Recipes and Economic Growth: A Combinatorial March Down an Exponential Tail

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Combinatorics and Pareto

• Weitzman (1998) and Romer (1993) suggest combinatorics important for growth.
  o Ideas are combinations of ingredients
  o The number of possible combinations from a child’s chemistry set exceeds the number of atoms in the universe
  o But absent from state-of-the-art growth models?

• Kortum (1997) and Gabaix (1999) on Pareto distributions
  o Kortum: Draw productivities from a distribution $\Rightarrow$ Pareto tail is essential
  o Gabaix: Pareto distribution (cities, firms, income) *results from* exponential growth

*Do we really need the fundamental idea distribution to be Pareto?*
Two Contributions

• A simple but useful theorem about extreme values
  ○ The increase of the max extreme value depends on
    (1) the way the number of draws rises, and
    (2) the shape of the upper tail
  ○ Applies to any continuous distribution

• Combinatorics and growth theory
  ○ Combinatorial growth: Cookbook of $2^N$ recipes from $N$ ingredients, with $N$ growing exponentially (population growth)

  Combinatorial growth with draws from thin-tailed distributions
  (e.g. the normal distribution) yields exponential growth

  ○ Pareto distributions are not required — draw faster from a thinner tail
Basic Foundations
Theorem (A Simple Extreme Value Result)

Let $Z_K$ denote the maximum value from $K$ i.i.d. draws from a continuous distribution $F(x)$, with $\bar{F}(x) \equiv 1 - F(x)$ strictly decreasing on its support. Then for $m \geq 0$

$$\lim_{K \to \infty} \Pr \left[ K\bar{F}(Z_K) \geq m \right] = e^{-m}$$

As $K$ increases, the max $Z_K$ rises so as to stabilize $K\bar{F}(Z_K)$.

The shape of the tail of $\bar{F}(\cdot)$ and the way $K$ increases determines the rise in $Z_K$. 
Graphically: Unpacking $K\bar{F}(Z_K)$

$\bar{F}(Z_K) = \Pr[\text{Next draw} > Z_K]$
Graphically: Unpacking $K\tilde{F}(Z_K)$

$\tilde{F}(Z_K) = \Pr[\text{Next draw} > Z_K]$

More draws $K$ means $Z_K$ increases and $\tilde{F}(Z_K)$ declines.

– Good ideas get harder to find!

$\tilde{F}(x)$

$0$

$Z_K$ $\rightarrow$ $Z'_K$

$\tilde{F}(Z_K)$ $\rightarrow$ $\tilde{F}(Z'_K)$
Graphically: Unpacking $K\bar{F}(Z_K)$

$\bar{F}(Z_K) = \Pr[\text{Next draw} > Z_K]$

More draws $K$ means $Z_K$ increases and $\bar{F}(Z_K)$ declines.

- Good ideas get harder to find!

Blowing up by $K \Rightarrow K\bar{F}(Z_K)$ will stabilize.

So the rate at which $K$ increases and the tail of $\bar{F}(\cdot)$ determine how $Z_K$ rises...
Proof of Theorem 1

- Given that $Z_K$ is the max over $K$ i.i.d. draws, we have

$$
\Pr [ Z_K \leq x ] = \Pr [ z_1 \leq x, z_2 \leq x, \ldots, z_K \leq x ]
= (1 - \bar{F}(x))^K
$$

- Let $M_K \equiv K\bar{F}(Z_K)$ denote a new random variable. Then for $0 < m < K$

$$
\Pr [ M_K \geq m ] = \Pr [ K\bar{F}(Z_K) \geq m ]
= \Pr [ \bar{F}(Z_K) \geq \frac{m}{K} ]
= \Pr [ Z_K \leq \bar{F}^{-1} \left( \frac{m}{K} \right) ]
= \left( 1 - \frac{m}{K} \right)^K \to e^{-m} \quad \text{QED.}
$$
Remarks

- Simpler and different from the standard EVT
  - If $\frac{Z_K - b_K}{a_K}$ converges in distribution, then it converges to one of three types
  - Which one depends on the tail properties of $F(\cdot)$

