An Overview of Floer Homologies

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Last updated: July 26, 2025

These are the course notes of MATH 286 - Topics in Differential Geometry (Winter 2022) at Stanford University, taught by Ciprian Manolescu.

Contents

Introduction

Course description

For closed manifolds, Morse homology is a way of computing the homology of the manifold by using the critical points and gradient flow lines of a Morse-Smale function. Floer homology is an infinite dimensional version of Morse homology. There are several versions of Floer homology, some used in symplectic geometry and some in low dimensional topology. They are important tools in both fields: the applications of Floer homology include the Arnol'd conjecture, the Weinstein conjecture, property P for knots, and Gordon's conjecture.

This course will give an overview of the different Floer homologies, explaining what they have in common and sketching some of their applications. The topics of the course might include:

- Morse theory and Morse homology;
- Hamiltonian Floer homology and the Arnold conjecture;
- Lagrangian Floer homology;
- contact homology and embedded contact homology;
- instanton (Yang–Mills) Floer homology;
- monopole (Seiberg–Witten) Floer homology;
- Heegaard Floer homology;
- symplectic instanton homology;
- symplectic Khovanov homology.

From Morse theory to Floer homology (Lecture 1)

We describe what will appear in our course very sketchily.

Morse homology assigns a homology group $HM_*(M, f)$ for a smooth manifold X with a Morse–Smale function $f\colon X \to \mathbb{R}$. It is the homology of the *Morse chain complex* $CM_*(f)$, which is defined by the following data:

- generators: critical points of f;
- coefficients: often $\mathbb{Z}$ in the oriented case, but can also others;
- differentials:
$$\partial x = \sum_{\mathrm{ind}(x)-\mathrm{ind}(y)=1} n_{xy} \cdot y,$$
  where n_{xy} counts flow lines
$$\dot{\gamma} = -\nabla f \circ \gamma$$

from x to y.

Under proper assumptions, the Morse homology is isomorphic of the singular homology of M.

Floer homology is an infinite-dimensional version of Morse homology. It typically produces Morse theory on an infinite dimensional space $\mathcal{B}$, called the *configuration space*, e.g. $C^\infty(X, Y)$, where X, Y are smooth manifolds, or $C^\infty(X, E)$, where E is a smooth vector bundle over X, with some suitable f to obtain Floer homology $HF_*(\mathcal{B}, f)$, which is in general not isomorphic to the singular homology of $\mathcal{B}$.

Floer theory appears in different places, but basically two branches: symplectic (and contact) geometry, and low-dimensional topology. There are several types of Floer homologies of different flavours. We will discuss them separately. For a general overview, see [?abbondandolo2019floer] for Floer theory in symplectic topology, and [?manolescu2015floer] for Floer theory in low-dimensional topology.

Floer theory in symplectic and contact geometry

Hamiltonian Floer homology

Let (M^{2n}, ω) be a symplectic manifold, which means ω is a closed 2-form on M satisfying ω^n is nowhere vanishing. Let $H_t \colon M \to \mathbb{R}$ be a time-dependent Hamiltonian, where $t \in \mathbb{R}/\mathbb{Z}$. For this, we can define the Hamiltonian vector field by

$$dH_t = \iota_{X_t}\omega.$$

The *Hamiltonian flow* is given by the integral curve of X_t such that $\phi_t \colon M \to M$ is a diffeomorphism for all t, and that $\phi_0 = \mathrm{id}$.

The Hamiltonian Floer homology was the first flavour of Floer homologies, defined by Floer [?floer1988morse]. The configuration space is given by

$$\mathcal{B} \subset C^\infty(S^1, M)$$

containing all null-homotopic loops. On $\mathcal{B}$ there is a functional $f \colon \mathcal{B} \to \mathbb{R}/d\mathbb{Z}$,

$$f(\gamma) = -\int_D \omega + \int_0^1 H_t(\gamma(t))\, dt.$$

Here D is a compressing disk for γ. We will see that critical points of f are time 1 periodic orbits of the Hamiltonian flow, and the flowlines are given by holomorphic cylinders between such orbits. We can then form the *Hamiltonian Floer homology* $HF_*(M, \omega, H_t)$.

Theorem 0.1. *Under certain assumptions, we have*

$$HF_*(M, \omega, H_t) \cong H_*(M),$$

the Morse homology of M.

This leads to a proof of the famous Arnold conjecture:

Theorem 0.2 (Arnold conjecture)**.** *The number of periodic orbits of (M, ω, H_t) is as least as the sum of Betti numbers of M.*

Lagrangian Floer homology

Let us keep the convention that (M^{2n}, ω) is a symplectic manifold. A submanifold L is said to be *Lagrangian* if it has dimension n and satisfies $\omega|_L = 0$. The study of Lagrangians is an important part of modern symplectic geometry.

Given two Lagrangians L_0 and L_1 that intersect transversely, we can form the configuration space as the path space

$$\mathcal{B} = P(L_0, L_1) = \{\gamma\colon [0,1] \to M\colon \gamma(0) \in L_0,\ \gamma(1) \in L_1\},$$

and f is some kind of "area functional" on $\mathcal{B}$. It is defined similar to the case of Hamiltonian Floer homology with $H_t = 0$. We will see that critical points of f are just intersections of L_0 and L_1, and the flowlines correspond to the holomorphic disks between two intersections. From this, we will produce the *Lagrangian Floer homology* $HF_*(L_0, L_1)$.

Theorem 0.3. *Under certain assumptions, we have*

$$HF_*(L_0, L_1) = HF_*(L_0, \psi(L_1)),$$

where ψ is a Hamiltonian transformation on M.

Theorem 0.4. *Under certain assumptions, we have*

$$HF_*(L, L; \mathbb{Z}/2\mathbb{Z}) \cong H_*(L; \mathbb{Z}/2\mathbb{Z}).$$

As a corollary, we can show the Arnold conjecture for Lagrangian intersections in good cases.

Conjecture 0.5 (Arnold–Givental conjecture)**.** *Let L be a Lagrangian, and ψ be a Hamiltonian transformation on M. Assume that L intersects $\psi(L)$ transversely. Then*

$$\#(L \cap \psi(L)) \geqslant \operatorname{rank} H_*(L; \mathbb{Z}/2\mathbb{Z}).$$

Another application of Lagrangian Floer homology is due to Gromov:

Theorem 0.6 (Gromov, [?gromov1985pseudo])**.** *There does not exist compact exact Lagrangians in $\mathbb{R}^{2n}$ with standard symplectic structure.*

The idea of the proof is that we can translate Lagrangians in the Euclidean space such that it doesn't intersect the original one, and then use it to compute the Lagrangian Floer homology.

Contact homology

Contact geometry is the odd-dimensional analog of symplectic geometry. A *contact manifold* is a pair (M^{2n-1}, α), where α is a 1-form on M such that $\alpha \wedge (d\alpha)^{n-1}$ is nowhere vanishing. Given a contact form α, we can talk about the *Reeb field* on M, which is a vector field R_α determined by the condition

$$\begin{cases} \iota_{R_\alpha} d\alpha = 0, \\ \iota_{R_\alpha} \alpha = 1. \end{cases}$$

As the Hamiltonian vector field, Reeb field also produces a flow on M. We can consider

$$\mathcal{B} = C^\infty(S^1, M)$$

and

$$f\colon \mathcal{B} \to \mathbb{R},\ f(\gamma) = \int_\gamma \alpha.$$

The critical points of f are closed Reeb orbits, and flowlines correspond to holomorphic cylinders in the symplectization $(M \times \mathbb{R}, d(e^t\alpha))$. From this, we can form the *(cylindrical) contact homology* $HC_*^{cyl}(M, \alpha)$.

There are many generalization of this construction. The *contact homology* $HC_*(M, \alpha)$ is defined as the homology of a chain complex generated by unions of Reeb orbits, and the differential counts holomorphic curves looking as in the left of Figure 1. A more general construction is the *symplectic field theory*, whose differential counts general holomorphic curves looking as in the right of Figure 1.

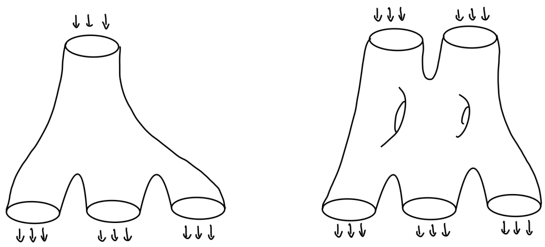

Figure 1: Holomorphic curve counting in the differentials of contact homology (left) and symplectic field theory (right).

One application is the following:

Theorem 0.7 (Ustilovsky, [?Ustilovsky1999InfinitelyMC]). *The sphere S^{4k+1} has infinitely many contact structures with the same contact plane field $\xi = \ker \alpha$ (up to smooth isotopy).*

One variation of contact homology in dimension 3 is the *embedded contact homology*, due to Hutchings–Sullivan [?hutchings2002index, ?Hutchings2004RoundingCO, ?Hutchings2004ThePF, ?jsg/1197491304, ?hutchings2009embedded]. Let (Y, α) be a closed contact 3-manifold, we can use the count of *embedded* holomorphic curves *of any genus* in the symplectization to define a differential, and produce the embedded contact homology $ECH(Y, \alpha)$. It is in fact independent to the choice of α, and is always nonzero. Using this, one can resolve the following:

Theorem 0.8 (Weinstein conjecture in dimension 3, Taubes [?taubes2007seiberg]). *There exists at least one closed Reeb orbit for any closed contact 3-manifold (Y, α).*

Floer theory in low-dimensional topology

Instanton (Yang–Mills) Floer homology

Instanton Floer homology was originally defined by Floer in 1988. Let Y^3 be a closed, oriented, smooth 3-manifold, and let $E \to Y$ be an SU(2)-bundle, or equivalently, a rank 2 complex vector bundle with Hermitian metric. The configuration space is

$$\mathcal{B} = \{(\text{nontrivial})\, \mathrm{SU}(2)\text{-connections on } E\} / \operatorname{Aut}(E).$$

Here the space of nontrivial SU(2) connections can be identified with 1-forms valued in Lie algebra $\Omega^1(Y; \mathfrak{su}(2))$, and $\operatorname{Aut}(E)$ is called the *gauge group*. The *Chern–Simons functional cs* on $\mathcal{B}$ is a $\mathbb{R}/\mathbb{Z}$-valued function defined by

$$cs(A) = \frac{1}{8\pi^2} \int_Y \operatorname{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$

The critical points of cs correspond to the nontrivial SU(2)-representation of $\pi_1(Y)$ (up to conjugacy). The gradient flow of cs corresponds to solutions of the Yang–Mills equation on $\mathbb{R} \times Y$. These define the instanton Floer homology group $I_*(Y)$.

There is a variant of $I_*(Y)$ defined for a pair (Y, K), where K is a knot in Y. The *singular instanton Floer homology* $I_*(Y, K)$ is defined by considering connections singular along K.

The most famous application of instanton Floer homology is the proof of property P for knots:

Theorem 0.9. *Let K be a knot in S^3. If $\pi_1(S^3_r(K)) = 1$, then K is the unknot.*

The idea of proof is to show that $I_*(Y) \neq 0$, where $Y = S^3_r(K)$ and K is not the unknot, to imply that the fundamental group has a nontrivial SU(2)-representation.

Monopole (Seiberg–Witten) Floer homology

Monopole Floer homology is another gauge-theoretic invariant for 3-manifolds. It has several versions, defined by Kronheimer–Mrowka [?Kronheimer2007MonopolesAT], Marcolli–Wang [?marcolli2001equivariant], and Manolescu [?manolescu2003seiberg]. Let Y^3 be a closed, oriented, smooth 3-manifold, and $S \to Y$ is a *spinor bundle* over Y. Let

$$\mathcal{B} = (\{U(1)\text{-connections on } S\} \oplus \Gamma(S)) \,/\text{gauge},$$

and f be the *Chern–Simons–Dirac functional*. The critical points of f correspond to solutions to the Seiberg–Witten equations on Y, and flowlines correspond to the Seiberg–Witten equations on $Y \times \mathbb{R}$. It outputs the *monopole Floer homology* $HM_*(Y)$.

One of the most famous application of monopole Floer homology is the Gordon conjecture:

Theorem 0.10 (Kronheimer–Mrowka–Ozsváth–Szabó [?kronheimer2007monopoles]). *Let $K \subset S^3$ be a knot, and let U be the unknot. If there is an orientation-preserving diffeomorphism*

$$S^3_r(K) \cong S^3_r(U)$$

for some rational number r, then $K = U$.

Another important application is a negative answer for the triangulation conjecture:

Theorem 0.11 (Manolescu [?manolescu2016pin]). *For every integer $n \geqslant 5$, there is a non-triangulable topological manifold of dimension n.*

Heegaard Floer homology

We continue considering a closed, oriented, smooth 3-manifold Y. It admits a *Heegaard splitting*, i.e. we can write

$$Y = U_0 \cup_{\Sigma_g} U_1,$$

where U_i are handlebodies of genus g. The symmetric product

$$M = \mathrm{Sym}^g(\Sigma) = \Sigma^g / S_g$$

has a symplectic structure, and the data of U_i gives us two Lagrangians T_α, T_β in M. Following Ozsváth–Szabó [?ozsvath2004holomorphic,?ozsvath2004holomorphic1], we then define the *Heegaard Floer homology* $HF_*(Y)$ as the Lagrangian Floer homology of T_α, T_β in M.

More or less surprisingly, three theories we have mentioned up to now are actually the same!

Theorem 0.12 (Taubes, Kutluhan–Lee–Taubes, Colin–Ghiggini–Honda). *We have*

$$HF_*(Y) = HM_*(Y) = ECH(Y, \alpha).$$

One variation is the *knot Floer homology* $HFK(Y, K)$ [?ozsvath2004holomorphic2, ?rasmussen2003floer], defined for knot $K \subset Y$. Knot Floer homology detects the unknot:

Theorem 0.13 (Ozsváth–Szabó [?ozsvath2004holomorphic3]). *If*

$$HFK(S^3, K) = HFK(S^3, U),$$

then $K = U$.

Comparing with other Floer-theoretic invariants, Heegaard Floer homology (knot Floer homology) might be the easiest to calculate, which makes the research very active.

Symplectic instanton homology

From the Heegaard splitting $Y = U_0 \cup_\Sigma U_1$, one can produce the Lagrangian Floer homology in another way. Let

$$M = \{\text{representations } \pi_1(\Sigma) \to \mathrm{SU}(2)\} / \mathrm{SU}(2),$$

and

$$L_i = \{\text{representations } \pi_1(U_i) \to \mathrm{SU}(2)\} / \mathrm{SU}(2).$$

We expect to define the *symplectic instanton homology* as the Lagrangian Floer homology of L_0 and L_1 in M. The name of this construction comes from the following conjecture:

Conjecture 0.14 (Atiyah–Floer conjecture, [?atiyah1988new]). *The symplectic instanton homology is isomorphic to the instanton homology of Y.*

It has been partially proved by Daemi–Fukaya–Lipyanskiy [?daemi2021lagrangians] in 2021.

Symplectic Khovanov homology

The last invariant we want to mention is the symplectic Khovanov homology, due to Seidel–Smith [?Seidel2004ALI]. Starting from a knot $K \subset S^3$ in a braid position, as in Figure 2. For each $t \in [0, 1]$, the slice gives $2n$ points on

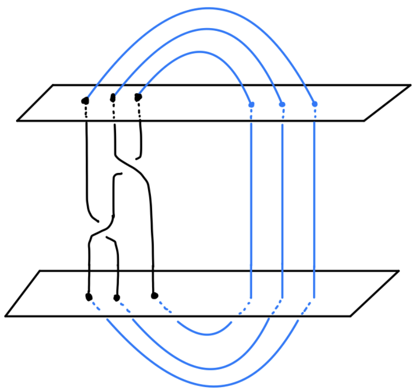

Figure 2: A braid position of the unknot

$\mathbb{R}^2$. The phases at $t = 0$ and 1 are the same, namely $2n$ points $z_1, z_2, \ldots, z_2 n$. Consider the space

$$S = \{u^2 + v^2 + \prod_{k=1}^{2n} (z - z_k) = 0\} \subset \mathbb{C}^3.$$

There is a distinguished Lagrangian L in an open set $Y_n \in \mathrm{Hilb}^n(S)$, the Hilbert scheme of n points in S. The data of the braid gives us a diffeomorphism β

on Y_n, and the symplectic Khovanov homology $\mathrm{Kh}^{symp}(K)$is defined as the Lagrangian Floer homology of L and $\beta(L)$ in Y_n. While the definition involves complicated symplectic geometry stuff, it actually coincides with the ordinary Khovanov homology, which is defined in a combinatorial way.

Theorem 0.15 (Abouzaid–Smith, [?abouzaid2019khovanov]). *We have*

$$\mathrm{Kh}^{symp}(K;\mathbb{Q}) \cong \mathrm{Kh}(K;\mathbb{Q}).$$

1 Morse theory and Morse homology

1.1 Classical Morse theory (lecture 2)

Let us begin with classical Morse theory. Roughly speaking, it studies properties of manifolds using "nice" smooth functions on them.

Definition 1.1. Let X be a closed, smooth n-manifold. A smooth function F is said to be *Morse* if every critical point p of f is non-degenerated, i.e.

$$\ker \operatorname{Hess}_p f = 0.$$

The following theorem claims that Morse functions are "generic":

Theorem 1.2. *Every smooth function $f \in C^\infty(X)$ can be approximated by Morse functions.*

Given that f is Morse, at each critical point p of f, we can decompose the tangent space T_pX as a direct sum of its eigenspaces of positive (negative) eigenvalue:

$$T_pX = T_p^+X \oplus T_p^-X.$$

The *index* of p is the dimension of the negative eigenspace T_p^-X. The local behaviour of Morse functions is determined by its index.

Theorem 1.3 (Morse lemma)**.** *Let p be a critical point of a Morse function f with index k. Then there are local coordinates $x_1, x_2, \ldots, x_n$ near p, such that locally f can be written as*

$$f(x) = f(p) - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2.$$

The importance of critical points is that the topology of X only changes when we cross critical points. More precisely:

Theorem 1.4. *Let*

$$X_c = f^{-1}(-\infty, c].$$

Let $c_1 < c_2$ be two real numbers.

- *If there is no critical value in $[c_1, c_2]$, then X_{c_1} and X_{c_2} are diffeomorphic.*
- *If there is exactly one critical value in $[c_1, c_2]$, namely $c \in (c_1, c_2)$, assume in further that there is exactly one critical point p such that $c = f(p)$. Then X_{c_2} is diffeomorphic to X_{c_1} with one k-handle attached, which means*

$$X_{c_2} \cong X_{c_1} \bigcup_{\partial D^k \times D^{n-k}} \left(D^k \times D^{n-k}\right).$$

See Figure 3 for a standard example. Recall that handles are the thickened analogues of cells. This theorem says that we can obtain a CW structure on the manifold from a Morse function on it.

As a corollary, we have the Morse inequality.

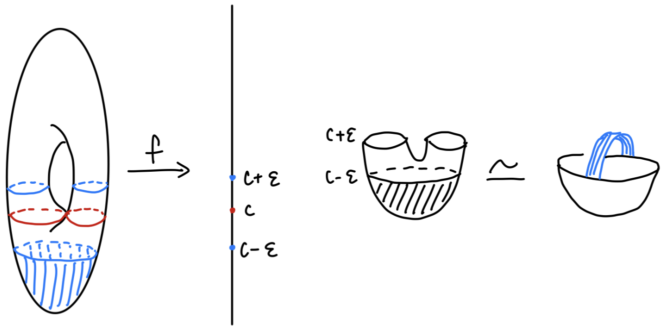

Figure 3: Passing an index 1 critical point of a Morse function on the torus leads to a two dimensional 1-handle attachment.

Corollary 1.5. *Let c_k be the number of critical points of f with index k. Let b_k be the k-th Betti number of X. Then for each integer $k \geqslant 0$, we have*

$$\sum_{l=0}^{k} (-1)^l c_{k-l} \geqslant \sum_{l=0}^{k} (-1)^l b_{k-l}.$$

Proof. The Morse function f gives CW structure on X. Consider the k-skeleton $X^{(k)}$ of X. That is, the union of all the cells with dimension at most k. Then we have

$$\sum_{l=0}^{k} (-1)^l c_{k-l} = (-1)^k \chi(X^{(k)}) = \sum_{l=0}^{k} (-1)^{k-l} \operatorname{rank} H_l(X^{(k)}).$$

On the other hand,

$$\sum_{l=0}^{k} (-1)^l b_{k-l} = \sum_{l=0}^{k} (-1)^{k-l} \operatorname{rank} H_l(X).$$

From the cellular homology, we know that

$$\operatorname{rank} H_l(X^{(k)}) = H_l(X)$$

for $l < k$, and $H_k(X^{(k)}) \geqslant H_k(X)$. The result follows. □

1.2 Morse homology

The idea of Morse homology originated from Milnor [?milnor2015lectures]. In 1980s, Witten wrote it in its modern form. Possible references for Morse homology include [?schwarz1993morse,?banyaga2004lectures,?audin2014morse].

We will focus on the part that can be generalized to infinite dimensions, i.e. the setting of Floer theories.

To define Morse homology, we first need to formulate the Morse–Smale condition. Fix a Riemannian metric g on X. We can make sense of the *gradient flow* of f. The (negative) gradient flowline equation for a curve $\gamma\colon \mathbb{R} \to X$ is

$$\gamma'(t) = -\nabla f(\gamma(t)).$$

It generates a flow $\phi_t\colon X \to X$.

Definition 1.6. Let p be a critical point of f. The *stable* and *unstable manifold* are defined respectively as

$$W^s(p) = \{x \in X\colon \lim_{t\to+\infty} \phi_t(x) = p\},$$

and

$$W^u(p) = \{x \in X\colon \lim_{t\to-\infty} \phi_t(x) = p\}.$$

Example 1.7. Consider an index 1 critical point on a surface. Locally the Morse function can be written as $f = x^2 - y^2 + c$, and the flow looks as in Figure 4.

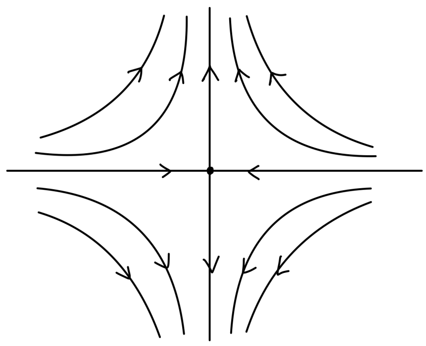

Figure 4: The local behaviour of a Morse function near an index 1 critical point.

Proposition 1.8. *We have*

$$W^s(p) \cong T_p^+(X),\, W^u(p) \cong T_p^-(X).$$

Definition 1.9. Let X be a closed, smooth n-manifold. A pair (f, g) is said to be *Morse–Smale* if under the Riemannian metric g, for all critical points p, q of f, we have $W^u(p)$ and $W^s(q)$ intersect transversely.

When fixing a metric g, we often say a function is Morse–Smale. We need to justify that the class of Morse–Smale functions is large enough to do perturbation. To make this precise, recall that a subset A in a complete metric space B is said to be *generic* if it contains a countable intersection of open dense sets. In particular, it is dense by Baire category theorem.

Theorem 1.10 (Kupka–Smale). *Let (X, g) be a Riemannian manifold. The class of Morse–Smale functions is generic in $C^\infty(X)$.*

Remark 1.11. One can also fix the function f and this is also true for generic g.

From now on we assume that (f, g) is Morse–Smale. The transversality condition tells us that

$$\mathcal{M}(p, q) = W^u(p) \cap W^s(q)$$

is a smooth manifold of dimension $\operatorname{ind} p - \operatorname{ind} q$. Unwrapping the definition, we see that $\mathcal{M}(p, q)$ is the union of flowlines from p to q. It carries a $\mathbb{R}$-action by translation. Hence we can consider the *unparametrized moduli space*

$$\widehat{\mathcal{M}}(p, q) = \mathcal{M}(p, q)/\mathbb{R}.$$

It is a smooth manifold of dimension

$$\dim \widehat{\mathcal{M}}(p, q) = \operatorname{ind} p - \operatorname{ind} q - 1.$$

Elements of this space are unparametrized flowlines, or *trajectories*.

An orientation on $W^u(p) \cong T_p^- X$ induces an orientation on $\widehat{\mathcal{M}}(p, q)$ since it also induces an orientation on $W^s(p) \cong T_p^+(X)$ by requiring the isomorphism

$$T_p^-(X) \oplus T_p^+(X) \cong T_p X$$

preserves orientations. In particular, when $\operatorname{ind} p = \operatorname{ind} q + 1$, $\widehat{\mathcal{M}}(p, q)$ is a compact, oriented, 0-manifold, i.e. points with signs. It gives a signed count

$$\#\widehat{\mathcal{M}}(p, q) \in \mathbb{Z}.$$

We can now define the Morse complex:

Definition 1.12. The *Morse complex* for X and a Morse–Smale pair (f, g) consists of the following data:

- for each integer k, an abelian group $CM_k(X, f, g)$, generated by index k critical points;
- for each integer k, a differential

  $$\partial \colon CM_k(X, f, g) \to CM_{k-1}(X, f, g),$$

  given by

  $$\partial p = \sum_q \#\widehat{\mathcal{M}}(p, q) \cdot q$$

  for each critical point p.

Remark 1.13. One can also do this in a more canonical way. Namely, we assign a rank 1 free abelian group Λ_p for each critical point p by

$$\Lambda_p = \langle \mathfrak{o}_1, \mathfrak{o}_2 \rangle / (\mathfrak{o}_2 = -\mathfrak{o}_1),$$

where $\mathfrak{o}_1, \mathfrak{o}_2$ are orientations of $T_p^- X$. We can then define the Morse complex as

$$CM_*(X, f, g) = \bigoplus_p \Lambda_p$$

and similarly define the differential.

While saying Morse *complex*, we have not really proved that it is a complex. The following theorem justifies this. It also shows that Morse homology is actually an invariant of X.

Theorem 1.14. *(a) The Morse complex $CM_*(X, f, g)$ is indeed a chain complex, i.e. we have $\partial^2 = 0$.*

(b) The homology $H_*(CM_*(X, f, g))$ is independent of the choice of (f, g).

(c) There is a natural isomorphism

$$H_*(CM_*(X, f, g)) \cong H_*(X).$$

Here (c) is specific to finite dimensional case, while (a) and (b) can be generalized to the Floer theoretic settings.

1.3 Invariance of Morse homology (lecture 3)

We sketch several proofs of Theorem 1.14, keeping an eye on whether these proofs can generalize to infinite dimensions.

The first proof

The first proof we sketch is from [?banyaga2004lectures], following Salamon. It is the most elementary one.

This proof relies on the following fact:

Proposition 1.15. *Let X be a topology space with a filtration*

$$\varnothing = U_{-1} \subset U_0 \subset U_1 \subset \cdots \subset X = \bigcup U_i.$$

Assume that $H_*(U_k, U_{k-1})$ is supported in dimension k. Let ∂ denote the composition

$$H_k(U_k, U_{k-1}) \xrightarrow{\delta} H_{k-1}(U_{k-1}) \to H_{k-1}(U_{k-1}, U_{k-2}),$$

where δ is the connecting homomorphism in the long exact sequence for pair (U_k, U_{k-1}). Then $\partial^2 = 0$, and the homology of the complex $(H_*(U_*, U_{*-1}), \partial)$ recovers $H_*(X)$.

It generalizes the machinery of cellular homology. We can apply this to

$$U_k = \{\phi_t(x) \colon t \geqslant 0,\ x \text{ is in a neighbourhood of a critical point } p \text{ with } \operatorname{ind} p \geqslant k\}.$$

It is clear that $H_*(U_k, U_{k-1})$ is a free abelian group supported in degree k, generated by critical points of index k. We can show that ∂ counts $\#\widehat{\mathcal{M}}(p, q)$, and hence it gives the homology of X.

Unfortunately, this proof makes no sense in Floer theory settings because we cannot talk about "neighbourhood of p", neither the index of p.

Proof from Audin–Damian [?audin2014morse]

The idea of the second proof is to compactify the moduli spaces of flowlines. To this end, we need to work with *pseudo-gradient*, which is vector field X that equals to the negative gradient near critical points of f, and such that $df_p(X_p) < 0$ elsewhere.

Theorem 1.16. *The unparametrized moduli space $\widehat{\mathcal{M}}(p,q)$ can be compactified by broken flowlines:*

$$\overline{\widehat{\mathcal{M}}(p,q)} = \coprod_{l \geqslant 0} \coprod_{r_1,\ldots,r_l \in \mathrm{Cr}(f)} \left(\widehat{\mathcal{M}}(p,r_1) \times \widehat{\mathcal{M}}(r_1,r_2) \times \cdots \times \widehat{\mathcal{M}}(r_l,q) \right).$$

In general, $\overline{\widehat{\mathcal{M}}(p,q)}$ is a manifold with corners. The codimension 1 stratum is given by

$$\coprod_r \widehat{\mathcal{M}}(p,r) \times \widehat{\mathcal{M}}(r,q),$$

as showed in Figure 5.

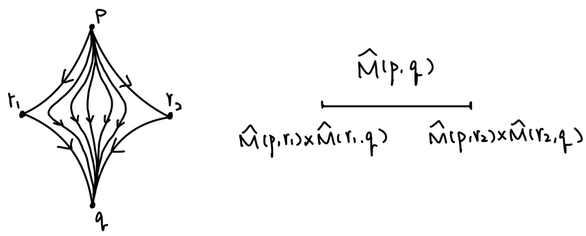

Figure 5: The compactified moduli space of flowlines.

The topology of $\overline{\widehat{\mathcal{M}}(p,q)}$ is given by looking at intersections with the boundaries of standard neighbourhoods of cirtical points. Using the compactness of boundaries of these neighbourhoods, we can show that $\overline{\widehat{\mathcal{M}}(p,q)}$ is indeed compact. Notice that this doesn't work in infinite dimensions since the unit sphere of an infinite dimensional Hilbert space is not compact! To show this gives a manifold with corners, we also need a *gluing theorem*, which says that unbroken trajectories can be arbitrarily close to every fixed broken trajectory.

