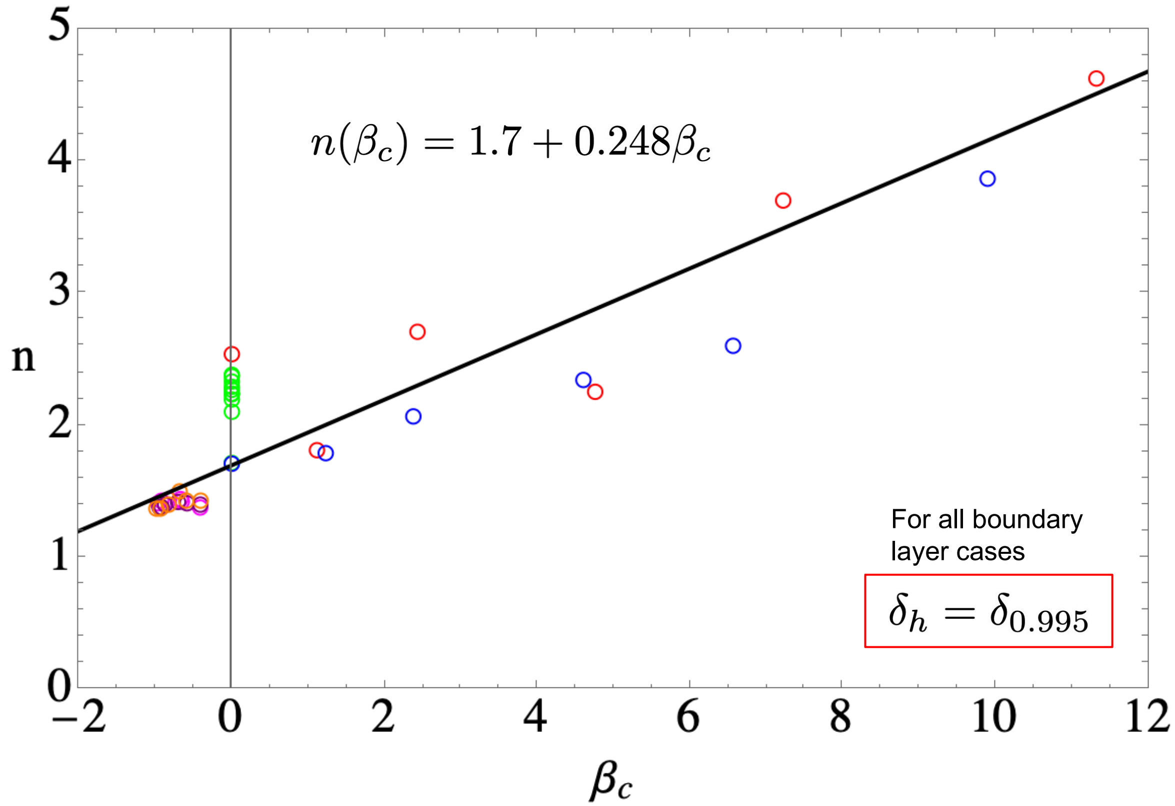
The boundary layer wall parameters *k* and *a* are approximately independent of Reynolds number and pressure gradient.





The boundary layer wake parameters b and n are approximately related through β_c





For $\beta_c = 0$, the parameter *b = 0.1760* which matches the correlation

For $\beta_c = 0$, the parameter *n = 2.217* which lies above the correlation

High Reynolds number

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Recall the UVP

$$u^+(y^+) = \int_0^{y^+} \left[\frac{2\left(1 - \frac{s}{R_\tau}\right)}{1 + \left(1 + 4\lambda^2 \left(1 - \frac{s}{R_\tau}\right)\right)^{1/2}} \right] ds$$

$$\lambda(k, a, b, n, R_\tau, y^+) = \frac{ky^+ \left(1 - \sigma\left(\frac{y^+}{a}\right)\right)}{\left(1 + \left(\frac{y^+}{bR_\tau}\right)^n\right)^{1/n}}$$

Carry out a scaling - Multiply and divide the damping and wake terms by *k*

Modified wall-wake mixing length function. The parameters *k* and *a* become one parameter *ka*.

$$\lambda(k, a, b, n, R_\tau, y^+) = \frac{ky^+ \left(1 - \sigma\left(\frac{y^+}{a}\right)\right)}{\left(1 + \left(\frac{y^+}{bR_\tau}\right)^n\right)^{1/n}} = \frac{ky^+ \left(1 - \sigma\left(\frac{ky^+}{ka}\right)\right)}{\left(1 + \left(\frac{ky^+}{bkR_\tau}\right)^n\right)^{1/n}} = \tilde{\lambda}(ka, b, n, kR_\tau, ky^+)$$

$y^+ \to ky^+$

$R_\tau \to kR_\tau$

Scaled velocity profile

$$ku^+(ka, b, n, kR_\tau, ky^+) = \int_0^{ky^+} \left[\frac{2\left(1 - \frac{s}{kR_\tau}\right)}{1 + \left(1 + 4\tilde{\lambda}^2 \left(1 - \frac{s}{kR_\tau}\right)\right)^{1/2}} \right] ds$$

