The End of Economic Growth? Unintended Consequences of a Declining Population

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Abstract

In many models, economic growth is driven by people discovering new ideas. These models typically assume either a constant or growing population. However, in high income countries today, fertility is already below its replacement rate: women are having fewer than two children on average. It is a distinct possibility that global population will decline rather than stabilize in the long run. In standard models, this has profound implications: rather than continued exponential growth, living standards stagnate for a population that vanishes. Moreover, even the optimal allocation can get trapped in this outcome if there are delays in implementing optimal policy.

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1. Introduction

In many growth models based on the discovery of new ideas, the size of the population plays a crucial role. Other things equal, a larger population means more researchers which in turn leads to more new ideas and to higher living standards. This basic feature is shared by the original endogenous growth models of Romer (1990), Aghion and Howitt (1992), and Grossman and Helpman (1991) as well as by the semi-endogenous growth models of Jones (1995), Kortum (1997), and Segerstrom (1998) in which standard policies have level effects instead of growth effects. It is a feature of numerous other models.[1]

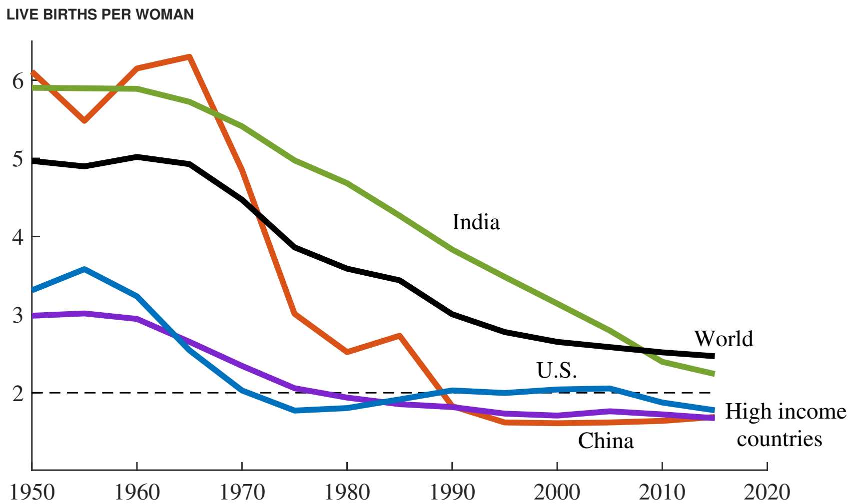
In a recent book entitled *Empty Planet*, Bricker and Ibbitson (2019) make the case based on a rich body of demographic research that global population growth in the future may not only fall to zero but may actually turn negative. For example, the natural rate of population growth (i.e. births minus deaths, ignoring immigation) is already negative in Japan and in many European countries such as Germany, Italy, and Spain (United Nations, 2019).

Figure 1 shows historical data on the total fertility rate for various regions. This measure is the average number of live births a cohort of women would have over their reproductive life if they were subject to the fertility rates of a given five-year period. To sustain a constant population requires a total fertility rate slightly greater than 2 in order to compensate for mortality. The graph shows that high income countries as a whole, as well as the U.S. and China individually, have been substantially below 2 in recent years. According to the U.N.'s *World Population Prospects 2019*, the total fertility rate in the most recent data is 1.8 for the United States, 1.7 for China and for High Income Countries on average, 1.6 for Germany, 1.4 for Japan, and 1.3 for Italy and Spain. In other words, fertility rates in the rich countries of the world are already consistent with negative long-run population growth: women are having fewer than two children throughout much of the developed world.

A sharp downward trend in India and for the world as a whole is also evident in the figure. As countries get richer, fertility rates appear to decline to levels consistent, not with a constant population, but actually with a declining population.

[1] Examples include Kremer (1993), Acemoglu (1998), Ngai and Samaniego (2011), Doepke and Zilibotti (2014), Acemoglu and Restrepo (2018), Akcigit and Kerr (2018), Atkeson and Burstein (2019), and Buera and Oberfield (2020).

Figure 1: The Total Fertility Rate (Live Births per Woman)



Note: The total fertility rate is the average number of live births a hypothetical cohort of women would have over their reproductive life if they were subject during their whole lives to the fertility rates of a given period and if they were not subject to mortality. Each data point corresponds to a five-year period. Source: United Nations (2019).

Conventional wisdom holds that in the future, global population will stabilize at something like 8 or 10 billion people. But maybe this is not correct. The fact that so many rich countries already have fertility below replacement indicates that a future with negative population growth is a possibility that deserves further consideration. Because ideas diffuse across countries, the relevant population for each country's living standards in the long run is the total population of the idea-creating countries, which perhaps encompasses the global population eventually. Either way, the evidence just discussed suggests that declines may be relevant.

The models of economic growth cited above assume a constant or growing population, and for understanding economic growth historically, that is clearly appropriate. The demographic evidence, however, suggests that this may not be the case in the future. Hence the focus of this paper: what happens to economic growth if population growth is negative?

We show below — first in models with exogenous population growth and then later in a model with endogenous fertility — that negative population growth can be particularly harmful. When population growth is negative, both endogenous and semi-endogenous growth models produce what we call the Empty Planet result: knowledge

and living standards stagnate for a population that gradually vanishes. In a model with endogenous fertility, a surprising result emerges: even the social planner can get stuck in this trap if society delays implementing the optimal allocation and suffers from inefficient negative population growth for a sufficiently long period. In contrast, if the economy switches to the optimal allocation soon enough, it can converge to a balanced growth path with sustained exponential growth: an ever-increasing population benefits from ever-rising living standards. Policies related to fertility may therefore determine whether we converge to an "Empty Planet" or to an "Expanding Cosmos"; they may be much more important than we have appreciated.

Literature Review. Many models feature endogenous fertility, modeled in a variety of ways. Becker and Barro (1988) and Barro and Becker (1989) take an altruistic approach in which the utility of children enters the utility function for parents, giving rise to a dynastic utility function. Papers that follow this approach include Doepke (2004) and Manuelli and Seshadri (2009). Other papers emphasize a "warm-glow" effect in which parents care about the number of their offspring; for example, see De La Croix and Doepke (2003), Hock and Weil (2012), and Doepke and Tertilt (2016). Finally, many papers feature a quantity-quality tradeoff and assign a key role to education, often in the context of explaining the demographic transition and the emergence of modern economic growth. These include Becker, Murphy and Tamura (1990), Galor and Weil (1996, 2000), Greenwood and Seshadri (2002), Kalemli-Ozcan (2002), Cervellati and Sunde (2015), and de Silva and Tenreyro (2020).

On the empirical side, Jones and Tertilt (2008) provide a detailed account of the decline in U.S. fertility using Census data, while Delventhal, Fernández-Villaverde and Guner (2021) study the demographic transition using data from 186 countries and 250 years. Chatterjee and Vogl (2018) use extensive microdata from 255 household surveys to study how fertility declines with economic growth. Song, Storesletten, Wang and Zilibotti (2015) note that urbanization can continue even if the population declines; if people in urban areas disproportionately invent ideas, urbanization could delay the onset of the Empty Planet result. Feyrer, Sacerdote and Stern (2008) highlight negative population growth in Japan and parts of Europe and raise the possibility that it could revert back to being positive as the status of women in society improves. Young (2005)

quantifies the neoclassical gain from higher capital-labor ratios that occurs when populations decline, in this case due to the shock from HIV and AIDS. Doepke and Tertilt (2016) and Greenwood, Guner and Vandenbroucke (2017) provide surveys of family macroeconomics, including fertility. This literature sometimes recognizes the possibility that population growth could ultimately be negative, but that is not its emphasis.

More generally, demographic forces are garnering broader attention in the macroeconomic literature. Several recent papers suggest that falling labor force growth may explain a substantial part of the decline in firm entry and dynamism in the U.S. economy, including Karahan, Pugsley and Sahin (2019), Hopenhayn, Neira and Singhania (2018), Engbom (2019), and Peters and Walsh (2021).

Galor and Moav (2002) suggest that evolutionary forces can play a key role in economic growth. In this context, one wonders if those forces might eventually favor groups with higher fertility, either for accidental genetic reasons or for cultural reasons. Berman (2000) suggests that the ultra-orthodox community in Israel might be one such group, with fertility rates above 7 in recent decades, which led a disproportionately larger fraction of school-age children to be from these communities. This is surely an important consideration to take into account in a broader study of fertility and growth. The point here is more narrow, namely to highlight some of the implications of negative population growth, were it to occur.

The literature that explicitly considers negative population growth in a growth context is much smaller. Manuelli and Seshadri (2009) explain the heterogeneity in international fertility rates by emphasizing that taxes and transfers in Europe may in part be responsible for low fertility. Sasaki and Hoshida (2017) study negative population growth in a semi-endogenous growth setting. They show that the rate of technological change falls to zero as people endogenously exit the research sector. More surprisingly, they provide a setting where negative population growth leads to *positive* steady-state growth in income per person because capital per person rises as the number of people declines. However, this result is incomplete in that they assume a zero depreciation rate for capital: if there is a fixed amount of capital but the population declines, then capital per person grows. One can easily generalize their result to positive depreciation rates using a Solow model. If the rate of population decline is η and capital depreciates at rate δ, then there are two possible regimes. If $\eta > \delta$, i.e. the rate of population

decline is faster than the depreciation rate of capital, then K/L rises asympotically along a balanced growth path. But when $\eta < \delta$, instead, you get the standard Solow result of constant K/L in steady state. Empirically, rates of population decline are perhaps 1% or smaller, whereas depreciation rates are 3% or 5% or more. The Sasaki and Hoshida (2017) case of exponential growth in capital per person from declining population therefore seems implausible as an empirical matter. Christiaans (2011) has results along these lines in a model with increasing returns that results from externalities to capital, showing the two possible regimes.

This motivates Sasaki (2019a) to consider a model with non-renewable resources, where a zero depreciation rate is more natural. In that case, though, one might wonder about elasticities of substitution: if a single Robinson Crusoe populated an earth full of land and natural resources, would her income be extremely high? Sasaki (2019b) considers a Solow model with CES production and finds that with an elasticity of substitution less than unity, the long-run growth rate is determined only by the rate of technological progress, with no contribution from the rising capital-labor ratio that results from negative population growth. Because capital is not essential, even an infinite capital-labor ratio gives finite output. These results suggests that capital and non-renewable resources can be omitted from the model without much loss in generality, which is what we do below.

Finally, related results can also be found in other idea-driven growth models. Kremer (1993) emphasizes the broad historical evidence linking population and rising living standards. Interestingly, he notes that the technological stock in Tasmania declined over thousands of years, while the small population of Flinders Island completely died off several thousand years after occupying the island. Kremer interprets these episodes as possibly indicating a role for the depreciation of knowledge but says they are likely "of limited importance when looking at the world as a whole" (Footnote 21). Romer (1990) notes that exponential growth requires a sufficiently large population. If the population is too small, the "market size" effect is too weak and the incentives for research disappear. This failure would not occur in Kremer (1993) or in the setup below, where all people create ideas (perhaps with some low probability) — these models feature "learning by working" instead of "learning or working." In that case, a constant population always produces a positive number of new ideas and living standards

rise without bound, albeit at a rate that slows over time. Thus, there is an important difference between low population and negative population growth.

2. The Empty Planet Result

How do idea-based growth models behave when the population declines? We begin by introducing exogenous, negative population growth into a simplified version of the Romer (1990), Aghion and Howitt (1992), and Grossman and Helpman (1991) endogenous growth models. This case turns out to be especially easy to analyze. Then we consider semi-endogenous growth models.

2.1 Fully Endogenous Growth as in Romer/AH/GH

Consider the following simplified version of idea-driven endogenous growth models:

$$Y_t = A_t^{\sigma} N_t \tag{1}$$

$$\frac{\dot{A}_t}{A_t} = \alpha N_t \tag{2}$$

$$N_t = N_0 e^{-\eta t}, \quad \eta > 0 \tag{3}$$

According to equation (1), a single consumption-output good is produced using people N_t and the stock of ideas ("knowledge") A_t. Crucially, as in Romer (1990), there is constant returns in this production function to rival inputs — here just people — and therefore increasing returns to people and ideas together. The degree of increasing returns is parameterized by σ.

Equation (2) is the endogenous growth equation. It says that the growth rate of knowledge is proportional to the population. The literature often distinguishes between researchers and workers who produce the consumption good, but not always. Here, we make the simplifying assumption that is closer in spirit to learning by doing: people can work to make consumption goods and get new ideas at the same time.

