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A Primer on Symmetry Analysis

Summary of Lectures 3, 4 and 5

Over the next three lectures we will be discussing certain geometric aspects of turbulent flow that can be described using methods that fall under the general topic of Symmetry Analysis. Much of the research described in these lectures took place in the 1980s and 1990s.

Lecture 3 - But before we launch into a discussion of the geometry of turbulent flow, it is necessary to review some of the basic properties of 2D and 3D vector fields. And it is necessary to provide a brief introduction to Lie groups and their use in solving physical problems. Note that when equation numbers appear on the slides, the material can be found in my Text; *Introduction to Symmetry Analysis.*

Lecture 4 - We will use groups to develop general similarity rules applicable to a wide variety of turbulent free shear flows including the scaling laws for integral measures of the flow as well as microscale motions responsible for dissipation of turbulent kinetic energy (TKE). The similarity rules developed for free shear flows will also be used to derive Kolmogorov scaling of the inertial subrange.

Lecture 5 - Finally, we will focus on one of the most fundamental problems in fluid mechanics that one can imagine: the flow created by an impulsive force applied at a point in an infinite, incompressible, viscous fluid. We will see that the fundamental nature of transition in the jet created by an impulsive point force is a sequence of bifurcations in the phase portrait of particle paths in similarity coordinates leading to the onset of a starting vortex.

The main goal of these three lectures is make the case for carrying out a direct numerical simulation of the starting jet up to a moderately large Reynolds number of several thousand or so. The data will be used to identify the geometry of the 3D flow, and the possible connection to classical descriptions of turbulence such as the energy cascade. It might also be possible to gain fundamental knowledge of the behavior of turbulent flow in the limit of infinite Reynolds number.

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Instantaneous velocity vector field in the wake of a circular cylinder as seen by two observers.

Notice the critical points defining the geometry of the flow pattern

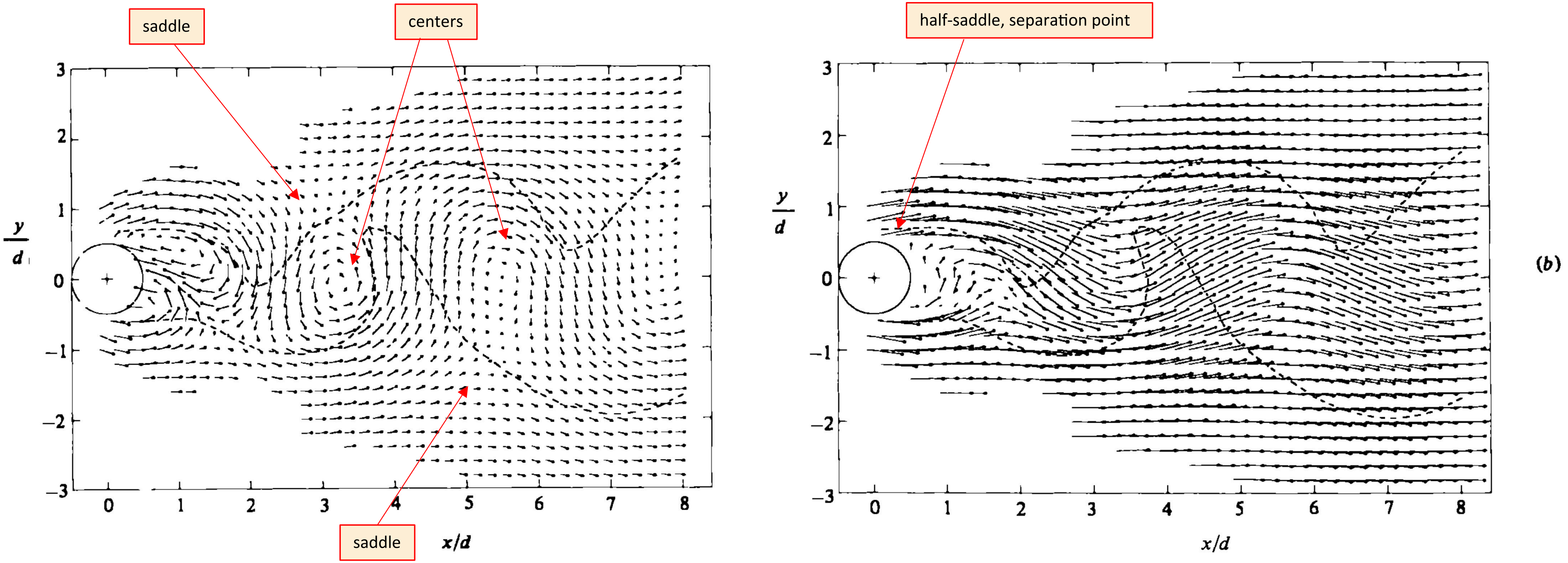

FIGURE 20. Interpolated velocity field at constant phase (7, 15) over 8 diameters of the wake as viewed from a frame of reference (*a*) moving downstream at $0.755u_\infty$. (*b*) fixed with respect to the cylinder. Dashed line is contour $\langle\gamma\rangle = 0.5$ from figure 23(*b*).

The Geometry of 2D and 3D vector fields

The trajectory of a fluid particle in a three-dimensional unsteady flow is governed by the system of first order ODEs,

$$\frac{dx^1}{dt} = U^1(\mathbf{x}, t) \qquad \frac{dx^2}{dt} = U^2(\mathbf{x}, t) \qquad \frac{dx^3}{dt} = U^3(\mathbf{x}, t)$$

At a given instant in time, $t = t_{\text{fixed}}$ the velocity field is frozen and instantaneous streamlines are determined by integrating the autonomous system

$$\frac{dx^1}{ds} = U^1(\mathbf{x}, t_{fixed}) \qquad \frac{dx^2}{ds} = U^2(\mathbf{x}, t_{fixed}) \qquad \frac{dx^3}{ds} = U^3(\mathbf{x}, t_{fixed})$$

where *s* is a pseudotime along an instantaneous streamline. The solution trajectories are

$$x^1 = f^1(\mathbf{x}, s, t_{fixed}) \qquad x^2 = f^2(\mathbf{x}, s, t_{fixed}) \qquad x^3 = f^3(\mathbf{x}, s, t_{fixed})$$

State-space analysis

Very often, the flow field can be completely understood without actually solving the particle path equations. Instead one looks at critical points, $\mathbf{x_c}$, in the flow field where,

$$U^j(\mathbf{x}_c, t_{fixed}) = 0 \qquad j = 1, 2, 3$$

Near a critical point the flow field can be expanded in a Taylor series

$$\frac{dx^j}{dt} = A^j{}_k(x^k - x^k{}_c) + O((x^k - x^k{}_c)^2) + \ldots \qquad j = 1, 2, 3$$

where the matrix of constants is

$$A^j{}_k = \left.\frac{\partial U^j}{\partial x^k}\right|_{\mathbf{x}=\mathbf{x}_c}$$

The geometry of the flow field, ie., the flow pattern, in the neighborhood of the critical point is determined by the eigenvalues of the matrix A^j_k.

Linear Flows in 2-Dimensions

In two dimensions the eigenvalues of A^j_k satisfy the quadratic

$$\lambda^2 + P\lambda + Q = 0, \qquad (3.166)$$

where P and Q are the matrix invariants

$$P = -A^j_j, \qquad Q = \mathrm{Det}\left(A^j_k\right). \qquad (3.167)$$

The eigenvalues are

$$\lambda = -\frac{P}{2} \pm \frac{1}{2}\sqrt{P^2 - 4Q}, \qquad (3.168)$$

and the character of the local flow is determined by the quadratic discriminant

$$D = Q - \frac{P^2}{4}. \qquad (3.169)$$

If $D > 0$, the eigenvalues are complex and a spiraling motion can be expected. Depending on the sign of P, the spiral may be stable or unstable. If $D < 0$, the eigenvalues are real and a predominantly straining flow can be expected. In this case the directionality of the local flow is defined by the two eigenvectors of A^j_k. The various possible flow patterns can be summarized on a crossplot of the invariants shown in Figure 3.5.