Now the proof of part (a) in Theorem 1.14 is clear. Let p be a critical point, and q be a critical point of index $\operatorname{ind} p - 2$. The coefficient of q appearing in $\partial^2 p$ is

$$\sum_{\operatorname{ind} r = \operatorname{ind} p - 1} \#\widehat{\mathcal{M}}(p,r)\#\widehat{\mathcal{M}}(r,q) = \#\left(\coprod_r \widehat{\mathcal{M}}(p,r) \times \widehat{\mathcal{M}}(r,q) \right).$$

By the compactification, we know that

$$\coprod_r \widehat{\mathcal{M}}(p,r) \times \widehat{\mathcal{M}}(r,q) = \partial \left(\overline{\widehat{\mathcal{M}}(p,q)} \right),$$

and the count is zero since it appears as the boundary of a 1-dimensional manifold.

To prove part (b), given two Morse–Smale pairs (f_0, g_0) and (f_1, g_1), we can choose a family of Morse–Smale pairs (f_t, g_t) $(t \in [0,1])$ connecting these two pairs, by the genericity of Morse–Smale functions, and extend to (f_t, g_t) $(t \in \mathbb{R})$ by constants. We construct a map

$$F \colon CM_*(X, f_0, g_0) \to CM_*(X, f_1, g_1)$$

as follows. Let p, q be critical points of f_0 and f_1 respectively. Consider

$$\mathcal{N}(p,q) = \{\gamma \colon \mathbb{R} \to X \colon \gamma'(t) = -\nabla f_t(\gamma(t)),\, \gamma(-\infty) = p,\, \gamma(+\infty) = q\}.$$

Unlike the previous $\mathcal{M}(p,q)$, now we don't have a $\mathbb{R}$-action. The space $\mathcal{N}(p,q)$ is an oriented smooth manifold of dimension $\operatorname{ind} p - \operatorname{ind} q$. It can be compactified with boundary

$$\partial\overline{\mathcal{N}(p,q)} = \left(\bigcup_r \left(\widehat{\mathcal{M}}_{f_0}(p,r) \times \mathcal{N}(r,q)\right)\right) \bigcup \left(\bigcup_r \left(\mathcal{N}(p,r) \times \widehat{\mathcal{M}}_{f_1}(r,q)\right)\right).$$

The two components reflect two possibilities of the index decreasing that it can happen on either the $t = 0$ side or the $t = 1$ side. We define

$$F(p) = \sum_{\operatorname{ind} q = \operatorname{ind} p} \#\mathcal{N}(p,q) \cdot q.$$

Exercise 1.17. Show that

$$F\partial = \partial F,$$

i.e. F is a chain map, by counting points in $\partial\overline{\mathcal{N}(p,q)}$.

We construct a chain map

$$G \colon CM_*(X, f_1, g_1) \to CM_*(X, f_0, g_0)$$

in the same spirit. One can show that F, G give chain homotopy inverses to each other. Hence

$$F_* \colon H_*(CM_*(X, f_0, g_0)) \to H_*(CM_*(X, f_1, g_1))$$

gives an isomorphism on Morse homologies.

The proof of part (c) in Theorem 1.14 is not so interesting to us because the result doesn't even hold in infinite dimensions. Hence we only list some approach to show this.

- Check explicitly that the differential ∂ in our first proof coincides with the cellular map of the cellular chain complex.
- Show that the map
$$CM_*(X, f, g) \to C_*(X)$$
given by
$$p \mapsto [\overline{W^u(p)}]$$
is a chain homotopy equivalence.
- Verify Eilenberg–MacLane axioms hold for Morse homology.

Proof from Schwartz [?schwarz1993morse]

The idea of our second proof is great, but as we have seen, its proof techniques cannot be generalized to infinite dimensions directly. We now introduce an alternative proof for the compactification theorem, which is the most relevant to Floer theory.

Let $\gamma_n \in \widehat{\mathcal{M}}(p,q)$ be a sequence of flowlines. For each γ_n, we can consider its *energy*

$$E(\gamma_n) = \int_{\mathbb{R}} |\gamma_n'(t)|^2 \, dt.$$

It turns out that

$$E(\gamma_n) = \int_{\mathbb{R}} \langle \gamma_n', -\nabla f(\gamma_n) \rangle \, dt = \int_{\mathbb{R}} \frac{d}{dt}(-f(\gamma_n(t))) \, dt = f(p) - f(q).$$

The sequence

$$\int_{\mathbb{R}} |\gamma_n - \gamma_0|^2 \, dt$$

is also uniformly bounded. Hence the L_1^2-norm of γ_n is uniformly bounded. By Alaoglu's theorem, there is a subsequence γ_{n_k} converging weakly to γ in L_1^2, and hence also converging strongly in L_{loc}^2. Now looking back on the flowline equation

$$\gamma_n'(t) = -\nabla f(\gamma_n),$$

we can see γ_{n_k}' also strongly converges in L_{loc}^2. In other words, γ_{n_k} converges to γ in $L_{1,loc}^2$.

This is a typical step of *bootstrapping*: from a sequence γ_n that has uniformly bounded L_1^2-norms, we find a subsequence converging in $L_{1,loc}^2$. Repeating this process, by passing to subsequences, we can eventually find a sequence, still called γ_n for simplicity, that converges to γ in C_{loc}^∞. It does satisfy the flowline equation, but it is not necessarily a flowline from p to q. Nonetheless, we can show that it does converge to some intermediate flowlines. Using this, we can establish the compactness theorem again.

Theorem 1.18. *There exists critical points $r_1, r_2, \ldots, r_m$ and translations $\tau_{n,j} \in \mathbb{R}$, such that $\tau_{n,j}\gamma_n$ converges to a flowline from r_{j-1} to r_j.*

1.4 Conley index

The classical Morse homology only treats closed manifold, i.e. compact manifold without boundary. It is natural to ask what happens if X is not compact or has boundary. We first discuss the case that X doesn't have boundary but might be non-compact.

The issue here is that $CM_*(f,g)$ might not be a complex.

Example 1.19. Let $X = \mathbb{R}^2$. Consider a function with level set showed in Figure 6. There are three critical points p, q, and r. Here p is a local maximum, and r is a local minimum. There are exactly two flowlines, one from p to q, and the other from q to r. We thus have

$$\partial p = \pm q,\ \partial q = \pm r,\ \partial^2 p = \pm r \neq 0.$$

In this case, $CM_*(X, f, g)$ is not a complex. The problem here is that

$$\widehat{\mathcal{M}}(p, r) = (0, +\infty),$$

while the "compactification"

$$\widehat{\mathcal{M}}(p, r) \bigcup \left(\widehat{\mathcal{M}}(p, q) \times \widehat{\mathcal{M}}(q, r) \right) = [0, +\infty)$$

is not compact!

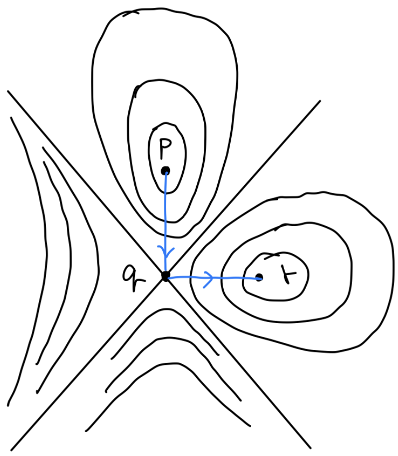

Figure 6: A function on $\mathbb{R}^2$. The black lines indicate level sets, and the blue lines are flowlines.

One way to resolve this is to consider only the good case. Let ϕ_t be the (partially-defined) negative gradient flow. Let

$$S = \{x \in X \colon \phi_t(x) \text{ exists for all } t,\ \phi_{\pm\infty}(x) \text{ are critical points of } f\}.$$

Assume that S is compact. Then the proof that $\partial^2 = 0$ still works, but now the question is that what is our homology $H_{Morse}(X, f, g)$?

Example 1.20. Consider $X = \mathbb{R}$, and flowlines are showed in Figure 7. In the first case, S is a point, and the Morse homology is $\mathbb{Z}$. In the second case, we have $S \cong [0, 1]$, and the Morse homology is zero.

This example indicates that the homology $H_{Morse}(X, f, g)$ is not necessarily the homology of X or S. To fix this, we need to take a compact neighbourhood of S together into account, which motivates the *Conley index* [?conley1978isolated].

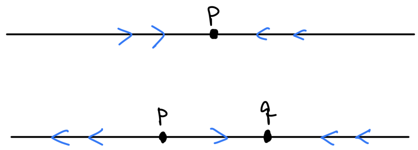

Figure 7: The non-compact manifold $\mathbb{R}$ with different flowlines.

Definition 1.21. Let ϕ_t be a flow on a smooth manifold X. For a compact subset $N \subset X$, the *invariant subset* of N is defined as

$$\text{Inv}(N, \phi) := \{x \in N \colon \phi_t(x) \in N, \text{ for all } t \in \mathbb{R}\}.$$

A compact subset $S \subset X$ is said to be an *isolated invariant set* if there exists a compact neighbourhood N such that $S \subset \text{int}\, N$, and that $S = \text{Inv}(N, \phi)$.

For example, let ϕ be the negative gradient flow, and S be as before. Then S is an isolated invariant set for any compact neighbourhood $N \supset S$.

Definition 1.22. Let S be an isolated invariant set for (X, ϕ_t). An *index pair* (N, L) for S is a pair of compact sets $L \subseteq N \subseteq X$, such that the following conditions hold.

(i) We have
$$\text{Inv}(N \backslash L, \phi) = S \subset \text{int}(N \backslash L).$$

(ii) L is an *exit set* for N, i.e. for any $x \in N$ and $t > 0$ with $\phi_t(x) \notin N$, there is a smaller $\tau \in [0, t)$ such that $\phi_\tau(x) \in L$.

(iii) L is *positive invariant* in N, i.e. for any $x \in L$ and $t > 0$ with $\phi_s(x) \in N$ for all $s \in [0, t]$, we have $\phi_s(x) \in L$ for all $s \in [0, t]$.

The following figure provides a schematic example and a counterexample.

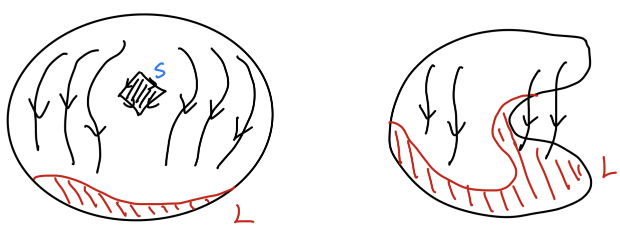

Figure 8: Left: a schematic example of an index pair. Right: L is an exit set but is not positive invariant.

Definition 1.23. The *Conley index* of S is the based homotopy type

$$I(\phi, S) = (N/L, [L]),$$

where (N, L) is n index pair for S.

The existence and invariance of Conley indices rely on the following theorems.

Theorem 1.24 (Conley)**.** *Every S admits an index pair.*

In fact, we can set N to be a manifold with boundary $L \cup L'$ satisfying that $\partial L = \partial L' = L \cap L'$, and that L is a codimension 0 submanifold in ∂N. We can think of L as the exit set and L' as the entrance set.

Theorem 1.25. *The Conley index $I(\phi, S)$ is independent of the choice of (N, L). Further, it is invariant under continuation maps. More precisely, let $\phi^\lambda = \{\phi_t^\lambda\}$ be a family of flows, and N be an isolating neighbourhood for all S^λ $(\lambda \in [0, 1])$. Then the Conley index $I(\phi^\lambda, S^\lambda)$ is independent of λ.*

Notice that in our definition, Conley index is a *homotopy type*, which means we actually obtain a family of homotopic equivalent spaces in the theorem above.

Let us see some examples.

Example 1.26. Consider a single isolated critical point p of a Morse-Smale function f, and let ϕ be the negative gradient flow of f. Locally we have

$$f = -\sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2 + f(p).$$

Here k is the Morse index of p. Now we can take $N = D^{n-k} \times D^k$, and $L = D^{n-k} \times \partial D^k$. See Figure 9. Then

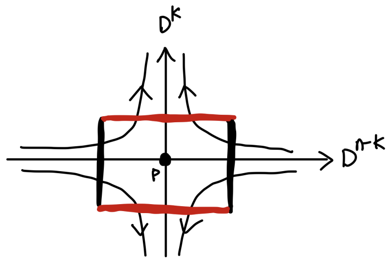

Figure 9: Conley index associated to a single isolated critical point.

$$I(\phi, \{p\}) \simeq D^k/\partial D^k \cong S^k.$$

Hence the Conley index indeed captures the information about critical points.

Example 1.27. Let X be a compact manifold now, and let $S = X$. Then we can take $N = X$ and $L = \varnothing$, and then $I = X^+ = X \coprod *$. In this case, we have

$$\widetilde{H}_*(I) = H_*(X).$$

Hence the Conley index indeed recovers the original Morse homology for compact manifolds.

Example 1.28. We now give a really interesting example, the *monkey saddle*. Consider the following vector field on $\mathbb{R}^2$ as in Figure 10. Let $S = \{p\}$ be

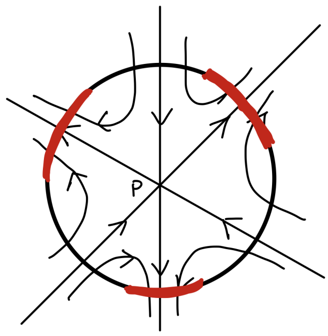

Figure 10: The monkey saddle.

the critical point. We can take N as a closed disk, and N is three arcs on the boundary. From this, we see that

$$I(S, \phi) \simeq S^1 \vee S^1.$$

In general, the Conley index does recover the Morse homology in good cases. The idea is that we attach a k-cell starting from L when passing an index k critical point, and show that it recovers X.

Theorem 1.29. *Let ϕ be the flow of a Morse–Smale pair (f, g) on X, and*

$$S = \{x \in X \colon \phi_t(x) \text{ exists for all } t,\ \phi_{\pm\infty}(x) \text{ are critical points of } f\}$$

be the set of points between critical points. Assume that S is compact. Then

$$\widetilde{H}_*(I(\phi, S)) \cong H_{Morse,*}(X, f, g).$$

The construction of Conley index is purely in finite dimensions. However, we will see it helps us to simplify some infinite-dimensional problems in the end of the class.

1.5 Morse homology for manifolds with boundaries (lecture 4)

We have treated the Morse theory when X is a manifold without boundary (but not necessarily compact). Now we turn to the case that X is compact but has boundary.

Let S be the set of points between critical points. If $S \subseteq \operatorname{int} X$, then we reduce to the previous case by using $\operatorname{int} X$, and obtain a homology $\widetilde{H}_*(I(S))$. This formalism can be slightly generalized:

Example 1.30. If f is maximal on the boundary, then $-\nabla f$ is inward on ∂X. We can simply choose $(N, L) = (X, \phi)$ to be an index pair, and obtain

$$H_{Morse}(X) = \widetilde{H}_*(X^+) = H_*(X).$$

Example 1.31. If f is minimal on the boundary, then $-\nabla f$ is outward on ∂X. In this case, we can choose $(N, L) = (X, \partial X)$, and the result homology is

$$\widetilde{H}_*(X/\partial X) = H_*(X, \partial X).$$

In general, we should have discrete critical points, but they might be on the boundary. Kronheimer and Mrowka treat this topic in detail in their book [?Kronheimer2007MonopolesAT], as a finite dimensional model for their construction of monopole Floer homology.

For a manifold with boundary X, we can consider its *double*

$$DX = X \cup_{\partial X} X,$$

which carries a natural involution ι. We assume that our function f comes from an ι-invariant Morse function on DX. In this case, ∇f is tangent to the boundary ∂X.

There are three types of cirtical points:

- critical points in the interior $\operatorname{int} X$;
- critical point $p \in \partial X$ such that

  $$\operatorname{Hess}(f)_p(\nu, \nu) > 0,$$

  where ν is the (outwards) normal vector. In this case, we have $W^u(p) \subset \partial X$, and we say p is *boundary-stable*;
- critical point $p \in \partial X$ such that

  $$\operatorname{Hess}(f)_p(\nu, \nu) < 0.$$

  In this case, we have $W^s(p) \subset \partial X$, and we say p is *boundary-unstable*;

Figure 11 provides an illustration of boundary-stable and boundary-unstable critical points.

We can divide the critical points into three parts:

$$\operatorname{Cr}(f) = C^o \cup C^s \cup C^u.$$

Because of the existence of the boundary, we cannot expect a complete transversality as the Morse–Smale condition. The replacement is the following:

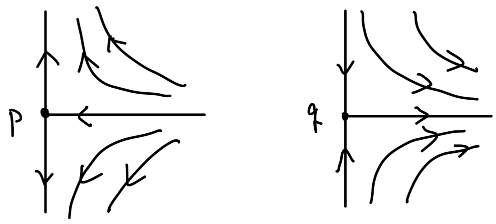

Figure 11: Critical points on the boundary: p is boundary-stable while q is boundary-unstable.

Definition 1.32. A smooth function f on X is said to be *regular* if $W^u(p)$ and $W^s(q)$ intersect transversely in X, except the "boundary obstructed" case that p is boundary-stable and q is boundary-unstable. In this case, we require $W^u(p)$ and $W^s(q)$ intersect transversely in ∂X, i.e.

$$T_x W^u(p) + T_x W^s(q) = T_x(\partial X).$$

One can easily see that f is regular implies f is Morse–Smale restricted to the boundary. Under this assumption, the moduli space

$$\mathcal{M}(p,q) = W^u(p) \cap W^s(q)$$

is a smooth manifold (possibly with boundary) of dimension

$$d = \begin{cases} \operatorname{ind} p - \operatorname{ind} q + 1, \ p, q \text{ are in the boundary-obstructed case;} \\ \operatorname{ind} p - \operatorname{ind} q, \text{ otherwise.} \end{cases}$$

It has boundary precisely when p is boundary-unstable, and q is boundary-stable. See Figure 12. In this case, we have

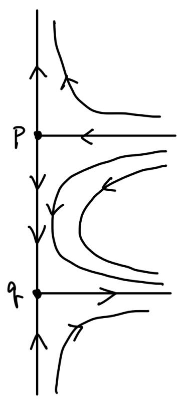

Figure 12: The boundary-obstructed case.

$$\mathcal{M}^{\partial}(p,q) := \partial \mathcal{M}(p,q) = W^u(p) \cap W^s(q) \cap \partial X.$$

These moduli spaces also carry an $\mathbb{R}$-action, and we define the unparametrized moduli spaces

$$\widehat{\mathcal{M}}(p,q) = \mathcal{M}(p,q)/\mathbb{R},\ \widehat{\mathcal{M}}^{\partial}(p,q) = \mathcal{M}^{\partial}(p,q)/\mathbb{R}.$$

Using these, we can define three flavours of Morse chain complexes.

- The "bar" version
$$\overline{C}_k = C_k^s \oplus C_{k+1}^u,$$
where C_k^s and C_k^u are sets of critical points of index k in C^s and C^u, respectively. The differential is defined by counting flowlines on the boundary:
$$\overline{\partial} p = \sum_q \#\widehat{\mathcal{M}}^{\partial}(p,q) \cdot q.$$

- The "to" version
$$\check{C}_k = C_k^o \oplus C_k^s$$
with differential
$$\check{\partial} = \begin{pmatrix} \partial_o^o & -\partial_o^u \overline{\partial}_u^s \\ \partial_s^o & \overline{\partial}_s^s - \partial_s^u \overline{\partial}_u^s \end{pmatrix}.$$
Here
$$\partial_\beta^\alpha \colon C_k^\alpha \to C_{k-1}^\beta$$
is defined for $\alpha, \beta \in \{o, s, u\}$, and counts $\#\widehat{\mathcal{M}}(p,q)$ in X, while the corresponding $\overline{\partial}_\beta^\alpha$s count flowlines in ∂X.

- The "from" version
$$\hat{C}_k = C_k^o \oplus C_k^u$$
with differential
$$\hat{\partial} = \begin{pmatrix} \partial_o^o & \partial_o^u \\ -\overline{\partial}_u^s \partial_s^o & -\overline{\partial}_u^u - \overline{\partial}_u^s \partial_s^u \end{pmatrix}.$$

Remark 1.33. Here these mysterious differentials are constructed by looking at all possible broken trajectories, and organized in a way such that $\partial^2 = 0$.

These constructions do give us the correct information of Morse homology, as the following theorem indicated.

Theorem 1.34. *We have isomorphisms*

$$H_*(\overline{C}_*) = H_*(\partial X),\ H_*(\check{C}_*) = H_*(X),\ H_*(\hat{C}_*) = H_*(X, \partial X).$$

Idea of the proof. The first one is nothing more than the original Morse homology. For the second, one can first show that $H_*(\check{C}_*)$ is independent of the choice of (f,g), using continuation maps. We then choose a Morse–Smale function f on ∂X and extend it on a collar neighbourhood by

$$f(x,t) = f|_{\partial X}(x) - t^2,\ t \in [0, \epsilon),$$

and then extend it to all of X. For this f, there is no boundary-stable critical points, and hence, the to version Morse chain complex is the same as the Morse chain complex in $\operatorname{int} X$. Thus it recovers $H_*(X)$. For the from version, the proof is similar: we choose

$$f(x,t) = f|_{\partial X}(x) + t^2,\ t \in [0,\epsilon).$$

□

1.6 Morse homology in infinite dimensions

To conclude the first part of our class, we briefly mention some early applications of Morse theory in infinite dimensions in the simplest situation, c.f. [?milnor1963morse]. Notice that this is *not* what is called Floer theory.

Let X be a Hilbert manifold, i.e. atlases of charts of X are modelled on open subsets of a separable Hilbert space, and transition functions are smooth. One can define the tangent space T_pX for $p \in X$, which is a Hilbert space; from this one can also define Riemannian metrics on X. We can also talk about smooth functions on X and the Hessian of a smooth function. Under suitable setting, many theorems in finite-dimensional differential topology, such as inverse function theorem and Sard's theorem, still work.

Definition 1.35. A smooth function f on a Hilbert manifold X is a *Morse function* if its Hessian

$$\mathrm{Hess}_p(f) \colon T_pX \to T_pX$$

is an isomorphism for all critical point p.

In infinite dimensions, we need to be careful to control the behaviour of convergence.

Definition 1.36. A smooth function f on a Hilbert manifold X satisfies the *Palais–Smale* condition if for any subset $A \subset X$ satisfying the following conditions:

- f is bounded on A;
- the closure of
$$\{\|\nabla f(x)\| : x \in A\}$$
in $\mathbb{R}$ contains 0,

we can find a sequence $x_n \in$ converging to a critical point of f.

Let A be $f^{-1}[a,b] \cap \mathrm{Cr}(f)$. We obtain:

Corollary 1.37. *Let f be a Morse, Palais–Smale function. Then for any finite real numbers $a < b$, there are only finitely many critical points in $f^{-1}[a,b]$.*

Theorem 1.38. *Let f be a Morse, Palais–Smale function, and let $p_1, \ldots, p_k$ be all critical points of f with critical values in $[a,b]$. Then $f^{-1}[a,b]$ is homeomorphic to the product $f^{-1}(a) \times [a,b]$, with a handle $D(T_p^- X) \times D(T_p^+ X)$ attached for each p_i along $\partial D(T_p^- X) \times D(T_p^+ X)$.*

Notice that here $T_p^{\pm}X$ can have infinite dimensions. If T_p^-X is finite-dimensional for all critical point p, and f is bounded below, we can make sense of the index of p. In this case, X is homotopy equivalent to a CW complex. When (f, g) is Morse–Smale, the homology again recovers the singular homology for X.

Application: the topology of path spaces

Let M be a finite-dimensional, closed, smooth Riemannian manifold, and let $p, q \in M$. Look at the path space

$$X = \Omega_1^2(M, p, q) := \{\gamma\colon [0,1] \to M\colon \gamma(0) = p,\ \gamma(1) = q,\ \|\gamma\|_{L_1^2} < \infty\}.$$

The tangent space of X at γ is

$$T_\gamma X = \{\lambda \in L_1^2([0,1], TM)\colon \lambda(t) \in T_{\gamma(t)}X,\ \lambda(0) = 0 = \lambda(1)\},$$

on which we have an inner product

$$\langle \lambda, \tau \rangle = \int_0^1 \langle \lambda'(t), \tau'(t) \rangle \, dt.$$

It defines a Riemannian metric on X. There is a natural functional on X, namely the *energy function*, defined by

$$E(\gamma) = \frac{1}{2} \int_0^1 \left\| \gamma'(t) \right\|^2 \, dt.$$

Theorem 1.39. *The energy functional E satisfies the Palais–Smale condition.*

Critical points of E correspond to geodesics from p to q as they have constant speeds. The dimension of $\ker \operatorname{Hess}_\gamma(E)$ equals to the dimension of Jacobi fields along γ, which is nonzero if p, q are conjugate. For example, when p, q are antipodal points on S^n. If this is zero, then E is Morse, and we can talk about the index

$$i(\gamma) = \#\{\gamma(t)\colon t \in (0,1) \text{ such that } \gamma(t) \text{ is conjugate to } \gamma(0) \text{ along } \gamma\}.$$

Homotopy theory tells us that $\Omega_1^2(M, p, q)$ is homotopy equivalent to the loop space ΩM. Hence using this machinery, we can calculate the homology of loop spaces.

Example 1.40. Let $X = S^n$. Choose p, q that are not antipodal points, and let p' be the antipodal point of p. The geodesics from p to q are pq, $pp'q$, $pqp'pq$, and so on. They have degree $0, n-1, 2(n-1), \ldots$ and generate the Morse chain complex for ΩX. When $n > 2$, there is no room for differentials. Hence we have

$$H_k(\Omega S^n) \cong \begin{cases} \mathbb{Z},\ n-1 \mid k; \\ 0, \text{ otherwise.} \end{cases}$$

One can also use Morse theory to study the stable homotopy group of Lie groups, and prove the famous Bott periodicity theorem.

2 General discussion on Floer homologies

Before starting the discussion on concrete theories, we first talk about some common features of Floer homologies. We are trying to make sense of some general principles rather than prove meta-theorems.

2.1 The basic setting (lecture 5)

Generally speaking, we perform Floer homology on an infinite-dimensional background space $\mathcal{B}$, called the *configuration space*. Typically, $\mathcal{B}$ can be $C^\infty(S^1, X)$, where X is some finite-dimensional Riemannian manifold, or $C^\infty(X; E)$, where E is a smooth vector bundle over a smooth manifold X. These are the two main families of configuration spaces we deal with.

These configuration spaces are somewhat "smooth", and we can make sense of tangent spaces. For example, for $\mathcal{B} = C^\infty(S^1, X)$, the tangent space at a loop γ is

$$T_\gamma \mathcal{B} = \{\text{vector fields on } X \text{ along } \gamma\};$$

for $\mathcal{B} = C^\infty(X; E)$, the tangent space is $C^\infty(X; TX \otimes E)$.

The second ingredient to perform Floer homology is a "Morse" function

$$f\colon \mathcal{B} \to \mathbb{R}.$$

The problem here is that $\mathcal{B}$ is *not* a Banach manifold (locally modelled on Banach spaces) in general, but only a Fréchet manifold. To resolve this, we complete $\mathcal{B}$ with respect to some Sobolev norm L^2_k on $\mathcal{B}$. Recall that this means we have a bound for all derivatives up to order k. Denote the Sobolev completion by $\mathcal{B}_k \supset \mathcal{B}$. The *Sobolev embedding theorem* tells us

$$\bigcap_{k=1}^{\infty} \mathcal{B}_k = \mathcal{B}.$$

The functional f extends to

$$f_k\colon \mathcal{B}_k \to \mathbb{R}$$

continuously for $k \gg 0$. We'll often omit the subscript in context. The completion spaces $\mathcal{B}_k$s are Hilbert manifolds, and we can look at the *formal gradient flow* of f with respect to the L^2-metric on $\mathcal{B}_0$:

$$\gamma'(t) = -\nabla f(\gamma(t)), \tag{2.1}$$

where

$$\langle \nabla f, v \rangle_{L^2} = df(v).$$

Ideally, the *Floer chain complex* CF_* is a chain complex generated by critical points of f, with the differential

$$\partial p = \sum_q \#\widehat{\mathcal{M}}(p,q) \cdot q.$$

The Floer homology is defined as the homology of CF_*. There are many issues to make sense of this definition.

First of all, the flowline equation involves the operator

$$\frac{d}{dt} + \nabla f \colon \{\gamma \colon \mathbb{R} \to \mathcal{B}\} \to \{\mathbb{R} \to T\mathcal{B}\}.$$

The source space can be written into a more familiar form. It is $C^\infty(S^1 \times \mathbb{R}, X)$ when $\mathcal{B} = C^\infty(X, S^1)$, and is $C^\infty(X \times \mathbb{R}; \pi^* E)$ when $\mathcal{B} = C^\infty(X; E)$. The point here is that:

> this operator should be an *elliptic* partial differential operator.

What does this mean? Let

$$P = \sum_{|\alpha| \leqslant m} a_\alpha(x) \partial^\alpha$$

be a pseudo-differential operator of degree m on an open set $\Omega \subset \mathbb{R}^n$, i.e. $a_\alpha \neq 0$ for some $|\alpha| = m$. It is said to be *elliptic* if its *symbol*

$$\sigma_P(\xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha$$

is nonzero for all $x \in \Omega$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \neq 0$. In general, a differential operator (on maps or sections on a smooth manifold X) is *elliptic* if it looks elliptic locally.

Examples 2.2. The Laplacian

$$\Delta = \sum_{i=1}^n \partial_i^2$$

on $\mathbb{R}^n$ is elliptic since it has symbol

$$\sigma = \sum_{i=1}^n \xi_i^2 = |\xi|^2.$$

The Cauchy–Riemann operator

$$\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$$

is elliptic since it has symbol $\sigma = z/2$ on $\mathbb{C}$.

If P is an elliptic operator not involving ∂_t, then $\partial_t + P$ is parabolic, which means solution to the corresponding equation has (short time) existence and uniqueness properties. For example, setting $P = \Delta$, we get the heat equation:

$$\frac{df}{dt} + \Delta f = 0.$$

Elliptic equations don't satisfy this. So this is an issue for $\frac{d}{dt} + \nabla f$. For example, the Cauchy–Riemann equation

$$\bar{\partial} u = 0,$$

viewed as an initial value problem on the upper half-plane $\mathbb{R} \times [0, +\infty)$ with given initial value $f \colon \mathbb{R} \to \mathbb{R}$, has a holomorphic extension only if u is real-analytic. In short, unlike in Morse theory, the flowline equation 2.1 might not have solutions!