Let's pause for a moment to recall the standard result from endogenous growth models. That is, ignore equation (3) and its negative population growth and instead assume that the population is constant at some value $\bar{N}$. In that case, equation (2) implies that the stock of ideas grows at a constant rate, $\alpha\bar{N}$, and equation (1) translates this

into growth in income per person. A constant population delivers constant exponential growth in living standards forever.

Now let's see what happens when population growth is negative. Equation (3) specifies that the population declines exogenously at the rate η. For example, $\eta = .005$ corresponds to a population that declines exponentially at a half a percent per year. We write the model here and throughout the paper so that all parameter values (Greek letters) are positive.

Combining (2) and (3) gives the following differential equation, in which the growth rate of knowledge declines exponentially:

$$\frac{\dot{A}_t}{A_t} = \alpha N_0 e^{-\eta t}.$$

This differential equation is easy to solve, yielding the following result (derived in Appendix A.1):

Result 1 (Romer/AH/GH with Negative Population Growth): *In the Romer/AH/GH model with negative population growth, the stock of knowledge A_t is given by*

$$\log A_t = \log A_0 + \frac{g_{A0}}{\eta}\left(1 - e^{-\eta t}\right)$$

Both A_t and income per person $y_t \equiv Y_t/N_t$ converge to constant values A^* and y^* as t goes to infinity, where

$$A^* = A_0 \exp\left(\frac{g_{A0}}{\eta}\right)$$

$$y^* = y_0 \exp\left(\frac{g_{y0}}{\eta}\right)$$

where g_{xt} denotes the exponential growth rate of some variable x at date t, and variables indexed by 0 denote initial values.

We refer to this as the Empty Planet result. Economic growth stagnates as the stock of knowledge and living standards asymptote to constant values. Meanwhile, the population itself falls at a constant rate, gradually emptying the planet of people. This outcome stands in stark contrast to the conventional result in growth models in which knowledge, living standards, and even population grow exponentially: not only do we get richer over time, but these higher living standards apply to an ever rising number of

people.

The last equation in Result 1 is amenable to calibration. For example, if $g_{y0} = g_{A0} = 1\%$ and $\eta = 1\%$, so that the population is falling at 1% per year, the long-run level of GDP per person will be $e^1 \approx 2.7$ times higher than current income. Slower declines in population would make this factor even higher.

In what follows, we explore the robustness of this finding. First, we see that it occurs in semi-endogenous growth models as well, and then we consider what happens when the population growth rate itself is an endogenous outcome.

2.2 Semi-Endogenous Growth with Declining Population

With positive rates of population growth, semi-endogenous growth models in the tradition of Jones (1995), Kortum (1997), and Segerstrom (1998) give very different results from the fully endogenous growth models. We see next that with negative population growth, the results are instead quite similar, and the Empty Planet result still emerges.

A simplified semi-endogenous growth model is obtained by changing the idea production function:

$$Y_t = A_t^\sigma N_t \tag{4}$$

$$\frac{\dot{A}_t}{A_t} = \alpha N_t^\lambda A_t^{-\beta} \tag{5}$$

$$N_t = N_0 e^{-\eta t}. \quad \eta > 0 \tag{6}$$

Specifically, we introduce the parameter $\beta > 0$, capturing the extent to which new ideas (proportional improvements in productivity) are getting harder to find.[2]

Once again, let's first remind ourselves of the standard result from semi-endogenous growth models. In particular, if we ignore the third equation and instead assume a constant *positive* rate of population growth, we have the standard semi-endogenous growth setup. The only way the left-hand side of equation (5) can be constant is if the right-hand side is constant, which requires N_t^λ / A_t^β to be constant. But this requires the growth rate of A_t to be proportional to the positive population growth rate. And that is the essence of semi-endogenous growth: the nonrivalry of ideas leads to increasing

[2] An alternative in the literature is to write the idea production function as $\dot{A}_t = \alpha N_t^\lambda A_t^\phi$ with $\phi < 1$. These are equivalent, with $\beta = 1 - \phi$.

returns, and the growth rate of the economy is the product of the degree of increasing returns to scale and the rate at which scale is growing — the population growth rate. This result reappears later in the paper so we postpone further discussion until that time and move on to the main point of this section.

Now assume population growth is negative instead of positive. Combining (5) and (6) gives the following differential equation:

$$\frac{\dot{A}_t}{A_t} = \alpha N_0^\lambda e^{-\lambda\eta t} A_t^{-\beta}.$$

Integrating this differential equation gives the next result (derived in Appendix A.2):

Result 2 (Semi-Endogenous Growth with Negative Population Growth): *In the semi-endogenous growth model with negative population growth, the stock of knowledge A_t is given by*

$$A_t = A_0 \left(1 + \frac{\beta g_{A0}}{\lambda\eta}\left(1 - e^{-\lambda\eta t}\right)\right)^{1/\beta}.$$

Defining $\gamma \equiv \lambda\sigma/\beta$ to capture the overall degree of increasing returns to scale in this economy, both A_t and income per person $y_t \equiv Y_t/N_t$ converge to constant values A^* and y^* as t goes to infinity, where

$$A^* = A_0 \left(1 + \frac{\beta g_{A0}}{\lambda\eta}\right)^{1/\beta}.$$

$$y^* = y_0 \left(1 + \frac{g_{y0}}{\gamma\eta}\right)^{\gamma/\lambda}. \tag{7}$$

Along the transition path, the growth rate satisfies

$$\frac{\dot{y}_t}{y_t} = g_{y0} \cdot \left(\frac{y_t}{y_0}\right)^{-\frac{\lambda}{\gamma}} e^{-\lambda\eta t} = \frac{g_{y0} e^{-\lambda\eta t}}{1 + \frac{g_{y0}}{\gamma\eta}\left(1 - e^{-\lambda\eta t}\right)}$$

In other words, the growth rate falls to zero slightly faster than $e^{-\lambda\eta t}$.

This result confirms that both endogenous growth and semi-endogenous growth lead to the Empty Planet outcome. Rather than sustained exponential growth in living standards and population, living standards stabilize for a vanishing number of people.

Quantitatively, however, the level at which lower living standards stagnate can be

much lower with semi-endogenous growth. To illustrate, we need to calibrate one additional parameter relative to what we had before. Across a range of different case studies, Bloom, Jones, Van Reenen and Webb (2020) find estimates of $\beta \approx 3$ when $\sigma = 1$ (a normalization when we do not observe ideas directly) and $\lambda = 1$. Alternatively, for $\lambda = 3/4$, they find $\beta \approx 2$. Both sets of parameters lead to $\gamma \approx 1/3$. Plugging these values into equation (7), along with an initial TFP growth rate of 1% and $\eta = 1\%$ as before, the long-run level of GDP per person would be around 60-90% higher than current income. In the endogenous growth case, the gain is two to three times larger: with $\beta = 0$ (so that $\gamma = \infty$), long run income is 170% higher than current income for the same parameter values.

We can also say something about how long it takes to reach the steady state. In particular, the amount of time it takes for $A(t)$ to rise half-way to its steady state value can be computed easily. For the parameter values just considered, the half lives range from 85 to 133 years for the semi-endogenous growth model and around 250 years for the Romer case.[3]

3. Endogenous Fertility and the Equilibrium Allocation

We now endogenize the population growth rate itself, with an eye toward answering two questions. First, can the equilibrium of an endogenous fertility model feature negative population growth in steady state? Second, how does the optimal allocation behave for such a model?

There are many related ways to endogenize fertility, and the literature has not converged on a single best practice; see the literature review at the start of this paper for references. Almost all approaches assume that having offspring is a time intensive activity, and this is at the center of the approach we take below.

In models of endogenous fertility, population growth in a decentralized equilibrium can be equal to, above, or below the optimal rate. In fact, because the number of people is endogenous, the definition of "optimal" is itself unclear: how should the welfare of people who exist in some allocations but not others be taken into account (Golosov, Jones and Tertilt, 2007; De la Croix and Doepke, 2021)? Here, we focus on a common

[3] Appendix A.2 contains the details.

welfare criterion considered in the literature, the welfare of a dynastic family. In the equilibrium we consider, parents do not fully internalize the fact that their offspring create nonrival ideas that benefit the entire economy, so equilibrium fertility is too low.

But there are also other possible nuances. For example, Farhi and Werning (2007) note that the social planner may care about future generations both because individuals care about their own children and because the social planner puts weight on each generation. This means that social welfare will generally put more weight on future generations than individuals do, also leading optimal fertility to be higher than equilibrium fertility. Externalities to human capital in models with a quality-quantity tradeoff can also give rise to inefficient fertility decisions. Alternatively, one can construct idea-based models in which optimal fertility is below equilibrium fertility; see Jones (2003) and Futagami and Hori (2010) for some discussion. Here, we do not attempt to draw any firm conclusion about the range of possible externalities that may exist. Instead, we focus on some general lessons that emerge when the equilibrium features negative population growth while the optimal allocation has positive population growth.

To simplify, we abstract from the demographic transition. That is, we are not focused on how fertility fell from 5 to 3 to 1.8 children per woman. Instead, the focus is on the stable fertility rate at the end of the demographic transition and what happens if it implies negative population growth.

3.1 Environment

The economic environment for the setup with endogenous fertility is in Table 1. It builds on our earlier model, with one enhancement. There is now a single allocative decision that has to be made at each date: each person is endowed with one unit of time that can be used to produce either consumption or offspring. Devoting ℓ_t units of time to producing children leads to a fertility rate of $b(\ell_t) = \bar{b}\ell_t$. The linear function is convenient analytically but not essential. There is a constant death rate, δ, and the population growth rate is $n_t = \bar{b}\ell_t - \delta$. Thus if ℓ_t is sufficiently small, the population growth rate can be negative.

This setup excludes many other considerations that would be interesting to explore in the future such as human capital, physical capital, and a quantity-quality tradeoff. We instead focus on the simplest model that allows us to highlight some important (and

Table 1: Economic Environment: Endogenous Fertility Model

Final output	$Y_t = A_t^\sigma (1-\ell_t) N_t$
Population growth	$\frac{\dot N_t}{N_t} = n_t = b(\ell_t) - \delta$
Fertility	$b(\ell_t) = \bar b \ell_t$
Ideas	$\frac{\dot A_t}{A_t} = N_t^\lambda A_t^{-\beta}$
Generation 0 utility	$U_0 = \int_0^\infty e^{-\rho t} u(c_t, N_t) dt,\ \ N_0 = 1,\ \rho \equiv \bar\rho + \delta$
Flow utility	$u(c_t, N_t) = \frac{(N_t^\varepsilon c_t)^{1-\theta}}{1-\theta}$
Consumption	$c_t = Y_t / N_t$

general) economic points.[4]

People obtain utility from consumption and from having descendants. The expected lifetime utility of a member of the generation born at date 0 is

$$U_0 = \int_0^\infty e^{-(\bar\rho+\delta)t} u(c_t, N_t) dt$$

where $\bar\rho$ is the pure rate of time preference, c_t is consumption, and $N_0 = 1$ so that N_t is the number of descendants of generation 0 at date t. Discounting also occurs because of the death rate, and we define $\rho \equiv \bar\rho + \delta$ as the overall discount rate.

Flow utility takes the form considered in the dynastic utility frameworks of Barro and Becker (1989) and Jones and Schoonbroodt (2010), which we will refer to as "gen-

[4] One question that comes up often is whether or not growth in the "quality" of people can make up for the lack of "quantity" of people. It is possible, but only in a knife-edge case that seems counterfactual. Let h_t denote human capital per person, and suppose the input into producing ideas is $h_t N_t$. Assume human capital evolves according to $\dot h_t = \alpha \ell_h h_t^\psi - \delta h_t$ where ℓ_h is time spent accumulating human capital. Clearly if $\psi < 1$, then h converges to a constant and growth in quality cannot make up for growth in quantity. On the other hand, if $\psi > 1$, quality growth explodes and you get faster than exponential growth. Only in the specific case of Lucas (1988) with $\psi = 1$ will quality grow exponentially. For that knife-edge case, you have to believe something like the following: can each generation spending a constant 16 years in school generate exponential growth in quality forever? If each generation starts with the same initial "de novo" amount of human capital, h_0, then this would require rising returns to education and/or experience across generations, which have not been documented in the micro data. It seems more natural to imagine that if knowledge is constant, 16 years of schooling allows each generation to achieve a constant high level of human capital, consistent with $\psi < 1$.

eralized Barro-Becker preferences":

$$u(c_t, N_t) = \frac{(N_t^\varepsilon c_t)^{1-\theta}}{1-\theta},$$

where ε and θ are greater than zero.[5] A Cobb-Douglas aggregator of people and consumption that sits inside CRRA preferences with a constant elasticity of intertemporal substitution $1/\theta$. For the traditional reasons well-known in macro, the Cobb-Douglas unitary elasticity of substitution is necessary so that a constant interior time allocation — here leading to a constant interior population growth rate — can coexist with exponential growth in consumption.