$u/u_\tau \to ku/u_\tau$

Define the shape function

$$\Phi(ka, b, n, kR_\tau, ky^+) = \int_0^{ky^+} \left[\frac{2\left(1 - \frac{s}{kR_\tau}\right)}{1 + \left(1 + 4\tilde{\lambda}^2\left(1 - \frac{s}{kR_\tau}\right)\right)^{1/2}} \right] ds - ln(ky^+)$$

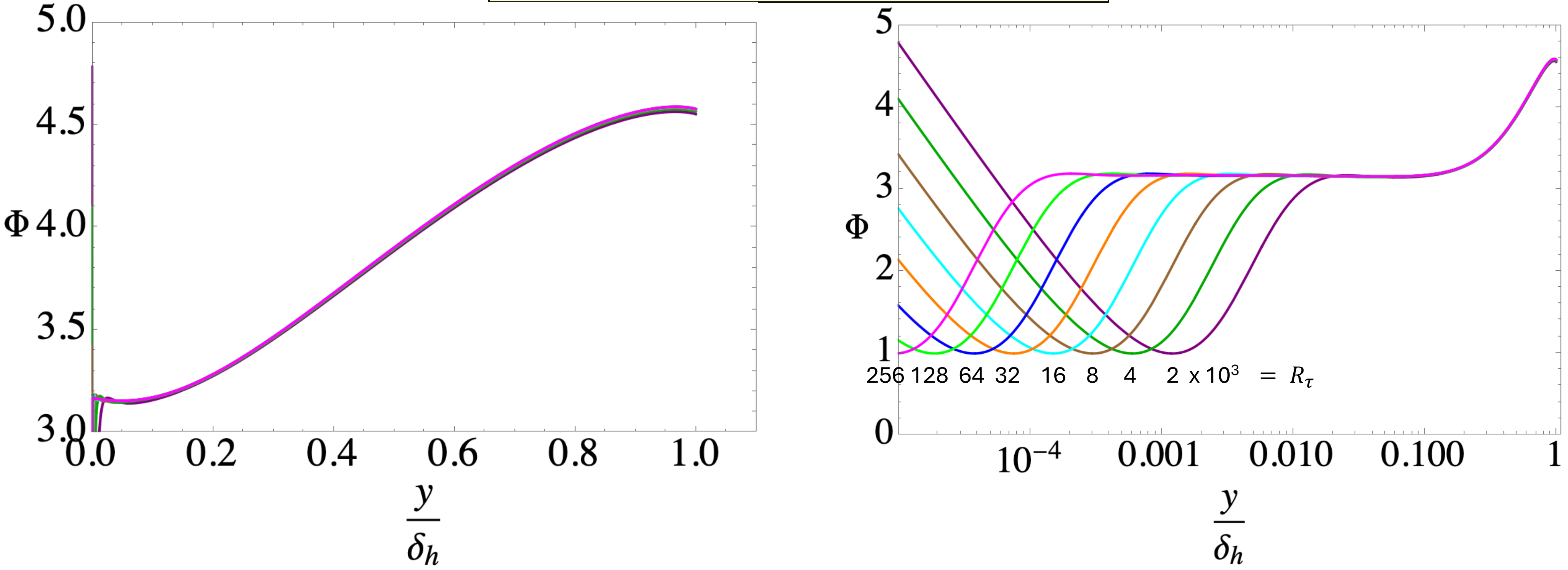
where

$$\tilde{\lambda}(ka, b, n, kR_\tau, ky^+) = \frac{ky^+\left(1 - \sigma\left(\frac{ky^+}{ka}\right)\right)}{\left(1 + \left(\frac{ky^+}{bkR_\tau}\right)^n\right)^{1/n}}$$

Note

$$ky^+ = \left(\frac{y}{\delta_h}\right) kR_\tau$$

Plot Φ versus y/δ_h for various R_τ.



