NILPOTENT SLICES, HILBERT SCHEMES, AND THE JONES POLYNOMIAL

CIPRIAN MANOLESCU

ABSTRACT. Seidel and Smith have constructed an invariant of links as the Floer cohomology for two Lagrangians inside a complex affine variety Y. This variety is the intersection of a semisimple orbit with a transverse slice at a nilpotent in the Lie algebra $\mathfrak{sl}_{2m}$. We exhibit bijections between a set of generators for the Seidel-Smith cochain complex, the generators in Bigelow's picture of the Jones polynomial, and the generators of the Heegaard Floer cochain complex for the double branched cover. This is done by presenting Y as an open subset of the Hilbert scheme of a Milnor fiber.

1. Introduction

Khovanov cohomology [9] is an invariant of links in the form of a bigraded abelian group $Kh^{*,*}(L)$ whose graded Euler characteristic is the unnormalized Jones polynomial of the link L :

$$\sum_{i,j\in\mathbb{Z}} (-1)^{i+j+1} t^{j/2} \dim(Kh^{i,j}(L)\otimes\mathbb{Q}) = (t^{1/2}+t^{-1/2})V_L(t).$$

By definition, the groups $Kh^{*,*}$ can be combinatorially computed starting from a specific diagram of the link. Nevertheless, they have been found to be quite powerful and to have much in common with more subtle invariants of links coming from gauge theory and symplectic geometry. For example, Rasmussen [28] has used Khovanov's theory to give a new proof of Milnor's conjecture on the slice genus of torus knots, a result proved previously by Kronheimer and Mrowka using gauge theory. Also, in [24], Ozsváth and Szabó have constructed a spectral sequence relating $Kh^{*,*}(L)$ to the Heegaard Floer homology of the double cover of S^3 branched over L. Heegaard Floer homology is a version of the Floer homology defined in symplectic geometry by counting pseudoholomorphic curves.

Seidel and Smith [30] have proposed a remarkable interpretation of Khovanov cohomology itself in terms of symplectic geometry. They represent a link L as the closure of an m-stranded braid $b\in Br_m$ or, equivalently, as the plat closure of $b\times 1^m \in Br_{2m}$. Let w be the writhe of the braid diagram. $b\times 1^m$ can be represented as a loop l in the configuration space $\mathrm{Conf}^{2m}(\mathbb{C})$ of $2m$ distinct points in the plane, with l starting at some basepoint τ. They construct a symplectic fibration over $\mathrm{Conf}^{2m}(\mathbb{C})$, whose fiber at τ is a symplectic manifold $Y=\mathcal{Y}_{m,\tau}$, and then introduce a Lagrangian $\mathcal{L}\subset Y$, well-defined up to isotopy. Applying to L the monodromy map along l yields another Lagrangian $\mathcal{L}'\subset Y$. They define the bigraded groups (the "symplectic Khovanov cohomology")

$$Kh^*_{symp}(L) = HF^{*+m+w}(\mathcal{L},\mathcal{L}')$$

as the Lagrangian Floer cohomology applied to $\mathcal{L}$ and $\mathcal{L}'$ and with a shift in degree.

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Seidel and Smith prove that Kh_{symp} is an invariant of the link L, and conjecture that it equals the original Khovanov cohomology, after a collapsing of its bigrading:

$$Kh^k_{symp}(L) \cong \bigoplus_{i-j=k} Kh^{i,j}(L) \qquad (?)$$

The Seidel-Smith cohomology does not come with a bigrading from which one can read off the Jones polynomial. The goal of the present paper is to define such a bigrading at the level of the cochain complex, as well as to shed some light on the rather mysterious Seidel-Smith construction by giving a more concrete characterization of the objects involved.

We start by describing the symplectic manifold Y as an open subset of a Hilbert scheme. $Y = \mathcal{Y}_{m,\tau}$ is in fact an affine variety, defined as the set of matrices in $\mathcal{S}_m$ with fixed characteristic polynomial $P_\tau(t)$. Here $\mathcal{S}_m$ is an affine subspace of the Lie algebra $\mathfrak{sl}_{2m}$ transverse to the orbit of a nilpotent element N_m with two Jordan blocks of size m, and the coefficients of P are described by $\tau \in \mathrm{Conf}^{2m}(\mathbb{C})$. More generally, instead of N_m we can consider a nilpotent with two Jordan blocks of sizes n and $2m-n$, respectively ($n \leq m$), and similarly construct an affine variety $\mathcal{Y}_{n,\tau}$. Note that m is implicit in this notation, τ being an unordered set of $2m$ points. We prove:

Theorem 1.1. *There is an injective holomorphic map from $\mathcal{Y}_{n,\tau}$ to the Hilbert scheme $\mathrm{Hilb}^n(S_\tau)$, where S_τ is the affine surface described by the equation $u^2 + v^2 + P_\tau(z) = 0$ in $\mathbb{C}^3$.*

Since $\mathcal{Y}_{n,\tau}$ and the Hilbert scheme have the same complex dimension $2n$, it follows that the former can be identified to an open subset of the second.

The case $n = m$ is the only one relevant for the construction of the Seidel-Smith homology. Nevertheless, Theorem 1.1 describes a phenomenon that can be interesting by itself. Indeed, both $\mathcal{Y}_{n,\tau}$ and $\mathrm{Hilb}^n(S_\tau)$ are quiver varieties in the sense of Nakajima [18]. One may ask whether there are other pairs of quiver varieties of the same dimension such that one is an open subset of the other.

Next, we give an explicit description of two Lagrangians that can be used to define the symplectic Khovanov cohomology. This is possible because the Hilbert scheme $\mathrm{Hilb}^m(S_\tau)$ is a certain iterated blow-up of the symmetric product $\mathrm{Sym}^m(S_\tau)$ along subsets of the diagonal Δ. Thus we can identify $\mathrm{Sym}^m(S_\tau) - \Delta$ to an open subset in $\mathrm{Hilb}^m(S_\tau)$, and on $\mathrm{Sym}^m(S_\tau)$ one has nice holomorphic coordinates. Indeed, a point in $\mathrm{Sym}^m(S_\tau) - \Delta$ is characterized as an unordered collection of m distinct points $(u_k, v_k, z_k) \in S_\tau, k = 1, \dots, m$.

Let us choose m disjoint arcs $\alpha_k : [0,1] \to \mathbb{C}$, joining together the $2m$ points of τ in pairs. The braid $b \times 1^m \in Br_{2m}$, whose plat closure is the link L, induces a diffeomorphism of the plane that maps the arcs α_k into $2m$ arcs $\beta_k : [0,1] \to \mathbb{C}$, again joining the points of τ. For each α_k, one can construct the Lagrangian 2-sphere:

$$\Sigma_{\alpha_k} = \{(u,v,z) \in S_\tau \ : z = \alpha_k(t) \text{ for some } t \in [0,1]; \ \ u, v \in \sqrt{-P_\tau(z)}\mathbb{R}\}$$

in S_τ with the standard Kähler metric. Of course, the same construction can be done for the beta curves, resulting in spheres Σ_{β_k}.

Theorem 1.2. *We can deform the Kähler metric on $\mathcal{Y}_{m,\tau}$ and the Lagrangians $\mathcal{L}, \mathcal{L}'$ in the Seidel-Smith construction in such a way that the Floer cohomology groups are preserved under this deformation, and the resulting Lagrangians are*

$$\mathcal{K} = \Sigma_{\alpha_1} \times \Sigma_{\alpha_2} \times \dots \times \Sigma_{\alpha_m} \ ; \ \mathcal{K}' = \Sigma_{\beta_1} \times \Sigma_{\beta_2} \times \dots \times \Sigma_{\beta_m}$$

in $\big(\mathrm{Sym}^m(S_\tau) - \Delta\big) \cap \mathcal{Y}_{m,\tau} \subset \mathrm{Hilb}^m(S_\tau)$.

This enables us to describe explicitly the intersection $\mathcal{K} \cap \mathcal{K}'$ and, in turn, this gives a set of generators for the Seidel-Smith cohomology. More precisely, we form a set $\mathcal{Z}$ in the following way: we assume that the α and β are simple curves that intersect transversely in their interior. For every intersection point $x \in \alpha_i \cap \beta_j$, we introduce an element $e_x \in \mathcal{Z}$ in the case when $x \in \tau$, and two elements e_x, e'_x when $x \notin \tau$. We define maps $A, B : \mathcal{Z} \to \{1, 2, \ldots, m\}$ by taking an element to the indices i, j of the corresponding α and β curves.

At this point, we note that this picture is very similar to the one given by Bigelow [1] in his definition of the Jones polynomial. By presenting a link as the plat closure of a braid, Bigelow obtains its Jones polynomial as a signed count of the intersection points of two half-dimensional submanifolds in a covering of a subset of $\mathrm{Sym}^m(\mathbb{C})$, with certain gradings.

We exploit this similarity and show:

Theorem 1.3. *There is a natural correspondence between a set of generators for $CF^*(\mathcal{K}, \mathcal{K}')$ and the set $\mathcal{G}$ of m-tuples $(z_1, \ldots, z_m)$ of unordered elements of $\mathcal{Z}$ with $A(z_i) \neq A(z_j)$ and $B(z_i) \neq B(z_j)$ for $i \neq j$. This set can also be identified with the set of intersection points in Bigelow's picture of the Jones polynomial.*

Bigelow defined two gradings $Q, T : \mathcal{G} \to \mathbb{Z}$ whose difference doubled and shifted by a constant is the "Jones grading" $J = 2(T - Q) + m + w$. The grading J of an element in $\mathcal{G}$ tells the coefficient in the Jones polynomial to which that generator contributes in the signed count. We define a third grading, $P : \mathcal{G} \to \mathbb{Z}$ (the "projective grading"), starting from the Maslov grading $\tilde{P}$ of the generator in $CF^*(\mathcal{K}, \mathcal{K}')$, and then normalizing to $P = \tilde{P} - (m + w)$. We also show how to read P from the concrete picture of the alpha and beta curves intersecting in the plane.

Our discussion can be related to the work of Ozsváth and Szabó [24], following the ideas of Seidel and Smith [31]. The involution $(u, v, z) \to (u, -v, z)$ on S_τ induces a corresponding involution on $\mathrm{Hilb}^m(S_\tau)$. This involution preserves the Lagrangians $\mathcal{K}, \mathcal{K}'$ from Theorem 1.2, and looking at its fixed point set we find two totally real tori $\mathbb{T}_{\hat{\alpha}}, \mathbb{T}_{\hat{\beta}}$ sitting inside the symmetric power of a Riemann surface. The Floer homology $HF_*(\mathbb{T}_{\hat{\alpha}}, \mathbb{T}_{\hat{\beta}})$ is exactly the Heegaard Floer homology of $\mathcal{D}(L)\#(S^1 \times S^2)$, where $\mathcal{D}(L)$ is the double branched cover of S^3 over L. We can also consider the restriction of the involution to the variety $\mathcal{Y}_{m,\tau}$. Its fixed point set W is an open subset of the symmetric product, and the complement of W is a codimension one subvariety ∇ called the "anti-diagonal." This point of view is useful because even though the homological grading on the Heegaard Floer chain complex in the symmetric product is only defined modulo 2, it can be improved to a $\mathbb{Z}$ grading by considering the restriction to W.

Theorem 1.4. *After some noncanonical choices (to be described in Section 7.3), the set $\mathcal{G}$ can be identified with a set of generators for the Heegaard Floer homology of $\mathcal{D}(L)\#(S^1 \times S^2)$. There is a well-defined integer grading on $\mathcal{G}$ induced by the Maslov grading of the intersection points of $\mathbb{T}_{\hat{\alpha}}$ and $\mathbb{T}_{\hat{\beta}}$ taken inside the open set W. This grading equals $\tilde{P} + T - Q$.*

Denote the Maslov grading from Theorem 1.4 by $\tilde{R} = \tilde{P} + T - Q$. If we renormalize to $R = \tilde{R} - (m + w)/2$, we obtain the simple formula $R = P + (J/2)$. This renormalization is similar to that of $P = \tilde{P} - (m + w)$. In the case of the identity braid $b = 1^m \in Br_m$, they make the P and R gradings of the resulting generators to be symmetric around zero. The fact that the correction term for R is half of that for P is related to the fact that restriction to the fixed point set cuts in half the dimension of the objects involved.

Let us end with some open questions. First, Theorem 1.3 does not say anything about the differentials in the Seidel-Smith cochain complex. We conjecture that these differentials preserve the J grading, and that therefore the bigrading (J, P) descends to the level of cohomology. This would imply the existence of a bigraded Floer theory analagous to Khovanov's. Some hope in the direction of this conjecture could come from looking at the localization of Floer cohomology under the involution considered above. As suggested by Seidel and Smith, localization could also give a geometric interpretation of the spectral sequence in [24]. While we cannot prove the conjecture at this point, a quick corollary of Theorem 1.4 is that the difference of the Maslov gradings in $\mathcal{Y}_{m,\tau}$ and in the fixed point set W, after some normalization, gives exactly the Jones polynomial i.e. the (conjectural) graded Euler characteristic for Kh_{symp} :

Corollary 1.5. *Given a bridge presentation of a link L, consider the corresponding set of generators $\mathcal{G}$ with the two gradings P and R as above. Then the Jones polynomial of L can be expressed in terms of symplectic geometric quantities by the formula:*

$$V_L(t) = -(t^{1/2} + t^{-1/2})^{-1} \cdot \sum_{\gamma \in \mathcal{G}} (-1)^{P(\gamma)} t^{R(\gamma) - P(\gamma)}.$$

Second, the Lagrangians $\mathcal{K}$ and $\mathcal{K}'$ are subsets of the Hilbert scheme $\mathrm{Hilb}^m(S_\tau)$. The possible relevance of $\mathrm{Hilb}^m(S_\tau)$ to low-dimensional topology was pointed out by Khovanov in [10, section 6.5]. One can equip the Hilbert scheme with a suitable Kähler metric and look at the Floer cohomology of $\mathcal{K}, \mathcal{K}'$ inside $\mathrm{Hilb}^m(S_\tau)$. An interesting question is whether this cohomology turns out to be a link invariant.

Finally, following [31], we can also look at the Floer homology of the tori $\mathbb{T}_{\hat{\alpha}}, \mathbb{T}_{\hat{\beta}}$ inside W. We conjecture that this homology, together with its R grading, is another link invariant.

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2. Nilpotent slices and Hilbert schemes

In this section we recall the definitions of the manifold $\mathcal{Y}_{m,\tau}$ appearing in the Seidel-Smith construction, of the Hilbert scheme $\mathrm{Hilb}^n(S_\tau)$, and then we prove Theorem 1.

Let us introduce some notation. For $m > 0$, $\mathrm{Sym}^{2m}(\mathbb{C})$ is the symmetric product of $\mathbb{C}$ and can be identified with $\mathbb{C}^{2m}$ via symmetric polynomials. $\mathrm{Sym}_0^{2m}(\mathbb{C}) \cong \mathbb{C}^{2m-1}$ is the space of unordered sets of $2m$ complex numbers with sum zero.

The configuration spaces

$$\mathrm{Conf}^{2m}(\mathbb{C}) \subset \mathrm{Sym}^{2m}(\mathbb{C}) \ ; \ \mathrm{Conf}_0^{2m}(\mathbb{C}) \subset \mathrm{Sym}_0^{2m}(\mathbb{C})$$

consist of the $2m$-tuples made of distinct complex numbers. The map

(1)
$$\begin{aligned} p : \mathrm{Sym}^{2m}(\mathbb{C}) &\rightarrow \mathrm{Sym}_0^{2m}(\mathbb{C}), \\ (a_1, \dots, a_{2m}) &\rightarrow (a_1 - \frac{1}{2m}\sum a_k, \dots, a_{2m} - \frac{1}{2m}\sum a_k) \end{aligned}$$

is a trivial $\mathbb{C}$-bundle, and its restriction to $\mathrm{Conf}^{2m}(\mathbb{C})$ also exhibits this space as a trivial $\mathbb{C}$-bundle over $\mathrm{Conf}_0^{2m}(\mathbb{C})$.

Throughout this section, we restrict our attention to $\tau \in \mathrm{Sym}_0^{2m}(\mathbb{C})$, with the understanding that everything extends trivially for $\tau \in \mathrm{Sym}^{2m}(\mathbb{C})$ by considering the objects associated to $p(\tau)$.

2.1. **Transverse slices at nilpotent orbits.** We start by presenting the definition of the manifold $\mathcal{Y}_{m,\tau}$ from [30]. As noted in the introduction, we work in a slightly more general setting and define a manifold $\mathcal{Y}_{n,\tau}$ for any $0 \leq n \leq m$ and $\tau \in \mathrm{Conf}_0^{2m}(\mathbb{C})$. In fact, we can do the same for every $\tau \in \mathrm{Sym}_0^{2m}(\mathbb{C})$, but in that case the result is a possibly singular variety $\mathcal{Y}_{n,\tau}$.

Consider the complex algebraic group $G = SL_{2m}$ and its Lie algebra $\mathfrak{g} = \mathfrak{sl}_{2m}$. The adjoint quotient map

$$\chi : \mathfrak{g} \to \mathfrak{g}/G = \mathbb{C}^{2m-1}$$

is defined by taking $A \in \mathfrak{g}$ to the coefficients of the characteristic polynomial $det(tI - A)$. The fiber of χ over $\tau \in \mathrm{Conf}_0^{2m}(\mathbb{C}) \subset \mathrm{Sym}_0^{2m}(\mathbb{C}) = \mathbb{C}^{2m-1}$ is a smooth manifold (the adjoint orbit of a semisimple element in $\mathfrak{g}$). In fact, the map χ can be shown to be a differentiable fiber bundle when restricted to the preimage of $\mathrm{Conf}_0^{2m}(\mathbb{C})$.

Choose a nilpotent element $N_n \in \mathfrak{g}$ with two Jordan blocks of sizes n and $2m - n$, respectively:

$$N_n = \left(\begin{array}{cccccc|cccccc}
0 & 1 & & & & & & & & & & \\
 & 0 & 1 & & & & & & & & & \\
 & & & \cdots & & & & & & & & \\
 & & & & & 1 & & & & & & \\
 & & & & & 0 & & & & & & \\
\hline
 & & & & & & 0 & 1 & & & & \\
 & & & & & & & 0 & 1 & & & \\
 & & & & & & & & & \cdots & & \\
 & & & & & & & & & \cdots & & \\
 & & & & & & & & & & & 1 \\
 & & & & & & & & & & & 0
\end{array}\right).$$

The orbit of N_n under the adjoint action of G is a manifold $\mathcal{O}_n$ whose tangent space at N_n can be described as:

$$T_{N_n}\mathcal{O}_n = N_n + ad(\mathfrak{g})N_n = \{N_n + [N_n, B] : B \in \mathfrak{g}\}.$$

Definition 2.1. *A* **transverse slice** *at $N_n \in \mathfrak{g}$ is a local complex submanifold $\mathcal{S} \subset \mathfrak{g}, N_n \in \mathcal{S}$, such that the tangent spaces of $\mathcal{S}$ and $\mathcal{O}_n$ at N_n are complementary.*

In our discussion we choose a particular slice $\mathcal{S}_n$, the affine subspace consisting of matrices of the form:

$$A = \left(\begin{array}{ccccc|ccccc}
a_1 & 1 & & & & b_1 & & & & \\
a_2 & 0 & 1 & & & b_2 & & & & \\
\cdots & & & \cdots & & \cdots & & & & \\
a_{n-1} & & & & 1 & b_{n-1} & & & & \\
a_n & & & & 0 & b_n & & & & \\
\hline
0 & & & & & d_1 & 1 & & & \\
\cdots & & & & & d_2 & 0 & 1 & & \\
0 & & & & & \cdots & & & \cdots & \\
c_1 & & & & & \cdots & & & \cdots & \\
\cdots & & & & & & & & & 1 \\
c_n & & & & & d_{2m-n} & & & & 0
\end{array}\right),$$

where $a_k, b_k, c_k, d_k \in \mathbb{C}, a_1 + d_1 = 0$.

