Diversity Multiplexing Tradeoff for Full-Duplex Relay Networks

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***Abstract*—Co-operative communication with relay assistance helps in enhancing the capacity of point to point links. The capacity of relay networks has been known to be upper bounded by the cut-set bound. Recently, the authors of [1] showed that it is possible to achieve the upper bound up to a constant additive gap, independent of the channel parameters. Presence of multiple relays provides multiple paths from the source to the destination. With no channel state information at the transmitter, these paths can be used to either obtain a high communication rate or a low probability of error. We can trade off one of these quantities for the other. This diversity multiplexing trade off has been studied extensively for MIMO systems. However, the trade off for full duplex Gaussian relay networks is not well understood. We shall study the diversity multiplexing trade off for multiple relay networks. This work provides a lower bound on the error probability at the destination as a function of the communication rate, assuming a finite time for communication. We see that if we choose block length corresponding to the optimal DMT for the min-cut, the probability of error is governed by one of the cuts with diversity gain less than the min-cut diversity gain. The error probability is exactly equal to the one obtained from the optimal DMT for the corresponding cut.**

I. Introduction

Recently, there has been a growing interest in analyzing the performance and trade offs such as the diversity multiplexing trade off (DMT) in multi-terminal wireless networks. DMT has been extensively studied for single transmitter, single receiver MIMO channels. Similar to a MIMO channel, a multi-terminal network provides multiple communication paths from the source to the destination. With additional degrees of freedom from these independent paths, we would expect to communicate at a rate higher than that achievable on any single path.

We know that the ergodic capacity of a fast fading MIMO channel with m transmitters and n antennas can be approximated by $\min\{m, n\} \log(\text{SNR})$ for large SNR[2]. On the other hand, we cannot achieve reliable communication over a slow fading MIMO channel when the channel state is not known at the transmitter. For any communication rate R, there exists a non-zero probability of channel outage, which induces an error at the receiver. Zheng and Tse have derived the trade off between the rate R and the probability that a message is received in error at the receiver [3]. The DMT gives the amount of multiplexing gain (an increase in the communication rate) that one can obtain for a certain diversity gain (a decrease in the error probability at the receiver). Additionally, they have shown that the optimal trade off can be achieved using codewords of length $l = m + n - 1$ at the receiver. This is an interesting result from the designer's perspective, since we cannot achieve any significant improvement in the system performance by using codes with words longer than l.

This work explores the following question - can we define a similar DMT result for multi-terminal networks, in a block fading setup and in the large SNR regime? The authors of [4] have studied the trade off for half duplex relay networks. However, the approach does not extend to full duplex networks. To study the DMT for full duplex relay networks, we first need to understand the capacity of these networks. Cover et al. established the information theoretic cut-set upper bound on the unicast and multi-cast capacity of wireless multi-terminal networks [5]. This bound is an extension of the max-flow min-cut theorem for flow networks. Avestimehr, Diggavi and Tse showed that it is possible to achieve the cut-set bound within a constant additive gap in Gaussian relay networks, using the Quantize-Map-Forward (QMF) strategy [1]. The gap depends only on the number of terminals in the network and is independent of the channel parameters. If we study the QMF strategy as the destination SNR $\to \infty$, the finite, additive gap becomes insignificant.

The QMF strategy involves coding over blocks of infinite length. We are interested in communicating over a finite time. As a result, we will face a non-zero probability of error at the receiver. However, since QMF approximately achieves the cut-set bound in the large SNR regime, it will also achieve the optimal DMT for finite block lengths. We will try to bound the length of the codes that can achieve the DMT.

The rest of this article is organized as follows. Section II describes the system model and the problem in detail. Section III describes a few ways to approach these questions, and provides a lower bound on the diversity-multiplexing trade off. Section IV concludes with some observations and directions for the future.