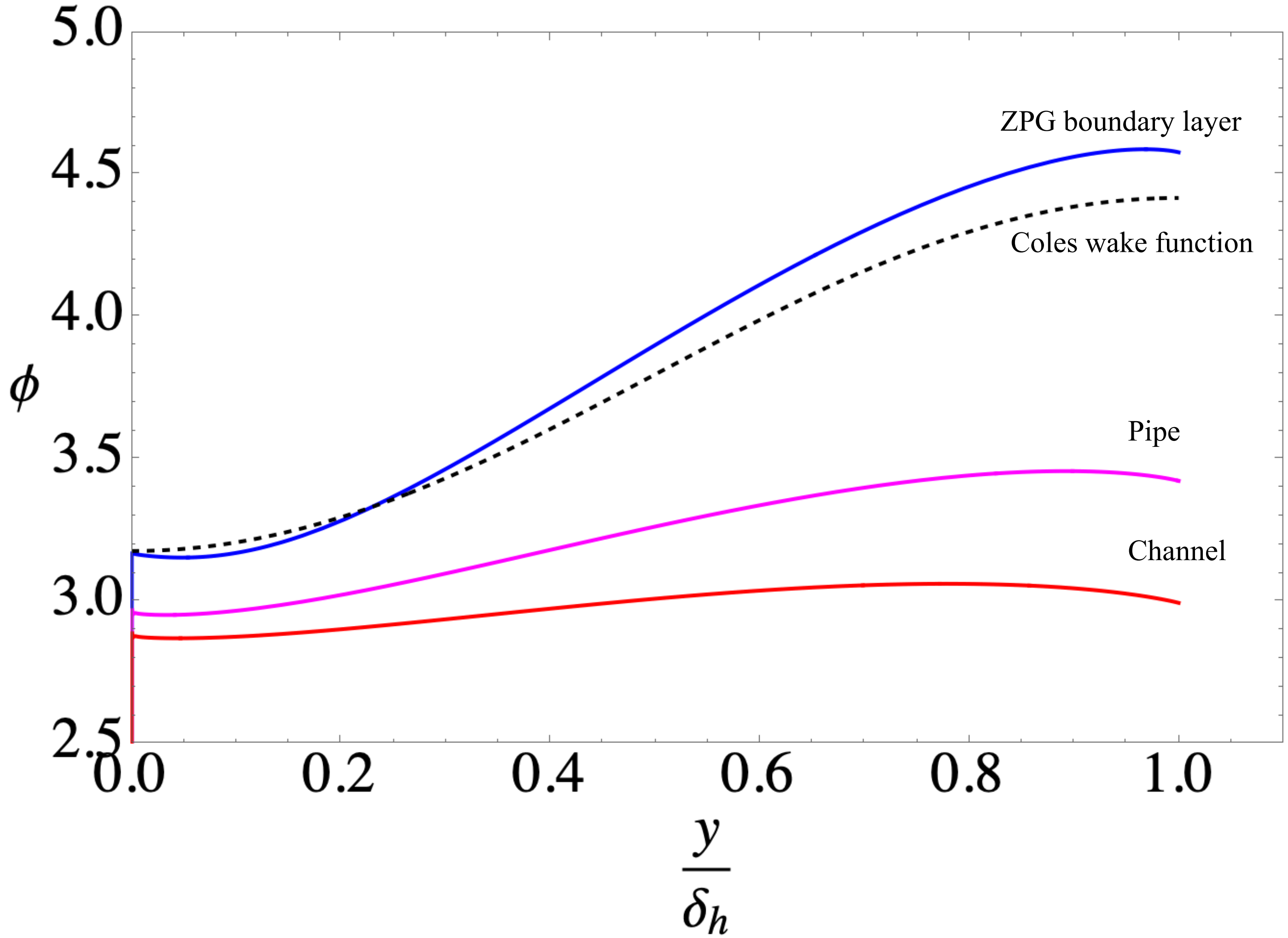
Above $kR_\tau \cong 2000,\ \Phi$ *is independent of* R_τ

$$\Phi(ka, b, n, kR_\tau, ky^+) = \phi(ka, b, n, \frac{y}{\delta_h})$$

For boundary layers, the wake parameters b and n are approximately related through β_c

$$\Phi(ka, b, n, kR_\tau, ky^+) = \phi(ka, \beta_c, \frac{y}{\delta_h})$$

ϕ versus y/δ_h for ZPGTBL, Pipe and Channel flow.



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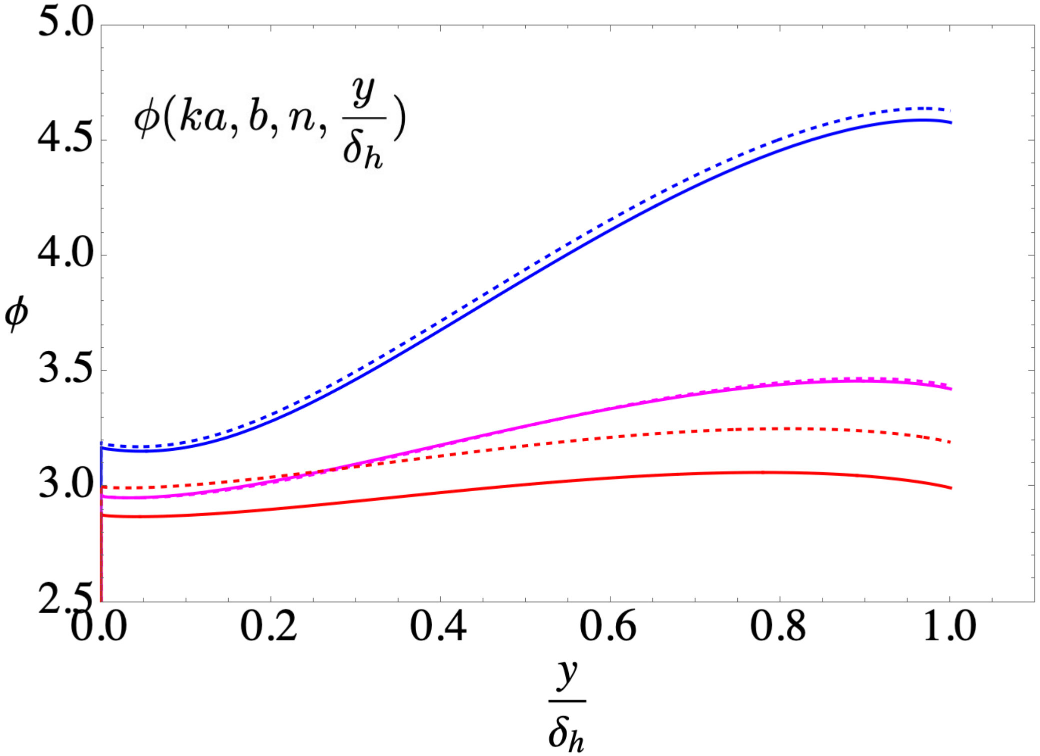
Average parameters before and after the introduction of the new wall damping function

