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Introduction

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1. Background

The common theme of the lecture series in this volume is the notion of a *topological quantum field theory*. This concept has its origin in physics, where *(classical) field theories* describe the dynamics (evolution in time) of fields on a manifold M. In general, a field can be a section of any given sheaf over M. In practice, most common are scalar fields (maps $M \to \mathbb{R}$), vector fields (sections of the tangent bundle $TM \to M$), and gauge fields (connections in a vector bundle, which appear for example in Maxwell's theory of electromagnetism).

A *quantum field theory* (QFT) is a more complicated physical theory, which typically combines:

- a classical field theory;
- quantum mechanics (including dependence on a parameter $\hbar$, such that the limit $\hbar \to 0$ recovers the classical theory);
- special relativity (including Lorentz invariance).

Quantum field theory is the basis of the Standard Model of particle physics, which has been successfully used to explain three of the four fundamental forces of nature: electromagnetism, the weak interactions, and the strong interactions. The fourth force, gravity, can be understood (at large scales) using Einstein's theory of general relativity. It remains an open problem to formulate a unified theory that combines relativity with the other three forces. The search for unification has led physicists to propose new theories, such as string theory. In the process they studied various quantum field theories that can serve as toy models for some aspects of the unified theory.

One offshoot of the study of QFT's has been the discovery of interesting *topological* quantum field theories (TQFT's), which have captured the attention of mathematicians. Typically, the classical or quantum field theories that appear in physics depend essentially on the choice of a Riemannian metric or Lorentzian metric on the underlying manifold. TQFTs depend only on the topology. On the other hand, dropping the requirement that the manifold is Lorentzian makes the theory interesting in a different way. Indeed, this allows for arbitrary topological models for space-time, i.e., cobordisms between the initial and the final manifold.

2. Definition

Let us sketch the mathematical definition of a TQFT. We emphasize that, while TQFT's arose from the study of QFT's, there is nothing quantum in the definition, so mathematicians sometimes call them simply *topological field theories* (TFT's).

A *symmetric monoidal category* $\mathfrak{C}$ is a category equipped with a notion of tensor product $\otimes$ and isomorphisms

$$A \otimes B \xrightarrow{\cong} B \otimes A,$$

satisfying certain properties which extend the symmetry to a symmetric group action permitting the factors in a product to be of any length. Examples of symmetric monoidal categories include $\mathrm{Vect}_{\mathbf{k}}$ (the category of vector spaces over a field $\mathbf{k}$), R-mod (modules over a ring R), and the cobordism category Cob_{d+1}. The objects of Cob_{d+1} are d-dimensional closed smooth manifolds, the morphisms are smooth $(d+1)$-dimensional cobordisms, and the tensor product is the disjoint union. Then, a *$\mathfrak{C}$-valued TQFT* is a symmetric monoidal functor

$$Z : \mathrm{Cob}_{d+1} \to \mathfrak{C}.$$

Concretely, in a TQFT (say, with values in $\mathrm{Vect}_{\mathbf{k}}$) we assign a vector space $Z(Y)$ to each closed d-manifold, and a homomorphism

$$Z(W) : Z(Y_0) \to Z(Y_1)$$

to each $(d+1)$-dimensional cobordism W between manifolds Y_0 and Y_1. These are required to satisfy some properties, for example functoriality under gluing along a common boundary:

$$Z(W_0 \cup W_1) = Z(W_1) \circ Z(W_0).$$

For the empty set we have $Z(\emptyset) = \mathbf{k}$, and therefore to a closed $(d+1)$-dimensional manifold X we assign an invariant

$$Z(X) \in \mathrm{End}(\mathbf{k}) \cong \mathbf{k}.$$

This definition of a TQFT can be tweaked in many ways, by considering manifolds and cobordisms equipped with various structures, such as: orientations, basepoints and paths, spin structures, spin^c structures, embeddings in some other fixed manifolds. For example, a TQFT for links in $\mathbb{R}^3$ would assign a vector space $Z(L)$ to each link $L \subset \mathbb{R}^3$, and a map $Z(\Sigma) : Z(L_0) \to Z(L_1)$ for each smoothly embedded cobordism $\Sigma \subset \mathbb{R}^3 \times [0,1]$ from L_0 to L_1.

