FOUR-DIMENSIONAL TOPOLOGY

CIPRIAN MANOLESCU

Abstract. We give a historical perspective on four-dimensional topology. We discuss the fundamental results of Freedman and Donaldson from the early 1980s, the rise of Seiberg-Witten theory in the mid 1990s, and more recent tools such as Heegaard Floer theory and Khovanov homology.

1. Introduction

Four dimensions are special in topology. Compact manifolds of dimension at most 2 admit a simple classification scheme, and those of dimension 3 can be understood through geometric methods (Thurston's geometrization program, proved to hold using the Ricci flow). In dimensions at least 4, a general classification was shown to be impossible, but one can restrict attention to manifolds that are simply connected, or have some other fixed (and relatively simple) fundamental group. In dimensions at least 5, we can study such manifolds using the h-cobordism theorem and surgery theory. The idea is to decompose the manifold into handles, and then to cancel the handles as much as possible. The key fact needed is Whitney's trick, which involves separating certain two-dimensional disks inside the manifold; since $2+2<5$, this can be done by a small perturbation.

By contrast, in dimension four the Whitney trick fails, and there is no known analogue of Thurston's geometrization. This makes it the most challenging dimension to study. It is also the lowest dimension where the distinction between smooth and topological manifolds appears. For example, we have the striking fact that, up to diffeomorphism, $\mathbb{R}^n$ admits a unique smooth structure for all $n \neq 4$, but uncountably many smooth structures when $n = 4$. Furthermore, smooth structures on the n-sphere S^n are unique for $n = 1, 2, 3, 5, 6$, and in higher dimensions their classification can be reduced to a problem in algebraic topology, involving the stable homotopy groups of spheres. On the other hand, the existence of exotic smooth structures on S^4 is a wide open problem, *the smooth four-dimensional Poincaré conjecture.*

Nevertheless, much progress has been made in understanding four-manifolds. A famous early result was Rokhlin's theorem, which constrained the intersection forms of smooth spin four-manifolds. Two major breakthroughs came in the early 1980s: the work of Freedman on topological 4-manifolds, which in particular resolved the topological four-dimensional Poincaré conjecture, and Donaldson's diagonalizability theorem, which showed that smooth four-manifolds are very different from the topological ones. Donaldson's work introduced an unexpected tool in smooth 4-dimensional topology: gauge theory, the study of certain PDEs (coming from physics) that admit a symmetry under the group of automorphisms of a bundle. Initially, the Yang-Mills equations were used, but in the 1990s they came to be supplanted by the Seiberg-Witten equations. This led to further progress, such as the resolution of the Thom conjecture by Kronheimer and Mrowka.

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In the last 20 years, the focus has shifted more towards understanding four-manifolds with boundary. Four-manifold topology has now become closely connected to three-dimensional topology and knot theory, through the perspective of topological quantum field theories such as Floer homology and Khovanov homology.

What follows is a short survey of these historical developments, starting with what was known before 1981, going through the work of Freedman and Donaldson, the introduction of Seiberg-Witten theory, and ending with an outline of more recent results. As with any such survey, we had to make a selection and leave out many important results; what is included may reflect the biases of the author.

2. Before 1981

2.1. **Algebraic geometry.** A few simple examples of four-manifolds can be easily provided: S^4, $\mathbb{RP}^4$, products of lower dimensional manifolds. Apart from these, a very rich source of examples is algebraic geometry. Compact complex surfaces include $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$, Abelian varieties (topologically T^4), the Hopf surface (topologically $S^1 \times S^3$), smooth hypersurfaces in $\mathbb{CP}^3$, etc.

Compact complex surfaces are an old subject. Their systematic study was started by Noether and Castelnuovo in the nineteenth century, and was continued in the twentieth century most notably by Enriques [Enr49] and Kodaira [Kod63]. The upshot was the Enriques-Kodaira classification, according to which every smooth compact complex surface is of one of the following 10 types, which are sometimes grouped according to the Kodaira dimension κ:

- rational surfaces, such as $\mathbb{CP}^2$ and Hirzebruch surfaces (with $\kappa = -\infty$);
- ruled surfaces (also with $\kappa = -\infty$), such as $\mathbb{CP}^1$-bundles over Riemann surfaces;
- class VII surfaces (non-algebraic, and also with $\kappa = -\infty$), such as the Hopf surface;
- Abelian surfaces ($\kappa = 0$);
- K3 surfaces ($\kappa = 0$);
- Enriques surfaces ($\kappa = 0$), quotients of K3 surfaces by an involution;
- Kodaira surfaces (non-algebraic, with $\kappa = 0$);
- hyperelliptic surfaces ($\kappa = 0$), quotients of a product of two elliptic curves by a finite group of automorphisms;
- elliptic surfaces ($\kappa \leq 1$);
- surfaces of general type (those with $\kappa = 2$).

Note this classification is up to biholomorphism, whereas in topology we are interested in the classification up to homotopy equivalence, homeomorphism or diffeomorphism. For example, there are many K3 surfaces up to biholomorphism, but all of them are diffeomorphic. Thus, in topology, we can think of K3 in terms of any algebraic model, for example as the Fermat quartic

$$K3 = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^4 \mid z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}.$$

We should also mention that the Enriques-Kodaira classification is not a full classification: while 9 of the 10 classes can be reasonably understood, surfaces of general type (and their moduli) are usually considered too complicated to fit into a simple list. Nevertheless, many families of surfaces of general type are well studied, and there are also constraints on their topology. For example, the Bogomolov-Miyaoka-Yau inequality [Yau78, Miy77] gives a constraint on the Chern numbers of surfaces of general type:

$$c_1^2 \leq 3c_2.$$

There is also the older Noether inequality [Noe75], which applies more generally to compact minimal complex surfaces:

$$5c_1^2 - c_2 + 36 \geq 0.$$

These are indeed topological constraints, because c_1^2 and c_2 can be expressed in terms of the Euler characteristic χ and the signature σ:

$$c_1^2 = 2\chi + 3\sigma, \quad c_2 = \chi.$$

2.2. **The impossibility of classification.** More examples of four-manifolds can be constructed using surgery techniques. For example, given a finite presentation of a group G, we can produce a smooth closed 4-manifold X with $\pi_1(X) = G$ as follows: We take a connected sum of $S^1 \times S^3$, one term per generator, and do surgery on loops corresponding to relations (that is, we replace $S^1 \times B^3$ with $B^2 \times S^2$).