Consider the polynomials

$$\begin{aligned} A(t) &= t^n - a_1 t^{n-1} + a_2 t^{n-2} - \cdots + (-1)^n a_n; \\ B(t) &= b_1 t^{n-1} - b_2 t^{n-2} + \cdots + (-1)^{n-1} b_n; \\ C(t) &= c_1 t^{n-1} - c_2 t^{n-2} + \cdots + (-1)^{n-1} c_n; \\ D(t) &= t^{2m-n} - d_1 t^{2m-n-1} + \cdots + (-1)^{2m-n} d_{2m-n}. \end{aligned}$$

It is easy to check that:

$$det(tI - A) = A(t)D(t) - B(t)C(t).$$

Also, a straightforward computation shows that $\mathcal{S}_n$ is complementary to $T\mathcal{O}_n$ at N_n, and thus is indeed a transverse slice. In the case $n = m$, $\mathcal{S}_m$ is the slice considered by Seidel and Smith in [30], with a reordering of the coordinates.

Let us look at the restriction of the adjoint map to the slice:

$$\chi|_{\mathcal{S}_n} : \mathcal{S}_n \to \mathrm{Sym}_0^{2m}(\mathbb{C}).$$

This is again a differentiable fiber bundle when restricted over $\mathrm{Conf}_0^{2m}(\mathbb{C})$, and its fibers are affine varieties of complex dimension $2n$.

For every $\tau \in \mathrm{Sym}_0^{2m}(\mathbb{C})$, we define:

$$\mathcal{Y}_{n,\tau} = \chi|_{\mathcal{S}_n}^{-1}(\tau).$$

Explicitly, $\mathcal{Y}_{n,\tau}$ is an affine variety in $\mathcal{S}_n = \mathbb{C}^{2m+2n-1}$. In terms of the coordinates a_k, b_k, c_k, d_k, it is described by a set of $2m-1$ algebraic equations that can all be grouped into one:

$$A(t)D(t) - B(t)C(t) = P_\tau(t). \tag{2}$$

Here P_τ is the polynomial with roots given by $\tau \in \mathrm{Conf}_0^{2m}(\mathbb{C})$, counted with multiplicities, and the equations correspond to identifying the $2m-1$ lowest coefficients of t in (2).

2.2. **An important example.** Let $n = 1$. Then $\mathcal{S}_n = \mathbb{C}^{2m+1}$ with coordinates $z = a_1 = -d_1, b_1, c_1, d_2, \ldots, d_{2m-1}$. Our manifold is described as:

$$\mathcal{Y}_{1,\tau} = \{(z, b_1, c_1, d_2, \ldots, d_{2m-1}) : (t - z)D(t) - b_1 c_1 = P_\tau(t)\}.$$

Note that once we know that $P_\tau(z) + b_1 c_1 = 0$, the polynomial $D(t)$ is determined uniquely. Hence, by making the coordinate change $b_1 = u + iv, c_1 = u - iv$, we can write:

$$\mathcal{Y}_{1,\tau} = \{(u, v, z) \in \mathbb{C}^3 : u^2 + v^2 + P_\tau(z) = 0\}$$

This is a complex surface, the Milnor fiber associated to the A_{2m} singularity, and it will play an important role in the discussion to follow. We denote it by S_τ. It is smooth if and only if the polynomial P_τ has no multiple roots, i.e. for $\tau \in \mathrm{Conf}_0^{2m}(\mathbb{C})$.

2.3. **The Hilbert scheme.** We recall a few standard results about Hilbert schemes of points on surfaces. The interested reader can consult the book by Nakajima [19] for a thorough exposition of the subject.

Let X be a quasi-projective scheme over an algebraically closed field k. Fix a polynomial $P \in \mathbb{Z}[t]$. A fundamental result of Grothendieck [6] is the existence of a quasi-projective scheme $\mathrm{Hilb}^P(X)$ that parametrizes flat families of closed subschemes of X with Hilbert polynomial P. In other words, $\mathrm{Hilb}^P(X)$ comes with a universal family $\mathcal{Z}$ such that for every flat family Z of closed subschemes of X with Hilbert polynomial P, parametrized by a scheme U, there is a unique morphism $\phi : U \to \mathrm{Hilb}^P(X)$ such that Z is the pullback $\phi^*(\mathcal{Z})$.

For our purposes we restrict to the case when $k = \mathbb{C}$ and P is the constant polynomial n. In this case $\mathrm{Hilb}^n(X)$ parametrizes closed 0-dimensional subschemes of X of length n. The typical example of a subscheme of this form is a subvariety consisting of n distinct points of n. We get nonreduced examples by letting some of the n points collide.

Definition 2.2. $\mathrm{Hilb}^n(X)$ *is called the Hilbert scheme of n points on X.*

Here are a few basic facts about Hilbert schemes of points, collected from [19]. We assume for simplicity that X is reduced.

Fact 2.3. *There is a natural morphism from the Hilbert scheme to the symmetric product of X, defined by*

$$\pi : \mathrm{Hilb}^n(X) \to \mathrm{Sym}^n(X),$$

$$\pi(Z) = \sum_{x \in X} \mathit{length}(Z_x)[x]. \tag{3}$$

This is called the Hilbert-Chow morphism.

Fact 2.4. *When* $\dim_{\mathbb{C}} X = 1$, π *is an isomorphism. Therefore,* $\mathrm{Hilb}^n(X) = \mathrm{Sym}^n(X)$.

Fact 2.5 (Fogarty's theorem [4]). *When* $\dim_{\mathbb{C}} X = 2$ *and X is smooth,* $\mathrm{Hilb}^n(X)$ *is smooth of complex dimension $2n$, and the Hilbert-Chow morphism is a resolution of singularities. If X is irreducible, then* $\mathrm{Hilb}^n(X)$ *is also irreducible.*

In higher dimensions there are examples when the Hilbert scheme is not smooth even for X smooth.

We are interested in the case when $\dim_{\mathbb{C}} X = 2$, so let us explain more carefully what happens then. The symmetric product $\mathrm{Sym}^n(X)$ is singular along its diagonal:

$$\Delta = \{(x_1, \ldots, x_n) \in \mathrm{Sym}^n(X) : x_i = x_j \text{ for some } i \neq j\}.$$

The diagonal has complex codimension 2 in $\mathrm{Sym}^n(X)$, and the Hilbert-Chow morphism $\pi : \mathrm{Hilb}^n(X) \to \mathrm{Sym}^n(X)$ is one-to-one over $\mathrm{Sym}^n(X) - \Delta$.

Fact 2.6. *When* $\dim X = 2$, *the preimage* $\pi^{-1}(\Delta)$ *has complex codimension* 1 *in* $\mathrm{Hilb}^n(X)$.

This is easiest to visualize when $n = 2$ and X is smooth. The Hilbert scheme $\mathrm{Hilb}^2(X)$ parametrizes 0-dimensional subschemes of length 2. The reduced ones are of the form $(x_1, x_2) \in \mathrm{Sym}^2(X) - \Delta$. The nonreduced ones are in the preimage $\pi^{-1}(\Delta)$. A point z which maps to (x, x) under π is a subscheme defined by $\mathcal{O}_X/\mathcal{I}$, where $\mathcal{I} \subset \mathcal{O}_X$ is an ideal of the form

$$\mathcal{I} = \{f \in \mathcal{O}_X : f(x) = 0, df_x(v) = 0\},$$

for $v \neq 0 \in T_xX$. In other words, points in $\pi^{-1}(\Delta)$ are described by a point $x \in X$ and a direction v at X. The fiber of π over $(x, x) \in \Delta$ is a copy of $\mathbb{P}^1$ and, in fact, the Hilbert scheme is a blow-up of the symmetric product:

$$\mathrm{Hilb}^2(X) = \mathrm{Blow}_{\Delta}(\mathrm{Sym}^2(X)) = \mathrm{Blow}_{\Delta}(X \times X)/\mathbb{Z}_2.$$

In general, $\mathrm{Hilb}^n(X)$ is more difficult to describe. We refer to [19] for an explicit stratification in the case $X = \mathbb{C}^2$. This is relevant for the local behaviour of $\mathrm{Hilb}^n(X)$ for any surface X.

2.4. **An open holomorphic embedding.** We are now ready to prove Theorem 1.1, which says that there is an injective holomorphic map j from $\mathcal{Y}_{n,\tau}$ into the Hilbert scheme of n points on the affine surface S_τ considered in section 2.2. We construct j as an algebraic morphism.

By the defining property of Hilbert schemes, a morphism into $\mathrm{Hilb}^n(S_\tau)$ is the same as a flat family of subschemes of S_τ with Hilbert polynomial n, parametrized by the domain $\mathcal{Y}_{n,\tau}$.

Recall from section 2.1 that $\mathcal{Y}_{n,\tau} = \mathrm{Spec}\ R$, where R is a commutative ring, the quotient of the polynomial ring in the $2m+2n-1$ coordinates a_k, b_k, c_k, d_k by the ideal generated by the algebraic relations in (2). We can think of $A(t) = t^n - a_1 t^{n-1} + \cdots + (-1)^n a_n$ as an element in $R[t]$, and the same is true for $B(t), C(t), D(t)$. Consider the polynomials

$$U(t) = \frac{1}{2}\big(B(t) + C(t)\big) \in R[t];\ \ V(t) = \frac{1}{2i}\big(B(t) - C(t)\big) \in R[t]. \tag{4}$$

Then (2) can be rewritten as:

$$U(t)^2 + V(t)^2 + P_\tau(t) = A(t)D(t). \tag{5}$$

Set

$$\mathcal{R} = R[u,v,z]/(u^2+v^2+P_\tau(z)).$$

Consider the map

$$\psi : \mathcal{R} \to R[t], \quad \psi(Q(u,v,z)) = Q(U(t),V(t),t)$$

and let us compose it with the natural projection $p : R[t] \to R[t]/(A(t))$.

It is easy to see that the composition

$$p \circ \psi : \mathcal{R} \to R[t]/(A(t))$$

is surjective. Let $\mathcal{I} \subset \mathcal{R}$ be its kernel. Then $\mathcal{R}/\mathcal{I}$ is isomorphic to $R[t]/(A(t)) \cong R^n$ as an R-module.

We define the closed subscheme

$$Z = \mathrm{Spec}\ \mathcal{R}/\mathcal{I} \subset \mathrm{Spec}\ \mathcal{R} = \mathcal{Y}_{n,\tau} \times S_\tau.$$

Since $\mathcal{R}/\mathcal{I}$ is a free n-dimensional module over R, it follows that the composition

$$Z \subset \mathcal{Y}_{n,\tau} \times S_\tau \to \mathcal{Y}_{n,\tau} = \mathrm{Spec}\ R$$

exhibits Z as a flat family of 0-dimensional subschemes of S_τ of length n. This defines the desired morphism

$$j : \mathcal{Y}_{n,\tau} \to \mathrm{Hilb}^n(S_\tau).$$

From now on we think of $\mathcal{Y}_{n,\tau}$ as an affine variety, with its reduced scheme structure. Then the points in $\mathcal{Y}_{n,\tau}$ are 4-tuples of polynomials (A,D,U,V) in $\mathbb{C}[t]$ satisfying (5), and the points in $\mathrm{Hilb}^n(S_\tau)$ can be identified with ideals in $\mathcal{O} = \mathbb{C}[u,v,z]/(u^2+v^2+P_\tau(z))$ such that $\dim_{\mathbb{C}}(\mathcal{O}/I) = n$. Explicitly, the morphism j is given by:

$$j(A,D,U,V) = \{Q(u,v,z) : A(t) \text{ divides } Q(U(t),V(t),t)\}. \tag{6}$$

Note that $R_0 = \mathbb{C}[z]$ is a subring of $\mathbb{C}[u,v,z]$, so $R_1 = R_0/(R_0 \cap (u^2+v^2+P_\tau(z))) \cong \mathbb{C}[z]$ is a subring of $\mathcal{O} = \mathbb{C}[u,v,z]/(u^2+v^2+P_\tau(z))$, the ring of functions on the affine variety S_τ. Given an ideal $I \subset \mathcal{O}$ describing a subscheme $X = \mathrm{Spec}\ \mathcal{O}/I$ in $\mathrm{Hilb}^n(S_\tau)$, the intersection $I \cap R_1$ corresponds to a subscheme of $\mathbb{C}$, the image of X under the map:

$$i : S_\tau \longrightarrow \mathbb{C},\ \ i(u,v,z) = z. \tag{7}$$

It is clear that $R_1/(I \cap R_1)$ injects into $\mathcal{O}/I$, hence $i(X)$ must have length at most n.

With this background in place, we prove the following:

Proposition 2.7. *The morphism j is an open embedding. The image of j in $\mathrm{Hilb}^n(S_\tau)$ consists of the subschemes X such that $i(X)$ is a subscheme of $\mathbb{C}$ of length exactly n.*

Proof. Since $\mathcal{Y}_{n,\tau}$ and $\mathrm{Hilb}^n(S_\tau)$ have the same dimension and $\mathrm{Hilb}^n(S_\tau)$ is irreducible (Fact 2.5), in order to prove that j is an open embedding it suffices to show that it is injective. Pick $(A_i, D_i, U_i, V_i), i = 1, 2$, that map to the same ideal $I \subset \mathcal{O}$ under j. Then

$$I \cap R_1 = \{Q \in \mathbb{C}[z] : A_i(z) \text{ divides } Q(z)\},$$

where i can be either 1 or 2. Since the A_i are monic polynomials of degree $n < 2m =$ degree (P_τ), it follows that $A_1 = A_2$.

Next, note that the polynomials $u - U_1(z)$ and $u - U_2(z)$ are in I, hence $U_1(z) - U_2(z) \in I \cap R_1$. But $U_1 - U_2$ has degree at most $n-1 < n =$ degree A, so A dividing $U_1 - U_2$ modulo P_τ implies $U_1 = U_2$. Similarly $V_1 = V_2$. Also, the relation (5) determines D uniquely from A, U, V and P_τ. Therefore, we must also have $D_1 = D_2$, and this shows that j is injective.

If I is in the image of j, then $I \cap R_1$ is of the form $\{Q \in \mathbb{C}[z] : A(z) \text{ divides } Q(z)\}$, where A is a monic polynomial of degree n. It follows that $\dim R_1/(I \cap R_1) = n$.

Conversely, let I be an ideal corresponding to a subscheme in $\mathrm{Hilb}^n(S_\tau)$ such that $\dim R_1/(I \cap R_1) = n$. We claim that I lies in the image of the embedding j. Since $R_1 \cong \mathbb{C}[z]$, the ideal $I \cap R_1$ is necessarily generated by a unique monic polynomial $A(z)$ of degree n. Because of the equality of dimensions, the inclusion $R_1/(I \cap R_1) \hookrightarrow \mathcal{O}/I$ is a bijection. Consider the projection

$$\mathcal{O} = \mathbb{C}[u, v, z]/(u^2 + v^2 + P_\tau(z)) \ \to \mathcal{O}/I \cong R_1/(I \cap R_1) \cong \mathbb{C}[z]/(A(z)).$$

The images of the elements $u, v \in \mathcal{O}$ under this projection can be represented by polynomials $U(z), V(z)$ of degree at most $n - 1$, while the image of z is obviously the polynomial z. Hence, we get a relation

$$U(z)^2 + V(z)^2 + P_\tau(z) = A(z)D(z),$$

where $D(z)$ is a uniquely determined monic polynomial of degree $2m - n$. Because of the form of P_τ, the second leading coefficients of A and D must add up to zero. We get that $I = j(A, D, U, V)$, so indeed I lies in the image of j. □

This completes the proof of Theorem 1.1. From now on we can think of $\mathcal{Y}_{n,\tau}$ as an open subset of the Hilbert scheme. Recall from section 2.3 that the Hilbert-Chow morphism

$$\pi : \mathrm{Hilb}^n(S_\tau) \to \mathrm{Sym}^n(S_\tau)$$

is 1-to-1 away from the diagonal Δ. We can easily write down the composition:

$$\pi \circ j : \mathcal{Y}_{n,\tau} \to \mathrm{Sym}^n(S_\tau).$$

This leads to probably the simplest way to think about the effect of j:

Remark 2.8. *Given a point (A, D, U, V) in $\mathcal{Y}_{n,\tau}$, its image under $\pi \circ j$ is the unordered collection of n points $(u_k, v_k, z_k) \in \mathrm{Sym}^n(S_\tau)$, $k = 1, \ldots, n$, where z_k are the roots of $A(t)$, $u_k = U(z_k)$ and $v_k = V(z_k)$.*

A quick corollary of this discussion is:

Corollary 2.9. *The intersection of $\mathcal{Y}_{n,\tau}$ with the open subset*

$$U_{n,\tau} = \pi^{-1}(\mathrm{Sym}^n(S_\tau) - \Delta) \subset \mathrm{Hilb}^n(S_\tau)$$

is the complement in $U_{n,\tau}$ of the codimension one subset

$$\Xi = \pi^{-1}\big(\{(u_k, v_k, z_k) \in \mathrm{Sym}^n(S_\tau) - \Delta : z_i = z_j \text{ for some } i \neq j\}\big).$$

Remark 2.10. *When $n = 1$, we have $\mathcal{Y}_{1,\tau} = \mathrm{Hilb}^1(S_\tau) = S_\tau$, and j is an isomorphism.*

2.5. **Families.** Until now we have described the open embedding j for a fixed τ. As we vary τ in $\mathrm{Sym}_0^{2m}(\mathbb{C})$, the varieties $\mathcal{Y}_{n,\tau}$ form the transverse slice $\mathcal{S}_n$. When $n = 1$, the family $\mathcal{Y}_{n,\tau} = S_\tau$ induces a family of Hilbert schemes

$$\mathcal{H}_n \to \mathrm{Sym}_0^{2m}(\mathbb{C}),$$

whose fiber over τ is $\mathrm{Hilb}^n(S_\tau)$. The same arguments used in section 2.4 carry over to show the following statement for families:

Proposition 2.11. *There is an open algebraic embedding $\hat{j} : \mathcal{S}_n \to \mathcal{H}_n$ whose restriction to each $Y_{n,\tau}$ is the embedding $j : \mathcal{Y}_{n,\tau} \to \mathrm{Hilb}^n(S_\tau)$ considered above.*

2.6. **Quiver varieties.** This section is not necessary for understanding the rest of the paper, and the reader may skip it if she wishes. Our only goal here is to put our discussion in a larger context, by noting that both $Y_{n,\tau}$ and $\mathrm{Hilb}^n(S_\tau)$ are particular examples of a more general concept. This concept is that of a quiver variety, introduced by Nakajima in [18]. Our discussion follows [18] closely, but is not intended as a substitute for that paper. We simply want to provide a rough idea of what is involved for the reader who is not familiar with this rich subject.

A *quiver* A is a finite oriented graph with no oriented cycles. Let K be the set of vertices and H the set of pairs consisting of an edge and an orientation on it, which may or may not be the one we chose initially. For $h \in H$, we set $\epsilon(h) = 1$ if the orientation agrees with the initial one, and $\epsilon(h) = -1$ otherwise. We also denote by $\mathrm{out}(h), \mathrm{in}(h) \in K$ the outgoing and incoming vertices of h, respectively, and by $\bar{h}$ the same edge with reversed orientation.

For each vertex $k \in K$, we choose a pair of Hermitian vector spaces V_k, W_k of dimensions v_k, w_k, respectively. We write the set of dimensions as vectors $\mathbf{v}, \mathbf{w}$ of length the cardinality of K. We form the complex vector space

$$\mathbb{M}(\mathbf{v},\mathbf{w}) = \Big(\bigoplus_{h\in H} \mathrm{Hom}(V_{\mathrm{out}(h)}, V_{\mathrm{in}(h)})\Big) \oplus \Big(\bigoplus_{k\in K} \mathrm{Hom}(W_k, V_k) \oplus \mathrm{Hom}(V_k, W_k)\Big).$$

An element of M consists of components B_h, i_k, j_k. The group $G_{\mathbf{v}} = \prod_{k\in K} U(V_k)$ with Lie algebra $\mathfrak{g}_{\mathbf{v}}$ acts on $\mathbb{M}(\mathbf{v},\mathbf{w})$ by:

$$(B_h, i_k, j_k) \to (g_{\mathrm{in}(h)} B_h g_{\mathrm{out}(h)}^{-1}, g_k i_k, j_k g_k^{-1}).$$

Consider the map

$$\mu = \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} : \mathbb{M}(\mathbf{v},\mathbf{w}) \to \mathfrak{g}_{\mathbf{v}} \oplus (\mathfrak{g}_{\mathbf{v}} \otimes \mathbb{C}),$$

$$\mu_{\mathbb{R}}(B,i,j) = \frac{i}{2}\Big(\sum_{h\in H: k=\mathrm{in}(h)} B_h B_h^* - B_{\bar{h}}^* B_{\bar{h}} - i_k i_k^* - j_k^* j_k\Big)_k \in \bigoplus_k \mathfrak{u}(V_k) = \mathfrak{g}_{\mathbf{v}},$$

$$\mu_{\mathbb{C}}(B,i,j) = \frac{i}{2}\Big(\sum_{h\in H: k=\mathrm{in}(h)} \epsilon(h) B_h B_{\bar{h}} + i_k j_k\Big)_k \in \bigoplus_k \mathfrak{gl}(V_k) = \mathfrak{g}_{\mathbf{v}} \otimes \mathbb{C}.$$

Let $Z_{\mathbf{v}} \subset \mathfrak{g}_{\mathbf{v}}$ be the center, and choose an element $\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}) \in Z_{\mathbf{v}} \oplus (Z_{\mathbf{v}} \otimes \mathbb{C})$.