3.2 A Competitive Equilibrium with Externalities

As in Romer (1990), the nonrivalry of ideas leads to increasing returns. Some departure from pure perfect competition is necessary, and the equilibrium in general will not be efficient. We consider a simple equilibrium in which the production of ideas is purely external. Also, we start with the equilibrium allocation because it is designed to be simple. Section 4 below considers an optimal allocation.

Firms produce final output in perfectly competitive markets, taking the stock of ideas A_t as exogenous. Each person chooses time spent raising children versus working in the market sector, ℓ_t versus $1-\ell_t$, in order to maximize utility, also taking the time path of A_t as exogenous. Hence ideas evolve according to the idea production function entirely as an externality: people do not recognize that by having children, their kids may produce new knowledge in the future that makes the entire economy more productive. Markets are perfectly competitive, subject to the idea externality, and the only price is the wage per unit of work, given by $w_t = A_t^\sigma$ in equilibrium.

Taking $\{w_t\}$ as given, people in the representative initial generation solve

$$\max_{\{\ell_t\}} \int_0^\infty e^{-\rho t} u(c_t, N_t) dt$$

subject to

$$\dot{N}_t = (b(\ell_t) - \delta) N_t$$

[5]Barro and Becker (1989) considered the case of $N^\alpha c^{1-\theta}$ with $\alpha > 0$ and $0 < \theta < 1$. Jones and Schoonbroodt (2010) extended the analysis to the equation in the main text with $\theta > 1$, which implicitly corresponds to $\alpha \equiv \varepsilon(1-\theta) < 0$.

$$c_t = w_t(1-\ell_t)$$

and given the function forms assumed in Table 1.

The Hamiltonian for this problem is

$$\mathcal{H} = u(c_t, N_t) + v_t[b(\ell_t) - \delta]N_t$$

where v_t is the shadow price (in utils) of another person.

The equilibrium allocation is then characterized in the following result (derived in Appendix A.3):

Result 3 (The Equilibrium with Endogenous Fertility): *The equilibrium allocation of labor to fertility is given by*

$$\ell_t = 1 - \frac{1}{\bar{b}\tilde{V}_t}, \quad \textit{where}\ \tilde{V}_t \equiv \frac{v_t N_t}{u_{ct} c_t}. \tag{8}$$

When there is an interior solution, population growth satisfies

$$n_t = \bar{b} - \delta - \frac{1}{\tilde{V}_t}. \tag{9}$$

Finally, there exists a steady state with a constant allocation of time devoted to offspring and therefore a constant population growth rate. The shadow value of the population in steady state is

$$\tilde{V}_{eq} = \frac{\varepsilon}{\rho - \varepsilon(1-\theta)n_{eq} - (1-\theta)g_c^{eq}} \tag{10}$$

The steady state population growth rate is

$$n_{eq} = \begin{cases} \frac{1}{\theta}\left(\bar{b} - \delta - \frac{\rho}{\varepsilon}\right) & \textit{if}\ \ \bar{b} - \delta - \frac{\rho}{\varepsilon} < 0 \\ \frac{\bar{b}-\delta-\rho/\varepsilon}{1-(1-\theta)\left(\frac{\varepsilon+\gamma}{\varepsilon}\right)} & \textit{otherwise} \end{cases} \tag{11}$$

Depending on parameter values, equilibrium population growth can be positive or negative. The Empty Planet result can therefore be supported as an equilibrium outcome with endogenous fertility.

The variable $\tilde{V}_t$ is important and has the following economic interpretation: it is the shadow value of the entire population $v_t N_t$, converted into output units by dividing by

the marginal utility of consumption $u_{ct} \equiv \partial u(c_t, N_t)/\partial c_t$, as a ratio to consumption per person. In other words, it is the shadow value of people measured in years of per capita consumption. With the generalized Barro-Becker preferences, it is constant in steady state.

Depending on parameter values, steady-state population growth can be positive or negative. The negative case is the one that is novel and of interest here. It can occur if ε is sufficiently small, for example, so that people do not care that much about their offspring. In that case, we have an equilibrium setup with endogenous fertility that feeds naturally into the results from Section 2. The negative population growth combined with the idea production function implies that the equilibrium with endogenous fertility features a growth rate that falls to zero so that output per person converges to a steady state, as in equation (7). Therefore, the Empty Planet result can be supported as an equilibrium outcome with endogenous fertility.

4. The Optimal Allocation

Now instead consider the optimal allocation in this economic environment. With endogenous fertility, there is no unique criterion for social welfare. Instead, we consider the allocation that maximizes the dynastic utility of a representative generation. The key reason this differs from the equilibrium allocation considered above is that the optimal allocation takes into account the fact that a larger population generates more nonrival ideas, raising everyone's income. This will lead optimal fertility to be higher than its equilibrium rate.

Defined this way, the optimal allocation solves

$$\max_{\{\ell_t\}} \int_0^\infty e^{-\rho t} u(c_t, N_t) dt$$

subject to

$$\dot{N}_t = (b(\ell_t) - \delta) N_t$$

$$\frac{\dot{A}_t}{A_t} = N_t^\lambda A_t^{-\beta}$$

$$c_t = Y_t/N_t = A_t^\sigma (1 - \ell_t).$$

The Hamiltonian for the optimal allocation is

$$\mathcal{H} = u(c_t, N_t) + \mu_t N_t^{\lambda} A_t^{1-\beta} + v_t[b(\ell_t) - \delta]N_t$$

where μ_t is the shadow price of an idea and v_t is the shadow price of another person. The first-order condition for this problem with respect to ℓ_t is

$$\underbrace{v_t N_t b'(\ell_t)}_{\text{MU of time in fertility}} = \underbrace{\frac{u_c(c_t, N_t)c_t}{1-\ell_t}}_{\text{MU of time in making goods}}. \tag{12}$$

Recall that the shadow value of the total population measured in years of per capita consumption is $\tilde{V}_t \equiv \frac{v_t N_t}{u_{ct} c_t}$. This first-order condition can be rewritten as

$$\ell_t = 1 - \frac{1}{\bar{b}\tilde{V}_t}$$

and therefore the population growth rate is

$$n_t = \bar{b} - \delta - \frac{1}{\tilde{V}_t} \tag{13}$$

where we've left implicit the constraint that $\ell \geq 0$ and therefore $n \geq -\delta$. The above two equations have the same form as the equilibrium solutions in Result 3; however, the shadow value of people, $\tilde{V}_t$, will be different. We abuse notation for now by not using a different letter for the equilibrium versus optimal $\tilde{V}_t$.

The first-order condition with respect to A_t can be expressed as an arbitrage equation:

$$\rho = \frac{\dot{\mu}_t}{\mu_t} + \frac{1}{\mu_t}\left(\sigma \frac{u_{ct} c_t}{A_t} + \mu_t(1-\beta)\frac{\dot{A}_t}{A_t}\right).$$

The required rate of return is ρ, and the production of ideas yields both a capital gain and a dividend. Continuing this analogy, this equation can be solved to yield the shadow price of an idea along a balanced growth path as the initial dividend divided by "r-g":

$$\mu_t = \frac{\sigma \frac{u_{ct} c_t}{A_t}}{\rho - g_{\mu t} - (1-\beta) g_{At}}. \tag{14}$$

It turns out to be very useful to define a new variable:

$$z_t \equiv \frac{\mu_t \dot{A}_t}{u_{ct} c_t} = \frac{\sigma g_{At}}{\rho - g_{\mu t} - (1-\beta) g_{At}}.$$

z_t is the social value of the new ideas produced in period t, measured in years of per capita consumption. It is constant along a BGP and given by[6]

$$z^* = \frac{\sigma g_A^*}{\rho - \varepsilon(1-\theta)n^* + (\beta - \sigma(1-\theta))g_A^*} \tag{15}$$

Importantly, notice that if $n^* < 0$ so that $g_A^* = 0$, the social value of the flow of ideas z^* is also zero: there are no ideas being produced in the Empty Planet, so the value is zero.

Returning to the Hamiltonian, the first-order condition for N_t in arbitrage form is

$$\rho = \frac{\dot{v}_t}{v_t} + \frac{1}{v_t}\left(u_{Nt} + \mu_t \lambda \frac{\dot{A}_t}{N_t} + v_t n_t\right).$$

Rearranging gives

$$v_t = \frac{u_{Nt} + \mu_t \lambda \frac{\dot{A}_t}{N_t}}{\rho - g_{vt} - n_t}$$

and therefore

$$\tilde{V}_t \equiv \frac{v_t N_t}{u_{ct} c_t} = \frac{\frac{u_{Nt} N_t}{u_{ct} c_t} + \frac{\lambda \mu_t \dot{A}_t}{u_{ct} c_t}}{\rho - g_{vt} - n_t} = \frac{\varepsilon + \lambda z_t}{\rho - g_{vt} - n_t}.$$

The optimal shadow value of the population $\tilde{V}$ is constant along a BGP and given by

$$\tilde{V}_{sp}^* = \frac{\varepsilon + \lambda z^*}{\rho - \varepsilon(1-\theta)n^* - (1-\theta)g_c^*}. \tag{16}$$

Comparing this equation to the equivalent condition in the equilibrium, equation (10), reveals that they differ because of the presence of z^*: the optimal allocation values people not only for the direct utility they provide (ε), but also because of the additional ideas they produce.

Finally, steady-state population growth is given by evaluating the first-order condition in (13) at this $\tilde{V}^*$:

$$n_{sp}^* = \bar{b} - \delta - \frac{1}{\tilde{V}_{sp}^*}. \tag{17}$$

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[6]The derivation just involves computing the growth rate of μ_t along a BGP from equation (14).

Then the three equations (15), (16), and (17) together determine the steady state for z, n, and $\tilde{V}$.

4.1 The Empty Planet Steady State

We can now solve these three equations and characterize the steady state. The major surprise that emerges is that when the equilibrium allocation features negative population growth in steady state, this Empty Planet steady state is also a solution to the planner problem. Moreover, the planner problem can feature multiple steady states.

The intuition underlying this result is tied to a fundamental nonconvexity in the math that we have highlighted since the beginning of the paper. In particular, steady state growth is

$$g_y^* = \sigma g_A^* = \sigma \left(\frac{N_t^\lambda}{A_t^\beta}\right)^* .$$

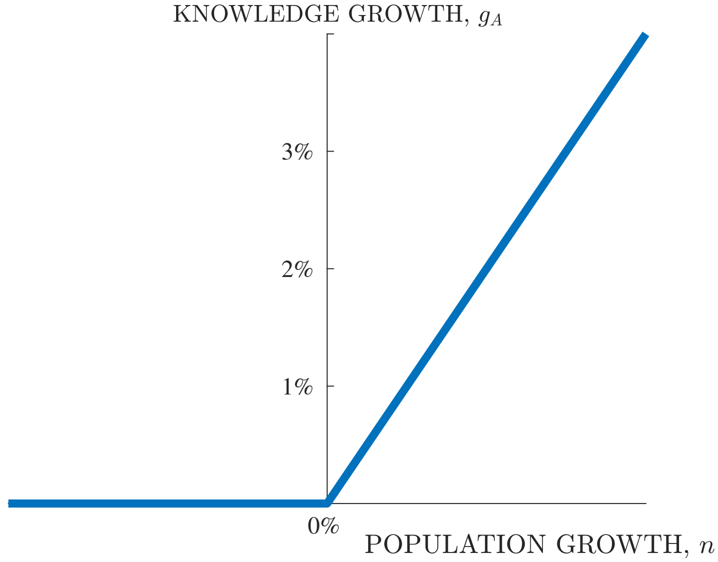
If population growth is positive, the constancy of the right-hand side of this equation requires N_t^λ and A_t^β to grow at the same rate, which requires the growth rate of A_t to be proportional to the rate of population growth: $g_A^* = \lambda n^*/\beta$. If population growth is negative, then y_t and A_t are bounded, as we saw in Section 2 of the paper. Therefore, steady state growth is given by

$$g_y^* = \begin{cases} \gamma n^* & \text{if } \; n_{sp}^* > 0 \\ 0 & \text{if } \; n_{sp}^* \leq 0 \end{cases} \tag{18}$$

In this semi-endogenous growth setup, the long-run growth rate is the product of the overall degree of increasing returns to scale, $\gamma \equiv \lambda\sigma/\beta$, and the rate at which scale is growing, n_{sp}^*. Alternatively, if the planner solution features zero or negative population growth in the steady state, then $g_A^* = g_y^* = 0$; see Figure 2.