Adyan and Rabin [Ady55, Rab58] showed that there is no algorithm that can be applied to finite group presentations to determine if they present the trivial group. In 1960, Markov leveraged this fact (together with the construction above) to prove:

Theorem 2.1 (Markov [Mar58]). *There is no algorithm that can tell whether two arbitrary closed* 4*-manifolds are diffeomorphic.*

In view of Markov's theorem, a classification scheme for general four-manifolds is not feasible. It is, however, reasonable to ask for such a scheme for 4-manifolds with fixed fundamental group, e.g. with $\pi_1(X) = 1$.

2.3. **Topological invariants.** Let us now focus on closed, simply connected, oriented 4-manifolds X. As a first attempt at distinguishing them, we can look at their classical invariants from algebraic topology. Using Poincaré duality and the universal coefficients theorem, we find that their homology groups take the form

$$H_0 = H_4 = \mathbb{Z}, \quad H_1 = H_3 = 0, \quad H_2 = \mathbb{Z}^b$$

for some $b \geq 0$. Furthermore, there is a symmetric, unimodular, bilinear intersection form

$$Q_X : \mathbb{Z}^b \times \mathbb{Z}^b \to \mathbb{Z}.$$

With regard to the classification of four-manifolds up to homotopy equivalence, we have the following:

Theorem 2.2 (Whitehead [Whi49], Milnor [Mil58]). *Let X be a closed, simply connected, oriented* 4*-manifold. Then, the intersection form Q_X determines the homotopy type of X.*

We are left to understand what intersection forms can be realized, and we can ask this about topological manifolds, or about smooth manifolds.

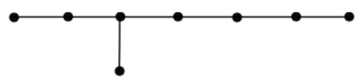
Let us first review what algebra tells us about the possible forms Q_X. Over $\mathbb{R}$, we can split Q_X as $m\langle 1\rangle \oplus n\langle -1\rangle$. In our case, we write

$$b_2^+(X) = m, \quad b_2^-(X) = n.$$

The signature of X is $\sigma(X) = b_2^+(X) - b_2^-(X)$ and the Euler characteristic is $\chi(X) = 2 + b_2^+(X) + b_2^-(X)$.

Over $\mathbb{Z}$, symmetric, unimodular, bilinear forms can be grouped into *indefinite* $(m, n > 0)$ and *definite* ($m = 0$ or $n = 0$). We can also distinguish between *even* forms, those such that the pairing of any element with itself is even; and the other forms, which are called *odd*.

In the indefinite case, there is a complete classification. Odd indefinite forms are isomorphic to $m\langle 1\rangle \oplus n\langle -1\rangle$ for some $m, n > 0$, and even indefinite forms to $p\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus qE_8$, for some $p > 0$ and $q \in \mathbb{Z}$. Here, E_8 is the matrix associated to the E_8 Dynkin diagram:



$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}$$

Definite unimodular forms admit no straightforward classification. There are the diagonal forms $n\langle 1\rangle$ for $n \in \mathbb{Z}$, and many examples of non-diagonal forms, even or odd (E_8, $E_8 \oplus E_8$, D_{16}^+, the Leech lattice, etc.)

Another result from algebra tells us that if Q_X is even, then its signature is divisible by 8. The first nontrivial constraint on the intersection forms of four-manifolds is the following celebrated theorem of Rokhlin, from 1952:

Theorem 2.3 (Rokhlin [Rok52])**.** *If X is a smooth, closed, spin 4-manifold (e.g. simply connected, with Q_X even), then the signature of X is divisible by* 16.

This shows that, for example, E_8 cannot appear as Q_X for X smooth.

Rokhlin's theorem is connected to many areas of mathematics. The original proof was based on cobordism theory, and has as corollary the calculation of the third stable homotopy group of spheres, $\pi_3^{st}(S^0) = \mathbb{Z}/24$. Rokhlin's theorem can also be deduced from differential geometry, as a consequence of the Atiyah-Singer index theorem. For other proofs of Theorem 2.3, we refer to [Mat86, FK78, Kir89].

Finally, Rokhlin's theorem has implications for the classification of high-dimensional manifolds. Kirby and Siebenmann [KS77] used it to determine the obstruction for a topological manifold M of dimension ≥ 5 to admit a piecewise linear structure. This happens if and only if an invariant $\mathrm{ks}(M) \in H^4(M; \mathbb{Z}/2)$, called the *Kirby-Siebenmann class*, vanishes.

2.4. **The generalized Poincaré conjecture.** An *h-cobordism* W between closed n-dimensional manifolds Y_0 and Y_1 is a compact $(n+1)$-dimensional manifold with $\partial W = (-Y_0) \cup Y_1$ such that the inclusions $Y_0 \hookrightarrow W$ and $Y_1 \hookrightarrow W$ are homotopy equivalences.

In dimensions $n \geq 5$, Smale [Sma62] proved that the *h-cobordism theorem* holds: Every h-cobordism between simply connected manifolds is a product $Y_0 \times [0, 1]$, and therefore Y_0 and Y_1 are diffeomorphic. A consequence is the *n-dimensional generalized Poincaré conjecture*: An n-manifold homotopy equivalent to S^n must be homeomorphic to S^n.

For non-simply connected manifolds, an analogue of the h-cobordism theorem still holds, called the s-cobordism theorem. For that, we need to strengthen the hypotheses by asking for the h-cobordism to have vanishing Whitehead torsion.

In dimension 4, the usual proofs of the h-cobordism and s-cobordism theorems break down, due to the failure of the Whitney trick. We still have the following weaker result, from 1964:

Theorem 2.4 (Wall [Wal64])**.** *Two simply connected, smooth, closed, oriented* 4*-manifolds with isomorphic intersection forms are h-cobordant, and become diffeomorphic after taking connected sums with* $k(S^2 \times S^2)$*, for some* $k \geq 0$.