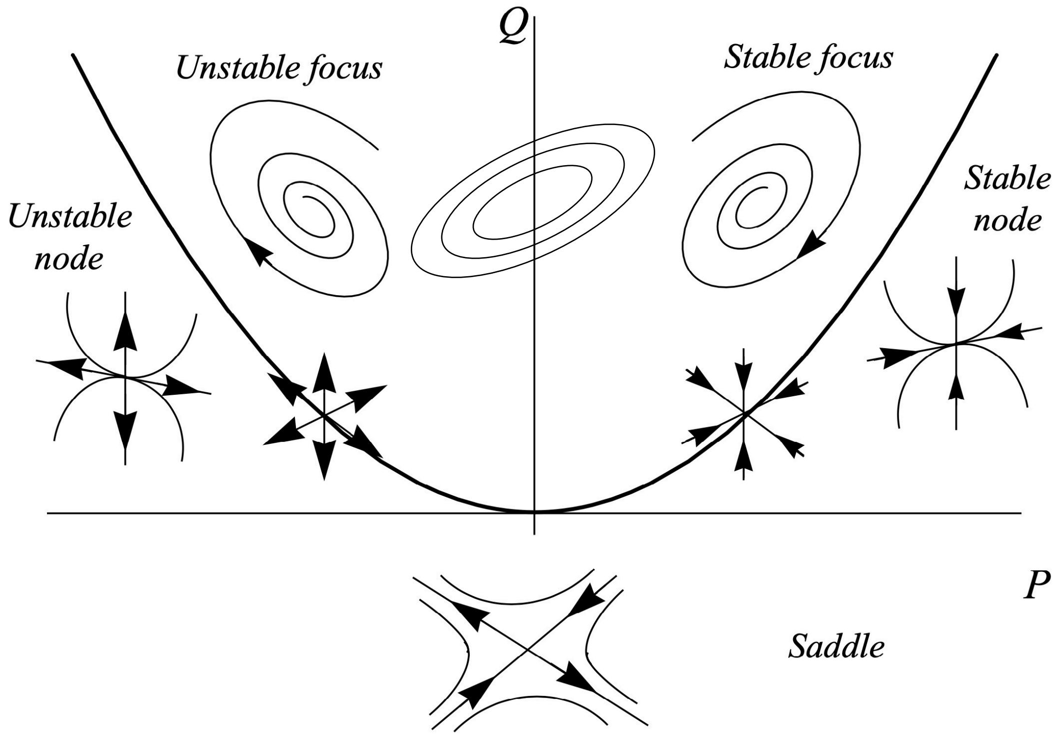

Fig. 3.5. Classification of linear solution trajectories in two dimensions.

Linear Flows in 3 - Dimensions

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In three dimensions the eigenvalues of A_k^j satisfy the cubic

$$\lambda^3 + P\lambda^2 + Q\lambda + R = 0, \tag{3.172}$$

where the invariants are

$$\begin{aligned} P &= -\mathrm{tr}[A] = -A_j^j, \\ Q &= \tfrac{1}{2}(P^2 - \mathrm{tr}[A^2]) = \tfrac{1}{2}\left(P^2 - A_k^j A_j^k\right), \\ R &= \tfrac{1}{3}(-P^3 + 3PQ - \mathrm{tr}[A^3]) = \tfrac{1}{3}\left(-P^3 + 3PQ - A_k^j A_m^k A_j^m\right). \end{aligned} \tag{3.173}$$

Any cubic can be simplified as follows. Let

$$\lambda = \alpha - \frac{P}{3}. \tag{3.174}$$

Then α satisfies

$$\alpha^3 + \hat{Q}\alpha + \hat{R} = 0, \tag{3.175}$$

where

$$\hat{Q} = Q - \tfrac{1}{3}P^2, \qquad \hat{R} = R - \tfrac{1}{3}PQ + \tfrac{2}{27}P^3. \tag{3.176}$$

Let

$$\begin{aligned} a_1 &= \left(-\frac{\hat{R}}{2} + \frac{1}{3\sqrt{3}}\left(\hat{Q}^3 + \tfrac{27}{4}\hat{R}^2\right)^{1/2}\right)^{1/3}, \\ a_2 &= \left(-\frac{\hat{R}}{2} - \frac{1}{3\sqrt{3}}\left(\hat{Q}^3 + \tfrac{27}{4}\hat{R}^2\right)^{1/2}\right)^{1/3} \end{aligned} \tag{3.177}$$

The real solution of (3.175) is expressed as

$$\alpha_1 = a_1 + a_2, \tag{3.178}$$

and the complex (or remaining real) solutions are

$$\begin{aligned} \alpha_2 &= -\tfrac{1}{2}(a_1 + a_2) + \frac{i\sqrt{3}}{2}(a_1 - a_2), \\ \alpha_3 &= -\tfrac{1}{2}(a_1 + a_2) - \frac{i\sqrt{3}}{2}(a_1 - a_2). \end{aligned} \tag{3.179}$$

Solving (3.172) for the eigenvalues leads to the cubic discriminant

$$D = \tfrac{27}{4}R^2 + \left(P^3 - \tfrac{9}{2}PQ\right)R + Q^2\left(Q - \tfrac{1}{4}P^2\right). \tag{3.180}$$

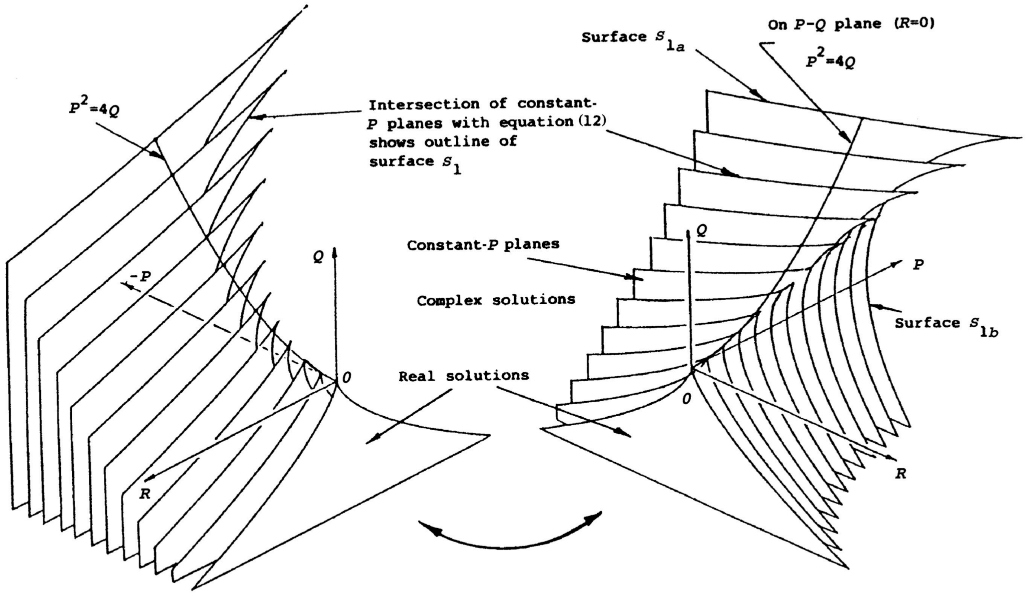

Fig. 3.8. The Cardano surface dividing real and complex eigenvalues in three dimensions (from Reference [3.10]).

If $D > 0$, the point (P, Q, R) lies above the surface and there is one real eigenvalue and two complex conjugate eigenvalues. If $D < 0$, all three eigenvalues are real. The invariants can be expressed in terms of the eigenvalues as follows. If the eigenvalues are real,

$$P = -(\lambda^1 + \lambda^2 + \lambda^3),$$
$$Q = \lambda^1\lambda^2 + \lambda^1\lambda^3 + \lambda^2\lambda^3, \quad (3.181)$$
$$R = -\lambda^1\lambda^2\lambda^3,$$

and if the eigenvalues are complex,

$$P = -(2\sigma + b),$$
$$Q = \sigma^2 + \omega^2 + 2\sigma b, \quad (3.182)$$
$$R = -b(\sigma^2 + \omega^2),$$

where b is the real eigenvalue and σ and ω are the real and imaginary parts of the complex conjugate eigenvalues.

A particularly interesting case occurs when $P = 0$. In this case the discriminant is

$$D = Q^3 + \tfrac{27}{4}R^2, \quad (3.183)$$

and the invariants are

$$Q = -\tfrac{1}{2}A^j_k A^k_j, \qquad R = -\tfrac{1}{3}A^j_k A^k_m A^m_j. \quad (3.184)$$

The various possible critical points in this case can be categorized on a plot of Q versus R. Figure 3.9 and Figure 3.5 are cuts through the Cardano surface (3.180) at $P = 0$ and $R = 0$ respectively.