Definition 2.12. *The quiver variety associated to the quiver A, the vectors $\mathbf{v}, \mathbf{w}$, and the parameter ζ is*

$$\mathfrak{M}_\zeta(\mathbf{v},\mathbf{w}) = \{(B,i,j) \in \mathbb{M}(\mathbf{v},\mathbf{w}) : \mu(B,i,j) = -\zeta\}/G_{\mathbf{v}}.$$

In fact, one can show that $\mathbb{M}(\mathbf{v}, \mathbf{w})$ is naturally a quaternionic vector space, and that the action of $G_{\mathbf{v}}$ respects its hyper-Kähler structure. The map μ is the hyper-Kähler moment map associated to that action, and the quiver variety is the so-called hyper-Kähler quotient. As such, $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ inherits a hyper-Kähler structure. It is a noncompact space and, in general, it can have singularities. It turns out that different choices of orientations on the same graph give rise to isomorphic quiver varieties.

When $\zeta_{\mathbb{R}} = 0$, $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ is homeomorphic to the affine algebraic quotient of $\mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}})$ by the action of the complexified group $G_{\mathbf{v}}^{\mathbb{C}} = \coprod_{k \in K} GL(V_k)$.

There are numerous interesting examples of quiver varieties: cotangent bundles of generalized flag manifolds, nilpotent adjoint orbits in the Lie algebra $\mathfrak{gl}_m$, and their intersections with various transverse slices. For the last two their quiver variety description follows from a result of Kronheimer [15], which should generalize for other orbits as well. In particular, we expect the variety $\mathcal{Y}_{n,\tau}$ from section 2.1 to be isomorphic to $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ for the extended Dynkin graph of type $\tilde{A}_{2m-1}$, i.e. a polygon with $2m$ vertices numbered $0, 1, \dots, 2m-1$, with the following data:

- $v_k = k$ for $k \leq n$; $v_k = n$ for $n \leq k \leq 2m-n$; $v_k = 2m-k$ for $k \geq 2m-n$;
- $w_k = 1$ for $k = n, 2m-n$ and $n < m$; $w_n = 2$ for $n = m$; $w_k = 0$ otherwise;
- $\zeta_{\mathbb{R}} = 0$; some $\zeta_{\mathbb{C}}$ depending on τ.

(As noted by Seidel and Smith [30], this result is only conjectural at this point.)

Another important class of examples comes from instanton theory and, in fact, this was the starting point for the whole subject. The moment map equations $\mu(B, i, j) = -\zeta$ above are modelled from the ADHM equations which describe the moduli space of instantons on $\mathbb{R}^4$. More generally, one can consider moduli spaces of instantons on the so-called ALE (asymptotically locally Euclidean) spaces [14]. An ALE space is a minimal resolution of a singularity of the type $\mathbb{C}^2/\Gamma, \Gamma \subset \mathrm{SU}(2)$ finite, endowed with a certain hyper-Kähler metric. An example is the Milnor fiber S_τ from section 2.2, which corresponds to the cyclic group $\Gamma = \mathbb{Z}/2m\mathbb{Z}$. The ALE spaces can be described as quiver varieties [14], and the same is true for various moduli spaces of instantons on them [16].

Furthermore, by a similar argument to that given by Kronheimer and Nakajima in [16], the Hilbert scheme of an ALE space is also a quiver variety. These Hilbert schemes have been studied by Qin and Wang in [27], [32]. In particular, $\mathrm{Hilb}^n(S_\tau)$ is isomorphic to $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ for the extended Dynkin graph of type $\tilde{A}_{2m-1}$ with the following data:

- $v_k = n$ for all k;
- $w_k = 1$ for $k = 0$, $w_k = 0$ otherwise;
- some $\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}$ depending on τ.

Thus, for every $n \leq m$, Theorem 1.1 describes an open holomorphic embedding of a quiver variety into another one, where both varieties are constructed from the same quiver but with different data $(\mathbf{v}, \mathbf{w})$. These are the only nontrivial examples of this kind known to the author. We leave the following as an open problem:

Question 2.13. *What are the pairs of quiver varieties (M, M') such that M admits an open holomorphic embedding into M' ?*

3. The Seidel-Smith construction

Seidel and Smith [30] defined a link invariant as the Floer cohomology of two specific Lagrangian submanifolds in $\mathcal{Y}_{n,\tau}$, for $n = m$. In this section we summarize their construction. We skip most of the proofs, and we refer the reader to [30] for more details.

3.1. **Kähler metrics and parallel transport.** Recall that the varieties $Y_{m,\tau}$ are fibers of the map:

$$\chi|_{\mathcal{S}_m} : \mathcal{S}_m \to \mathrm{Sym}_0^{2m}(\mathbb{C}).$$

Seidel and Smith endow the affine slice $\mathcal{S}_m$ with a Kähler form Ω, and give the fibers the Kähler form induced by restriction. The form Ω is constructed as $\Omega = -dd^c\psi$, for some smooth function $\psi : \mathcal{S}_m \to \mathbb{R}$. The function ψ is not unique, but there is a weakly contractible parameter space of possible choices, and in the end different choices give rise to the same link invariant.

The main requirements for ψ are related to its behaviour at infinity in $\mathcal{S}_m$. Basically, ψ is asked to satisfy the following four conditions:

(8) • $-dd^c\psi > 0,$ so that Ω is Kähler.

(9) • ψ is proper and bounded below.

(10) • Outside a compact set of $\mathcal{S}_m$, $\|\nabla\psi\| \leq C\psi$ for some $C > 0$;

(11) • The fiberwise critical set of ψ maps properly to $\mathrm{Sym}_0^{2m}(\mathbb{C})$ under χ.

The construction of ψ with these properties starts with the observation that there is a natural action λ of the multiplicative group $\mathbb{R}_+$ on $\mathcal{S}_m$. In terms of the coordinates a_k, b_k, c_k, d_k from section 2.1, $k = 1, \dots, m$, the element $r \in \mathbb{R}_+$ acts by:

$$\lambda_r : (a_k, b_k, c_k, d_k) \to (r^k a_k, r^k b_k, r^k c_k, r^k d_k). \tag{12}$$

This action takes the fiber $Y_{m,\tau}$ to $Y_{m,r\tau}$, where $\tau = (\tau_1, \dots, \tau_{2m}) \in \mathrm{Sym}_0^{2m}(\mathbb{C})$ and r acts on τ by multiplication on each component.

Fix some real number $\alpha > m$. For each $k = 1, \dots, m$ consider the function $\xi_k(z) = |z|^{\alpha/k}$ on $\mathbb{C}$, and add to it a compactly supported function η_k on $\mathbb{C}$ such that $\psi_k = \eta_k + \xi_k$ is C^∞ and satisfies $-dd^c\psi_k > 0$ on $\mathbb{C}$. Apply ψ_k to the coordinates a_k, b_k, c_k, d_k for each k and sum up these functions to obtain the function ψ on $\mathcal{S}_m$. Conditions (8), (9) and (10) can be checked immediately, while for condition (11) one needs to use the asymptotical homogeneity of ψ with respect to the $\mathbb{R}_+$ action (see Lemma 4.1 below).

The reasons for the choice of ψ are that conditions (8) – (11) allow one to define *rescaled parallel transport maps,* in the following sense.

The map χ is a fibration over $\mathrm{Conf}_0^{2m}(\mathbb{C})$ when restricted to

$$W = \chi^{-1}(\mathrm{Conf}_0^{2m}(\mathbb{C}) \cap \mathcal{S}_m).$$

Take a path $\gamma : [0,1] \to \mathrm{Conf}_0^{2m}(\mathbb{C})$ on the base. The parallel transport vector field H_γ on the pullback $\gamma^* W \to [0,1]$ consists of the sections of $TW|_{W_{\gamma(s)}}$ which project to $\gamma'(s)$, and are orthogonal to the tangent space along the fibers in the given Kähler metric. We would like to define a symplectic isomorphisms between the fibers by integrating H_γ. However, this is not possible, because the fibers are not compact, and integral lines of H_γ may go to infinity in finite time. The only thing we can say is that for every compact $P \subset W_{\gamma(s)}$ and $\epsilon > 0$ small (depending on P), there is a symplectic embedding

$$h_\gamma : P \to W_{\gamma(s+\epsilon)},$$

called *naive parallel transport*.

However, Seidel and Smith show that one can say more when the Kähler metric on $\mathcal{S}_m$ has well-chosen behaviour at infinity, such as our $-dd^c\psi$, with ψ satisfying (8)-(11). In this case for every compact $P \subset W_{\gamma(0)}$, we can define a symplectic embedding:

$$h_\gamma^{\mathrm{resc}} : P \to W_{\gamma(1)}.$$

This is the rescaled parallel transport, and is defined (roughly) by subtracting from the usual parallel transport vector field a multiple of a certain Liouville vector field on the fibers, integrating the resulting vector field for all times, and then rescaling back on the image. For a bigger compact set P' containing P, there is another rescaled parallel transport map to $W_{\gamma(1)}$, but its restriction to P is isotopic to the previous one, in the class of symplectic embeddings. Therefore, given a closed Lagrangian submanifold $\mathcal{L} \subset W_{\gamma(0)}$, there is a closed Lagrangian $h_\gamma^{\text{resc}}(\mathcal{L}) \subset W_{\gamma(1)}$, well-defined up to Lagrangian isotopy.

It is not hard to see from the definition in [30] that rescaled parallel transport behave nicely under composition of paths: if $\gamma : [0,2] \to \text{Conf}_0^{2m}(\mathbb{C})$ is smooth, then $h^{\text{resc}}_{\gamma|_{[1,2]}} \circ h^{\text{resc}}_{\gamma|_{[0,1]}}$ is isotopic to the h_γ^{resc} for the full path.

Also, rescaled parallel transport has the same result as the naive parallel transport for small paths, where the latter is defined. It follows that in some cases, we can describe the image $h_\gamma^{\text{resc}}(\mathcal{L})$ of a closed Lagrangian $\mathcal{L} \subset W_{\gamma(0)}$ in terms of naive parallel transport as follows: we break the path γ into small pieces corresponding to a partition $0 = t_0 < t_1 < \cdots < t_n = 1$. We assume that $h_{\gamma|_{[t_0,t_1]}}(\mathcal{L})$ is well-defined, and we use a Lagrangian isotopy to deform it into another Lagrangian $\mathcal{L}_1 \subset W_{\gamma(t_1)}$, such that $h_{\gamma|_{[t_1,t_2]}}(\mathcal{L}_1)$is well-defined. We iterate the process and end up with a Lagrangian $\mathcal{L}_n \subset W_{\gamma(1)}$, the same as $h_\gamma^{\text{resc}}(\mathcal{L})$ up to isotopy. Of course, there is no guarantee that the partition and the Lagrangians $\mathcal{L}_k$ considered above exist, but we will see that in some cases they do. This is the strategy that we use in section 4.4 below.

3.2. A_1 **fibered singularities.** The following result describes the local structure of the singular part of $\mathcal{Y}_{m,\tau}$ when two of the points in τ become zero. It is seen to have a fibered singularity of (A_1) type.

Let $\bar{\tau} = (\mu_3, \ldots, \mu_{2m}) \in \text{Conf}_0^{2m-2}(\mathbb{C})$ and consider the disk $D \subset \text{Sym}_0^{2m}(\mathbb{C})$ corresponding to eigenvalues $\tau(\zeta) = (-\sqrt{\zeta}, \sqrt{\zeta}, \mu_3, \ldots, \mu_{2m})$ with ζ small. We have:

Lemma 3.1. *(i) The singular set* $\text{Sing}(\mathcal{Y}_{m,\tau(0)})$ *of* $\mathcal{Y}_{m,\tau(0)}$ *is canonically isomorphic to* $\mathcal{Y}_{m-1,\bar{\tau}}$.

(ii) There is a neighborhood of $\text{Sing}(\mathcal{Y}_{m,\tau(0)})$ *inside* $\chi^{-1}(D) \cap \mathcal{S}_m$*, and an isomorphism of that with a neighborhood of* $(\mathcal{Y}_{m-1,\bar{\tau}}) \times \{0\}^3$ *inside* $(\mathcal{Y}_{m-1,\bar{\tau}}) \times \mathbb{C}^3$*. This isomorphism is compatible with the one in (i) and fits into a commutative diagram:*

$$\begin{array}{ccc} \chi^{-1}(D) \cap \mathcal{S}_m & \xrightarrow{\text{local}\cong} & (\mathcal{Y}_{m-1,\bar{\tau}}) \times \mathbb{C}^3 \\ \chi \big\downarrow & & \big\downarrow u^2+v^2+z^2 \\ D & \xrightarrow{\zeta} & \mathbb{C} \end{array}$$

where u, v, z *are the coordinates on* $\mathbb{C}^3$.

In terms of the coordinates a_k, b_k, c_k, d_k on $\mathcal{Y}_{m,\tau(0)} \subset \mathcal{S}_m$, the isomorphism in (i) corresponds to simply setting $a_m = b_m = c_m = d_m = 0$, and identifying the other coordinates with the corresponding ones on $\mathcal{Y}_{m-1,\bar{\tau}} \subset \mathcal{S}_m$. Note that if we choose the same α and the same functions ψ_k in the construction of the Kähler form for both m and $m-1$, and give the singular set the Kähler form induced by restriction, then the isomorphism in (i) is in fact a symplectomorphism.

3.3. **Relative vanishing cycles.** The next lemma deals with the construction of relative vanishing cycles near the singularity considered above. Let X be a complex manifold. (In our applications X will be taken to be $\mathcal{Y}_{m-1,\bar{\tau}}$.) Give $Y = X \times \mathbb{C}^3$ any Kähler metric, and consider the map

$$q : Y = X \times \mathbb{C}^3 \to \mathbb{C}, \quad q(x, u, v, z) = u^2 + v^2 + z^2. \tag{13}$$

We equip the fibers $Y_w = q^{-1}(w)$ with the induced metrics. The critical point set of q is $\{u = v = z = 0\} \cong X$, and the real part $Re(q)$ is a Morse-Bott function. Let $Q \subset q^{-1}(\mathbb{R}^{\geq 0})$ be the stable manifold of $Re(q)$, and $l : Q \to X$ the map which assigns to a point in Q its limit under the negative gradient flow of $Re(q)$.

Lemma 3.2. *Let $K \subset X \cong X \times \{0\}$ be a compact Lagrangian submanifold. Then for sufficiently small $w > 0$, $L_w = l^{-1}(K) \cap Y_w$ is a Lagrangian submanifold of Y_w diffeomorphic to $K \times S^2$, and called the relative vanishing cycle associated to K. L_w can also be described in terms of parallel transport along the linear path $\gamma : [0, 1] \to \mathbb{C}$, $\gamma(s) = (1 - s)t$, as the set of points in Y_w which are taken to a point in K by the naive parallel transport along γ, in the limit $s \to 1$.*

Note that by multiplying q with some constant in S^1, we can define stable manifolds which lie over other half-lines in $\mathbb{C}$, and corresponding relative vanishing cycles in Y_w for all sufficiently small $w \in \mathbb{C}^*$.

3.4. **The Lagrangians in terms of parallel transport.** As mentioned in the introduction, Seidel and Smith defined their link invariant with the help of two Lagrangians $\mathcal{L}, \mathcal{L}' \subset \mathcal{Y}_{m,\tau}$. Both $\mathcal{L}$ and $\mathcal{L}'$ are described in [30] in terms of the following construction, which makes use of the rescaled parallel transport maps and of vanishing cycles.

Let $\tau = (\mu_1, \dots, \mu_{2m}) \in \mathrm{Conf}^{2m}(\mathbb{C})$. In section 2.1 we made the remark that all our constructions can be done for $\mathrm{Conf}^{2m}(\mathbb{C})$ and $\mathrm{Sym}^{2m}(\mathbb{C})$ via their projections p to $\mathrm{Conf}^{2m}(\mathbb{C})$ and $\mathrm{Sym}^{2m}(\mathbb{C})$, respectively. In particular, $\mathcal{Y}_{m,\tau}$ is well-defined. We will associate a Lagrangian $\mathcal{L}(\delta) \subset \mathcal{Y}_{m,\tau}$ to a a collection of m disjoint oriented arcs $\delta = (\delta_1, \dots, \delta_m)$ in $\mathbb{C}$, joining together the points of τ in pairs. Without loss of generality, we assume that δ_k runs from μ_{2k-1} to μ_{2k}. $\mathcal{L}(\delta)$ will be well-defined up to Lagrangian isotopy.

Consider the path $\tilde{\gamma} : [0, 1] \to \mathrm{Sym}^{2m}(\mathbb{C})$ starting at τ which keeps $\mu_3, \dots, \mu_{2m}$ fixed, and moves μ_1 and μ_2 towards each other following δ_1. They collide at the midpoint $\delta_1(1/2)$. We assume that the arc δ_1 is a straight horizontal line near its midpoint, and the two points move towards each other with equal speed for s close to 1. We compose that with the natural projection to get a path $\gamma : [0, 1] \to \mathrm{Sym}^{2m}_0(\mathbb{C}), \gamma(s) = p \circ \tilde{\gamma}(s)$.

The construction of $\mathcal{L}(\delta)$ is done inductively on m. We could start with the trivial case of a point as subset of a point for $m = 0$, but let us do the case $m = 1$ for completeness. Then the adjoint quotient map is

$$\chi : \mathcal{S}_1 = \mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}^3 \to \mathrm{Sym}^2_0(\mathbb{C}) \cong \mathbb{C}$$

given by $\chi(u, v, z) = u^2 + v^2 + z^2$. Here we identify the base with $\mathbb{C}$ by $\tau = (\mu, -\mu) \to \mu^2$, and let $P_\tau(z) = z^2 - \mu^2$. The map χ has a unique critical point in the fiber over $\gamma(0) = 0$. In the nearby fibers $\gamma(s)$ for s close to 1 we have an associated vanishing cycle, a Lagrangian two-sphere. We then use the reverse rescaled parallel transport maps along γ to move this to a Lagrangian $\mathcal{L}(\delta) \subset \mathcal{Y}_{1,\tau}$.

The inductive step is similar. Let $\tau' = (\mu'_1, \mu'_2, \mu'_3, \dots, \mu'_{2m}) = \gamma(1) \in \mathrm{Sym}^{2m}_0(\mathbb{C})$, with $\mu'_1 = \mu'_2$. For small s we have $\gamma(1 - s) = \tau'_s = (\mu'_1 - s, \mu'_1 + s, \mu'_3, \dots, \mu'_{2m})$. Let also $\bar{\tau} = (\mu''_3, \dots, \mu''_{2m}) \in \mathrm{Sym}^{2m-2}_0(\mathbb{C})$ be the image of $(\mu_3, \dots, \mu_{2m})$ under the projection p. It follows that $\tau''_s = (-s, s, \mu''_3, \dots, \mu''_{2m})$ is in $\mathrm{Sym}^{2m}_0(\mathbb{C})$ as well. Assume we already have a compact Lagrangian in $\mathcal{Y}_{m-1,\bar{\tau}}$ as in section 3.3. Using Lemmas 3.1 and 3.2 we get a

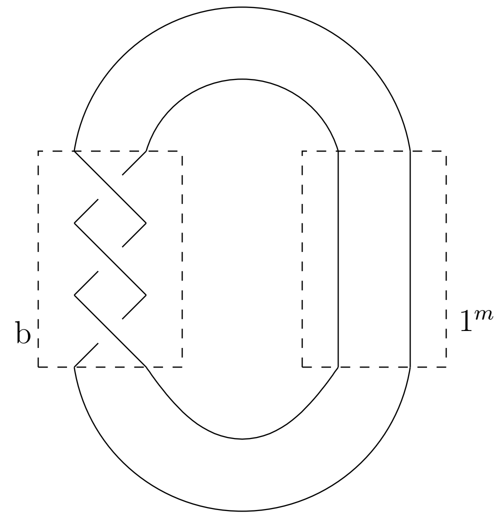

FIGURE 1. The left-handed trefoil.

