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The Shape of Production Functions and the Direction of Technical Change

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Introduction

- Macro/growth literatures: strong assumptions on PF and direction of technical change. Justification?
- What is a production function? $y = f(k, t)$
  - Leontief example.
  - Switching from low $k$ to high $k$ may involve very different production techniques/ideas.
  - A production function is not a single technology, but rather represents the substitution possibilities across different techniques.
- The global shape of the production function is determined by the distribution of ideas.
Overview


- Results:
  1. A production function with
     - low EofS for any given technique
     - Cobb-Douglas global production function.
  2. A theory of LATC
     - Possibility of KATC in model, but
     - Economy “chooses” LATC only in LR.
     - cf Acemoglu (2003)
Outline

1. Baseline Model
2. Model w/ Microfoundations
3. Discussion: Role of Pareto
4. Embed in a growth model: LATC
5. Simulation Results
Baseline Model: Preliminaries

- Idea = \((a_i, b_i)\). Production with technique \(i\):
  \[ Y = \tilde{F}(b_i K, a_i L) \quad \leftarrow \text{the local production function} \]
  where \(\tilde{F}\) is a neoclassical PF with EofS < 1.

- Rewrite in per worker terms as
  \[ y = a_i \tilde{F}\left(\frac{b_i}{a_i}, 1\right), \]

- Define \(y_i = a_i\) and \(k_i = a_i / b_i\). Then
  \[ y = y_i \tilde{F}\left(\frac{k}{k_i}, 1\right) \]
  so that \(k = k_i \Rightarrow y = y_i\).
The Global Production Function
Simple Model

- Firm has a stock of knowledge, $N$, that generates a menu of ideas
  \[ H(a, b) = N, \quad H_a > 0, \quad H_b > 0. \]  
  (1)

- Associated with any idea $(a, b)$ is a local production technique, as above.

- The global production function gives the highest output that can be produced using this menu:
  \[ Y = F(K, L; N) \equiv \max_{b, a} \tilde{F}(bK, aL) \]

subject to the technology menu constraint in (1).
Fig. 2: Direction of Technical Change

\[ H(a, b) = N \]

\[ Y = Y^* \]
First-order condition:

\[
\frac{\theta_K}{\theta_L} = \frac{\eta b}{\eta a},
\]

where \(\theta_K(a, b; K, L) \equiv \tilde{F}_1 bK/Y\), \(\theta_L = 1 - \theta_K\), \(\eta_x \equiv \frac{\partial H}{\partial x} \frac{x}{H}\).

Key special case: Constant elasticity menu

\[
H(a, b) \equiv a^\alpha b^\beta = N.
\]

\[\Rightarrow \theta_K = \beta/\alpha + \beta.\]

i.e. Capital share is constant for any \(K, L\), and \(N\).

This leads to two results.
Result 1. Cobb-Douglas

- The capital share is constant for any $K, L, N$

  $\Rightarrow$ The global production function is Cobb-Douglas.

- Derive exact form:

  $$ y_i = a_i $$

  $$ k_i = \frac{a_i}{b_i} $$

  Technology menu then implies:

  $$ y_i = (N k_i^\beta)^{\frac{1}{\alpha+\beta}}. $$

- The global production function equals this menu:

  $$ Y = \left(NK^\beta L^\alpha\right)^{\frac{1}{\alpha+\beta}}. $$
Result 2. LATC

- Embed this production setup in a standard neoclassical growth model
- Global Cobb-Douglas implies BGP exists if \( N \) grows exponentially.
- Steady-State Growth Theorem: In a steady state, either
  - Production is Cobb-Douglas, or
  - Technical change is labor augmenting.
- Production always occurs with some local PF, and the local is not Cobb-Douglas. Therefore LATC.
Proving LATC

- Rewrite the FOC as

\[
\frac{bK \tilde{F}_1(bK, aL)}{aL \tilde{F}_2(bK, aL)} = \frac{\beta}{\alpha}.
\]

- Define \( x \equiv bK/aL \). \( \tilde{F} \) CRS \( \Rightarrow \) the marginal products are HD0:

\[
\frac{x \tilde{F}_1(x, 1)}{\tilde{F}_2(x, 1)} = \frac{\beta}{\alpha}.
\]

\( \Rightarrow x \) must be constant.
Proof (continued)

- To show: $x$ constant requires $b$ constant in SS. Recall

$$Y_t = F(K_t, L_t; N_t) = \tilde{F}(b_t K_t, a_t L_t),$$

where $b_t$ and $a_t$ are the optimal choices of the technology levels.

- Because $\tilde{F}$ exhibits constant returns, we have

$$\frac{Y_t}{a_t L_t} = \tilde{F} \left( \frac{b_t K_t}{a_t L_t}, 1 \right).$$

- $x = bK/aL$ constant $\Rightarrow$ $Y/aL$ constant $\Rightarrow$ $bK/Y$ constant.

- $K/Y$ is constant in SS $\Rightarrow$ $b$ constant. QED.
Intuition

Because local PF is not Cobb-Douglas, balanced growth requires \( bK \) and \( aL \) to grow at the same rate.