These two growth regimes were the focus of the first part of the paper. If steady-state population growth is positive, then steady-state knowledge growth is proportional to n. But if population growth is negative, then steady-state knowledge growth is zero. This kink — these two regimes — is ultimately responsible for one of the key results of the paper, which we state now:

Figure 2: Knowledge Growth and Population Growth in Steady State



Note: There is a fundamental "kink" in the technology for generating steady-state growth in the model. If population growth is positive, then steady-state knowledge growth is proportional to n. But if population growth is negative, then steady-state knowledge growth is zero. This kink — these two regimes — gives rise to the possibility of multiple steady states.

Result 4 (The Empty Planet Result as an Optimal Steady State): *Consider the case where $\bar{b} - \delta - \rho/\varepsilon < 0$ — that is, the case where the equilibrium allocation features negative population growth (the Empty Planet result). Then this Empty Planet steady state is also a steady state of the optimal allocation problem.*

The result is easy to see from equations (15) and (16). Guess that $n^* < 0$ is a solution. Then $g_A^* = 0$. As we noted above, this means that $z^* = 0$ as well: if no ideas are being produced, the social value of the new idea flow is zero. But when $z^* = 0$, $\tilde{V}_{sp}^* = \tilde{V}_{eq}^*$ as can be seen by comparing equations (16) and (10): when the idea value of people is zero, both the planner and the households value people solely through the Barro-Becker preference associated with ε. They therefore choose the same population growth rate, verifying that the Empty Planet steady state is a solution to the optimal allocation.

4.2 Multiple Steady States

What we just showed is that if the equilibrium allocation has negative population growth in the steady state, then this Empty Planet steady state is also a solution of the planner

problem. This is true despite the fact that in the equilibrium, there is a possibly substantial externality: individuals do not take into account that their fertility decisions influence the overall production of ideas and therefore long-run growth. It is just that this externality shrinks to zero when there is negative population growth because the flow of new ideas vanishes relative to the stock; that is, $z \to 0$.

We now show that, provided the idea externality (γ) is sufficiently large, an alternative steady state with positive population growth will also be a solution to the planner problem. That is, the planner problem will involve multiple steady states. This turns out to be easiest to see in the case in which $\theta = 1$, the log case, though it holds more generally as we show in the appendix. We begin by motivating the case of $\theta = 1$ as one of interest and then proceed with it.

Motivating the Log Case ($\theta = 1$). Our model so far has considered generalized Barro-Becker preferences $u(c, N) = (N^{\varepsilon}c)^{1-\theta}/(1-\theta)$. Barro and Becker (1989) originally considered only the case of $0 < \theta < 1$. Jones and Schoonbroodt (2010) extended the analysis to $\theta > 1$. On the one hand, this is the case that economists typically focus on. However, $\theta > 1$ means that the cross-partial derivative $\partial^2 u(c, N)/\partial c \partial N < 0$. That is, the marginal utility of adding more people declines as consumption increases when $\theta > 1$. Instead, when $\theta < 1$, this cross partial is positive: the value of adding more people is higher whenever consumption per person is higher, which seems like the "natural" case. This raises a quandary: on the one hand, we often like $\theta \geq 1$ in thinking about intertemporal tradeoffs. On the other hand, we also like the cross-partial to be positive, which requires $\theta < 1$.

What all of this points out is that the $\theta = 1$ case is intermediate and so in some sense balances these tradeoffs. Moreover, in Section 5, we will see that the $\theta = 1$ case is tremendously helpful in simplifying the analysis of transition dynamics and gaining intuition. For all these reasons we will now focus on this case. The case of $\theta = 1$ corresponds to log preferences, so $u(c, N) = \varepsilon \log N + \log c$.[7]

Multiple Steady States in the Log Case. A useful property of the log case is that the effects of the growth rate on the discount rate in computing present values drops out.

[7] As usual, seeing this limit involves specifying flow utility as $(k^{1-\theta} - 1)/(1-\theta)$ where $k \equiv N^{\varepsilon}c$.

This is readily seen by rewriting (15) and (16) when $\theta = 1$:

$$\tilde{V}^*_{sp} = \frac{\varepsilon + \lambda z^*}{\rho} \tag{19}$$

where

$$z^* = \frac{\sigma g^*_A}{\rho + \beta g^*_A}. \tag{20}$$

Combining these equations gives us one of the relationships between $\tilde{V}$ and n in the steady state. Dropping the asterisks:

$$\tilde{V}_{sp}(n) = \begin{cases} \frac{1}{\rho}\left(\varepsilon + \frac{\gamma}{1+\frac{\rho}{\lambda n}}\right) & \text{if } n > 0 \\ \frac{\varepsilon}{\rho} & \text{if } n \leq 0 \end{cases} \tag{21}$$

where the presence of two distinct cases is precisely driven by the kink shown above in Figure 2.

In contrast, the equilibrium value of $\tilde{V}$ from equation (10) is very simple when $\theta = 1$:

$$\tilde{V}_{eq} = \frac{\varepsilon}{\rho}. \tag{22}$$

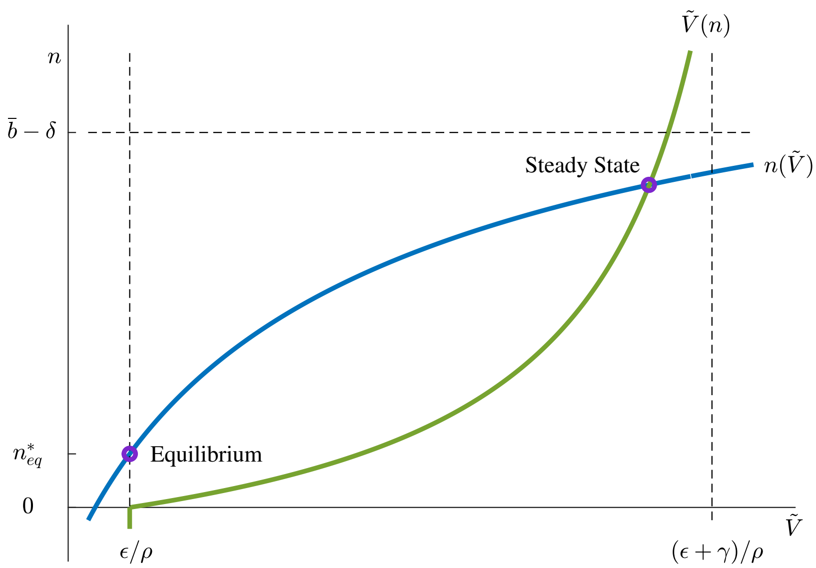
That is, $\tilde{V}_{eq}$ is constant over time, even along the transition path. And of course for both allocations, population growth satisfies a second relation

$$n(\tilde{V}) = \bar{b} - \delta - \frac{1}{\tilde{V}}. \tag{23}$$

These equations determine the steady states for the equilibrium and optimal allocations in the case of $\theta = 1$. They are characterized graphically in Figures 3 and 4.

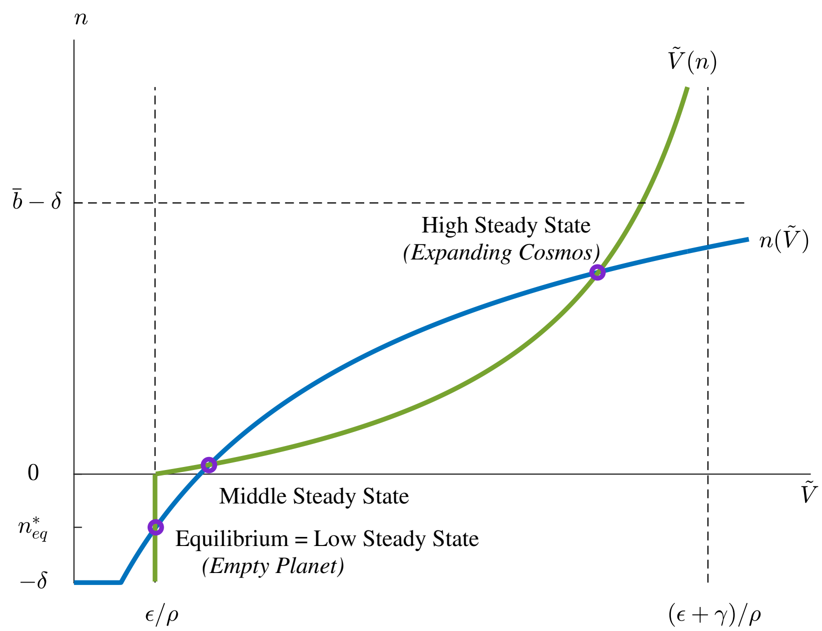
A Conventional Case when $n^*_{eq} > 0$. Figure 3 considers the case in which equilibrium population growth is positive. In this case, there is a unique solution to equations (21) and (23). The optimal allocation features a unique steady state in which optimal population growth exceeds the equilibrium rate, i.e. $n^*_{sp} > n^*_{eq}$. In some sense, this is exactly what one would expect in a model like this. There is a positive externality in equilibrium in that households when choosing their fertility ignore the effect of having more kids on the production of future ideas. The planner takes this into account and chooses a

Figure 3: A Unique Steady State for the Optimal Allocation when $n^*_{eq} > 0$



Note: In the case of $\theta = 1$, when $\bar{b} - \delta - \rho/\epsilon > 0$, equilibrium fertility is positive. There is then a unique solution to equations (21) and (23). That is, the optimal allocation features a unique steady state with $n^*_{sp} > n^*_{eq}$.

Figure 4: Multiple Steady States in the Optimal Allocation when $n_{eq}^* < 0$



Note: When equilibrium fertility is negative and there exists an "Expanding Cosmos" steady state for the planner problem, there are three solutions to equations (21) and (23) that characterize the steady state for γ sufficiently large. We will see later that the middle steady state is unstable and can be ruled out.

higher population growth rate and therefore a higher growth rate for the economy.

When the Equilibrium Features Negative Population Growth. The case of interest in this paper, however, is when equilibrium population growth is negative; relative to Figure 3, consider lowering the value of $\bar{b} - \delta$, for example. This case gives rise to a rich set of outcomes, as suggested by Figure 4. With $n_{eq}^* < 0$, the optimal allocation then features *three* steady states. A high "Expanding Cosmos" steady state has positive population growth. The low steady state has the same negative population growth rate as the equilibrium allocation; this is the Empty Planet outcome. Finally, there is a middle steady state in between. We will see shortly that this steady state is unstable and would never be reached along the optimal path. Appendix A.5 solves analytically for the multiple steady states and the conditions under which they occur, but the formulas are not especially helpful.

Using the graph in Figure 4, however, some key comparative statics can be appreciated. First, as we have just seen, changes in $\bar{b} - \delta$ — the maximum possible fertility rate if 100% of time was devoted to fertility — shift the blue $n(\tilde{V})$ schedule up and down. A higher $\bar{b} - \delta$ raises both the equilibrium and optimal fertility rates, and it is only for $\bar{b} - \delta$ sufficiently small that the equilibrium fertility rate can be negative and therefore the Empty Planet result emerges.

Second, a higher γ — that is, a larger degree of increasing returns associated with ideas and therefore the larger the "idea value of people" — rotates the green $\tilde{V}(n)$ schedule down and to the right and therefore increases the population growth rate associated with the Expanding Cosmos steady state. Intuitively, the size of the gap between the Empty Planet and the Expanding Cosmos is pinned down by the importance of ideas in the economy. As the important of ideas vanishes to zero, the green $\tilde{V}(n)$ schedule rotates backwards and eventually the planner problem features a unique steady state that is the same as the equilibrium.

As discussed in Appendix A.5, a similar graph characterizes the steady states when $\theta \neq 1$, but it is more complicated. When $\theta > 1$, the $\tilde{V}(n)$ schedule eventually "bends backward" heading back to $\tilde{V} = 0$ as n goes to infinity. When instead $\theta < 1$, the $\tilde{V}(n)$ schedule flattens out at a finite n as $\tilde{V}$ goes to infinity. The three steady states exist for a range of plausible parameter values. At some level, we know this is intuitively true. We showed above in Result 4 that the Empty Planet steady state is a solution of the optimal allocation for any θ. And by the "positive externality of people" argument, one would expect a high steady state to exist as long as ideas are sufficiently important, i.e. for γ sufficiently large.