On the other hand, Cappell and Shaneson [CS76] produced examples of *fake* $\mathbb{RP}^4$*'s*: manifolds that are homotopy equivalent, and in fact smoothly s-cobordant, to $\mathbb{RP}^4$, but are not diffeomorphic to $\mathbb{RP}^4$. (They were later shown to be homeomorphic to $\mathbb{RP}^4$.) Thus, the s-cobordism theorem fails in dimension 4.

With regard to the generalized Poincaré Conjecture in dimension 4, its topological version was proved by Freedman. (See Section 3.1 below.) The smooth version is still open:

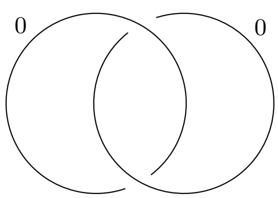
Question 2.5. *If* X^4 *is homotopy equivalent to* S^4*, then is it diffeomorphic to* S^4*?*

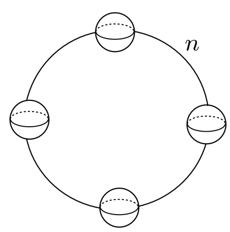
Among experts, opinions are split on whether we should expect the answer to be yes or no. Over time, many potential counterexamples have been proposed (manifolds that are homotopy equivalent to S^4, but not known to be diffeomorphic to it). Here are three famous families:

- *Andrews-Curtis examples* [AC65]: constructed from presentations of the trivial group;
- *Gluck twists* [Glu62]: take out a neighborhood embedded sphere (2-knot) $S^2 \hookrightarrow S^4$, and glue it back using the map $f : S^1 \times S^2 \to S^1 \times S^2,\ f(e^{i\theta}, x) = (e^{i\theta}, \mathrm{rot}_\theta(x))$;
- *Cappell-Shaneson spheres* [CS76]: two-fold covers of fake $\mathbb{RP}^4$'s.

Some of these were later shown to be standard S^4's; see for example [Akb10], [Gom10].

2.5. **Kirby calculus.** In 1978, Kirby [Kir78] developed a "calculus for links in S^3", which consists of moves that relate any two presentations of a 3-manifold in terms of surgery on links. His calculus was later extended to representations of 4-manifolds. We visualize 4-manifolds in terms of links in $\mathbb{R}^3$ as follows. By Morse theory, a smooth 4-manifold (possibly with boundary) can be decomposed into handles. A *Kirby diagram* for the 4-manifold shows the attaching spheres of the 1-handles and 2-handles. For example, here are pictures representing $S^2 \times S^2$ (left) and a D^2-bundle over T^2 of Euler number n (right):





Furthermore, two Kirby diagrams represent the same 4-manifold if and only if they are related by a sequence of certain moves (handle cancellations and handleslides). We refer to the book [GS99] for an extensive treatment of the subject.

3. Freedman's and Donaldson's results, and consequences (1981-1994)

3.1. **Freedman's results.** An important advance in our understanding of four-manifolds was made by Friedman in 1981, when he proved the topological h-cobordism theorem in dimension 4:

Theorem 3.1 (Freedman [Fre82])**.** *If* W *is a topological h-cobordism between closed topological* 4*-manifolds* M *and* N*, and* $\pi_1(M) = 1$*, then we have a homeomorphism* $W \cong M \times [0, 1]$.

The proof uses the technique of *Casson handles* [Cas86] to cancel intersection points of Whitney disks. A key ingredient is the result that Casson handles are homeomorphic to standard 2-handles ($S^2 \times D^2$). We refer to the books [FQ90] and [BKPR21] for details of the proof.

A corollary of Theorem 3.1 is the 4D topological Poincaré Conjecture:

Theorem 3.2 (Freeedman [Fre82]). *If a topological* 4*-manifold* M *is homotopy equivalent to* S^4*, then it is homeomorphic to* S^4*.*

More generally, Theorem 3.1 gives the classification of simply connected, closed, topological 4-manifolds.

Theorem 3.3 (Freedman [Fre82]). *(a) For every unimodular symmetric bilinear form* Q *there exists a simply connected, closed, topological* 4*-manifold* X *such that* $Q_X \cong Q$*.*

(b) If Q *is even, the manifold* X *is unique up to homeomorphism.*

(c) If Q *is odd, there are exactly two homeomorphism types of manifolds with the given* Q*, and at most one is smoothable.*

For example, there exists a closed, simply connected topological manifold M_{E_8} with intersection form E_8. By Rokhlin's theorem, the manifold M_{E_8} cannot admit a smooth structure. Further, there exists a fake $\mathbb{CP}^2$, denoted $*\mathbb{CP}^2$, which is homotopy equivalent but not homeomorphic to $\mathbb{CP}^2$. (It is distinguished from the usual $\mathbb{CP}^2$ by the Kirby-Siebenmann invariant.)

As a consequence of Theorem 3.3, simply connected, smooth 4-manifolds are determined up to homeomorphism by their intersection forms.

Freedman's work was further extended by Quinn [Qui82], who showed (among other results) that all noncompact, connected topological 4-manifolds are smoothable; and that topological 5-manifolds admit handle decompositions.

One can also ask about the classification of topological 4-manifolds with fixed fundamental group $\pi_(X) = G$. Freedman's results are based on the existence of suitable Whitney disks, which exist for a class of groups G called *good*. Good groups include finite groups, and finitely generated abelian groups. See for example [FQ90, Theorem 10.7A] for a classification result in the case $G = \mathbb{Z}$.

3.2. **Gauge theory.** In particle physics, the differential equations that underlie the standard model (electromagnetic + weak + strong interactions) admit *gauge symmetry*, i.e. they are invariant under an infinite-dimensional *gauge group* consisting of automorphisms of a vector bundle; e.g. $C^\infty(X, G)$ for a trivial G-bundle over X, where G is a Lie group.

The simplest example of equations with a gauge symmetry are Maxwell's equations for electromagnetism, which are invariant under $U(1)$ gauge. Electroweak interactions are modeled by the *Yang-Mills equations*, which have $SU(2)$ gauge symmetry:

$$d_A^* F_A = 0. \tag{1}$$

Here A is a connection in an $SU(2)$-bundle E over a four-manifold X^4. In physics, the manifolds come equipped with Lorentzian metrics, but one can also consider the same equations over Riemannian manifolds.

