

# A.I., Automation, and Explosive Growth: A Condition for Finite-Time Blow-Up

ChatGPT 5.2 Pro  
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**Chad:** In Aghion, Jones, and Jones (2019), there is an error in the proof of Example 3 on page 256–257. Phil Trammell originally pointed this out to me and graciously provided a correct proof ([here](#)). I was happy to see that the proof was not trivial, but I always suspected there must be a cleaner way of showing the result. This was one of my “Can A.I. solve this” questions that I have long felt is an excellent test of A.I. models. The document below gives the elegant response from ChatGPT 5.2 Pro from a conversation we had on December 13, 2025. I wish I’d come up with this myself back in 2017!

## 1. Setup and statement of the problem (Prompted by Chad)

[The text below is pasted into the prompt. It provides the first key paragraphs of the example for context and then asks for a derivation.]

Suppose the production function for goods is

$$Y_t = A_t^\sigma K_t^\alpha L^{1-\alpha}$$

(where constant population  $L$  simplifies the analysis; exogenous population growth would not change the core logic). The capital accumulation equation and the idea production function are

$$\dot{K}_t = \bar{s}L^{1-\alpha} A_t^\sigma K_t^\alpha - \delta K_t, \quad (1)$$

$$\dot{A}_t = K_t^\beta S^\lambda A_t^\phi, \quad \phi < 1 \quad (2)$$

where  $0 < \alpha < 1$  and  $0 < \beta < 1$ , and where we take research effort  $S$  to be constant. Follow-

ing the [Zeira \(1998\)](#) interpretation, we interpret  $\alpha$  as the fraction of goods tasks that have been automated and  $\beta$  as the fraction of tasks in idea production that have been automated.

The standard endogenous growth result requires “constant returns to accumulable factors.” To see what this means, define the parameter

$$\gamma := \frac{\sigma}{1 - \alpha} \cdot \frac{\beta}{1 - \phi}. \quad (3)$$

In this setup, the endogenous growth case corresponds to  $\gamma = 1$ . The singularity case occurs if  $\gamma > 1$ .

In growth rates, the laws of motion for capital and ideas are

$$\frac{\dot{K}_t}{K_t} = \bar{s}L^{1-\alpha} \frac{A_t^\sigma}{K_t^{1-\alpha}} - \delta, \quad (4)$$

$$\frac{\dot{A}_t}{A_t} = S^\lambda \frac{K_t^\beta}{A_t^{1-\phi}}. \quad (5)$$

**Question.** What is the key condition on parameter values for explosive growth? How do you show formally that this is the key condition? And what is the intuition for the key condition?

## 2. ChatGPT 5.2 Pro's Answer

### 2.1 Key condition for explosive growth

The key condition is

$$\boxed{\gamma > 1} \iff \boxed{\sigma\beta > (1 - \alpha)(1 - \phi).}$$

When  $\gamma = 1$  the model is on the constant-returns knife-edge (endogenous growth with asymptotically constant growth rates). When  $\gamma < 1$  there is diminishing returns to accumulable factors overall and one does not obtain finite-time blow-up.

### 2.2 Formal argument: why $\gamma > 1$ is the right condition

The main idea is to (i) remove the diminishing-returns exponents in each accumulation equation via a change of variables, and then (ii) apply a standard blow-up criterion for differential inequalities.

### 2.2.1 Step 1: Remove diminishing returns via a change of variables

Assume  $K(0) > 0$  and  $A(0) > 0$  so that the state remains positive. Define

$$k(t) := K(t)^{1-\alpha}, \quad a(t) := A(t)^{1-\phi}.$$

Differentiate  $k$  using (1):

$$\begin{aligned} \dot{k} &= (1-\alpha)K^{-\alpha}\dot{K} \\ &= (1-\alpha)K^{-\alpha}(\bar{s}L^{1-\alpha}A^\sigma K^\alpha - \delta K) \\ &= (1-\alpha)\bar{s}L^{1-\alpha}A^\sigma - (1-\alpha)\delta K^{1-\alpha} \\ &=: c_1 A^\sigma - \delta_1 k, \end{aligned} \tag{6}$$

where  $c_1 := (1-\alpha)\bar{s}L^{1-\alpha}$  and  $\delta_1 := (1-\alpha)\delta$ .