- We will see later that Theorem 1 covers cases not covered by EVT

- Intuition for why so few conditions on $F(\cdot)$ are required:
  - For any distribution of $x$, $\bar{F}(x)$ is Uniform[0,1]
  - Min over $K$ draws from a uniform, scaled up by $K$, is exponential $= K\bar{F}(Z_K)$ (from standard EVT)
  - Barton and David (1959), Galambos (1978, Chapter 4), and Embrechts et al (1997, Prop 3.1.1) have related results
Example: Kortum (1997)

- Pareto: $\bar{F}(x) = x^{-\beta}$

- Apply Theorem 1:
  \[
  K\bar{F}(Z_K) = \varepsilon + o_p(1) \\
  KZ_K^{-\beta} = \varepsilon + o_p(1) \\
  \frac{K}{Z_K^\beta} = \varepsilon + o_p(1) \\
  \frac{Z_K}{K^{1/\beta}} = (\varepsilon + o_p(1))^{-1/\beta}
  \]

- Exponential growth in $K$ leads to exponential growth in $Z_K$

  \[g_z = g_K/\beta\]

  $\beta$ = how thin is the tail = rate at which ideas become harder to find
Example: Drawing from an Exponential Distribution

- Exponential: \( \bar{F}(x) = e^{-\theta x} \)

\[
K\bar{F}(Z_K) = \varepsilon + o_p(1)
\]
\[
Ke^{-\theta Z_K} = \varepsilon + o_p(1)
\]

\[\Rightarrow \log K - \theta Z_K = \log(\varepsilon + o_p(1))\]

\[\Rightarrow Z_K = \frac{1}{\theta} \left[ \log K - \log(\varepsilon + o_p(1)) \right]\]

\[\Rightarrow \frac{Z_K}{\log K} = \frac{1}{\theta} \left( 1 - \frac{\log(\varepsilon + o_p(1))}{\log K} \right)\]

\[\frac{Z_K}{\log K} \overset{p}{\longrightarrow} \text{Constant}\]
Drawing from an Exponential (continued)

\[ \frac{Z_K}{\log K} \xrightarrow{p} \text{Constant} \]

- \( Z_K \) grows with \( \log K \)
  - If \( K \) grows exponentially, then \( Z_K \) grows linearly

- Definition of **combinatorial growth**: \( K_t = 2^{N_t} \) with \( N_t = N_0 e^{g_N t} \)

\[ g_z = g_{\log K} = g_N \]

*Combinatorial growth with draws from a thin-tailed distribution delivers exponential growth!*
Growth Model
Setup

- Cookbook is a collection of $K_t$ recipes

- At a point in time, researchers have evaluated all recipes from $N_t$ ingredients
  - Each ingredient can either be included or excluded, so $K_t = 2^{N_t}$
    - (which equals $\sum_{k=0}^{N_t} \binom{N_t}{k}$, the sum of all combinations)

- Research = learning the “productivity” of the new recipes that come from adding a new ingredient

- $\dot{N}_t = \alpha R_t \Rightarrow$ each researcher can evaluate $\alpha$ new ingredients each period
  - $R_t$ grows with population $\Rightarrow$ so does $N_t$

*Combinatorial growth: Cookbook of $K = 2^N$ recipes from $N$ ingredients, with $N$ growing exponentially*
Corollary (Poisson version of Theorem 1)

Let $Z_K$ denote the maximum over $P$ independent draws from a distribution with a strictly decreasing and continuous tail cdf $\bar{F}(x)$ and suppose $P$ is distributed as Poisson with parameter $K$. Then for $0 < m < K$

$$\Pr \left[ K\bar{F}(Z_K) \geq m \right] = \frac{e^{-m} - e^{-K}}{1 - e^{-K}}.$$ 

- Applies at each point in time, not just asymptotically
- Integrate across a continuum of sectors to make aggregate growth deterministic
  - New recipe applies to a randomly chosen (uniform) sector
- (Thanks to Sam Kortum for this suggestion)
Economic Environment: Like Kortum (1997) but with Weibull/Combinatorial Growth