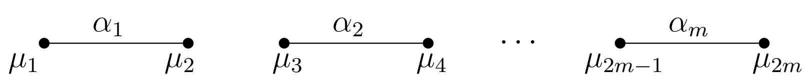

FIGURE 2. Standard crossingless matching.

relative vanishing cycle in $\mathcal{Y}_{m,\tau_s''}$ for small s. We move it to $\mathcal{Y}_{m,\gamma(1-s)}$ using rescaled parallel transport along the linear path going from τ_s'' to τ_s'. Rescaled parallel transport is then used again in reverse along γ to give a Lagrangian submanifold in $\mathcal{Y}_{m,\tau}$, which is the desired $\mathcal{L}(\delta)$.

Note: Seidel and Smith interpolate between τ_s'' and τ_s' for $s = 0$ in the space composed of the singular sets of $\mathcal{Y}_m$'s, then use the vanishing cycle construction at τ_s' rather than τ_s''. However, the two constructions are easily seen to be equivalent up to isotopy.

3.5. **Links as braid closures.** Given a link L, we can present it as the closure of an m-stranded braid $b \in Br_m$. (See Figure 1 for a presentation of the left-handed trefoil.)

Equivalently, L is the plat closure of the braid $b \times 1^m \in Br_m \times Br_m \hookrightarrow Br_{2m}$. We represent $b \times 1^m$ by a loop $l : [0,1] \to \mathrm{Conf}^{2m}(\mathbb{C})$. Our convention is that braids act on the $2m$ punctured plane on the right, with geometric braids reading from top to bottom, in the sense that the first generator of the braid from the top acts first, and so forth.

Consider the standard crossingless matching of $2m$ points in the plane in Figure 2. The endpoints of the m segments $\alpha_1, \ldots, \alpha_m$ are $\mu_1, \ldots, \mu_{2m} \in \mathbb{R} \subset \mathbb{C}$, with $\tau = (\mu_1, \ldots, \mu_{2m}) \in \mathrm{Conf}_0^{2m}(\mathbb{C})$. The odd points $\mu_1, \mu_3, \ldots, \mu_{2m-1}$ correspond to the strands on the left side of Figure 1, and the even points $\mu_2, \ldots, \mu_{2m}$ corresponds to the vertical strands on the right.

Note that Seidel and Smith consider a different crossingless matching to be standard in [30], but their picture is equivalent to ours after isotopy and conjugation with a fixed braid, and conjugation does not change the link type.

Given our loop $l : [0,1] \to \mathrm{Conf}^{2m}(\mathbb{C})$, we can find a smooth family of crossingless matchings in the plane with endpoints $l(s), s \in [0,1]$. Note that $l(s)$ are $2m$-tuples of points in $\mathbb{C}$, and $\mu_2, \ldots, \mu_{2m}$ always appear among these $2m$ points. At time $s = 1$ we get a crossingless matching composed of m segments $\beta_1, \ldots, \beta_m$ joining the points of τ in pairs.

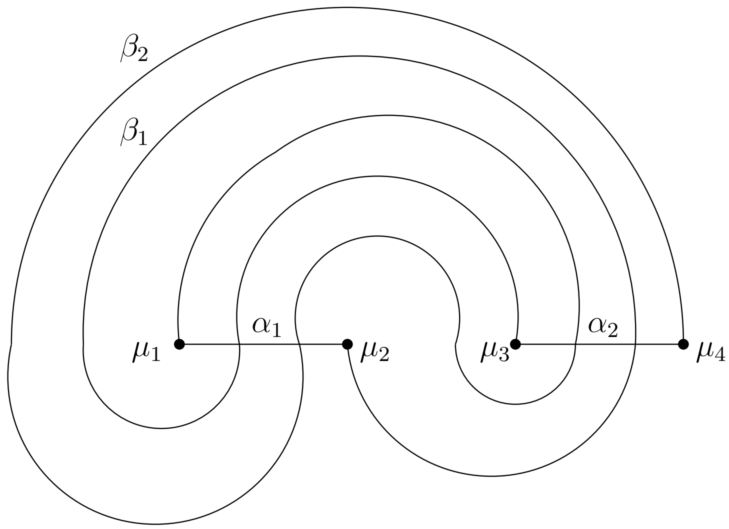

FIGURE 3. Flattened braid diagram of the trefoil.

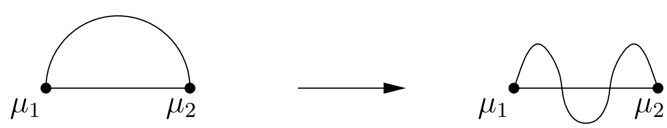

FIGURE 4. Two flattened diagrams of the 1-braided unknot.

We assume that the α and β are simple curves that intersect transversely in their interior. Then the original link L can be recovered from the diagram of the alpha and beta curves intersecting in the plane, which we call a *flattened braid diagram* for L. Indeed, if we choose the alpha curves to serve as underpasses at each crossing in a flattened braid diagram, we obtain a usual plane diagram for the link. This is shown in Figure 3 for the trefoil.

The Lagrangians $\mathcal{L}$ and $\mathcal{L}'$ considered by Seidel and Smith are then $\mathcal{L}(\delta)$ when the crossingless matching δ is given by the alpha and beta curves, respectively.

Remark 3.3. *The braid b determines the flattened diagram up to isotopies that keep the μ_k's fixed. Note that such isotopies can introduce new intersections between the alpha and beta curves, as shown in Figure 4 for the* 1*-braided unknot.*

3.6. **Floer cohomology.** Lagrangian Floer cohomology [3] is a very rich subject. The version used in [30] takes place in a Kähler manifold (Y, Ω) such that

$$\Omega \text{ is exact } ; \; Y \text{ is Stein } ; \; c_1(Y) = 0 \; ; \; H^1(Y) = 0. \tag{14}$$

We also need two closed connected Lagrangian submanifolds $\mathcal{L}, \mathcal{L}' \subset Y$ such that

$$H_1(\mathcal{L}) = H_1(\mathcal{L}') = 0 \; ; \; w_2(\mathcal{L}) = w_2(\mathcal{L}') = 0 \tag{15}$$

One can check that $Y = \mathcal{Y}_{m,\tau}$ and $\mathcal{L} = \mathcal{L}(\alpha), \mathcal{L}' = \mathcal{L}(\beta)$ satisfy these conditions. In general, for any Y and $\mathcal{L}, \mathcal{L}'$ satisfying (14) and (15) there is a well-defined abelian group with a relative $\mathbb{Z}$ grading:

$$HF^*(\mathcal{L}, \mathcal{L}') = H(CF^*(\mathcal{L}, \mathcal{L}'), d).$$

This is called Floer cohomology and is obtained from a cochain group $CF^*(\mathcal{L}, \mathcal{L}')$ using a differential d. The $\mathbb{Z}$ grading is relative in the sense that it is well-defined only up to an overall constant shift.

A short overview of the main properties of HF^* relevant to their construction is given by Seidel and Smith in [30, section 4(D)]. The most important property is that HF^* is invariant under smooth deformations of the objects involved, provided that one remains within a class where HF^* is well-defined. For example, HF^* is invariant under Lagrangian isotopies of $\mathcal{L}$ and $\mathcal{L}'$. In this paper we are mostly interested in understanding the cochain complex CF^*. Let us just quickly note that the differential d is defined by counting "pseudo-holomorphic disks" for a family of almost complex structures taming Ω. These are solutions to some PDE's similar to the Cauchy-Riemann equations, with certain boundary conditions.

A set of generators over $\mathbb{Z}$ for the cochain complex CF^* is obtained by isotoping one Lagrangian so that the intersection $\mathcal{L} \cap \mathcal{L}'$ is transverse, and then taking a generator for each point in $\mathcal{L} \cap \mathcal{L}'$. The relative grading is obtained from a Maslov index calculation. In some cases, including the one of interest to us, it can be improved to an absolute $\mathbb{Z}$ grading. This was done by Seidel in [29], following the ideas of Kontsevich [13].

Let $\mathfrak{L} \to Y$ be the natural fiber bundle over Y whose fibers $\mathfrak{L}_x$ are the manifolds of Lagrangian subspaces of T_xY. Then $\pi_1(\mathfrak{L}_x) \cong \mathbb{Z}$ has a canonical generator called the Maslov class.

Since $c_1(Y) = 0$, we can pick a complex volume form Θ on Y, i.e. a nowhere vanishing section of the canonical bundle. This determines a square phase map

$$\theta : \mathfrak{L} \to \mathbb{C}^*/\mathbb{R}_+, \ \theta(V) = \Theta(e_1 \wedge \cdots \wedge e_N)^2 \tag{16}$$

for any orthonormal basis $e_1, \ldots, e_N$ of $V \subset T_xY$. Here $2N = \dim_{\mathbb{R}} Y$. We can identify $\mathbb{C}^*/\mathbb{R}_+$ with S^1 by the contraction $z \to z/|z|$. Let $\tilde{\mathfrak{L}} \to \mathfrak{L}$ be the infinite cyclic covering obtained from the universal covering $\mathbb{R} \to S^1$ by pulling it back under the map θ.

Note that the Lagrangian submanifold $\mathcal{L}$ gives a canonical section $s_{\mathcal{L}} : \mathcal{L} \to \mathfrak{L}, s_{\mathcal{L}}(x) = T_x\mathcal{L}$. This produces a map

$$\theta_{\mathcal{L}} = \theta \circ s_{\mathcal{L}} : \mathcal{L} \to \mathbb{C}^*/\mathbb{R}_+ \cong S^1. \tag{17}$$

The condition $H^1(\mathcal{L}) = 0$ allows us to lift $s_{\mathcal{L}}$ to a section $\tilde{s}_{\mathcal{L}} : \mathcal{L} \to \tilde{\mathfrak{L}}$. This is equivalent to lifting $\theta_{\mathcal{L}}$ to a map $\tilde{\theta}_{\mathcal{L}} : \mathcal{L} \to \mathbb{R}$.

Definition 3.4. *A* **grading** *on $\mathcal{L}$ is a choice of a lift $\tilde{\theta}_{\mathcal{L}} : \mathcal{L} \to \mathbb{R}$.*

If we choose a grading for $\mathcal{L}'$ as well, then every point x in the intersection $\mathcal{L} \cap \mathcal{L}'$ (which was assumed to be transverse) has a well-defined absolute Maslov index $\mu(x) \in \mathbb{Z}$ [29]. To define it, take a path $\tilde{\lambda} : [0, 1] \to \tilde{\mathfrak{L}}_x$ with endpoints $\tilde{\lambda}(0) = \tilde{s}_{\mathcal{L}}(x)$ and $\tilde{\lambda}(1) = \tilde{s}_{\mathcal{L}'}(x)$. The projection of $\tilde{\lambda}$ to $\mathfrak{L}_x$ is a path $\lambda_0 : [0, 1] \to \mathfrak{L}_x$ joining $T_x\mathcal{L}$ and $T_x\mathcal{L}'$. Consider also the constant path $\lambda_1 : [0, 1] \to \mathfrak{L}_x, \lambda_1(t) = T_x\mathcal{L}'$. Set

$$\mu(x) = \mu^{\text{paths}}(\lambda_0, \lambda_1) + \frac{N}{2}, \tag{18}$$

where $\mu^{\text{paths}}(\lambda_0, \lambda_1)$ is the Maslov index for paths in $\mathfrak{L}_x$ defined in [26].

The Maslov grading μ induces an absolute $\mathbb{Z}$ grading on the cochain complex and on cohomology. The result does not depend on the choice of θ, because the condition $H^1(Y) = 0$ ensures that different choices are homotopic in the class of smooth trivializations of the canonical bundle.

If $\mathcal{L} \to \mathcal{L}[1]$ denotes the process which subtracts the constant 1 from the grading, we have

$$CF^*(\mathcal{L}, \mathcal{L}'[1]) = CF^*(\mathcal{L}[-1], \mathcal{L}') = CF^{*+1}(\mathcal{L}, \mathcal{L}') \tag{19}$$

and the same holds for cohomology.

In our case $Y = \mathcal{Y}_{m,\tau}$ and $\mathcal{L} = \mathcal{L}(\alpha), \mathcal{L}' = \mathcal{L}(\beta)$ there is a way of eliminating even this last $\mathbb{Z}$ indeterminacy in HF^* by choosing the gradings consistently on $\mathcal{L}$ and $\mathcal{L}'$. Specifically, we start with arbitary complex volume forms on the slice $\mathcal{S}_m \cong \mathbb{C}^{4m-1}$ and on the base $\mathrm{Sym}_0^{2m}(\mathbb{C}) \cong \mathbb{C}^{2m-1}$. This induces a smooth family of complex volume forms on the smooth fibers $\mathcal{Y}_{m,\tau}, \tau \in \mathrm{Conf}_0^{2m}(\mathbb{C})$. We know that $\mathcal{L}(\beta)$ is obtained from $\mathcal{L}(\alpha)$ by following the family of crossingless matchings in the plane determined by the braid $b \times 1^m$. Given an arbitrary grading of $\mathcal{L} = \mathcal{L}(\alpha)$, we can continue it uniquely to a smooth family of gradings on the respective Lagrangians, and end with a grading on $\mathcal{L}' = \mathcal{L}(\beta)$. Adding a constant to the grading on $\mathcal{L}$ affects the one on $\mathcal{L}'$ in the same way, so by (19) the grading on CF^* remains the same. Thus it makes sense to write $HF^k(\mathcal{L}, \mathcal{L}')$ for any $k \in \mathbb{Z}$.

As noted in the introduction, the main result proved in [30] is:

Theorem 3.5 (Seidel-Smith)**.** *Let us denote by w the writhe of the braid $b \in Br_m$ i.e. the number of positive minus the number of negative crossings (in Figure 1, not in the flattened braid diagram!). Then the Floer cohomology groups*

$$Kh^*_{symp} = HF^{*+m+w}(\mathcal{L}, \mathcal{L}')$$

are link invariants.

Conditions (14) and (15) can be weakened in various ways. In many cases we can still define a version of Lagrangian Floer cohomology at the expense of giving up some nice properties: for example, the Floer groups can be only $\mathbb{Z}/2$-graded, or only defined over a Novikov ring, etc. Also, typically Floer cohomology is only invariant under a restricted class of deformations, e.g. Hamiltonian isotopies of $\mathcal{L}, \mathcal{L}'$ instead of all Lagrangian isotopies. For a discussion of Lagrangian Floer cohomology in a very general setting, we refer to [5].

Furthermore, sometimes we can define Floer cohomology for half-dimensional submanifolds $\mathcal{T}, \mathcal{T}' \subset Y$ which are not Lagrangian, but only totally real.

Definition 3.6. *A real subspace $V \subset \mathbb{C}^N$ is called* **totally real** *(with respect to the standard complex structure) if $\dim_{\mathbb{R}} V = N$ and $V \cap iV = 0$. A half-dimensional submanifold $\mathcal{T}$ of an almost complex manifold (Y, J) is called* **totally real** *if $T_x\mathcal{T} \cap J(T_x\mathcal{T}) = 0$ for all $x \in \mathcal{T}$.*

Pseudo-holomorphic disks with boundary on totally real submanifolds were studied in [21]. Unlike in the Lagrangian case where certain energy bounds come for free, here one needs to make sure that these bounds still hold. In some cases they do and then a Floer cohomology can be defined. Most notably, this is the setting for the Heegaard Floer theory of Ozsváth and Szabó [22].

On the other hand, the problem of defining the Maslov index is no more difficult in the totally real case than in the Lagrangian case. The relative index is treated in [20] and [8], and then this can easily be improved to an absolute index in the spirit of [29]. Let $\mathbb{C}^N$ be endowed with its standard symplectic form and complex structure. Denote by $\mathfrak{L}_n$ and $\mathfrak{T}_n$ the spaces of Lagrangian and totally real subspaces of $\mathbb{C}^N$, respectively. Then the inclusion

$$\mathfrak{L}_n \cong U(N)/O(N) \hookrightarrow GL(N, \mathbb{C})/GL(N, \mathbb{R}) \cong \mathfrak{T}_n \tag{20}$$

is a homotopy equivalence (see [8] or [20]).

We discuss here the case which will be relevant for us in section 7.4. Y is a Kähler manifold with $c_1(Y) = 0$, and Θ a complex volume form as before. Let $\mathcal{T}, \mathcal{T}' \subset Y$ be totally real and intersecting transversely. We do not want to assume neither that $H^1(\mathcal{T}) = H^1(\mathcal{T}') = 0$ nor that $H^1(Y) = 0$. Just like in the Lagrangian case, there is a natural bundle $\mathfrak{T} \to Y$ whose fibers $\mathfrak{T}_x$ are the manifolds of totally real subspaces of T_xY. There is also a section $s_{\mathcal{T}} : \mathcal{T} \to \mathfrak{T}$ and the square phase maps $\theta : \mathfrak{T} \to \mathbb{C}^*/\mathbb{R}_+ \cong S^1$ and $\theta_{\mathcal{T}} : \mathcal{T} \to S^1$ are still well-defined.

We construct the infinite cyclic covering $\tilde{\mathfrak{T}} \to \mathfrak{T}$ as before. Since $H^1(\mathcal{T})$ may be nonzero, it is possible that $s_{\mathcal{T}}$ does not lift to a section in $\tilde{\mathfrak{T}}$. However, let us assume it does. A *grading* on $\mathcal{T}$ is a choice of such a lift. Assuming that $\mathcal{T}'$ also has a grading, we can define an absolute Maslov grading $\mu(x) \in \mathbb{Z}$ for every $x \in \mathcal{T} \cap \mathcal{T}'$. Indeed, we can construct paths λ_0 and λ_1 in $\mathfrak{T}_x$ as before, get corresponding paths in $\mathfrak{L}_x$ using the homotopy equivalence (20), and then use formula (18) for those. Note that this time the result can depend on θ, because $H^1(Y)$ may be nonzero.

4. A different Kähler metric

In this section we go through the Seidel-Smith construction again, but using a different choice of the Kähler metric on $\mathcal{Y}_{m,\tau}$. This will make it easier to write down the Lagrangians in terms of the embedding of $\mathcal{Y}_{m,\tau}$ into the Hilbert scheme. The goal is to prove Theorem 1.2.

4.1. **A choice of Kähler form.** Corollary 2.9 says that via the Hilbert-Chow morphism π from (3), we can identify an open subset of $\mathcal{Y}_{m,\tau}$ with

$$\mathcal{V}_{m,\tau} = \{(u_j, v_j, z_j) \in \mathrm{Sym}^m(S_\tau) : z_i \neq z_j \text{ for all } i \neq j\}. \tag{21}$$

The subsets $\mathcal{V}_{m,\tau}$ form a family $\mathcal{V}_m$ over $\mathrm{Sym}_0^{2m}(\mathbb{C})$. Using Proposition 2.11, we can identify $\mathcal{V}_m$ with an open subset of $\mathcal{S}_m$. For a point $(\mu_1, \dots, \mu_{2m}) \in \mathrm{Sym}_0^{2m}(\mathbb{C})$, we form the symmetric polynomials

$$\nu_j = \sum_{i_1 < \dots < i_j} \mu_{i_1}\mu_{i_2}\dots\mu_{i_j}$$

for $2 \leq j \leq 2m$. The ν_j are coordinates on $\mathrm{Sym}_0^{2m}(\mathbb{C})$ viewed as an affine space $\mathbb{C}^{2m-1}$.

We construct a new Kähler form $\tilde{\Omega} = -dd^c\tilde{\psi}$ on $\mathcal{S}_m$, guided by the following requirement: We fix a large relatively compact, open subset U of $\mathcal{V}_m$. We want the restriction of $\tilde{\Omega}$ to U to have the form

$$\tilde{\Omega}|_U = \frac{i}{2}\sum_{j=1}^{m}\big(du_j \wedge d\bar{u}_j + dv_j \wedge d\bar{v}_j + dz_j \wedge d\bar{z}_j\big) + \frac{i}{2}\sum_{j=2}^{2m} d\nu_j \wedge d\bar{\nu}_j. \tag{22}$$

This corresponds to $\tilde{\psi}$ of the form

$$\sum_{j=1}^{m}(|u_j|^2 + |v_j|^2 + |z_j|^2) + \sum_{j=2}^{2m}|\nu_j|^2.$$

We also want $\tilde{\psi}$ to satisfy conditions (8)-(11), so that $\tilde{\Omega}$ has well-defined rescaled parallel transport maps and the construction of the Lagrangians can proceed as before.