II. System Model and Assumptions

The wireless network is defined by a set of vertices $\mathcal{V}$ and edges $\mathcal{E}$ that represent the nodes and the links between the nodes. Broadcast and superposition are inherent properties of a wireless network, but we restrict these to the links represented by $\mathcal{E}$. The received signal y_j at a node j is given by,

$$y_j = \sum_i h_{ij} x_i + z_j \tag{1}$$

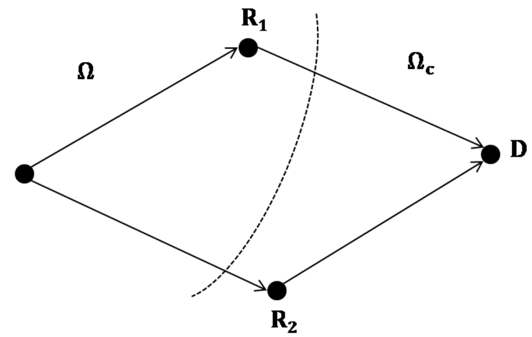

Fig. 1. The diamond relay network with 2 relays. R_1 and R_2 help in transmitting message from S to D. Due to the presence of two different paths, we expect diversity and multiplexing gain for this channel.

Here, $h_{ij} \sim \mathcal{CN}(0,1)$ are the fading co-efficients for the links between nodes i and j and x_i represents the signal transmitted by the node i. We assume a block fading model so that each h_{ij} stays constant over an entire block of communication. The channel state information is not known at any transmitters. The noise z_j at each node is assumed to be white and Gaussian with variance σ^2, i.e., $\mathcal{CN}(0,\sigma^2)$. All transmitted signals have the same average power constraint $\mathbf{E}[|x_i|^2] \leq$ P. Define SNR as,

$$\text{SNR} = \frac{\text{P}}{\sigma^2} \tag{2}$$

Taking a cue from [1], we will first study networks with layered structure where all paths from the source to the destination have equal lengths. The analysis can be extended to non-layered networks. We will restrict attention to cases with only one transmitter and one receiver and use the simplified two relay diamond network of Figure 1 for illustration purposes. If the source S is transmitting information at a rate R, we define the multiplexing gain r as,

$$r = \lim_{\text{SNR}\to\infty} \frac{R}{\log(\text{SNR})} \tag{3}$$

The diversity gain $d(r)$, in terms of the probability of error P_e at the destination D, is defined as,

$$d = -\frac{\log(P_e)}{\log(\text{SNR})} \tag{4}$$

The relay nodes are causal with delay one. The transmitted signal at each node is a function of the received signal. Some of the analysis and conclusions in the next section will be based on the QMF strategy, which works as follows. Divide the communication period into blocks of length T. The source S maps each block of RT message bits to a random Gaussian code word of length T and transmits the word over T channel uses. There are 2^{RT} such messages and hence, possible code words. The relays receive y_{R_i}, the attenuated message corrupted by noise, and quantize them to $[y_{R_i}]$ at the noise level σ. Each relay then maps the quantized versions of the received vector to a new random Gaussian codeword of length T, and transmits it in the next time block. In a multi-layered network, this process continues till the message reaches the destination D. Given the random mapping at each relay node, D then attempts to decode the transmitted message.

III. Bounds on Trade off for Full Duplex Relay Networks

Consider a cut of the layered network that divides the nodes into two parts – Ω denoting the set which contains the the source and Ω_c, the set which contains the destination. This is illustrated in Figure 1. The vector of transmitted signals by the nodes in Ω and Ω_c is denoted by x_Ω and x_{Ω_c}, and the received signals are represented by y_Ω and y_{Ω_c} respectively. We know that the information theoretic cut-set bound for the capacity of this network, $\bar{C}$ is given by,

$$\bar{C} = \max_{p(\{\mathbf{x_i}\}_{i\in\mathcal{V}})} \min_{\Omega\in\Lambda_D} I(x_\Omega; y_{\Omega_c}|x_{\Omega_c}) \tag{5}$$

Here, the maximization is over all the joint distributions of the transmitted signals and Λ_D represents the set of Ω for all possible cuts. Since QMF employs a random coding strategy with transmitted signals drawn from a Gaussian distribution, x_i are uncorrelated, and hence independent. We assume each $x_i \sim$ i.i.d. $\mathcal{CN}(0,P)$. It has specifically been proved in [1] that the gap between the cut-set bound and the rate from the i.i.d. Gaussian assumption is a constant, and hence becomes insignificant for high SNR analysis.