Even worse, sometimes ∇f itself may not exist. Recall that the original gradient is

$$\nabla f \colon \mathcal{B} \to T\mathcal{B}.$$

However, after completions, it becomes $\mathcal{B}_k \to T\mathcal{B}_{k-1}$ (not to $T\mathcal{B}_k$!). Hence even the gradient field does not exist on a single $\mathcal{B}_k$. We need to consider all the completions $\mathcal{B}_k$. Nevertheless, one can still define the moduli space

$$\mathcal{M}(p,q) = \{\gamma \colon \mathbb{R} \to \mathcal{B} \colon \gamma'(t) = -\nabla f(\gamma(t)),\, \gamma(-\infty) = p,\, \gamma(+\infty) = q\}.$$

Secondly, the Hessian

$$\mathrm{Hess}_p(f) \colon (T_p\mathcal{B})_0 \to (T_p\mathcal{B})_0$$

is a self-adjoint *unbounded* operator and iot has unbounded spectrum in both directions. As a result, $T_p^+\mathcal{B}$ and $T_p^+\mathcal{B}$ are infinite-dimensional for all critical point p of f. In this case, there is no well-defined index for critical points!

Nevertheless, we can still define a *relative index* $\mu(p,q) \in \mathbb{Z}$ for critical points p, q, satisfying

$$\mu(p,q) + \mu(q,r) = \mu(p,r).$$

It plays the role of $\mathrm{ind}(p) - \mathrm{ind}(q)$ in Morse theory. The dimension of $\mathcal{M}(p,q)$ is expected to be $\mu(p,q)$ (notice that this presumes $\mathcal{M}(p,q)$ is a finite-dimensional manifold). We then define the unparametrized moduli space

$$\widehat{\mathcal{M}}(p,q) = \mathcal{M}(p,q)/\mathbb{R}.$$

Now the differential can be formalized as

$$\partial p = \sum_{\mu(p,q)=1} \#\widehat{\mathcal{M}}(p,q) \cdot q.$$

As a result of the relatively-defined index, the Floer homology group is only relatively graded in general. If we have some preferred critical point p_0 of f, then we can set an absolute grading by $\mathrm{ind}(p_0) = 0$ and $\mathrm{ind}(p) = \mu(p, p_0)$.

2.2 Analysis on moduli spaces (lecture 6)

To make everything perform well, there are analysis problems that needs to be done in any Floer theory.

Fredholmness

Let H_1, H_2 be Hilbert spaces. Recall that a bounded linear operator

$$F\colon H_1 \to H_2$$

is said to be a *Fredholm operator* if it has closed image and finite-dimensional kernel and cokernel. For Fredholm operator F, we can define its *index* by

$$\operatorname{ind}(F) = \dim \ker F - \dim \operatorname{coker} F.$$

Example 2.3. All linear operators are Fredholm if H_1 and H_2 are finite-dimensional. In this case, we always have

$$\operatorname{ind}(F) = \dim H_1 - \dim H_2.$$

Example 2.4. Elliptic operators on a closed manifold are Fredholm. For example, let X be a closed smooth manifold. Consider

$$d + d^*\colon \Omega^{even}(X) \to \Omega^{odd}(X).$$

It is a Fredholm operator. By Hodge theory, it has index $\chi(X)$.

Proposition 2.5. *The indices of Fredholm operators are invariant under continuous deformation.*

Fredholmness allows us to mimic some classical theorems in an infinite-dimensional setting.

Theorem 2.6 (Sard–Smale theorem)**.** *Let*

$$F\colon X \to Y$$

be a map between Hilbert manifolds. Assume that X is connected, and the differential

$$dF_x\colon T_xX \to T_{f(x)}Y$$

is Fredholm for all $x \in X$. Then $F^{-1}(y)$ is a smooth manifold of dimension $\operatorname{ind}(F)$ *for generic $y \in Y$.*

Notice that here the index of F makes sense because of the last proposition. In our case, we view $\mathcal{M}(p,q)$ as a subset of $\mathcal{P}_k(p,q)$, the L^2_k-completion of

$$\mathcal{P} = \{\gamma\colon \mathbb{R} \to \mathcal{B}\colon \gamma(-\infty) = p,\ \gamma(+\infty) = q\}.$$

Then $\mathcal{M}(p,q)$ is the zero set of

$$F\colon \mathcal{P}_k(p,q) \to T\mathcal{P}_{k-1},\ F(\gamma) = (\frac{d}{dt} + \nabla f)(\gamma).$$

We want to use the Sard–Smale theorem to show that $\mathcal{M}(p,q)$ is a manifold, so the task is to examine the differential dF_γ. We have

$$dF_\gamma(\tau) = d(\frac{d}{dt} + \nabla f)_\gamma(\tau) = (\frac{D}{dt} + \operatorname{Hess}(f)_\gamma)(\tau),$$

where $\frac{D}{dt}$ means the covariant derivative. Hence, we need:

Requirement the operator

$$L = \frac{D}{dt} + \mathrm{Hess}(f)$$

is Fredholm.

Assuming this, we can make sense of the index of a flowline. Notice that the space $\mathcal{P}$ may not be connected. When applying Sard–Smale theorem, we must restrict ourself to connected components of $\mathcal{P}$. As a result, there might be issues to talk about index between critical points.

The question now is how to compute the Fredholm index. For a path γ, let $H_t = \mathrm{Hess}(f)_{\gamma(t)}$, which should be a self-adjoint elliptic compact operator, and has a discrete spectrum of eigenvalues. Assume that p, q are non-degenerated, we can define the *spectral flow* $SF(H_t)$ as the number of eigenvalues crossing the zero-line as t runs from $-\infty$ to $+\infty$, counted with sign. See Figure TBD. The following theorem states that the spectral flow of H_t reflects the Fredholm index.

Theorem 2.7. *We have*

$$\mathrm{ind}(\frac{D}{dt} + H_t) = SF(H_t).$$

Idea of the proof. For simplicity, we assume that H_t has spectrum $\lambda_k(t)$ with eigenvectors u_k that are constant in t. Let

$$u(t) = \sum_{k=1}^{\infty} c_k(t) u_k.$$

Then

$$(\frac{d}{dt} + H_t)u(t) = u'(t) + \sum_{k=1}^{\infty} c_k(t)\lambda_k(t)u_k = \sum_{k=1}^{\infty}(c_k'(t) + c_k(t)\lambda_k(t))u_k.$$

If $u \in \ker(\frac{D}{dt} + H_t)$, then

$$c_k'(t) = -c_k(t)\lambda_k(t),$$

and

$$c_k(t) = a_k e^{-\int \lambda_k(s)\, ds}.$$

This is asymptotically $e^{-t\lambda_k(\pm\infty)}$ as $t \to \pm\infty$, and thus, the dimension of the kernel equals the number of eigenvalues that change from negative to positive when t runs from $-\infty$ to $+\infty$. Similarly, the dimension of the cokernel equals the number of eigenvalues that from positive to negative. □

Transversality

Recall that the moduli space $\mathcal{M}(p,q) = F^{-1}(0)$, where

$$F\colon \mathcal{P}_k(p,q) \to T\mathcal{P}_{k-1}(p,q),\ F = \frac{D}{dt} + \nabla f.$$

Assuming Fredholmness, then a generic point in $T\mathcal{P}_{k-1}(p,q)$ is a regular value. To make $\mathcal{M}(p,q)$ become a smooth manifold of finite dimension, ideally we want 0 is a regular value. This is too much. In practice, we want to find a large set S of perturbations $\{\beta_s \colon \mathcal{B} \to \mathbb{R} \colon s \in S\}$ such that for generic $s \in S$, the operator

$$F_S = \frac{D}{dt} + \nabla f_S,\ f_s := f + \beta_s$$

has 0 as a regular value. We can then define the Fredholm index $\mu(p,q)$ through a perturbation s. The space $\mathcal{M}_s(p,q) = F_s^{-1}(0)$ may depend on s, but a continuation argument will show that the Floer homology group is independent to the choice of s.

Compactness and bubbling

To count the number of solutions in the moduli spaces, we need to have some compactness result to ensure the finiteness. This is the hardest thing in Floer theory. It really depends on which equation we are studying.

In general, for a sequence of solutions $\gamma_n \in \mathcal{M}(p,q)$, we want to find a subsequence converging (in C^∞) to a broken trajectory. The rough plan is as follows. Starting from the energy argument

$$\int_{\mathbb{R}} |\gamma_n'|^2 = \int \langle \gamma_n', -\nabla f(\gamma_n) \rangle = f(p) - f(q),$$

we obtain a uniform bound for γ_n in L^2_1. Using the Arzela–Ascoli lemma, we can find a subsequence converging in L^2. Then we do elliptic bootstrapping to get a subsequence converging in C^∞_{loc}.

Sometimes there is an issue: we might lost energy in the limit. As a result, the moduli space may have *bubbles*. In this case, the compactified moduli space typically looks like

$$\bigcup \left(\widehat{\mathcal{M}}(p,r_1) \times \cdots \times \widehat{\mathcal{M}}(r_n,q) \times \{\text{space of bubbles}\} \right).$$

To ensure $\partial^2 = 0$, we need to show that bubbles only happen on high codimensions (at least 2).

Gluing

To compactify the moduli spaces as a manifold (with boundary or with corner), we also need to show that near any broken trajectory (possibly with bubbles), there are unbroken trajectories. This involves some implicit function theorem in infinite dimensions.

Orientations

To count the number of points with signs, we need to orient $\mu(p,q)$. Recall that in Morse homology, we orient $\mu(p,q)$ by orientations on X and $T_p^- X$ for all critical points p. Now $T_p^- X$ is infinite-dimensional, and

$$T_\gamma \mathcal{M}(p,q) = \ker\left(L_\gamma = \frac{D}{dt} + \text{Hess}(\gamma(t)) \right),$$

assuming all moduli spaces cut out transversely. We can define a "virtual bundle" over all $\mathcal{P}(p,q)$, thought as $[\ker(L_\gamma)] - [\text{coker}(L_\gamma)]$, which has a well-defined (actual!) determinant line bundle

$$\Lambda_{p,q,\gamma} = \bigwedge^{top}(\ker(L_\gamma)) \otimes \bigwedge^{top}(\text{coker}(L_\gamma)).$$

We want to orient this over $\mathcal{M}(p,q)$ such that this is compatible with gluing, which is needed to prove $\partial^2 = 0$. This translates to a pushout diagram

$$\begin{array}{ccc} \Lambda_{p,q} \boxtimes \Lambda_{q,r} \boxtimes \mathbb{R} & \longrightarrow & \Lambda_{p,r} \\ \downarrow & & \downarrow \\ \mathcal{P}_{p,q} \times \mathcal{P}_{q,r} \times [0,\epsilon) & \hookrightarrow & \mathcal{P}_{p,r} \end{array}.$$

Such orientations may or may not exist, depending on the topology of $\mathcal{B}$. For example, they do exist if $\mathcal{M}$ is contractible. In this case we just need to trivialize all $\Lambda_{p,p_0} \to \mathcal{P}_{p,p_0}$, where the base space is contractible. It such orientation doesn't exists, then we can just use $\mathbb{Z}/2\mathbb{Z}$-coefficients.

3 Hamiltonian Floer homology

3.1 Review of symplectic geometry

The first kind of Floer theory we introduce is the Hamiltonian Floer homology. Historically, it was developed to solve the Arnold conjecture. We start by recalling some basic symplectic geometry.

Definition 3.1. A *symplectic manifold* is a pair (M^{2n}, ω), where M is a smooth manifold of dimension $2n$, and ω is a non-degenerated closed 2-form. Recall that non-degeneracy means that ω^n gives a volume form.

Examples 3.2. We give some examples of symplectic manifolds.

- The simplest example is
$$(\mathbb{R}^{2n}, \omega_{can} = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n).$$
In fact, Darboux theorem claims that every symplectic manifold locally looks like this.
- A surface Σ together with an area form on it gives a symplectic manifold.
- The complex projective space $\mathbb{CP}^n$ together with the *Fubini–Study form* ω_{FS} forms a symplectic manifold.
- Any smooth projective algebraic variety $V \subset \mathbb{CP}^n$ is a symplectic manifold (equipped with the restriction of ω_{FS}).
- For any smooth manifold M, the cotangent bundle T^*M together with the canonical form ω_{can} gives a symplectic manifold. This generalizes the case of $\mathbb{R}^{2n}$.

Definition 3.3. A symplectic manifold (M, ω) is said to be *exact* if ω is exact, i.e. $\omega = d\lambda$ for some $\lambda \in \Omega^1(M)$.

For example, T^*M is an exact symplectic manifold for any M. On the other hand, closed symplectic manifolds can never be exact.

We can now consider the time-dependent *Hamiltonians*, which are simply smooth functions $H_t \colon M \to \mathbb{R}$. We assume that they are 1-*periodic*, i.e. $H_t = H_{t+1}$ for all t, and hence, induce
$$H \colon M \times S^1 \to \mathbb{R},\ H(x,t) = H_t(x).$$
From this, we can define the *Hamiltonian vector field* X_t by the condition that
$$\omega(X_t, -) = dH_t(-) \in \Omega^1(M),$$
and the *Hamiltonian flow* ϕ_t by
$$\frac{d}{dt}\phi_t(x) = X_t(\phi_t(x)).$$

Example 3.4. Consider $(\mathbb{R}^{2n}, \omega_{can})$. Then the Hamiltonian flow equation becomes

$$\dot{x}_i = \frac{\partial H}{\partial y_i}, \ \dot{y}_i = -\frac{\partial H}{\partial x_i},$$

which indicates the motion of particles.

Definition 3.5. The *Hamiltonian transformation* is defined as the time-1 map $\Phi = \phi_1$.

Exercise 3.6. Verify that Φ is a *symplectomorphism*, i.e. $\Phi^*\omega = \omega$.

We are interested in the *closed orbits* of the Hamiltonian flow, or equivalently, fixed points of Φ. We say a closed orbit of X_t is *non-degenerated* if

$$\det(I - d\Phi(x(0))) \neq 0.$$

Example 3.7. To justify the meaning of this, let us consider the constant Hamiltonian $H_t \equiv H_0$. Then the non-degeneracy of closed orbits is same as the non-degeneracy of critical points of H_0. In this case, we have

$$\#\{\text{closed orbits of } X_t\} \geqslant \#\mathrm{Cr}(H_0) \geqslant \sum_{i=0}^{2n} b_i(M)$$

by Morse inequality.

This example also motivates the following famous conjecture.

Conjecture 3.8 (Arnold conjecture (weak form))**.** *Let (M^{2n}, ω) be a closed symplectic manifold, and let H_t be a periodic Hamiltonian on it. Then*

$$\#\{\textit{closed orbits}\} \geqslant \sum_{i=0}^{2n} b_i(M).$$

Many mathematicians made great contributions to this conjecture! It was first proved by Conley–Zehnder [?conley1983birkhoff] for T^{2n}. Floer [?floer1988morse] invented Hamiltonian Floer homology and established the Arnold conjecture for *monotone* symplectic manifolds, which was generalized by Hofer–Salamon [?hofer1995floer] and Ono [?ono1995arnold]. For general case, Fukaya–Ono [?fukaya1999arnold], Liu–Tian [?liu1998floer], and Ruan [?ruan1999virtual] showed this conjecture (in this form). More recent progresses include works of Pardon [?pardon2016algebraic], Abouzaid–Blumberg [?abouzaid2021arnold], and Bai–Xu [?bai2022arnold].

To establish the Hamiltonian Floer homology, we need to be able to do "complex analysis" on symplectic manifolds, which needs the concept of almost complex structure.

Definition 3.9. An *almost complex structure* J on a manifold M is a smooth section $T \in \Gamma(\mathrm{End}(TM))$ such that $J^2 = -\mathrm{Id}$. A symplectic form ω on M is said to be *compatible* with J if

$$g_J(v, w) := \omega(v, Jw)$$

defines a Riemannian metric on M.

Roughly speaking, the action of an almost complex structure J can be thought as "multiplication by i" on M.

Example 3.10. Consider $(\mathbb{R}^{2n}, \omega_{can})$. Then $J(x_i) = y_i$, $J(y_i) = -x_i$ gives a compatible almost complex structure. The corresponding metric is the standard metric on $\mathbb{R}^{2n}$.

Theorem 3.11. *Let (M, ω) be a symplectic manifold. Then the space of compatible almost complex structures is non-empty and contractible.*

A choice of the almost complex structure J gives a structure of complex vector bundle on the tangent bundle TM. We can then talk about the first Chern class c_1 of TM, which is the Poincaré dual of the zero set of a generic section of the determinant line bundle $\Lambda_{\mathbb{C}}^{top}(TM)$. It is independent of the choice of J (but still depends on ω), and hence we denote it by $c_1(M, \omega)$, or c_1 for short. Let $g = g_J$ be the corresponding compatible metric.

3.2 Hamiltonian Floer homology: the simplest case (lecture 7)

We are now ready to define the Hamiltonian Floer homology.

Settings and assumptions

The configuration space is given by

$$\mathcal{B} := \{x \in C^\infty(S^1, M) \colon x \text{ is null-homotopic,}\}$$

with the tangent space

$$T_x\mathcal{B} = \{\xi \colon S^1 \to TM : \xi(t) \in T_{x(t)}M\}.$$

It naturally equips an L^2 inner product induced from g:

$$\langle \xi, \nu \rangle = \int_0^1 (\xi(t), \nu(t))\, dt.$$

From which we can perform Sobolev completions on $\mathcal{B}$ to obtain $\mathcal{B}_k$, which have their own inner products, but we only care about the norm.

The functional $F \colon \mathcal{B} \to \mathbb{R}$ is given by

$$F(x) = -\int_{D^2} u^*\omega + \int_0^1 H_t(x(t))\, dt, \tag{3.12}$$

where $u \colon D^2 \to M$ is a smooth extension of x (recall that x is null-homotopic).

The immediately coming question is that does this depend on the choice of u? Let us choose two extension disks $u, v \colon D^2 \to M$. Then we have

$$\int_{D^2} (u^*\omega - v^*\omega) = \int_{S^2} (u\#(-v))^*\omega = ([\omega], f_*[S^2]),$$

where $f = u\#(-v)$ is the concatenation, living in $\pi_2(M)$. Hence the functional can differ by $\omega(\pi_2(M))$. To make it well-defined, the simplest way is to ask $\pi_2(M) = 0$, which includes all the surface Σ_g $(g > 0)$ and their products. In fact, it is enough to require that (M, ω) is *symplectic aspherical*, i.e. $\omega\big|_{\pi_2(M)} = 0$.

To explore the idea of Hamiltonian Floer homology, let us make the following assumption in this subsection:

Assumption (the simplest case) We assume that (M, ω) is symplectic aspherical and in further $c_1\big|_{\pi_2(M)} = 0$.

We will see why we need the second condition later.

Floer's equation

To find the critical points of F, we calculate the gradient:

$$\begin{aligned}(dF)_x(\xi) &= \int_0^1 \omega(\dot{x}(t), \xi_t)\, dt - \int_0^1 (dH_t)_{x(t)}(\xi_t)\, dt \\ &= \int_0^1 \omega(\dot{x}(t) - X_t(x(t)), \xi_t)\, dt \\ &= \int_0^1 \left(J(\dot{x}(t) - X_t(x(t))), \xi_t \right)\, dt.\end{aligned}$$

It is zero for all ξ at a critical point x, and we have $\dot{x}(t) = X_t(x(t))$. Therefore:

Proposition 3.13. *The critical points of F are exactly the periodic orbits of X_t.*

We now look at the gradient flow lines. They are maps from $\mathbb{R}$ to $\mathcal{B} \subset C^\infty(S^1, M)$, or equivalently, map $u \colon \mathbb{R} \times S^1 \to M$, satisfying the gradient equation

$$\frac{\partial u}{\partial s} = -(\nabla_{L^2} F)(u(s)).$$

From this, we obtain *Floer's equation*:

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = J(u) X_t(u). \tag{3.14}$$

Here s and t are coordinates on $\mathbb{R}$ and S^1, respectively.

Remark 3.15. Floer's equation can be viewed as a *perturbed Cauchy–Riemann equation*. To see this, we introduce the partial bar operator $\bar{\partial}_J = \frac{\partial}{\partial s} + J\frac{\partial}{\partial t}$, and Floer's equation can be written as

$$\bar{\partial}_J u = J(u) X_t(u).$$

If we set the right hand side to be zero, then it is precisely the Cauchy-Riemann equation, and its solutions are called *J-holomorphic cylinders*.

From our general discussion of Floer homologies, we need to study the Fredholmness, compactness, transversality, and orientation. Let us do these in order.

Fredholmness

We first discuss the Fredholmness, which will give us grading on the Floer chain complex.

Theorem 3.16. *Let x, y be two critical points of F, and γ be a path connecting x and y. Then the operator*

$$\frac{D}{dt} + \mathrm{Hess}_{\gamma(t)}$$

is Fredholm of index $\mu(x,y) = \mu_{CZ}(x) - \mu_{CZ}(y)$. Here μ_{CZ} is the Conley–Zehnder index.

So actually we have an absolute grading on critical points, but what is the Conley–Zehnder index? Roughly speaking, it counts the rotation number of a symplectic path. In general, for a non-degeneracy contractible orbit

$$x\colon S^1 \to M,\ x(t) = \phi_t(x(0)),$$

we can trivialize the symplectic bundle (x^*TM, ω) through an extending disk $u\colon D^2 \to M$. The differential of the flow

$$d\phi_t(x(0))\colon T_{x(0)}M \to T_{x(t)}M$$

gives a linear symplectomorphism $\psi_t\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. In other words, we obtain a symplectic path $\psi_t \in \mathrm{Sp}(2n)$. The non-degeneracy of x means that

$$\det(I - \psi_1) \neq 0.$$

We first define the Conley–Zehnder index for such symplectic paths. Notice that the set

$$\{A \in \mathrm{Sp}(2n)\colon \det(I - A) = 0\}$$

divides $\mathrm{Sp}(2n)$ into two simply connected components. We can concatenate ψ with a path $\psi_1\colon [1,2] \to \mathrm{Sp}(2n)$, from ψ_1 to either $B^+ = -I$ or

$$B^- = \mathrm{diag}(2, -1, \ldots, -1, \frac{1}{2}, -1, \ldots, -1),$$

to obtain a path $\tilde{\psi}\colon [0,2] \to \mathrm{Sp}(2n)$. Let ρ be that composition of the following maps

$$\mathrm{Sp}(2n) \xrightarrow{r} \mathrm{U}(n) \xrightarrow{\det} S^1.$$

Then the *Conley–Zehnder index* of ϕ_t is defined as

$$\mu_{CZ}(\phi_t) = \deg(-\rho^2 \circ \tilde{\psi}).$$

We then define the Conley–Zehnder index of x as $\mu_{CZ}(x,u) := \mu_{CZ}(\phi_t)$. It doesn't depend on the choice of the extending disk u under our assumption that $c_1\big|_{\pi_2(M)} = 0$, thanks to the following proposition.

Proposition 3.17. *Let $A \in \pi_2(M)$. Then*

$$\mu_{CZ}(x, u\#A) - \mu_{CZ}(x,u) = 2c_1([A]).$$

Compactness

To discuss the compactness, we need the concept of energy.

Definition 3.18. Let $u\colon \mathbb{R} \times S^1 \to M$ be a J-holomorphic cylinder. The *energy* of u is defined as

$$E(u) = \int_{\mathbb{R}} \left\| \frac{\partial u}{\partial s} \right\|^2_{L^2(S^1)} ds.$$

Here s is the coordinate on the $\mathbb{R}$ factor.

Simple calculation gives

$$\begin{aligned} E(u) &= \int_{\mathbb{R}\times S^1} \langle \frac{\partial u}{\partial s}, -J\frac{\partial u}{\partial t} + J(u)X_t(u)\rangle \\ &= \int_{\mathbb{R}\times S^1} \left(\omega(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}) - \omega(\frac{\partial u}{\partial s}, X_t(u)) \right) \\ &= \int_{\mathbb{R}\times S^1} u^*\omega - \int_{\mathbb{R}\times S^1} dH_t(\frac{\partial u}{\partial s}). \end{aligned}$$

Hence:

Proposition 3.19. *Let $x_\pm$ be two critical points of F. Assume that $u \to x_\pm$ as $s \to \pm\infty$. Then*

$$E(u) = F(x_-) - F(x_+).$$

We can now give the statement of the compactness theorem, which is due to Gromov [?gromov1985pseudo].

Theorem 3.20 (Gromov compactness theorem)**.** *Any sequence of solutions of Floer's equation 3.14 from x to y with bounded energy has a convergent subsequence, which converges to a union of broken trajectories and sphere bubbles.*

[TBD] Sphere bubbles

Example 3.21. We give an example of the bubbling phenomenon. Consider a sequence of holomorphic maps $u_n\colon \mathbb{CP}^1 \to \mathbb{CP}^2$, given by

$$[x : y] \mapsto [xy : y^2 : \frac{1}{n}x^2].$$

Then $(u_n)_*[\mathbb{CP}^1] = 2 \in H_2(\mathbb{CP}^2)$. Away from $y = 0$, u_n can be written as

$$[x : y] \mapsto [\frac{x}{y} : 1 : \frac{1}{n}(\frac{x}{y})^2],$$

and it tends to a $\mathbb{CP}^1 \subset \mathbb{CP}^2$, given by $[a : 1 : 0]$. By reparametrizing away from $x = 0$, it tends to another sphere given by $[b : 0 : 1]$. Hence the limit of u_n is a union of two spheres.

Bubbles may affect the proof on $\partial^2 = 0$. However, in our case, there is no sphere bubble! This can be seen by considering the energy: if such a bubble u exists, then

$$E(u) = \int_{S^2} \left\| \frac{\partial u}{\partial z} \right\|^2 = \int_{S^2} u^*\omega = 0.$$

Transversality, gluing, and orientations

We don't discuss too much about transversality, but it can be ensured by perturbing J to be time dependent. Alternatively, one can also perturb the Hamiltonian.

For the orientation on the moduli space $\mathcal{M}(x,y)$, which is the preimage of 0 under the map $\bar{\partial}_J - JX_t$, we consider the linearization

$$T\mathcal{M}(x,y) = \ker(D(\bar{\partial}_J - JX_t)).$$

Here $D(\bar{\partial}_J - JX_t)$ is a Fredholm operator of index $\mu_{CZ}(x) - \mu_{CZ}(y)$, and we consider the determinant of the index bundle

$$\det(\text{index bundle}) = \bigwedge^{\text{top}}(\ker) \otimes \left(\bigwedge^{\text{top}}(\text{coker})\right)^*.$$

One can deform $D(\bar{\partial}_J - JX_t)$ into a simply $\bar{\partial}_J$, which gives canonical orientations on the kernel and cokernel.

Hamiltonian Floer homology

Finally we can define the Hamiltonian Floer homology! At the chain level, the Hamiltonian Floer chain complex $CF_*(M, \omega, H_t, J_t)$ is generated by critical points of F with coefficient $\mathbb{Z}$, and the differential is given by

$$\partial x = \sum_{\mu(x,y)=1} \#\widehat{\mathcal{M}}(x,y) \cdot y.$$

The homology of it, denoted by $HF_*(M, \omega, H_t, J_t)$ temporally, is called the *Hamiltonian Floer homology* of (M, ω).

Theorem 3.22 (Floer)**.** *The graded group $HF_*(M, \omega, H_t, J_t)$ is independent of the choice of H_t and J_t. In fact, it is isomorphic to the ordinary homology $H_*(M)$.*

Sketch of the proof. The independence is proved by considering the continuation maps, as in the finite-dimensional case. After that, we can pick a Morse–Smale function $f\colon M \to \mathbb{R}$ and choose the Hamiltonian $H_t = \epsilon f$. When $\epsilon > 0$ is small enough, one can imagine that the flow is too slow to have periodic orbits other than the critical points of f. One can also show that solutions to Floer's equations are one-to-one corresponding to negative gradient flow-lines of f. Hence the Hamiltonian Floer homology, calculated using ϵf, is isomorphic to the Morse homology of M. □

Recall Proposition 3.13 states that critical points of F are exactly the periodic orbits of the Hamiltonian flow. Now the Arnold conjecture (in its weak homological form) is an immediate corollary of Theorem 3.22.

Corollary 3.23. *The number of periodic orbits of (M, ω, H_t) is as least as the sum of Betti numbers of M.*

3.3 The monotone case (lecture 8)

We now turn to the discussion of a slightly more general case that (M, ω) is monotone.

Definition 3.24. A symplectic manifold (M, ω) is said to be *monotone* if there exists a positive constant τ such that $c_1(A) = \tau \cdot \omega(A)$ for all $A \in \pi_2(M)$.

Remark 3.25. The requirement that τ is positive is essential. We will see this later.

Examples 3.26. The sphere S^2 together with an area form dA, and more generally, all the complex projective spaces with the Fubini–Study form ω_{FS}, are monotone. Much more generally, all complex Grassmannians, and even all Fano varieties are examples of monotone symplectic manifolds.

But how does it help? Recall that in the simplest case, we exploit the conditions to give the well-defined $\mathbb{R}$-valued functional F and the absolute $\mathbb{Z}$-grading μ_{CZ}. We will see that the monotonicity helps us to give a well-defined F and μ_{CZ} in an acceptable manner.

Definition 3.27. The *minimal Chern number* of (M, ω) is an integer $N > 0$ such that $N\mathbb{Z} = \operatorname{im}(c_1(\pi_2(M))) \subset \mathbb{Z}$.

From the monotonicity, we have

$$\operatorname{im}([\omega](\pi_2(M))) = \frac{1}{\tau} N\mathbb{Z}.$$

By rescaling ω, we can assume $[\omega](\pi_2(M)) \subset \mathbb{Z}$. Recall that when defining the function F by 3.12, different choices of the extending disk differ the value of F by $[\omega]|_{\pi_2(M)}$. Hence we have a well-defined functional $F\colon \mathcal{B} \to \mathbb{R}/\mathbb{Z}$. We can lift it to an $\mathbb{R}$-valued functional, but it is not necessary.

For the Conley–Zehnder index, the situation is similar. Recall from Proposition 3.17 that for $A \in \pi_2(M)$, we have

$$\mu_{CZ}(x, u\#A) - \mu_{CZ}(x, u) = 2c_1([A]) \in 2N\mathbb{Z}.$$

Hence we have a well-defined $\mathbb{Z}/2N\mathbb{Z}$ grading on CF_*.