At this point, a key question remains: what then determines which steady state is reached? Would the optimal allocation ever involve going to the Empty Planet steady state? To answer these questions, we turn to the transition dynamics of the model.

5. Stability and Transition Dynamics

Given the presence of three steady states in the optimal allocation, the transition dynamics are subtle. Moreover, because our model has two state variables, A_t and N_t, as well as one control variable, ℓ_t, transition dynamics can be hard to visualize. However,

an advantage of the log case ($\theta = 1$) is that a redefinition of the state variables simplifies the dynamics.

In particular, redefine the state variables as $p_t \equiv \log N_t$ and $x_t \equiv A_t^{\beta}/N_t^{\lambda}$. We will refer to x_t as "knowledge per person," which is a slight abuse of language in that it ignores the exponents. The optimal allocation of labor can be expressed solely as a function of x_t, which means that population growth can as well. This allows us to study the transition dynamics in a simple two-dimensional plane. It is also useful to keep in mind that with this definition of the state, $\dot{A}_t/A_t = 1/x_t$. That is, the state-like variable x_t can also be interpreted as the inverse of the growth rate of knowledge. Along any balanced growth path, x_t will be constant. Finally, notice that a bigger x is not necessarily better: it is the ratio of two state variables that are each good for welfare, A and N. The results are summarized more precisely below (derived in Appendix A.6):

Result 5 (Transition Dynamics for the Log Case): *When $\theta = 1$, we can define new state variables $x_t \equiv A_t^{\beta}/N_t^{\lambda}$ and $p_t \equiv \log N_t$ and then the following results emerge:*

1. *The value function $\mathcal{H}(x_t, p_t)/\rho$ can be expressed as $W(x_t) + \nu p_t$ where $\nu \equiv (\varepsilon + \gamma)/\rho$.*
2. *The policy function for optimal population growth depends only on x_t and not on p_t.*
3. *The first order necessary conditions governing the optimal allocation can be expressed as two differential equations in x_t and n_t:*

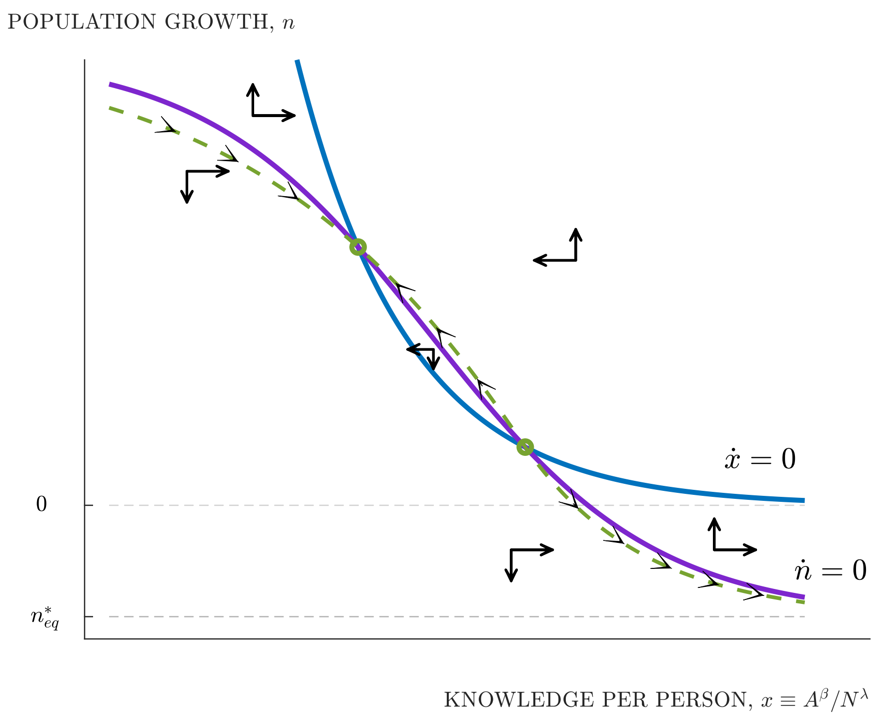
$$\dot{x}_t = \beta - \lambda n_t x_t \tag{24}$$

$$\dot{n}_t = -(\bar{b} - \delta - n_t)^2 \left[\left(\rho + \frac{\beta}{x_t} \right) \left(\nu - \frac{1}{\bar{b} - \delta - n_t} \right) - \gamma \right] \tag{25}$$

These differential equations — together with an initial condition x_0 and a transversality condition — pin down the optimal path of population growth. Figure 5 shows the phase diagram corresponding to the differential equations, with the green dashed line showing the path for the optimal allocation. The high steady state is saddle-path stable. The middle steady state is unstable. And the asymptotic Empty Planet steady state is stable as well.

Rather than discuss these dynamics in detail now, it proves helpful to first calibrate the parameters of the model and solve for the transition dynamics numerically. That

Figure 5: The Phase Diagram for the Optimal Allocation



Note: This figure shows the phase diagram for the optimal allocation in (x, n) space based on the differential equations in (24) and (25) when parameters are such that equilibrium population growth is negative. Arrows indicate dynamics and there are two interior steady states where the curves intersect. The dashed green line shows the path for the optimal allocation.

way we can discuss the transition dynamics in the context of somewhat realistic numbers.

5.1 Numerical Solution of the Transition Dynamics

Table 2 summarizes our parameter choices. Values are chosen to be realistic, but the general results are robust to a range of alternative values.

Because we do not observe ideas directly, it is convenient to normalize $\sigma = 1$ so that A has the units of total factor productivity. The extensive evidence on idea production functions in Bloom, Jones, Van Reenen and Webb (2020) suggests that $\beta > 0$ so that the exponential growth of ideas is getting harder to achieve. With some decreasing returns to research at a point in time ($\lambda = 0.75$), their evidence is consistent with $\beta = 2$. These choices imply that the overall implied degree of increasing returns to scale is $\gamma \equiv \lambda\sigma/\beta = 0.375$.

We assume a conventional death rate of $\delta = 1/90 \approx 1.1\%$, corresponding to a life expectancy of 90 years. We then set the overall rate of time preference as $\rho = \delta \approx 1.1\%$ as well.

Motivated by the recent fertility experience in the OECD, Japan, and the United States, we assume $n^*_{eq} = -0.5\%$, so that in equilibrium, the population will decline at half a percent per year. Finally, we assume that the typical person spends about 1/8 of his or her time endowment producing and raising children.[8]

Given these assumptions, the following four equations determine the values of $\bar{b}$, ε, n^*_{sp}, and ℓ^*_{sp}:

$$n^*_{eq} = \bar{b}\ell^*_{eq} - \delta = -0.5\% \tag{26}$$

$$\ell^*_{eq} = 1 - \frac{\rho}{\bar{b}\varepsilon} = 1/8 \tag{27}$$

$$n^*_{sp} = \bar{b}\ell^*_{sp} - \delta \tag{28}$$

$$\ell^*_{sp} = 1 - \frac{\rho}{\bar{b}(\varepsilon + \lambda z^*)} \tag{29}$$

[8]Taking a broad interpretation of time, i.e. including education, this value is reasonable. Smaller values produce qualitatively similar results. However, for $\ell_{eq} \leq 1/10$, for example, the dynamics around the middle steady state feature spirals and jumps; these are discussed in more detail in Jones (2020). Given that this makes the discussion more complicated without adding much insight, we've chosen the slightly higher value of 1/8.

Table 2: Parameter Values and Steady-State Results

Key Assumed Values as Inputs to Quantitative Analysis

Parameter/Moment	Value	Comment
θ	1	Dramatically simplifies analysis
σ	1	Normalization
λ	0.75	Duplication effects
β	2	Bloom, Jones, Van Reenen and Webb (2020)
δ	1/90	Death rate, life expectancy is 90 years
ρ	δ	All discounting is from mortality, $\rho = \delta = 1.11\%$
n^*_{eq}	-0.5%	Suggested by fertility rates in Europe, Japan, U.S.
ℓ^*_{eq}	1/8	Time spent raising children

Implied Parameter Values and Steady-State Results

Result	Value	Comment
$\bar{b}$	.049	$n^*_{eq} = \bar{b}\ell^*_{eq} - \delta = -0.5\%$
ε	.260	From equation (27) for ℓ^*_{eq}
γ	0.375	Overall degree of IRS, $\gamma \equiv \lambda\sigma/\beta$
High "Expanding Cosmos" SS		
n^*_{sp}	1.16%	From equations (28) and (29) for ℓ^*_{sp} and n^*_{sp}
ℓ^*_{sp}	0.46	From equations (28) and (29) for ℓ^*_{sp} and n^*_{sp}
g^{sp}_y	0.43%	Equals γn^*_{sp}
Unstable Middle SS		
n^*_{sp}	0.26%	From equations (28) and (29) for ℓ^*_{sp} and n^*_{sp}
ℓ^*_{sp}	0.28	From equations (28) and (29) for ℓ^*_{sp} and n^*_{sp}
g^{sp}_y	0.10%	Equals γn^*_{sp}

Note: The first panel in the table shows key assumptions that are an input into the numerical examples. The second panel shows implied parameter values and steady-state results given these assumptions.

where z^* is given by (20).

Implied Parameter Values and Steady-State Results. The implied parameter values and steady-state outcomes are shown in the bottom part of Table 2. For the high steady state, the optimal population growth rate given these values is substantially higher than the equilibrium rate: 1.16% versus -0.5%. Even with sharp dynamic diminishing returns in the idea production function ($\beta = 2$), there is a large positive externality to offspring in this calibration. The associated steady-state growth rate of income per person is 0.43%. This is lower than growth rates observed for the past century in the U.S. because this model omits other — transitory — sources of growth such as rising educational attainment and declining misallocation that have been important. See Jones (2022) for a discussion and for broader evidence suggesting that this seemingly low long-run growth rate, driven primarily by $\gamma = 0.375$, is a plausible calibration.

The last part of Table 2 shows the values associated with the unstable middle steady state. We will discuss these values more shortly.

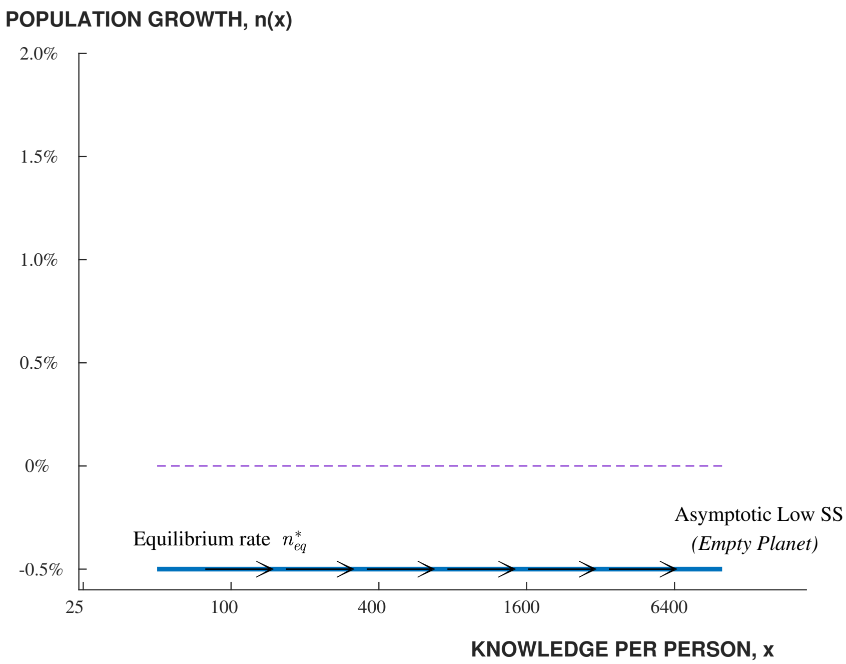
Stability and Dynamics. As a warm-up exercise, Figure 6 shows the transition dynamics for the equilibrium allocation. When $\theta = 1$, the equilibrium actually features a constant value of $\tilde{V}_t = \tilde{V}_{eq} = \frac{\varepsilon}{\rho}$. Because the shadow value of people is constant over time, so is the population growth rate. In other words, the population growth rate is always equal to -0.5% per year in this calibration. (Details are in Appendix A.3.)