3D Incompressible Flow, P = 0

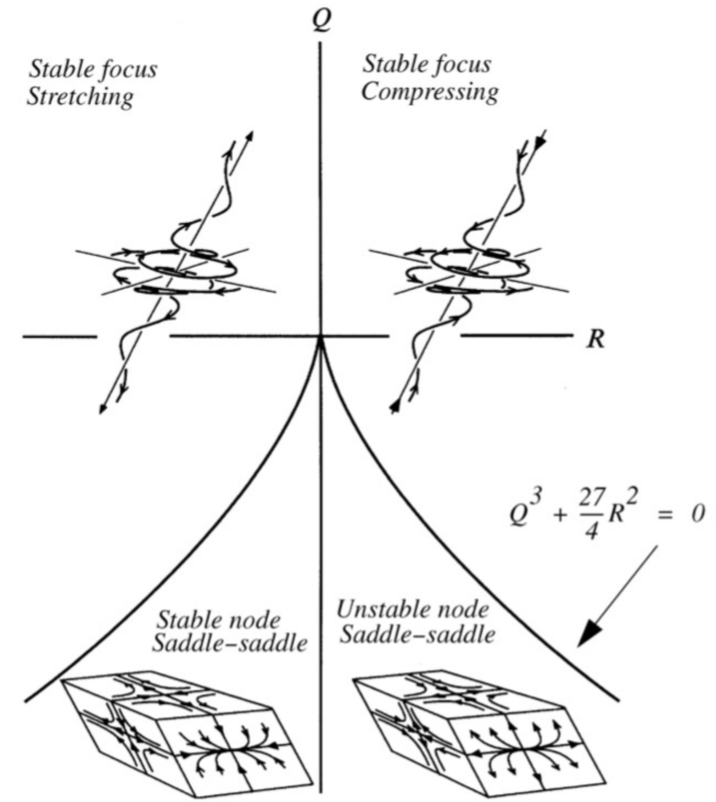

Fig. 3.9. Three-dimensional flow patterns in the plane $P = 0$ (from Reference [3.11]).

The smooth vector fields generated by fluid flow are Lie Groups. This subject will be central to our later discussion of the flow generated by an impulsive point force.

A primer on one parameter Lie groups

Lie groups arise as the solution families of autonomous systems of first order ODEs

$$\frac{dx^j}{ds} = \xi^j[\mathbf{x}] \qquad j = 1, ..., n$$

Lie groups are diffeomorphisms. That is, they are invertible maps that preserve the tangent structure of the transformation $\boldsymbol{F}$ to all orders.

with continuous, invertible solutions of the form

$$x^j = F^j[\tilde{\mathbf{x}}, s] \qquad j = 1, ..., n$$

and initial conditions

$$\tilde{x}^j = F^j[\tilde{\mathbf{x}}, 0] \qquad j = 1, ..., n$$

i) The group is a real analytic function of the group parameter s.
ii) Every s has an inverse s_{inv}.
iii) s=0 is the identity element.

Definition 5.3. *A function* $\Psi[\boldsymbol{x}]$ *is said to be invariant under the Lie group* $T^s : \{x^j = F^j[\tilde{\boldsymbol{x}}, s],\ \ j = 1, \ldots, n\}$ *if and only if*

$$\Psi[\boldsymbol{x}] = \Psi[\boldsymbol{F}[\tilde{\boldsymbol{x}}, s]] = \Psi[\tilde{\boldsymbol{x}}]. \tag{5.19}$$

Sophus Lie's great advance was to recognize that this difficult to apply *nonlinear* invariance condition could be replaced by a much simpler *linear* condition.

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The Lie group ***F*** can be expanded in a Taylor Series about the identity element of the group, s=0.

$$\tilde{x}^j = x^j + s\left[\frac{\partial F^j}{\partial s}\right]_{s=0} + O(s^2) \qquad j = 1, ..., n$$

The infinitesimals of the group are

$$\xi^j[\mathbf{x}] = \left[\frac{\partial F^j[\mathbf{x}, s]}{\partial s}\right]_{s=0}$$

also called the *vector field* of the group.

One may notice that, up to this point, we have used the tilde to denote the source point of the transformation, and here we have suddenly switched the role of the tilde to denote the target point of the transformation. Of course it is immaterial which value of ***x*** is assigned the tilde, which is merely a distinguishing mark. We are about to develop the infinitesimal theory of groups, which involves expanding the transformation ***F*** about the source point, and for convenience it is simpler to assign the tilde to the target value of the transformation. This avoids having to carry tildes along in all our formulas.

$$F^j[\mathbf{x}, s] \approx x^j + s\xi^j[\mathbf{x}] \qquad j = 1, ..., n$$

Expand the right-hand-side of $\Psi[\tilde{\mathbf{x}}] = \Psi[\mathbf{F}[\mathbf{x}, s]]$ about s=0. The result is the Lie series expansion for Ψ.

$$\Psi[\tilde{\mathbf{x}}] = \Psi[\mathbf{x}] + s(X\Psi) + \frac{s^2}{2!}X(X\Psi) + \frac{s^3}{3!}X(X(X\Psi)) + ... = e^{sX}\Psi[\mathbf{x}]$$

This is sometimes called the exponential map of Ψ.

$$\Psi[\tilde{\mathbf{x}}] = \Psi[\mathbf{x}] + s(X\Psi) + \frac{s^2}{2!}X(X\Psi) + \frac{s^3}{3!}X(X(X\Psi)) + ... = e^{sX}\Psi[\mathbf{x}]$$

The group operator is

$$X = \xi^j \frac{\partial}{\partial x^j}$$

A function Ψ is invariant ($\Psi[\tilde{\mathbf{x}}] = \Psi[\mathbf{x}]$) under the group $\mathbf{F}$ if and only if

$$\boxed{X\Psi = 0}$$

When this condition is satisfied, the right-hand side of the Lie series expansion is zero to all orders beyond the zeroeth.

This is a much simpler invariance condition to apply as it just requires the solution of a first order linear PDE. Equivalently one usually just solves the characteristic equations of the group, $dx^j/ds = \xi^j\,[\mathbf{x}]$, for up to n-1 invariants, Ψ^i, *i = 1,.....,n-1* . When applied to PDEs and ODEs, the invariants become the similarity variables used to reduce the dimensionality or the order of an equation.

Solve the characteristic equations

$$\frac{dx^1}{\xi^1[\boldsymbol{x}]} = \frac{dx^2}{\xi^2[\boldsymbol{x}]} = \frac{dx^3}{\xi^3[\boldsymbol{x}]} = \cdots = \frac{dx^n}{\xi^n[\boldsymbol{x}]}$$

Recall

$$\frac{dx^j}{ds} = \xi^j[\mathbf{x}] \qquad j = 1, ..., n$$

For integrals

$$\psi^i = \Psi^i[\boldsymbol{x}], \qquad i = 1, \ldots, n-1$$

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Theorem 9.2. *The pth-order system of partial differential equations* $\psi^i = \Psi^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_p] = 0$ *is a vector of locally analytic functions of the differential variables* $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_p$. *Expand* $\Psi^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_p]$ *in a Lie series*

$$\Psi^i[\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}_1, \tilde{\boldsymbol{y}}_2, \ldots, \tilde{\boldsymbol{y}}_p] = \Psi^i[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_p] + sX_{\{p\}}\Psi^i + \frac{s^2}{2!}X_{\{p\}}\left(X_{\{p\}}\Psi^i\right) + \cdots, \quad (9.51)$$

where $X_{\{p\}}$ *is the pth extended group operator*

$$X_{\{p\}} = \xi^j \frac{\partial}{\partial x^j} + \eta^i \frac{\partial}{\partial y^i} + \eta^i_{\{j_1\}} \frac{\partial}{\partial y^i_{j_1}} + \eta^i_{\{j_1 j_2\}} \frac{\partial}{\partial y^i_{j_1 j_2}} + \cdots + \eta^i_{\{j_1 j_2 \ldots j_p\}} \frac{\partial}{\partial y^i_{j_1 j_2 \ldots j_p}}. \quad (9.52)$$

The system Ψ^i *is invariant under the group* (ξ^j, η^i) *if and only if*

$$\boxed{X_{\{p\}}\Psi^i = 0, \qquad i = 1, \ldots, m.} \quad (9.53)$$

The characteristic equations corresponding to (9.53) are

$$\boxed{\frac{dx^j}{\xi^j} = \frac{dy^i}{\eta^i} = \frac{dy^i_{j_1}}{\eta^i_{\{j_1\}}} = \frac{dy^i_{j_1 j_2}}{\eta^i_{\{j_1 j_2\}}} = \cdots = \frac{dy^i_{j_1 j_2 \ldots j_p}}{\eta^i_{\{j_1 j_2 \ldots j_p\}}}.} \quad (9.54)$$

The main thing to recognize when applying Lie theory to ODEs and PDEs is that the function Ψ can be essentially any differential object. The mechanics of Lie theory gets complicated because in order to transform an ODE or PDE one needs to work out the transformations of derivatives to whatever order appears in the equations or system of equations under study.