Aggregate output

\[ Y_t = \left( \int_0^1 Y_{it}^{\frac{\sigma-1}{\sigma}} \, di \right)^{\frac{\sigma}{\sigma-1}} \text{ with } \sigma > 1 \]

Variety \( i \) output

\[ Y_{it} = Z_{Kit} \left( M_{it}^{-\frac{1}{\rho}} \sum_{j=1}^{M_{it}} x_{ijt}^{\rho-1} \right)^{\frac{\rho}{\rho-1}} \text{ with } \rho > 1 \]

Production of ingredients

\[ x_{ijt} = L_{ijt} \]

Best recipe

\[ Z_{Kit} = \max_c z_{ic} \]

Weibull distribution of \( z_{ic} \)

\[ z_{ic} \sim F(x) = 1 - e^{-x^\beta} \quad \beta = \text{how thin is tail} \]

Number of ingredients evaluated

\[ \dot{N}_t = \alpha R_t^\lambda N_t^\phi, \quad \phi < 1 \]

Cookbook (Poisson parameter)

\[ K_t = 2^{N_t} \]

Resource constraint: workers

\[ L_{it} = \sum_{j=1}^{M_i} L_{ijt} \quad \text{and} \quad \int_0^1 L_{it} \, di = L_{yt} \]

Resource constraint: R&D

\[ R_t + L_{yt} = L_t \]

Population growth (exogenous)

\[ L_t = L_0 e^{\delta t} \]
Allocation

- Consider the allocation of labor that maximizes $Y_t$ at each date with a constant fraction of people working in research
  - $L_{ijt}$ maximizes $Y_t$
  - $R_t = \bar{s}L_t$

- Number of ingredients evaluated (eventually) grows at a constant rate
  \[
  \frac{\dot{N}_t}{N_t} = \frac{R_t^\lambda}{N_t^{1-\phi}} \implies \dot{g}_N = \frac{\lambda g_L}{1 - \phi}
  \]

- And we have combinatorial growth in the number of recipes in the cookbook
  \[
  K_t = 2^{N_t} \implies \dot{g}_{\log K} = g_N
  \]
Applying Theorem 1 to the Weibull Distribution

• Suppose \( y \sim \text{Exponential} \). Let \( y \equiv x^\beta \). Then \( x \sim \text{Weibull} \): \( \bar{F}(x) = e^{-x^\beta} \)

\[
\frac{\max_y}{\log K} \xrightarrow{p} \text{Constant} \\
\Rightarrow \frac{\max x^\beta}{\log K} \xrightarrow{p} \text{Constant} \\
\Rightarrow \frac{\max x}{(\log K)^{1/\beta}} \xrightarrow{p} \text{Constant}
\]

• Therefore

\[
g_{\log K} = \frac{g_{\log K}}{\beta} = \frac{g_N}{\beta} = \frac{1}{\beta} \frac{\lambda g_L}{1 - \phi}
\]

(slightly more complicated with Poisson process, but same idea)
Remarks

\[ g_{ZK} = \frac{g_{\log K}}{\beta} = \frac{g_N}{\beta} = \frac{1}{\beta} \frac{\lambda g_L}{1 - \phi} \]

- This is the growth rate of output per person in the growth model
- Combinatorial march down a Weibull tail
- Growth rate depends on
  - Population growth = growth rate of researchers
  - \( \lambda \) and \( \phi \) = how researchers evaluate ingredients
  - Allows \( \phi > 0 \): it may get easier (or harder) to evaluate ingredients
  - While \( \beta \) captures the degree to which good ideas get harder to find
Can the distribution shift out over time?

- Consider all the technologies that could ever be invented. They are recipes.
  - Let $\bar{F}(x)$ be the associated distribution of productivities
  - That doesn’t shift...

- What’s behind the question: some technologies cannot be invented before others
  - The smartphone could not come before electricity, radio, and semiconductors

- Answer: Suppose new ideas are future ingredients
  - Ingredients must be evaluated in a specific order
  - Nothing changes...
Generality?