For a sequence of solutions $u_n\colon \mathbb{R} \times S^1 \to M$ with the same index, their homotopy classes may differ by $A \in \pi_2(M)$ such that $c_1(A) = 0$, which implies $[\omega](A) = 0$, and the energy is bounded. Hence Gromov compactness applies in this case.

Even though, bubbles may exist, which may threaten $\partial^2 = 0$. To explore this, we consider moduli spaces of index 2. Assume that a bubble class A appears in the limit of a sequence of index 2 J-holomorphic cylinders. Then we have

$$2 = \mu_{CZ}(\gamma\#A) = \mu_{CZ}(\gamma) + 2c_1(A) \geqslant 2c_1(A),$$

and hence $1 \geqslant c_1(A) \in N\mathbb{Z}$. The monotonicity implies $c_1(A) = \tau[\omega](A) > 0$ (notice that here we use the condition that $\tau > 0$ in an essential way!). If

$N > 1$, we are already done. The only remaining case is that $N = 1$ and $c_1(A) = 1$, and $\mu_{CZ}(\gamma) = 0$, i.e. γ is a constant path.

So what happens next? We have the following index formula (thought as a variant of Riemann–Roch theorem).

Theorem 3.28. *The expected dimension of the moduli space of parametrized J-holomorphic spheres in class A is $2n + 2c_1(A)$, where $\dim M = 2n$.*

By "parametrized", we mean different maps may represent the same sphere. They differ by a biholomorphic map $\phi\colon S^2 \to S^2$, i.e. an action of $\mathrm{PSL}(2, \mathbb{C})$. Hence the expected dimension of unparametrized spheres is $2n + 2c_1(A) - 6$.

In our case, $c_1(A) = 1$, so the expected dimension is $2n - 4$. The union of all such spheres then has expected dimension $2n - 2$, i.e. codimension 2. The periodic orbits have dimension 1, so they don't intersect generically! In summary:

Proposition 3.29. *We can choose the almost complex structure J_t such that no sphere bubble exists.*

We can then define the Hamiltonian Floer homology as in the easiest case! The only difference is that we can only get a $(\mathbb{Z}/2N\mathbb{Z})$-graded theory now. Theorem 3.22 still holds in this case:

Theorem 3.30. *$HF_*(M, \omega) \simeq H_*(M)$ as $(\mathbb{Z}/2N\mathbb{Z})$-graded abelian groups.*

3.4 The general case

We conclude this section by discussing the most general case that (M, ω) is an arbitrary closed symplectic manifold. We will see that things become much more complicated in the general setting.

Compactness and the Novikov ring

The first issue is that bounded index doesn't imply bounded energy, which is needed to apply Gromov compactness. This can be resolved by considering the Novikov ring.

Definition 3.31. The *Novikov ring* and *Novikov field* are respectively defined as

$$\Lambda_0 = \{\sum_{i=1}^{+\infty} a_i T^{\lambda_i} \colon a_i \in \mathbb{Q},\ \lambda_i \in \mathbb{R}\ \lambda_i \geqslant 0,\ \lambda_i \to +\infty\}$$

and

$$\Lambda = \{\sum_{i=1}^{+\infty} a_i T^{\lambda_i} \colon a_i \in \mathbb{Q},\ \lambda_i \in \mathbb{R},\ \lambda_i \to +\infty\}.$$

We can think of Λ_0 and Λ as replacements of $\mathbb{Z}$ and $\mathbb{Q}$. In the definition in the general case, we use Λ_0 as the coefficient ring, and the differential is given by

$$\partial x = \sum_{y \in \mathrm{Cr}(F)} \sum_{\gamma \in \pi_0(P(x,y)),\, \mu_{CZ}(\gamma)=1} \#\widehat{\mathcal{M}}(x,y) T^{E(\gamma)} \cdot y.$$

Here

$$P(x,y) = \{\gamma\colon \mathbb{R} \to \mathcal{B} : \gamma(-\infty) = x,\ \gamma(+\infty) = y\}.$$

The energy $E(\gamma) = F(x,u) - F(y, \gamma\#u)$, where u is an extending disk for x. One can similar define the index of γ and show that they are independent of the choice of u.

The usage of the Novikov ring allows us to define a finite differential, and to "separate" different levels of energy, and then we can apply Gromov compactness at each level. However, sphere bubbles may exist, and in fact, transversality is a severe issue.

Transversality issue

We try to construct the moduli space of bubbles and find its expected dimension. Suppose that γ is a trajectory and u is a J-holomorphic sphere such that $\mu(\gamma) = 2k+1 > 1$, and $c_1(u) = -k$. Then we have

$$\mu(\gamma\#u) = \mu(\gamma) + 2c_1(u) = 1,$$

and hence, $\widehat{\mathcal{M}}(\mu\#u)$ has expected dimension 0. This means that there are solutions given by the union of a cylinder and a sphere bubble generically. If we precompose it with the degree two map $\psi\colon \mathbb{CP}^1 \to \mathbb{CP}^1$, $\psi(z) = z^2$, then $c_1(u \circ \psi) = -2k$, and

$$\mu(\gamma\#(u \circ \psi)) = 1 - 2k < 0,$$

i.e. the expected dimension is negative, but solution exists generically! Hence the transversality is impossible in general!

People invented many new tools to deal with this phenomenon, such as "multivalued perturbations" and "Kuranishi chart". We treat $\widehat{\mathcal{M}}(\gamma)$ as branched manifolds. In this setting, a multi-ply cover bubble v, as described in the last paragraph, should be counted with a weight $1/\#\,\mathrm{Aut}(v)$, where

$$\mathrm{Aut}(v) = \{\psi \in \mathrm{PSL}(2,\mathbb{C})\colon v \circ \psi = v\}.$$

The weight is a rational number, and that is why we have to use $\mathbb{Q}$-coefficient in this case. Using this construction, we can make sense of $HF_*(M,\omega)$ as a module over Λ_0 and resolve the Arnold conjecture over $\mathbb{Q}$.

Rational coefficient kills all torsions and gives a somewhat weak lower bound. More recently, people can also deal with the torsion part using deep machineries: Abouzaid and Blumberg [?abouzaid2021arnold] established the $\mathbb{F}_p$-coefficient case using Morava K-theory, and Bai and Xu [?bai2022arnold] managed to resolve the integral case.

4 Lagrangian Floer homology

4.1 Basic settings (lecture 9)

We now turn to the discussion of Lagrangian Floer homology, which can be thought as a relative version of Hamiltonian Floer homology.

Through out the section, assume that (M^{2n}, ω) is a symplectic manifold (not necessarily closed).

Definition 4.1. A *Lagrangian* in M is a n-dimensional submanifold $L \subset M$ such that $\omega\big|_L = 0$.

One can show that the isotropic subspace of a symplectic form has dimension at most n. That Lagrangians are the isotropic submanifolds of maximal dimension.

Examples 4.2. We give some examples of Lagrangians:

- $\mathbb{R}^n \times \{0\} \subset T^*\mathbb{R}^n$. The physical meaning is the "position space" lying in the "phase space".
- More generally, the zero section of a tangent bundle: $Q \subset T^*Q$;
- curves on surfaces.

Roughly speaking, Lagrangian Floer homology concerns the number of intersection points of Lagrangians. It corresponds to another Arnold conjecture ("Arnold conjecture for Lagrangian intersections").

Let L_0, L_1 be two Lagrangians in (M, ω). Assume that L_0 and L_1 are compact, and intersect transversely. So they intersect at finitely many points. We are going to construct a homology theory $HF_*(L_0, L_1)$.

The configuration space is the path space between L_0 and L_1:

$$\mathcal{B} = \{x\colon [0,1] \to M : x(0) \in L_0,\ x(1) \in L_1\}.$$

It may have many components, and we choose a basepoint x_0 (i.e. a path from L_0 to L_1) in each component. On $\mathcal{B}$ there is a functional $F\colon \mathcal{B} \to \mathbb{R}$, defined by

$$F(x) = -\int_{D^2} u^*\omega.$$

Here u is a path from x_0 to x in $\mathcal{B}$, i.e. a homotopy $u\colon I \times I \to M$, as showed in Figure TBD. The value of this integral may depend on the choice of u, and a general solution is to consider a cover

$$\widetilde{\mathcal{B}} = \{(x, \alpha)\colon x \in \mathcal{B},\ \alpha \text{ is a homotopy class from } x_0 \text{ to } x \text{ in } \mathcal{B}\}.$$

Then $F\colon \widetilde{\mathcal{B}} \to \mathbb{R}$ is well-defined. In fact, assume that u_0, u_1 are homotopic paths from x_0 to x in $\mathcal{B}$. Then we have a map $C\colon I \times I \times I \to M$, such that

$C(t,0,r) = x_0(t)$, $C(t,1,r) = x(t)$ for all r, $C(t,s,i) = u_i(t,s)$ for $i = 0,1$. See Figure TBD. Notice that $C(i,s,r) \in L_i$ for $i = 0,1$, and Stokes theorem gives

$$
\begin{aligned}
0 &= \int_{I^3} C^*(d\omega) \\
&= \int_{I\times I\times\{1\}\cup(-I\times I\times\{0\})\cup\{1\}\times I\times I\cup(-\{0\}\times I\times I)} C^*\omega \\
&= \int_{D^2} u_1^*\omega - \int_{D^2} u_0^*\omega.
\end{aligned}
$$

After choosing a compatible almost complex structure J, we can form the gradient of f. This is similar and even easier than the Hamiltonian case: the gradient is given by

$$\nabla F(x) = J(x)\frac{\partial x}{\partial t}.$$

Hence:

Proposition 4.3. *A path x is a critical point of F if and only if x is a constant path i.e. $x \in L_0 \cap L_1$.*

We can also derive the equation of flowlines. A flowline from x to y is a map $u\colon \mathbb{R} \times [0,1] \to M$, with $u(s,i) \in L_i$ for $i = 0,1$, $u(-\infty,t) = x$, and $u(+\infty,t) = y$, which satisfies the *J-holomorphic strip* condition:

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0. \tag{4.4}$$

Ideally, we can form the chain complex by $CF_*(L_0,L_1) = \mathbb{Z}\langle L_0 \cap L_1\rangle$ and

$$\partial x = \sum_{\mu(x,y)=1} \#\widehat{\mathcal{M}}(x,y)\cdot y.$$

We then show that $\partial^2 = 0$ and obtain the Lagrangian Floer homology group $HF_*(L_0,L_1)$. However, as usual, we still have many works to be done: compactness, Fredholmness, grading, transversality, bubbling, etc. We need to add conditions on (M,ω,L_0,L_1) to make life simpler.

4.2 The exact case (Lecture 10)

In the simplest case, we make the following assumptions.

Definition 4.5. A symplectic manifold (M,ω) is said to be *exact* if ω is exact i.e. $\omega = d\lambda$ for some $\lambda \in \Omega^1(M)$. In this case, we say a Lagrangian $L \subset M$ is *exact* if $[\lambda|_L] = 0 \in H^1(L)$.

Definition 4.6. A symplectic manifold (M,ω) is said to be *convex at infinity* if for every compact set $K \subset M$, there exists a compact neighbourhood $K' \supset K$ such that all J-holomorphic curves in M with boundary on K live in K'.

Assumptions In this subsection, we assume that

- (M, ω) is exact and convex at infinity;
- L_0, L_1 are exact in M.

Before exploring how do these assumptions help, let us first get some intuition of these conditions by looking at examples.

Example 4.7. *Stein manifolds* are analog of affine algebraic varieties. A Stein manifold is a complex manifold (M, J) together with a *plurisubharmonic* function $h\colon M \to \mathbb{R}$ such that $h^{-1}((-\infty, a])$ is compact for all a. Here "plurisubharmonic" means that the complex Jacobian $(\partial^2 h/\partial z_i \bar{\partial} z_j)$ is positive semidefinite. The standard example is $\mathbb{C}^n$ together with

$$h = \sum_{k=1}^{n} |z_k|^2.$$

It can be showed that all Stein manifolds are convex at infinity. Roughly speaking, it argues that any J-holomorphic cannot be tangent to level sets of h in its interior by a version of maximal principle.

Example 4.8. Any level set $h^{-1}((-\infty, a)$ of a Stein manifold (M, J, h) is convex at infinity. This is an example of *Liouville domains*. That is a compact manifold (V, ω) with boundary such that there is a primitive λ (i.e. $d\lambda = \omega$), such that the vector field X defined by $\iota_X \omega = \lambda$ is transverse with the boundary ∂V. The standard example is the ball in $\mathbb{R}^{2n}$: $V = B^{2n}(r)$. In this case, we have

$$\lambda = \frac{1}{2} \sum_{k=1}^{n} (x_k dy_k - y_k dx_k),$$

and $X = \frac{1}{2} r \partial_r$. One can show that the interior of a Liouville domain is convex at infinity.

Example 4.9. Another example for convexity at infinity is the cotangent bundle T^*Q. The idea is that the function $h(q, p) = |p|^2$ prevents J-holomorphic curves from escaping.

Examples 4.10. We also give some (non-)example of exact Lagrangians.

- The zero section of cotangent bundle T^*Q is exact.
- Consider a section $L\colon S^1 \hookrightarrow T^*S^1$. It is exact if and only if the (signed) area bounded by L and the zero section is zero.
- More generally, a section $\sigma\colon Q \to T^*Q$ is exact if and only if it is exact as a 1-form.
- A non-example is given by any circle $S^1 \subset \mathbb{R}^2$.

So why do we care about exactness? First, it gives a well-defined functional $F\colon \mathcal{B} \to \mathbb{R}$. In fact, assume that u and v are paths in $\mathcal{B}$ from x_0 to x. Let $\alpha\colon S^1 \times [0,1]$ be the concatenation $u\#v$. Then the exactness of M and L_i gives

$$\begin{aligned}\int_{I\times I} u^*\omega - \int_{I\times I} v^*\omega &= \int_{S^1\times I} \alpha^*\omega = \int_{S^1\times I} d(\alpha^*\lambda)\\ &= \int_{S^1\times\{1\}\cup(-S^1\times\{0\})} \alpha^*\lambda = 0.\end{aligned}$$

Compactness

To discuss the compactness, we need to introduce Gromov compactness in the strip case.

Theorem 4.11 (Gromov compactness for strips)**.** *A sequence of J-holomorphic strips with bounded energy has a subsequence converging to a broken strip, possibly with trees of sphere and disk bubbles. See Figure TBD.*

Example 4.12. Example as in Heegaard Floer homology. TBD

Another distinguished advantage of exactness is that there is no bubble at all! To see this, consider a sphere bubble $u\colon S^2 \to M$. Then by the exactness of M, we have

$$E(u) = \int_{S^2} u^*\omega = 0.$$

Similarly, exactness of L_i implies there is also no disk bubble. Great!

Fredholmness and relative grading

We now turn to the problem of grading. Let $x, y \in L_0 \cap L_1$, and $P_{x,y}$ be the path space

$$\{\gamma\colon \mathbb{R} \to \mathcal{B} : \gamma(-\infty) = x,\ \gamma(+\infty) = y\}.$$

Then the moduli space decomposes as

$$\widehat{\mathcal{M}}(x,y) = \coprod_{\phi\in\pi_0(P_{x,y})} \widehat{\mathcal{M}}(\phi).$$

For $\phi \in \pi_0(P_{x,y})$, pick a representative $\gamma \in \phi$, and we have

$$\mu(\phi) = \operatorname{ind}(D_\gamma\bar{\partial}_J).$$

Here $D_\gamma\bar{\partial}_J$ is an elliptic operator since it can be written as $\bar{\partial}$+(zero order terms), and $\mu(\phi)$ is the *Maslov index* defined similarly to μ_{CZ}, as follows.

We trivialize the symplectic bundle $\gamma^*(TM,\omega)$ and its Lagrangian subbundle γ^*TL_0 simultaneously as $(\mathbb{R}^{2n}, \omega_{can})$ and $\mathbb{R}^n \times \{0\}$. We cannot trivialize γ^*TL_1 at the same time, but it gives a path in the *Lagrangian Grassmannian* $L(n)$, which collects Lagrangian subspaces of $\mathbb{R}^{2n}$ and has homotopy type $\mathrm{U}(n)/\mathrm{O}(n)$. We then define $\mu(\phi) = \mu(\gamma)$ as an element in $\pi_1(L(n)) \cong \mathbb{Z}$.

Example 4.13. TBD

Maslov index shares similar properties as the Conley–Zehnder index:

Proposition 4.14. *Let ϕ_1, ϕ_2, ϕ be strips, D be a disk bubble, and A be a sphere bubble. Then we have*

$$\mu(\phi_1 \# \phi_2) = \mu(\phi_1) + \mu(\phi_2),\ \mu(\phi \# D) = \mu(\phi) + \mu(D),\ \mu(\phi \# A) = \mu(\phi) + 2c_1(A).$$

In summary, we have a map $\mu\colon \pi_0(P_{x,y}) \to \mathbb{Z}$. The image of μ has the form $k + N\mathbb{Z}$, and hence we can define a $\mathbb{Z}/N\mathbb{Z}$ grading on $L_0 \cap L_1$ by

$$\mu(x, y) = \mu(\phi) \pmod N,$$

given that $P_{x,y}$ is nonempty.

Remark 4.15. This seems not so satisfactory, but at least there is something: $\mu(D)$ is even if L_i are orientable, and N is an even number. So at least we obtain a $\mathbb{Z}/2\mathbb{Z}$ grading.

Absolute grading

Sometimes we can get an absolute $\mathbb{Z}$ grading, due to Seidel [?seidel2000graded]. The idea is to encode the uncertainty on gradings into the Lagrangian Grassmannian itself.

More specifically, we can consider the universal cyclic cover $\widetilde{L(n)}$ of $L(n)$, called the *graded Lagrangian Grassmannian*. The tangent bundle $TM \to M$ induces a bundle of Lagrangian Grassmannians $L(TM) \to M$ in an obvious way. If $2c_1(M, \omega) = 0 \in H^2(M; \mathbb{Z})$, then $L(TM)$ has an infinite cyclic cover $\widetilde{L}(TM)$ with fibers $\widetilde{L(n)}$.

A Lagrangian $L \subset M$ induces a section of $L(TM)\big|_L$. Seidel showed that there is an obstruction class $\mu_L \in H^1(L)$, called the *Maslov class*, such that TL lifts to a section $\theta \in \Gamma(\widetilde{L}(TM)\big|_L)$ if and only if μ_L vanishes. In this case, the pair $\widetilde{L} = (L, \theta)$ is called a *graded Lagrangian submanifold*.

Given two graded Lagrangian submanifold $\widetilde{L}_0, \widetilde{L}_1$ and $x \in L_0 \cap L_1$, there is a preferred path from T_xL_0 in T_xL_1 given by the graded structure. Concatenating this with a canonical shortest path gives a loop in $L(T_xM)$, and hence an absolute grading $\mu(x) \in \mathbb{Z}$.

Remark 4.16. There are also weaker conditions to ensure the existence of an absolute $\mathbb{Z}/N\mathbb{Z}$ grading.

Transversality

We again don't talk too much about transversality. The idea is still to make J time-dependent. Since there is no bubbling issue in the exact case, we immediately obtain the moduli space $\widehat{\mathcal{M}}(\phi)$ of dimension $\mu(\phi)$ for $\phi \in \pi_0(P_{x,y})$.

Orientation

The moduli spaces may not be oriented even though we assumed the exactness. Hence, we can only get a $\mathbb{Z}/2\mathbb{Z}$ coefficient theory in general.

Sometimes we can do better. Recall that $\mathrm{Pin}(n)$ is the double cover of the orthogonal group $\mathrm{O}(n)$.

Definition 4.17. A $\mathrm{Pin}(n)$-*structure* on a manifold L is a lift of its frame $\mathrm{O}(n)$-bundle $\mathrm{Fr}(L)$ to a $\mathrm{Pin}(n)$-bundle.

Proposition 4.18. $\mathrm{Pin}(n)$*-structure exists if and only if the second Stiefel–Whitney class w_2 vanishes. If it exists, then there is a non-canonical 1-1 correspondence between* $\mathrm{Pin}(n)$ *structures and* $H^1(L;\mathbb{Z}/2\mathbb{Z})$.

Remark 4.19. Pin-structure is a non-oriented analogue of Spin-structure which might be more familiar to readers.

Theorem 4.20. Pin-*structures on L_0 and L_1 give orientations on $\widehat{\mathcal{M}}(x,y)$ for all $x,y \in L_0 \cap L_1$.*

Remark 4.21. In fact, we just need *relative* Pin-*structure*, corresponding to the condition

$$w_2(TL) \in \mathrm{im}(H^2(M;\mathbb{Z}/2\mathbb{Z}) \to H^2(L;\mathbb{Z}/2\mathbb{Z})).$$

In the best case, we consider *Lagrangian frame* $L^\sharp = (L,\theta,P)$, where θ is a grading on L, and P is a Pin-structure on L. For two Lagrangian frames $L_0^\sharp, L_1^\sharp$, we can define a homology theory $HF_*(L_0^\sharp, L_1^\sharp)$, which is absolutely $\mathbb{Z}$-graded with coefficient in $\mathbb{Z}$. In general, we can only expect a relatively $\mathbb{Z}/N\mathbb{Z}$-graded theory with coefficient in $\mathbb{Z}/2\mathbb{Z}$.

Lagrangian Floer homology and an application

We finally establish our Lagrangian Floer homology $HF_*(L_0,L_1)$. Using continuation maps, one can show that:

Theorem 4.22. *The Lagrangian Floer homology $HF_*(L_0,L_1)$ is invariant under exact perturbations, i.e. perturbation through a smooth family of exact Lagrangians.*

In spirit of Theorem 4.22, if L_0 and L_1 don't intersect transversely, we can still define $HF_*(L_0,L_1)$ by exact perturbations. This is particularly important when $L_0 = L_1$:

Theorem 4.23. *We have*

$$HF_*(L,L) \cong H_*(L).$$

Here both sides are considered as $\mathbb{Z}/N\mathbb{Z}$-graded groups with coefficients in $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}$ if Pin*-structures exist.*

Sketch of the proof. By a Darboux-type theorem for Lagrangian neighbourhoods, we have isomorphism

$$HF_*(L,L;M) \cong HF_*(L,L;T^*L).$$

Pick a Morse–Smale function $f\colon L \to \mathbb{R}$ and consider the exact deformation L_t, given by $L_t = \Gamma(tdf)$. Then

$$L_0 \cap L_t = \{df = 0\} = \mathrm{Cr}(f).$$

When t is small, J-holomorphic strips correspond to negative gradient flow-lines, and hence $HF_*(L, L)$ is isomorphic to the Morse homology of L. □

Remark 4.24. Hamiltonian Floer homology can be viewed as a special case of Lagrangian Floer homology. Given a Hamiltonian transformation ψ, we can consider the graph

$$\Gamma(\psi) \subset (M \times M, \pi_1^*\omega - \pi_2^*\omega),$$

which is a Lagrangian submanifold of the product. Notice that the graph of the identity is the diagonal Δ, and the intersection $\Delta \cap \Gamma(\psi)$ gives the fixed point set of ψ. One can also show that J-holomorphic strips between Δ and $\Gamma(\psi)$ are the same as J-holomorphic cylinders in $M \times M$. Hence the Hamiltonian Floer homology $HF_*(M, \omega, \psi)$ is isomorphic to the Lagrangian Floer homology $HF_*(\Delta, \Gamma(\psi))$ (as long as the latter makes sense).

One can consider an intermediate case: *Floer homology for symplectomorphisms*. That is, for symplectomorphism $\phi\colon M \to M$, we can produce the Lagrangian Floer homology $HF_*(\Delta, \Gamma(\phi))$. It is not isomorphic to $H_*(M)$ in general, and can be used to show that some symplectomorphisms are not Hamiltonian isotopic to the identity.

As an application, we can prove the following theorem.

Theorem 4.25 (Gromov)**.** *There is no compact exact Lagrangian L in $(\mathbb{R}^{2n}, \omega_{can})$.*

Proof. Otherwise consider the exact perturbation $L_t = L + t$, $t \in [0, T]$. When T is large, $L \cap L_T = \varnothing$. Hence

$$HF_*(L, L) = HF_*(L, L_T) = 0$$

since there is no generator for Lagrangian Floer chain complex. This implies $H_*(L) = 0$ by Theorem 4.23, which is impossible as L is compact. □

Remark 4.26. Gromov's original proof doesn't use this Floer-theoretic formalism, but it is still based on analysis of J-holomorphic curves.

Remark 4.27. Any one of the two conditions in Theorem 4.25 cannot be removed: $S^1 \subset \mathbb{C}$ is compact but not exact; $\mathbb{R} \subset \mathbb{C}$ is exact but non-compact.

4.3 General cases (lecture 11)

We now breifly discuss some more general case.

Case II: M is compact with $[\omega]|_{\pi_2(M,L_i)} = 0$

Similar to the exact setting, we still have no bubbles in this case. Hence after dealing with transversality, $\partial^2 = 0$ is guaranteed, and we can form the Lagrangian Floer homology $HF_*(L_0, L_1)$.

Recall that we have exact perturbation invariance in the exact case. We want to explore a similar result in this setting. Assume that L_i $(i = 0, 1, 2)$ are Lagrangians, and $L_t|_{t\in[1,2]}$ is a Lagrangian isotopy such that "the area swept out by the isotopy is 0 for all $t \in [1, 2]$". More precisely, let

$$L_1 \times [1, 2] \to M$$

be the Lagrangian isotopy, which produces a time-dependent vector field X_t for $t \in [1, 2]$. We require that

$$\int_{S^1} \iota_{X_t}\omega = 0$$

for any $S^1 \to L_t$ to control the energy of the isotopy. In this case, we can use the continuation map to show that

$$HF_*(L_0, L_1) \cong HF_*(L_0, L_2).$$

This happens e.g. if

$$\iota_{X_t}\omega = dH_t$$

for some Hamiltonian H_t, i.e. $L_t = \phi_t(L_1)$, where ϕ_t is the corresponding Hamiltonian flow. In conclusion, we have:

Theorem 4.28. *Let L_0, L_1 be Lagrangians and ϕ be a Hamiltonian transformation. Then*

$$HF_*(L_0, L_1) \cong HF_*(L_0, \phi(L_1)).$$

Example 4.29. Torus with two Lagrangians. TBD

Question from the class How to classify Hamiltonian isotopies on a surface?

A: Basically by area conditions. So there is nothing interesting in the surface case.

We can similarly define $HF_*(L, L)$ by Hamiltonian perturbations in this case, and Theorem 4.23 also holds. Roughly speaking, Lagrangian Floer homology categorifies the number of intersection points of Lagrangians. As a corollary of this isomorphism, we have the Arnold–Givental conjecture.

Theorem 4.30 (Arnold–Givental conjecture, Floer)**.** *Let L be a Lagrangian in (M, ω), and $\psi\colon M \to M$ be a Hamiltonian transformation. Assume that M is compact, $[\omega]|_{\pi_2(M,L)} = 0$, and that L is transverse to $\psi(L)$. Then*

$$\#(L \cap \psi(L)) \geq \sum_{k=0}^{n} \dim H_k(L; \mathbb{Z}/2\mathbb{Z}) \geq \sum_{k=0}^{n} b_k(L).$$

Case III: (M, L_i) **is monotone**

Recall that monotonicity helps us to control the energy for fixed index in Hamiltonian Floer homology. In Lagrangian Floer homology, we can do similar things.

Definition 4.31. We say (M, L_i) is *monotone* if there is a constant $\tau > 0$, such that

$$c_1|_{\pi_2(M)} = \tau \cdot [\omega]|_{\pi_2(M)},$$

and also

$$\mu|_{\pi_2(M,L_i)} = 2\tau \cdot [\omega]|_{\pi_2(M,L_i)}$$

for $i = 0, 1$.

Here the coefficient 2 comes from the behaviour of Maslov index under concatenations, Proposition 4.14. In the equality in the definition, the left hand side corresponds to the expected dimension of holomorphic disks, and the right hand side corresponds to the energy. Hence, the energy is bounded for a fixed index, and Gromov compactness applies.

There might be disk and sphere bubbles. Nonetheless, as in the Hamiltonian Floer case, we can avoid them. For a Lagrangian L, we define $N_L \geqslant 0$ such that

$$\mu_L\colon \pi_2(M, L) \to \mathbb{Z}$$

has image $N_L\mathbb{Z}$.

Proposition 4.32. *Assume that* $N_{L_i} > 2$ *for* $i = 0, 1$. *Then bubbles can only happen in codimension* 3 *or higher.*

In this case, they don't affect $\partial^2 = 0$, and we have a well-defined Lagrangian Floer homology.

The most general case

We discuss the most general case informally. There are many issues:

- Now index doesn't control the energy. The standard solution is to work with Novikov ring Λ_0, as we did in Hamiltonian Floer homology. In this case, we have
$$\partial x = \sum_y \sum_{\substack{\phi \in \pi_0(P_{x,y}) \\ \mu(\phi)=1}} \#\widehat{\mathcal{M}}(x, y) T^{E(\phi)} \cdot y.$$
Gromov compactness works in this setting.
- Transversality issues also exist, as there might by multiply covered bubbles, but they can be dealt with for sphere bubbles.

- In general $\partial^2 \neq 0$. Instead, we have

$$\partial^2 = (\beta_{L_0} - \beta_{L_1}) \cdot \mathrm{id}\,.$$

Here β_{L_i} is the number of disk bubbles of index 2 with boundary in L_i, counted with coefficient $T^{E(D)}$.

Example 4.33. Let us calculate an example for $\partial^2 \neq 0$. Let L_0, L_1 be two circles in punctured complex plane $\mathbb{C}\backslash\{p\}$, as in Figure TBD. They intersect at two points x and y.We have

$$CF_*(L_0, L_1) = \Lambda_0\langle x, y\rangle,\ \partial x = \pm T^A y\ \partial y = \pm T^B x,$$

where A, B are the areas of two regions bounded by L_0 and L_1. Hence

$$\partial^2 x = (\beta_{L_0} - \beta_{L_1})x,\ \partial^2 y = (\beta_{L_0} - \beta_{L_1})y,$$

where $\beta_{L_0} = T^{A+B}, \beta_{L_1} = 0$.