Example 5.1 (Invariance of a parabola under dilation). Transform

$$\Psi[x, y] = y/x^2 \tag{5.14}$$

using the dilation (or stretching) group

$$T^{\text{dil}} : \{x = s\tilde{x},\ \ y = s^n \tilde{y},\ \ s > 0\}. \tag{5.15}$$

The restriction on s in (5.15) arises because $s = 0$ has no inverse. Once a point has been mapped to $(x, y) = (0, 0)$, there is no way to return to the original point by some choice of s.

The important property of a group that makes it so useful is that it is always possible to transform points smoothly and invertibly along the pathlines traced out by the group.

Note that the identity element of (5.15) is $s = 1$. A more natural way of expressing the dilation group is to write it in the form

$$T^{\text{dil}} : \{x = e^s \tilde{x},\ \ y = e^{ns} \tilde{y}\}. \tag{5.16}$$

Now the parameter s can take on the full range of values from $-\infty$ to $+\infty$, and the identity element is $s = 0$. This is the form of the dilation group that was used in the discussion of dimensional analysis in Chapter 2. Use (5.16) to transform (5.14):

$$\Psi[x, y] = y/x^2 = e^{s(n-2)}(\tilde{y}/\tilde{x}^2). \tag{5.17}$$

For general n the function is not invariant under the group; however, if we set $n = 2$, then

$$\Psi[x, y] = y/x^2 = \tilde{y}/\tilde{x}^2 = \Psi[\tilde{x}, \tilde{y}]. \tag{5.18}$$

The parameter s does not appear in (5.18), and the function is said to be invariant under (5.16) for $n = 2$.

Example 5.2 (The rotation group in two dimensions). Consider the rotation group

$$T^{rot} : \left\{ \begin{aligned} \tilde{x} &= x\cos[s] - y\sin[s] \\ \tilde{y} &= x\sin[s] + y\cos[s] \end{aligned} \right\}. \tag{5.35}$$

The infinitesimals of the group are $(\xi, \eta) = (-y, x)$, and the invariance condition is

$$-y\frac{\partial \Psi}{\partial x} + x\frac{\partial \Psi}{\partial y} = 0 \tag{5.36}$$

with corresponding characteristic equation

$$\frac{dy}{x} = -\frac{dx}{y}. \tag{5.37}$$

Equation (5.37) is particularly simple in that the two terms are uncoupled. The integral invariant [the integral of (5.36), invariant under (5.35)] is the family of circles

$$\psi = \Psi(x, y) = x^2 + y^2. \tag{5.38}$$

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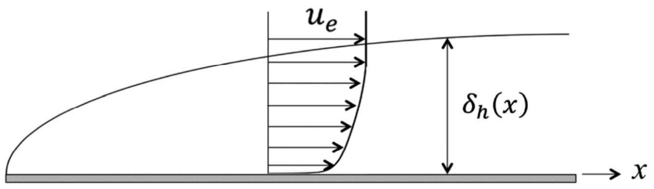
What do we mean when we say a differential equation is invariant under a group?



Consider the PDE that governs the Blasius boundary layer streamfunction.

$$\tilde{\psi}_{\tilde{y}}\tilde{\psi}_{\tilde{x}\tilde{y}} - \tilde{\psi}_{\tilde{x}}\tilde{\psi}_{\tilde{y}\tilde{y}} - \nu\tilde{\psi}_{\tilde{y}\tilde{y}\tilde{y}} = 0$$

Test the invariance under a three-parameter dilation group.

$$\tilde{x} = e^a x, \qquad \tilde{y} = e^b y, \qquad \tilde{\psi} = e^c \psi$$

Substitute the group into the equation

$$\tilde{\psi}_{\tilde{y}}\tilde{\psi}_{\tilde{x}\tilde{y}} - \tilde{\psi}_{\tilde{x}}\tilde{\psi}_{\tilde{y}\tilde{y}} - \nu\tilde{\psi}_{\tilde{y}\tilde{y}\tilde{y}}$$
$$= e^{2c-a-2b}\psi_y\psi_{xy} - e^{2c-a-2b}\psi_x\psi_{yy} - \nu e^{c-3b}\psi_{yyy} = 0$$

For invariance the group must map the equation to itself. For this we must have

$$2c - a - 2b = c - 3b$$

The equation is invariant under a two parameter dilation group

$$\tilde{x} = e^a x, \qquad \tilde{y} = e^b y, \qquad \tilde{\psi} = e^{a-b}\psi$$

It is necessary to work out all the derivatives that appear in the equation. This is easy for dilation groups.

$$\tilde{\psi}_{\tilde{y}} = e^{c-b}\psi_y$$

$$\tilde{\psi}_{\tilde{x}\tilde{y}} = e^{c-a-b}\psi_{xy}$$

$$\tilde{\psi}_{\tilde{x}} = e^{c-a}\psi_x$$

$$\tilde{\psi}_{\tilde{y}\tilde{y}} = e^{c-2b}\psi_{yy}$$

$$\tilde{\psi}_{\tilde{y}\tilde{y}\tilde{y}} = e^{c-3b}\psi_{yyy}$$

Example – Flow induced by a heated vertical plate

10.6 Consider the buoyancy induced flow produced by a heated flat plate sketched in Figure 10.15. This flow is governed by a coupled system of convection–diffusion equations for the momentum and temperature. Changes in density are related to changes in temperature by a thermal expansion coefficient:

$$\rho - \rho_\infty = \beta(T - T_\infty). \tag{10.152}$$

If changes in density are small $[(\rho - \rho_\infty)/\rho_\infty \ll 1]$, the fluid behaves incompressibly ($\nabla \cdot \boldsymbol{u} = 0$) with a local body force equal to $(\rho - \rho_\infty)g$. The governing equations are $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$,

$$\begin{aligned} \frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} - \nu\frac{\partial^3\psi}{\partial y^3} &= \frac{\beta(T - T_\infty)g}{\rho_\infty}, \\ \frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y} &= \kappa\frac{\partial^2 T}{\partial y^2}. \end{aligned} \tag{10.153}$$

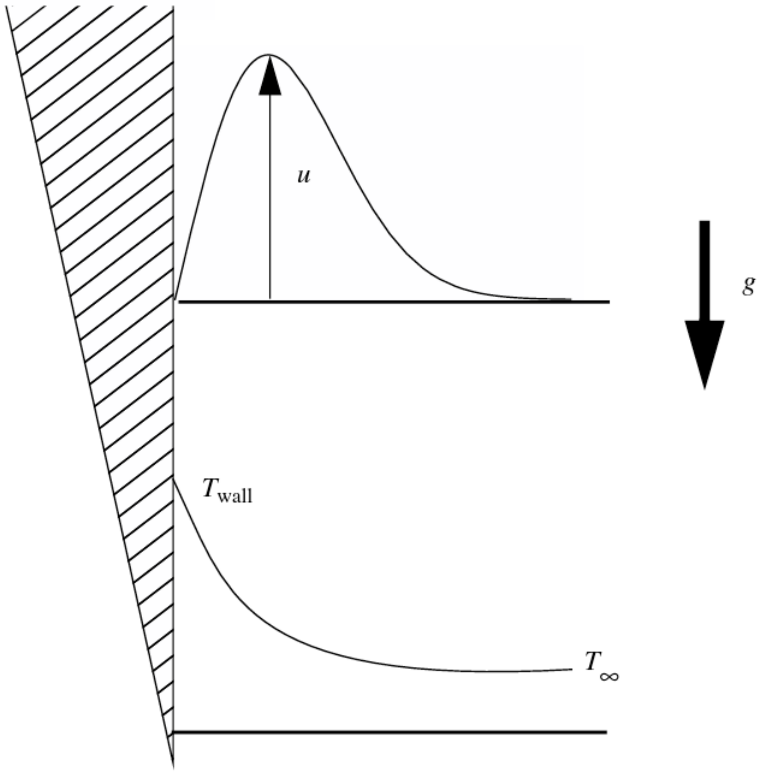


Fig. 10.15.