Differentiate  $a$  using (2):

$$\begin{aligned} \dot{a} &= (1-\phi)A^{-\phi}\dot{A} \\ &= (1-\phi)A^{-\phi}(S^\lambda K^\beta A^\phi) \\ &= (1-\phi)S^\lambda K^\beta =: c_2 K^\beta, \end{aligned} \tag{7}$$

where  $c_2 := (1-\phi)S^\lambda$ .

Now express  $A$  and  $K$  in terms of  $a$  and  $k$ :

$$A = a^{1/(1-\phi)}, \quad K = k^{1/(1-\alpha)}.$$

Ignoring (for the moment) the linear drag  $-\delta_1 k$  in (6), the “core” positive-feedback system becomes

$$\dot{k} = c_1 a^n, \quad \dot{a} = c_2 k^m, \tag{8}$$

where

$$n := \frac{\sigma}{1-\phi}, \quad m := \frac{\beta}{1-\alpha}.$$

Crucially,

$$mn = \frac{\beta}{1-\alpha} \cdot \frac{\sigma}{1-\phi} = \frac{\sigma}{1-\alpha} \cdot \frac{\beta}{1-\phi} = \gamma.$$

### 2.2.2 Step 2: Derive an inequality of the form $\dot{a} \geq Ca^\eta$

Differentiate  $\dot{a} = c_2 k^m$  and substitute  $\dot{k} = c_1 a^n$ :

$$\ddot{a} = c_2 m k^{m-1} \dot{k} = c_2 m k^{m-1} (c_1 a^n). \quad (9)$$

Using  $\dot{a} = c_2 k^m$ , we can write  $k = (\dot{a}/c_2)^{1/m}$ . Plugging this into (9) yields

$$\ddot{a} = m c_1 c_2^{1/m} a^n (\dot{a})^{(m-1)/m}. \quad (10)$$

Define

$$u := (\dot{a})^{(m+1)/m}.$$

Then

$$\frac{du}{dt} = \frac{m+1}{m} (\dot{a})^{1/m} \ddot{a}.$$

Substitute (10):

$$\frac{du}{dt} = (m+1) c_1 c_2^{1/m} a^n \dot{a}.$$

Divide by  $\dot{a} > 0$  to get an ODE in  $a$ :

$$\frac{du}{da} = (m+1) c_1 c_2^{1/m} a^n. \quad (11)$$

Integrate:

$$(\dot{a})^{(m+1)/m} = (\dot{a}_0)^{(m+1)/m} + \frac{(m+1) c_1 c_2^{1/m}}{n+1} (a^{n+1} - a_0^{n+1}). \quad (12)$$

Hence there exist constants  $\tilde{C} > 0$  and  $\bar{a} > 0$  such that for all  $a \geq \bar{a}$ ,

$$\dot{a} \geq \tilde{C} a^\eta, \quad \eta := \frac{m(n+1)}{m+1}. \quad (13)$$

### 2.2.3 Step 3: Apply the blow-up criterion and recover $\gamma > 1$

From (13), the time required for  $a$  to diverge is bounded above by

$$T - t_0 \leq \int_{a_0}^{\infty} \frac{da}{\tilde{C} a^\eta}.$$

This improper integral is finite if and only if  $\eta > 1$ . Check:

$$\eta > 1 \iff \frac{m(n+1)}{m+1} > 1 \iff m(n+1) > m+1 \iff mn > 1 \iff \gamma > 1.$$

Therefore, when  $\gamma > 1$ , the transformed variable  $a(t) = A(t)^{1-\phi}$  reaches  $+\infty$  in finite time, and so does  $A(t)$ . Since  $\dot{k} = c_1 a^n$  in (8),  $k(t)$  and hence  $K(t)$  also diverge in finite time.