When $c_2(E)[X] > 0$, the minimal energy solutions to (1) are the ASD (anti-self-dual) solutions, satisfying

$$*F_A = -F_A.$$

In the 1970's, mathematicians started paying attention to these. In 1978, Atiyah, Drinfeld, Hitchin and Manin [AHDM78] described the instantons (solutions to ASD) on S^4. Uhlenbeck [Uhl82] proved a compactness theorem for ASD connections, and Taubes [Tau82] proved an existence / gluing theorem. Using their work, Donaldson [Don83] studied the moduli space of ASD connections on a definite 4-manifold and surprised the mathematical world by giving a topological application of gauge theory, his diagonalizability theorem:

Theorem 3.4 (Donaldson [Don83]). *If the intersection form Q_X of a smooth, simply connected, closed 4-manifold X is definite, then $Q_X \cong n\langle 1\rangle$ for some $n \in \mathbb{Z}$.*

This means that not only E_8 cannot appear as Q_X for such X smooth (which followed from Rokhlin's theorem); neither does $E_8 \oplus E_8$. Hence, $M_{E_8} \# M_{E_8}$ is not smoothable.

3.3. **Applications of Yang-Mills theory.** After Freedman's and Donaldson's beakthroughs, it was almost immediately realized that the imply the existence of an exotic $\mathbb{R}^4$. One way to construct it is to take $X = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$, with $Q_X = \langle 1\rangle \oplus 9\langle -1\rangle = (-E_8) \oplus \langle -1\rangle \oplus \langle 1\rangle$. The generator α of the last $\langle 1\rangle$ cannot be represented by a smoothly embedded sphere (by a consequence of Donaldson's theorem), but can be represented by a topological one (by a consequence of Freedman's work). A neighborhood U of this sphere Σ can be embedded in $\mathbb{CP}^2$, with $[\Sigma]^2 = 1$. Then, the complement $\mathbb{CP}^2 \setminus \Sigma$ is homeomorphic but not diffeomorphic to $\mathbb{R}^4$.

Building on these ideas, Gompf [Gom85] proved that $\mathbb{R}^4$ has infinitely many smooth structures, and Taubes [Tau87] proved that it has uncountably many.

In a different direction, Donaldson [Don90] turned his attention to the moduli space of ASD connections on four-manifolds X with $b_2^+(X) \geq 1$. For $b_2^+(X) \geq 3$ odd, "counting" solutions (with some constraints) yields homogeneous polynomial functions

$$q_d(X) : H^2(X; \mathbb{R}) \to \mathbb{R}$$

of degree $d = 4k - (3b^+ + 1)$, where $k \in \mathbb{Z}$ is the instanton number. There is a generalization of this to $b_2^+(X) = 1$, which involves "wall crossing."

The Donaldson polynomials are invariants of smooth four-manifolds that are sensitive to the smooth structure. Among their properties we mention:

- We have $q_d = 0$ when $X = X_1 \# X_2$ with $b_2^+(X_i) > 0, i = 1, 2$
- For complex projective surfaces, the invariants can be understood in terms of counts of stable vector bundles, and often do not vanish;
- There is a blow-up formula relating the invariants of X and $X \# \overline{\mathbb{CP}^2}$;
- Assuming a certain condition (simple type), the polynomials q_d satisfy some recurrence relations; see [KM94b].

Starting from here, Yang-Mills theory yielded a number of striking new applications:

- The smooth h-cobordism theorem fails in dimension 4; see [Don87]: for example, $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ and the Dolgachev surface are homotopy equivalent (hence, h-cobordant and homeomorphic) but not diffeomorphic;
- Many other complex surfaces (K3, $\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ for $n \geq 8$, many surfaces of general type) admit exotic smooth structures; in some cases, infinitely many of them; see [FM88], [OVdV89], [Kot89], etc.;
- Gompf and Mrowka [GM93] showed that irreducible simply connected 4-manifolds need not be complex;

- Existence of *corks*, i.e. compact contractible 4-manifolds W with boundary, with involutions on ∂W that extend to self-homeomorphisms but not self-diffeomorphisms inside. The first such example was found by Akbulut in [Akb91]. It was later shown that the failure of the h-cobordism theorem is always due to such corks; see [CFHS96], [Mat96].

3.4. **Embedded surfaces.** Apart from four-manifolds *per se*, topologists are also interested in studying surfaces smoothly embedded in them. A typical problem is to determine the minimal genus of an embedded surface that represents a given homology class $h \in H_2(X;\mathbb{Z})$.

By studying the ASD equations on 4-manifolds X with singularity along a surface $\Sigma \subset X$, Kronheimer and Mrowka [KM93] proved genus bounds for surfaces in a given homology class, provided the Donaldson invariants of X are non-zero. Here are two applications:

Theorem 3.5 (Kronheimer-Mrowka [KM93])**.** *Smooth complex curves in the $K3$ surface are genus minimizing in their homology class.*

Corollary 3.6 (Local Thom Conjecture; cf. Kronheimer-Mrowka [KM93])**.** *Algebraic curves in $\mathbb{C}^2$ are locally genus minimizing.*

There is a variant of the genus minimization problem for surfaces with boundary. Given a knot $K \subset S^3$, we ask about its *slice genus*:

$$g_s(K) = \min\{\text{genus}(\Sigma) \mid \Sigma \text{ oriented}, \Sigma \subset B^4, \partial\Sigma = \Sigma \cap S^3 = K\}$$

Knots with $g_s(K) = 0$ are called *slice*. As of now, there is no known algorithm to determine the slice genus of a knot, or even an algorithm that would tell us if it is slice.