Recall the construction of ψ from section 3.1. In it we use smooth functions $\psi_k : \mathbb{C} \to \mathbb{R}$ $(k = 1, \dots, m)$ with $-dd^c\psi_k > 0$ everywhere and $\psi_k(z) = \xi_k(z) = |z|^{\alpha/k}$ for z near infinity. We define similar functions ξ_k, ψ_k for $k = m+1, \dots, 2m$ as well. If α is chosen so that $\alpha > 4m$, then given any relatively compact open subset of $\mathbb{C}$, we can construct ψ_k with these properties so that $\psi_k(z) = |z|^2$ on the specified open subset. We take this choice of ψ_k and define a function on the Milnor fiber $S_\tau : (u^2 + v^2 + P_\tau(z) = 0)$ by

$$(u, v, z) \to \psi_m(u) + \psi_m(v) + \psi_1(z).$$

Summing up these functions for all coordinates gives a function on the symmetric product $\mathrm{Sym}^m(S_\tau)$. We pull it back to the Hilbert scheme $\mathrm{Hilb}^m(S_\tau)$ via the Hilbert-Chow morphism π, then add the terms

$$\psi_2(\nu_2) + \psi_3(\nu_3) + \cdots + \psi_{2m}(\nu_{2m})$$

to obtain a plurisubharmonic function on the family of Hilbert schemes $\mathrm{Hilb}^m(S_\tau)$ over $\mathrm{Sym}^0_{2m}(\mathbb{C})$. By Proposition 2.11, $\mathcal{S}_m$ is an open subset of this family, hence by restriction we obtain a plurisubharmonic function $\rho : \mathcal{S}_m \to \mathbb{R}$.

We cannot use ρ to construct a Kähler metric, because $-dd^c\rho$ is degenerate on the preimages $\pi^{-1}(\Delta) \cap \mathcal{Y}_{m,\tau}$ of the diagonals $\Delta \subset \mathrm{Sym}^m(S_\tau)$, i.e. outside $\mathcal{V}_m = \cup_\tau \mathcal{V}_{m,\tau}$. The function ψ which was used to construct Ω in section 3.1 comes to the rescue. Let $\beta : \mathcal{S}_m \to \mathbb{R}$ be a bump function that is identically 1 on a neighborhood of $\mathcal{S}_m - \mathcal{V}_m$ and identically 0 outside a slightly larger neighborhood of $\mathcal{S}_m - \mathcal{V}_m$. Then for $\epsilon > 0$ sufficiently small, the function

$$\tilde{\psi} = \rho + \epsilon\beta \cdot \psi \tag{23}$$

is strictly plurisubharmonic and defines a Kähler metric $\tilde{\Omega} = -dd^c\tilde{\psi}$ on $\mathcal{S}_m$, and has the form (22) on U.

Note that if we had used the functions ξ_k instead of ψ_k in the construction of $\tilde{\psi}$, we would have obtained a corresponding C^1 function $\tilde{\xi} : \mathcal{S}_m \to \mathbb{R}$ that is identical to $\tilde{\psi}$ near infinity.

4.2. **Rescaled parallel transport.** Let us check that $\tilde{\psi}$ satisfies conditions $(8)-(11)$ and therefore gives rise to well-defined rescaled parallel transport maps. This runs parallel to the corresponding proof for ψ given in section 5(A) of [30].

Conditions (8) and (9) are immediate from the definition. To check (10), note that $\|\nabla\psi_k\| \leq c + d\psi_k$ for some $c, d > 0$. Adding up these inequalities, together with the fact that $\psi \gg 0$ outside a compact subset, shows that $\|\nabla\psi\| < C\psi$ for some $C > 0$.

For condition (11), we need to use the $\mathbb{R}_+$ action (12) on $\mathcal{S}_m$. On the open set $\mathcal{V}_m \subset \mathcal{S}_m$ the action of $r \in \mathbb{R}_+$ can be written in terms of u_j, v_j, z_j, ν_j as

$$\lambda_r : (u_j, v_j, z_j, \nu_j) \to (r^m u_j, r^m v_j, r z_j, r^j \nu_j).$$

Because the set $\mathcal{V}_m$ is preserved by this action, we can choose the bump function β to satisfy $\beta \circ \lambda_r = \beta$. Then $\tilde{\xi}$ is homogeneous of weight α, i.e. $\tilde{\xi} \circ \lambda_r = r^\alpha \tilde{\xi}$.

Lemma 4.1. *The function $\tilde{\psi}$ is asymptotically homogeneous with respect to the action λ, in the sense that*

$$\frac{\tilde{\psi} \circ \lambda_r}{r^\alpha} \to \tilde{\xi} \text{ as } r \to \infty$$

where the convergence is uniform in C^1 norm.

Proof. This is the analogue of Lemma 40 in [30]. The differences $\psi_k - \xi_k$ are compactly supported, and after rescaling the support of $\psi_k(r^k z) - \xi_k(r^k z)$ is getting smaller as $r \to \infty$. This gives a uniform bound on their C^0 norms. For their derivatives we use the fact that $(\psi_k(r^k z) - \xi_k(r^k z))/r^\alpha \to 0$ uniformly for $\alpha > 4m > k$. □

Now, to see that condition (11) is satisfied, we can apply the argument used by Seidel and Smith in Lemma 41 from [30]. Here is a sketch: there is a simultaneous resolution of the map $\chi : \mathcal{S}_m \to \mathrm{Sym}^{2m}_0(\mathbb{C})$ in the form of a differentiable fiber bundle $\hat{\chi} : \hat{\mathcal{S}}_m \to \mathbb{C}^{2m-1}$. The $\mathbb{R}_+$ action lifts to one on $\hat{\mathcal{S}}_m$. It preserves the fiber over 0, and the asymptotic homogeneity of the lift of $\tilde{\psi}$ (Lemma 4.1) shows that the critical set of the restriction of this lift to $\hat{\chi}^{-1}(0)$ is compact. Then one shows that the whole fiberwise critical set of the lift of $\tilde{\psi}$ to $\hat{\mathcal{S}}_m$ maps properly to the base by using a rescaling argument.

4.3. **Restriction and interpolation.** The transverse slice $\mathcal{S}_{m-1}$ sits embedded in $\mathcal{S}_m$ by forgetting the coordinates a_m, b_m, c_m, d_m. This is compatible with the isomorphism in Lemma 3.1 (i). In terms of the coordinates u_j, v_j, z_j, ν_i on $\mathcal{V}_m \subset \mathcal{S}_m$, the corresponding coordinates on $\mathcal{V}_{m-1} \subset \mathcal{S}_{m-1}$ are

$$u'_j = \frac{u_j}{z_j} \ ; \ v'_j = \frac{v_j}{z_j} \ ; \ z'_j = z_j \ ; \ \nu'_i = \nu_i$$

for $j = 1, \ldots, m-1$ and $i = 2, \ldots, 2m-2$.

One disadvantage that our Kähler form $\tilde{\Omega}$ has over Seidel and Smith's Ω is that its restriction to $\mathcal{S}_{m-1}$ is not of the same form. For example, on a big open subset $U' \subset U \cap \mathcal{S}_{m-1}$ avoiding $z'_j = 0$, instead of (22) we have

$$\tilde{\Omega}|_{U'} = \frac{i}{2} \sum_{j=1}^{m-1} \big(d(u'_j z'_j) \wedge d(\bar{u}'_j \bar{z}'_j) + d(v'_j z'_j) \wedge d(\bar{v}'_j \bar{z}'_j) + dz'_j \wedge d\bar{z}'_j \big) + \frac{i}{2} \sum_{j=2}^{2m-2} d\nu'_j \wedge d\bar{\nu}'_j. \tag{24}$$

Therefore, if in the recursive construction of the Lagrangians from section 3.4 we used the forms $\tilde{\Omega}_k$ on $\mathcal{S}_k$ for all $k \leq m$, where $\tilde{\Omega}_k$ are constructed just like $\tilde{\Omega}_m = \tilde{\Omega}$ in section 4.1, we could run into the problem that the Lagrangians constructed in the singular set of $\mathcal{Y}_{k,\tau}$ may not be Lagrangians for the Kähler metric on $\mathcal{S}_k$. We deal with this problem by using the restrictions of $\tilde{\Omega}_m = \tilde{\Omega}$ to each $\mathcal{S}_k$ instead. Their properties are not very different from those of $\tilde{\Omega}_k$; in particular, they have a similar behaviour at infinity.

Let us interpolate linearly between the two functions $\psi, \tilde{\psi} : \mathcal{S}_m \to \mathbb{R}$, by defining $\psi_t = (1-t)\psi + t\tilde{\psi}$ for $t \in [0,1]$. The proof of the following lemma is entirely similar to the argument in section 4.2 above, so we omit it.

Lemma 4.2. *The functions ψ_t and their restrictions to any $\mathcal{S}_k \subset \mathcal{S}_m, k < m$ satisfy the conditions* (8)$-$(11). *Thus the corresponding Kähler forms have well-defined rescaled parallel transport maps over the respective configuration spaces.*

Since Floer cohomology is invariant under deformation, it follows that we can define Lagrangians $\tilde{\mathcal{L}}, \tilde{\mathcal{L}}'$ in $\mathcal{S}_m$ with the form $\tilde{\Omega}$ and

$$HF^*(\tilde{\mathcal{L}}, \tilde{\mathcal{L}}') = HF^*(\mathcal{L}, \mathcal{L}'). \tag{25}$$

Therefore, the link invariant Kh^*_{symp} can also be defined using $\tilde{\Omega}$ rather than Ω.

4.4. **The new Lagrangians in an explicit form.** In section 3.4 we associated a Lagrangian $\mathcal{L}(\delta) \subset \mathcal{Y}_{m,\tau}$ to every crossingless matching δ consisting of arcs $\delta_1, \ldots, \delta_m$ joining the points of τ in pairs. Let $\tilde{\mathcal{L}}(\delta)$ be the corresponding Lagrangian constructed using the Kähler form $\tilde{\Omega}$ in place of Ω.

To each arc δ_k we associate the Lagrangian 2-sphere in the Milnor fiber S_τ:

$$\Sigma_{\delta_k} = \{(u, v, z) \in S_\tau \ : z = \delta_k(t) \text{ for some } t \in [0,1]; \ \ u, v \in \sqrt{-P_\tau(z)}\mathbb{R}\}.$$

These Lagrangians have appeared before in the work of Khovanov and Seidel [12].

Let us change coordinates on S_τ from (u, v, z) to (x, y, z) where

$$x = \frac{u + iv}{\sqrt{2}} \ ; \ y = \frac{u + iv}{\sqrt{2}}.$$

Then the equation for S_τ changes to $2xy + P_\tau(z) = 0$ and

$$\Sigma_{\delta_k} = \{(x, y, z) \in S_\tau \ : |x| = |y|, \ z = \delta_k(t) \text{ for some } t \in [0,1]\}.$$

We change coordinates on $\mathcal{V}_{m,\tau}$ accordingly, from (u_k, v_k, z_k) to (x_k, y_k, z_k) for $k = 1, \ldots, m$.

Proposition 4.3. *$\tilde{\mathcal{L}}(\delta)$ is Lagrangian isotopic to $\mathcal{K}(\delta) = \Sigma_{\delta_1} \times \Sigma_{\delta_2} \times \cdots \times \Sigma_{\delta_m} \subset \mathcal{V}_{m,\tau} \subset \mathcal{Y}_{m,\tau}$, where $\mathcal{V}_{m,\tau}$ is as in (21).*

Proof. First of all let us note that we can only be sure that $\mathcal{K}(\delta)$ is a Lagrangian for $\tilde{\Omega}$ if we know that it lives inside the chosen subset $U \subset \mathcal{V}_m$ where (22) holds. We will arrange so that all of our constructions take place inside U.

We do induction on m starting with the trivial case $m = 0$. Let us explain the inductive step. We return to the notations from section 3.4. Assume the recursive procedure had already given us the Lagrangian $\mathcal{K}(\bar{\delta}) \subset \mathcal{Y}_{m-1,\bar{\tau}}$, where $\bar{\delta}$ is the crossingless matching obtained from δ by erasing δ_1 and then translating everything from $(\mu_3, \dots, \mu_{2m})$ to $\bar{\tau} = (\mu''_3, \dots, \mu''_{2m})$ by (1).

We apply the vanishing cycle procedure and then rescaled parallel transport to obtain a Lagrangian in $\mathcal{Y}_{m,\tau}$. To control the form of the Lagrangians we use a moment map for a torus action. The argument is similar to the one used by Seidel and Smith in Lemma (32) of [30].

The torus $T^m = (S^1)^m$ acts on $\mathcal{V}_m$ preserving the fibers $\mathcal{V}_{m,\tau}$ in the following way. The action of an element $(e^{i\theta_1}, \dots, e^{i\theta_m}) \in T^m$ is

$$(x_k, y_k, z_k) \to (e^{i\theta_k} x_k, e^{-i\theta_k} y_k, z_k) \ k = 1, \dots, m.$$

If we restrict to the open set U, this action is Hamiltonian with moment map $f : U \to \mathbb{R}^m$,

$$f(\{(x_k, y_k, z_k); k = 1 \dots m\}) = (|x_1|^2 - |y_1|^2, \dots, |x_m|^2 - |y_m|^2).$$

Furthermore, an easy computation shows that the naive parallel transport vector fields for $\chi : (U, \tilde{\Omega}) \to \mathrm{Sym}^{2m}_0(\mathbb{C})$ are invariant with respect to the torus action and df vanishes on them. Since $\mathcal{K}(\bar{\delta}) \subset \mathcal{Y}_{m-1,\bar{\tau}}$ is invariant under the action and lies in $f^{-1}(0)$, using Lemma 3.2 we get that the vanishing cycle in $\mathcal{Y}_{m,\tau''_s}$ is also invariant and part of $f^{-1}(0)$. We also know that it is diffeomorphic to $(S^2)^m$. It follows that it must be of the form $\mathcal{K}(\delta''_s)$, where δ''_s is a matching of the points of τ''_s. Since it must lie close to $\mathcal{K}(\bar{\delta})$ for s small, its isotopy class is determined uniquely. We can take δ''_s to consist of the linear path from $-s$ to s together with the paths in $\bar{\delta}$.

The next step is moving $\mathcal{K}(\delta''_s)$ from $\mathcal{Y}_{m,\tau''_s}$ to $\mathcal{Y}_{m,\tau'_s}$ by rescaled parallel transport along a linear path $\zeta : [0, 1] \to \mathrm{Sym}^{2m}_0(\mathbb{C})$. We choose a partition $0 = t_0 < t_1 < \cdots < t_N = 1$ of the interval $[0, 1]$ and consider the corresponding partition of ζ. If each piece is sufficiently small, we can apply naive parallel transport to the Lagrangian so that we again know that it is invariant under the T^m action and lies in $f^{-1}(0)$. Its isotopy class is determined uniquely as before. In particular, we can isotope it into K of the matching consisting of the linear path between the first two coordinates and the respective translation of $\bar{\delta}$ joining the others. Then we continue the process using naive parallel transport. If the partition is sufficiently fine, everything is kept inside U. By the discussion at the end of section 3.1, the result is the same as that of the rescaled parallel transport.

We arrived at some Lagrangian in $\mathcal{Y}_{m,\gamma(1-s)}$. We choose a fine partition of the path $\gamma : [0, 1-s] \to \mathrm{Sym}^{2m}_0(\mathbb{C})$ and use naive parallel transport and isotopies to move the Lagrangian into $\mathcal{Y}_{m,\gamma(0)}$ such that at each step we have $\mathcal{K}$ of some matching and everything is kept inside U. The isotopy classes of the matchings are uniquely determined, and at the end we get $\mathcal{K}(\delta) \subset \mathcal{Y}_{m,\tau}$ as desired.

There is one caveat about our inductive argument. Strictly speaking, the inductive hypothesis gives us the Lagrangian $\mathcal{K}(\bar{\delta})$ in $\mathcal{Y}_{m-1,\bar{\tau}}$ with the Kähler form $\tilde{\Omega}_{m-1}$. However, as explained in section 4.3, we would like to use the restriction of the form $\tilde{\Omega} = \tilde{\Omega}_m$ instead. Let us interpolate linearly between the two forms by letting $\omega(t)$ be the restriction of

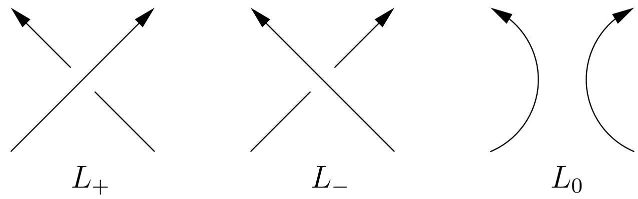

FIGURE 5. The links in the skein relation.

$(1-t)\tilde{\Omega}_{m-1} + t\tilde{\Omega}$ to $\mathcal{Y}_{m-1,\bar{\tau}}$, for $t \in [0,1]$. We get a corresponding family of Lagrangians $\ell(t)$ in $(\mathcal{Y}_{m-1,\bar{\tau}}; \omega(t))$ for all t, with $\ell(0) = \mathcal{K}(\bar{\delta})$. Note that $\omega(1)$ is of the form (24) on a big open set U' that can be assumed to contain $\mathcal{K}(\bar{\delta})$. Furthermore, $\mathcal{K}(\bar{\delta})$ and $\mathcal{K}$ of other matchings in U' are Lagrangians for $\omega(t)$ for all t. Also, the torus action is still Hamiltonian with the same moment map for all $\omega(t)$.

We break the deformation into small pieces. On the first piece $[0,\epsilon] \subset [0,1]$, Moser's lemma gives a family of symplectic embeddings ϕ_{t_1,t_2} from a neighborhood of $\mathcal{K}(\bar{\delta})$ with the form ω_{t_1} into U' with the form ω_{t_2}, for any $0 \le t_1 \le t_2 \le \epsilon$. Using the moment map as before we get that $\phi_{0,t}(\ell(0))$ must be of the form $\mathcal{K}$ of some crossingless matching close to $\bar{\delta}$ for all $t \in [0,\epsilon]$. In particular $\phi_{0,\epsilon}(\ell(0))$ is Lagrangian isotopic to $K(\bar{\delta})$ with respect to ω_ϵ. On the other hand, $\phi_{0,\epsilon}(\ell(0))$ is isotopic to $\ell(\epsilon)$ via the isotopy $\phi_{t,\epsilon}(\ell(t))$, $t \in [0,\epsilon]$. It follows that $\ell(\epsilon)$ is isotopic to $\mathcal{K}(\bar{\delta})$ and by iterating this procedure we get that the same is true for all $\ell(t), t \in [0,1]$. Thus we can safely start the inductive step with $\omega(1)$ rather than $\omega(0)$. □

Theorem 1.2 is now a direct consequence of (25) and Proposition 4.3.

Remark 4.4. *We assumed that the variety $\mathcal{Y}_{m,\tau} \subset \mathcal{S}_m$ came with the restriction of the Kähler form $\tilde{\Omega}$. In fact, since $\mathcal{K} = \mathcal{K}(\alpha), \mathcal{K}' = \mathcal{K}(\beta)$ are in the chosen open set U, by doing linear interpolation we see that the Floer cohomology groups are the same for any exact Kähler form that has the form (22) on $U \cap \mathcal{Y}_{m,\tau}$.*

5. Bigelow's definition of the Jones polynomial

The Jones polynomial [7] is an invariant of oriented links in S^3. It takes a link L to a Laurent polynomial $V_L(t) \in \mathbb{Z}[t^{\pm 1/2}]$ and is usually defined by the normalization $V_{\text{unknot}} = 1$ and the skein relation:

$$t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (t^{1/2} - t^{-1/2})V_{L_0}(t).$$

Here L_+, L_- and L_0 are links that are identical except in a ball, where they look as in Figure 5.