Suggestion: use $\dfrac{T - T_\infty}{T_w - T_\infty}$ as the appropriate temperature variable.

This is OK because of the invariance under translation of the temperature.

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Dimensioned Governing Equations

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} - \nu \frac{\partial^3 \psi}{\partial y^3} = \theta \frac{\beta (T_{wall} - T_\infty) g}{\rho_\infty}$$

where

$$\theta = \frac{T - T_\infty}{T_{wall} - T_\infty}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y} = \kappa \frac{\partial^2 \theta}{\partial y^2}$$

The parameters of the problem can be used to define length and time scales.

$$L = \left[\frac{\rho_\infty}{\beta g (T_{wall} - T_\infty)}\right]^{1/3} \nu^{2/3} \qquad T = \left[\frac{\rho_\infty}{\beta g (T_{wall} - T_\infty)}\right]^{2/3} \nu^{1/3}$$

and dimensionless variables

$$x \rightarrow Lx, \qquad y \rightarrow Ly, \qquad \psi \rightarrow (L^2/T)\psi$$

Use this mapping to produce the dimensionless forms of the original PDEs.

Dimensionless Governing Equations

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^3\psi}{\partial y^3}=\theta$$

$$\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}=P_r\frac{\partial^2\theta}{\partial y^2}$$

where

$$P_r=\frac{\kappa}{\nu}$$

is the Prandtl number

Use the Mathematica package IntroToSymmetry.m on my website https://web.stanford.edu/~cantwell/ to work out the invariant groups of the governing equations. Screenshots of the notebook are provided after the Main References.

The equations are invariant under a 5 parameter Lie group

Three translation groups in x, y, ψ and θ (*s* is the group parameter)

$$\tilde{x} = x + s \quad \tilde{y} = y \quad \tilde{\psi} = \psi \quad \tilde{\theta} = \theta$$

$$\tilde{x} = x \quad \tilde{y} = y + f(x) \quad \tilde{\psi} = \psi \quad \tilde{\theta} = \theta$$

f(x) is an arbitrary translation of the *y* coordinate.

$$\tilde{x} = x \quad \tilde{y} = y \quad \tilde{\psi} = \psi + s \quad \tilde{\theta} = \theta$$

and two dilation (stretching) groups.

$$\tilde{x} = e^s x \quad \tilde{y} = y \quad \tilde{\psi} = e^s \psi \quad \tilde{\theta} = e^s \theta$$

$$\tilde{x} = e^s x \quad \tilde{y} = e^s y \quad \tilde{\psi} = \psi \quad \tilde{\theta} = e^{-3s} \theta$$

Note that translation and dilation groups can usually be found by inspection.

The two finite dilation groups, their infinitesimals and group operators are as follows.

$$\tilde{x} = e^s x \quad \tilde{y} = e^s y \quad \tilde{\psi} = \psi \quad \tilde{\theta} = e^{-3s}\theta$$

$$\tilde{x} = e^s x \quad \tilde{y} = y \quad \tilde{\psi} = e^s \psi \quad \tilde{\theta} = e^s\theta$$

$$\xi_1 = x \quad \xi_2 = y \quad \eta_1 = 0 \quad \eta_2 = -3\theta$$

$$\xi_1 = x \quad \xi_2 = 0 \quad \eta_1 = \psi \quad \eta_2 = \theta$$

$$X^a = x\frac{d}{dx} + y\frac{d}{dy} + 0\frac{d}{d\psi} - 3\theta\frac{d}{d\theta}$$

$$X^b = x\frac{d}{dx} + 0\frac{d}{dy} + \psi\frac{d}{d\psi} + \theta\frac{d}{d\theta}$$

The fact that there are two dilations, a consequence of the algebraic structure of the equations and the diffusive nature of the two coupled equations is very significant. The reason is, that the equation is invariant under, not just the two specific dilation groups X^a and X^b , but any linear combination of the two.

This means we are free to choose the linear combination that matches some boundary condition. In this case we need to generate similarity variables that preserve the constant temperature condition at the wall.

Choose

$$X^c = X^a + 3X^b = 4x\frac{d}{dx} + y\frac{d}{dy} + 3\psi\frac{d}{d\psi} + 0\frac{d}{d\theta}$$

Finite Group

$$\tilde{x} = e^{4s}x \qquad \tilde{y} = e^{s}y \qquad \tilde{\psi} = e^{3s}\psi \qquad \tilde{\theta} = \theta$$

The characteristic equations are

$$\frac{dx}{4x} = \frac{dy}{y} = \frac{d\psi}{3\psi} = \frac{d\theta}{0}$$

The three invariants of this group are

$$\alpha = \frac{y}{x^{1/4}}, \qquad F = \frac{\psi}{x^{3/4}} \qquad G = \theta$$

And so we seek solutions of the form

$$\psi = x^{3/4}F(\alpha) \qquad \theta = G(\alpha)$$

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Reduction to two ODEs in the similarity variables α, F and G.

```
In[1]:= (************ DEFINE VARIABLES ****************)
In[2]:= α[x_, y_] := y / x^(1/4)
In[3]:= ψ[x_, y_] := x^(3/4) * F[α[x, y]]
In[4]:= θ[x_, y_] := G[α[x, y]]
In[5]:= (************ SUBSTITUTE INTO THE STREAM FUNCTION EQUATION ****************)
In[6]:= D[ψ[x, y], y, y, y] + θ[x, y] + D[ψ[x, y], x] * D[ψ[x, y], y, y] - D[ψ[x, y], y] * D[ψ[x, y], x, y]
```

Out[6]=

$$G\left[\frac{y}{x^{1/4}}\right] + x^{1/4}\left(\frac{3F\left[\frac{y}{x^{1/4}}\right]}{4x^{1/4}} - \frac{yF'\left[\frac{y}{x^{1/4}}\right]}{4\sqrt{x}}\right)F''\left[\frac{y}{x^{1/4}}\right] - \sqrt{x}\,F'\left[\frac{y}{x^{1/4}}\right]\left(\frac{F'\left[\frac{y}{x^{1/4}}\right]}{2\sqrt{x}} - \frac{yF''\left[\frac{y}{x^{1/4}}\right]}{4x^{3/4}}\right) + F^{(3)}\left[\frac{y}{x^{1/4}}\right]$$

In[7]:=

$$G\left[\frac{y}{x^{1/4}}\right] + x^{1/4}\left(\frac{3F\left[\frac{y}{x^{1/4}}\right]}{4x^{1/4}} - \frac{yF'\left[\frac{y}{x^{1/4}}\right]}{4\sqrt{x}}\right)F''\left[\frac{y}{x^{1/4}}\right] - \sqrt{x}\,F'\left[\frac{y}{x^{1/4}}\right]\left(\frac{F'\left[\frac{y}{x^{1/4}}\right]}{2\sqrt{x}} - \frac{yF''\left[\frac{y}{x^{1/4}}\right]}{4x^{3/4}}\right) + F^{(3)}\left[\frac{y}{x^{1/4}}\right] \;/.\; \frac{y}{x^{1/4}} \to \alpha$$

Out[7]=

$$G[\alpha] + x^{1/4}\left(\frac{3F[\alpha]}{4x^{1/4}} - \frac{yF'[\alpha]}{4\sqrt{x}}\right)F''[\alpha] - \sqrt{x}\,F'[\alpha]\left(\frac{F'[\alpha]}{2\sqrt{x}} - \frac{yF''[\alpha]}{4x^{3/4}}\right) + F^{(3)}[\alpha]$$

In[8]:=

$$G[\alpha] + x^{1/4}\left(\frac{3F[\alpha]}{4x^{1/4}} - \frac{yF'[\alpha]}{4\sqrt{x}}\right)F''[\alpha] - \sqrt{x}\,F'[\alpha]\left(\frac{F'[\alpha]}{2\sqrt{x}} - \frac{yF''[\alpha]}{4x^{3/4}}\right) + F^{(3)}[\alpha] \;/.\; y \to \alpha * x^{1/4}$$