With a constant negative population growth rate, $x_t \equiv A_t^{\beta}/N_t^{\lambda}$ increases over time: intuitively, A_t rises to an upper bound, while N_t is falling, which causes x_t to increase. The Empty Planet steady state occurs as x_t goes to infinity, so that $\dot{A}_t/A_t = 1/x_t$ falls to zero. These dynamics are shown in Figure 6 and are very simple, but this is a good way to introduce the figure and it complements the dynamics of the optimal allocation, which we turn to now.

Figure 7 shows the more complicated dynamics of the optimal allocation; the numerical solution method is discussed in Appendix B. The high Expanding Cosmos steady state is saddle-path stable. There are a wide range of potential starting points for knowledge per person, x, such that the optimal allocation ultimately converges to the high steady state. This is what one would generally expect in a problem like this.

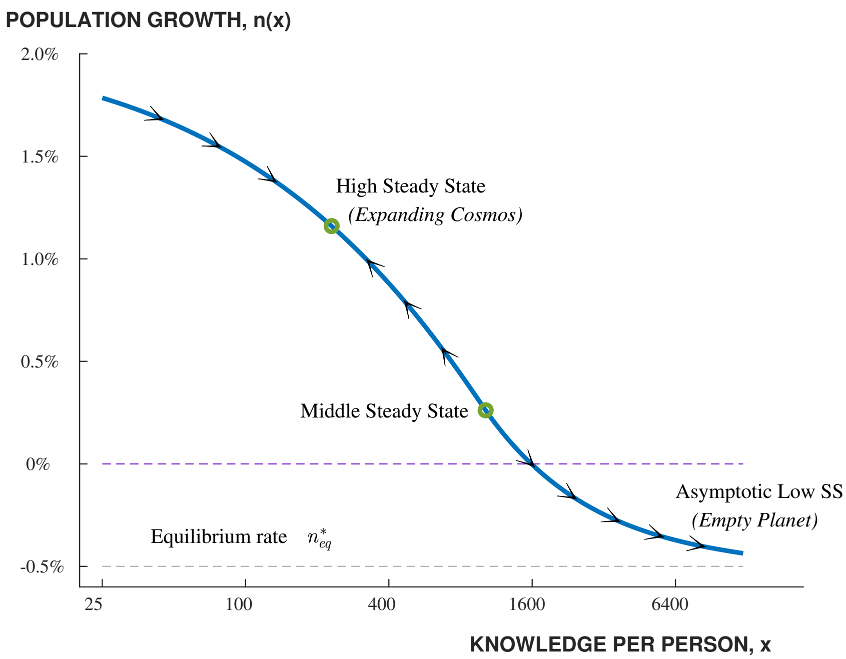
As suggested by the phase diagram back in Figure 5, the middle steady state is

Figure 6: Transition Dynamics for the Equilibrium Allocation



Note: This figure shows the transition dynamics in the equilibrium allocation with $\theta = 1$ and negative population growth. The state variable on the horizontal axis is $x_t \equiv A_t^{\beta}/N_t^{\lambda}$, which we somewhat loosely refer to as "knowledge per person." The law of motion for x_t is $\dot{x}_t = \beta - \lambda n_t x_t$. Clearly, if n_t is negative, then $\dot{x} > 0$. Arrows indicate these transition dynamics. The equilibrium features a constant negative rate of population growth, which causes x_t to increase over time.

Figure 7: Transition Dynamics for the Optimal Allocation



Note: This figure shows the transition dynamics in the optimal allocation with $\theta = 1$. The state variable on the horizontal axis is $x \equiv A^\beta / N^\lambda$, which we somewhat loosely refer to as "knowledge per person." Arrows indicate transition dynamics. If the economy begins with a stock of knowledge per person that is not too high, it converges to the stable "high" steady state. Alternatively, if knowledge per person is sufficiently high, the economy converges to the Empty Planet steady state with negative population growth, which equals the equilibrium rate. The "middle steady state" is unstable and divides the two regions.

unstable. For values of x just to the left of the middle steady state, the dynamics take the economy to the high steady state. Alternatively, if x is just above the middle steady state, the dynamics take the economy to the right, ultimately converging to the Empty Planet steady state.[9]

The intuition for the stability of the steady states is easiest to see if we begin at the Empty Planet. Close to the Empty Planet steady state — i.e. for large x_t — n_t is negative so that population is declining. Because $x_t \equiv A_t^{\beta}/N_t^{\lambda}$, x_t will increase when N_t is declining (because A_t is always increasing). This means that the Empty Planet steady state is stable. Interestingly, x can also increase even when n is positive, provided n is sufficiently small. Recall that its law of motion is $\dot{x}_t = \beta - \lambda n_t x_t$. Clearly if n_t is negative, then $\dot{x} > 0$, but this is sufficient rather than necessary: $\dot{x}_t > 0$ can occur with positive population growth as long as n_t is sufficiently small. This explains how the transition dynamics feature rising x_t to the right of the middle steady state.

Further intuition goes as follows. An increase in knowledge per person x_t causes optimal fertility to decline because the extra ideas produced by offspring have a diminishing marginal benefit; this explains the negative slope of $n(x)$ in Figure 7. If equilibrium fertility were positive, then optimal fertility would also remain positive — the planner values people at least as much as the equilibrium. But if equilibrium fertility is negative, then for x_t high enough, optimal fertility becomes negative as well. This is because as x_t goes to infinity, the stock of knowledge divided by the number of people is so high that the "knowledge value" of additional offspring falls to zero. But once population growth is negative, x_t increases over time rather than decreases since the denominator of $x \equiv A^{\beta}/N^{\lambda}$ is falling. That causes x_t to increase, reinforcing the change. That is the intuition for the bifurcation in Figure 7 and the perhaps surprising stability of the Empty Planet outcome.

What do the transition dynamics look like for alternative parameter values? The intuition for the answer can be found by looking at the $n(\tilde{V})$ and $\tilde{V}(n)$ figure that characterized the multiple steady states several pages back in Figure 4. If we reduce the importance of ideas in the economy — i.e. if we reduce γ, say by making ideas even harder to find via a higher β — the $V(n)$ curve rotates back to the left. This pushes down

[9]For higher values of γ than we assume in our baseline, the unstable middle steady state can become a spiral "Skiba point" instead of a source. The dynamics are slighly more complicated than here, but the bottom line points are unchanged. See Jones (2020) for the analysis of this case.

the Expanding Cosmos steady state and raises the middle steady state. That is, the two points get closer together. In Figure 7, this shrinks the range of x values for which the transition dynamics lead x_t to decline. If we continue to lower γ and reduce the importance of ideas, eventually the $n(\tilde{V})$ and $\tilde{V}(n)$ curves are tangent — the "high" and "middle" steady states become the same point. If we reduce the importance of ideas even further, so that the $n(\tilde{V})$ curve in Figure 4 lies entirely below the $\tilde{V}(n)$ schedule, the transition dynamics then *always* involve x increasing, and the only steady state is the Empty Planet outcome. Intuitively, if ideas are not very important, both the equilibrium and the optimal allocations will feature negative population growth.

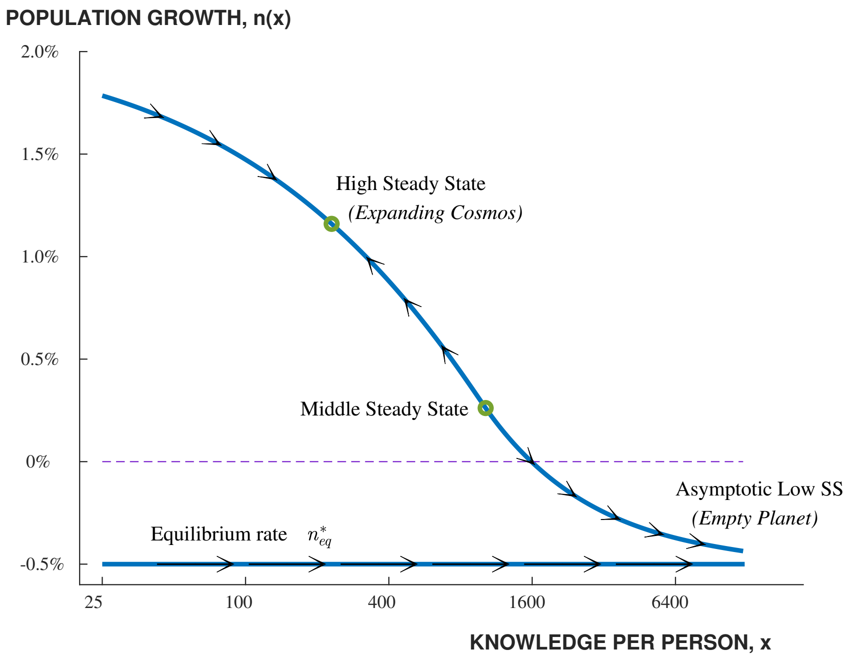
5.2 The Economics of the Transition Dynamics

The transition dynamics lead to an important economic point, summarized in Figure 8. Consider an economy that is governed by the equilibrium allocation. It features negative population growth at rate n^*_{eq}, and suppose the economy is initially endowed with a certain population and stock of knowledge such that knowledge per person, x_0, equals 50 (and therefore $\dot{A}_0/A_0 = 1/x_0 = 2\%$). The social planner would like the economy to have a much higher fertility rate and converge to the Expanding Cosmos steady state with positive population growth and positive economic growth: both the number of people and income per person would rise exponentially forever. In contrast, the equilibrium allocation will simply move the economy steadily to the right, to higher values of x, along the lower line: there will be a constant negative rate of population growth, so knowledge per person, x, will rise as the number of people declines.

At any point in time, society may adopt better policies, such as a fertility subsidy, that move the economy to the optimal allocation. If this occurs at $x = 100$ or $x = 400$, then the economy will eventually transition to the high steady state and exhibit exponential growth forever. Notice that the TFP growth rate of the economy is just $1/x_t$, so these values of x correspond to TFP growth rates of 1% or 0.25%, helping us to think about mapping this diagram into our actual economy.

However — and this is the surprising point — if the economy delays adopting good policies for too long, eventually knowledge per person x will rise *above* the value associated with the middle steady state. In our calibration, this occurs at $x^*_{middle} = 1020$

Figure 8: Transition Dynamics: Summary



Note: The bottom line in the figure shows the transition dynamics for the equilibrium allocation while the curved lines show the dynamics for the optimal allocation. An economy governed by the equilibrium dynamics can get trapped in the Empty Planet outcome if it waits too long to switch to the optimal allocation. In this calibration, that would occur if knowledge per person, x, rises above about 1020 or, equivalently, when TFP growth slows to less than 0.1%.

and therefore when TFP growth is $\dot{A}_t/A_t = 1/x^*_{middle} = 0.1\%$.[10] Once this happens, the optimal regime changes. Adopting good policies that deliver the optimal allocation now puts the economy along a path that takes it to the right. Knowledge per person continues to grow and the economy will converge to the low steady state. Population growth eventually turns permanently negative, the population declines, knowledge will remain below an upper bound, and income per person will stagnate. This is the Empty Planet outcome. **The surprise is that if society waits too long to adopt good policies, the optimal allocation switches from one of sustained exponential growth in population, knowledge, and living standards to one of stagnation and an empty planet.**

In our quantitative analysis, we can also compute how much growth remains once

[10] In exploring different plausible parameter values in the calibration, TFP growth at the middle steady state was typically very low. As discussed above, higher values of γ push population growth and TFP growth at the middle steady state even lower.

the optimal population growth rate turns negative (at around $x = 1600$). From that point forward, growth is already so low that income per person only increases by a further 31% before stagnating at the Empty Planet steady state.

This discussion is summarized in our last main result:

Result 6 (The Optimal Allocation with Endogenous Fertility): *The allocation that maximizes the welfare of each generation converges to one of two steady states. If the economy adopts the optimal allocation while knowledge per person, x, is sufficiently low, it leads to the Expanding Cosmos outcome of sustained exponential growth in population, knowledge, and living standards. Alternatively, if the economy waits too long to switch to the optimal path, it converges to the Empty Planet outcome: living standards stagnate as the population gradually declines toward zero.*

6. Conclusion

Historically, fertility rates in high-income countries have fallen from 5 children per women to 4, 3, 2, and now even fewer. From a family's standpoint, there is nothing special about "above two" versus "below two" and the demographic transition may lead families to settle on fewer than two children. The macroeconomics of the problem, however, make this distinction one of critical importance: it is the difference between an Expanding Cosmos of exponential growth in both population and living standards and an Empty Planet, in which incomes stagnate and the population vanishes.