- \( Y = \tilde{F}(bK, aL) \) suggests new interpretation of “balanced”
- \( bK \) and \( aL \) must balance to keep factor shares stable.

Can only happen with \( b \) constant.
- Recall, \( b \) constant means \( K/aL \) constant.
- If \( b \) grew, so would \( bK/aL \).
Clarifying the Result

- Well-known that with Cobb-Douglas production, the direction of technical change has no meaning.
- So how can we have both?
- Recall:

\[ Y_t = F(K_t, L_t; N_t) = \tilde{F}(b_t K_t, a_t L_t). \]

\text{global pf} \quad \text{local pf}

- Global production function \( F(K, L; N) \) is Cobb-Douglas. Local production function \( \tilde{F}(bK, aL) \) has LATC.
Discussion

- Acemoglu (2003) has related results in a Romer-type model:
  - LATC if production function for ideas is “just so”
  - Capital share in LR is invariant to policy
Model with Microfoundations

Assume the local production function is Leontief:

\[ Y = \hat{F}(b_i K, a_i L) = \min\{b_i K, a_i L\} \]

Ideas drawn from independent Pareto distributions:

\[ \text{Prob} [a_i \leq a] = 1 - \left( \frac{a}{\gamma_a} \right)^{-\alpha}, \quad a \geq \gamma_a > 0 \]

\[ \text{Prob} [b_i \leq b] = 1 - \left( \frac{b}{\gamma_b} \right)^{-\beta}, \quad b \geq \gamma_b > 0. \]

Then, \( G(b, a) \equiv \text{Prob} [b_i > b, a_i > a] = \left( \frac{b}{\gamma_b} \right)^{-\beta} \left( \frac{a}{\gamma_a} \right)^{-\alpha} \)
Distribution of Output from Idea \( i \)

Let \( Y_i(K, L) \) denote output with idea \( i \). Since \( \tilde{F} \) is Leontief, the distribution of \( Y_i \) is

\[
H(\tilde{y}) \equiv \text{Prob}[Y_i > \tilde{y}] = \text{Prob}[b_i K > \tilde{y}, a_i L > \tilde{y}]
\]

\[
= G \left( \frac{\tilde{y}}{K}, \frac{\tilde{y}}{L} \right)
\]

\[
= \gamma K^\beta L^\alpha \tilde{y}^{-(\alpha+\beta)},
\]

where \( \gamma \equiv \gamma_a^\alpha \gamma_b^\beta \).

That is, the distribution of \( Y_i \) is also Pareto.
The Global Production Function

- Assume only one technique can be used at a time.
- Let $N$ denote the number of ideas, drawn independently.
- The global production function $F(K, L; N)$ is given as

$$F(K, L; N) \equiv \max_{i \in \{1, \ldots, N\}} \tilde{F}(b_iK, a_iL).$$

- Let $Y = F(K, L; N)$. Then

$$\text{Prob}[Y \leq \tilde{y}] = (1 - H(\tilde{y}))^N.$$

$$= \left(1 - \gamma K^{\beta} L^{\alpha} \tilde{y}^{-(\alpha + \beta)} \right)^N.$$
(continued)

\[
\text{Prob} [Y \leq \tilde{y}] = \left( 1 - \gamma K^\beta L^\alpha \tilde{y}^{-(\alpha+\beta)} \right)^N.
\]

- As \( N \) gets large, this probability goes to zero. 
  \[ \Rightarrow \text{normalize to get a stable distribution} \]

\[ z_N \equiv \left( \gamma N K^\beta L^\alpha \right)^{\frac{1}{\alpha+\beta}}. \]

- Then,

\[
\text{Prob} [Y \leq z_N \tilde{y}] = \left( 1 - \gamma K^\beta L^\alpha \left( z_N \tilde{y} \right)^{-(\alpha+\beta)} \right)^N
\]

\[ = \left( 1 - \frac{\tilde{y}^{-(\alpha+\beta)}}{N} \right)^N. \]
The Cobb-Douglas Result

- Now, let $N$ get large

$$\lim_{N \to \infty} \text{Prob}[Y \leq z_N \tilde{y}] = \exp(-\tilde{y}^{-(\alpha+\beta)})$$

- Or,

$$Y \quad \frac{Y}{(\gamma N K^\beta L^\alpha)^{1/\alpha+\beta}} \overset{a}{\sim} \text{Fréchet}(\alpha + \beta).$$

- And therefore, for large $N$,

$$Y \approx \left(\gamma N K^\beta L^\alpha\right)^{\frac{1}{\alpha+\beta}} \epsilon$$
Remarks

\[ Y \approx \left( \gamma N K^\beta L^\alpha \right)^{\frac{1}{\alpha+\beta}} \epsilon \]

1. Appendix: Poisson process for discovery of ideas yields the result for finite \( N \).

2. Cobb-Douglas exponent depends on parameters of search distributions
   - Easier to find ideas → lower exponent.
   - Intuition: EofS < 1.

3. \( \epsilon \) is an iid shock drawn from a Fréchet distribution.