A. Rate in terms of MIMO capacity

Defining R_Ω as $I(x_\Omega; y_{\Omega_c}|x_{\Omega_c})$ and taking the transmitted signals $x_i \sim$ i.i.d. $\mathcal{CN}(0,P)$, we have,

$$R_\Omega = I(x_\Omega; y_{\Omega_c}|x_{\Omega_c}) = I(x_\Omega; y_{\Omega_c}) \tag{6}$$

The above expression says that we can calculate R_Ω for each cut by treating the network as a MIMO channel with $|\Omega|$ transmitters and $|\Omega_c|$ receivers. This can be justified as follows,

$$\begin{aligned} I(x_\Omega; y_{\Omega_c}|x_{\Omega_c}) &= H(x_\Omega|x_{\Omega_c}) - H(x_\Omega|y_{\Omega_c}, x_{\Omega_c}) \\ &\overset{(a)}{=} H(x_\Omega) - H(x_\Omega|y_{\Omega_c}, x_{\Omega_c}) \\ &\overset{(b)}{=} H(x_\Omega) - H(x_\Omega|y_{\Omega_c}) \\ &= I(x_\Omega; y_{\Omega_c}) \end{aligned}$$

Step (a) follows from the independence of x_Ω and x_{Ω_c}, and (b) from the fact that $x_\Omega \to y_{\Omega_c} \to x_{\Omega_c}$ forms a Markov chain. Using the result for MIMO channels in [2] and Equation 5, 6, we can then approximate $\bar{C}$ as $\min_{\Omega\in\Lambda_D} R_\Omega \approxeq (\min_{\Omega\in\Lambda_D} \min\{|\Omega|, |\Omega_c|\}) \log(\text{SNR})$ for large SNR. These rates are achieved reliably, i.e., with diminished small error probability, only when we are willing to communicate over large blocks of time.

B. Lower Bound on the Probability of Error

We now restrict communication to blocks of length T. We are interested in finding the probability that a message u_i sent by S is confused for another message $u_j \neq u_i$. The message consists of RT bits, and the corresponding code word consists of T channel uses. Thus, if $\mathbf{y_D}$ represents the received signal over T channel uses and $\hat{U}$, the decoding function, we are interested in $P(\hat{U}(\mathbf{y_D}(u_i)) \neq u_i)$. Define P_{e_Ω} as the probability of error for a MIMO system corresponding to a cut Ω. The error probability P_e corresponding to the relay network can only be greater than each P_{e_Ω}. This is because while the nodes (in sets Ω and Ω_c) in the relay network in each cut are constrained to communicate in a particular manner, the nodes on a MIMO transmitter or a MIMO receiver can completely co-operate and hence, communicate in any manner they want. Additionally, there is also some outage associated with the each channel in the relay network, which gets eliminated when we make the nodes in each set one entity. Let $d_\Omega(r)$ represent the diversity gain, and r_Ω the multiplexing gain, for the MIMO system corresponding to the cut Ω.

$$P_e \geq \max_{\Omega \in \Lambda_D} P_{e_\Omega} \doteq \max_{\Omega \in \Lambda_D} \text{SNR}^{-d_\Omega(r)} \tag{7}$$

Consider the network in Figure 1. There are 4 possible cuts, corresponding to 4 different MIMO channels – $1 \times 2, 2 \times 2, 2 \times 2$ and 2×1. If we constrain T by the length of the code corresponding to the optimal DMT of the min-cut (1×2 or 2×1), we have $T = 2$. For this length, the error probability is dominated by the min-cut (1×2 or 2×1). For a multiplexing gain $r \leq 1$, $\max_\Omega d(r) = (1-r)(2-r)$. The 2×2 MIMO channels always perform better than the other 2 channels and hence, never play a role in the trade off obtained from Equation 7.