**Remark 1** (Role of depreciation). *The preceding steps ignore the linear drag term  $-\delta_1 k$  in (6). Intuitively (and formally via comparison arguments), a linear term cannot prevent blow-up driven by superlinear positive feedback: for sufficiently large  $a$ , the forcing term  $c_1 a^n$  dominates  $\delta_1 k$ , so the dynamics eventually dominate a scaled version of the core system (8). Hence the threshold condition for explosive growth remains  $\gamma > 1$ .*

### 2.3 Intuition for $\gamma > 1$

The parameter  $\gamma$  is the *loop gain* of the positive feedback between capital accumulation and idea accumulation, after accounting for diminishing returns in each equation.

- **$A \rightarrow K$  channel (goods side).** Higher ideas raise goods output and thus investment. Because the capital accumulation equation has diminishing returns in  $K$  when  $\alpha < 1$ , translating improvements in  $A$  into sustained expansion of  $K$  is governed by an effective elasticity  $\sigma/(1 - \alpha)$ .
- **$K \rightarrow A$  channel (ideas side).** Higher capital raises research productivity. Because the idea production equation has diminishing returns in  $A$  when  $\phi < 1$ , translating improvements in  $K$  into sustained expansion of  $A$  is governed by an effective elasticity  $\beta/(1 - \phi)$ .

Multiplying the two amplification factors gives

$$\gamma = \left( \frac{\sigma}{1 - \alpha} \right) \left( \frac{\beta}{1 - \phi} \right).$$

If  $\gamma < 1$ , one loop through “ $A$  raises  $K$ , which raises  $A$ ” attenuates rather than amplifies; diminishing returns dominate and growth does not run away. If  $\gamma = 1$ , the system has constant returns to the accumulable pair  $(K, A)$  and admits endogenous growth with asymptotically constant growth rates. If  $\gamma > 1$ , each round of feedback is amplifying strongly enough to create effective increasing returns to accumulable factors: the system ultimately behaves like a differential inequality  $\dot{a} \gtrsim C a^\eta$  with  $\eta > 1$ , and such inequalities imply *finite-time blow-up*.

### 2.4 Example from the prompt

If  $\alpha = \beta = \phi = 1/2$ , then

$$\gamma = \frac{\sigma}{1 - 1/2} \cdot \frac{1/2}{1 - 1/2} = 2\sigma.$$

Hence explosive growth occurs if and only if  $\sigma > 1/2$ .

### 3. Why the change of variables $k = K^{1-\alpha}$ and $a = A^{1-\phi}$ is the “right” one

**Chad’s Prompt.** In the automation-growth system, a key step is defining

$$k(t) := K(t)^{1-\alpha} \quad \text{and} \quad a(t) := A(t)^{1-\phi}.$$

Is there a general principle at work here, and how can one recognize such a change of variables in future problems?

**Answer: the Bernoulli / constant-elasticity power transform.** The substitution is an instance of a general and widely useful idea: when a state variable appears with a *constant elasticity* (a power) in its own law of motion, a *power transform* often removes that nonlinearity.

To see the structure, write the capital law of motion (suppressing constants for clarity):

$$\dot{K} = c A^\sigma K^\alpha - \delta K. \quad (14)$$

If we treat  $A(t)$  as given for the moment, this is a Bernoulli-type ODE in  $K$ :

$$\dot{K} + \delta K = u(t) K^\alpha, \quad u(t) := c A(t)^\sigma, \quad \alpha \neq 1.$$

A standard result for Bernoulli equations,

$$\dot{x} + p(t)x = q(t)x^n, \quad n \neq 1,$$

is that the change of variables  $z = x^{1-n}$  transforms the equation into a linear ODE in  $z$ .

In our case,  $n = \alpha$ , so  $k := K^{1-\alpha}$  is the natural choice. Similarly, in the idea-production law,

$$\dot{A} = \tilde{c} K^\beta A^\phi, \quad (15)$$

treating  $K(t)$  as given makes this Bernoulli-type in  $A$  with exponent  $\phi$ , so  $a := A^{1-\phi}$  is the natural counterpart.