Out[8]=

$$G[\alpha] + x^{1/4}\left(\frac{3F[\alpha]}{4x^{1/4}} - \frac{\alpha F'[\alpha]}{4x^{1/4}}\right)F''[\alpha] - \sqrt{x}\,F'[\alpha]\left(\frac{F'[\alpha]}{2\sqrt{x}} - \frac{\alpha F''[\alpha]}{4\sqrt{x}}\right) + F^{(3)}[\alpha]$$

```
In[9]:= (************ THIRD ORDER MOMENTUM ODE ****************)
```

In[10]:=

$$\text{Simplify}\left[G[\alpha] + x^{1/4}\left(\frac{3F[\alpha]}{4x^{1/4}} - \frac{\alpha F'[\alpha]}{4x^{1/4}}\right)F''[\alpha] - \sqrt{x}\,F'[\alpha]\left(\frac{F'[\alpha]}{2\sqrt{x}} - \frac{\alpha F''[\alpha]}{4\sqrt{x}}\right) + F^{(3)}[\alpha]\right]$$

Out[10]=

$$G[\alpha] - \frac{1}{2}F'[\alpha]^2 + \frac{3}{4}F[\alpha]\,F''[\alpha] + F^{(3)}[\alpha]$$

```
In[11]:= (************ SUBSTITUTE INTO THE TEMPERATURE EQUATION ****************)
In[12]:= Pr * D[θ[x, y], y, y] - D[ψ[x, y], y] * D[θ[x, y], x] + D[ψ[x, y], x] * D[θ[x, y], y]
```

Out[12]=

$$\frac{yF'\left[\frac{y}{x^{1/4}}\right]G'\left[\frac{y}{x^{1/4}}\right]}{4x^{3/4}} + \frac{\left(\frac{3F\left[\frac{y}{x^{1/4}}\right]}{4x^{1/4}} - \frac{yF'\left[\frac{y}{x^{1/4}}\right]}{4\sqrt{x}}\right)G'\left[\frac{y}{x^{1/4}}\right]}{x^{1/4}} + \frac{\Pr G''\left[\frac{y}{x^{1/4}}\right]}{\sqrt{x}}$$

In[13]:=

$$\frac{yF'\left[\frac{y}{x^{1/4}}\right]G'\left[\frac{y}{x^{1/4}}\right]}{4x^{3/4}} + \frac{\left(\frac{3F\left[\frac{y}{x^{1/4}}\right]}{4x^{1/4}} - \frac{yF'\left[\frac{y}{x^{1/4}}\right]}{4\sqrt{x}}\right)G'\left[\frac{y}{x^{1/4}}\right]}{x^{1/4}} + \frac{\Pr G''\left[\frac{y}{x^{1/4}}\right]}{\sqrt{x}} \;/.\; \frac{y}{x^{1/4}} \to \alpha$$

Out[13]=

$$\frac{yF'[\alpha]G'[\alpha]}{4x^{3/4}} + \frac{\left(\frac{3F[\alpha]}{4x^{1/4}} - \frac{yF'[\alpha]}{4\sqrt{x}}\right)G'[\alpha]}{x^{1/4}} + \frac{\Pr G''[\alpha]}{\sqrt{x}}$$

In[14]:=

$$\frac{yF'[\alpha]G'[\alpha]}{4x^{3/4}} + \frac{\left(\frac{3F[\alpha]}{4x^{1/4}} - \frac{yF'[\alpha]}{4\sqrt{x}}\right)G'[\alpha]}{x^{1/4}} + \frac{\Pr G''[\alpha]}{\sqrt{x}} \;/.\; y \to \alpha * x^{1/4}$$

Out[14]=

$$\frac{\alpha F'[\alpha]G'[\alpha]}{4\sqrt{x}} + \frac{\left(\frac{3F[\alpha]}{4x^{1/4}} - \frac{\alpha F'[\alpha]}{4x^{1/4}}\right)G'[\alpha]}{x^{1/4}} + \frac{\Pr G''[\alpha]}{\sqrt{x}}$$

```
In[15]:= (************ SECOND ORDER TEMPERATURE ODE ****************)
```

In[16]:=

$$\text{Simplify}\left[\frac{\alpha F'[\alpha]G'[\alpha]}{4\sqrt{x}} + \frac{\left(\frac{3F[\alpha]}{4x^{1/4}} - \frac{\alpha F'[\alpha]}{4x^{1/4}}\right)G'[\alpha]}{x^{1/4}} + \frac{\Pr G''[\alpha]}{\sqrt{x}}\right]$$

Out[16]=

$$\frac{3F[\alpha]\,G'[\alpha] + 4\Pr G''[\alpha]}{4\sqrt{x}}$$

When the (dimensionless) ψ and θ are substituted into the original PDEs the result is a pair of coupled ODEs in the similarity variables F, G and α.

$$\frac{d^3F}{d\alpha^3} + \frac{3}{4}F\frac{d^2F}{d\alpha^2} - \frac{1}{2}\left(\frac{dF}{d\alpha}\right)^2 + G = 0 \qquad 0 \le \alpha < \infty$$

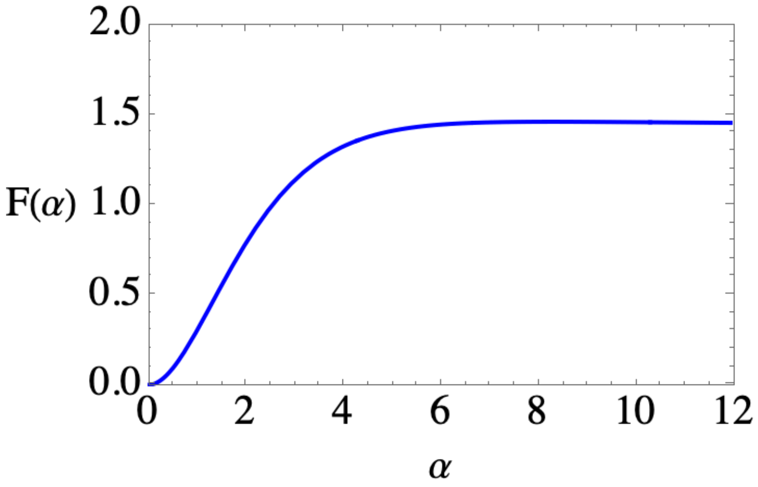
$$\frac{d^2G}{d\alpha^2} + \frac{3}{4P_r}F\frac{dG}{d\alpha} = 0$$

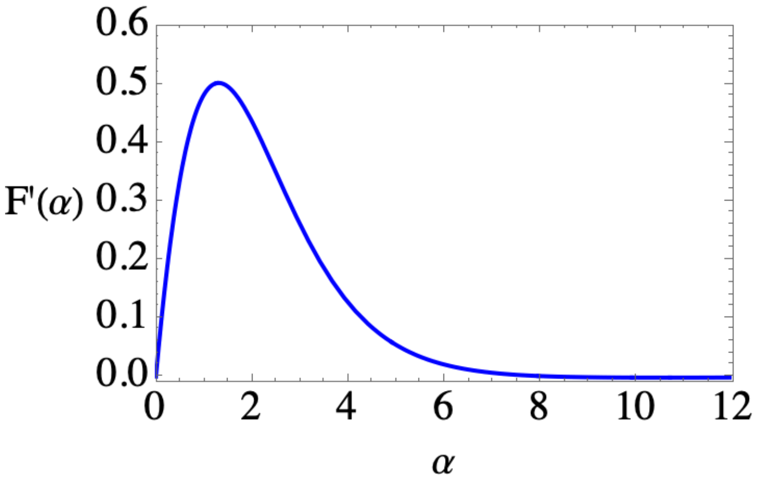
With boundary conditions

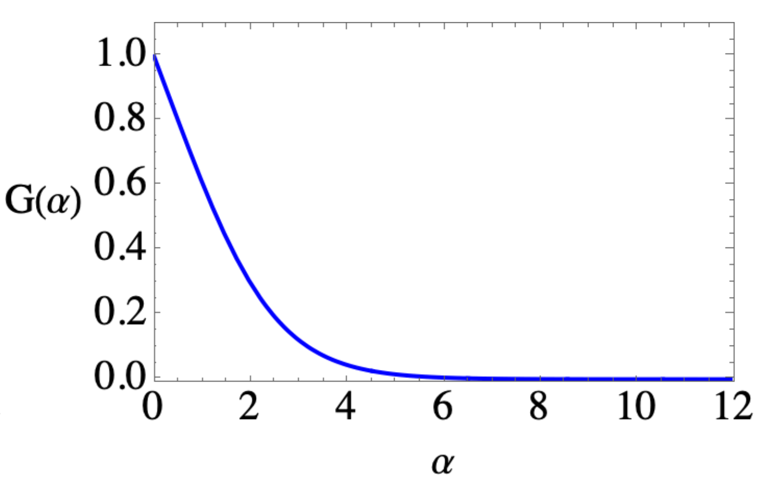
$$F(0) = 0 \quad F'(0) = 0 \quad F(\infty) = 0 \quad G(0) = 1 \quad G(\infty) = 0$$

In practice the equations are most easily solved as an initial value problem and for Pr=1 a good choice is

$$F(0) = 0 \quad F'(0) = 0 \quad F''(0) = 0.907925 \quad G(0) = 1 \quad G'(0) = -0.40093$$







Concluding Remarks

This chapter represents a brief introduction to the methods of Symmetry Analysis.