Nevertheless, for some families of knots, Yang-Mills theory was successfully used to compute the slice genus. For example, the torus knot

$$T_{p,q} = S^3 \cap \{(x,y) \in \mathbb{C}^2 \mid x^p - y^q = \epsilon\}$$

bounds an algebraic curve, which must be genus minimizing by the local Thom conjecture. From here, Kronheimer and Mrowka obtained a proof of a conjecture of Milnor:

Corollary 3.7 (Kronheimer-Mrowka [KM93])**.** *The slice genus of the torus knot $T_{p,q}$ is $(p-1)(q-1)/2$.*

4. Seiberg-Witten theory (1994-2000)

4.1. **The Seiberg-Witten equations.** In 1994, Seiberg and Witten introduced a new set of gauge-invariant equations over four-manifolds:

$$F_A^+ = \sigma(\Phi), \quad D_A\Phi = 0$$

where A is a Spin^c connection (in a Spin^c bundle S over X), Φ is a spinor, σ is a certain quadratic form, and D_A is the Dirac operator.

Compared to Yang-Mills, the Seiberg-Witten equations have some advantages: the group is abelian (being locally modeled on $U(1)$ instead of $SU(2)$) and the moduli space of solutions (monopoles) is compact. By counting solutions (with certain constraints) we get the *Seiberg-Witten invariants* of 4-manifolds X:

$$SW_X : \text{Spin}^c(X) \to \mathbb{Z}$$

where $\mathrm{Spin}^c(X)$ is the set of Spin^c structures on X. The invariants are usually defined for $b_2^+(X) \geq 3$ odd. When $b_2^+(X) = 1$, we have two different functions SW related by a "wall-crossing formula."

Witten [Wit94] conjectured that the Seiberg-Witten invariants are a repackaging of the Donaldson invariants, by a specific formula. Witten's conjecture was proved by Feehan and Leness [FL18] in many cases, and a completely general proof is now in sight.

Many results obtained with Yang-Mills theory (e.g. Donaldson's diagonalizability theorem) can be re-proved with Seiberg-Witten theory, and some can be improved. A famous early application was the Thom Conjecture:

Theorem 4.1 (Kronheimer-Mrowka [KM94a])**.** *Smooth algebraic curves in $\mathbb{CP}^2$ are genus minimizing in their homology class. Therefore,*

$$\min\{g(\Sigma) \mid \Sigma \subset \mathbb{CP}^2, [\Sigma] = d[\mathbb{CP}^1]\} = (d-1)(d-2)/2.$$

4.2. **Symplectic four-manifolds.** In a series of foundational papers, Taubes [Tau94, Tau95] studied the Seiberg-Witten equations on symplectic manifolds, proving for example that they are nonvanishing for the canonical class K, and later showing that the Seiberg-Witten invariants equal the Gromov-Witten invariants (counts of pseudo-holomorphic curves).

This led to several applications to symplectic geometry:

- Taubes [Tau95] showed that $\mathbb{CP}^2$ has a unique symplectic structure, up to scaling and symplectomorphism;
- Lalonde and McDuff [LM96] classified ruled symplectic manifolds (S^2-bundles over compact surfaces);
- Szabó [Sza98] showed the existence of simply connected irreducible 4-manifolds that are not symplectic;
- Ozsváth and Szabó [OS00] proved the *Symplectic Thom Conjecture*: symplectic surfaces in symplectic manifolds are genus minimizing in their homology class.

4.3. **Fintushel-Stern knot surgery.** In 1996, Fintushel and Stern [FS98] introduced the following surgery operation on 4-manifolds. Let X be a simply connected smooth 4-manifold with $b_2^+ > 1$. Let $T \subset X$ be a c-embedded torus (e.g. a fiber of the elliptic fibration in the K3 surface), such that $\pi_1(X - T) = 1$. Write the Seiberg-Witten invariants as a formal power series

$$SW_X = \sum_{s \in \mathrm{Spin}^c(X)} SW_X(s) e^{c_1(s)}.$$

Let $K \subset S^3$ be a knot. Let

$$X_K = (X \setminus \mathrm{nbhd}(T)) \cup_{T^3} (S^1 \times (S^3 \setminus \mathrm{nbhd}(K)))$$

This is homeomorphic to X, and we have

$$SW_{X_K} = SW_X \cdot \Delta_K(e^{2[T]})$$

where Δ_K is the Alexander polynomial of K. For example, if X is the K3 surface, then $SW_X = 1$ and SW_{X_K} is given by the Alexander polynomial. As we vary the knot K, this can be any symmetric Laurent polynomial $p(t) \in \mathbb{Z}[t, t^{-1}]$ with $p(1) = 1$. This gives many exotic smooth structures on the K3 surface: if two knots have different Alexander polynomials, then the corresponding four-manifolds X_K are homeomorphic but not diffeomorphic. To what extent the knot K is determined by the diffeomorphism type of X_K remains an open problem.

4.4. **Furuta's 10/8-Theorem.** For a 4-manifold with a spin structure (e.g. $\pi_1(X) = 1$ and Q_X even), the Seiberg-Witten equations admit a symmetry under the group

$$Pin(2) = S^1 \cup jS^1 \subset \mathbb{C} \oplus j\mathbb{C} = \mathbb{H}.$$

Furuta [Fur01] introduced the finite dimensional approximation to the Seiberg-Witten map: an equivariant map between $Pin(2)$-representation spheres, with certain properties. Its existence implies:

Theorem 4.2 (Furuta [Fur01])**.** *Let X be a smooth, spin, closed 4-manifold. If the intersection form Q_X is not definite, then $b_2(X) \geq \frac{10}{8}|\sigma(X)| + 2$.*

The inequality $b_2(X) \geq \frac{11}{8}|\sigma(X)|$ is Matsumoto's 11/8-*conjecture* [Mat82]. The conjecture is still open, but Furuta's theorem represents significant progress in its direction. If proved, the 11/8-conjecture would complete the classification of smooth simply connected 4-manifolds up to *homeomorphism*, by an application of Theorem 3.3.

5. The last twenty years (2000-2020)

In the last twenty years, several new research directions have emerged in four-dimensional topology. We sketch them here, along with some open problems and topics of current interest.

5.1. **More applications of finite dimensional approximation.** Furuta's finite dimensional approximation technique yielded a refinement of the Seiberg-Witten invariants, the *Bauer-Furuta invariants* from [BF04]. While the Seiberg-Witten invariants are numbers, the Bauer-Furuta invariants are elements in an S^1-equivariant stable homotopy group of spheres. (For spin manifolds, we can also get elements in a $Pin(2)$-equivariant stable homotopy group of spheres.) One can think of the Bauer-Furuta invariants as capturing the Seiberg-Witten moduli space as a framed manifold, via the Pontryagin-Thom construction.