Exponential damping

Flow	$\overline{k}$	σ_k	$\overline{a}$	σ_a	$\overline{m}$	σ_m	$\overline{b}$	σ_b	$\overline{n}$	σ_n
Pipe (21 profiles)	0.4092	0.0057	20.0950	0.381	1.6210	0.0379	0.3195	0.0157	1.6190	0.1204
Channel (7 profiles)	0.4086	0.0179	22.8673	1.599	1.2569	0.0292	0.4649	0.0485	1.3972	0.1213
ZPG Boundary Layer (11 profiles)	0.4233	0.0068	24.9583	0.663	1.1473	0.0373	0.1752	0.0060	2.1707	0.2238

UWDF

Flow	$\overline{k}$	σ_k	$\overline{a}$	σ_a	$\overline{m}$	σ_m	$\overline{b}$	σ_b	$\overline{n}$	σ_n
Pipe (21 profiles)	0.4082	0.0055	54.47	1.21	——	——	0.3315	0.0245	1.564	0.156
Channel (7 profiles)	0.3994	0.0030	53.56	1.53	——	——	0.4938	0.0406	1.414	0.095
ZPG Boundary Layer (11 profiles)	0.42050	0.00059	58.17	0.71	——	——	0.1760	0.0050	2.217	0.184




```
In[328]:= Log[0.4233] / 0.4233 - Log[0.4205] / 0.4205
Out[328]= 0.0293059
```

```
In[326]:= Log[0.4092] / 0.4092 - Log[0.4082] / 0.4082
Out[326]= 0.0113436
```

```
In[327]:= Log[0.4086] / 0.4086 - Log[0.3994] / 0.3994
Out[327]= 0.107475
```

$$u^+ = \frac{1}{k} ln(y^+) + \frac{1}{k} ln(k) + \frac{1}{k}\phi\left(ka, \beta_c, \frac{y}{\delta_h}\right)$$

Differences in the shape function before and after are almost entirely due to small differences in *k* before and after the incorporation of the new damping function.

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Explicit high Reynolds number form of the UVP

$$u^+(k,a,b,n,R_\tau,y^+) = \int_0^{y^+} \left[\frac{2\left(1-\frac{s}{R_\tau}\right)}{1+\left(1+4\lambda^2\left(1-\frac{s}{R_\tau}\right)\right)^{1/2}} \right] ds$$

$0 < y^+ < R_\tau$

At Reynolds numbers larger than $kR_\tau \cong 2000$ the boundary layer velocity profile above $y^+ = 132$ is accurately approximated by

$$u^+ = \frac{1}{k} ln(ky^+) + \frac{1}{k}\phi\left(ka,b,n,\frac{y}{\delta_h}\right)$$