The normalization and the skein relation completely determine the polynomial. In this paper we will work mainly with a different normalization. Specifically, we make the change of variable $q = -t^{1/2}$ and set

$$J_L(q) = (q + q^{-1}) \cdot V_L \tag{26}$$

This is usually called the *unnormalized Jones polynomial.* For example, the normalized Jones polynomial of the left-handed trefoil is $V_L = t^{-1} + t^{-3} - t^{-4}$ and the unnormalized one $J_L = q^{-1} + q^{-3} + q^{-5} - q^{-9}$.

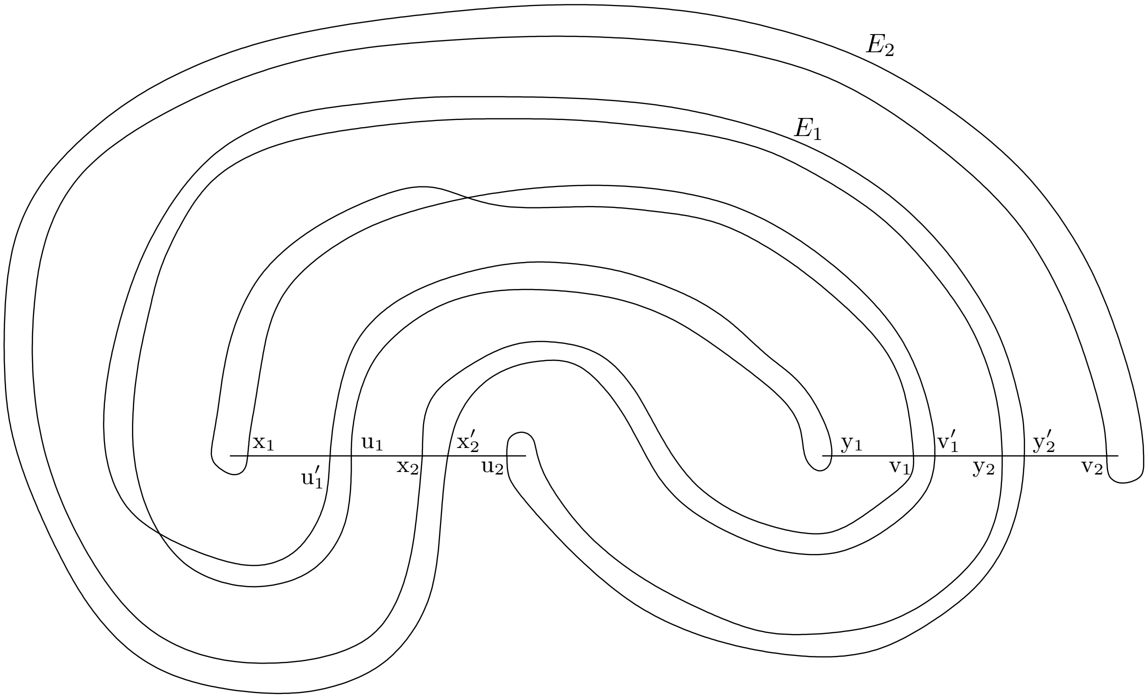

FIGURE 6. Replacing the beta curves with figure-eights in a flattened braid diagram.

5.1. **Bigelow's picture.** Bigelow [1] gave a different definition of the Jones polynomial. His construction is basically a reinterpretation of the work of Lawrence [17], with the resulting picture made more concrete and geometric.

We alert the reader to the fact that Bigelow's conventions are different from ours, because he considers braids to act on the punctured plane on the left rather than on the right. Our conventions are more in line with Seidel and Smith's paper [30]. Correspondingly, the gradings Q and T defined below are in fact minus the gradings in Bigelow's paper.

His construction starts with an arbitrary plat representation of the link, but for our purposes it suffices to consider braid diagrams as in section 3.5. We represent L as the closure of $b \in Br_m$ and consider the resulting flattened braid diagram as in Figure 3 for the trefoil. We replace every beta curve with a figure-eight immersed loop running around it. This is shown in Figure 6 for the trefoil in Figure 3. We denote by E_i be the figure-eight going around the curve β_i for $i = 1, \dots, m$.

Consider a big disk $D \subset \mathbb{C}$ containing all the α, β and E curves. Let D^* be the punctured disk

$$D^* = D - \{\mu_1, \dots, \mu_{2m}\}.$$

We can think of the braid group $Br_{2m} = \pi_1(\mathrm{Conf}^{2m}(D))$ as the mapping class group of D^*. Thus $b \times 1^m$ induces a boundary preserving homeomorphism of D^*. Because of the 1^m factor, this homeomorphism can in fact be extended over the even puncture points $\mu_2, \mu_4, \dots, \mu_{2m}$.

We denote by α^o the alpha curves with the endpoints removed. Then

$$M_1 = \alpha_1^o \times \cdots \times \alpha_m^o \quad \text{and} \quad M_2 = E_1 \times \cdots \times E_m$$

are two manifolds in $\mathcal{C} = \mathrm{Conf}^m(D^*)$, the first one embedded, the second immersed. The Jones polynomial will come up as a graded intersection number of the lifts of M_1 and M_2 in a certain covering of $\mathcal{C}$, which we now proceed to define.

The composition of the map induced by the inclusion $D^* \hookrightarrow D$ with the natural abelianization map of the braid group is a homomorphism:

$$\Phi_1 : \pi_1(\mathrm{Conf}^m(D^*)) \to \pi_1(\mathrm{Conf}^m(D)) \cong Br_m \to \mathbb{Z}.$$

There is also an inclusion $\mathrm{Conf}^m(D^*) \hookrightarrow \mathrm{Conf}^{3m}(D)$ obtained by adding the $2m$ puncture points $\mu_1, \dots, \mu_{2m}$. We get a corresponding homomorphism

$$\Phi_2 : \pi_1(\mathrm{Conf}^m(D^*)) \to \pi_1(\mathrm{Conf}^{3m}(D)) \cong Br_{3m} \to \mathbb{Z}.$$

It is easy to check that the image of $\Phi_1 - \Phi_2$ lies in $2\mathbb{Z}$. The homomorphism

$$\Phi = \big(\frac{1}{2}(\Phi_2 - \Phi_1), \Phi_1\big) : \pi_1(\mathcal{C}) \to \mathbb{Z} \oplus \mathbb{Z}$$

is surjective and determines a $\mathbb{Z} \oplus \mathbb{Z}$ covering $\tilde{\mathcal{C}}$ of $\mathcal{C}$. Let us denote by $\phi[a, b] : \tilde{\mathcal{C}} \to \tilde{\mathcal{C}}$ the covering transformation corresponding to the element $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$.

A more intuitive interpretation of the morphism Φ is the following. A loop ℓ in $\mathcal{C}$ can be represented as a set of m arcs in D^*. The image of ℓ in the first $\mathbb{Z}$ factor counts the total winding number of the m arcs around the puncture points. The image in the second $\mathbb{Z}$ factor counts twice the winding number of the arcs around each other, which is basically the linking number of ℓ with the diagonal in $\mathrm{Sym}^m(D^*) \hookrightarrow \mathrm{Sym}^m(D)$.

In order to lift the manifolds M_1 and M_2 from $\mathcal{C}$ to $\tilde{\mathcal{C}}$, we need to specify some basepoints. We do this by attaching handles to each alpha curve. A *handle* h_i is a segment in the lower half plane that starts from a point η_i on the boundary of D and ends in the midpoint of α_i, as shown in Figure 7.

We take $\eta = (\eta_1, \dots, \eta_m)$ as a basepoint in $\mathcal{C}$ and also fix a lift $\tilde{\eta} \in \tilde{\mathcal{C}}$ of η. The collection of all handles $h = (h_1, \dots, h_m)$ describes a path in $\mathcal{C}$ from η to a point $m \in M_1$. We lift h to a path in $\tilde{\mathcal{C}}$ from $\tilde{\eta}$ to a point $\tilde{m}$ in the preimage of m. Let $\tilde{M}_1 \subset \tilde{\mathcal{C}}$ be the lift of M_1 which contains $\tilde{m}$.

Similarly we can defined a lift $\tilde{M}_2$ of the m-torus M_2. Consider the images of the handles under the homeomorphism of the disk determined by $b \times 1^m$ which takes the alpha curves to the beta curves. This gives a handle set for the beta curves that can be turned into one for the figure eights and used to specify the lift of M_2.

There are well-defined algebraic intersection numbers $(\phi[a, b]\tilde{M}_1, \tilde{M}_2) \in \mathbb{Z}$ between the translates of $\tilde{M}_1$ and $\tilde{M}_2$. We denote by w the writhe of the braid b as in Theorem 3.5. The following result is proved in [1]:

Theorem 5.1. *The unnormalized Jones polynomial of the link L can be expressed as*

$$J_L(q) = (-1)^m q^{m+w} \sum_{a,b \in \mathbb{Z}} (-1)^b q^{2(b-a)} (\phi[a, b]\tilde{M}_1, \tilde{M}_2). \tag{27}$$

5.2. **The Bigelow generators.** Looking at the formula (27), it is clear that each intersection point between M_1 and M_2 contributes only once to a certain coefficient j_n in $J_L(q) = \sum_{n \in \mathbb{Z}} j_n q^n$.

Definition 5.2. *The elements of $\mathcal{G} = M_1 \cap M_2$ are called* **Bigelow generators.** *The integer n corresponding to a Bigelow generator is called its* **Jones grading.**

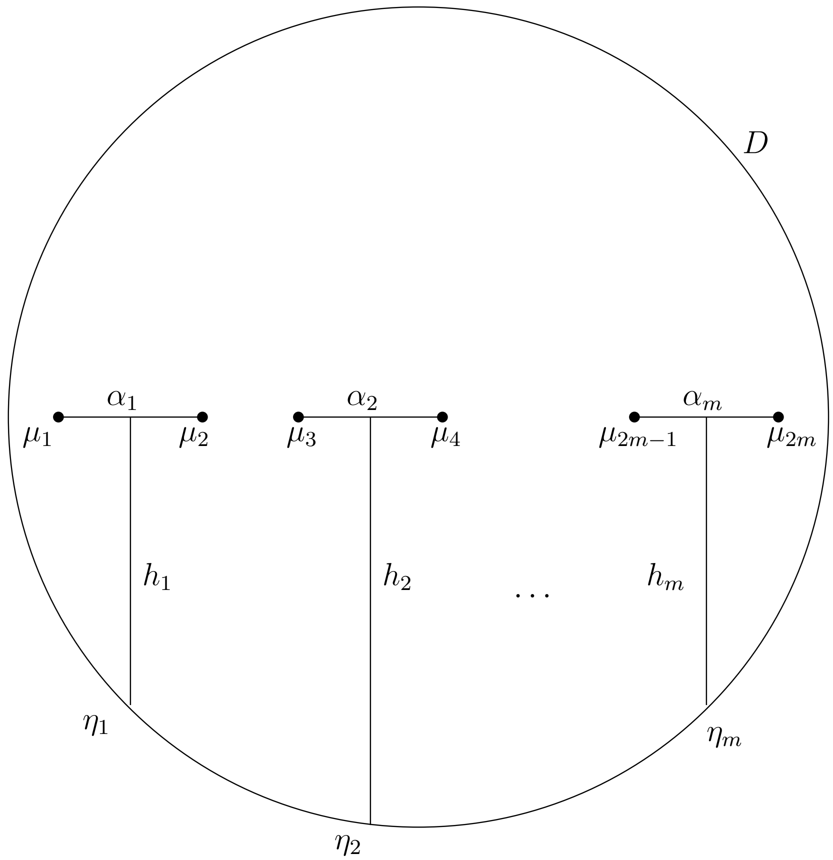

FIGURE 7. Handles for the alpha curves.

The formula (27) can be rewritten as:

$$J_L(q) = \sum_{\gamma \in \mathcal{G}} \sigma(\gamma) q^{J(\gamma)}, \tag{28}$$

where σ is a sign function $\sigma : \mathcal{G} \to \{\pm 1\}$.

Let us explain more carefully the structure of the set $\mathcal{G}$ and fix some notation. We started with a flattened braid diagram for the link L. For the sake of concreteness, we will always write down what happens for the case of the left-handed trefoil in Figure 3. Let $\bar{\mathcal{Z}}$ be the set of all intersection points between an alpha curve and a beta curve. Figure 8 shows them for the trefoil:

$$\bar{\mathcal{Z}} = \{\bar{\mathrm{x}}_1, \bar{\mathrm{x}}_2, \bar{\mathrm{y}}_1, \bar{\mathrm{y}}_2, \bar{\mathrm{u}}_1, \bar{\mathrm{u}}_2, \bar{\mathrm{v}}_1, \bar{\mathrm{v}}_2\}.$$

Note that the set of puncture points $\tau = \{\mu_1, \dots, \mu_{2m}\}$ is in fact a subset of $\bar{\mathcal{Z}}$. As explained in the introduction, we construct a set $\mathcal{Z}$ from $\bar{\mathcal{Z}}$ by doubling the points of $\bar{\mathcal{Z}} - \tau$. That is, for every $x \in \tau$ we introduce an element $e_x \in \mathcal{Z}$ and for every $x \in \mathcal{Z} - \tau$ we introduce two elements $e_x, e'_x \in \mathcal{Z}$. There is a natural map

$$f : \mathcal{Z} \to \bar{\mathcal{Z}} \tag{29}$$

which takes e_x and e'_x to x.

The set $\mathcal{Z}$ can be thought of as the set of intersection points between the alpha curves and the figure eights E_i. Indeed, near each $x \in \alpha_i \cap \beta_j$ there is one point $e_x \in \alpha_i \cap E_j$

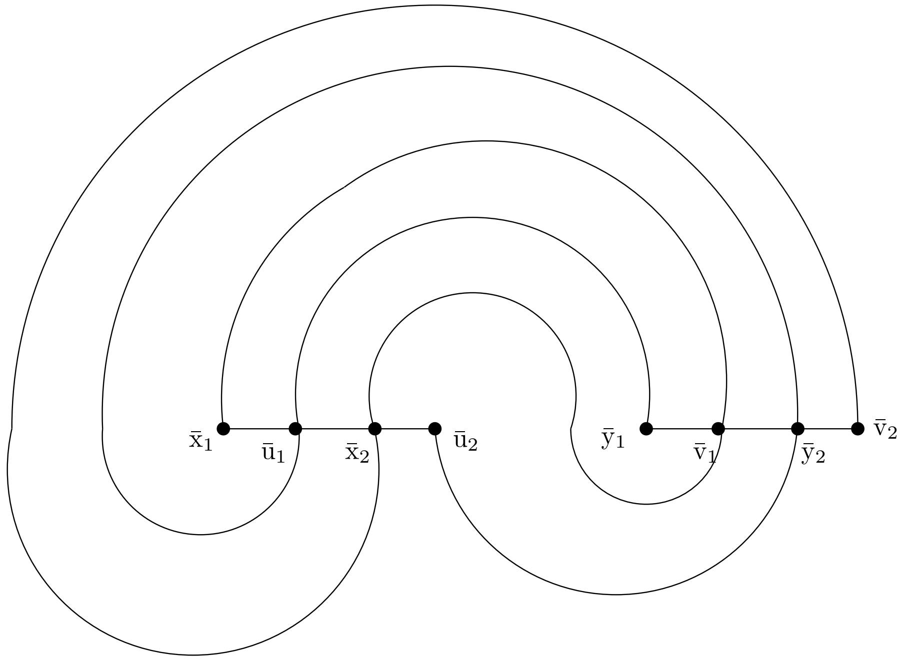

FIGURE 8. Intersections of the alpha and beta curves.

when $x \in \tau$ and two points e_x, e'_x otherwise. We distinguish between e_x and e'_x by requiring that the loop in the plane starting at e_x, following the part of the figure eight with no self-intersections around a puncture point up to e'_x and then going back to e_x along the alpha curve has winding number $+1$ with the puncture.

In our trefoil example (Figure 6) we have

$$\mathcal{Z} = \{\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}'_2, \mathrm{y}_1, \mathrm{y}_2, \mathrm{y}'_2, \mathrm{u}_1, \mathrm{u}'_1, \mathrm{u}_2, \mathrm{v}_1, \mathrm{v}'_1, \mathrm{v}_2\}$$

and $f(\mathrm{x}_1) = \bar{\mathrm{x}}_1, f(\mathrm{x}_2) = f(\mathrm{x}'_2) = \bar{\mathrm{x}}_2$, etc.

We define maps $\bar{A}, \bar{B} : \bar{\mathcal{Z}} \to \{1, 2, \dots, m\}$ by taking an intersection point in $\alpha_i \cap \beta_j$ to (i, j). The set

$$\bar{\mathcal{G}} = (\alpha_1 \times \cdots \times \alpha_m) \cap (\beta_1 \times \cdots \times \beta_m) \subset \mathrm{Conf}^m(\mathbb{C})$$

consists of unordered m-tuples $(x_1, \dots, x_m)$ of elements of $\bar{\mathcal{Z}}$ with $A(x_i) \neq A(x_j)$ and $B(x_i) \neq B(x_j)$ for all $i \neq j$.

Similarly, we can consider the compositions $A = \bar{A} \circ f, B = \bar{B} \circ f : \mathcal{Z} \to \{1, 2, \dots, m\}$. The elements of the resulting set $\mathcal{G}$ are exactly the Bigelow generators. The map (29) induces natural map from $\mathcal{G}$ to $\bar{\mathcal{G}}$, which we still denote by f.

In the trefoil example we have 18 Bigelow generators:

$$\mathcal{G} = \{\ \mathrm{x}_1\mathrm{y}_1,\ \mathrm{x}_1\mathrm{y}_2,\ \mathrm{x}_1\mathrm{y}'_2,\ \mathrm{x}_2\mathrm{y}_1,\ \mathrm{x}_2\mathrm{y}_2,\ \mathrm{x}_2\mathrm{y}'_2,\ \mathrm{x}'_2\mathrm{y}_1,\ \mathrm{x}'_2\mathrm{y}_2,\ \mathrm{x}'_2\mathrm{y}'_2\ \}$$

$$\cup\ \{\ \mathrm{u}_1\mathrm{v}_1,\ \mathrm{u}_1\mathrm{v}'_1,\ \mathrm{u}_1\mathrm{v}_2,\ \mathrm{u}'_1\mathrm{v}_1,\ \mathrm{u}'_1\mathrm{v}'_1,\ \mathrm{u}'_1\mathrm{v}_2,\ \mathrm{u}_2\mathrm{v}_1,\ \mathrm{u}_2\mathrm{v}'_1,\ \mathrm{u}_2\mathrm{v}_2\ \}.$$

5.3. **Gradings.** An absolute grading on the Bigelow generators is a map $F : \mathcal{G} \to \mathbb{Z}$. An affine grading is an equivalence class of absolute gradings under the equivalence relation $F_1 \sim F_2$ if $F_1 = F_2 + k$ for some $k \in \mathbb{Z}$. The following two definitions make sense for both absolute and affine gradings.

Definition 5.3. *A grading $F : \mathcal{G} \to \mathbb{Z}$ is called* **additive** *if it comes as a summation of gradings on $\mathcal{Z}$, i.e. there is a grading $F^* : \mathcal{Z} \to \mathbb{Z}$ and F is defined by $F(e_1, \dots, e_m) = F^*(e_1) + \cdots + F^*(e_m)$.*

Definition 5.4. *A grading $F : \mathcal{G} \to \mathbb{Z}$ is called* **stable** *if it can be expressed as a composition $\bar{F} \circ f$ for a grading $\bar{F} : \bar{\mathcal{G}} \to \mathbb{Z}$.*

Our interest lies in the absolute Jones grading from Definition 5.2, which we denote by $J : \mathcal{G} \to \mathbb{Z}$. In section 3 of [1] Bigelow indicated how to compute J from the flattened braid diagram. If we look at equation (27), we see that affinely J is minus twice the difference of two gradings Q and T which correspond to the pair $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$ describing the covering transformation. To fix Q, T and J as absolute gradings we use a distinguished element $\nu \in \mathcal{G}$. Specifically, we let $\nu_1, \dots, \nu_m \in \mathcal{Z}$ be the preimages of the m even puncture points $\mu_2, \dots, \mu_{2m} \in \tilde{\mathcal{Z}}$ under f, and set $\nu = (\nu_1, \dots, \nu_m)$. In our trefoil example, the distinguished element is $\nu = (\mathrm{u}_2\mathrm{v}_2)$.