Hence, we cannot expect a homology theory in the most general case. One solution is to introduce more algebra: a graded abelian group together with a map ∂ such that $\partial^2 = C \cdot \mathrm{id}$ is called a *curved chain complex*. One can also study the homological algebra of curved chain complexes.

On the other hand, if $\beta_{L_0} = \beta_{L_1}$ and L_0, L_1 are spin (which allows us to define the Novikov ring over $\mathbb{Q}$), we can from a homology group $HF_*(L_0, L_1)$ as a Λ_0-module. It still has Hamiltonian isotopy invariance, but Theorem 4.23 doesn't hold because of the existence of disk bubbles.

4.4 Application: symplectic Dehn surgery

We now describe another application of Lagrangian Floer homology, due to Seidel [?seidel1999lagrangian]. It exemplifies that two Lagrangians can be smoothly isotopic but not Hamiltonian isotopic. To do this, we first introduce an operation called *symplectic Dehn twist*.

In the simplest case, a Dehn twist for surface is the following procedure: given an embedded circle S^1 in a surface S, pick a tubular neighbourhood $S^1 \times [0, 2\pi] \subset S$, and rotate $S^1 \times \{\theta\}$ by θ, as showed in Figure TBD. This gives a diffeomorphism of S to itself, given by a twist on the neck. We will see that this can be generalized to the high-dimensional and symplectic setting.

Let (M^{2n}, ω) be a symplectic manifold, and $S^n \subset M$ be a Lagrangian sphere. Lagrangian neighbourhood theorem gives us a tubular neighbourhood of S^n that is symplectomorphic to a neighbourhood N of the zero section in T^*S^n. We choose the coordinate as $(u, v) \in T^*S^n \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, and let

$$H(u, v) = \frac{1}{2}|v|$$

be a Hamiltonian on $T^*S^n\backslash S^n$. The Hamiltonian flow induced by H is the geodesic flow on $T^*S^n\backslash S^n$, identified with $TS^n\backslash S^n$ by the round metric on S^n. This gives a map

$$\sigma\colon \mathbb{R}/2\pi\mathbb{Z} \to \mathrm{Aut}(T^*S^n\backslash S^n),$$

thought as the rotation. In particular, $\sigma(\pi)$ is the antipodal map, which can be extended to the zero section $S^n \subset T^*S^n$. We denote this extension by A.

Now pick a bump function $\psi\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}$, such that $\psi(0) = \pi$, $\psi(x) \in (0, \pi)$ for $x \in (0, \epsilon)$, $\operatorname{supp}\psi = [0, \epsilon]$. The *symplectic Dehn twist* $\tau\colon M \to M$ is a symplectomorphism defined by

$$\tau(p, \xi) = \begin{cases} \sigma(e^{i\psi(|\xi|)})(p, \xi), \ \xi \neq 0; \\ A(\xi), \ \xi = 0; \\ \text{id}, \ \text{away from the } \epsilon\text{-neighbourhood.} \end{cases}$$

Proposition 4.34. *In the case of $n = 2$, τ^2 is smoothly isotopic to the identity.*

Sketch of the proof. The idea is that we can use cross product in $\mathbb{R}^3$. The map σ involves rotations about the axis $u \times v$ for $(u, v) \in T^*S^2$. We can interpolate from this to the rotation about u via $tu + (1 - t)(u \times v)$. At $t = 1$, it is well-defined even for $v = 0$. Now τ^2 is a Dehn twist in each fiber around a contractible circle and hence isotopic to the identity. □

Now let M be the plumbing of 3 spheres. Recall that it means gluing disk bundles over these spheres by switching two factors of $D^2 \times D^2$.There are three Lagrangian spheres L_i $(i = 0, 1, 2)$ living in M, as the zero section of S^2 in T^*S^2. A schematic picture is showed in Figure TBD. From the picture, we can see L_1 intersects L_0 and L_2 in one point respectively, and L_0 and L_2 are disjoint. Hence $HF_*(L_0, L_2) = 0$.

Let L_0' be the image of L_0 under the symplectic Dehn twist with respect to L_1. We want to investigate the Lagrangian Floer homology $HF_*(L_0', L_2)$. We can arrange that $L_0 \cap L_1$ is the antipode of $L_1 \cap L_2$. Recall that τ^2 is a rotation by $2\psi(|\xi|)$, and τ interchanges two S^1 on L_0 and L_2, which corresponds to the rotation by π. Hence $L_0' \cap L_2 = S^1$, i.e. they don't intersect transversely. Nevertheless, they have *clean intersection*, which means that $L_0' \cap L_2$ is a smooth submanifold of M and satisfies

$$T(L_0' \cap L_2) = (TL_0'|_{S^1}) \cap (TL_2|_{S^1}).$$

In this case, a theorem of Pozniak holds, which generalizes Floer's Theorem 4.23.

Theorem 4.35 (Pozniak)**.** *Assume that Lagrangians L and L' intersects cleanly. Then*

$$HF_*(L, L') \cong H_*(L \cap L)$$

with appropriate coefficients and gradings.

Hence $HF_*(\tau(L_0), L_2) = HF_*(L_0', L_2) \cong H_*(L_0' \cap L_2) \neq 0$. In other words, L_0 and L_0' are smoothly isotopic but are not Hamiltonian isotopic. In the case that $H_1(L_i) = 0$, it is equivalent to say they are not isotopic through Lagrangians. Hence we have found an isotopy class that contains two different Lagrangian representatives. We also conclude that τ^2 is isotopic to the identity smoothly but not symplectically.

4.5 Fukaya categories (lecture 12)

We conclude this section by introducing Fukaya categories very sketchily. The standard reference is Seidel's book [?seidel2008fukaya]. Another (maybe more friendly) reference is Auroux's notes [?auroux2014beginner].

Let (M, ω) be a symplectic manifold that is compact or convex at infinity. The *Fukaya category* $\mathrm{Fuk}(M, \omega)$ consists of the following data:

- objects: graded Lagrangians, or "branes", $L^\sharp = (L, \theta, P)$, where L is a Lagrangian in M, θ is a grading on L (in the sense in Subsection 4.2), and P is a Pin-structure on L.
- operations: for positive integer k and graded Lagrangians $L_0^\sharp, L_1^\sharp, \ldots, L_k^\sharp$, we have map

  $$\mu_k \colon CF^*(L_{k-1}^\sharp, L_k^\sharp) \otimes \cdots \otimes CF^*(L_0^\sharp, L_1^\sharp) \to CF^{*-2+k}(L_0^\sharp, L_1^\sharp),$$

  defined by

  $$\mu^k(x_k, \ldots, x_1) = \sum_y \#\widehat{\mathcal{M}}(x_1, x_2, \ldots, x_k, y) \cdot y,$$

  where $\widehat{\mathcal{M}}(x_1, x_2, \ldots, x_k, y)$ is the moduli space of J-holomorphic polygons with boundaries on $L_0, \ldots, L_k$ and vertices $x_1, \ldots, x_k, y$.

The maps μ_k satisfy the following A_∞*-relations*:

$$\sum_{l=1}^{k} \sum_{j=0}^{k-l} (-1)^{j+\sum \deg x_j} \mu^{k+1-l}(x_k, \ldots, x_{j+l+1}, \mu^l(x_{j+l}, \ldots, x_{j+1}), \ldots, x_1) = 0.$$

In general, any collection of objects and operations satisfying these relations is called an A_∞*-category*.

Example 4.36. The first A_∞ relation is just $(\mu^1)^2 = 0$, i.e. μ^1 serves as the differential on Floer chain complexes. Hence we write $\mu^1 = \partial$.

The second relation reads as

$$\partial[a, b] = \pm[\partial a, b] + [a, \partial b],$$

where $[a, b] = \mu^2(a, b)$. Hence μ^2 should be thought as a multiplication.

This multiplication is not strictly associative, but it is associative *up to homotopy*, which is encoded in the third A_∞ relation:

$$\pm[[a, b], c] \pm [a, [b, c]] = \pm \partial \mu^3(a, b, c) \pm \mu^3(a, \partial b, c) \pm \mu^3(a, b, \partial c).$$

The first and the most important motivation to study Fukaya categories is *homological mirror symmetry* (HMS), due to Kontsevich. Given a symplectic manifold (M, ω), people conjectured that there is a *mirror* Calabi–Yau manifold $\check{M}$, such that the derived Fukaya category $\mathrm{D^b Fuk}(M, \omega)$ is equivalent (in some sense) to $\mathrm{D^b Coh}(\check{M})$, the derived category of coherent sheaves on $\check{M}$. This conjecture has been showed for many cases, but remains open in general.

5 Instanton Floer homology

We now discuss of Floer homology in low-dimensional topology. The first invariant we are going to talk about is the *instanton Floer homology*. Given a closed 3-manifold Y with a SO(3) or SU(2) bundle (with some additional conditions), it outputs a group $I_*(Y)$.

The standard reference for instanton Floer homology is Donaldson's book [?donaldson2002floer].

5.1 Connections and curvatures

We first review some basic notions. In this section, let $E \to X$ be a (real or complex) vector bundle. We'll focus on the case that E is an Hermitian rank 2 complex vector bundle.

Connections

We denote the collection of E-valued p-forms by

$$\Omega^p(X; E) = \Gamma(\bigwedge^p T^*X \otimes E).$$

Definition 5.1. A *connection* on E is a linear map

$$\nabla \colon \Gamma(E) \to \Omega^1(X; E),$$

satisfying the Leibniz role:

$$\nabla(fs) = f\nabla s + df \otimes s$$

for $f \in C^\infty(M)$, $s \in \Gamma(E)$.

Remark 5.2. Given a connection on E, there is a standard process to form a connection on $E \oplus E$, $E \otimes E$, $\wedge^k E$, $\det E$, etc.

Remark 5.3. As in the case of Riemannian geometry, connection helps us make sense of covariant derivative along a vector field $V \in \mathcal{X}(X)$ by

$$\nabla_V s = (\nabla s)(V) \in \Gamma(E)$$

for $s \in \Gamma(E)$. In particular, given a path γ on X, we can define the *parallel transport* along γ.

Let ∇, ∇' be two connections, and let $a = \nabla - \nabla'$. Then we have

$$a(f \cdot s) = f \cdot a(s)$$

for all $f \in C^\infty(M)$, $s \in \Gamma(E)$. It means that a is tensorial, i.e. a $C^\infty(X)$-module map. Hence $a \in \Omega^1(X, \mathrm{End}(E))$. In general, we have:

Proposition 5.4. *There is a non-canonical one-to-one correspondence between connections on E and forms in* $\Omega^1(X, \mathrm{End}(E))$. *Fixed a base connection* A_0, *this correspondence is given by* $a \mapsto A_0 + a$.

On a local chart U, we always have the trivial connection d, and hence we can write a connection as $\nabla = d + A$, where $A \in \Omega^1(U, \mathrm{End}(E|_U))$. Because of this, sometimes we write a connection ∇ as d_A or ∇_A.

There is a similar notion of connections for principal bundles. Let G be a Lie group. Recall that a principal G-bundle is a fiber bundle $P \to X$ together with a right G action on P that is free and transitive on fibers. Let $\mathfrak{g}$ be the Lie algebra of G.

Definition 5.5. A *connection* on P is a $\mathfrak{g}$-valued 1-form $A \in \Omega^1(P, \mathfrak{g})$, which is invariant under the G action, and restricts to the canonical right invariant form on fibers.

Remark 5.6. A connection A on P defines a family of *horizontal spaces* $H_p = \ker A_p$, i.e. a choice of splitting

$$T_p P = H_p \oplus T_p F_p,$$

where F_p is the fiber of P over p. From this, we can also form the notion of parallel transport.

A vector bundle is the same as it frame bundle, which is a principal bundle with the corresponding structure group G. According to our need, we restrict ourself to the case of $\mathrm{SU}(2)$-bundles.

Proposition 5.7. *We have the following correspondences:*

complex rank 2 *vector bundle E*	$\leftrightsquigarrow$	$\mathrm{GL}(2, \mathbb{C})$-*bundle* $P = Fr(E)$
connections on E	$\leftrightsquigarrow$	*connections on P*
Hermitian metrics on E	$\leftrightsquigarrow$	$\mathrm{U}(2)$ *structures on P*
Hermitian metrics with trivializations of det *E*	$\leftrightsquigarrow$	$\mathrm{SU}(2)$ *structures on P*
compatible connections on E	$\leftrightsquigarrow$	$\mathrm{SU}(2)$-*connections on P*

Here in the last row, the compatibility requires an additional condition that the connection induces the trivial connection d on det E.

We can hence think a connection on E as a $\mathfrak{su}(2)$-valued 1-form on X. Recall that $\mathfrak{su}(2)$ contains traceless antisymmetric endomorphisms of a rank 2 complex space. In our case, the 1-form a as in Proposition 5.4 actually lies in $\Omega^1(X; \mathfrak{su}(2))$, or written as $\Omega^1(X; \mathfrak{g}_E)$ for more generality.

Gauge theory considers gauge equivalent principal bundles as the same.

Definition 5.8. The *gauge group* $\mathcal{G}_E$ is defined as the collection of structure-preserving automorphisms of E.

In our case, the gauge group $\mathcal{G}$ contains fiber-preserving automorphisms $u: E \to E$ that preserves the Hermitian metric on E and induces the identity on $\det E$. It naturally acts on the sections of E by pointwise multiplication, and on SU(2)-connections by

$$\nabla_{u(A)} s = u \nabla_A (u^{-1} s),$$

or locally

$$u(A) = uAu^{-1} - (du)u^{-1}.$$

Curvatures

Given a connection A, we can extend its action to higher degree E-valued forms, and hence form a sequence

$$\Omega^0(X;E) \xrightarrow{d_A} \Omega^1(X;E) \xrightarrow{d_A} \Omega^2(X;E) \xrightarrow{d_A} \cdots$$

by

$$d_A(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^{\deg \omega} \omega \wedge d_A \theta,$$

where ω is an ordinary form on X, and θ is an E-valued form.

Definition 5.9. The *curvature* F_A is given by $d_A^2 \colon \Omega^0(X;E) \to \Omega^2(X;E)$.

We list some basic properties of curvature.

Proposition 5.10. *Let A be a connection on E, and F_A is the curvature of A. Then:*

- $F_A(fs) = f \cdot F_A(s)$ *for* $f \in C^\infty(X)$ *and* $s \in \Gamma(E)$*, i.e.* $F_A \in \Omega^2(X; \mathrm{End}(E))$.
- $F_{A+a} = F_A + d_A a + a \wedge a$ *for* $a \in \Omega^1(X; \mathfrak{g}_E)$.
- *(Bianchi identity)* $d_A F_A = 0$.
- *Locally we have* $F_A = dA + A \wedge A$*, where* $\nabla = d + A$.
- *For* $u \in \mathcal{G}_E$*, we have* $F_{uA} = uF_A u^{-1}$.

Recall that for an oriented Riemannian manifold X of dimension n, we have the *Hodge star operator*

$$\star \colon \Omega^p(X) \to \Omega^{n-p}(X).$$

It can be easily generalized to E-valued forms

$$\star \colon \Omega^p(X;E) \to \Omega^{n-p}(X;E).$$

For this, we can define the "codifferential"

$$d_A^* \colon \Omega^p(X;E) \to \Omega^{p-1}(X;E)$$

by

$$d_A^* = (-1)^{np+1} \star d \star.$$

It is the adjoint of d_A under the L^2-norm:

$$\int_X \langle d_A s, t \rangle = \int_X \langle s, d_A^* t \rangle.$$

Remark 5.11. TBD

The Yang–Mills functional

Definition 5.12. The *Yang–Mills equation* is the following:

$$d_A^* F_A = 0.$$

We focus on the case that X has dimension 4 and E is an $\mathrm{SU}(2)$-bundle on X. In this case, the Hodge star operator acts as an involution on $\Omega^2(X)$, and hence gives eigenspace decomposition $\Omega^2(X) = \Omega^+(X) \oplus \Omega^-(X)$, given by

$$a \mapsto (\frac{a + \star a}{2}, \frac{a - \star a}{2}).$$

Similar result holds for $\Omega^2(X; E)$.

A particular family of solutions to the Yang–Mills equation is the *anti-self-dual (ASD) connections*. That is, a connection A satisfying the anti-self-dual equation

$$\star F_A = -F_A,$$

or equivalently,

$$F_A^+ = 0.$$

Why do we care about ASD solutions?

Definition 5.13. For $\mathrm{SU}(2)$-connection A on E, the *Yang–Mills functional* is defined as

$$YM(A) = \int_X |F_A|^2.$$

When X is closed, the Yang–Mills functional can be written as

$$YM(A) = \int_X |F_A^+|^2 + \int_X |F_A^-|^2.$$

On the other hand, we have

$$\int_X (|F_A^-|^2 - |F_A^+|^2) = \int_X \mathrm{tr}(F_A \wedge F_A) = 8\pi^2 k,$$

where $k = c_2(E)[X] \in \mathbb{Z}$ is the *instanton number*. Hence

$$YM(A) = 2\int_X |F_A^+|^2 + 8\pi^2 k(E),$$

and:

Proposition 5.14. *ASD solutions are corresponding to the minimal points of the Yang–Mills functional.*

Remark 5.15. Similarly, we can also consider the self-dual solutions, which are the maximal points of the Yang–Mills functional. We'll see that solutions of the Yang–Mills equation correspond to critical points of the Yang–Mills functional.

Remark 5.16. What do $A \wedge A$ and $\mathrm{tr}(F_A \wedge F_A)$ actually mean? In general, let $A, B \in \Omega^*(X; \mathfrak{g}_E)$, locally written as

$$A = A_{i_1 \ldots i_m} dx^{i_1} \ldots dx^{i_m},\ B = B_{j_1 \ldots j_n} dx^{j_1} \ldots dx^{j_n}.$$

Then

$$A \wedge B = A_{i_1 \ldots i_m} B_{j_1 \ldots j_n} dx^{i_1} \ldots dx^{i_m} dx^{j_1} \ldots dx^{j_n} \in \Omega^{n+m}(X; \mathrm{End}(E)).$$

Notice that it is not valued in $\mathfrak{g}_E$ in general. However, when $A = B$ is a 1-form, then

$$A \wedge A = \sum_{i<j} (A_i A_j - A_j A_i) dx^i dx^j = \sum_{i<j} [A_i, A_j] dx^i dx^j$$

is valued in $\mathfrak{g}_E$. This is not true if $A \neq B$ or A is a 2 form. It makes sense to talk about the trace of the components in $F_A \wedge F_A$.

5.2 Yang–Mills equation in dimension 4 (lecture 13)

We briefly introduce the moduli space of solutions to the Yang–Mills equation in dimension 4. As usual, to form a nice moduli space, we need to discuss Fredholmness, transversality, compactness, orientation, etc.

Proposition 5.17. *The ASD equation*

$$F_A^+ = 0$$

is elliptic modulo gauge.

Recall that the gauge group $\mathcal{G}_E$ acts on $\mathrm{SU}(2)$-connections by conjugation. It descends onto the collection of Yang–Mills connections naturally. Proposition 5.17 states that after choosing a gauge, it becomes an elliptic PDE (system). The most common choice is the *Columb gauge*

$$d^*(A - A_0) = 0.$$

Yang–Mills equation together with the Columb gauge equation form an elliptic system of PDEs.

Theorem 5.18. *Fixed an instanton number k. For generic Riemannian metric on X, the moduli space*

$$\mathcal{M}_{X,k} = \{\text{solutions to } F_A^+ = 0 \text{ for } k(E) = k\} / \mathcal{G}_E$$

is a smooth manifold of dimension $8k - 3(1 - b_1 + b_2^+)$.

Here b_1 is the first Betti number of X, and b_2^+ is the maximal possible dimension of subspaces of $H^2(X)$ with the restricted intersection form positive definite.

Remark 5.19. Here the situation is nice - we don't need to perturb the equation. However, we'll need to do this for 3-dimensional case.

For compactness, we have the following:

Theorem 5.20 (Uhlenbeck compactness)**.** *A sequence of ASD connections with bounded energy has a convergent subsequence up to gauge. More precisely, let A_k be a sequence of ASD connections with bounded energy. Then we can find a subsequence A_{n_k} and gauge transformations u_k such that $u_k A_{n_k}$ converges to an ASD solution A, possibly with some bubbles.*

Here bubbles mean the phenomenon that energy concentrates near a point. After changing the metric, it can be viewed as a splitting $X \cong X \# S^4$, and the connection restricts to a standard solution on S^4 with instanton number 1. From this, we can get the following:

Theorem 5.21. *We have compactification*

$$\overline{\mathcal{M}_{X,k}} = \mathcal{M}_{X,k} \coprod (\mathcal{M}_{X,k-1} \times X) \coprod \left(\mathcal{M}_{X,k-2} \times \mathrm{Sym}^2(X)\right) \coprod \cdots .$$

Here the second term corresponds to the case that there is one bubble point (which causes the instanton number decreases by 1), and so on. Recall from Theorem 5.18 that $\mathcal{M}_{X,k-1}$ has codimension 8 in $\mathcal{M}_{X,k}$, and $\mathcal{M}_{X,k-1} \times X$ has codimension 4. In other words:

bubbles only happen in codimension at least 4.

In particular, the moduli space is compact if its expected dimensional is at most 3.

The orientation of the moduli space is given by the trivialization of the index bundle of

$$d_A^+ \oplus d_A^* : \Omega^1(X;E) \to \Omega^0(X;E) \oplus \Omega^2(X;E).$$

By Hodge theory, it is same as choosing an orientation for $H_1(X) \oplus H_2^+(X)$.

Definition 5.22. Let k be the instanton number such that the expected dimension of $\mathcal{M}_{X,k}$ is zero. Then the *Donaldson invariant* $\mathcal{D}_X \in \mathbb{Z}$ is defined as the oriented count of points in $\mathcal{M}_{X,k}$.

Remark 5.23. In general, we can define Donaldson invariants for higher dimensional moduli spaces by integration on certain hypersurfaces.

5.3 The idea of instanton Floer homology

Donaldson invariant is independent on the choice of metrics, and hence is an invariant for smooth manifolds. It can detect exotic smooth structures. However, it is difficult to compute directly from the definition. That was the original motivation of Floer to define instanton Floer homology, and a relative version of Donaldson invariants.

The plan is as follows. For a closed manifold X^4 with an embedded hypersurface Y^3, we can cut X along Y to obtain two pieces X_1 and X_2, with boundary Y and $-Y$ respectively as in Figure TBD. For a manifold X_i with boundary Y, we can define a quantity $\mathcal{D}_{X_i}$, the *relative Donaldson invariant*, living in the instanton Floer homology group of the boundary $I_*(Y)$, such that the following gluing formula holds.

Theorem 5.24. *Let $X = X_1 \cup_Y X_2$. Then*

$$\mathcal{D}_X = \langle \mathcal{D}_{X_1}, \mathcal{D}_{X_2} \rangle,$$

where $\mathcal{D}_{X_2}$ lives in $I_*(-Y)$, which is canonically isomorphic to the dual $I_*(Y)^\vee$, and the pairing is the natural pairing.

But we have not even defined the group $I_*(Y)$! OK, let's give a sketch here first. Let Y be a closed 3-manifold. The instanton Floer chain complex $CI_*(Y)$ consists of the following data:

- the coefficient ring can be $\mathbb{Z}$;
- generators are time-independent solutions to the ASD equation on $\mathbb{R} \times Y$, modulo gauge;
- the differential of an ASD connection x is

$$\partial x = \sum_y \#\widehat{\mathcal{M}}(x, y) \cdot y,$$

  where $\#\widehat{\mathcal{M}}(x, y)$ counts solutions to the ASD equation asymptotic to x as $t \to -\infty$ and to y as $t \to +\infty$, modulo gauge.

We then define the instanton Floer homology $I_*(Y)$ as the homology of $CI_*(Y)$. We will see many problems in this process, but now we just pretend everything is fine for now. Let X be a 4-manifold with boundary Y. We can add an infinity cylindrical end to form a manifold-without-boundary

$$\hat{X} = X \cup_Y (Y \times \mathbb{R}_{\geqslant 0}).$$

Define

$$\Phi_X = \sum_x \#\widehat{\mathcal{M}}(\hat{X}, x) \cdot x \in CI_*(Y),$$

where we count 0-dimensional moduli spaces of solutions to the ASD equation on $\hat{X}$ asymptotic to x. The element Φ_X is closed because we have compactification

$$\partial\overline{\mathcal{M}(\hat{X}, x)} = \bigcup (\mathcal{M}(\hat{X}, y_n) \times \widehat{\mathcal{M}}(y_n, y_{n-1}) \times \cdots \times \widehat{\mathcal{M}}(y_1, x)).$$

When $\dim \mathcal{M}(\hat{X}, x) = 1$, we have

$$\partial\overline{\mathcal{M}(\hat{X}, x)} = \bigcup_y (\mathcal{M}(\hat{X}, y) \times \widehat{\mathcal{M}}(y, x)).$$

We then define

$$\mathcal{D}_X = [\Phi_X] \in I_*(Y).$$

5.4 Chern–Simons functional

We now come into details on defining $I_*(Y)$.

The first task is to understand ASD solutions on the infinity cylinder $\mathbb{R} \times Y$. Recall that SU(2)-bundles on Y^3 are trivial, and the same holds on $Y \times \mathbb{R}$. This can be seen by looking at the classifying map $Y^3 \to \mathbb{HP}^4 = BSU(2)$, and notice that $\mathbb{HP}^4$ has a trivial 3-skeleton. From now on, we fix a trivialization $E = Y \times \mathbb{C}^2$, and SU(2)-connections are now canonically corresponding to elements in $\Omega^1(Y; \mathfrak{g}_E)$ by comparing with the trivial connection on E.

Let A be a connection on $\pi_* E$, where π is the projection from $\mathbb{R} \times Y$ to Y. Write A as

$$A = A(t) + \alpha(t) dt,$$

where $A(t)$ is a family in $\Omega^1(Y; \mathfrak{su}(2))$, and $\alpha(t) \in C^\infty(Y; \mathfrak{su}(2))$.

Lemma 5.25. *We can arrange $\alpha(t) = 0$ after gauge. In this case, we say the connection is* in temporal gauge.

Proof. Let $u \colon \mathbb{R} \times Y \to SU(2)$ be a gauge transformation. Its action on A is given as

$$A \mapsto uAu^{-1} - (du)u^{-1},$$

which has dt component

$$\left(u(t)\alpha(t)u^{-1}(t) - \frac{\partial u}{\partial t} u^{-1}(t) \right) dt.$$

Eliminating this is a first order ODE, and hence can be done. Furthermore, the choice of the gauge is unique if $u(0)$ is fixed. □

We can now expand the ASD equation on the cylinder in temporal gauge. To avoid confusion, we denote the Hodge star and differential on $\mathbb{R} \times Y$ by $\star_4$ and d_4, and use the original $\star$ and d to denote operations in dimension 3. Under this convention, we have

$$F_A = d_4 A + A \wedge A = dA(t) + \frac{\partial A(t)}{\partial t} dt + A(t) \wedge A(t),$$

and

$$\star_4 F_A = -dt \wedge (\star dA(t) + \star(A(t) \wedge A(t))) - \star \frac{\partial A(t)}{\partial t} = -dt \wedge \star F_{A(t)} - \star \frac{\partial A(t)}{\partial t}.$$

Therefore the ASD equation is equivalent to

$$\frac{\partial A(t)}{\partial t} = \star F_{A(t)}. \tag{5.26}$$

Solutions constant in t are just flat connections modulo gauge, or equivalently, SU(2)-representations of the fundamental group $\pi_1(Y)$ modulo conjugation. They don't depend on the metric on Y.

We claim that Equation 5.26 is a negative gradient flowline equation on the configuration space

$$\mathcal{B} = \{\text{SU}(2)\text{-connections on } Y\}/\mathcal{G}.$$

Recall that on a closed 4-manifold X we have

$$\int_X \text{tr}(F_A \wedge F_A) = 8\pi^2 k(E) \in 8\pi^2\mathbb{Z}. \tag{5.27}$$

The Chern–Simons functional is defined as a relative version of this:

Definition 5.28. Let A be an SU(2)-connection on Y^3. Choose X^4 such that $\partial X = Y$, and extend A to a connection on X. The *Chern–Simons functional* is defined as

$$CS(A) = \frac{1}{8\pi^2}\int_X \text{tr}(F_A \wedge F_A).$$

Remark 5.29. The existence of such null-cobordism X comes from general cobordism theory.

Given two choices of the extensions, we can close them up to obtain a closed 4-manifold with separated hypersurface Y. We then deduce from Equation 5.27 that:

Proposition 5.30. *The quantity $CS(A)$ is well-defined in $\mathbb{R}/\mathbb{Z}$, i.e. it is independent of the choice of X and the extension of the connection.*

Lemma 5.31. *Let A be a connection in temporal gauge on $[0,1]\times Y$. Then*

$$\text{tr}(F_A \wedge F_A) = d\left(\text{tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A)\right).$$

Proof. This follows from direct calculation:

$$\begin{aligned}\text{LHS} &= \text{tr}\left((dA + A\wedge A)\wedge(dA + A \wedge A)\right)\\ &= \text{tr}\left((dA\wedge dA + 2dA\wedge A + A\wedge A\wedge A\wedge A)\right)\\ &= \text{tr}\left((dA\wedge dA + 2dA\wedge A)\right)\\ &= \text{RHS}.\end{aligned}$$

Here the points are $\text{tr}(A \wedge B) = \text{tr}(B\wedge A)$, and $A\wedge A\wedge A\wedge A = 0$ since there is no dt component appearing in A. □

Using this lemma, we can actually write CS in a more "intrinsic" form.

Proposition 5.32. *We have*

$$CS(A) = \frac{1}{8\pi^2}\int_Y \text{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right).$$

Proof. Fix a null-cobordism X. We choose a bump function β such that $\beta = i$ near $t = i$, $i = 0, 1$. We then extend A by $\beta(t)A$ on $X = X \cup_Y (Y \times [0,1])$. Now by Lemma 5.31, we have

$$\begin{aligned} CS(A) &= \frac{1}{8\pi^2}\int_{Y\times[0,1]} \mathrm{tr}(F_A\wedge F_A) \\ &= \frac{1}{8\pi^2}\int_{Y\times[0,1]} d\left(\mathrm{tr}(A\wedge dA+\frac{2}{3}A\wedge A\wedge A)\right) \\ &= \frac{1}{8\pi^2}\int_Y \mathrm{tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right). \end{aligned}$$

□

Exercise 5.33. The Chern–Simons function, valued in $\mathbb{R}/\mathbb{Z}$, is gauge invariant.