Motivated by this example, let us study what happens when we constrain the communication time for a general layered network by the length corresponding to the min-cut. Let Ω_i and Ω_{i_c} represent various cuts of the network. Let $|\Omega_i| = m_i$ and $|\Omega_{i_c}| = n_i$. Let $i = 0$ correspond to the min-cut as in Section III-A. This implies that $\min\{m_0, n_0\} \leq \min\{m_i, n_i\} \; \forall \; i$. Let $i = 1$ correspond to the cut with minimum $m_i n_i$ so that $m_1 n_1 \leq m_i n_i \; \forall \; i$. Represent other cuts by other values of i. If we choose to transmit at $r \log(\text{SNR})$ using code words of length $T = m_0 + n_0 - 1$, how does the dominating term in the expression for error probability (Equation 7) vary? To study this, let us consider the following 2 cases – $m_i n_i \leq m_0 n_0$ and $m_i n_i \geq m_0 n_0$.

Claim: If $m_i n_i \leq m_0 n_0$, then $l_i = m_i + n_i - 1 \leq l_0 = m_0 + n_0 - 1$. Hence, with $T = l_0$, the DMT is achieved for each of these i cuts. Therefore, if the dominating term in Equation 7 comes from one of these terms, then the diversity gain is exactly equal to $(m_i - r)(n_i - r)$.

Justification: Assume that $l_i \geq l_0$. Without loss of generality, we can assume that $n_0 < m_0$. We have,

$$m_i n_i \leq m_0 n_0 \text{ and } m_i + n_i > m_0 + n_0$$

Assume $n_0 = 1$. If $n_0 \neq 1$, we can always make it 1 by dividing first expression by n_0^2 and the second by n_0. Therefore, $m_i, n_i, m_0 \geq 1$. Substituting $n_0 = 1$,

$$\begin{gathered} n_i \leq \frac{m_0}{m_i} \\ m_i + n_i \leq m_i + \frac{m_0}{m_i} \\ \therefore m_i + \frac{m_0}{m_i} > m_0 + 1 \\ \therefore m_i^2 - (m_0+1)m_i + m_0 > 0 \end{gathered}$$

Solving the above quadratic, we get $m_i < 1$ or $m_i > m_0$. Discarding the first solution since it contradicts our assumption, we have $n_i \leq \frac{m_0}{m_i} < 1$. This again contradicts our assumption. Hence, we cannot have $l_i > l_0 \Rightarrow l_i \leq l_0$.

Claim: If $m_i n_i \geq m_0 n_0$, and if we choose $T = m_0 + n_0 - 1$, then the dominating term in the expression for error probability in Equation 7, or the least diversity gain corresponds to the min-cut, i.e., $d_{\Omega_i}(r) \geq d_{\Omega_0}(r)$.

Justification: If $m_0 = n_0$, we can always build a $m_0 \times n_0$ MIMO system from the $m_i \times n_i$ system, since $m_i, n_i \geq n_0$. WLOG, take $m_0 > n_0$, and $m_0 + n_0 = a$. If $m_i + n_i = b \leq a$,

$$\begin{aligned} m_i n_i &\geq m_0 n_0 \\ m_i + n + i &\leq m_0 + n_0 \\ \therefore m_i n_i - (m_i + n_i) r &\geq m_0 n_0 - (m_0 + n_0) r \\ \therefore (m_i - r)(n_i - r) &\geq (m_0 - 2)(n_0 - r) \end{aligned}$$

Hence, if we set $T = a$, the term corresponding to the min-cut dominates. Now, if $b \geq a$, we can always take a $b' = a$, such that, $m'_i \leq m_i$, $n'_i \leq n_i$ (consider only m'_i transmitter nodes in Ω_i and n'_i receiver nodes in Ω_{c_i}), $m'_i, n'_i \geq n_0$ and $m'_i n'_i \geq m_0 n_0$. This is because we assumed $n_0 < m_0$, and the fact that $x(a-x)$ increases as we increase x towards $\frac{x}{2}$. The case now reduces to the previous case of $b \leq a$, and hence, the term corresponding to the min-cut dominates.