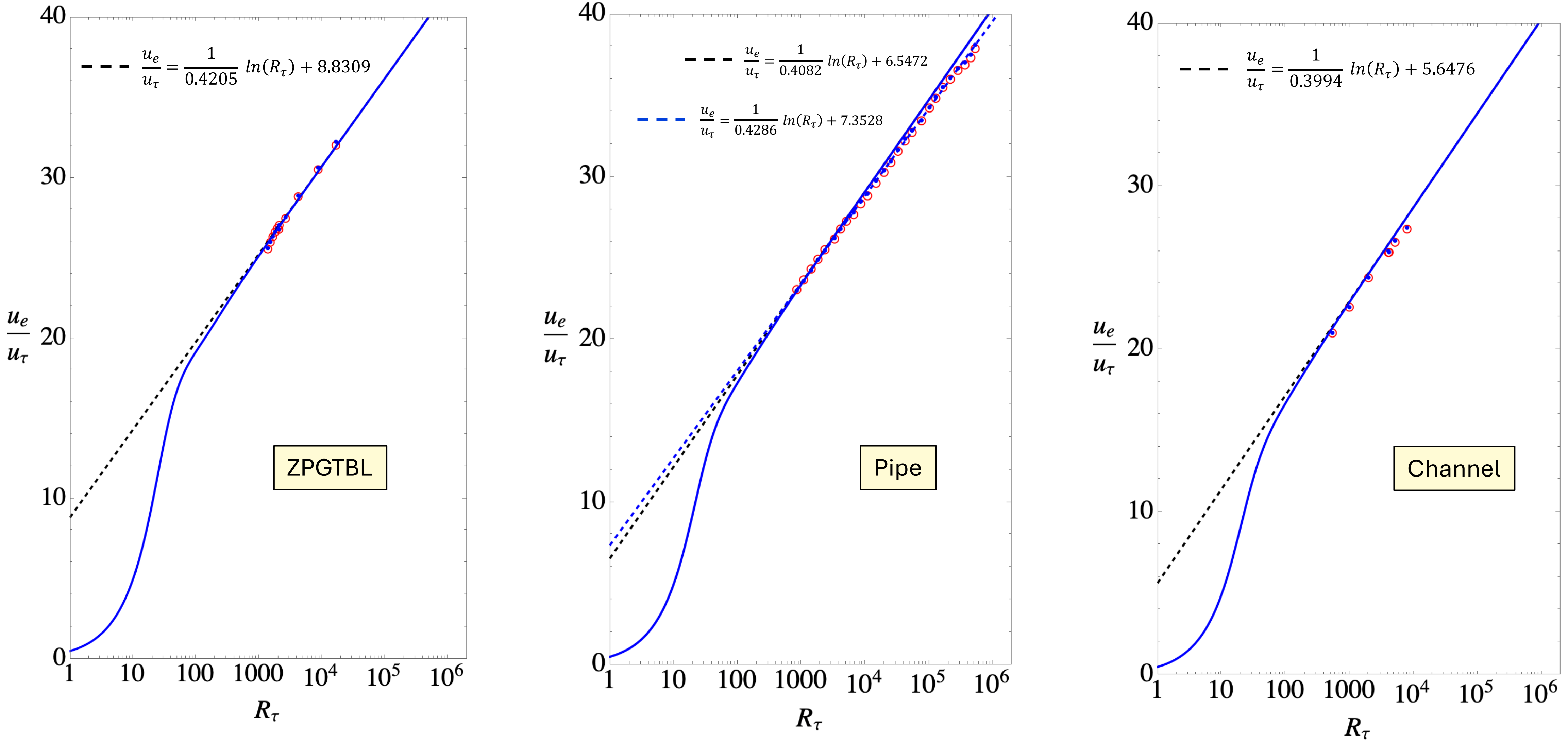
$y^+ > 132$

Evaluate at the boundary layer edge to determine the friction law.

$$u^+ = \frac{1}{k} ln(ky^+) + \frac{1}{k}\phi\left(ka,b,n,1\right)$$

The UVP friction law (blue line) based on averaged parameters

$$\frac{u_e}{u_\tau} = \frac{1}{k}\ ln(R_\tau) + C$$



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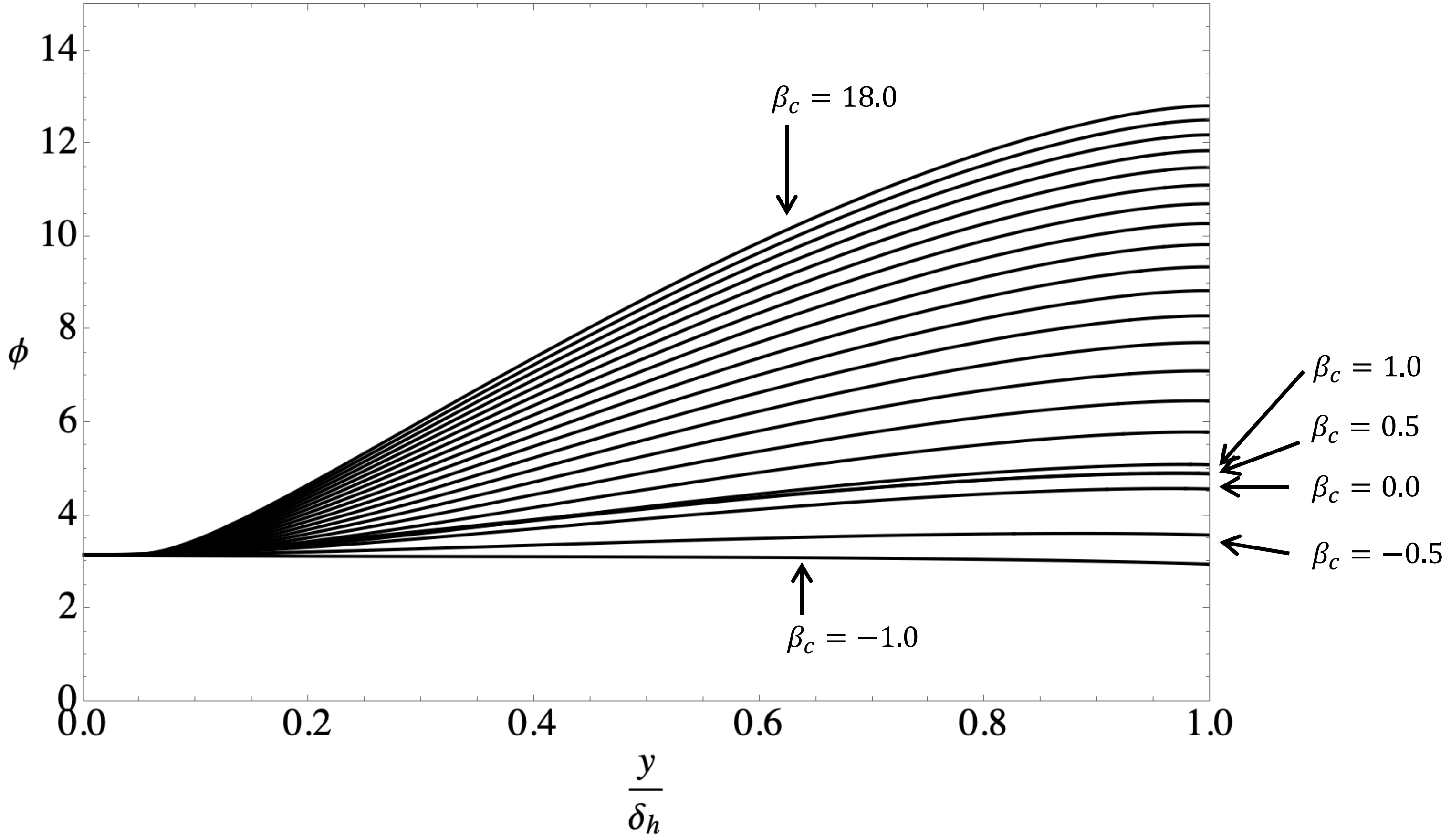
The boundary layer shape function for various β_c is essentially unchanged