Proposition 5.34. *The gradient flow of CS is given by*

$$\nabla CS(A) = \frac{1}{4\pi^2}\star F_A.$$

Proof. This is also direct calculation:

$$\begin{aligned} dCS_A(a) &= \lim_{h\to 0}\frac{CS(A+ha)-CS(A)}{h} \\ &= \frac{1}{8\pi^2}\int_Y \mathrm{tr}\left(a\wedge dA + A\wedge da + 2A\wedge A\wedge a\right) \\ &= \frac{1}{4\pi^2}\int_Y \mathrm{tr}\left(a\wedge(dA+A\wedge A)\right) \\ &= \frac{1}{4\pi^2}\langle a,\star F_A\rangle. \end{aligned}$$

Here the third line uses

$$d\,\mathrm{tr}(A\wedge a) = \mathrm{tr}(dA\wedge a - A\wedge da).$$

□

In conclusion:

ASD connections on $\mathbb{R}\times Y \leftrightsquigarrow$ downward gradient flowlines of $-4\pi^2 CS$.

5.5 Instanton Floer homology (lecture 14)

Formally, the instanton Floer homology is the Floer homology of

$$-4\pi^2 CS\colon \mathcal{B}\to\mathbb{R}/4\pi^2\mathbb{Z}.$$

However, there will be many issues. Let us discuss them in order.

Completion

Ideally, we want to complete the configuration space $\mathcal{B}$ by a L^2_k-completion of the affine space

$$\mathcal{A} = \{\mathrm{SU}(2)\text{-connections}\} = \Omega^1(Y, \mathfrak{su}(2)),$$

quotient by the L^2_{k+1}-completion of the gauge group $\mathcal{G}_{k+1}$. However, the resulting $\mathcal{B}_k$ is *not* a Hilbert manifold. This is because the gauge group action is not free. In fact, it only acts freely on irreducible connections.

Definition 5.35. Let A be an $\mathrm{SU}(2)$-connection on Y. Let

$$\Gamma_A = \{u \in \mathcal{G} : u^*A = A\}$$

be its stabilizer. It has three possibilities: $\{\pm 1\}$ (the center of $\mathrm{SU}(2)$), S^1, or the whole $\mathrm{SU}(2)$. We say A is *irreducible* if $\Gamma_A = \{\pm 1\}$; otherwise we say it is *reducible*.

Remark 5.36. Reducibility of A means that the parallel transport induced by A preserves a line subbundle of E. For flat connection, it corresponds to the reducibility of the corresponding $\mathrm{SU}(2)$-representation.

Hence we only want to deal with irreducibles. Let

$$\mathcal{A}^* \subset \mathcal{A}$$

be the open set of irreducibles, and let

$$\mathcal{B}^* = \mathcal{A}^*/\mathcal{G}.$$

We can now perform the L^2_k-completion and get a Hilbert manifold $\mathcal{B}^*_k$.

Fredholmness and grading

Recall from Proposition 5.17 that the ASD equation is elliptic modulo gauge. For a general path $A(t)$ in $\mathcal{B}$, we can make sense of its index

$$\mu(A) = \mathrm{ind}\left(D_A(\frac{d}{dt} + \star F)\right).$$

In particular, for irreducible flat connections (which are generators of the Floer chain complex $CI_*(Y)$) x and y, and an ASD solution A on $\mathbb{R} \times Y$ that asymptotic to x and y as $t \to \pm\infty$ respectively, the index $\mu(A)$ gives the expected dimension of the moduli space $\widehat{\mathcal{M}}(x, y)$ near A.

For such x, y, we have

$$\frac{1}{8\pi^2}E(A) = \frac{1}{8\pi^2}\int_{\mathbb{R}\times Y} \mathrm{tr}(\mathrm{F_A} \wedge \mathrm{F_A}) \equiv CS(x) - CS(y) \pmod{\mathbb{Z}}.$$

Recall that on a closed manifold $\frac{1}{8\pi^2}E(A) = k \in \mathbb{Z}$, and Theorem 5.18 tells us that the mod 8 expected dimension of the ASD moduli space doesn't depend on k. This also holds in our case. There is an infinite cyclic cover

$$\widetilde{\mathcal{B}^*} = \mathcal{A}/\mathcal{G}_0 \to \mathcal{B}^*,$$

where $\mathcal{G}_0$ consists of gauge transformations that homotopic to the identity, and the Chern–Simons function and the index are well defined on $\widetilde{\mathcal{B}^*}$ as $\mathbb{R}$-valued functions under the convention that the index of the trivial connection is zero. Furthermore, let u_0 be a generator of $\pi_0(\mathcal{G}) \cong \mathbb{Z}$. Then we have

$$CS(u_0 \cdot A) = CS(A) + 1, \; \mu(u_0 \cdot A) = \mu(A) + 8.$$

In conclusion, we have a well-defined $\mathbb{Z}/8$-grading on $CI_*(Y)$, given by $\mu(x, y) = \mu(A)$. In further, we also see that the index controls the energy, as in the monotone case in Lagrangian Floer homology.

Transversality

The first issue on transversality is that the generators of $CI_*(Y)$ may not be discrete (for example, on Brieskorn sphere $\Sigma(2, 3, 5, 7)$). Recall that they are irreducible flat connections on E and are independent of the choice of the metric. Hence one cannot make them discrete by perturbing the metric. Nonetheless, we can still achieve this by perturbing the equation.

Theorem 5.37. *The space of solutions of the equation*

$$\star F_A = \eta$$

modulo gauge is discrete for generic η.

The second issue is that we require the moduli space $\mathcal{M}(x, y)$ is a smooth manifold of the expected dimension $\mu(x, y) \in \mathbb{Z}/8$. This needs more general perturbation, called the "holonomy perturbation", which we will not discuss here. It is complicated but at least doable!

Compactness and gluing

As in the closed case Theorem 5.21, we can form a compactification of $\mathcal{M}(x, y)$ that bubbles only appear in moduli spaces of dimension 8 or higher. Hence, we can expect that bubbles don't hurt $\partial^2 = 0$, which only needs moduli spaces of dimension at most 2.

The serious problem is that flowlines may involve reducible connections, which are excluded from the generators of $CI_*(Y)$. More precisely, the compactification has the form

$$\overline{\mathcal{M}}(x, y) = \mathcal{M}(x, y) \coprod (\mathcal{M}(x, z_1) \times \cdots \times \mathcal{M}(z_k, y)) \coprod (\text{bubbles}),$$

and the z_k's may be reducible.

This is indeed a problem when $H_1(Y) \neq 0$. For now, we assume that $H_1(Y) = 0$, i.e. Y is an integral homology sphere. This has already contained many interesting examples, such as the $1/n$-surgery on knots.

In this case, let A be a reducible flat connection. Then the corresponding representation $\rho\colon \pi_1(Y) \to \mathrm{SU}(2)$ factors through a circle S^1, and hence factors through the abelization $H_1(Y) = 0$. Hence, the only reducible is the trivial connection θ.

We split the quotient process into two steps. Let $\widetilde{\widetilde{\mathcal{M}}}(x,y)$ be the space of ASD solutions quotient a free action by *based* gauge transformations. Then

$$\widehat{\mathcal{M}}(x,y) = \widetilde{\widetilde{\mathcal{M}}}(x,y) / \mathrm{SO}(3).$$

The point is that the gluing process can be done on $\widetilde{\widetilde{\mathcal{M}}}(x,y)$. Therefore, we have

$$\dim\left(\widetilde{\widetilde{\mathcal{M}}}(x,\theta) \times \widetilde{\widetilde{\mathcal{M}}}(\theta,y)\right) = \dim \widetilde{\widetilde{\mathcal{M}}}(x,y) - 1.$$

After quotient SO(3), we have

$$[\dim\left(\widehat{\mathcal{M}}(x,\theta) \times \widehat{\mathcal{M}}(\theta,y)\right) = \dim \widehat{\mathcal{M}}(x,y) - 4.$$

After perturbation, θ can only appear in non-negative dimensions. Hence it doesn't affect $\partial^2 = 0$!

Question from the class Why do we need the connection θ to be trivial?

A: Because SO(3) acts trivially in this case. Others might have stabilizer S^1, and it will drop the dimension only by 1, which is not enough for proving the invariance using continuation map.

Orientations

Recall that we need an orientation of $H_1(X) \oplus H_2^+(X)$. In our case that $X = \mathbb{R} \times Y$ and $H_1(Y) = 0$, this is automatic.

Finally we can form the instanton Floer homology group $I_*(Y)$ for integral homology sphere Y! It is a $\mathbb{Z}/8$-graded, $\mathbb{Z}$-coefficient invariant of Y.

Example 5.38. We have $I_*(S^3) = 0$ because there is no irreducible connection.

Example 5.39. Let P be the Poincaré homology sphere. Then

$$\pi_1(P) = \langle a, b, c \mid a^2 = b^3 = c^5 = abc \rangle,$$

which has 2 irreducible representations modulo conjugation. Calculation on indices then gives $I_*(P) = (\mathbb{Z}, 0, 0, 0, \mathbb{Z}, 0, 0, 0)$.

Another invariant of 3-manifold related to the representation of fundamental groups is the *Casson invariant* $\lambda(Y)$. It is related to the instanton Floer homology we just constructed:

Theorem 5.40. *We have*

$$\chi(I_*(Y)) = 2\lambda(Y).$$

Extra structure on $I_*(Y)$

We can extract more information on $I_*(Y)$. The first two maps are from the trivial connection θ. Let x be a generator of $CI_1(Y)$. Define $D_1(x)$ to be the count of 0-dimensional moduli space from x to θ. One can show that $D_1\partial = 0$, and it gives a function

$$D_1\colon I_1(Y) \to \mathbb{Z}.$$

Dually, we can define a map

$$D_2\colon \mathbb{Z} \to CI_4(Y)$$

by

$$D_2 = \sum_x \#\widehat{\mathcal{M}}(\theta, x) \cdot x,$$

which gives an element in $I_4(Y)$.

There is also an action on the Floer chain complex $CI_*(Y;\mathbb{Q})$ as follows. Recall that for irreducibles x and y, we have

$$\widehat{\mathcal{M}}(x,y) = \widetilde{\widetilde{\mathcal{M}}}(x,y) / \operatorname{SO}(3).$$

The suspension of it gives the parametrized moduli space

$$\Sigma\widehat{\mathcal{M}}(x,y) \subset \mathcal{B}^* = \widetilde{\mathcal{B}^*} / \operatorname{SO}(3).$$

Here the last quotient is free since we are only considering irreducibles, and it gives an SO(3)-bundle on $\Sigma\widehat{\mathcal{M}}(x,y)$, i.e. a map

$$f\colon \Sigma\widehat{\mathcal{M}}(x,y) \to \operatorname{BSO}(3).$$

Using $\mathbb{Q}$-coefficient, it is easy to compute the cohomology of the classifying space:

$$H^*(\operatorname{BSO}(3);\mathbb{Q}) = H^*(\operatorname{BSU}(2);\mathbb{Q}) = \mathbb{Q}[u],$$

where u is an element of degree 4. It is a universal characteristic class: the pullback f^*u gives information about the SO(3)-bundle on $\Sigma\widehat{\mathcal{M}}(x,y)$. Let

$$t_{x,y} = (f^*u)[\Sigma\widehat{\mathcal{M}}(x,y)] \in \mathbb{Q}.$$

Here we only pair f^*u with 4-dimensional moduli spaces, i.e. $\mu(x,y) = 4$. We then have a map

$$U\colon CI_*(Y;\mathbb{Q}) \to CI_{*-4}(Y;\mathbb{Q}),$$

given by

$$x \mapsto \sum_y t_{x,y} \cdot y.$$

It is not always a chain map because of the existence of reducibles.

Proposition 5.41. *Under this setting, we have*

$$\partial U - U\partial = -\frac{1}{4} D_2 D_1.$$

Remark 5.42. We can also do this with $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ coefficients, but rational coefficient is the easiest to deal with.

5.6 Variants of instanton Floer homology (lecture 15)

Up to now we only establish the instanton Floer homology for integral homology spheres, which is Floer's original work. In the case of $H_1(Y) \neq 0$, we have several modifications.

Floer homology for admissible bundles

The first solution is that we can consider an SO(3)-bundle P over Y with flat connections on P. The second Stiefel–Whitney class $w_2(P) \in H^2(Y; \mathbb{Z}/2\mathbb{Z})$ characterizes such bundle P. Recall that there is a map

$$\mathrm{U}(2) = \mathrm{SU}(2) \times_{\mathbb{Z}/2\mathbb{Z}} S^1 \to \mathrm{SU}(2)/\{\pm 1\} = \mathrm{SO}(3),$$

given by projection to the first factor.

Proposition 5.43. *Any such P can be lifted to a* U(2)*-bundle.*

Proof. Recall that U(2)-bundles are characterized by the first Chern class $c_1(P) \in H^2(Y; \mathbb{Z})$, which restricts to the second Stiefel–Whitney class $w_2(P)$. Now the result follows from the fact that the connecting homomorphism b in the long exact sequence

$$\cdots \to H^2(Y; \mathbb{Z}) \to H^2(Y; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{b} H^3(Y; \mathbb{Z}) \to \cdots$$

associated to the coefficient change

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

is zero, since $H^3(Y; \mathbb{Z}) = \mathbb{Z}$ is torsion-free. □

We continue the discussion in the language of vector bundles. We choose a U(2) lift, which produces a rank 2 Hermitian vector bundle E. Let

$$\mathcal{A} = \{\text{connections on } E \text{ with fixed induced connection on } \det E\}.$$

The gauge group $\mathcal{G}$ acts on $\mathcal{A}$ in a similar way.

Lemma 5.44. *Reducible solution doesn't exist if and only if $h(w_2(E)) \neq 0$, where*

$$h\colon H^2(Y; \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}(H_2(Y); \mathbb{Z}/2\mathbb{Z}).$$

We say E is *admissible* if $h(w_2(E)) \neq 0$. The lemma tells us admissible bundles don't admit no reducible solution.

Example 5.45. Let $Y = S^1 \times S^2$. Then $H^2(Y; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. There are two SO(3)-bundles over Y: one is trivial and another is admissible.

We can form the instanton homology group $I_*(Y, P)$. It also carries a U-map of degree -4. Because of the avoidance of reducibles, U is a chain map now, and it descends onto an action on $I_*(Y, P)$.

When does admissible bundle exist?

Proposition 5.46. *Admissible bundle exists if and only if $b_1(Y) > 0$, i.e. Y is not a rational homology sphere.*

Hence we can define instanton homology for all 3-manifolds but rational homology spheres that are not integral homology sphere. Weird!

Framed instanton Floer homology

We can use a trick to define Floer homology for all 3-manifolds.

Definition 5.47. Let Y be a 3-manifold. Let P_0 be the trivial SO(3)-bundle over Y, and let P_{adm} be an admissible bundle over T^3. The *framed instanton Floer homology* is defined as the instanton homology for admissible pairs

$$I_*^\#(Y) = I_*(Y\#T^3, P_0\#P_{adm}).$$

Remark 5.48. There is essentially one admissible bundle over T^3 because of the homogeneous action on T^3.

Remark 5.49. For integral homology sphere Y, $H^\#(Y)$ can be expressed in terms of $(I_*(Y), D_1, D_2, U)$, due to Scaduto [?scaduto2015instantons].

Equivariant instanton homology

Another way to define instanton homology of rational homology spheres is to take all the reducibles into account in an equivariant way. This is due to Miller [?miller2019equivariant].

We first recall some basics of equivariant homology. Let G be a Lie group acting on a space X. The *Borel homology* is defined as the "homotopy quotient"

$$H_*^G(X) = H_*(X \times_G EG),$$

where $EG \to BG$ is the universal G-bundle. It gives a $H^*(BG)$-module.

Example 5.50. We have

$$H_*^G(X) = H_*(X/G)$$

if G acts freely on X. When the action is trivial, we have

$$H_*^G(X) = H_*(X \times BG).$$

Recall that under the SO(3)-action by constant gauge transformations, reducibles can have stabilizers SO(3) or S^1, and irreducibles have stabilizers $\{1\}$. Hence the orbit can be a point, a sphere $S^2 = \mathrm{SO}(3)/S^1$, or the whole $\mathrm{SO}(3) = \mathbb{RP}^3$.

We now construct the Floer chain complex $\widetilde{CI}_*(Y)$ by

$$\widetilde{CI}_*(Y) = \bigoplus_a C_*(\mathcal{O}_a),$$

where the sum takes for all flat connection on the trivial SU(2)-bundle over Y modulo gauge, $\mathcal{O}_a$ refers to the orbit of a under the SO(3)-action, and $C_*(\mathcal{O}_a)$ is the "geometric homology" chain complex [?lipyanskiy2014geometric]. We can define differential $\widetilde{CI}_*(Y)$ that counts solutions from a chain to another, by the geometric homology construction, which generates the chain complex by manifolds with corners. It then produces the tilde version instanton Floer homology $\widetilde{I}_*(Y)$, which is also $\mathbb{Z}/8\mathbb{Z}$-graded.

The equivariant version is defined through the dg tensor product with the dg module of the universal bundle:

$$I_*^+(Y) = H_*(\widetilde{CI}_*(Y) \otimes_{C_*(\mathrm{SO}(3))} C_*(E\,\mathrm{SO}(3))).$$

Similar to the ordinary Borel homology, it carries an action of $H^*(B\,\mathrm{SO}(3);\mathbb{Z})$. These invariants can also be recovered from $(I_*(Y), D_1, D_2, U)$ when Y is an integral homology sphere.

5.7 Applications of instanton Floer homology

We sketch some topological applications of instanton homology.

Frøyshov invariant

Let Y be an integral homology sphere. From the data $I_*(Y), D_1, D_2, U)$, one can extract a numerical invariant $h(Y) \in \mathbb{Z}$, called the *Frøyshov invariant* [?froyshov2002equivariant]. Roughly speaking, it measures how much irreducible solutions affect the reducible solution.

Frøyshov invariant can be used to study the structure of homology cobordism group. Recall that a *homology cobordism* between two integral homology spheres Y_0, Y_1 is a cobordism W, such that $\partial W = (-Y_0) \coprod Y_1$, and $H_*(W, Y_i) = 0$. The *homology cobordism group*

$$\Theta^3_{\mathbb{Z}} = \{\text{integral homology spheres}\}/\text{homology cobordisms}.$$

Theorem 5.51. *The Frøyshov invariant is stable under homology cobordisms. Hence it descends onto a homomorphism*

$$h\colon \Theta^3_{\mathbb{Z}} \to \mathbb{Z}.$$

Further, we have

$$h(P) = 1$$

for Poincaré sphere P. Therefore h is surjective, and this gives a $\mathbb{Z}$-summand of $\Theta^3_{\mathbb{Z}}$.

People also uses other invariants, such as (equivariant) monopole Floer homology and involutive Heegaard Floer homology, to study the homology cobordism group, but this is the first known infinity summand of $\Theta^3_{\mathbb{Z}}$.

Khovanov homology detects unknot

Khovanov homology [?khovanov2000categorification] is a combinatorial invariant for knots. It outputs a bigraded abelian group $\mathrm{Kh}(K)$ for a knot K.

Theorem 5.52 (Kronheimer–Mrowka [?kronheimer2011khovanov]). *There is a spectral sequence with E_2 page isomorphic to $\mathrm{Kh}(K)$ and converging to $I^\sharp(K^r)$). Here $I^\sharp$ is the singular instanton homology, defined using connections singular along the knot, and K^r denotes K with orientation reversed.*

As a corollary of the spectral sequence, the rank of Khovanov homology is always at least the rank of singular instanton homology. Kronheimer and Mrowka also showed that $I^\sharp$ detects unknot:

Theorem 5.53 (Kronheimer–Mrowka [?kronheimer2011khovanov]). *If* $\operatorname{rank} I^\sharp(K) = 2$*, then K is the unknot.*

Hence if $\operatorname{rank}\mathrm{Kh}(K) = 2$, then K is the unknot. That is to say, Khovanov homology is an unknot-detector.

Knot surgeries

Generators of the instanton Floer chain complex correspond to the $\mathrm{SU}(2)$-representations of the fundamental group. Thus, instanton Floer homology can give us information about representations of the fundamental group.

The first example is Kronheimer–Mrowka's proof of the property P for knots, which can be viewed as a special case of the Poincaré conjecture.

Theorem 5.54 (Kronheimer–Mrowka [?kronheimer2004witten]). *Assume that $Y = S^3_r(K)$ is simply connected. Then K is the unknot, and $Y = S^3$.*

They argued by showing that $I_*(Y) \neq 0$ if K is not the unknot, and hence it admits a nontrivial $\mathrm{SU}(2)$-representation. Similar technique is used in their sequent paper:

Theorem 5.55 (Kronheimer–Mrowka [?Kronheimer2004DehnST]). *Let $r \in [0,2]$ be a rational number, and let K be a nontrivial knot. Then the fundamental group of $Y = S^3_r(K)$ admits a nontrivial* $\mathrm{SU}(2)$*-representation (i.e. a representation with non-abelian image).*

The same result holds for $r = 4$ by Baldwin–Sivek [?baldwin2022instantons] and $r = 3$ by Baldwin–Li–Sivek–Ye [?baldwin2021small]. It is not true for $r = 5$: the 5-surgery on the trefoil yields a lens space $L(5,2)$, which has abelian fundamental group $\mathbb{Z}/5$. It remains open for many rationals in $(2,5)$.

5.8 Atiyah–Floer conjecture (lecture 16)

Recall that each 3-manifold has a *Heegaard decomposition*

$$Y = U_0 \cup_\Sigma U_1,$$

where U_0 and U_1 are genus g handlebodies with common boundary Σ. One can stretch the neck by inserting a cylinder $\Sigma \times [-T, T]$. Atiyah's observation is that the ASD equation on $\mathbb{R} \times Y$, as $T \to +\infty$, is the same as the Cauchy–Riemann equation for strips

$$\mathbb{R} \times [0,1] \to \mathcal{M}(\Sigma)$$

with certain boundary conditions on $\mathcal{M}(U_i)$. It hence builds a relation between gauge theory (ASD equation) with symplectic geometry (Cauchy–Riemann equation).

More precisely, $\mathcal{M}(\Sigma)$ contains flat SU(2)-connections on the trivial SU(2)-bundle on Σ modulo gauge, or equivalently, contains rank 2 degree 0 stable holomorphic bundles on Σ, or SU(2)-representations of $\pi_1(\Sigma)$ modulo conjugations. It has dimension $6g-6$. One can similar formulate $\mathcal{M}(U_i)$ as the collection of SU(2) representations of $\pi_1(U_i)$ modulo conjugations, which has dimension $3g-3$. We focus on the irreducible part: in this case,

$$\mathcal{M}^*(\Sigma) \subset \mathcal{M}(\Sigma)$$

is a smooth manifold of dimension $6g-6$. It admits a canonical symplectic form

$$\omega(a,b) = \int_\Sigma \mathrm{tr}(a \wedge b)$$

for $a, b \in \Omega^1(\Sigma, \mathfrak{su}(2))$. The irreducible part $\mathcal{M}^*(U_i)(i=0,1)$ are Lagrangians in $\mathcal{M}^*(\Sigma) \subset \mathcal{M}(\Sigma)$. Ideally, we can consider their Lagrangian Floer homology, and the Atiyah–Floer conjecture claims that it is the same as the instanton Floer homology!

Conjecture 5.56 (Atiyah–Floer conjecture, [?atiyah1988new]). *Let Y be an integral homology sphere. Then*

$$I_*(Y) \cong HF_*(\mathcal{M}^*(U_0), \mathcal{M}^*(U_1)).$$

Here the right hand side is the Lagrangian Floer homology in $\mathcal{M}^*(\Sigma)$.

The right hand side is called the *symplectic instanton homology*. However, the issue is that it makes no sense in general! Recall that to define Lagrangian Floer homology, we need the symplectic manifold to be compact or convex at infinity, and the Lagrangians should be compact. Neither of them is satisfied in our setting: $\mathcal{M}^*(\Sigma)$ is not compact and not convex at infinity, and $\mathcal{M}^*(U_i)$ are not compact. The problem is that the action of ignoring all reducibles is too crude to preserve nice properties of the manifold.

For admissible pairs

One solution is to consider the admissible pairs. Assume that $b_1(Y) > 0$, and let $P \to Y$ be an admissible bundle. Recall that $I_*(Y, P)$ is defined using flat connections in a lift of P to a U(2)-bundle.

We can decompose Y as

$$U_0 \bigcup_{\Sigma_0 \coprod \Sigma_1} U_1,$$

as showed in Figure TBD. Here U_i are called *compression bodies*. We require $P|_{\Sigma_i}$ are nontrivial, i.e. $c_1(P) \in H^2(\Sigma_i;\mathbb{Z})$ are odd. From this, we can define $\mathcal{M}'(\Sigma_i)$ as the collection of flat U(2)-connections on an odd degree U(2)-bundle with fixed determinant on Σ_i modulo gauge, or equivalently, stable holomorphic bundles on Σ_i with fixed determinant bundle and odd degree, or representations

$$\rho\colon \pi_1(\Sigma_i \backslash \{z\}) \to \mathrm{SU}(2)$$

such that $\rho(\gamma_z) = -I$ modulo conjugations. Here z is a fixed basepoint on Σ, and γ_z is a loop around z.

As above, $\mathcal{M}'(\Sigma_i)$ is a smooth symplectic manifold, and similarly formed $\mathcal{M}'(U_i)$ are Lagrangians in $\mathcal{M}'(\Sigma_0) \times \mathcal{M}'(\Sigma_1)$. The point is that now $\mathcal{M}'(\Sigma_0) \times \mathcal{M}'(\Sigma_1)$ is compact, and $\mathcal{M}'(U_i)$ are monotone. Hence the Lagrangian Floer homology

$$HF_*(\mathcal{M}'(U_0), \mathcal{M}'(U_1))$$

is sensible as a $\mathbb{Z}/4$-graded theory. Now we can state the Atiyah–Floer conjecture in a rigorous way, which has been proved recently:

Theorem 5.57 (Daemi–Fukaya–Lipyanskiy [?daemi2021lagrangians]). *Under this setting, we have*

$$I_*(Y,P) \cong HF_*(\mathcal{M}'(U_0), \mathcal{M}'(U_1))$$

as $\mathbb{Z}/4$-graded abelian groups.

For equivariant instanton homology

In the case of $b_1(Y) = 0$, recall that we can define equivariant instanton homology $\widetilde{I}(Y), I^+(Y)$. We expect to find the symplectic instanton counterpart for them.

Recall that

$$\mathcal{M}(\Sigma) = \{\pi_1(\Sigma) \to \mathrm{SU}(2)\}/\,\mathrm{SO}(3),$$

where the space $\{\pi_1(\Sigma) \to \mathrm{SU}(2)\}$ has dimension $6g-3$, but not smooth. We can construct $\mathcal{M}(\Sigma)$ from another point of view. Consider

$$\pi\colon W = \mathrm{SU}(2)^{2g} \to \mathrm{SU}(2),\ (A_i, B_i) \mapsto \prod_{i=1}^{g} [A_i, B_i].$$

Then $\mathcal{M}(\Sigma) = \pi^{-1}(I)/\,\mathrm{SO}(3)$. The idea of Huebschmann–Jeffrey is to consider the "extended moduli space". More explicitly, consider

$$\mathcal{N}(\Sigma) = \{(A_i, B_i)\colon \prod_{i=1}^{g} [A_i, B_i] = e^{2\pi i \theta},\ \theta \in \mathfrak{su}(2),\ \mathrm{tr}(\theta) > 0\}$$

as a subspace of W. There is a natural map

$$\psi\colon \mathcal{N}(\Sigma) \to \mathfrak{su}(2)$$

sending an element $(A_i, B_i) \in \mathcal{N}(\Sigma)$ to the corresponding $\theta \in \mathfrak{su}(2)$(recall that the exponential function is injective on a neighbourhood of I). The idea is that the larger space $\mathcal{N}(\Sigma)$ is an appropriate replacement of $\mathcal{M}(\Sigma)$. In fact, one can show that $\mathcal{N}(\Sigma)$ is a monotone symplectic manifold, and ψ is a Hamiltonian moment map generating the action of $\mathrm{SO}(3)$ on $\mathcal{N}(\Sigma)$. From this, we can recover $\mathcal{M}(\Sigma)$ as the *symplectic quotient* $\psi^{-1}(0) / \mathrm{SO}(3)$.

We similarly define $L_i = \mathcal{N}(U_i)$, which are Lagrangians in $\mathcal{N}(\Sigma)$. Manolescu and Woodward [?manolescu2011floer] showed that we can make sense of $HF(L_0, L_1)$ inside $\mathcal{N}(\Sigma)$. Now we can state the Atiyah–Floer conjecture for rational homology spheres:

Conjecture 5.58. *We have*

$$\widetilde{I}(Y) \cong HF(L_0, L_1).$$

In the equivariant setting, we have

$$I^+(Y) \cong HF^{\mathrm{SO}(3)}(L_0, L_1).$$

Here the right hand side is a version of equivariant Lagrangian Floer homology, defined by Cazassus [?cazassus2022equivariant].

6 Monopole Floer homology

We now turn to our last main topic: monopole Floer homology. Kronheimer and Mrowka's book [?Kronheimer2007MonopolesAT] is a comprehensive reference for this topic, and another shorter reference is Lin's notes [?lin2016lectures].

As instanton Floer homology, monopole (Seiberg–Witten) invariants also have a 4-dimensional counterpart. For closed smooth oriented 4-manifold X with $b^+(X) > 1$ and a spin^c-structure $\mathfrak{s}$, Seiberg and Witten [?seiberg1994electric, ?witten1994monopoles] formulate the monopole (Seiberg–Witten) equations from some duality in physics. They contain two equations and are an elliptic system modulo gauge. By counting solutions in 0-dimensional moduli spaces, we can define the Seiberg–Witten invariant $SW_{X,\mathfrak{s}} \in \mathbb{Z}$.

For closed 3-manifold Y with a spin^c-structure $\mathfrak{s}$, after choosing a Riemannian metric g, we can form the monopole Floer chain complex $CM_*(Y, \mathfrak{s})$. Its generators are $\mathbb{R}$-invariant solutions to the Seiberg–Witten equations on $\mathbb{R} \times Y$, i.e. solutions to the Seiberg–Witten equations on Y. The differential from x to y counts solutions to the Seiberg–Witten equations on $\mathbb{R} \times Y$, asymptotic to x and y as $t \to \pm\infty$, and it can be interpreted as the gradient flowline of the *Chern–Simons–Dirac functional*.