The grading Q is defined to be additive. Let us explain what is the corresponding $Q^* : \mathcal{Z} \to \mathbb{Z}$. We start by setting $Q^*(\nu_k) = 0$ for all $k = 1, \dots, m$. Then, for every $e \in \alpha_i \cap E_j \subset \mathcal{Z}$, we set $Q^*(e)$ to be the total winding number around the $2m$ puncture points of the following loop in $\mathbb{C}$: we start at $\nu_j \in E_j$, we follow one branch of the figure eight E_j up to e, then we go along α_i to its midpoint, we follow the handle h_i down to η_i, then move to η_j along the lower half of the boundary ∂D. Next we go up the handle h_j to the midpoint of α_j and back to ν_j along α_j. This completes the loop, and since every figure eight has total winding number $+1 - 1 = 0$ with the punctures, it doesn't really matter which route on E_j we followed from ν_j to e.

Figure 9 shows the corresponding loop for $\nu_2 = \mathrm{v}_2$ and $e = \mathrm{x}_1$ in the trefoil example. The loop has total winding number five around the puncture points, so $Q^*(\mathrm{x}_1) = 5$. If we do similar computations for all the intersection points we find that $Q^*(\mathrm{y}_2) = -1$; $Q^*(\mathrm{v}_2) = Q^*(\mathrm{y}_2') = Q^*(\mathrm{u}_2) = 0$; $Q^*(\mathrm{y}_1) = 1$; $Q^*(\mathrm{u}_1) = Q^*(\mathrm{v}_1) = 2$; $Q^*(\mathrm{v}_1') = Q^*(\mathrm{x}_2) = Q^*(\mathrm{u}_1') = 3$; $Q^*(\mathrm{x}_2') = 4$; $Q^*(\mathrm{x}_1) = 5$.

This gives the Q grading on Bigelow generators:

$Q = 6$	:	$\mathrm{u}_1'\mathrm{v}_1'$	$\mathrm{x}_1\mathrm{y}_1$		
$Q = 5$	:	$\mathrm{u}_1\mathrm{v}_1'$	$\mathrm{u}_1'\mathrm{v}_1$	$\mathrm{x}_2'\mathrm{y}_1$	$\mathrm{x}_1\mathrm{y}_2'$
$Q = 4$	:	$\mathrm{u}_1\mathrm{v}_1$	$\mathrm{x}_2\mathrm{y}_1$	$\mathrm{x}_2'\mathrm{y}_2'$	$\mathrm{x}_1\mathrm{y}_2$
$Q = 3$	:	$\mathrm{u}_2\mathrm{v}_1'$	$\mathrm{u}_1'\mathrm{v}_2$	$\mathrm{x}_2\mathrm{y}_2'$	$\mathrm{x}_2'\mathrm{y}_2$
$Q = 2$	:	$\mathrm{u}_2\mathrm{v}_1$	$\mathrm{u}_1\mathrm{v}_2$	$\mathrm{x}_2\mathrm{y}_2$	
$Q = 1$	:	$-$			
$Q = 0$	:	$\mathrm{u}_2\mathrm{v}_2$			

The second grading T on $\mathcal{G}$ is not additive but is stable. The corresponding $\bar{T} : \bar{\mathcal{G}} \to \mathbb{Z}$ is defined as follows. Let $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$ be two elements of $\bar{\mathcal{G}}$. Each of them is formed from m intersections of the α and β curves. To compute the difference $\bar{T}(x) - \bar{T}(y)$ we consider the following loop in in $\mathrm{Sym}^m(D) \subset \mathrm{Sym}^m(\mathbb{C}) \cong \mathbb{C}^m$. We start at x, go along the alpha curves to y, then go back to x along the beta curves. Then $\bar{T}(x) - \bar{T}(y)$ is the linking number of this loop with the diagonal Δ in $\mathrm{Sym}^m(\mathbb{C})$, where $\mathrm{Sym}^m(\mathbb{C})$ and the diagonal are taken with their complex orientations. Once we know $\bar{T}$, we get T by composing with the natural map $f : \mathcal{G} \to \bar{\mathcal{G}}$. To fix T as an absolute grading, we set $T = 0$ on the distinguished element ν.

In practice, the linking number with the diagonal records the twisting of the points around each other. For example, a half twist such as the one between $\bar{\mathrm{x}}_1\bar{\mathrm{y}}_1$ and $\bar{\mathrm{u}}_1\bar{\mathrm{v}}_1$ in Figure 8

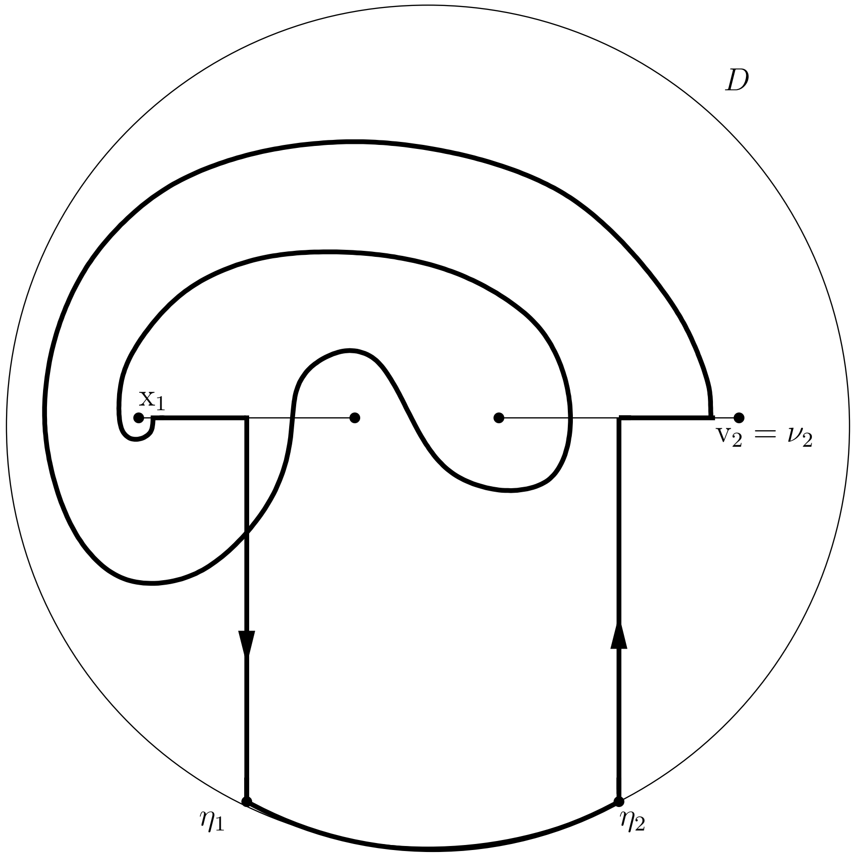

FIGURE 9. Loop used to compute the Q^* grading for x_1.

gives a difference of 1 in their $\bar{T}$ gradings. In fact there are enough half twists of this form in Figure 8 to relate any two elements of $\bar{\mathcal{G}}$. Thus it is easy to write down the T grading of the Bigelow generators by calculating $\bar{T}$ and then composing with f. We obtain:

$T = 3$	:	x_1y_1							
$T = 2$	:	u_1v_1'	$u_1'v_1$	u_1v_1'	$u_1'v_1'$				
$T = 1$	:	x_2y_1	$x_2'y_1$	x_1y_2	x_1y_2'	x_2y_2	$x_2'y_2$	x_2y_2'	$x_2'y_2'$
$T = 0$	:	u_2v_1	u_2v_1'	u_1v_2	$u_1'v_2$	u_2v_2			

Looking back at (27), we get that the T and Q gradings determine J by the formula $J = 2(T - Q) + m + w$. Doing this computation for the trefoil, and taking into account the factor $m + w = 2 - 3 = -1$, we get:

$J = -1$	:	u_2v_2							
$J = -3$	:	x_2y_2							
$J = -5$	:	u_1v_1	u_1v_2	u_2v_1	$x_2'y_2$	x_2y_2'			
$J = -7$	:	u_2v_1'	$u_1'v_2$	u_1v_1'	$u_1'v_1$	x_1y_1	x_1y_2	$x_2'y_2'$	x_2y_1
$J = -9$	:	$u_1'v_1'$	x_1y_2'	$x_2'y_1$					

5.4. **The sign.** Now that we understand J, let us turn our attention to the sign $\sigma : \mathcal{G} \to \{\pm 1\}$ appearing in the formula (28). Looking at (27), we see that σ has a contribution $(-1)^{b+m}$ from the T grading and the number of strands, and a contribution from comparing the orientations in the intersection product. The sign of an intersection point $\gamma \in \mathcal{G}$ is also composed of two parts σ' and σ''. The sign $\sigma' = (-1)^b$ represents the parity of the permutation of $\{1, \ldots, m\}$ associated to the Bigelow generator γ. The second factor σ'' is obtained by giving orientations to the figure eights, checking whether the figure eight hits the alpha curve from above or from below at each point in $\mathcal{Z}$, and then multiplying the local intersection signs of all the m points that form γ. In [1] the alpha curves are oriented from left to right, and the figure eights are oriented so that they hit the alpha curves from above at the endpoints ν_j. This means that $\sigma''(\nu) = (-1)^m$.

Putting these together we get the formula:

$$\sigma(\gamma) = (-1)^{b+m}\sigma'(\gamma)\sigma''(\gamma) = (-1)^m\sigma''(\gamma). \tag{30}$$

Note that this overall sign σ is always $+$ on the distinguished element ν.

6. Floer cohomology and a bijective correspondence

In this section we prove Theorem 1.3 and define a new grading on the Bigelow generators. Some care needs to be taken about the transversality of intersections.

6.1. Clean intersections and Floer cochains.

We work with the two Lagrangians

$$\mathcal{K} = \Sigma_{\alpha_1} \times \Sigma_{\alpha_2} \times \cdots \times \Sigma_{\alpha_m}\ ,\ \mathcal{K}' = \Sigma_{\beta_1} \times \Sigma_{\beta_2} \times \cdots \times \Sigma_{\beta_m} \subset Y = \mathcal{Y}_{m,\tau}$$

from the statement of Theorem 1.2. We would like to describe a set of generators for the Floer cochain $CF^*(\mathcal{K}, \mathcal{K}')$.

However, the intersection $\mathcal{N} = \mathcal{K} \cap \mathcal{K}'$ is not transverse, and we will need to isotope one of the Lagrangians in order to achieve transversality. Let us understand the structure of $\mathcal{N}$. The 2-spheres Σ_{α_i} are Lagrangians in $S_\tau = (u^2 + v^2 + P_\tau(z) = 0)$ and they map to the corresponding alpha curve under the projection to the $z \in \mathbb{C}$ coordinate. Similarly Σ_{β_j} map to the beta curves. Therefore the intersection $\Sigma_{\alpha_i} \cap \Sigma_{\beta_j}$ is a disjoint union of circles and points: each point $x \in \alpha_i \cap \beta_j \subset \bar{\mathcal{Z}}$ contributes a point e_x to $\Sigma_{\alpha_i} \cap \Sigma_{\beta_j}$ if $x = \mu_k$ for some k and a circle S^1_x otherwise. It follows that the intersection $\mathcal{N} = \mathcal{K} \cap \mathcal{K}'$ consists of a disjoint union of tori $T^k = S^1 \times \cdots \times S^1$ of various dimensions, one for each point in $\bar{\mathcal{Z}}$. Usually not all tori are trivial, so the intersection is not transverse.

In order to control the isotopy necessary for transversality, we observe that $\mathcal{N} = \mathcal{K} \cap \mathcal{K}'$ is a *clean intersection* in the sense of Pozniak [25], i.e. $T\mathcal{N} = (T\mathcal{K}|_{\mathcal{N}}) \cap (T\mathcal{K}'|_{\mathcal{N}})$. A model for isotoping a Lagrangian in a neighborhood V of a clean intersection is due to Weinstein [33] and was used by Khovanov and Seidel in the proof of Proposition 5.15 in [12]. Basically, if one starts with a Morse-Smale function g on $\mathcal{N}$, then $\mathcal{K}'$ can be isotoped into a Lagrangian $\mathcal{K}''$ which is identical to $\mathcal{K}'$ outside V and intersects $\mathcal{K} \cap V$ exactly at the critical points of g. Moreover, given two critical points x, y of g, the difference in their cohomological grading in $CF^*(\mathcal{K}, \mathcal{K}'')$ is the same as the difference in their Morse indices on $\mathcal{N}$.

We apply this to our case. Take the standard height function on every circle S^1_x with two critical points: a minimum e_x and a maximum e'_x. Deform the Lagrangian spheres Σ_{β_j} into Σ''_{β_j} accordingly and set

$$\mathcal{K}'' = \Sigma''_{\beta_1} \times \Sigma''_{\beta_2} \times \cdots \times \Sigma''_{\beta_m} \subset Y = \mathcal{Y}_{m,\tau}.$$

This makes the intersection $\mathcal{N}'' = \mathcal{K} \cap \mathcal{K}''$ transverse. The intersection points in $\Sigma_{\alpha_i} \cap \Sigma''_{\beta_j}$ are now all of the form e_x, e'_x, exactly in 1-to-1 correspondence with the elements of $\mathcal{Z}$. In turn this induces a 1-to-1 correspondence between the points in $\mathcal{N}''$ and the Bigelow generators. Thus we can think of $CF^*(\mathcal{K}, \mathcal{K}') = CF^*(\mathcal{K}, \mathcal{K}'')$ as being generated by the elements of $\mathcal{G}$. This proves Theorem 1.3.

6.2. **The projective grading.** Recall from section 3.6 that the Floer cochain complex $CF^*(\mathcal{K},\mathcal{K}')$ is absolutely $\mathbb{Z}$-graded.

Definition 6.1. *We denote by $\tilde{P}$ the cohomological grading on Bigelow generators, induced by the identification in Theorem 1.3. We also set $P = \tilde{P} - m - w$.*

The renormalization by $m+w$ is made to account for the shift in degree in Theorem 3.5. Of course, P and $\tilde{P}$ are identical as affine gradings. We would like to understand P in terms of a flattened braid diagram.

To define the absolute grading we started with complex volume forms for the slice $\mathcal{S}_m$ and the base $\mathrm{Sym}_0^{2m}(\mathbb{C})$, which gave us a family of forms on $\mathcal{Y}_{m,\tau}$ for $\tau \in \mathrm{Conf}_0^{2m}(\mathbb{C})$. We want to specify the volume form on the open sets $U_{m,\tau} \cap \mathcal{Y}_{m,\tau}$ from Corollary 2.9, which we can identify with open subsets (21) in $\mathrm{Sym}^m(S_\tau - \Delta)$ using the Hilbert-Chow morphism π.

We begin by considering the standard volume form $\omega = (dv \wedge dz)/2u$ on the hypersurface $S_\tau \in \mathbb{C}^3$ given by the equation $u^2 + v^2 + P_\tau(z) = 0$. Then we use the following lemma, which is subsumed in the proof of Theorem 1.17 in [19]. The argument there is in fact due to Beauville [2].

Lemma 6.2. *Let ω be a complex volume form on a smooth surface S. Clearly ω^m is a complex volume form on the Cartesian product $S^{\times m}$ invariant under the action of the symmetric group, and therefore descends to a complex volume form ω_m on $\mathrm{Sym}^m(S) - \Delta$.*

Then there is an extension of the pullback $\pi^*\omega_m$ from $\pi^{-1}(\mathrm{Sym}^m(S) - \Delta)$ to a complex volume form on the whole Hilbert scheme $\mathrm{Hilb}^m(S)$.

Taking $S = S_\tau$ as τ varies over $\mathrm{Conf}_0^{2m}(\mathbb{C})$ we get a family of volume forms on the Hilbert schemes $\mathrm{Hilb}^m(S_\tau)$, and by restriction one on the family of $\mathcal{Y}_{m,\tau} \subset \mathrm{Hilb}^m(S_\tau)$. We can use these volume forms to define the absolute grading. On the open set (21), in terms of the coordinates u_j, v_j, z_j, the volume form is

$$\Theta = \prod_{j=1}^{m} \frac{dv_j \wedge dz_j}{2u_j}. \tag{31}$$

At a point $x \in \mathcal{K} = \Sigma_{\alpha_1} \times \cdots \times \Sigma_{\alpha_m}$ the coordinates u_j, v_j, z_j satisfy $z_j = \alpha_j(t_j)$ for some $t_j \in [0,1]$ and $u_j, v_j \in \sqrt{-P_\tau(\alpha_j(t_j))}\mathbb{R}$. Hence the resulting square phase map (17) on the Lagrangian $\mathcal{K}$ is given by

$$\theta_{\mathcal{K}} : \mathcal{K} \to \mathbb{C}/\mathbb{R}_+,\ \theta_{\mathcal{K}}(x) = \prod_{j=1}^{m} \frac{-P_\tau(\alpha_j(t_j)) \cdot \alpha'(t_j)^2}{-P_\tau(\alpha_j(t_j))} = \prod_{j=1}^{m} \alpha_j'(t_j)^2. \tag{32}$$

This is of course constant because the alpha curves are horizontal. On the other hand, the square phase map on $\mathcal{K}'$ is

$$\theta_{\mathcal{K}'} : \mathcal{K}' \to \mathbb{C}/\mathbb{R}_+,\ \theta_{\mathcal{K}'}(x) = \prod_{j=1}^{m} \beta_j'(t_j)^2, \tag{33}$$

which is nontrivial. Equation (33) and the standard additivity properties of the Morse index imply that the $\tilde{P}$ grading on the Bigelow generators is additive in the sense of Definition 5.3 and can be computed from the flattened braid diagram. The corresponding grading $\tilde{P}^* : \mathcal{Z} \to \mathbb{Z}$ on the intersection points is described by the phase map $x = \beta(t) \to \beta'(t)^2$ on the beta curves in the plane.

This is the projective grading considered by Khovanov and Seidel in [12, section 3d]. If we take into account the difference in grading of 1 between e_x' and e_x coming from the

Morse-Smale function on the clean intersections, we get a very simple description of the affine grading $\tilde{P}^* = P^*$ in terms of Figure 6, where we replaced the beta curves by figure-eights. Specifically, we assume that all the figure eights intersect the alpha curves at 90° angles. Each figure eight is the image of an immersion $\gamma_j : S^1 \to \mathbb{C}$. We get a map

$$\varepsilon_j : S^1 \to S^1, \ t \to \gamma_j'(t)^2/|\gamma_j'(t)|^2.$$

This map has degree 0 so it can be lifted to a real valued map $\tilde{\varepsilon}_j : S^1 \to \mathbb{R}$. We choose the lift so that $\tilde{\varepsilon}_j(t) = 0$ when $\gamma_j(t)$ is the puncture point μ_{2j}. Then $\varepsilon_j(t)$ is an integer whenever $\gamma_j(t) \in \mathcal{G}$, and that integer is its $\tilde{P}^*$ grading.

In our trefoil example we have $\tilde{P}^*(\mathrm{u}_2) = \tilde{P}^*(\mathrm{v}_2) = \tilde{P}^*(\mathrm{y}_2) = 0$; $\tilde{P}^*(\mathrm{y}_2') = 1$; $\tilde{P}^*(\mathrm{x}_2) = \tilde{P}^*(\mathrm{v}_1) = \tilde{P}^*(\mathrm{u}_1) = \tilde{P}^*(\mathrm{y}_1) = 2$; $\tilde{P}^*(\mathrm{x}_2') = \tilde{P}^*(\mathrm{v}_1') = \tilde{P}^*(\mathrm{u}_1') = 3$ and $\tilde{P}^*(\mathrm{x}_1) = 4$.