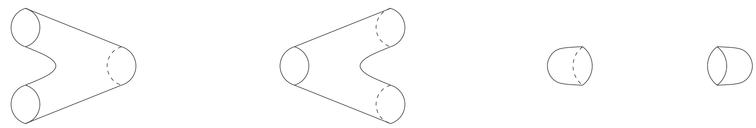
3. Examples

Here are a few examples of TQFT's:

(1) $(1+1)$-*dimensional TQFT's* with values in $\mathrm{Vect}_{\mathbf{k}}$. These consist of a vector space $A = Z(S^1)$ and four operations

$$\mu : A \otimes A \to A, \quad \Delta : A \to A \otimes A, \quad \eta : \mathbf{k} \to A, \quad \varepsilon : A \to \mathbf{k}$$

corresponding to the four cobordisms shown at the top of the next page.



The four operations are called the multiplication, comultiplication, unit, and counit. The TQFT structure imposes certain properties on them, which turns A into a *Frobenius algebra*. Conversely, any Frobenius algebra produces a $(1+1)$-dimensional TQFT.

(2) *Khovanov homology* is a TQFT for links in $\mathbb{R}^3$. This is built by starting from the Frobenius algebra

$$A = \mathbf{k}[x]/(x^2) \cong H^*(S^2; \mathbf{k})$$

with the usual polynomial multiplication μ, the unit 1, and the comultiplication and counit given by

$$\Delta(1) = 1 \otimes x + x \otimes 1,\ \Delta(x) = x \otimes x,\ \varepsilon(1) = 0,\ \varepsilon(x) = 1.$$

The Frobenius algebra A is what is assigned to the unknot. For a general link, Khovanov described a procedure to construct a chain complex by taking a cube of resolutions for a planar diagram, and using the operations μ and Δ to define the chain maps. The homology of this complex is an interesting link invariant, whose Euler characteristic is the celebrated Jones polynomial. Khovanov homology and its generalizations ($\mathfrak{sl}(n)$ Khovanov-Rozansky homologies, HOMFLY-PT homology, colored homologies) are the focus of the Jacob Rasmussen's lectures.

(3) *Invertible TQFT's* are those where the objects and the morphisms are invertible under the tensor product operation. For example, in dimension $1+1$, they correspond to Frobenius algebras such that $A \cong \mathbf{k}$, $\eta = 1$, $\mu : \mathbf{k} \to \mathbf{k} \otimes \mathbf{k} \cong \mathbf{k}$ is given by a unit $\mu \in \mathbf{k}^*$, and Δ by μ^{-1}. In higher dimensions, general TQFT's are hard to classify, whereas it is easier to get a handle on invertible TQFT's, using the tools of homotopy theory. Invertible TQFT's are related to classifying spaces of diffeomorphism groups, and thus to the classification of manifold bundles. These topics are discussed in the lectures of Søren Galatius.

More interesting examples of TQFT's come from actual quantum field theories studied in physics. There are two major kinds of TQFT's that arise from physics, sometimes called *Schwarz-type* and *Witten-type*.

4. TQFT's from path integrals

Schwarz-type TQFT's are obtained via path integrals, i.e., integrals over an (infinite-dimensional) space of paths or connections. It is hard to make mathematical sense of such integrals, but they are fundamental in QFT. The typical example of a Schwarz-type TQFT is Chern-Simons theory, in dimension $2+1$. To a closed 3-manifold X, this theory associates the path integral

$$Z(X) = \int_{\mathcal{A}} e^{2\pi i k \, CS(A)} dA.$$

Here, $\mathcal{A}$ is the space of gauge equivalence classes of connections in the (trivial) $SU(2)$-bundle over X, and CS is the Chern-Simons functional

$$CS(A) = \frac{1}{4\pi^2} \int_X \operatorname{tr}\big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\big) \in \mathbb{R}/\mathbb{Z}.$$

This definition of $Z(X)$ is due to Witten. Later, Reshetikhin and Turaev gave a rigorous mathematical definition of $Z(X)$, using representation theory and surgery presentations of 3-manifolds in terms of links in $\mathbb{R}^3$. The Witten-Reshetikhin-Turaev (WRT) invariants $Z(X)$ are the main object of study in a branch of mathematics called quantum topology. They admit an extension for links $L \subset X$, where the path integral incorporates Wilson loops. For example, for a link $L \subset S^3$, the resulting invariant is the Jones polynomial.

Recent developments in Chern-Simons theory are the focus of the lectures of Pavel Putrov.

5. TQFT's from supersymmetry

Witten-type TQFT's are obtained from some QFT's by a process called "topological twisting." We will not describe the process in any detail here, but let us mention a few key terms that are appear in this context.