The Bauer-Furuta invariants contain more information than the Seiberg-Witten invariants. For example, they can be nontrivial for some manifolds with b_2^+ even, such as $K3\#K3$, and have been used to show the existence of exotic smooth structures on $\#^n K3$ for $n \leq 4$; see [Bau04].

In a different direction, the 10/8 inequality (Theorem 4.2) was improved (by a constant term) by several authors, culminating in the $10/8 + 4$ theorem of Hopkins, Lin, Shi and Xu [HLSX18]. In that paper, the authors characterized exactly which stable $Pin(2)$-maps between representation spheres with the required properties exist.

5.2. **Cut-and-paste techniques: Floer homology.** The last twenty years have seen a surge in research activity aimed at understanding 4-manifolds with boundary. While for closed 4-manifolds the Donaldson and Seiberg-Witten invariants take numerical values, for manifolds X with boundary their analogues take values in a group associated to ∂X, called *Floer homology.* This allows computing the closed 4-manifold invariants by cut-and-paste techniques, using gluing formulas of the type

$$X = X_0 \cup_Y X_1 \;\Rightarrow\; \Phi(X) = \langle \Phi(X_0), \Phi(X_1) \rangle.$$

Here, $\partial X_0 = -\partial X_1 = Y$, the symbol Φ denotes a 4-manifold invariant, and $\langle \cdot, \cdot \rangle$ is a natural pairing between the Floer homologies of Y and $-Y$.

The first version of Floer homology was *instanton homology*, constructed by Floer [Flo88] using the Yang-Mills equations. It is defined for integer homology 3-spheres (that is, 3-manifolds Y with $H_*(Y) = H_*(S^3)$), and gives a gluing formula for Donaldson invariants, for 4-manifolds cut along an integer homology 3-sphere.

Since then, many other Floer homologies for 3-manifolds were developed:

- More versions of *instanton homology*, cf. [AB96], [Frø02], [KM10], [Mil19];
- *Monopole (a.k.a. Seiberg-Witten) Floer homology*, cf. [KM07], [MW01], [Man03], [Frø10];
- *Heegaard Floer homology*, cf. [OS04b];
- *Symplectic instanton homology*, cf. [WW20], [MW12], [Hor16];
- *Embedded contact homology* [Hut14], which is associated to a 3-manifold equipped with a contact structure.

Monopole Floer homology, Heegaard Floer homology and embedded contact homology are now known to be isomorphic, by the work of Taubes [Tau10], Kutluhan-Lee-Taubes [KLT20] and Colin-Ghiggini-Honda [CGH12]. Instanton homology and symplectic instanton homology are conjectured to be isomorphic to each other, while the relation between these and the other three theories is less clear.

Most of the Floer homologies above have analogues associated to knots in 3-manifolds. These are of interest both in the study of knots per se, and as a stepping stone towards understanding the 3-manifold invariants, via surgery formulas. (Every 3-manifold is known to be obtained by surgery on a link in S^3.) Indeed, Floer homology has many 3-dimensional applications, particularly to questions related to surgery. For example, in 2003, Kronheimer and Mrowka [KM04] used instanton Floer homology to prove Property P for knots (that surgeries on knots cannot produce counterexamples to the Poincaré conjecture). In the same year, Kronheimer, Mrowka, Ozsváth and Szabó used monopole Floer homology to show that the unknot is characterized by any of its surgeries [KMOS07].

5.3. **Heegaard Floer theory.** Heegaard Floer homology is a 3-manifold invariant constructed by Ozsváth and Szabó [OS04b] using symplectic geometry (counts of pseudo-holomorphic curves in symmetric products of Riemann surfaces). Its knot theory counterpart is called *knot Floer homology*, and was independently constructed by Ozsváth-Szabó and Rasmussen [OS04a, Ras03].

Heegaard Floer theory has become popular among researchers because it is more computationally tractable than the invariants from gauge theory. Indeed, there are:

- concrete formulas for the Heegaard Floer homology of many families of 3-manifolds, such as Seifert fibrations, negative definite plumbings, or surgeries on alternating and torus knots; see for example [OS03c, OS03b, OS05];
- combinatorial descriptions of knot Floer homology, that make it algorithmically computable for any knot; see [MOS09], [OS09], [BL12];
- general combinatorial descriptions of the Heegaard Floer homology of 3-manifolds, and of the related 4-manifold invariants (conjecturally equal to the SW invariants); see [SW10], [MOT09];
- versions of Heegaard Floer homology for 3-manifolds with boundary (bordered Floer homology) which give effective algorithms for computation; see [LOT18, LOT14, OS19].

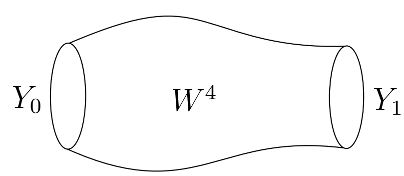
Furthermore, Heegaard Floer homology has numerous applications to questions about knots and 3-manifolds, and to contact geometry. Since this is a survey on four-manifolds, however, let us mention a few of its four-dimensional applications:

- Constraints on the intersection forms of smooth 4-manifolds with boundary a specific 3-manifold; see [OS03a];
- New calculations of the invariants of closed 4-manifolds, e.g. for knot concordance surgery (a variant of Fintushel-Stern knot surgery) see [JZ18];
- A proof of the existence of compact 4-manifolds X homotopy equivalent to S^2, such that the homotopy equivalence cannot be realized by a piecewise linear embedding $S^2 \hookrightarrow X$; see [LL19].