$$u^+ = \frac{1}{k} ln(ky^+) + \frac{1}{k}\phi\left(ka, \beta_c, \frac{y}{\delta_h}\right)$$

$y^+ > 132$
$kR_\tau > 2000$



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The high Reynolds number form of the UVP can be used to study all of the integral measures of the boundary layer as functions of R_τ

$$u^+ = \frac{1}{k} ln(ky^+) + \frac{1}{k}\phi\left(ka, \beta_c, \frac{y}{\delta_h}\right)$$

$$\frac{u_e}{u_\tau} = \frac{1}{k} ln(kR_\tau) + \frac{1}{k}\phi\left(ka, \beta_c, 1\right)$$

$$\frac{u}{u_e} = \frac{ln(kR_\tau) + ln\left(\frac{y}{\delta_h}\right) + \phi\left(ka, \beta_c, \frac{y}{\delta_h}\right)}{ln(kR_\tau) + \phi\left(ka, \beta_c, 1\right)}$$

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Integral measures of the ZPGTBL as functions of R_τ

For all boundary layer cases

$$\delta_h = \delta_{0.995}$$

This insures consistency when mean values of the parameters are used in applications.

For $\beta_c = 0$, the parameter $b = 0.1760$ which matches the correlation

For $\beta_c = 0$, the parameter $n = 2.217$ which does not match the correlation

Required integrals exponential damping

$$C_1 \equiv \phi(ka, m, b, n, 1) = 4.530802,$$
$$C_2 \equiv \int_0^1 \phi(ka, m, b, n, \eta) d\eta = 3.861861,$$
$$C_3 \equiv \int_0^1 \phi(ka, m, b, n, \eta)^2 d\eta = 15.149878,$$
$$C_4 \equiv \int_0^1 \phi(ka, m, b, n, \eta)^3 d\eta = 60.327378,$$
$$C_5 \equiv \int_0^1 \ln(\eta) \phi(ka, m, b, n, \eta) d\eta = -3.459397,$$
$$C_6 \equiv \int_0^1 \ln(\eta) \phi(ka, m, b, n, \eta)^2 d\eta = -12.092035,$$
$$C_7 \equiv \int_0^1 \ln(\eta)^2 \phi(ka, m, b, n, \eta) d\eta = 6.578894,$$
$$C_8 \equiv \int_0^1 \ln(\eta)^2 \phi(ka, m, b, n, \eta)^2 d\eta = 21.719529.$$

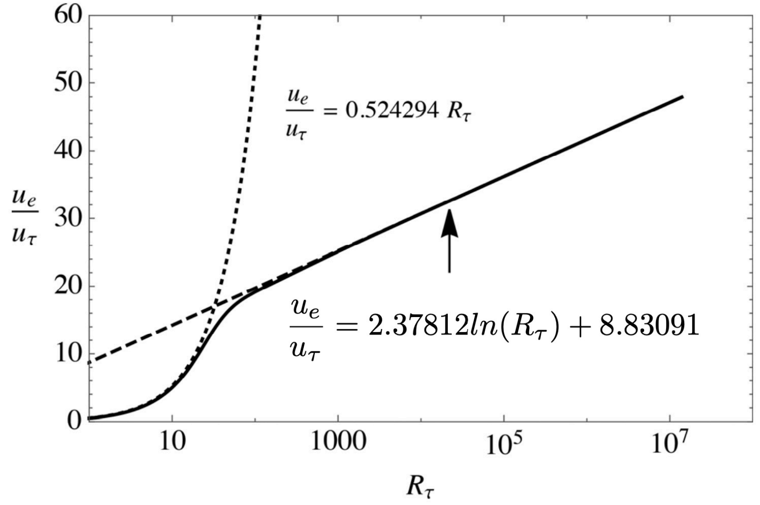
Required integrals new damping function

$$C_1 \equiv \phi(ka, 0, 1) = 4.57971$$
$$C_2 \equiv \int_0^1 \phi(ka, 0, \eta)\, d\eta = 3.88251$$
$$C_3 \equiv \int_0^1 \phi(ka, 0, \eta)^2\, d\eta = 15.3355$$
$$C_4 \equiv \int_0^1 \phi(ka, 0, \eta)^3\, d\eta = 61.5649$$
$$C_5 \equiv \int_0^1 ln(\eta) \phi(ka, 0,\ eta)\, d\eta = -3.45378$$
$$C_6 \equiv \int_0^1 ln(\eta) \phi(ka, 0, \eta)^2\, d\eta = -12.0726$$
$$C_7 \equiv \int_0^1 ln(\eta)^2 \phi(ka, 0, \eta)\, d\eta = 6.53181$$
$$C_8 \equiv \int_0^1 ln(\eta)^2 \phi(ka, 0, \eta)^2\, d\eta = 21.4279$$

The new integrals are very close to the old integrals

Integral measures of the ZPGTBL as functions of R_τ using the new damping function.