For what distributions do combinatorial draws $\Rightarrow$ exponential growth?
Theorem (A general condition for combinatorial growth)

Consider the growth model above but with $z_i \sim F(z)$ as a general continuous and unbounded distribution, where $F(\cdot)$ is monotone and differentiable. Let $\eta(x)$ denote the elasticity of the tail cdf $\overline{F}(x)$; that is, $\eta(x) \equiv -\frac{d \log \overline{F}(x)}{d \log x}$. Then

$$\lim_{t \to \infty} \frac{\dot{Z}_{Kt}}{Z_{Kt}} = \frac{g^N}{\alpha}$$

if and only if

$$\lim_{x \to \infty} \frac{\eta(x)}{x^\alpha} = \text{Constant} > 0$$

for some $\alpha > 0$. 
Remarks

\[
\frac{\dot{Z}_{Kt}}{Z_{Kt}} \rightarrow \frac{\delta N}{\alpha} \iff \lim_{x \to \infty} \frac{\eta(x)}{x^\alpha} = \text{Constant} > 0
\]

- Thinner tails require faster draws but still require power functions:
  - It's just that the elasticity itself is now a power function!

- Examples
  - Weibull: \( \bar{F}(x) = e^{-x^\beta} \Rightarrow \eta(x) = x^\beta \)
  - Normal: \( \bar{F}(x) = 1 - \int_{-\infty}^{x} e^{-u^2/2} du \Rightarrow \eta(x) \sim x^2 \) – like Weibull with \( \beta = 2 \)

- Intuition
  - Kortum (1997): \( \bar{F}(x) = x^{-\beta} \Rightarrow \eta(x) = \beta \) so \( K_t = e^{nt} \) is enough
  - Here: \( \bar{F}(x) = e^{-x^\beta} \) so must march down tail exponentially faster, \( K_t = 2^{e^{nt}} \)
For what distributions do combinatorial draws $\Rightarrow$ exponential growth?

- Combinatorial draws lead to exponential growth for many familiar distributions:
  - Normal, Exponential, Weibull, Gumbel
  - Gamma, Logistic, Benktander Type I and Type II
  - Generalized Weibull: $\bar{F}(x) = x^\alpha e^{-x^\beta}$ or $\bar{F}(x) = e^{-(x^\beta + x^\alpha)}$
  - Tail is dominated by “exponential of a power function”

- When does it not work?
  - Lognormal: If it works for normal, then $\log x \sim$ Normal means percentage increments are normal, so tail will be too thick!
  - Logexponential = Pareto
  - Surprise: Does not work for all distributions in the Gumbel domain of attraction (not parallel to Kortum/Frechet).
Scaling of $Z_K$ for Various Distributions

<table>
<thead>
<tr>
<th>Distribution</th>
<th>cdf</th>
<th>$Z_K$ behaves like</th>
<th>Growth rate of $Z_K$ for $K = 2^N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential</td>
<td>$1 - e^{-\theta x}$</td>
<td>log $K$</td>
<td>$g_N$</td>
</tr>
<tr>
<td>Gumbel</td>
<td>$e^{-e^{-x}}$</td>
<td>log $K$</td>
<td>$g_N$</td>
</tr>
<tr>
<td>Weibull</td>
<td>$1 - e^{-x^\beta}$</td>
<td>$(\log K)^{1/\beta}$</td>
<td>$\frac{g_N}{\beta}$</td>
</tr>
<tr>
<td>Normal</td>
<td>$\frac{1}{\sqrt{2\pi}} \int e^{-x^2/2}dx$</td>
<td>$(\log K)^{1/2}$</td>
<td>$\frac{g_N}{2}$</td>
</tr>
<tr>
<td>Lognormal</td>
<td>$\frac{1}{\sqrt{2\pi}} \int e^{-(\log x)^2/2}dx$</td>
<td>$\exp(\sqrt{\log K})$</td>
<td>$\frac{g_N}{2} \cdot \sqrt{N}$</td>
</tr>
<tr>
<td>Gompertz</td>
<td>$1 - \exp(-e^{\beta x} - 1)$</td>
<td>$\frac{1}{\beta} \log(\log K)$</td>
<td>Arithmetic</td>
</tr>
<tr>
<td>Log-Pareto</td>
<td>$1 - \frac{1}{(\log x)^\alpha}$</td>
<td>$\exp(K^{1/\alpha})$</td>
<td>Romer!</td>
</tr>
</tbody>
</table>
Microfoundations for Romer (1990)