5.4. **Homology cobordism.** An easy exercise in algebraic topology shows that if X is a compact four-manifold with boundary and $H_*(X) = H_*(B^4)$, then the boundary ∂X is a homology 3-sphere. Conversely, every homology 3-sphere bounds a topological 4-manifold X with $H_*(X) = H_*(B^4)$, by a result of Freedman [Fre82]. (In fact, X can be taken to be contractible.) In the world of smooth manifolds, however, the question of which homology 3-spheres bound homology 4-balls is a difficult one. Observe that a homology 3-sphere Y has this property if and only if it represents the zero class in the following abelian group:

$$\Theta^3_{\mathbb{Z}} = \{Y^3 \text{ oriented}, H_*(Y) = H_*(S^3)\} \ / \ \sim$$

where the equivalence relation is given by $Y_0 \sim Y_1 \iff$ there exists compact, oriented, smooth four-manifold W with $\partial W = (-Y_0) \cup Y_1$ and $H_*(W, Y_i; \mathbb{Z}) = 0$:



The group $\Theta^3_{\mathbb{Z}}$ is called the *homology cobordism group*, and is an important object of study at the interface of 3- and 4-dimensional topology. It also has applications to the theory of triangulations of manifolds in dimensions ≥ 5.

One can show that $\Theta^3_{\mathbb{Z}} \neq 0$ by considering the *Rokhlin homomorphism*

$$\mu : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}/2, \quad \mu(Y) = \sigma(W)/8 \pmod 2$$

where W is any compact, smooth, spin 4-manifold with boundary Y. Theorem 2.3 shows that this is a well-defined map.

For example, we have $\mu(S^3) = 0$, but $\mu(\text{Poincaré sphere}) = 1$ and hence $\Theta^3_{\mathbb{Z}} \neq 0$.

Using Yang-Mills theory, in the 1990s it was proved that $\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}^\infty$ subgroup [Fur90], [FS85]. Later, Frøyshov [Frø02] constructed an epimorphism

$$h : \Theta^3_{\mathbb{Z}} \to \mathbb{Z},$$

showing that $\Theta^3_{\mathbb{Z}}$ has at least a $\mathbb{Z}$ summand. Similar epimorphisms can be defined using Seiberg-Witten or Heegaard Floer theory.

In 2013, the author used a $Pin(2)$-equivariant version of Seiberg-Witten Floer homology to show that the exact sequence

$$0 \longrightarrow \ker(\mu) \longrightarrow \Theta^3_{\mathbb{Z}} \xrightarrow{\mu} \mathbb{Z}/2 \longrightarrow 0$$

does not split [Man16]. Combined with the work of Galewski-Stern [GS80] and Matumoto [Mat78] from the 1970's, this showed the existence of non-triangulable manifolds in dimensions ≥ 5.

Similar ideas gave rise to an involutive version of Heegaard Floer homology [HM17], which was recently used to prove that $\Theta^3_{\mathbb{Z}}$ admits a $\mathbb{Z}^\infty$ summand [DHST18]. Understanding the structure of $\Theta^3_{\mathbb{Z}}$ remains an active area of research.

5.5. **Khovanov homology.** For links $K \subset S^3$, Khovanov [Kho00] defined a bigraded homology theory

$$Kh(K) = \bigoplus_{i,j} Kh_{i,j}(K)$$

whose Euler characteristic is the Jones polynomial. The construction is purely combinatorial, and is inspired by ideas from representation theory (categorification).

However, Khovanov homology is formally similar to Floer homologies for knots; e.g. it is functorial under knot cobordisms in $S^3 \times [0,1]$. In fact, Khovanov homology was later given a symplectic interpretation in terms of Lagrangian Floer homology; see [SS06] and [AS19]. Furthermore, in 2011, Witten [Wit12] proposed an interpretation of Khovanov homology in terms of gauge theory, based on the Kapustin-Witten and Haydys-Witten equations with certain boundary conditions.

Given these similarities, an important problem is to extend Khovanov homology to knots in arbitrary 3-manifolds, and to see if this leads to interesting new 4-manifold invariants. In 2019, Morrison, Walker and Wedrich [MWW19] proposed a candidate theory, which generalizes Khovanov homology to links in the boundaries of 4-manifolds.

In fact, four-dimensional applications of Khovanov homology, particularly related to knots in S^3, already exist. In 2004, using a deformation of Khovanov homology, Ramussen extracted a numerical knot invariant denoted s, which gives a lower bound for the slice genus

$$|s(K)| \leq 2g_s(K).$$

He then used s to give a combinatorial proof of Milnor's Conjecture, that $g_s(T_{p,q}) = (p-1)(q-1)/2$; see [Ras10]. One can also use s to show the existence of topologically slice knots that are not smoothly slice, which gives a new proof (without gauge theory) of the existence of exotic smooth structures on $\mathbb{R}^4$.

Building on these ideas, in 2018, Lambert-Cole [LC20a] gave a new proof of the Thom Conjecture (Theorem 4.1), about the minimal genus of a surfaces in $\mathbb{CP}^2$. While the original proof (due to Kronheimer and Mrowka) used Seiberg-Witten theory, Lambert-Cole's proof is based on Khovanov homology, together with some arguments from contact geometry, and involving a new way of visualizing 4-manifolds, in terms of *trisections*, due to Gay and Kirby [GK16]. Moreover, in [LC20b], Lambert-Cole gave a new proof (in the same spirit, using Khovanov homology) of an adjunction inequality in symplectic 4-manifolds. Corollaries of this were the symplectic Thom conjecture, and the existence of exotic smooth structures on some closed four-manifolds, such as $K3\#\mathbb{CP}^2$. Again, these were known before (as applications of gauge theory), but it is remarkable that they can be reproved in this way.

Thus, to a significant extent, Khovanov homology is a replacement for gauge theory. Nevertheless, there are still results proved with gauge theory (such as Donaldson's diagonalizability theorem, Theorem 3.4) which do not yet have Khovanov-theoretic proofs.

There are also new applications of Khovanov homology, for which no gauge theoretic proofs are known. For example, Piccirillo [Pic19] showed that the slice genus of a knot can differ from a related invariant called the shake genus. In [Pic20], she showed that the

Conway knot C is not slice; this was a knot for which all previously known obstructions to sliceness vanished; in particular, the Rasmussen invariant is $s(C) = 0$. Piccirillo constructed a partner knot C' such that C is slice $\iff$ C' is slice, and she showed that C' is not slice by calculating that $s(C') \neq 0$.