Endogenizing fertility leads to an additional subtlety. When the equilibrium fertility rate is negative, the optimal allocation typically features two stable steady states. If the economy adopts the optimal allocation soon enough, it converges to the Expanding Cosmos. But if the economy waits too long to switch, even the optimal allocation can be trapped by the Empty Planet outcome.

We've presented our results in the context of simple models that omit many other considerations related to fertility, including the demographic transition, a quality-quantity tradeoff, urbanization, and rising female labor force participation. This is not because these other forces are unimportant but rather reflects an effort to highlight the key mechanisms in the paper as cleanly as possible. The "idea value of people" is tied to the flow of new ideas that are created at each point in time and is, at least partially, a

positive externality that would lead optimal population growth to exceed the equilibrium rate in many models. With negative population growth, however, the flow of new ideas goes to zero. It is this force that allows the optimal allocation to be trapped in an Empty Planet, and this mechanism would also be at work in richer models of fertility and growth. In that sense, we believe the key results in this paper would generalize to richer setups.

One force we have abstracted from here is the possible depreciation of knowledge. It is well-known by historians that fundamental ideas have been lost with the decline of some civilizations. That may not be a problem here in that living standards continue to increase in this model, so that our technologies for storing knowledge may remain effective. However, if knowledge were to depreciate at a constant exogenous rate, it is easy to show in the simple models at the start of this paper that this would lead to declining living standards in the presence of negative population growth, an even more dire outcome.

Of course, the results in this paper are not a forecast — the paper is designed to suggest that a possibility we have until now not considered carefully deserves more attention. There are ways in which this model could fail to predict the future even though the forces it highlights are operative. Automation and artificial intelligence could enhance our ability to produce ideas sufficiently that growth in living standards continues even with a declining population, for example. Or new discoveries could eventually reduce the mortality rate to zero, allowing the population to grow despite low fertility. Or evolutionary forces could eventually favor groups with high fertility rates (Galor and Moav, 2002). Nevertheless, the emergence of negative population growth in many countries and the possible consequences for the future of economic growth make this a topic worthy of further exploration.

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Appendix to "The End of Economic Growth?
Unintended Consequences of a Declining Population"

A. Derivation of Results

A.1 Derivation of Result 1. Romer/AH/GH with Negative Population Growth

Integrate the differential equation:

$$\int \frac{dA_t}{A_t} = \alpha N_0 \int e^{-\eta t} dt$$

which gives

$$\log A_t = C_0 - \frac{\alpha N_0}{\eta} e^{-\eta t}$$

Setting $t = 0$ to solve for the constant gives

$$C_0 = \log A_0 + \frac{\alpha N_0}{\eta}$$

Next, note that $g_{A0} = \alpha N_0$. Then the time path for the stock of ideas over time:

$$\log A_t = \log A_0 + \frac{g_{A0}}{\eta} \left(1 - e^{-\eta t}\right)$$

so that as $t \to \infty$,

$$\log A_t \to \log A^* \equiv \log A_0 + \frac{g_{A0}}{\eta}$$

In other words, an exponentially declining growth rate leads to a steady state level of technology and income per person.

$$y_t \to y^* \equiv \left(A_0 e^{g_{A0}/\eta}\right)^\sigma$$

Finally, converting fully into output terms using $g_y = \sigma g_A$:

$$\frac{y^*}{y_0} = e^{g_{y0}/\eta} = \exp\left(\frac{g_{y0}}{\eta}\right).$$

The time it takes A_t to reach $(A^* + A_0)/2$ is then

$$t_{1/2} = -\frac{1}{\eta}\log\left(1 - \frac{\eta}{g_{A0}}\log\left(\frac{1}{2}\frac{A_0 + A^*}{A_0}\right)\right).$$

A.2 Derivation of Result 2. Semi-Endogenous Growth with Negative Population Growth

Integrate the differential equation:

$$\int A_t^{\beta-1} dA_t = \alpha N_0^\lambda \int e^{-\lambda\eta t} dt$$

which gives

$$\frac{1}{\beta} A_t^\beta = C_0 - \frac{\alpha N_0^\lambda}{\lambda\eta} e^{-\lambda\eta t}$$

Setting $t = 0$ to solve for the constant gives

$$C_0 = \frac{1}{\beta} A_0^\beta + \frac{\alpha N_0^\lambda}{\lambda\eta}$$

Then the time path for the stock of ideas over time:

$$A_t^\beta = A_0^\beta + \frac{\beta\alpha N_0^\lambda}{\lambda\eta}\left(1 - e^{-\lambda\eta t}\right)$$

Dividing by A_0^β and noting that $g_{A0} = \alpha N_0^\lambda A_0^{-\beta}$ gives

$$\frac{A_t}{A_0} = \left(1 + \frac{\beta g_{A0}}{\lambda\eta}\left(1 - e^{-\lambda\eta t}\right)\right)^{1/\beta}$$

Converting to output using $y = A^\sigma$ and defining $\gamma \equiv \lambda\sigma/\beta$ to measure the overall degree of increasing returns to scale:

$$\frac{y_t}{y_0} = \left(1 + \frac{g_{y0}}{\gamma\eta}\left(1 - e^{-\lambda\eta t}\right)\right)^{\gamma/\lambda} \tag{30}$$

Taking the limit as $t \to \infty$,

$$\frac{y^*}{y_0} = \left(1 + \frac{g_{y0}}{\gamma\eta}\right)^{\gamma/\lambda}$$

Taking logs and derivatives of equation (30) gives the growth rate over time:

$$\frac{\dot{y}_t}{y_t} = g_{y0} \cdot \left(\frac{y_t}{y_0}\right)^{-\frac{\lambda}{\gamma}} e^{-\lambda\eta t} = \frac{g_{y0}e^{-\lambda\eta t}}{1 + \frac{g_{y0}}{\gamma\eta}\left(1 - e^{-\lambda\eta t}\right)}$$

The time it takes A_t to reach $(A^* + A_0)/2$ is then

$$t_{1/2} = -\frac{1}{\lambda\eta} \log\left[1 + \frac{\lambda\eta}{\beta g_{A0}}\left(1 - \left[\frac{1}{2}\frac{A_0 + A^*}{A_0}\right]^{\beta}\right)\right].$$

A.3 Derivation of Result 3. The Equilibrium with Endogenous Fertility

The Hamiltonian for this problem is

$$\mathcal{H} = u(c_t, N_t) + v_t[b(\ell_t) - \delta]N_t$$

where v_t is the shadow price (in utils) of another person.

The first-order condition for this problem with respect to ℓ_t is

$$\underbrace{u_c(c_t, N_t)w_t}_{\text{MU of time in making goods}} = \underbrace{v_t N_t b'(\ell_t)}_{\text{MU of time in fertility}} \tag{31}$$

On the right side, spending a little more time on fertility leads each of N_t people to have $b'(\ell_t)$ additional offspring, valued at shadow price v_t. Alternatively, the time could be spent working to earn the wage w_t, which is converted to utility units using the marginal utility of consumption. At the maximum, individuals are indifferent between these two options.

Using our function form assumptions with some algebra, this condition can be rewritten as

$$\ell_t = 1 - \frac{1}{\bar{b}\tilde{V}_t}, \text{ where } \tilde{V}_t \equiv \frac{v_t N_t}{u_{ct} c_t}. \tag{32}$$

The variable $\tilde{V}_t$ is important and has the following economic interpretation: it is the shadow value of the entire population $v_t N_t$, converted into output units by dividing by the marginal utility of consumption $u_{ct} \equiv \partial u(c_t, N_t)/\partial c_t$, as a ratio to consumption per person. In other words, it is the shadow value of the population measured in years

of per capita consumption. Time spent having kids, ℓ_t, and therefore overall fertility $n_t = \bar{b}\ell_t - \delta$, depends on this key variable:

$$n_t = \bar{b} - \delta - \frac{1}{\tilde{V}_t} \tag{33}$$

The first-order condition for N_t gives an arbitrage-like equation for the shadow price of another person:

$$\rho = \frac{\dot{v}_t}{v_t} + \frac{1}{v_t}\left(u_{Nt} + v_t n_t\right) \tag{34}$$

where $u_{Nt} \equiv \partial u(c_t, N_t)/\partial N_t$ is the marginal utility of having another person in the dynasty. Rearranging to solve for v_t gives

$$v_t = \frac{u_{Nt}}{\rho - g_{vt} - n_t.}$$

Along a BGP, g_{vt} and n_t are constant, so the growth rate of v_t is given by the growth rate of u_{Nt}. To see its value, notice that the Cobb-Douglas structure gives $u_{Nt}N_t = \varepsilon(1-\theta)u(c_t, N_t)$ and $u_{ct}c_t = (1-\theta)u(c_t, N_t)$. Therefore $g_v = g_u - n$ and $g_u = \varepsilon(1-\theta)n + (1-\theta)g_c$. Substituting these equations into the equation for v_t along the BGP gives

$$\tilde{V}^* = \frac{\varepsilon}{\rho - \varepsilon(1-\theta)n^* - (1-\theta)g_c^*} \tag{35}$$

Fertility along the BGP is then

$$n^* = \bar{b} - \delta - \frac{1}{\tilde{V}^*}. \tag{36}$$

The equilibrium allocation in steady state is given by the solution of these two equations in n^* and $\tilde{V}^*$.

If $n^* < 0$, then $g_c^* = 0$. Combining these two equations to solve for n_{eq}^* then leads to

$$n_{eq}^* = \frac{1}{\theta}\left(\bar{b} - \delta - \frac{\rho}{\varepsilon}\right) \tag{37}$$

which is negative if $\bar{b} - \delta - \rho/\varepsilon < 0$.

Alternatively, if $\bar{b} - \delta - \rho/\varepsilon > 0$, then $n^* > 0$ and $g_c^* = \gamma n^*$. In this case, solving the

two equations in two unknowns gives[11]

$$n_{eq}^{*} = \frac{\bar{b} - \delta - \rho/\varepsilon}{1 - (1-\theta)\left(\frac{\varepsilon+\gamma}{\varepsilon}\right)}. \tag{38}$$

The Special Case of $\theta = 1$. In the special case of $\theta = 1$, it is obvious from (35) that $\tilde{V}_{eq}^{*} = \frac{\varepsilon}{\rho}$. But in fact, it turns out that $\tilde{V}_t = \tilde{V}_{eq}^{*}$ at all points in time. This can be seen by noting that the utility function when $\theta = 1$ is $u(c, N) = \log c + \varepsilon \log N$ (see Section 4.2) so that $u_N = \varepsilon/N$. The law of motion for v_t in equation (34) can then be written as

$$\dot{\tilde{V}}_t = \rho\tilde{V}_t - \varepsilon.$$

This differential equation has a rest point at $\tilde{V}_{eq}^{*}$ which is unstable. Any solution other than $\dot{\tilde{V}}_t = 0$ for all t turns out to violate the transversality condition or a resource constraint.

The constancy of $\tilde{V}_t$ when $\theta = 1$ means that equilibrium population growth is also constant over time:

$$n_t = \bar{b} - \delta - \frac{\rho}{\varepsilon} \quad \text{for all } t. \tag{39}$$

A.4 Derivation of Result 4. The Empty Planet Result as an Optimal Steady State

These results are derived in the main text.

A.5 Multiple Steady States with Barro-Becker Preferences

The three equations (15), (16), and (17) together determine the steady state for z, n, and $\tilde{V}$. Result 4 already showed that there can exist an Empty Planet steady state that involves negative population growth. We show in this section the conditions under which the Expanding Cosmos steady state and a middle unstable steady state can exist. In particular, our focus is on the case shown in Figure 4 when the two interior steady states involve positive population growth. Relative to that figure, though, we do this for the generalized Barro-Becker preferences that permit $\theta \neq 1$. We see below that two

[11] This case requires further conditions on parameters. For example, $n_{eq}^{*} < \frac{\rho}{(1-\theta)(\varepsilon+\gamma)}$ to keep the denominator in (35) positive.

conditions that support the presence of multiple steady states are (i) $n^*_{eq} < 0$ and (ii) γ sufficiently large.