Relations and differences between Yang–Mills theory and Seiberg–Witten theory are worthwhile to discuss:

- The key feature of Seiberg–Witten equations is that they have *compact* moduli spaces, which means there is no issue on compactness and bubbles. Hence the analysis used in Seiberg–Witten theory is somewhat easier than Yang–Mills theory.
- Many results, originally proved by Yang–Mills theory, such as Donaldson's diagonalization theorem and exotic smooth structure detection, can be showed using Seiberg–Witten theory.
- Witten's conjecture relates Seiberg–Witten invariants to Donaldson (instanton) invariants for closed 4-manifolds. However, it is unclear what happens in dimension 3.
- Each theory has its own feature. For example, instanton Floer homology helps us study the representations of fundamental groups, and monopole Floer homology has a stable homotopy refinement.

6.1 Spin^c structures and Dirac operators

Spin^c structures

Recall that $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2\mathbb{Z}$ for $n \geqslant 3$. The *spin group* $\mathrm{Spin}(n)$ is defined as the universal (double) cover of $\mathrm{SO}(n)$. It carries an involution τ by deck transformation. The spin^c group $\mathrm{Spin}^c(n)$ is defined by

$$\mathrm{Spin}^c(n) = \mathrm{Spin}(n) \times_{\mathbb{Z}/2\mathbb{Z}} S^1 = \{(g,\theta)\}/((g,\theta) \sim (\tau g, -\theta)).$$

Definition 6.1. Let (X^n, g) be an oriented Riemannian manifold. A *spin^c-structure* on X is a lift of its frame bundle $Fr(X)$ as an SO(n)-bundle to a Spin$^c(n)$-bundle.

In the case of $n = 3$, we have

$$\mathrm{Spin}^c(3) = \mathrm{SU}(2) \times_{\mathbb{Z}/2\mathbb{Z}} S^1 = \mathrm{U}(2).$$

Hence a spin^c-structure on Y^3 is a rank 2 Hermitian vector bundle $S \to Y$ together with an orientation preserving bundle isometry

$$\rho\colon TY \to \mathfrak{su}(S) = \{A \in \mathrm{End}(S)\colon \operatorname{tr} A = 0,\ A + A^* = 0\},$$

called the *Clifford multiplication*. Locally speaking, we can find frame $e_i (i = 1, 2, 3)$ such that $\rho(e_i) = \sigma_i$. Here σ_i are the *Pauli matrices*:

$$\sigma_1 = \begin{pmatrix} i & \\ & -i \end{pmatrix}, \sigma_2 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \sigma_3 = \begin{pmatrix} & i \\ i & \end{pmatrix}.$$

They satisfy relations $\sigma_i\sigma_j = -\sigma_j\sigma_i\ (i \neq j)$, $\sigma_i^2 = -I$. Hence we can use the Riemannian metric to extend ρ to a map

$$\Lambda^* TY \otimes \mathbb{C} \to \mathrm{End}(S).$$

Basic existence and classification results of spin^c structures are collected in the following proposition.

Proposition 6.2. *Let Y be a closed oriented 3-manifold. Then:*

- *spin^c-structure on Y always exists since TY is trivial;*
- *spin^c structures on Y are one-to-one, non-canonically corresponding to complex line bundles over Y, or equivalently, elements in $H^2(Y;\mathbb{Z})$;*
- *Given a spin^c structure (S, ρ) on Y, then a line bundle L gives another spin^c structure $(S \otimes L, \rho \otimes \mathrm{id})$, and their Chern classes are related by*

$$c_1(S \otimes L) = c_1(S) + 2c_1(L);$$

- *in particular, S is determined by its first Chern class if $H_1(Y;\mathbb{Z})$ has no 2-torsion.*

Definition 6.3. A connection A on a spinor bundle S is *spin^c* if it is compatible with the Hermitian metric, and satisfies the Leibniz rule

$$\nabla_A(\rho(X)\psi)(v) = \rho(\nabla_v(X))\psi + \rho(X)\nabla_A\psi(v).$$

Here X and v are vector fields on Y, $\psi \in \Gamma(S)$ is a spinor, and ∇ is the Levi-Civita connection on TY.

As general connections, spin^c connections are corresponding to imaginary-valued 1-forms in a non-canonical way.

Proposition 6.4. *We have the following non-canonical correspondences:*

$$\begin{array}{lcl} \textit{connections on } S & \leftrightsquigarrow & \Omega^1(Y;\mathrm{End}(S)) \\ \textit{Hermitian connections on } S & \leftrightsquigarrow & \Omega^1(Y;\mathfrak{u}(S)) \\ \textit{spin}^c \textit{ connections on } S & \leftrightsquigarrow & \Omega^1(Y;i\mathbb{R}) \end{array} .$$

Further, a spin^c connection A on S induces a connection A^t on $\det S$*. Under the last correspondence, we have*

$$(A+a)^t = A^t + 2a.$$

Dirac operators

To write down the Seiberg–Witten equation, we need the notion of Dirac operators.

Definition 6.5. Let $\mathfrak{s} = (S, \rho)$ be a spin^c structure on Y, A be a spin^c connection on S, and g be a Riemannian metric on Y. The *Dirac operator D_A* is the composition

$$\Gamma(S) \xrightarrow{\nabla_A} \Gamma(S \otimes T^*Y) \xrightarrow{g} \Gamma(S \otimes TY) \xrightarrow{\rho} \Gamma(S).$$

Locally the Dirac operator can be written as

$$D_A = \sum_{i=0}^{3} \sigma_i \partial_i.$$

6.2 Chern–Simons–Dirac functional (Lecture 17)

We construct monopole Floer homology followed the usual procedure. We need to specify the configuration space and the functional on it first.

The configuration space

The ordinary configuration space of monopole Floer homology is

$$\mathcal{C} = \{(A, \phi) \colon A \text{ is a spin}^c \text{ connection on } Y,\ \phi \in \Gamma(S)\}.$$

It possesses an action by the gauge group

$$\mathcal{G} = \{u \colon Y \to S^1\}$$

by

$$uA = uAu^{-1} - (du)u^{-1} = A - u^{-1}du,$$

and $u\phi$ by pointwise multiplication. Here we can see that one advantage of the U(1)-gauge group is that the form of action is simper than the non-abelian setting.

As before, the action is not free, and the quotient space

$$\mathcal{B} = \mathcal{C}/\mathcal{G}$$

is not a manifold.

Definition 6.6. A configuration (A, ϕ) is *reducible* if if has nontrivial stabilizer under the gauge group action; otherwise it is *irreducible*.

It is easy to see (A, ϕ) is reducible if and only if $\phi = 0$, and in this case it has stabilizer S^1, the constant gauges. One can first take the quotient by the based gauge group

$$\mathcal{G}_0 = \{u \colon Y \to S^1,\, u(p) = 1\}$$

to obtain

$$\widetilde{\mathcal{B}} = \mathcal{C}/\mathcal{G}_0.$$

Recall that

$$[Y, S^1] = H^1(Y; \mathbb{Z}),$$

and an element $u \in \mathcal{G}_0$ is null-homotopic if and only if $u = e^{\theta}$ for some $\theta \in C^\infty(Y; i\mathbb{R})$. We have a non-canonical identification

$$\widetilde{\mathcal{B}} \cong \Omega^1(Y; i\mathbb{R}) \oplus \Gamma(S)/(C^\infty(Y; i\mathbb{R}) \oplus H^1(Y; \mathbb{Z})).$$

Here the first factor of the quotient acts by

$$\theta(A, \phi) = (A - d\theta, e^{\theta}\phi),$$

and the second factor corresponds to harmonic functions $u \colon Y \to S^1$. Recall Hodge decomposition gives

$$\Omega^1(Y; i\mathbb{R}) = \operatorname{im} d \oplus \operatorname{im} d^* \oplus H^1(Y; i\mathbb{R}).$$

Therefore

$$\widetilde{\mathcal{B}} \cong (\operatorname{im} d^*) \oplus (\Gamma(S) \oplus H^1(Y; \mathbb{R}))/H^1(Y; \mathbb{Z}).$$

We get a vector bundle (with infinite dimensional fibers) over the torus $T^{b_1(Y)}$. Now we can do L^2_k-completion to obtain a Hilbert manifold $\widetilde{\mathcal{B}}_k$, which possesses an S^1-action. One can also do this directly by

$$\widetilde{\mathcal{B}}_k = \mathcal{C}_l/\mathcal{G}_{0,k+1},$$

where $\mathcal{G}_{0,k+1}$ is the L^2_{k+1}-completion of the based gauge group.

The Chern–Simons–Dirac functional

We are now ready to define the functional on the configuration space.

Definition 6.7. Fixed a spin^c connection A_0. The *Chern–Simons–Dirac functional $CSD \colon \mathcal{C} \to \mathbb{R}$* is defined as

$$CSD(A, \phi) = \frac{1}{8}\int_Y (A^t - A_0^t) \wedge (F_{A^t} + F_{A_0^t}) + \frac{1}{2}\int_Y \langle D_A\phi, \phi\rangle.$$

Exercise 6.8. For $u \in \mathcal{G}$, we have

$$CSD(u(A,\phi)) = CSD(A,\phi) + 2\pi^2([u] \cup c_1(S))[Y].$$

Therefore, CSD descends onto a functional

$$\widetilde{\mathcal{B}} \to \mathbb{R}/2\pi^2 d\mathbb{Z},$$

where d is the greatest common divisor of $(c_1(S) \cup \alpha)[Y]$ for $\alpha \in H^1(Y;\mathbb{Z})$. In particular, it is $\mathbb{R}$-valued if $c_1(S)$ is torsion.

The next task is to compute the formal gradient of CSD (with respect to the L^2 product). Let (A,Φ) be a configuration, and (a,ϕ) be a tangent vector. Here $a \in \Omega^1(Y;i\mathbb{R})$, and $\phi \in C^\infty(Y;i\mathbb{R})$. Then

$$\begin{aligned} d(CSD_{A,\Phi})(a,\phi) =& \lim_{h\to 0}\frac{1}{h}(CSD(A+ha, e^{i\phi}\Phi) - CSD(A,\Phi)) \\ =& \frac{1}{8}\int_Y \Big(2a \wedge (F_{A^t} + F_{A_0^t}) + (A^t - A_0^t) \wedge 2da\Big) \\ &+ \frac{1}{2}\int_Y (2\langle D_A\Phi, \phi\rangle + 2\langle \rho(a)\Phi, \Phi\rangle) \\ =& \frac{1}{4}\int_Y a \wedge 2F_{A^t} + \int_Y \langle D_A\Phi, \phi\rangle + \int_Y \langle a, \rho^{-1}(\Phi\Phi^*)_0\rangle \\ =& \frac{1}{2}\langle a, \star F_{A^t}\rangle + \langle D_A\Phi, \phi\rangle + \langle a, \rho^{-1}(\Phi\Phi^*)_0\rangle. \end{aligned}$$

Here $(\Phi\Phi^*)_0 \in \mathfrak{su}(S)$ is the traceless part of the endomorphism $\Phi\Phi^* \in \mathrm{End}(S)$. Locally it is given by

$$\Phi = (\alpha,\beta)^T,\ (\Phi\Phi^*)_0 = \begin{pmatrix} \frac{|\alpha|^2-|\beta|^2}{2} & \alpha\overline{\beta} \\ \overline{\alpha}\beta & \frac{-|\alpha|^2+|\beta|^2}{2} \end{pmatrix}.$$

The map ρ^{-1} sends $\mathfrak{su}(S)$ to $\Omega^1(Y;i\mathbb{R})$. In conclusion, we have:

Proposition 6.9. *The gradient of CSD is given by*

$$\nabla CSD(A,\Phi) = (\frac{1}{2}\star F_{A^t} + \rho^{-1}(\Phi\Phi^*)_0, D_A\Phi) \in \Omega^1(Y;i\mathbb{R}) \oplus \Gamma(S).$$

Now the Seiberg–Witten equation on $\mathbb{R}\times Y$ is the same as the gradient flow equation on Y:

$$\frac{d}{dt}(A(t),\Phi(t)) = -\nabla CSD(A(t),\Phi(t)).$$

The critical points of CSD correspond to the t-invariant solutions on $\mathbb{R}\times Y$:

$$\nabla CSD(A,\Phi) = 0.$$

On Y, it can be written as

$$\begin{cases} \frac{1}{2}\star F_{A^t} + \rho^{-1}(\Phi\Phi^*)_0 = 0 \\ D_A\Phi = 0 \end{cases}.$$

Its solutions are called *monopoles*. Notice that unlike the instanton case, they depend on the Riemannian metric on Y.

Reducibles are flat $\mathrm{U}(1)$ connections on $\det S$ modulo gauge. Recall Chern–Weil theory tells us

$$c_1(S) = [-\frac{1}{2\pi i}F_{A^t}] \in \operatorname{im}\left(H^2(Y;\mathbb{Z}) \to H^2(Y;\mathbb{R})\right).$$

In particular, there is no reducibles if $c_1(S)$ is non-torsion. If $c_1(S)$ is torsion, the reducibles form a torus $H^1(Y;\mathbb{R})/H^1(Y;\mathbb{Z})$ of dimension b_1. One can imagine the quotient space as an infinite-dimensional cone over this torus, as showed in Figure TBD. We have the quotient space

$$\mathcal{B}_k = \widetilde{\mathcal{B}}_k/S^1.$$

6.3 The Blown-up configuration space (lecture 18)

The naïve idea is to do Floer homology on the irreducible part $\mathcal{B}_k^* \subset \mathcal{B}_k$. Recall that in the instanton case, we can do this when Y is an integral homology sphere and there is only one reducible connection θ. It succeeds because the phenomenon that broken flowlines with θ involved only appear in codimension $4 = \dim \mathrm{SU}(2) + 1$. In the monopole case, it may happen in codimension $\dim \mathrm{U}(1) + 1 = 2$. It may not affect $\partial^2 = 0$, for which only considers unparametrized moduli spaces of dimension 1, but it definitely affects the invariance of the homology group under the change of metrics, which involves 2-dimensional moduli spaces with no $\mathbb{R}$-action. It does become a trouble when $c_1(\mathfrak{s})$ is torsion.

Hence, we must develop a machinery to treat the reducibles. This can be down by *blowing up*.

Near a reducible, the quotient $\mathcal{B} = \mathcal{C}/\mathcal{G}$ looks locally like

$$\mathbb{C}^\infty/S^1 \oplus \mathbb{R}^\infty.$$

The first factor is a cone over $\mathbb{CP}^\infty$. We can blow up the quotient along the singular point. That is, replace the cone by $\mathbb{CP}^\infty \times [0,+\infty)$. Globally, we have

$$\mathcal{C}^\sigma = \{(A, r, \Phi) \colon A \text{ is a spin}^c \text{ connection}, r \geqslant 0, \Phi \in \Gamma(S), \|\Phi\|_{L^2} = 1\}.$$

It carries a map

$$\pi \colon \mathcal{C}^\sigma \to \mathcal{C}, \ (A, r, \Phi) \mapsto (A, r\Phi).$$

One can easily see that π is an isomorphism when restricted to the irreducible part, and the preimage of a reducible is the unit sphere in $\mathbb{C}^\infty$. The action of $\mathcal{G}$ is now free on $\mathcal{C}^\sigma$. The plan is to perform Floer homology on

$$\mathcal{B}^\sigma = \mathcal{C}^\sigma/\mathcal{G},$$

or more precisely, on the completion

$$\mathcal{B}_k^\sigma = \mathcal{C}_k^\sigma/\mathcal{G}_{k+1},$$

which is a Hilbert manifold with boundary, locally like H or $H \times [0,+\infty)$, where H is a Hilbert space.

Question from the class Can we do the same construction in the instanton case?

A: Maybe! However, the advantage of the U(1) gauge is that there is only one type of reducibles, and we can resolve them in one time, while the stabilizer can be SU(2) or S^1 in the instanton case.

One cannot directly pull back the Chern–Simons–Dirac functional to the quotient of the blown-up configuration space. Nonetheless, we have the following:

Fact 6.10. The gradient ∇CSD pulls back to a gradient-like vector filed $(\nabla CSD)^\sigma$ on $\mathcal{B}^\sigma$. Stationary points of $(\nabla CSD)^\sigma$ have two types:

- irreducible solutions to the Seiberg–Witten equations, which correspond to the ordinary irreducible solutions;
- pairs (A, ψ), where $(A, 0)$ is a reducible solution to the Seiberg–Witten equation, and ψ is an eigenvector of D_A.

Remarks 6.11. The Dirac operator D_A is a self-adjoint Fredholm operator with infinitely many eigenvalues

$$\cdots \leqslant \lambda_{-2} \leqslant \lambda_{-1} \leqslant 0 \leqslant \lambda_0 \leqslant \lambda_1 \leqslant \ldots.$$

The model of the definition above is $\mathbb{CP}^\infty$ with the function

$$f(z) = \frac{1}{2}\langle z, Lz\rangle,$$

where L is a self-adjoint bounded linear map on $\mathbb{C}^\infty$.

6.4 Monopole Floer homology

Now we are ready to define the monopole Floer homology through the blown-up configuration space.

We first briefly recall the finite dimensional case, Morse theory for manifold with boundary. Let X be a (finite dimensional) manifold with boundary, and f be a Morse–Smale function on X. There are three types of critical points:

$$\mathrm{Cr}(f) = C^s \cup C^u \cup C^o,$$

which are collections of boundary stable, boundary unstable, and interior critical points. We then define three chain complexes

$$(\check{C} = C^o \oplus C^s, \check{\partial}),\ (\hat{C} = C^o \oplus C^u, \hat{\partial}),\ (\bar{C} = C^u \oplus C^s, \bar{\partial}),$$

whose homology recover

$$H_*(X),\ H_*(X, \partial X),\ H_*(\partial X)$$

respectively.

We want to do the same with monopoles. A boundary critical point is stable if and only if it has positive eigenvalue; otherwise it is unstable. For generic metric g, the Dirac operator D_A has nonzero simple spectrum, i.e. we have

$$\cdots < \lambda_{-2} < \lambda_{-1} < 0 < \lambda_0 < \lambda_1 < \dots.$$

For critical points $x = (A, \phi)$ and $y = (B, \psi)$, we can define the relative grading

$$\mathrm{gr}(x, y) \in \mathbb{Z}/d(\mathfrak{s})\mathbb{Z},$$

where $d(\mathfrak{s})$ is the greatest common divisor of $(c_1(S) \cup \alpha)[Y]$ for $\alpha \in H^1(Y; \mathbb{Z})$, as the index of the linearization of ∇CSD from x to y along a path. Here we only have a $\mathbb{Z}/d\mathbb{Z}$ grading because the configuration space $\mathcal{B}$ or $\mathcal{B}^\sigma$ has nontrivial topology:

$$\mathcal{B} \cong T^{b_1(Y)} \times \mathrm{Cone}(\mathbb{CP}^\infty) \times \mathbb{R}^\infty,$$

$$\mathcal{B}^\sigma \cong T^{b_1(Y)} \times \mathbb{CP}^\infty \times [0, +\infty) \times \mathbb{R}^\infty.$$

One the blown-up space, we have

$$\mathrm{gr}(((A, 0), \psi_k), (B, \psi)) = \mathrm{gr}((A, 0), (B, \psi)) + 2k.$$

Compactness is particularly easy: the moduli spaces are a priori compact, and there is automatically no bubbles.

To ensure transversality, we need to perturb the Seiberg–Witten equations by a 1-form on Y, and we also need more complicated perturbation for $\mathbb{R}\times$. At least it is doable!

As the finite dimensional case, we can now obtain three chain complexes

$$\widecheck{CM}(Y, \mathfrak{s}),\ \widehat{CM}(Y, \mathfrak{s}),\ \overline{HM}(Y, \mathfrak{s}),$$

and the homologies of them give three flavours of monopole Floer homology

$$\widecheck{HM}(Y, \mathfrak{s}),\ \widehat{HM}(Y, \mathfrak{s}),\ \overline{HM}(Y, \mathfrak{s}),$$

read as "HM to", "HM from", and "HM bar".

The U-action

Recall in the instanton setting, we have an SO(3)-bundle over $\mathcal{B}$, which gives an action of $H^*(B\,\mathrm{SO}(3); \mathbb{Q}) = \mathbb{Q}[U]$ on $I_*(Y)$, with $\deg U = -4$. There is a similar construction in the monopole setting. Namely, there is an S^1 bundle

$$\widetilde{\mathcal{B}}^\sigma = \mathcal{C}^\sigma/\mathcal{G}_0 \to \mathcal{B}^\sigma.$$

From this, we have an action of $H^*(BS^1; \mathbb{Z}) = \mathbb{Z}[U]$ on HM° ($\circ$ refers to any of the from, to, bar versions of monopole Floer homology) with $\deg U = -2$. More concretely, let P be the Poincaré dual of the generator of

$$\mathrm{im}\left(H^2(\mathbb{CP}^\infty) \to H^2(\mathcal{B}^\sigma)\right).$$

Then the action is given by

$$Ux = \sum_{\mathrm{gr}(x,y)=2} \#(\mathcal{M}(x,y) \cap P) \cdot y.$$

Similarly, there is an action by $H_k(T^{b_1(Y)})$ of degree $-k$.

The long exact sequence

Recall in the finite dimensional case, three flavours of Morse homology can be organized into the long exact sequence for the pair $(B, \partial B)$. Similar result holds in infinite dimensional setting:

Theorem 6.12. *The monopole Floer homology groups fit into a long exact sequence*

$$\cdots \xrightarrow{i_*} \widecheck{HM}_*(Y) \xrightarrow{j_*} \widehat{HM}_*(Y) \xrightarrow{p_*} \overline{HM}_*(Y) \xrightarrow{i_*} \widecheck{HM}_*(Y) \xrightarrow{j_*} \cdots .$$

Example 6.13. Consider the simplest case $Y = S^3$. There is a unique spin^c structure $\mathfrak{s}$ since $H^2(Y;\mathbb{Z}) = 0$. We equip S^3 with the round metric, which has positive scalar curvature. An application of the Weitzenböck formula shows that if Y admits positive scalar curvature, then there is no irreducible solutions to the Seiberg–Witten equation on Y. For S^3, there is a unique reducible since $b_1(Y) = 0$. After blowing up, there is a generator in each even degree, which gives the homotopy type of $\mathbb{CP}^\infty$. In the end, we obtain

$$\widecheck{HM}(S^3) = \mathbb{Z}[U],\ \overline{HM}(S^3) = \mathbb{Z}[U, U^{-1}],\ \widehat{HM}(S^3) = \mathbb{Z}[U, U^{-1}]/\mathbb{Z}[U].$$

Functoriality

One important feature of monopole Floer homology is that it forms a (3+1)d-TQFT. More precisely, for a cobordism W^4 with $\partial W = (-Y_0) \coprod Y_1$ and a spin^c structure on W, we can construct a map

$$F_{W,\mathfrak{s}} \colon HM^\circ(Y_0, \mathfrak{s}|_{Y_0}) \to HM^\circ(Y_1, \mathfrak{s}|_{Y_1})$$

by

$$F_{W,\mathfrak{s}}(x_0) = \sum_{x_1} \#\mathcal{M}(x_0, W, x_1) \cdot x_1,$$

where $\#\mathcal{M}(x_0, W, x_1)$ counts solutions to the Seiberg–Witten equations on W with two cylindrical ends

$$W^* = (Y_0 \times (-\infty, 0]) \bigcup_{Y_0} W \bigcup_{Y_1} (Y_1 \times [0, +\infty))$$

with asymptotic conditions given by x_0 and x_1. See Figure TBD.

Assume that there is two cobordisms $W_0 \colon Y_0 \to Y_1$, $W_1 \colon Y_1 \to Y_2$. We need to compose cobordisms to ensure the functoriality:

Theorem 6.14. *Let $\mathfrak{s}_i$ be spin^c structures on W_i respectively ($i = 0, 1$). Assume that $\mathfrak{s}_0|_{Y_1} = \mathfrak{s}_1|_{Y_1}$. Then*

$$F_{W_1,\mathfrak{s}_1} \circ F_{W_0,\mathfrak{s}_0} = \sum_{\substack{\mathfrak{s}\in \mathrm{spin}^c(W) \\ \mathfrak{s}|_{W_i}=\mathfrak{s}_i}} F_{W,\mathfrak{s}}.$$

Remark 6.15. Here we start from spin^c structures on each components, and the choice of $\mathfrak{s}$ may not be unique since the map

$$H^2(W) \to H^2(W_0) \oplus H^2(W_1)$$

may not be injective.

Recall that HM admits a relative $\mathbb{Z}/d\mathbb{Z}$-grading. When $c_1(\mathfrak{s})$ is torsion, it not only has a relative $\mathbb{Z}$-grading, but also an absolute $\mathbb{Q}$-grading, which behaves well under cobordisms.

Theorem 6.16. *Let $\mathfrak{s}$ be a spin^c structure on $W\colon Y_0 \to Y_1$ such that $c_1(\mathfrak{s}|_{Y_i})$ are torsion. Then $F_{W,\mathfrak{s}}$ changes the absolute grading by*

$$\frac{1}{4}(c_1(\mathfrak{s})^2 - \sigma(W)) - \iota(W),$$

where

$$\iota(W) = \frac{1}{2}(\chi(W) + \sigma(W) + b_1(Y_1) - b_1(Y_0)),$$

$\sigma(W)$ is the signature of W, and $\chi(W)$ is the Euler characteristic of W.

Remark 6.17. The mysterious quantity

$$\frac{1}{4}(c_1(\mathfrak{s})^2 - \sigma(W)) - \iota(W)$$

is the expected dimension of Seiberg–Witten moduli spaces on closed 4-manifold, which can be calculated by Atiyah–Singer index theorem.

Exercise 6.18. Verify that for composition of cobordisms, we have

$$\iota(W_1 \circ W_0) = \iota(W_1) + \iota(W_0).$$

In fact, one can *define* the absolute grading in spirit of Theorem 6.16. We set $\widetilde{HM}(S^3)$ have lowest degree 0. For a general 3-manifold Y, pick a cobordism $W\colon S^3 \to Y$ with a spin^c structure $\mathfrak{s}$ with $c_1(\mathfrak{s}|_Y)$ torsion. Fir $x \in \widetilde{HM}(Y, \mathfrak{s}|_Y)$, we define

$$\mathrm{gr}(x) = -\,\mathrm{gr}(x_0, W, x) + \frac{1}{4}(c_1(\mathfrak{s})^2 - \sigma(W)) - \iota(W).$$

Here x_0 is a generator of the lowest degree summand of $\widetilde{HM}(S^3)$, and $\mathrm{gr}(x_0, W, x)$ is the expected dimension of $\mathcal{M}(x_0, W, x)$.

6.5 Applications of monopole Floer homology

Frøyshov invariant

If $b_1(Y) = 0$, i.e. Y is a rational homology sphere, for each spin^c structure $\mathfrak{s}$ on Y, there is a unique reducible solution θ. In the instanton theory, we can simply set $\mathrm{gr}(\theta) = 0$, but here we have had an absolute grading already, which means that the quantity

$$n(Y, \mathfrak{s}, g) = \mathrm{gr}(\theta) \in \mathbb{Q}$$

is *not* an invariant of $(Y, \mathfrak{s})$. When Y is an integral homology sphere, we get an even number $n(Y, g) \in 2\mathbb{Z}/2\mathbb{Z}$. This leads to the following definition:

Definition 6.19. Let Y be an integral homology sphere. The *Frøyshov invariant* of Y is defined by

$$h(Y) = -\frac{1}{2} \min \mathrm{gr}\{x \in \widetilde{HM}(Y) \colon x \in \mathrm{im}(U^n), \text{ for all } n > 0\}.$$

Remark 6.20. The Frøyshov invariant consists of the information from the reducible, i.e. the *U-tower* in $\widehat{HM}(Y)$. The remaining part is called the *reduced monopole Floer homology*, denoted by HM^{red}. See Figure TBD.

One interesting property of Frøyshov invariant is that it controls the intersection form of 4-manifolds. To clarify this, we first introduce:

Proposition 6.21. *Let $W \colon Y_0 \to Y_1$ be a cobordism. Assume that W is negative definite, i.e. the intersection form*

$$Q \colon H_2(W) \times H_2(W) \to \mathbb{Z}$$

is negatively definite. Then the induced map

$$\widetilde{HM}(Y_0; \mathbb{Z}/2\mathbb{Z}) \to \widetilde{HM}(Y_1; \mathbb{Z}/2\mathbb{Z})$$

is nonzero on the tower.

The idea is that showing there is a unique reducible solution on W.

Corollary 6.22. *We have*

$$h(Y_0) \geqslant h(Y_1) + \rho(Q).$$

Here

$$\rho(Q) = \frac{1}{8}(\mathrm{rank}\, Q - \inf |Q(c, c)|),$$

where the infimum runs through characteristic elements of Q.

Proof. The degree shifts by

$$\frac{1}{4}(c_1(\mathfrak{s})^2 - \sigma(W)) - \iota(W)$$

by Theorem 6.16. For simplicity we assume that $b_1(W) = 0$, and then $\iota(W) = 0$ since $b_2^+(W) = 0$. We also have $\sigma(W) = -\mathrm{rank}\, Q$. One can choose $\mathfrak{s}$ such that this shift attains $2\rho(Q)$. The $\mathbb{Z}/2\mathbb{Z}[U]$-module structure ensures that the degree cannot decrease, and the result follows. See Figure TBD. □

Example 6.23. Let X be a closed smooth 4-manifold with $b_1(X) = b_2^+(X) = 0$. Remove two balls from X, and we get a cobordism $W = X \backslash (B^4 \cup B^4)$ from S^3 to S^3. Corollary 6.22 says

$$\rho(Q) \leqslant 0.$$

By a nontrivial theorem on bilinear forms due to Elkies, it implies that Q is diagonalizable. This recovers Donaldson's diagonalization theorem!

Example 6.24. Consider a smooth 4-manifold with boundary $\partial X = P$, the Poincaré sphere. We have $h(P) = -1$. In the similar vein, we obtain

$$\rho(Q) \leqslant 1$$

from Corollary 6.22. This is a weaker constraint, and Q might not be diagonal in this case. For example, we can take X be the E_8-plumbing, which has intersection form E_8.