Friction law



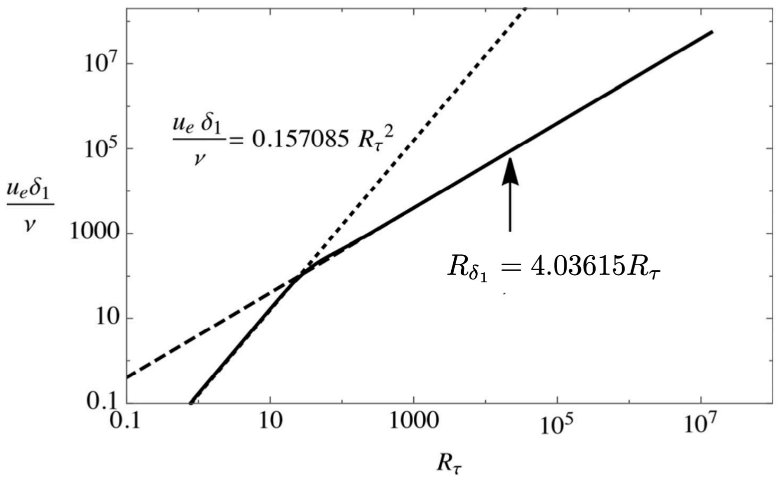
$$\frac{u_e}{u_\tau} = \frac{1}{k} ln(kR_\tau) + \frac{1}{k}\phi\left(ka, 0, 1\right)$$

$$\frac{u_e}{u_\tau} = \frac{1}{k} ln(R_\tau) + \frac{1}{k} ln(k) + \frac{1}{k} C_1$$

$$\frac{u_e}{u_\tau} = \frac{1}{0.4205} ln(R_\tau) + \frac{1}{0.4205} ln(0.4205) + \frac{1}{0.4205} 4.57971$$

$$\boxed{\frac{u_e}{u_\tau} = 2.37812 ln(R_\tau) + 8.83091}$$

Displacement thickness

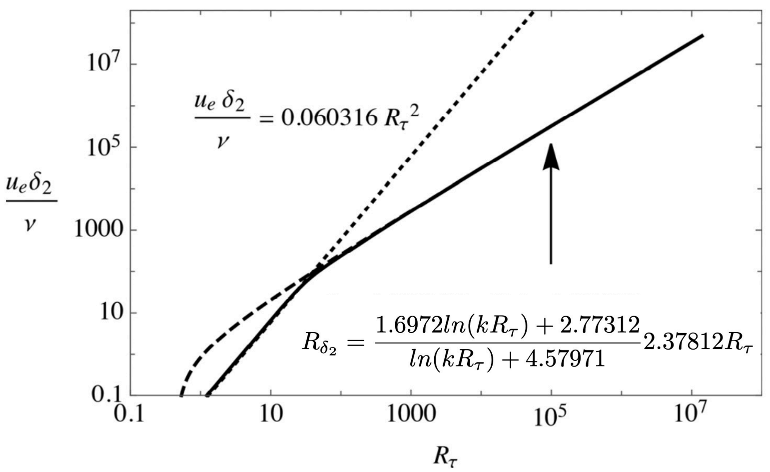


$$R_{\delta_1} = \frac{\delta_1 u_e}{\nu} = \int_0^1 \left(1 - \frac{u}{u_e}\right) d\eta$$