There is even some hope that Khovanov homology may help disprove the smooth Poincaré Conjecture in dimension 4 (SPC4); i.e. give a negative answer to Question 2.5. This is based on a strategy proposed in 2009 by Freedman, Gompf, Morrison, and Walker [FGMW10]: Find a knot $K \subset S^3$ such that $s(K) \neq 0$ (hence $g_s(K) \neq 0$, i.e. K does not bound a smooth disk in B^4) but such that K bounds a smooth disk in some homotopy ball Z. This would imply that $Z \ncong B^4$, and hence $Z \cup B^4$ would be a nontrivial homotopy 4-sphere.

It is worth noting that gauge theoretic invariants cannot distinguish between sliceness in B^4 and in a homotopy 4-ball. It is unclear whether s can do so. In [MMSW19], it was proved that that if K bounds a smooth disk in a homotopy 4-ball obtained from B^4 by a Gluck twist, then $s(K) = 0$. (The proof involved showing properties of s with respect to surfaces in $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$.) Thus, the strategy in [FGMW10] fails for Gluck twists.

Still, there are other examples of homotopy 4-balls where the strategy could conceivably work. In [MP21], the author and Piccirillo produced homotopy 4-spheres from pairs of knots with the same 0-surgery. By computer experimentation, they found 5 examples of topologically slice knots such that, if any of them were slice, then SPC4 would be false.

5.6. **New constructions of four-manifolds.** Recall from Section 2.1 that many interesting examples of 4-manifolds come from algebraic geometry. The study of complex surfaces has continued to develop. Particularly striking was the progress in constructing *fake projective planes*, complex surfaces that have the same homology as $\mathbb{CP}^2$ but are not biholomorphic to it. Note that, by a result of Yau [Yau78], such surfaces cannot be homeomorphic to $\mathbb{CP}^2$. Rather, they are algebraic surfaces of general type with nontrivial fundamental group. The first example of a fake projective plane was given by Mumford in 1979 [Mum79], but a full understanding came only in the 2000's, thanks to the work of Prasad and Young [PY07], Keum [Keu08] and Cartwright and Steger [CS10]. The conclusion is that there are exactly 100 fake projective planes up to biholomorphism.

Every complex projective surface is Kähler and hence symplectic. On the other hand, symplectic geometry is a more general source of constructions of 4-manifolds than complex geometry. In 1995, Gompf [Gom95] introduced the fiber sum operation on symplectic manifolds, and used it to show the existence of simply connected symplectic 4-manifolds that are not complex. He also showed that there exist symplectic 4-manifolds with arbitrary (finitely presented) fundamental group. A combination of the work of Gompf [Gom95] and Donaldson [Don99] provided a topological characterization of symplectic 4-manifolds, in terms of *Lefschetz pencils*. Since then, much research has been done on the geography problem for simply connected symplectic 4-manifolds: deciding which triples (χ, σ, t) can be realized by such manifolds, where χ is the Euler characteristic, σ the signature, and t the parity of the intersection form.

In the non-simply connected case, it is worth mentioning the result of Friedl and Vidussi [FV11], who completely characterized which four-manifolds of the form $M \times S^1$ are symplectic: those such that the 3-manifold M fibers over the circle.

With regard to smooth four-manifolds per se, there was a flurry of activity around 2003-2007 aimed at finding exotic smooth structures on simply connected 4-manifolds with small b_2. Following work of J. Park, Akhmedov, D. Park, Fintushel, Stern, Stipsicz and Szabó, it was established that $\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$ admits infinitely many exotic smooth structures, provided that $n \geq 2$; see [AP10]. The existence of exotic smooth structures on simpler four-manifolds such as S^4, $\mathbb{CP}^2$, $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$, $S^1 \times S^3$, T^4 or $S^2 \times S^2$ is still unknown. Also open is the case of the definite manifolds $\#^n\mathbb{CP}^2$, for any n.

5.7. **Diffeomorphism groups.** Another active area of research concerns the diffeomorphism groups of 4-manifolds, and their relation to homeomorphism groups. Gauge theory has proved useful here. Ruberman [Rub98] used Seiberg-Witten theory to give the first examples of simply connected four-manifolds X such that the map $\pi_0(\mathit{Diff}(X)) \to \pi_0(\mathit{Homeo}(X))$ is not injective. When X is the $K3$ surface, Baraglia and Konno [BK19] showed that the map $\pi_1(\mathit{Diff}(X)) \to \pi_1(\mathit{Homeo}(X))$ is not surjective, and Baraglia [Bar21b] showed the nontriviality of $\pi_1(\mathit{Diff}(X))$. More generally, Baraglia [Bar21a] showed that the map $\mathit{Diff}(X) \to \mathit{Homeo}(X)$ is not a weak homotopy equivalence for any closed, smooth, simply connected indefinite 4-manifold with signature of absolute value greater than 8.

In 2018, by completely different methods (based on Kontsevich's characteristic classes for disk bundles), Watanabe [Wat18] disproved the *4D Smale Conjecture*: He showed that $\mathit{Diff}(S^4)$ is not homotopy equivalent to $\mathrm{O}(5)$.

5.8. **The lightbulb theorem.** The classical three-dimensional lightbulb theorem says that if a knot $K \subset S^1 \times S^2$ intersects $\{1\} \times S^2$ transversely and exactly once, then K is isotopic to $S^1 \times y_0$ for some $y_0 \in S^2$. This has an elementary proof. Much more difficult is the four-dimensional version of this result, which was established in 2017 by Gabai:

Theorem 5.1 (Gabai [Gab20])**.** *If a two-sphere $R \subset S^2 \times S^2$ satisfies $[R] = [x_0 \times S^2]$ and R intersects $S^2 \times y_0$ transversely at a single point, then $R \cong x_0 \times S^2$.*

Note that invariants from gauge theory, Heegaard Floer homology, and Khovanov homology typically lead to "negative" results (e.g. two manifolds are not diffeomorphic, or a surface of a given genus and homology class does not exist). Theorem 5.1 is a result of a different kind; we could call it "positive." Its proof has given researchers renewed hope that topological methods can lead to progress in smooth four-dimensional topology.

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Department of Mathematics, Stanford University, Stanford, CA 94305
E-mail address: cm5@stanford.edu