**A quick derivation without memorizing Bernoulli.** One can “derive” the exponent by inspection using the identity

$$\frac{d}{dt} K^r = r K^{r-1} \dot{K}.$$

In (14), the problematic term is  $K^\alpha$ . Multiply both sides by  $K^{-\alpha}$  to cancel it:

$$K^{-\alpha} \dot{K} = c A^\sigma - \delta K^{1-\alpha}.$$

Now choose  $r$  so that the left-hand side becomes an exact derivative:

$$K^{-\alpha} \dot{K} = \frac{1}{1-\alpha} \frac{d}{dt} (K^{1-\alpha}).$$

This pins down  $r = 1 - \alpha$  immediately, hence  $k = K^{1-\alpha}$ . The same logic applied to (15) yields  $a = A^{1-\phi}$ .

**What this transform buys in the present model.** Using  $k = K^{1-\alpha}$  and  $a = A^{1-\phi}$ , one obtains (up to constants)

$$\dot{k} = (1 - \alpha) \bar{s} L^{1-\alpha} A^\sigma - (1 - \alpha) \delta k, \quad (16)$$

$$\dot{a} = (1 - \phi) S^\lambda K^\beta. \quad (17)$$

The key point is that each equation becomes “linear” (or at least affine) in its *own* transformed state: the  $K^\alpha$  term has been eliminated from the right-hand side of  $\dot{k}$ , and the  $A^\phi$  term has been eliminated from the right-hand side of  $\dot{a}$ .

Substituting back  $A = a^{1/(1-\phi)}$  and  $K = k^{1/(1-\alpha)}$  reveals the clean cross-power feedback:

$$\dot{k} \propto a^{\sigma/(1-\phi)} - \delta k, \quad (18)$$

$$\dot{a} \propto k^{\beta/(1-\alpha)}. \quad (19)$$

The product of the cross-elasticities is then

$$\left( \frac{\sigma}{1-\phi} \right) \left( \frac{\beta}{1-\alpha} \right) = \frac{\sigma}{1-\alpha} \cdot \frac{\beta}{1-\phi} = \gamma,$$

which is exactly the “loop gain” parameter governing whether the positive feedback from  $A \rightarrow K \rightarrow A$  is strong enough to yield explosive growth.

**A reusable recipe for future problems.** When confronted with nonlinear growth laws built from monomials (Cobb–Douglas terms), the following checklist is useful:

1. **Power-transform to remove own-state curvature.** If the law of motion has the form

$$\dot{x} + \delta x = f(t, \text{others}) x^q \quad (q \neq 1),$$

try the change of variables  $z = x^{1-q}$ . Equivalently, multiply the equation by  $x^{-q}$  and recognize  $x^{-q}\dot{x}$  as an exact derivative.

2. **Homogeneity / normalization tricks for balanced growth.** If the system is homogeneous in levels (e.g.  $Y = A^\sigma K^\alpha$ ), ratio variables like  $K/A^\theta$  (or  $K/(AL)$ ) may remove trends and expose stationary dynamics; choose  $\theta$  to cancel the dominant scaling terms.
3. **Scaling guesses if blow-up is suspected.** For finite-time singularities, it is often effective to guess asymptotic forms such as

$$K(t) \sim (T - t)^{-p}, \quad A(t) \sim (T - t)^{-q},$$

and match exponents. The power-transform approach can be viewed as a way to reach the same scaling logic without guessing  $p$  and  $q$  upfront.

**Economic intuition.** When a production or accumulation equation exhibits diminishing returns—for example,  $K^\alpha$  with  $\alpha < 1$ —the transform  $K^{1-\alpha}$  is the *natural coordinate* that “integrates out” that curvature:

$$\int K^{-\alpha} dK \propto K^{1-\alpha}.$$

In this coordinate, the law of motion separates the pure forcing term (here, proportional to  $A^\sigma$ ) from the linear drag term (here, proportional to  $k$ ), making the strength of positive feedback and the possibility of runaway dynamics transparent.

## References

Aghion, Philippe, Benjamin F. Jones, and Charles I. Jones, “Artificial Intelligence and Economic Growth,” in Ajay Agrawal, Joshua Gans, and Avi Goldfarb, eds., *The Economics of Artificial Intelligence: An Agenda*, University of Chicago Press, 2019, pp. 237–282.

Zeira, Joseph, “Workers, Machines, And Economic Growth,” *Quarterly Journal of Economics*, November 1998, 113 (4), 1091–1117.