Result: From the above 2 claims, we see that if we transmit at $r \log(\text{SNR})$ and choose $T = m_0 + n_0 - 1$, then $\min_{\Omega_i \in \Lambda_D} d_{\Omega_i}(r)$ is exactly equal to $\min_{\Omega_i \in \Lambda_D} (m_i - r)(n_i - r)$. This worst cut, in terms of error probability, comes from the set of cuts with maximum diversity gain $m_i n_i \leq m_0 n_0$. The maximum multiplexing gain for the complete relay network is constrained to $\min\{m_0, n_0\}$, while the maximum diversity is constrained to $m_1 n_1$.

This agrees with the intuition that if $m_i + n_i - 1$ is large for any MIMO link, with large diversity and multiplexing gain, the system will do at least as good as a system with a lower diversity and multiplexing gain, when smaller code lengths are used. If a system has a low diversity gain but a high multiplexing gain, then the code length required to achieve the corresponding DMT is small, compared to a system with high diversity gain but low multiplexing gain.

IV. Future Directions and Conclusions

We derived a lower bound on the error probability in a relay network, by treating each cut as a MIMO link, and finding the worst such MIMO link. This is motivated by the cut-set bound achieving property of the QMF strategy in the high SNR regime. Modeling the cuts as MIMO links, to get a bound on the error probability is motivated from random coding and a similar modeling to get the capacity bound for the relay network. With tranmissions of length $m_0 + n_0 - 1$, the error probability is lower bounded by optimal DMT for the worst cut. The worst cut comes from the set of cuts which have maximum diversity gain $m_i n_i$ less than $m_0 n_0$.

We can see if there is a better lower bound on the error probability. Also, a lower bound, without an upper bound, is not very meaningful. The authors of [1] have found a capacity bound for the QMF strategy by finding the mutual information between the transmitted and the quantized, received signal. They have done this by writing the total mutual information term as the mutual information expression for a deterministic network (conditioning the noise), plus an entropy term which considers the effect of noise. The mutual information for the relay network, conditioned on noise, comes from an error analysis for deterministic networks. With codes of finite block lengths, we cannot apply same analysis to upper bound the error probability for Gaussian relay networks, since we need to consider the effect of noise. The upper bound should include the effect of both, the noise, and the interference.

One way to get an upper bound is to treat the cut as a set of MISO systems – the nodes in Ω represent a single transmitter, as in our analysis of lower bound. However instead of taking all nodes in Ω_c as one receiver with multiple antennas, we can take each node in Ω_c as a single receiver. Assuming no error at the nodes in Ω, we can find the probability of error at all nodes in Ω_c. We can find the probability of such an event by treating the transmissions from nodes in Ω_c as noise. We can than add up such events corresponding to all valid cuts. However, the bound from this may be very weak. The upper and lower bounds should be tight so that we have an approximate expression for the error probability.

Another question that can be considered, is finding a bound on probability when we choose $T \leq m_0 + n_0 - 1$. How does the DMT change, when the codes have lengths less than $m_i + n_i - 1$ for some i.

Also, can we say something specific about the minimum d, more than what is mentioned in the claims? For example, we can show how the diversity gain $d(r)$ transitions occur as the multiplexing gain r is varied. Does the term with least $m_i n_i$ always dominate d, or is the dominance only for small values of r? We can study this specifically for a few realizations of relay networks and compare the analytical lower bound to the bound obtained from QMF scheme with finite block lengths, through simulations.

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