For steady states that feature positive population growth, $g^*_A = \lambda n^*/\beta$ and $g^*_c = \gamma n^*$. Making these substitutions, equations (15), (16), and (17) can be written as

$$z^* = \frac{\sigma\gamma n^*}{\rho + \lambda n^* - (1-\theta)(\varepsilon+\gamma)n^*} \tag{40}$$

$$\tilde{V}^* = \frac{\varepsilon + \lambda z^*}{\rho - (1-\theta)(\varepsilon+\gamma)n^*} \tag{41}$$

$$n^* = \bar{b} - \delta - \frac{1}{\tilde{V}^*}. \tag{42}$$

Defining $\phi \equiv (1-\theta)(\varepsilon+\gamma)$ to simplify the notation, these equations can be written as a quadratic equation in n^*:

$$\underbrace{[\lambda(\varepsilon+\gamma) - (\varepsilon+\lambda)\phi + \phi^2]}_{\equiv a}(n^*)^2 - \underbrace{\left[(\varepsilon+\gamma)(\bar{b}-\delta) - \frac{\varepsilon\phi}{\lambda}n^{log}_{eq} - \rho\left(1 + \frac{\varepsilon-\phi}{\lambda}\right)\right]}_{\equiv b} n^* \underbrace{-\rho\varepsilon n^{log}_{eq}}_{\equiv c} = 0 \tag{43}$$

where $n^{log}_{eq} \equiv \bar{b} - \delta - \rho/\varepsilon$ denotes the equilibrium population growth rate in the log preference case $\theta = 1$.

There exist parameter values such that this quadratic equation has two positive real roots. To see this, notice that examples would feature $a > 0$, $b > 0$, and $c > 0$. This last piece, $c > 0$, is true when the equilibrium population growth rate is negative, illustrating the important role played here by a negative equilibrium population growth rate. Next, notice that $\theta = 1$ implies $\phi = 0$. So $a > 0$ is guaranteed when $\theta = 1$ and in fact is true for any $\theta > 1$ and more generally as long as θ is not too small.

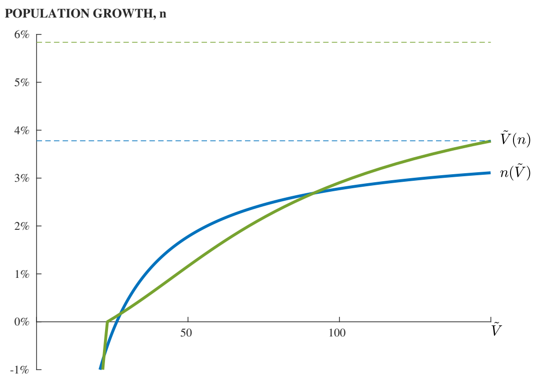
Third, we need $b > 0$. To see that this can occur notice that when $\theta = 1$ so that $\phi = 0$, we have

$$b > 0 \iff \bar{b} - \delta - \frac{\rho}{\lambda}\frac{\varepsilon+\lambda}{\varepsilon+\gamma} > 0$$

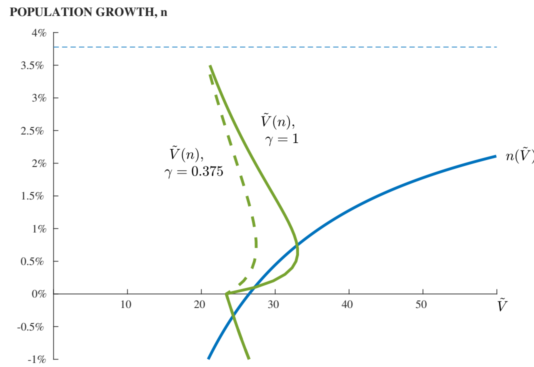
Clearly this condition will hold for γ sufficiently large.

Figure 9 shows examples with multiple steady states for different values of θ starting from the baseline parameters we use in the main paper given in Table 2. Panel (a) shows that our baseline parameter values deliver multiple steady states when $\theta = 0.7$. When $\theta = 1.5$ in panel (b), there is a unique steady state at the Empty Planet for our baseline

Figure 9: Multiple Steady States with Barro-Becker Preferences



(a) $\theta = 0.7$



(b) $\theta = 1.5$

Note: The graphs show the possibility of multiple steady states for the generalized Barro-Becker preferences when $\theta \neq 1$. Steady states occur where the $n(\tilde{V})$ and the $\tilde{V}(n)$ schedules intersect. These graphs are based on our baseline parameters given in Table 2. When $\theta = 1.5$ in panel (b), there is a unique steady state with negative population growth (the Empty Planet) for our baseline value of $\gamma = 0.375$; the idea value of people is too small to create the high steady state in this case. But for $\gamma = 1$, the multiple steady states reappear.

value of $\gamma = 0.375$; the idea value of people is too small to create the high steady state in this case. But when ideas are more important — say for $\gamma = 1$ — the multiple steady states reappear. This illustrates the basic message that when equilibrium population growth is negative and when ideas are sufficiently important (i.e. when γ is sufficiently high), the optimal allocation features multiple steady states.

A.6 Derivation of Result 5. Transition Dynamics for the Log Case ($\theta = 1$)

Define new state variables $x_t \equiv A_t^{\beta}/N_t^{\lambda}$ and $p_t \equiv \log N_t$. Using the laws of motion for A_t and N_t, these state variables evolve according to

$$\dot{x}_t = \beta - \lambda n_t x_t$$
$$\dot{p}_t = n_t = \bar{b}\ell_t - \delta$$

Consumption per person is $c_t = x_t^{\gamma/\lambda} N_t^{\gamma}(1 - \ell_t)$. Taking logs, flow utility when $\theta = 1$

is

$$u(c_t, N_t) = \log c_t + \varepsilon \log N_t$$
$$= \frac{\gamma}{\lambda} \log x_t + (\varepsilon + \gamma) p_t + \log(1 - \ell_t)$$

and therefore the Hamiltonian for the optimal allocation is linear in p_t:

$$\mathcal{H} = \frac{\gamma}{\lambda} \log x_t + (\varepsilon + \gamma) p_t + \log(1 - \ell_t) + \mu_t(\beta - \lambda n(\ell_t) x_t) + \nu_t n(\ell_t)$$

The first-order condition with respect to ℓ_t, $\mathcal{H}_\ell = 0$, can be written as

$$\ell_t = 1 - \frac{1}{\bar{b}(\nu_t - \lambda \mu_t x_t)}$$

which then implies

$$n_t = \bar{b} - \delta - \frac{1}{(\nu_t - \lambda \mu_t x_t)}. \tag{44}$$

The first-order condition with respect to x_t is

$$\frac{\dot{\mu}_t}{\mu_t} = \rho + \lambda n_t - \frac{\gamma}{\lambda} \cdot \frac{1}{\mu_t x_t}. \tag{45}$$

The first-order condition with respect to p_t is

$$\frac{\dot{\nu}_t}{\nu_t} = \rho - \frac{1}{\nu_t}(\varepsilon + \gamma).$$

This can be rewritten as

$$\dot{\nu}_t = \rho \nu_t - (\varepsilon + \gamma).$$

This differential equation has a rest point which is unstable. Any solution other than $\dot{\nu} = 0$ for all t turns out to violate the transversality condition or a resource constraint. Therefore we have

$$\nu_t = \nu \equiv \frac{\varepsilon + \gamma}{\rho}. \tag{46}$$

This means the Hamiltonian can be written as

$$\mathcal{H} = \frac{\gamma}{\lambda} \log x_t + \log(1 - \ell_t) + \mu_t(\beta - \lambda n(\ell_t) x_t) + \nu n(\ell_t) + (\varepsilon + \gamma) p_t \tag{47}$$

Moreover, the first-order conditions above in (44) and (45) are independent of p given the constant ν. Since the value function is $\mathcal{H}/\rho$, it can therefore be written as $W(x_t)+\nu p_t$ once we recognize that n is some function of x_t only.

The remaining piece of the result is to derive the law of motion for n_t in terms of n_t and x_t only. To do this, it useful to define an intermediate variable $m_t \equiv \lambda \mu_t x_t$. Note that

$$\begin{aligned}\frac{\dot{m}_t}{m_t} &= \frac{\dot{\mu}_t}{\mu_t} + \frac{\dot{x}_t}{x_t} \\ &= \left(\rho + \lambda n_t - \frac{\gamma}{m_t}\right) + \left(\frac{\beta}{x_t} - \lambda n_t\right) \\ &= \rho + \frac{\beta}{x_t} - \frac{\gamma}{m_t}\end{aligned}$$

so that

$$\dot{m}_t = \left(\rho + \frac{\beta}{x_t}\right) m_t - \gamma$$

Next, from (44), we have

$$n_t = \bar{b} - \delta - \frac{1}{(\nu - m_t)}. \tag{48}$$

Taking time derivatives:

$$\begin{aligned}\dot{n}_t &= -(\nu - m_t)^{-2}\dot{m}_t \\ &= -(\bar{b} - \delta - n_t)^2 \dot{m}_t \\ &= -(\bar{b} - \delta - n_t)^2 \left[\left(\rho + \frac{\beta}{x_t}\right) m_t - \gamma\right]\end{aligned}$$

Finally, we can replace m_t in this equation by rewriting (48) as

$$m_t = \nu - \frac{1}{\bar{b} - \delta - n_t}.$$

A.7 Derivation of Result 6. The Optimal Allocation with Endogenous Fertility

These results are derived in the main text and in Appendix B on transition dynamics.

B. Solving Numerically for Transition Dynamics

The key differential equations characterizing the optimal allocation were given in Result 5 in equations (24) and (25):

$$\dot{x}_t = \beta - \lambda n_t x_t \tag{49}$$

$$\dot{n}_t = -(\bar{b} - \delta - n_t)^2 \left[\left(\rho + \frac{\beta}{x_t} \right) \left(\nu - \frac{1}{\bar{b} - \delta - n_t} \right) - \gamma \right] \tag{50}$$

The steady state of this system is

$$x^* = \frac{\beta}{\lambda n^*}$$

$$n^* = \frac{1}{2}\tilde{b} \pm \sqrt{\frac{1}{4}\tilde{b}^2 + \frac{\varepsilon n_{eq}^{\log}}{\lambda \nu}}$$

where

$$\tilde{b} \equiv \bar{b} - \delta - \frac{\varepsilon + \lambda}{\lambda \nu} > 0$$

and

$$n_{eq}^{\log} \equiv \bar{b} - \delta - \frac{\rho}{\varepsilon}.$$

The numerical solution of the transition dynamics for the optimal allocation proceeds in two steps, first listed here and then described in more detail below:

1. We linearize the transition dynamics associated with these two differential equations to see that the high steady state is stable while the middle steady state is unstable.

2. We solve the original nonlinear system of differential equations in (x_t, n_t) numerically using a "reverse shooting" approach together with a starting guess close to the steady state from the linearized system; see the Matlab program `OptimalDynamics.m`.

B.1 Linearizing around the Interior Steady States

Linearizing this differential system around the steady state gives

$$\dot{x}_t = -\lambda n^*(x_t - x^*) - \lambda x^*(n_t - n^*)$$
$$\dot{n}_t = \beta \left(\frac{\bar{b} - \delta - n^*}{x^*}\right)^2 \left(\nu - \frac{1}{\bar{b} - \delta - n^*}\right)(x_t - x^*)$$
$$+ \left[2(\bar{b} - \delta - n^*)(\varepsilon + \lambda\nu n^*) - \rho - \lambda n^*\right](n_t - n^*)$$

Expressing the linearized system in matrix form with $X \equiv [x\ n]'$ allows us to write it as $\dot{X}_t = B(X_t - X^*)$ where B is the matrix of coefficients, which in turn depends on various steady-state values.

We can now evaluate this linearized system around the steady states using the parameter values in Table 2. First, consider the "high" steady state. The matrix B_{high} has one negative eigenvalue and one positive eigenvalue, both real, indicating a saddle-path stable steady state. In contrast, the "middle" steady state is unstable: both its eigenvalues are positive and real. These results are computed in the Matlab program `OptimalDynamics.m`.

This general characterization is broadly robust to the parameter values and shows our first point: the high steady state is stable, while the middle steady state is an unstable source.

B.2 Numerical Transition Dynamics using the Original Hamiltonian

We solve the full nonlinear system numerically using a "reverse shooting" approach. To begin, we start from the high steady state, move a tiny amount away according to the negative eigenvalue and corresponding eigenvector of the linearized system, and solve the full nonlinear dynamics backwards to characterize the optimal path. In one direction, this takes us "up and to the left" in Figure 7 while in the other it takes us toward the middle steady state. Finally, to get the dynamics between the middle steady state and the Empty Planet steady state, we begin with $x_0 = 26,000$ (corresponding to an initial growth rate of $1/26,000 \approx 0\%$) and $\ell_0 = \ell_{eq}$, i.e. close to the "low" steady state. We then solve backwards so this takes us along the optimal path back to the unstable middle steady state.