One important property of Frøyshov invariant is that it descends onto a homomorphism from the homology cobordism group $\Theta^3_{\mathbb{Z}}$. To see this, consider a homology cobordism W with boundary components Y_0 and Y_1. We have $\rho(Q) = 0$ since $Q = 0$. Now Corollary 6.22 gives

$$h(Y_0) \geqslant h(Y_1).$$

Reversing W gives the reverse inequality. Hence $h(Y_0) = h(Y_1)$. It indeed gives a homomorphism essentially because $\mathbb{F}[U]$ is a PID, and we can apply Künneth formula. As in Theorem 5.51, this gives a $\mathbb{Z}$-summand of $\Theta^3_{\mathbb{Z}}$, generated by the Poincaré sphere.

We breifly mention some of the other famous applications of monopole Floer homology.

The Gordon conjecture

One of the first application of monopole Floer homology is the proof of the Gordon conjecture, due to Kronheimer, Mrowka, Ozsváth, and Szabó, which gives a surgery characterization of the unknot:

Theorem 6.25 (Kronheimer–Mrowka–Ozsváth–Szabó [?kronheimer2007monopoles]). *Let $K \subset S^3$ be a knot, and let U be the unknot. If there is an orientation-preserving diffeomorphism*

$$S^3_r(K) \cong S^3_r(U)$$

for some rational number r, then $K = U$.

The idea of the proof is to reduce the problem to the case of $r \in \mathbb{Z}$ and then to $r = 0$, using exact triangles in monopole Floer homology. The case of $r = 0$ was proved previously by Gabai [?gabai1987foliations] in a completely different approach, using machinery of foliations and sutured manifolds.

The Weinstein conjecture

Another famous application is the proof of the Weinstein conjecture, due to Taubes.

Let (Y^3, α) be a contact 3-manifold. This means α is a 1-form on Y such that $\alpha \wedge d\alpha$ is nowhere vanishing. We consider the *Reeb flow* associated to α, which is the flow generated by the flow R_α, determined by

$$\begin{cases} \iota_{R_\alpha}\alpha = 1, \\ \iota_{R_\alpha}d\alpha = 0. \end{cases}$$

Theorem 6.26 (The Weinstein conjecture, [?taubes2007seiberg]). *For every contact 3-manifold (Y, α), the Reeb flow ϕ_α has at least one periodic orbit.*

The idea of the proof is to deform the Seiberg–Witten equations using the contact form α. More specifically, we consider the equations

$$\begin{cases} \frac{1}{2} \star F_A + r(\rho^{-1}(\Phi\Phi^*)_0 + 2i\alpha = 0, \\ D_A\Phi = 0 \end{cases}$$

for $r \geqslant 0$. This recovers the ordinary (perturbed) Seiberg–Witten equations when $r = 1$, and the moduli space keeps stable when r varies by continuation maps. Kronheimer and Mrowka shows that $\overline{HM}(Y, \mathfrak{s}) \neq 0$, which produces solutions to the equations. In particular, we get rid of the curvature term in the first equation when $r \to +\infty$, and Taubes showed that the zero set of Φ gives a closed orbit of ϕ_α.

Remark 6.27. After the theory of *embedded contact homology* had been developed, people showed that "HM=ECH", which gives another description of the proof of the Weinstein conjecture.

6.6 The Seiberg–Witten Floer stable homotopy type (Lecture 19)

We now introduce an alternative construction of monopole Floer homology, due to Manolescu [?manolescu2003seiberg].

In this (and the next) subsection, we assume that $b_1(Y) = 0$, i.e. Y is a rational homology sphere, unless otherwise specified. The advantage of this assumption is that there is a unique reducible solution up to gauge, after fixing the spin^c structure $\mathfrak{s}$. We choose this reducible solution, or the flat connection A_0, as the base point of the space of connections, and we now have a canonical isomorphism between $\Omega^1(Y; i\mathbb{R})$ and spin^c-structures on Y.

Recall that

$$\widetilde{\mathcal{B}} = \mathcal{C}/\mathcal{G}_0 \cong \mathbb{R}^\infty \times \mathbb{C}^\infty,$$

where the first factor contains connections, and the second factor contains spinors. We have an S^1-action on this space. Let $\widetilde{\mathcal{B}}_k$ be the L^2_k-completion of $\widetilde{\mathcal{B}}$.

We take $k \gg 0$ (in fact, $k > 5$) such that the multiplication is continuous under L^2_k-norms. The Seiberg–Witten map is given by

$$SW\colon \widetilde{\mathcal{B}}_k \to \widetilde{\mathcal{B}}_{k-1},\ (A,\Phi) \mapsto (\frac{1}{2}\star F_A + \rho^{-1}(\Phi\Phi^*)_0, D_A\Phi),$$

or equivalently,

$$(a,\phi) \mapsto (\frac{1}{2}\delta a, D_{A_0}\phi) + (\rho^{-1}(\phi\phi^*)_0, \rho(a)\phi) = l(a,\phi) + c(a,\phi).$$

The point here is that l is a linear, elliptic (and hence, Fredholm), self-adjoint operator, and c is a quadric compact operator. In the case of $b_1(Y) = 0$, the L^2_k-completion $\widetilde{\mathcal{B}}_k$ is a Hilbert *space* (not just a Hilbert manifold). Hence we can do eigenspace decomposition for l.

Let τ, ν be real numbers, $\tau \ll 0 \ll \nu$. Define

$$\widetilde{\mathcal{B}}^\nu_\tau = \bigoplus (\text{eigenspace of } l \text{ with eigenvalue in } (\tau,\nu])\,.$$

It is a finite-dimensional vector space, and we can think of it as a finite dimensional approximation of $\widetilde{\mathcal{B}}$. Let

$$p^\nu_\tau\colon \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}^\nu_\tau$$

be the L^2 projection. We then have the restricted Chern–Simons–Dirac functional

$$SW^\nu_\tau = \nabla(CSD|_{\widetilde{\mathcal{B}}^\nu_\tau}) = l + p^\nu_\tau \circ c.$$

It doesn't preserve c, but it is "small" as a compact operator. It inherits many good properties from the original Seiberg–Witten equations. The following is a compactness theorem for the finite-dimensional approximation:

Theorem 6.28. *Fix $k > 5$.*

- *All critical points of CSD and finite energy flowlines are inside a ball*

$$\mathcal{B}(R) \subset \widetilde{\mathcal{B}}_k$$

  for some $R \gg 0$.

- *For $\nu \gg 0 \gg \tau$, all critical points and flowlines of SW^ν_τ that stay in $\mathcal{B}(2R)$ actually stay in $\overline{\mathcal{B}(R)}$.*

In other words, there is no critical point in $\mathcal{B}(2R)\backslash\mathcal{B}(R)$, and there is no flowline with two endpoint in $\mathcal{B}(R)$ gets rid of $\mathcal{B}(R)$ but not $\mathcal{B}(2R)$. See Figure TBD. Intuitively, we only care about solutions to SW^ν_τ in $\mathcal{B}(R)$ because this is the behaviour of the original SW.

Now $\widetilde{\mathcal{B}}^\nu_\tau \cap \mathcal{B}(2R)$ is a non-compact, finite-dimensional manifold, and CSD^ν_τ is a function on it. We can hence do Morse homology without referring to any more infinite-dimensional technique!

Recall from Subsection 1.3 that for Morse theory on a non-compact manifold, we have the machinery of Conley index. We denote the resulted Conley index by I^ν_τ. We can do $\widetilde{H}_*(I^\nu_\tau)$, but recall that Conley index itself is an invariant of the space, so why not do this on the level of stable homotopy types?

We need to examine the dependence on the choice of ν, τ and the metric g.

Proposition 6.29. *The Conley index I^ν_τ is unchanged when increasing ν.*

Let $\tau' < \tau \ll 0$. Then

$$I^\nu_{\tau'} = (\widetilde{\mathcal{B}}^\tau_{\tau'})^+ \wedge I^\nu_\tau = \Sigma^{\widetilde{\mathcal{B}}^\tau_{\tau'}} I^\nu_\tau.$$

The idea is that for a simple critical point of index k, the Conley index outputs a sphere S^k. Besides, the Conley index is invariant under deformations of operators, and hence we only need to consider the affect of eigenvalues of the linear part l.

This suggests us consider the formal desuspension $\Sigma^{-\widetilde{\mathcal{B}}^0_\tau} I^\nu_\tau$, which is independent of ν and τ. It remains to investigate the dependence on the metric g on Y. Recall that the reducible should live in degree $n(Y, \mathfrak{s}, g)$, but it now in degree 0, which suggests us add another shift on $\Sigma^{-\widetilde{\mathcal{B}}^0_\tau} I^\nu_\tau$.

Definition 6.30. Let $(Y, \mathfrak{s})$ be a rational homology sphere with a spin^c structure. The *Seiberg–Witten–Floer stable homotopy type* of $(Y, \mathfrak{s})$ is a suspension spectrum

$$SWF(Y, \mathfrak{s}) = \Sigma^{-\frac{n(Y,\mathfrak{s},g)}{2}\mathbb{C}} \Sigma^{-\widetilde{\mathcal{B}}^0_\tau} I^\nu_\tau.$$

According to the discussion above, we see that $SWF(Y, \mathfrak{s})$ is an invariant for $(Y, \mathfrak{s})$. Hence its homology gives a version of monopole Floer homology. It is natural to compare this definition with Kronheimer–Mrowka's construction. To proceed, we remark that SWF carries a natural S^1-action from the gauge group action. Using it, we can recover all the flavours of monopole Floer homology in Kronheimer–Mrowka's setting from SWF as different versions of equivariant homology.

Theorem 6.31 (Lidman–Manolescu, [?lidman2016equivalence])**.** *We have isomorphisms*

$$\widetilde{HM}(Y, \mathfrak{s}) = H^{S^1}_*(SWF(Y, \mathfrak{s})),\ \widehat{HM}(Y, \mathfrak{s}) = cH^{S^1}_*(SWF(Y, \mathfrak{s})),$$

$$\overline{HM}(Y, \mathfrak{s}) = tH^{S^1}_*(SWF(Y, \mathfrak{s})),\ \widetilde{HM}(Y, \mathfrak{s}) = H_*(SWF(Y, \mathfrak{s})).$$

Here the right hand sides are Borel homology, coBorel homology, Tate homology, and ordinary homology respectively. The tilde version of monopole Floer homology appearing in the last isomorphism is defined as

$$\widetilde{HM}(Y, \mathfrak{s}) = H_*(\text{Cone}(\widehat{CM}(Y, \mathfrak{s}) \xrightarrow{U} \widehat{CM}(Y, \mathfrak{s}))).$$

We don't really define these equivariant homologies seriously. Instead, we give some remarks. First, they carry an action of $H^{S^1}_*(*) = \mathbb{Z}[U]$, which recovers the module structure on the left hand sides. Second, for coBorel homology we have a the following duality:

$$cH^{S^1}_*(SWF(Y, \mathfrak{s})) = H^{-*}_{S^1}(SWF(-Y, \mathfrak{s})),$$

where the right is the Borel cohomology. Last, in fact we have

$$\overline{HM}(Y,\mathfrak{s}) = tH_*^{S^1}(SWF(Y,\mathfrak{s})) = \mathbb{Z}[U,U^{-1}].$$

We conclude this subsection by discussing the pros and cons of SWF.

Advantages of SWF over HM

The first advantage of SWF is that we have had a space (or a spectrum), so we don't need to do Morse homology again. We can just apply singular (equivariant) homology to extract algebraic invariants. Furthermore, we can apply generalized cohomology theories, such as (equivariant) K-theory or KO-theory, or complex bordism theory, to SWF to obtain other invariants. In particular, we can get analog of Frøyshov invariant with generalized cohomology coefficient.

The second advantage is that it is easier to incorporate symmetries. This is in two ways.

- When Y carries a finite group action by G, the spectrum $SWF(Y,\mathfrak{s})$ inherits a G-action, and we can then consider the $G \times S^1$-equivariant Floer homology. One example is the branched double cover $\Sigma(K)$ along a knot K, which has a natural $\mathbb{Z}/2\mathbb{Z}$-action. Using this construction, we can produce some invariants for the knot.
- When the spin^c structure is actually *spin*, $SWF(Y,\mathfrak{s})$ carries a $\mathrm{Pin}(2)$ symmetry, and we can form the $\mathrm{Pin}(2)$-equivariant monopole Floer homology. We will discuss this in more detail in the next lecture.

Disadvantage of SWF, and how to overcome

The foremost disadvantage of SWF is that it is defined only for rational homology spheres originally. For $b_1(Y) > 0$, $\widetilde{\mathcal{B}}$ is a Hilbert bundle over a torus T^{b_1}, which makes the finite dimensional approximation hard to produce.

Here are some works in this direction.

- Khandhawit–Lin–Sasahira [?khandhawit2018unfolded] defined a spectrum $\underline{SWF}(Y,\mathfrak{s})$ by working with the universal cover of T^{b_1}, but it only recovers the monopole Floer homology with twisted coefficients:

$$H_*^{S^1}(\underline{SWF}(Y,\mathfrak{s})) = \underline{\widetilde{HM}}_*(Y,\mathfrak{s}).$$

- Sasahira–Stoffregen [?sasahira2021seiberg] defined $SWF(Y,\mathfrak{s})$ when the Hilbert bundle has a *spectral section*, which is defined as follows. For the Hilbert bundle W over T^{b_1} and a point $x \in T^{b_1}$, we have eigenspace decomposition

$$W_x = W_x^+ \oplus W_x^-$$

through the Dirac operator D_{A_x}, where A_x is the flat connection corresponding to the reducible solution x. A spectral section is a subbundle $V \subset W$ such that the projection $V_x \to W_x^-$ is Fredholm, and $V_x \to W_x^-$ is compact. They showed that such a section exists if and only if for any $a_1, a_2, a_3 \in H^1(Y)$, we have

$$a_1 \cup a_2 \cup a_3 = 0.$$

Hence their definition is valid for all manifolds Y with $b_1(Y) \leqslant 2$, and connected sum of such manifolds. On the other hand, it doesn't include manifolds like $S^1 \times \Sigma_g$ for $g \geqslant 1$.

- For general Y, we can only expect a "twisted parametrized spectrum". This is work in progress by Behrens–Hedenlund–Kraph.

6.7 Pin(2)-equivariant monopole Floer homology and the triangulation conjecture (lecture 20)

Let Y be an integral homology sphere, and $\mathfrak{s}$ is the unique spin^c structure on Y. In this case, $\mathfrak{s}$ is actually coming from a spin structure. As we premised in the last lecture, we describe the Pin(2)-equivariant theory for $(Y, \mathfrak{s})$.

Pin(2)-equivariant monopole Floer homology

Definition 6.32. A *spin structure* on an oriented Riemannian manifold (Y, g) is a lift of the SO(n) frame bundle to a Spin(n) bundle.

Recall that Spin(n) is the universal cover of SO(n). We are particularly interested in the case of $n = 3$, and Spin(3) $=$ SU(2), which embeds into U(2) $=$ Spin$^c(3)$ in a natural way.

Proposition 6.33. *Let Y be a 3-manifold. Then spin structures on Y exist and form an affine space modelled by $H^1(Y; \mathbb{Z}/2\mathbb{Z})$.*

In particular, there is a unique spin structure when Y is an integral homology sphere.

If $\mathfrak{s} = (S, \rho)$ is a spin^c structure on Y coming from on spin structure, then we can make S as a quaternionic line bundle, which gives an action of j on the configuration space by

$$j(a, \phi) = (-a, j\phi).$$

Exercise 6.34. Verify that Seiberg–Witten equations are invariant under the action of j.

Let

$$\text{Pin}(2) = S^1 \cup jS^1 \subset \mathbb{C} \cup j\mathbb{C} = \mathbb{H}.$$

We can repeat the whole story of the last subsection in a Pin(2)-equivariant setting. In particular, we obtain a Pin(2)-equivariant spectrum $SWF(Y, \mathfrak{s})$. This leads to the following:

Definition 6.35. Let $(Y, \mathfrak{s})$ be an integral homology sphere with the unique spin structure $\mathfrak{s}$. The $\mathrm{Pin}(2)$-*equivariant monopole Floer homology* $SWFH_*^{\mathrm{Pin}(2)}(Y)$ is defined as

$$SWFH_*^{\mathrm{Pin}(2)}(Y) = H_*^{\mathrm{Pin}(2)}(SWF(Y, \mathfrak{s})).$$

Here we use coefficient $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. As in the general story of equivariant homology, $SWFH_*^{\mathrm{Pin}(2)}(Y)$ is a module over $H^*_{\mathrm{Pin}(2)}(*; \mathbb{F})$.

Proposition 6.36. *We have*

$$H^*_{\mathrm{Pin}(2)}(*; \mathbb{F}) = \mathbb{F}[q, v]/(q^3),$$

where $\deg q = 1$, $\deg v = 4$.

Proof. We have natural inclusion $\mathrm{Pin}(2) \subset \mathrm{SU}(2)$, which gives fiber sequence

$$\mathrm{SU}(2)/\mathrm{Pin}(2) \to E\,\mathrm{SU}(2)/\mathrm{Pin}(2) \to B\,\mathrm{SU}(2) = E\,\mathrm{SU}(2)/\mathrm{SU}(2).$$

The first term is

$$(S^3/S^1)/\mathbb{Z}/2\mathbb{Z} \cong \mathbb{RP}^2,$$

and the second term is a $B\,\mathrm{Pin}(2)$. Hence the fiber sequence can be rewritten as

$$\mathbb{RP}^2 \to B\,\mathrm{Pin}(2) \to \mathbb{HP}^\infty.$$

Using the Serre spectral sequence: TBD. There is no room for the differential, so the spectral sequence collapses at E_2 page. Hence

$$H^*_{\mathrm{Pin}(2)}(*; \mathbb{F}) = H^*(B\,\mathrm{Pin}(2); \mathbb{F}) = \mathbb{F}[q, v]/(q^3)$$

as graded rings. Further discussion shows it is actually a ring isomorphism. □

Remark 6.37. There is an alternative definition of $\mathrm{Pin}(2)$-equivariant monopole Floer homology in a Kronheimer–Mrowka style, due to Lin [?lin2018morse]. It produces a group $HS_*(Y, \mathfrak{s})$ for all spin 3-manifold $(Y, \mathfrak{s})$.

The triangulation conjecture

A topological space is said to be *triangulable* if is homeomorphic to the geometric realization of a simplicial complex.

Question 6.38 (Kneser)**.** *Is every topological manifold triangulable?*

It took people about 90 years to get the answer! Here are some related results. For a more comprehensive survey, see [?manolescu2016lectures].

- It's true for smooth manifolds.
- It's true for dimension $d \leqslant 3$. This is trivial when $d = 0, 1$, due to Radó when $d = 2$, and due to Moise when $d = 3$.

- It's false for some manifolds of $d = 4$. The first example is Freedman's E_8-manifold, which is non-triangulable by Casson's work.
- It's false for some manifolds of $d \geqslant 5$. Galewski–Stern and Matumoto reduced this question to a problem in $3+1$ dimensions, and Manolescu [?manolescu2016pin] resolved the latter problem using Pin(2)-equivariant monopole Floer homology.

The remaining part of this subsection is devoted to explain the situation when $d \geqslant 5$.

Obstruction for triangulability

To study the triangulation, we introduce the notion of link for simplices.

Definition 6.39. Let K be a simplicial complex. For a simplex $\sigma \in K$, the *star* of σ is the closure of the union of simplices that intersect σ. The *link* $lk(\sigma)$ is the union of simplices in the star of σ that don't contain σ.

For example, TBD.

Let M be a topological manifold of dimension $n \geqslant 5$. Assume that M admits a triangulation K. We expect that $lk(\sigma)$ is a sphere for each simplex $\sigma^{\in} K$. This only happens in the case that the triangulation is "nice". In practice, one can show that $lk(\sigma)$ is an integral homology sphere, but may not be the standard S^{n-l-1}.

Example 6.40. The Poincaré sphere P admits a triangulation since it is smooth, which induces a triangulation on the suspension ΣP, and a triangulation on the double suspension $\Sigma^2 P$. The *double suspension theorem* asserts that $\Sigma^2 P$ is homeomorphic to S^5. However, the link of the cone point is ΣP, which is not even a manifold. See Figure TBD.

One can show that
$$\sum_{\sigma \in K^{n-4}} [lk(\sigma)] \cdot \sigma]$$
is a closed chain valued in $\Theta^3_{\mathbb{Z}}$. Let

$$c(K) = \left[\sum_{\sigma \in K^{n-4}} [lk(\sigma)] \cdot \sigma \right] \in H_{n-4}(M; \Theta^3_{\mathbb{Z}}) \cong H^4(M; \Theta^3_{\mathbb{Z}}).$$

Remark 6.41. The significance of $c(K)$ comes from the fact that the n-dimensional homology cobordism group $\theta^n_{\mathbb{Z}}$ is trivial for $n \neq 3$. Hence the obstruction is about the non-triviality of $\Theta^3_{\mathbb{Z}}$.

To proceed, recall that the *Rokhlin homomorphism*

$$\mu \colon \Theta^3_{\mathbb{Z}} \to \mathbb{Z}/2\mathbb{Z}$$

is defined by

$$\mu(Y) = \frac{\sigma(W)}{8} \pmod 2,$$

where W^4 is a spin filling of Y, and $\sigma(W)$ is the signature of W. It is well-defined because of the Rokhlin theorem.

Example 6.42. We have $\mu(S^3) = 0$, $\mu(P) = 1$.

We have a short exact sequence

$$0 \to \ker \mu \to \Theta^3_{\mathbb{Z}} \xrightarrow{\mu} \mathbb{Z}/2\mathbb{Z} \to 0. \tag{6.43}$$

It induces a long exact sequence

$$\cdots \to H^4(M; \Theta^3_{\mathbb{Z}}) \xrightarrow{\mu} H^4(M; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\delta} H^5(M; \ker \mu) \to \cdots .$$

The class $c(K)$ lives in the leftmost term of the long exact sequence.

Theorem 6.44. *We have*

$$\mu(c(K)) = \Delta(M) \in H^4(M; \mathbb{Z}/2\mathbb{Z}).$$

Here $\Delta(M)$ is the Kirby–Siebenmann class of M, which obstructs the existence of PL structure on M.

Hence $\delta(\Delta(M)) = \delta(\mu(c(K))) = 0$ when such triangulation K exists. Galewski–Stern and Matumoto showed that the converse is also true. That is, for manifold M with dimension $n \geqslant 5$, M is triangulable if and only if $\delta(\Delta(M)) = 0$. This can always happen when the short exact sequence 6.43 splits, i.e. there exists homomorphism

$$\eta \colon \mathbb{Z}/2\mathbb{Z} \to \Theta^3_{\mathbb{Z}}$$

such that $\mu\eta = 1$. By Galewski–Stern and Matumoto, the converse is also true. More precisely:

Theorem 6.45. *The following statements are equivalent.*

1. *The short exact sequence 6.43 doesn't split.*
2. *For every $n \geqslant 5$, there exists a manifold M^n with $\delta(\Delta(M)) \neq 0$, i.e. M is not triangulable.*

In conclusion, to find non-triangulable manifold, we only need to show that 6.43 doesn't split. Equivalently, there doesn't exists integral homology sphere Y such that $\mu(Y) = 1$, and $2[Y] = 0$ in $\Theta^3_{\mathbb{Z}}$.

The lift of μ

The idea of showing the assertion above is to lift μ to $\mathbb{Z}$. More precisely, it suffices to find a map

$$\beta\colon \Theta^3_{\mathbb{Z}} \to \mathbb{Z}$$

such that $\beta \pmod 2 = \mu$, and $\beta(-Y) = -\beta(Y)$. In fact, given such a β, if there were an integral homology Y such that $\mu(Y) = 1$, and $2[Y] = 0$ in $\Theta^3_{\mathbb{Z}}$, we should have $[Y] = -[Y]$ in $\Theta^3_{\mathbb{Z}}$, and hence,

$$\beta(-Y) = -\beta(Y) \implies \mu(Y) = \beta(Y) \mod 2 = 0,$$

which is a contradiction.

There are some previous works inspiring the construction of β. The first one is the Casson invariant $\lambda(Y)$, which is a lift of μ to $\mathbb{Z}$ by counting representations of the fundamental group. However, it doesn't descend to a map from $\Theta^3_{\mathbb{Z}}$. For example, $\Sigma(2,3,13)$ is homology cobordant to S^3, but they have different Casson invariant.

The second is the Frøyshov invariant h, which we have defined previously. It is a $\mathbb{Z}$-valued homomorphism from $\Theta^3_{\mathbb{Z}}$, but it doesn't recover μ when modulo 2. For example, we have $h(\Sigma(2,3,7)) = 0$ while $\mu(\Sigma(2,3,7)) = 1$.

Recall that Frøyshov invariant is defined as the lowest degree of the U-tower. While it is irrelevant to the Rokhlin invariant, the degree of the reducible θ in the chain complex does relate to it. Recall that we denote $\deg\theta$ by $n(Y,g) \in 2\mathbb{Z}$. It relates to $\mu(Y)$: we have

$$n(Y,g) = -\operatorname{gr}(x_0, W, \theta) + \frac{c_1(\mathfrak{s})^2 - \sigma(W)}{4} - \iota(W).$$

Here W is a spin cobordism from Y to S^3, and x_0 is the reducible on S^3, lying in degree 0. We have

$$\operatorname{gr}(x_0, W, \theta) = \operatorname{ind} D_{A_0} + \operatorname{ind}(d^+ + d^*) = \operatorname{ind} D_{A_0} - \iota(W),$$

and $c_1(\mathfrak{s})^2 = 0$ since $\mathfrak{s}$ is spin. Therefore

$$n(Y,g) = -\operatorname{ind} D_{A_0} - \frac{\sigma(W)}{4} \equiv 2\mu(Y) \pmod 4.$$

Here we use the fact that $\operatorname{ind} D_{A_0} \in 4\mathbb{Z}$ since it's a connection on a quaternionic line bundle. However, $n(Y,g)$ depends on g.

We need to exploit the Pin(2)-equivariant structure. The complex that calculates $SWFH^{\mathrm{Pin}(2)}_*(Y)$ is as follows. TBD Taking differential kills some elements from the reducible, but there are still three infinite v-towers, lying in degree $2\mu \pmod 4$, $2\mu + 1 \pmod 4$, and $2\mu + 2 \pmod 4$ respectively. Assume that these towers have lowest degree 2α, $2\beta + 1$, $2\gamma + 2$ respectively. Then

$$\alpha, \beta, \gamma \equiv \frac{n(Y,g)}{2} \equiv \mu(Y) \pmod 2.$$

Hence each of them gives a lift of μ. We can show that they descend to maps

$$\alpha, \beta, \gamma \colon \Theta^3_{\mathbb{Z}} \to \mathbb{Z}$$

following the same approach for h. They are not homomorphisms since $H^*_{\mathrm{Pin}(2)}(*; \mathbb{F})$ is not a PID. Nonetheless, they do satisfy a duality theorem, coming from a version of Spanier–Whitehead duality for equivariant spectra.

Theorem 6.46. *We have $\alpha(-Y) = -\gamma(Y)$, and $\beta(-Y) = -\beta(Y)$.*

Therefore our β works! This confirms the existence of non-triangulable manifold in every dimension $n \geqslant 5$!

Question from the class Is there an explicit construction for such manifolds?

A: Yes! It uses mysterious techniques, such as attaching infinitely handles. Some people do understand this.

6.8 Heegaard Floer homology

In the end of the class, we introduce the "symplectic monopole homology", or more commonly known as Heegaard Floer homology, due to Ozsváth and Szabó [?ozsvath2004holomorphic,?ozsvath2004holomorphic1].

The origin of Heegaard Floer homology is to find a symplectic analog of monopole Floer homology in the framework of Atiyah–Floer conjecture. Starting from a Heegaard splitting $U_0 \cup_\Sigma U_1$ of a 3-manifold Y, we insert a long neck $\Sigma \times [-T, T]$ in the middle. As $T \to \infty$, the Seiberg–Witten equations approximate to the Cauchy–Riemann equation for strips

$$u \colon \mathbb{R} \times [0,1] \to \mathrm{Sym}^k(\Sigma).$$

Here $\mathrm{Sym}^k(\Sigma)$ is the *symmetric product* of Σ:

$$\mathrm{Sym}^k(\Sigma) = \Sigma^k / S_k.$$

It is the moduli space of $\mathbb{R}$-invariant solutions to the Seiberg–Witten equations on $\Sigma \times \mathbb{R}$, or *vortices* on Σ.

Definition 6.47. Let Σ be a closed surface, and L is a holomorphic line bundle over Σ. The *vortex equations* are the following:

$$\begin{cases} \bar{\partial}_A \phi = 0, \\ \star F_A = -i(1 - |\phi|^2). \end{cases}$$

Here A is an Hermitian connection on L, and ϕ is a section of L.

Jaffe and Taubes [?jaffee1980vortices] showed that solutions to the vortex equations correspond to points on the symmetric product $\mathrm{Sym}^k(\Sigma)$, where k is an integer only depends on L, by $(A, \phi) \mapsto Z(\phi)$, the zero set of ϕ.

Definition 6.48. The *plus version of Heegaard Floer homology* is defined as

$$HF^+(Y, \mathfrak{s}) = HF(L_0, L_1).$$

Here L_0, L_1 are Lagrangians constructed from U_0 and U_1, g is the genus of Σ, and the right side is the Lagrangian Floer homology in $\mathrm{Sym}^g(\Sigma)$.

The discussion above on vortex equations is just the motivation of Heegaard Floer homology: Oszváth and Szabó didn't really prove an analog of Atiyah–Floer conjecture. However, we do have a huge theorem relating Heegaard Floer homology and Monopole Floer homology:

Theorem 6.49 ("HF=HM", [?kutluhan2020hf1, ?kutluhan2020hf2, ?kutluhan2020hf3, ?kutluhan2021hf4, ?kutluhan2021hf5]). *We have*

$$HF^+(Y, \mathfrak{s}) \cong \widecheck{HM}(Y, \mathfrak{s}).$$

Similar isomorphisms hold for other flavours on both sides.

This completes our course! We have explored many flavours of Floer homology, but we don't have time to talk about contact homology, embedded contact homology, and symplectic Khovanov homology.