- Kortum (1997) found there is no process satisfying EVT that delivers Romer (1990) result of exponential growth with constant flow of draws

- But Theorem 1 shows us how to get it:
  - If \( x \sim \text{logpareto with } \alpha = 1 \), then linear growth in \( K \) (e.g. \( \dot{K}_t = \bar{L} \)) gives exponential growth in max

- Implies \( Z_K = \exp\left(\frac{1}{\alpha}(\tilde{\epsilon} + o_p(1))\right) \)
  - No affine transformation of \( Z_K \) works, which is why EVT fails (need to take logs)
  - Implies that log productivity is Fréchet in cross-section
    - much thicker tail than we observe in the data
    - variance of log productivity would rise over time
Evidence from Patents

Combinatorial growth matches the patent data
Rate of Innovation?

- Kortum (1997) was designed to match a key “fact”: that the flow of patents was stationary
  - Never clear this fact was true (see below)

- Flow of patents in the model?
  - Theory of record-breaking: \( p(K) = \frac{1}{K} \) is the fraction of ideas that are improvements  
    [cf Theorem 1: \( \bar{F}(Z_K) = \frac{1}{K}(\varepsilon + o_p(1)) \)]
  - Since there are \( \dot{K} \) recipes added to the cookbook every instant, the flow of patents is
    \[
p(K)\dot{K} = \frac{\dot{K}_t}{K_t}
    \]
  - This is constant in Kortum (1997) \( \Rightarrow \) constant flow of patents
Flow of Patents in Combinatorial Growth Model?

- Simple case: $\dot{N}_t = \alpha R_t$ (i.e. $\lambda = 1$ and $\phi = 0$).

- Then

  $$K_t = 2^{N_t}$$

  $$\Rightarrow \frac{\dot{K}_t}{K_t} = \log 2 \cdot \dot{N}_t$$

  $$= \log 2 \cdot \alpha R_t$$

  $$= \log 2 \cdot \alpha \bar{s} L_0 e^{g_L t}$$

- That is, the combinatorial growth model predicts that the number of new patents should grow exponentially over time.
Annual Patent Grants by the U.S. Patent and Trademark Office

Total in 2019: 390,000
U.S. origin: 186,000
Foreign share: 52%
Annual Academic Publication Counts, 1970–2020

MILLIONS

Google Scholar
CAGR=3.3%

Web of Science
CAGR=4.3%

Scopus
CAGR=4.4%
Remarks

• In Kortum (1997), rise in patents should correspond to a rise in growth rates.
  o Data seem more consistent with the combinatorial growth model
  o (Important caveat: meaning of a “patent” is not stable over time)

• Can researchers evaluate a combinatorially growing list of recipes?
  o Maybe it is only the “good” ideas that take time
  o With $\lambda = 1$ and $\phi = 0$, the number of good ideas per researcher is constant
  o Chess players find the best line from an exploding set of possibilities
Conclusion

• $K\tilde{F}(Z_K) \sim \varepsilon$ links $K$ and the shape of the tail cdf to how the max increases

• **Weitzman meets Kortum:** Combinatorial growth in recipes whose productivities are draws from a thin-tailed distribution gives rise to exponential growth

• Suggests that exponential growth is the fundamental, not Pareto distributions
  - Then Gabaix/Luttmer mechanism of exponential growth generates Pareto distributions that we see in the data

• Other applications: wherever Pareto has been assumed in the literature, perhaps we can use thin tails?
  - Many literatures at this point
  - E.g. models of technology diffusion, search, markups, productivity
Can we allow for the productivities to be correlated?

- The productivity of flour+tomatoes+cheese+pepperoni is probably correlated with the productivity of flour+tomatoes+cheese+sausage!

- Thinking about this. Seems like it should work...

- Galambos (1978, Theorem 4.1.1)