In the next lecture we will use Lie groups to develop general similarity rules for turbulent shear flows.

Then in the third lecture we will we will focus on one case and use these methods to determine the nature of transition in an impulsively started viscous jet.

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Main References

Cantwell, Brian J., *Introduction to Symmetry Analysis,* Cambridge University Press, Cambridge Texts in Applied Mathematics, 2002.

Schlichting and Gersten, *Boundary Layer Theory,* Springer 9th Edition, 2017.

Chong, M. S., Perry, A. E., and Cantwell, B. J. 1990. A general classification of three-dimensional flow fields. *Phys. Fluids A* **2**(5):765–777.

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```
In[1]:= Off[General::spell]
```

Clear all symbols in the current context.

```
In[2]:= ClearAll[Evaluate[Context[] <> "*"]]
```

First read in the package which is located in User Home Folder/Library/*Mathematica*/Applications/SymmetryAnalysis.

```
In[3]:= Needs["SymmetryAnalysis`IntroToSymmetry`"]
```

Enter the list of independent variables.

```
In[4]:= independentvariables = {"x", "y"};
```

Enter the list of dependent variables.

```
In[5]:= dependentvariables = {"Ψ", "θ"};
```

Enter the input equations as strings. Don't include the ==0 at the end. The ambient reference temperature is tinf.

```
In[6]:= equation1 = "D[Ψ[x,y],y,y,y]+(1/ν)*θ[x,y]+(1/ν)*D[Ψ[x,y],x]*D[Ψ[x,y],y,y]-(1/ν)*D[Ψ[x,y],y]*D[Ψ[x,y],x,y]";

In[7]:= equation2 = "D[θ[x,y],y,y]-(1/κ)*D[Ψ[x,y],y]*D[θ[x,y],x]+(1/κ)*D[Ψ[x,y],x]*D[θ[x,y],y]";

In[8]:= rulesarray =
   {"D[Ψ[x,y],y,y,y]->-(1/ν)*θ[x,y]-(1/ν)*D[Ψ[x,y],x]*D[Ψ[x,y],y,y]+(1/ν)*D[Ψ[x,y],y]*D[Ψ[x,y],x,y]",
    "D[θ[x,y],y,y]->(1/κ)*D[Ψ[x,y],y]*D[θ[x,y],x]-(1/κ)*D[Ψ[x,y],x]*D[θ[x,y],y]"};
```

The input equations are expressed internally in terms of generic variables, (x1,x2,x3,..., y1,y2,y3,...) using the function StringReplace. When this is done it is important that function names and constant names that may appear in the equation are not modified. Enter these protected names in string format.

```
In[9]:= frozennames = {"ν", "κ"};
```

Enter the maximum derivative order of the input equation set.

```
In[10]:= p = 3;
```

The maximum derivative order that the infinitesimals are assumed to depend on is specified by the input parameter r. This parameter is only nonzero when the user is looking for Lie contact groups or Lie-Backlund groups. For the usual case where one is searching for point groups set r=0.

```
In[11]:= r = 0;
```

When searching for Lie-Backlund groups (r=1 or greater) one can, without loss of generality, leave the independent variables untransformed. The corresponding infinitesimals (the xse's) are set to zero by setting xseon=0. If one is searching for point groups then set xseon=1. The choice xseon=1 is also an option when looking for Lie-Backlund groups and this can be useful when looking for contact symmetries.

```
In[12]:= xseon = 1;
```

When searching for Lie-Backlund groups it is necessary to differentiate the input equation to produce derivatives of order p+r and append these higher order differential consequences to the set of rules applied to the invariance condition. This process is carried out automatically when internalrules=1. For point groups the equation or equation system is the only rule or set of rules needed and one sets internalrules=0.

```
In[13]:= internalrules = 0;
```

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Now work out the determining equations of the Lie point group that leaves equation1 invariant. The output is available as a table of strings called zdeterminingequations. Notice that rulesarray contains both equations and that therefore both equations in the set are applied to the invariance condition for each equation.

```
In[14]:= Timing[FindDeterminingEquations[
   independentvariables, dependentvariables, frozennames,
   p, r, xseon, equation1, rulesarray, internalrules]]

The function FindDetermining Equations has
  begun, the memory in use = 155874976, the time used = 10.318586`

The function FindDeterminingEquations is nearly complete. The invariance condition has been
  created with all rules applied. The final step in the generation of the determining
  equations is to sum together terms in the table of invariance condition terms (called
  infinitesimaltable) that are multiplied by the same combination of products of free y
  derivatives. The result is the table infinitesimaltablesums corresponding to matching
  y-derivative expressions. If the invariance condition is long as it often is this process
  could take a long time since it requires sorting through the table infinitesimaltable
  once for each possible combination of y derivative products. This is the rate limiting
  step in the function FindDeterminingEquations.Virtually all other steps are quite
  fast including the generation of the extended derivatives of the infinitesimals.

The determining equations have been expressed in terms of z-variables, the
  length of zdeterminingequations = 73, the byte count of zdeterminingequations
  = 10560, the memory in use = 157570840, the time used = 10.569338000000002`

FindDeterminingEquations is done. The memory in use = 157572480, the time used = 10.569648`

FindDeterminingEquations has finished executing. You can look at the output in the
  table zdeterminingequations. Each entry in this table is a determining equation
  in string format expressed in terms of z-variables. Rules for converting between
  z-variables and conventional variables are contained in the table ztableofrules. To
  view the determining equations in terms of conventional variables use the command
  ToExpression[zdeterminingequations]/.ztableofrules. There are two other items the user
  may wish to look at; the equation converted to generic (x1,x2,...,y1,y2,...) variables is
  designated equationgenericvariables and the various derivatives of the equation that appear in
  the invariance condition can be viewed in the table invarconditiontable. Rules for converting
  between z-variables and generic variables are contained in the table ztableofrulesxy.

Out[14]= {0.277017, Null}

In[15]:= equationgenericvariables

Out[15]= D[y1[x1,x2],x2,x2,x2]+(1/ν)*y2[x1,x2]+(1/ν)*D[y1[x1,x2],x1]*D[y1[x1,x2],x2,x2]-(1/ν)*D
    [y1[x1,x2],x2]*D[y1[x1,x2],x1,x2]

In[16]:= invarconditiontable
```

Out[16]= $\left\{0, 0, 0, \frac{1}{\nu}, \frac{y1^{(0,2)}[x1, x2]}{\nu}, -\frac{y1^{(1,1)}[x1, x2]}{\nu}, 0, -\frac{y1^{(0,1)}[x1, x2]}{\nu}, \frac{y1^{(1,0)}[x1, x2]}{\nu}, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\}$

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Here are the determining equations in terms of z-variables. Set them equal to a new table called zdeterminingequations1. Remove the semicolon at the end if you want to look at the equations.

```
In[17]:= zdeterminingequations1 = zdeterminingequations;
```

Now work out the determining equations of the point group which leaves equation2 invariant.

```
In[18]:= Timing[FindDeterminingEquations[independentvariables, dependentvariables,
    frozennames, p, r, xseon, equation2, rulesarray, internalrules]]

The function FindDetermining Equations has
  begun, the memory in use = 158327776, the time used = 10.610982`

The function FindDeterminingEquations is nearly complete. The invariance condition has been
  created with all rules applied. The final step in the generation of the determining
  equations is to sum together terms in the table of invariance condition terms (called
  infinitesimaltable) that are multiplied by the same combination of products of free y
  derivatives. The result is the table infinitesimaltablesums corresponding to matching
  y-derivative expressions. If the invariance condition is long as it often is this process
  could take a long time since it requires sorting through the table infinitesimaltable
  once for each possible combination of y derivative products. This is the rate limiting
  step in the function FindDeterminingEquations.Virtually all other steps are quite
  fast including the generation of the extended derivatives of the infinitesimals.