4. Higher \( N \) implies Higher \( Y \).

5. Obviously Pareto assumption is crucial to result. More on this shortly.
Discussion: 1. Baseline Model

- Baseline model: constant elasticity in technology menu.
- Here, stochastic version. Consider iso-probability curve:

\[ \text{Prob} \left[ b_i > b, a_i > a \right] \equiv G(b, a) = C. \]

With Pareto,

\[ b^\beta a^\alpha = \frac{\gamma}{C}. \]

- Stochastic version of the baseline technology menu.
  - Pareto delivers \( \eta_b = \beta \) and \( \eta_a = \alpha \)
  - \( 1/C \) plays the role of \( N \)
  - Get the same form for the production function.
2. Comparison to Houthakker (1955)

- Pareto+Leontief = Cobb-Douglas is Houthakker
- Houthakker’s result is an aggregation result
  - Continuum of firms with capacity constraints.
  - Firm PF: Leontief, with requirements ~ Pareto.
  - Aggregate PF: Cobb-Douglas with DRS
- Result here:
  - Result applies for a firm/industry/country
  - Applies to global production function, i.e. across techniques.
  - No restriction to Leontief for SR PF (technique)
  - Nonrivalry of ideas ⇒ CRS
3. Evidence for Pareto Distributions

- Key property: $\text{Prob} \left[ X \geq \gamma x \mid X \geq x \right]$ for $\gamma > 1$ is independent of $x$.

- Empirical evidence for incomes, patent values, profitability, citations, firm size, stock returns.
  - Benchmark in literature is to test Pareto
  - Findings: Pareto (sometimes hard to distinguish from Lognormal)

- Kortum (1997):
  - Assume a production function and draw $a_i$ only
  - Iff ideas are from a Pareto distribution, then we get exponential growth
Why is Pareto so important?

- Steady-state growth requires probability the new best idea exceeds frontier by 5% is invariant to $y$.
- Gabaix (1999) shows the reverse. Exponential growth delivers a Pareto distribution for city sizes (Zipf).

This suggests that Pareto Distributions and exponential growth are two sides of the same coin.

- What I add is that this same basic assumption delivers two additional results:
  1. Cobb-Douglas production
  2. Labor-augmenting technical change (next).
Embed this setup in a neoclassical growth model

\[ Y_t = \left( \gamma N_t K_t^\beta L_t^\alpha \right)^{\frac{1}{\alpha + \beta}} \epsilon_t. \]

\[ K_{t+1} = (1 - \delta) K_t + sY_t \]

\[ N_t = N_0 e^{gt} \]

Therefore, steady-state growth in \( Y/L \):

\[ E[\log \frac{y_{t+1}}{y_t}] \approx \frac{g}{\alpha}. \]

Note: depends on \( \alpha \) but not \( \beta \).
Model exhibits a stable balanced growth path, because of global Cobb-Douglas production.

However, production at date \( t \) occurs with some technique \( i(t) \):

\[
Y_t = \tilde{F}(b_i(t) K_t, a_i(t) L_t).
\]

Now use Steady-State Growth Theorem:
- The production function for a technique is not Cobb-Douglas,
- so Steady State implies that \( b_i(t) \) is stationary!

That is, technical change in this model is (asymptotically) labor-augmenting.
- This is true even though \( \max_i b_i \to \infty \).
Simulating the Model

- Relax Leontief and allow multiple techniques
- CES production technique:

\[ Y_t = \tilde{F}(b_i K_t, a_i L_t) = (\lambda (b_i K_t)^\rho + (1 - \lambda)(a_i L_t)^\rho)^{1/\rho} \]

- First, show Cobb-Douglas.
  - \( N = 500, \alpha = 5, \beta = 2.5, \rho = -1. \)
    \[ \Rightarrow \frac{\beta}{\alpha + \beta} = \frac{1}{3}. \]
  - Compute convex hull and sample a \((k, y)\) point randomly.
  - Repeat 1000 times and plot the sample.
Fig. 3: The Cobb-Douglas Result

\[ \log y \]

![Graph showing log y vs. log k]

- OLS Slope = 0.320
- Std. Err. = 0.006
- \( R^2 \) = 0.75

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Full Dynamic Simulation

Parameter Values: $N_0 = 50$, $g = .10$, $\alpha = 5$, $\beta = 2.5$, $\gamma_a = 1$, $\gamma_b = 0.2$, $k_0 = 2.5$, $s = 0.2$, $\lambda = 1/3$, $\delta = .05$, and $\rho = -1$.

- Growth should average 2 percent
- Cobb-Douglas capital share 1/3
Production Functions and Technical Change

Fig. 4: Production

Output per Worker, $y$

Capital per Worker, $k$
Fig. 5: Output per Worker

Output per Worker (log scale)

Time

Production Functions and Technical Change
Fig. 6: The Capital Share over Time
Fig. 7: Technology Choices

Capital–Aug. Technology, $b$

Labor–Augmenting Technology, $a$
Conclusions

- Houthakker + Kortum =
  - Exponential growth
  - Cobb-Douglas (global) production function
  - Labor-augmenting technical change.

The Pareto distribution buys us a lot!

Extensions and future work:
- Skilled versus unskilled labor?
- What about computers and ISTC? GHK, Whelan, etc.?