$$R_{\delta_1} = \frac{1 + C_1 - C_2}{k} R_\tau$$

$$\boxed{R_{\delta_1} = 4.03615 R_\tau}$$

Momentum thickness

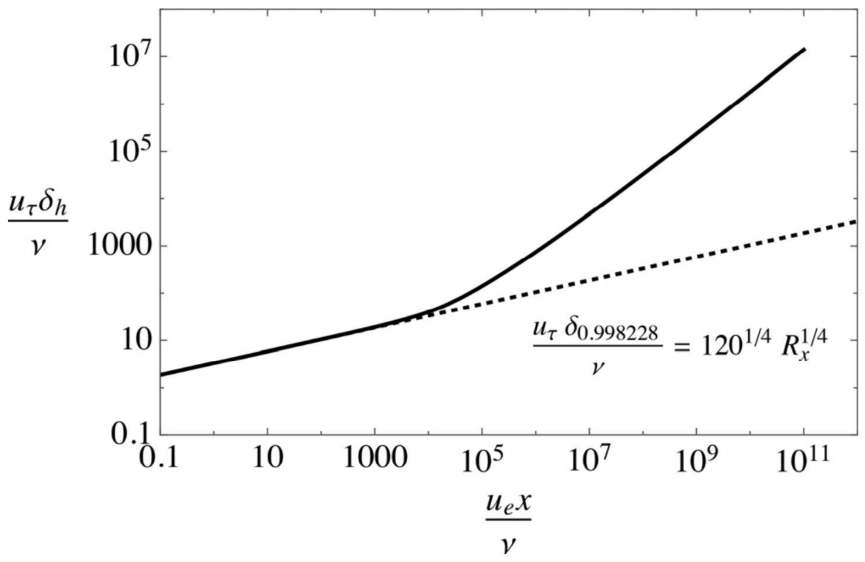


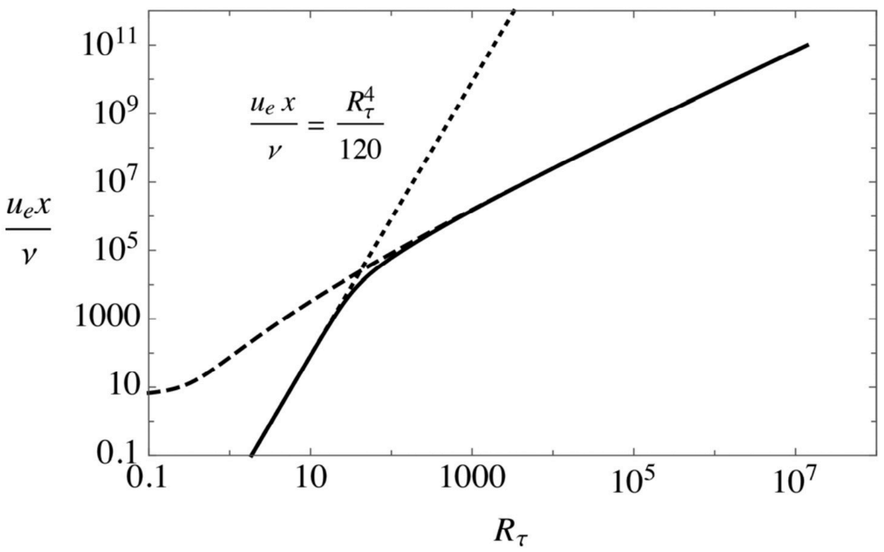
$$R_{\delta_2} = \frac{\delta_1 u_e}{\nu} = \int_0^1 \frac{u}{u_e}\left(1 - \frac{u}{u_e}\right) d\eta$$

$$R_{\delta_2} = \frac{(1 + C_1 - C_2) ln(kR_\tau) - 2 - C_1 - C_3 - 2C_5 + C_1 C_2}{ln(kR_\tau) + C_1} \frac{R_\tau}{k}$$

$$\boxed{R_{\delta_2} = \frac{1.6972 ln(kR_\tau) + 2.77312}{ln(kR_\tau) + 4.57971} 2.37812 R_\tau}$$

Relationship between R_τ and R_x





Karman integral equation for zero pressure gradient

$$\frac{d\delta_2}{dx} - \left(\frac{u_\tau}{u_e}\right)^2 = 0$$

$$\frac{dR_\tau}{dR_x} = \frac{U}{dR_{\delta_2}/dR_\tau}\left(\frac{u_\tau}{u_e}\right)^2$$

$$\frac{dR_x}{dR_\tau} = \alpha + \beta \ln(R_\tau) + \gamma ln(R_\tau)^2$$

$$R_x = (\alpha - \beta + 2\gamma + (\beta - 2\gamma)\ln(R_\tau) + \gamma ln(R_\tau)^2)R_\tau$$

$$\alpha = \frac{1}{k^3}(2 - 2C_1C_2 + C_1^2C_2 + C_3 - C_1C_3 + 2C_5 - 2C_1C_5 - (2 - C_1{}^2 + C_3 + 2C_5)\ln(k) + (1 + C_1 - C_2)\ln(k)^2) = 132.307$$

$$\beta = \frac{1}{k^3}(-2 + C_1^2 - C_3 - 2C_5 + 2(1 + C_1 - C_2)\ln(k)) = 102.285$$

$$\gamma = \frac{1}{k^3}(1 + C_1 - C_2) = 22.8263$$

$$R_x = (3.31525 + 2.48103 ln(R_\tau) + ln(R_\tau)^2) 22.8263 R_\tau$$

Conclusions

1) Over the past year, a new wall damping function has been derived for the UVP that improves the agreement with data while reducing the number of parameters in the UVP model from 5 to 4 for pipes and channels and from 4 to 3 for boundary layers.

2) The new damping function decays faster than the van Driest-type exponential and oscillates about zero several times.

3) The agreement with DNS data for the derivative of the mean velocity is much better near the wall. This leads to better agreement between the Reynolds shear stress generated from the UVP and DNS data near the wall.

4) The variation in optimal model parameter values from case to case is reduced, especially for boundary layer flows.

5) The Kármán constant, k, and the wall length scale, a, cannot be thought of as independent parameters. They act together through the product, *ka*, that appears in the shape function. If ka is fixed, then changing the Karman constant applies a pure scaling to the velocity.

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Main References

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Cantwell, B.J. 2021 Integral measures of the zero pressure gradient boundary layer over the Reynolds number range $0 \leq R_\tau < \infty$. *Phys. Fluids* 33, 085108.

M. A. Subrahmanyam, B. J. Cantwell, and J. J. Alonso, “A universal velocity profile for turbulent wall flows including adverse pressure gradient boundary layers,”JFM. 933, A16 (2022).