The determining equations have been expressed in terms of
  z-variables, the length of zdeterminingequations = 27, the byte count of
  zdeterminingequations = 3056, the memory in use = 157574584, the time used = 10.722915`

FindDeterminingEquations is done. The memory in use = 157575656, the time used = 10.723146`

FindDeterminingEquations has finished executing. You can look at the output in the
  table zdeterminingequations. Each entry in this table is a determining equation
  in string format expressed in terms of z-variables. Rules for converting between
  z-variables and conventional variables are contained in the table ztableofrules. To
  view the determining equations in terms of conventional variables use the command
  ToExpression[zdeterminingequations]/.ztableofrules. There are two other items the user
  may wish to look at; the equation converted to generic (x1,x2,...,y1,y2,...) variables is
  designated equationgenericvariables and the various derivatives of the equation that appear
  in the invariance condition can be viewed in the table invarconditiontable. Rules for
  converting between z-variables and generic variables are contained in the table ztableofrulesxy.

Out[18]= {0.13223, Null}

In[19]:= equationgenericvariables

Out[19]= D[y2[x1,x2],x2,x2]-(1/κ)*D[y1[x1,x2],x2]*D[y2[x1,x2],x1]+(1/κ)*D[y1[x1,x2],x1]*D[y2[x1,x2
   ],x2]

In[20]:= invarconditiontable
```

Out[20]= $\left\{0, 0, 0, 0, \frac{y2^{(0,1)}[x1, x2]}{\kappa}, -\frac{y2^{(1,0)}[x1, x2]}{\kappa}, 0, 0, 0, 0, 0, 0, 0, -\frac{y1^{(0,1)}[x1, x2]}{\kappa}, \frac{y1^{(1,0)}[x1, x2]}{\kappa}, 0, 0, 1, 0, 0, 0, 0\right\}$

Here are the determining equations in terms of z-variables. Set them equal to a new table called zdeterminingequations2. Remove the semicolon at the end if you want to look at the equations.

```
In[21]:= zdeterminingequations2 = zdeterminingequations;
```

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Now concatenate these two sets of determining equations together to form the table zdeterminingequations3 that contains the determining equations for the entire equation set.

```
In[22]:= zdeterminingequations3=Join[
   zdeterminingequations1,zdeterminingequations2];
```

How many determining equations are we dealing with?

```
In[23]:= Length[zdeterminingequations3]

Out[23]= 100
```

Here is the correspondence between z-variables and conventional variables.

```
In[24]:= ztableofrules

Out[24]= {z1 → x, z2 → y, z3 → Ψ[x, y], z4 → θ[x, y]}
```

Now solve the determining equations in terms of multivariable polynomials of some selected order.

```
In[25]:= Timing[SolveDeterminingEquations[
    independentvariables, dependentvariables, r, xseon, zdeterminingequations3, order = 5]]

The variable powertablelength is the number of terms required for each multivariate
  polynomial used for the infinitesimals. This number is determined by the choice
  of polynomial order and the number of zvariables. The time needed to solve the
  determining equations increases as powertable increases. powertablelength = 126

The polynomial expansions have been substituted into the
  determining equations. It is now time to collect the coefficients of various
  powers of zvariables into a table called table of coefficientsall. This step
  uses the function CoefficientList and is a fairly time consuming procedure.

The memory in use = 163741896, The time = 11.243573999999999`

The number of unknown polynomial coefficients = 504

The number of equations for the polynomial coefficients = 4563

Now it we are ready to use the function Solve to find the nonzero polynomial coefficients
  corresponding to the symmetries of the input equation(s). This can take a while.

The memory in use = 167011496, The time = 11.356684999999999`

Solve has finished.

The function SolveDeterminingEquations is finished executing.

The memory in use = 168683560, The time = 13.102214`

You can look at the output in the tables xsefunctions and etafunctions. Each entry in these
  tables is an infinitesimal function in string format expressed in terms of z-variables
  and the group parameters. The output can also be viewed with the group parameters
  stripped away by looking at the table infinitesimalgroups. In either case you may wish
  to convert the z-variables to conventional variables using the table ztableofrules.

Keep in mind that this function only finds solutions of the determining equations
  that are of polynomial form. The determining equations may admit solutions that
  involve transcendental functions and/or integrals. Note that arbitrary functions
  may appear in the infinitesimals and that these can be detected by running the
  package function SolveDeterminingEquations for several polynomial orders. If
  terms of ever increasing order appear, then an arbitrary function is indicated.

Out[25]= {2.34685, Null}
```

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Here are the infinitesimal transformation functions for the independent variables.

```
In[26]:= xsefunctions
Out[26]= {xse1[z1_, z2_, z3_, z4_]=a10 + (a26 + b121)*z1,
  xse2[z1_, z2_, z3_, z4_]=a20 + a21*z1 +
    a22*z1^2 + a23*z1^3 + a24*z1^4 + a25*z1^5 + a26*z2}
```

and for the dependent variables.

```
In[27]:= etafunctions
Out[27]= {eta1[z1_, z2_, z3_, z4_]=b10 + b121*z3, eta2[z1_, z2_, z3_, z4_]=(-3*a26 + b121)*z4}
```

```
In[28]:= ToExpression[xsefunctions]/.ztableofrules
```

Out[28]= $\{a10 + (a26 + b121)\, x,\ a20 + a21\, x + a22\, x^2 + a23\, x^3 + a24\, x^4 + a25\, x^5 + a26\, y\}$

```
In[29]:= ToExpression[etafunctions]/.ztableofrules
Out[29]= {b10 + b121 Ψ[x, y], (-3 a26 + b121) θ[x, y]}
```

```
In[30]:= Column[infinitesimalgroups /.ztableofrules]
Out[30]=
{{1, 0}, {0, 0}}
{{0, 1}, {0, 0}}
{{0, x}, {0, 0}}
{{0, x^2}, {0, 0}}
{{0, x^3}, {0, 0}}
{{0, x^4}, {0, 0}}
{{0, x^5}, {0, 0}}
{{x, y}, {0, -3 θ[x, y]}}
{{0, 0}, {1, 0}}
{{x, 0}, {Ψ[x, y], θ[x, y]}}
```

Check the infinitesimals in the determining equations.

```
In[31]:= Simplify[ToExpression[zdeterminingequations3]]
Out[31]= {True, True, True, True, True, True, True, True, True, True, True, True, True, True, True,
  True, True, True, True, True, True, True, True, True, True, True, True, True, True, True,
  True, True, True, True, True, True, True, True, True, True, True, True, True, True,
  True, True, True, True, True, True, True, True, True, True, True, True, True, True,
  True, True, True, True, True, True, True, True, True, True, True, True, True, True,
  True, True, True, True, True, True, True, True, True, True, True, True, True, True,
  True, True, True, True, True, True, True, True, True, True, True, True, True, True}
```

Notice that the infinite dimensional translation group on the wall normal coordinate still comes out, ie, this symmetry is not lost when the temperature equation is coupled to the boundary layer equation. This invariance allows the solution on a flat plate to be mapped to an arbitrary shaped wall (within reason).

Now generate the commutator table of the Lie algebra. Convert z-variables to conventional variables. Note that in the conversion the functional dependence of the conventional variables is left out. The procedure that creates the commutator table requires that the table of groups be in this form.

```
In[32]:= infinitesimalgroupsxy1 = infinitesimalgroups /. {z1 → x, z2 → y, z3 → Ψ, z4 → θ};

In[33]:= infinitesimalgroupsxy =
   Drop[infinitesimalgroupsxy1, {3, 7}];

In[34]:= Column[infinitesimalgroupsxy]
Out[34]=
{{1, 0}, {0, 0}}
{{0, 1}, {0, 0}}
{{x, y}, {0, -3 θ}}
{{0, 0}, {1, 0}}
{{x, 0}, {Ψ, θ}}

In[35]:= MakeCommutatorTable[independentvariables, dependentvariables, infinitesimalgroupsxy]

MakeCommutatorTable has finished executing. You can look at the output
  in the table commutatortable. To present the output in the most readable
  form you may want view it as a matrix using MatrixForm[commutatortable].
  Occasionally the entries in the commutatortable will have terms that cancel.
  To get rid of these terms use the function Simplify before viewing the table.

In[36]:= MatrixForm[Simplify[commutatortable]]
```

ut[36]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$