For a QFT to admit a topological twist, one needed ingredient is *supersymmetry* (SUSY). This means that the theory contains fields of two types:

- bosons (those whose spin is an integer), e.g. scalar fields or vector fields. In particle physics, the photon is the standard example of a boson;
- fermions (with half-integer spin), e.g. spinors. In particle physics, electrons, neutrons and protons are all fermions.

Furthermore, the QFT should admit a symmetry that interchanges the bosons and the fermions; this is called a supersymmetry. A QFT may have several supersymmetries. A *supercharge* is an odd element of the Lie algebra of SUSY transformations.

When physicists talk about a 3d $\mathcal{N} = 2$ or 4d $\mathcal{N} = 4$ theory (for example), the number d refers to the dimension of the underlying spacetime, and $\mathcal{N}$ captures the number of independent supercharges. More precisely, the number of independent supercharges is $k\mathcal{N}$, where $k = k(d)$ depends only on d, being the dimension of the minimal super-Poincaré algebra in dimension d. For example, we have $k = 2$ in dimension $d = 3$, and $k = 4$ for $d = 4$. Other cases mentioned below are $d = 2$ and $d = 6$; for those, supersymmetries come in two types, so one talks about $\mathcal{N}$ being a pair of numbers, such as $(2, 2)$.

Another concept familiar to physicists, but sometimes mysterious to mathematicians, is that of *BPS states*, named after Bogomolnyi, Prasad, and Sommerfeld. These are the states in a supersymmetric QFT that are annihilated by a supercharge. Mathematically, they are described as solutions of a first order PDE, and they minimize some energy functional. As a typical example, in Yang-Mills theory, the Yang-Mills equations are of order two:

$$d_A^* F_A = 0,$$

whereas the anti-self-dual (ASD) equations

$$\star F_A = -F_A$$

are of order one and minimize the energy $\int_M |F_A|^2 dvol$. The solutions to the ASD equations are BPS states, as are many other interesting mathematical objects: Seiberg-Witten monopoles, J-holomorphic curves, etc.

It is worth mentioning here that the limit as $\hbar \to 0$ of a quantum system is a classical system, described by solutions to differential equations. In a TQFT of Witten-type, since there is no metric dependence, there is also no dependence on $\hbar$. That is why such theories are formulated in terms of differential equations.

Supersymmetry and BPS states are discussed further in Andrew Neitzke's lectures. In particular, these lectures include a detailed description of $\mathcal{N} = (2, 2)$ supersymmetry in dimension $d = 2$.

In the next subsections we mention various other supersymmetric theories that are of interest to mathematicians.

Pure $SU(2)$ super Yang-Mills theory This is a 4d $\mathcal{N} = 2$ theory, which admits a topological twist giving rise to a Witten-type TQFT. At high energy (i.e., small distance scales), the topological theory is called *Donaldson theory*, and is described by the ASD equations. At low energy (high distance scales), it gives rise to *Seiberg-Witten theory*, described by the Seiberg-Witten equations.

The Donaldson and Seiberg-Witten theories are fundamental tools in low dimensional topology. They give rise to numerical invariants of smooth 4-manifolds, which are able to detect exotic smooth structures (pairs of manifolds homeomorphic, but not diffeomorphic to each other). Since they are two aspects of the same theory, the Seiberg-Witten and Donaldson invariants are closely related, by a formula called Witten's Conjecture (proved in many cases by Feehan and Leness).

The Seiberg-Witten and ASD equations have similar properties, such as invariance under an infinite-dimensional gauge group. The study of these and other gauge-invariant equations forms the object of *mathematical gauge theory*, which is discussed (with a particular focus on Seiberg-Witten theory) in the lectures of Andriy Haydys. In dimension 3, the Donaldson and Seiberg-Witten TQFT's associate to a 3-manifold Abelian groups called *instanton Floer homology* and *monopole Floer homology*, respectively. These have many topological applications, to questions about surgery, cobordisms, contact structures, foliations, etc. Unlike for the 4-dimensional invariants, the relation between instanton and monopole Floer homology is still a mystery. Instanton Floer homology is the topic of the lectures of Tomasz Mrowka.

Seiberg-Witten theory has a symplectic-geometric counterpart developed by Ozsváth and Szabó. To a three-manifold Y, they associated an invariant called *Heegaard Floer homology*, defined by decomposing Y into two handlebodies (glued along a surface Σ), and then using J-holomorphic curves on a symmetric product of Σ. It is now known that the Heegaard Floer and monopole Floer homology are isomorphic. Ozsváth and Szabó also associated numerical invariants to four-manifolds, which are conjecturally the same as the Seiberg-Witten invariants. There is also a version of Heegaard Floer homology for knots in 3-manifolds, called *knot Floer homology*. Heegaard Floer theory is the topic of Jennifer Hom's lectures.

More 4d $\mathcal{N} = 2$ **theories** In addition to pure super Yang-Mills theory, there is a whole family of 4d $\mathcal{N} = 2$ theories, called *of class S*. These are parametrized by triples $(\mathfrak{g}, C, D)$ where $\mathfrak{g}$ is a Lie algebra of ADE type, C is a Riemann surface with punctures, and D is certain data at the punctures. For example, pure $SU(2)$ super Yang-Mills theory corresponds to $\mathfrak{g} = \mathfrak{su}(2)$, the planar surface $C = \mathbb{P}^1 \setminus \{0, \infty\}$, and some data D.

Theories of class S are all derived from a superconformal 6d $\mathcal{N} = (2, 0)$ theory, called "theory $\mathcal{X}$" or "fivebrane theory". This involves considering theory $\mathcal{X}$ on $M \times C$ ("compactifying on C"), where M is a 4-manifold and C is the punctured Riemann surface above.

If we write a theory of class S on a product $M = Y \times S^1$, we get a 3d theory that can be described in terms of maps $Y \to \mathfrak{M}_{\text{Higgs}}(C)$, where $\mathfrak{M}_{\text{Higgs}}(C)$ is the *Hitchin moduli space* of Higgs bundles on C. The Hitchin moduli space has a hyperkähler structure, and appears in various guises in different branches of mathematics. If C is a closed Riemann surface and $\mathfrak{g} = \mathfrak{su}(2)$, then the Hitchin moduli space is homeomorphic to the $SL(2, \mathbb{C})$ character variety of C:

$$\mathfrak{M}_{\text{Higgs}}(C) \cong \{\pi_1(C) \to SL(2, \mathbb{C})\}/\text{ conjugation}$$

The mathematics around the Hitchin moduli space is discussed in the lectures of by Laura Schaposnik.

The count of BPS states in the 3d theory (coming from the 4d theory class of S) gives information about the hyperkähler metric on $\mathfrak{M}_{\text{Higgs}}(C)$. A discussion of this appears in Andy Neitzke's lectures.

4d $\mathcal{N} = 4$ **super Yang-Mills theory** This is obtained from the 6d $\mathcal{N} = (2, 0)$ theory $\mathcal{X}$ by writing it on $M^4 \times T^2$, i.e., compactifying it on a torus T^2. The 4d $\mathcal{N} = 4$ super Yang-Mills theory admits several topological twists, giving rise to some interesting gauge-invariant PDE such as the Kapustin-Witten and Vafa-Witten equations. These have recently been studied by mathematicians, with an eye toward new topological applications (beyond those coming from Donaldson, Seiberg-Witten, or Heegaard Floer theory). A specific proposal in this direction was made by Witten: interpret the coefficients of the Jones polynomial (of knots in $\mathbb{R}^3$) as counts of solutions to the Kapustin-Witten equations on $\mathbb{R}^3 \times (0, \infty)$, with some boundary conditions depending on the knot. Furthermore, Khovanov homology (the object of Rasmussen's res) should be obtained in a similar way, but using the Haydys-Witten equations, a 5-dimensional extension of the Kapustin-Witten equations.

Let us mention here that the 2-dimensional reduction of both the Kapustin-Witten and Vafa-Witten equations are the *Hitchin equations*. Their moduli space is the space of Higgs bundles $\mathfrak{M}_{\text{Higgs}}(C)$, the focus of Schaposnik's lectures.

3d $\mathcal{N} = 2$ **theory** This is obtained from the 6-dimensional theory $\mathfrak{X}$ by writing it on $Y \times S^1 \times D^2$, where Y is a 3-manifold. Gukov, Pei, Putrov and Vafa explained how this theory should give rise to new invariants of 3-manifolds $\widehat{Z}_a(Y)$, in the form of power series in a variable q, with integer coefficients. The power series converge in the unit disk $|q| < 1$, and as q goes to some roots of unity (and one takes certain linear combinations), one should get the Witten-Reshetikhin-Turaev invariants. There is no rigorous mathematical definition of $\widehat{Z}_a(Y)$ yet, but physicists can compute them for several families of examples.

The invariants $\widehat{Z}_a(Y)$ are discussed in Putrov's lectures. One reason they are of interest is to mathematicians is that they are expected to admit a categorification, a "Khovanov homology for 3-manifolds", given by counting BPS states in the 3d $\mathcal{N} = 2$ theory.