We can add these together and get the $\tilde{P}$ grading on $\mathbb{Z}$. Taking into account the factor $m + w = 2 - 3 = -1$ we can list the P grading on Bigelow generators:

$P = 7$	:	$\mathrm{u}_1'\mathrm{v}_1'$	$\mathrm{x}_1\mathrm{y}_1$		
$P = 6$	:	$\mathrm{u}_1\mathrm{v}_1'$	$\mathrm{u}_1'\mathrm{v}_1$	$\mathrm{x}_2'\mathrm{y}_1$	$\mathrm{x}_1\mathrm{y}_2'$
$P = 5$	:	$\mathrm{u}_1\mathrm{v}_1$	$\mathrm{x}_2\mathrm{y}_1$	$\mathrm{x}_2'\mathrm{y}_2'$	$\mathrm{x}_1\mathrm{y}_2$
$P = 4$	:	$\mathrm{u}_2\mathrm{v}_1'$	$\mathrm{u}_1'\mathrm{v}_2$	$\mathrm{x}_2\mathrm{y}_2'$	$\mathrm{x}_2'\mathrm{y}_2$
$P = 3$	:	$\mathrm{u}_2\mathrm{v}_1$	$\mathrm{u}_1\mathrm{v}_2$	$\mathrm{x}_2\mathrm{y}_2$	
$P = 2$	:	$-$			
$P = 1$	:	$\mathrm{u}_2\mathrm{v}_2$			

Remark 6.3. *In section 5.4 we described the sign $\sigma = (-1)^m \sigma''$ appearing in front of a Bigelow generator in the formula (28) for the Jones polynomial. It is now straightforward to see that:*

$$\sigma = (-1)^{\tilde{P}} = (-1)^{P+m+w}.$$

6.3. **The $P - Q$ grading.** The P and Q gradings are additive but not stable. If we look at the corresponding tables for our trefoil example we may get the impression that P and Q are identical as affine gradings. This is just a coincidence. If we replaced the beta curve with a figure eight on the right hand side of Figure 4, for example, we would get several Bigelow generators for the unknot that have the same Q grading but different projective gradings.

Nevertheless, we can still say something about the difference $P - Q$ for any diagram:

Lemma 6.4. *The grading $P - Q : \mathcal{G} \to \mathbb{Z}$ is stable in the sense of Definition 5.4.*

Proof. Since P and Q are additive it suffices to show that $(P^* - Q^*)(e_x) = (P^* - Q^*)(e_x')$ for every $x \in \mathcal{Z}$ which is not a puncture. But this follows from the fact that $P^*(e_x') - P^*(e_x) = Q^*(e_x') - Q^*(e_x) = 1$ for any such x. □

6.4. **Comparison with Khovanov cohomology.** In the introduction we mentioned the conjecture made by Seidel and Smith about the equivalence between their theory and the cohomology theory of Khovanov [9]:

$$Kh^k_{symp}(L) \cong \bigoplus_{i-j=k} Kh^{i,j}(L) \quad (?) \tag{34}$$

The grading j in Khovanov cohomology is the "Jones grading" which describes which coefficient in the unnormalized Jones polynomial comes from the Euler characteristic:

$$\sum_{i,j\in\mathbb{Z}} (-1)^i q^j \dim(Kh^{i,j} \otimes \mathbb{Q}) = J_L(q).$$

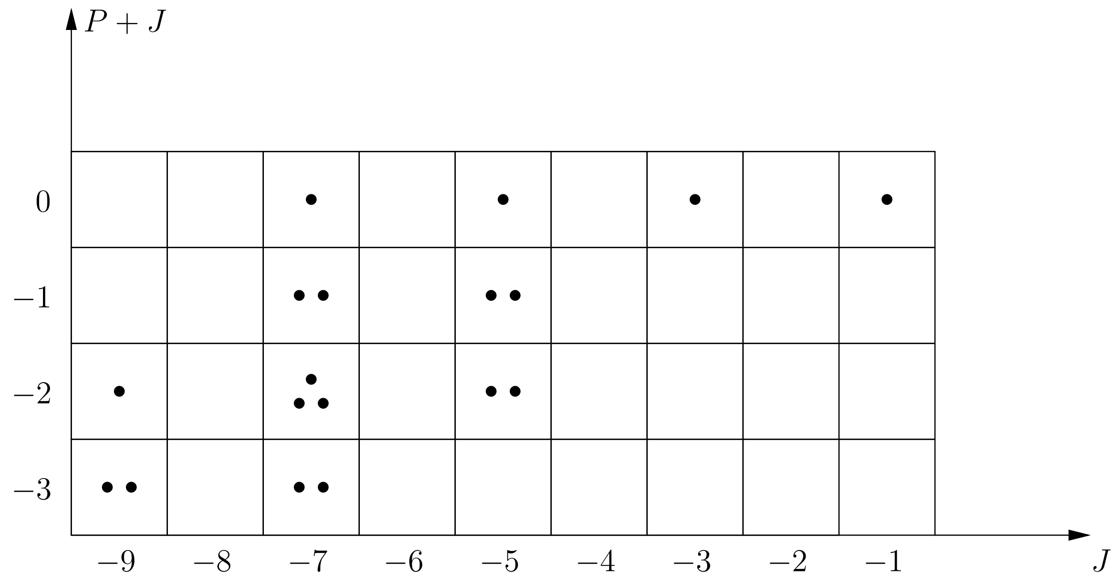

FIGURE 10. Bigelow generators for the trefoil.

The grading i describes the cohomological degree, while k on the Seidel-Smith side of (34) is the projective grading P, which is supposed to correspond to $i-j$.

Let us plot the Bigelow generators in our example for the left-handed trefoil, with the J and $P+J$ grading on the axes. The result is Figure 10, where each dot represents a generator.

Let us also plot the Khovanov cohomology of the trefoil, calculated in [9, section 7]. We get Figure 11.

Note that Conjecture (34) was verified for the trefoil by Seidel and Smith in [30]. They computed Kh^*_{symp} of the trefoil to be $(\mathbb{Z}, 0, \mathbb{Z}^2, 0, \mathbb{Z}/2, \mathbb{Z})$ in degrees from 1 to 6 respectively, and 0 in all other degrees.

Let us compare Figures 10 and 11. We know that the Seidel-Smith cochain complex is generated by the dots in Figure 10, and that the differentials have to increase the P grading by 1. Although we do not deal with differentials in this paper, let us observe that there is enough room for them to produce the abelian groups in Figure 11, even assuming that they preserve the J grading. This lends support to the conjecture that the bigrading (P, J) on Bigelow generators should descend to a bigrading on Kh^*_{symp} similar to that on Kh^*, with the Jones grading J playing the role of j.

7. Heegaard Floer homology of the double branched cover

Seidel and Smith ([30], [31]) have considered an interesting involution acting on the manifold $\mathcal{Y}_{m,\tau}$ and described how this could be used to relate their theory to the Heegaard Floer homology of $\mathcal{D}(L)$, the double cover of S^3 branched over L. In this section we explore this relation in terms of the Bigelow generators.

7.1. **An involution.** On the Milnor fiber S_τ given by the equation $u^2+v^2+P_\tau(z)=0$ there is an involution

$$(u, v, z) \to (u, -v, z). \tag{35}$$

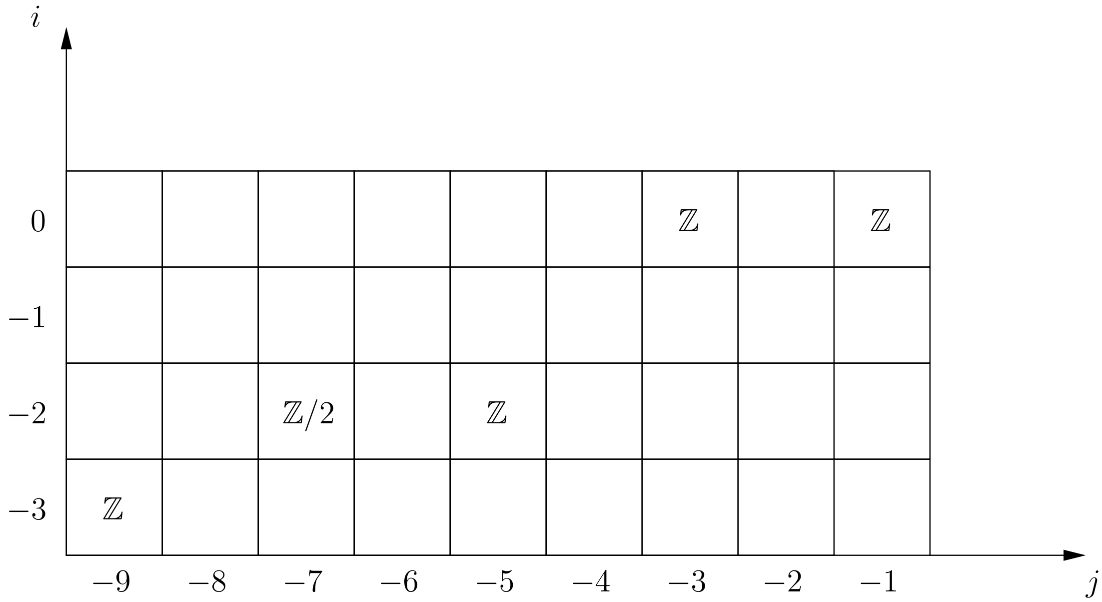

FIGURE 11. Khovanov cohomology of the trefoil.

Its fixed point set $\hat{S}_\tau$ is the affine complex curve in $\mathbb{C}^2$ with equation $u^2 + P_\tau(z) = 0$. The map $\hat{S}_\tau \to \mathbb{C}, (u, z) \to z$ presents $\hat{S}_\tau$ as a double branched cover of the plane, with branch points at the $2m$ components of τ.

The involution (35) induces one on the Hilbert scheme $\mathrm{Hilb}^m(S_\tau)$. We denote it by σ. The fixed point set of σ parametrizes the closed 0-dimensional subschemes in S_τ which are invariant under the involution. In particular $\mathrm{Fix}(\sigma)$ contains $\mathrm{Hilb}^m(\hat{S}_\tau)$, which by Fact 2.4 is the same as the symmetric product $\mathrm{Sym}^m(\hat{S}_\tau)$.

Theorem 1.1 presents $\mathcal{Y}_{m,\tau}$ as an open subset of $\mathrm{Hilb}^m(S_\tau)$ via the morphism (6). The involution σ maps $\mathcal{Y}_{m,\tau}$ to itself. In terms of the polynomials $A(t), U(t), V(t)$ appearing in (6) we have

$$\sigma : (A(t), U(t), V(t)) \to (A(t), U(t), -V(t)).$$

Going back to (4) and (2) we can infer the effect of σ on the polynomials $A(t)$, $B(t)$, $C(t), D(t)$ which record the coefficients of a matrix in the slice $\mathcal{S}_m$:

$$(A(t), B(t), C(t), D(t)) \;\to\; (A(t), C(t), B(t), D(t)).$$

This is exactly the involution considered by Seidel and Smith [31].

The following result appears in [31]:

Proposition 7.1. *The fixed point set of* $\sigma|_{\mathcal{Y}_{m,\tau}}$ *is the complement in* $\mathrm{Sym}^m(\hat{S}_\tau)$ *of the anti-diagonal*

$$\nabla = \{(u_k, z_k),\ k = 1, \dots, m :\ u_k^2 + P_\tau(z_k) = 0,\ (u_i, z_i) = (-u_j, z_j) \textit{ for some } i \neq j\}.$$

Proof. Looking at equation (2), the fixed point set is given by $B(t) = C(t)$, so its points can be described as a triad of polynomials $A(t), B(t), D(t)$ satisfying

$$A(t)D(t) - B(t)^2 = P_\tau(t), \tag{36}$$

with $B(t)$ of degree $m-1$, $A(t)$ and $D(t)$ monic of degree m and such that the coefficients of t^{m-1} in $A(t)$ and $D(t)$ sum up to zero.

Recall that the embedding $j : \mathcal{Y}_{m,\tau} \hookrightarrow \mathrm{Hilb}^m(S_\tau)$ from (6) is given by taking the roots z_k of $A(t)$ and then setting $u_k = B(z_k)$ for all $k = 1, \ldots, m$. It follows that the image of $\mathrm{Fix}(\sigma|_{\mathcal{Y}_{m,\tau}})$ under this embedding lies in $\mathrm{Sym}^m(\hat{S}_\tau)$. To find the image, pick a point $x \in \mathrm{Sym}^m(\hat{S}_\tau)$ given by coordinates $(u_k, z_k), k = 1, \ldots, m$. This lies in the image of j if and only if we can find a polynomial $B(t)$ of degree $\leq m-1$ such that $B(z_k) = u_k$ for all k. The necessary and sufficient condition for the existence of B is that u_i must equal u_j whenever $z_i = z_j$. This is the same as saying that x is not on the anti-diagonal. □

Let us turn our attention to the Lagrangians $\mathcal{K}$ and $\mathcal{K}'$ from Theorem 1.2. Note that the Lagrangian 2-spheres Σ_{α_k}, are preserved by (35) and the resulting fixed point sets are the simple closed curves

$$\hat{\alpha}_k = \{(u, z) \in \mathbb{C}^2 : z = \alpha_k(t) \text{ for some } t \in [0, 1];\ u = \pm\sqrt{-P_\tau(z)}\} \subset \hat{S}_\tau. \tag{37}$$

The same holds true for the beta curves and gives a set of other m simple closed curves $\hat{\beta}_k,\ k = 1, \ldots, m$ on $\hat{S}_\tau$.

Consequently, we have:

Proposition 7.2. *The Lagrangians $\mathcal{K}, \mathcal{K}' \subset \mathcal{Y}_{m,\tau} \subset \mathrm{Hilb}^m(S_\tau)$ are mapped into themselves by the involution σ. The fixed point sets of σ restricted to $\mathcal{K}$ and $\mathcal{K}'$ are the tori*

$$\mathbb{T}_{\hat{\alpha}} = \hat{\alpha}_1 \times \hat{\alpha}_2 \times \cdots \times \hat{\alpha}_m;\ \mathbb{T}_{\hat{\beta}} = \hat{\beta}_1 \times \hat{\beta}_2 \times \cdots \times \hat{\beta}_m \subset \mathrm{Sym}^m(\hat{S}_\tau) - \nabla,$$

respectively.

7.2. **Heegaard Floer homology.** Heegaard Floer homology is a powerful tool in low-dimensional topology introduced by Ozsváth and Szabó in [22], [23]. We will be interested in only one aspect of their theory, namely the invariant $\widehat{HF}$ of 3-manifolds. We will work with cohomology so that we are consistent with our previous conventions.

Let M be a closed, oriented 3-manifold. Then M can be described by a *Heegaard diagram*, i.e. a triple $(\Sigma, \hat{\alpha}, \hat{\beta})$ with $\Sigma = \Sigma_g$ a closed oriented surface of genus g and $\hat{\alpha} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_g)$, $\hat{\beta} = (\hat{\beta}_1, \ldots, \hat{\beta}_g)$ two collections of g simple closed curves on Σ, such that the g curves in each collection are linearly independent in $H_1(\Sigma; \mathbb{Z})$ and disjoint from the other curves in the same collection.

Given a Heegaard diagram, we can reconstruct M with the help of a Heegaard splitting, i.e. a decomposition $M = H \cup_\Sigma H'$ into two handlebodies with oriented boundary $\partial H = \Sigma = -\partial H'$. The handlebody H is obtained from Σ by first attaching g two-handles along the $\hat{\alpha}$ curves, and then attaching one three-handle (this last step can be done in an essentially unique way). Similarly, H' is constructed from Σ by first attaching g two-handles along the $\hat{\beta}$ curves, and then attaching one three-handle.

Consider the tori

$$\mathbb{T}_{\hat{\alpha}} = \hat{\alpha}_1 \times \hat{\alpha}_2 \times \cdots \times \hat{\alpha}_g;\ \mathbb{T}_{\hat{\beta}} = \hat{\beta}_1 \times \hat{\beta}_2 \times \cdots \times \hat{\beta}_g \subset \mathrm{Sym}^g(\Sigma)$$

similar to the ones appearing in Proposition 7.2. If we pick $w \in \Sigma$ a basepoint disjoint from the $\hat{\alpha}$ and $\hat{\beta}$ curves, we can also think of $\mathbb{T}_{\hat{\alpha}}$ and $\mathbb{T}_{\hat{\beta}}$ as living in $\mathrm{Sym}^g(\Sigma - w)$. We call $(\Sigma, \hat{\alpha}, \hat{\beta}, w)$ a *pointed Heegaard diagram.*

Of course, there are many Heegaard diagrams that give rise to the same three-manifold. Ozsváth and Szabó have defined an abelian group $\widehat{HF}(M)$ by applying a version of Lagrangian Floer cohomology to the submanifolds $\mathbb{T}_{\hat{\alpha}}$ and $\mathbb{T}_{\hat{\beta}}$ in $\mathrm{Sym}^g(\Sigma - w)$. Then they proved that $\widehat{HF}(M)$ is a well-defined invariant of the three-manifold M, in the sense that it does not depend on the Heegaard diagram chosen to represent M.

Let us outline the aspects in the construction of $\widehat{HF}$ which are of interest to us. The first observation is that $\mathbb{T}_{\hat\alpha}$ and $\mathbb{T}_{\hat\beta}$ are actually not quite Lagrangians. If we have a Kähler form η on Σ, then $\eta^{\times m}$ is a Kähler form on Σ^m invariant under the action of the symmetric group. Hence it descends to a Kähler form ω_0 on $\Sigma^m - \Delta$, and the tori $\mathbb{T}_{\hat\alpha}$ and $\mathbb{T}_{\hat\beta}$ are Lagrangian for ω_0.

However, in general ω_0 cannot be extended over the diagonal Δ, so a modification of the usual construction is needed. This is done by using a class of almost complex structures J_s on $\mathrm{Sym}^g(\Sigma)$, which are chosen to tame ω_0 in a neighborhood of the tori. It follows that $\mathbb{T}_{\hat\alpha}$ and $\mathbb{T}_{\hat\beta}$ are totally real submanifolds of $(\mathrm{Sym}^g(\Sigma - w), J_s)$. Under a certain admissibility condition for the pointed Heegaard diagram (explained below), one can still count pseudo-holomorphic disks in $(\mathrm{Sym}^g(\Sigma - w), J_s)$ and get a Floer cohomology group

$$HF(\mathbb{T}_{\hat\alpha}, \mathbb{T}_{\hat\beta}) = \widehat{HF}(M).$$

The generators of the cochain complex $CF^*(\mathbb{T}_{\hat\alpha}, \mathbb{T}_{\hat\beta})$ are still given by the intersection points between $\mathbb{T}_{\hat\alpha}$ and $\mathbb{T}_{\hat\beta}$, provided that their intersection is made transverse after a small isotopy. The Floer cohomology groups $\widehat{HF}(M) = \widehat{HF}^*(M)$ admit a $\mathbb{Z}/2$ grading, given by the sign of these intersection points.

The admissibility condition for pointed Heegaard diagrams needed in the construction of $\widehat{HF}(M)$ is called *weak admissibility* in [22, Definition 4.10]. The definition there is given in terms of a decomposition of $\widehat{HF}(M)$ according to spin^c structures on M. It is then proved that for every spin^c structure $\mathfrak{s}$ on M there is a pointed Heegaard diagram which is weakly admissible with respect to $\mathfrak{s}$ and that diagram can be used to define a group $\widehat{HF}(M, \mathfrak{s})$. Then we take the direct sum of $\widehat{HF}(M, \mathfrak{s})$ over all $\mathfrak{s}$ to obtain $\widehat{HF}(M)$.

For our purposes, it is only important that a pointed Heegaard diagram is weakly admissible for all $\mathfrak{s}$ if it satisfies the requirements in the following definition:

Definition 7.3. *Let $(\Sigma, \hat\alpha, \hat\beta, w)$ be a pointed Heegaard diagram. We denote by $D_1, \dots, D_s$ the closures of the components of $\Sigma - \hat\alpha_1 - \cdots - \hat\alpha_g - \hat\beta_1 - \cdots - \hat\beta_g$, with the convention that D_s is the component containing w. We say that a two-chain $\mathcal{P} = \sum_{i=1}^{s-1} n_i D_i, n_i \in \mathbb{Z}$ is a* **periodic domain** *if its boundary is a sum of $\hat\alpha$ and $\hat\beta$ curves.*

The pointed Heegaard diagram $(\Sigma, \hat\alpha, \hat\beta, w)$ is called **admissible** *if every nontrivial periodic domain admits both negative and positive coefficients among the n_i's.*

7.3. **Double branched covers.** The collection of curves $\hat\alpha = (\hat\alpha_1, \dots, \hat\alpha_m)$ from (37) and its analogue $\hat\beta$ can be used as a set of attaching curves in a Heegaard diagram. We need to be careful though. Note that $\hat{S}_\tau$ is the complement of two points at infinity $\pm\infty$ in the closed surface Σ_{m-1}, the double cover of S^2 branched over $2m$ points. A collection of m curves needs to be put on a surface of genus m, so the cure is to add an extra handle. We do this by removing two small disks from neighborhoods of $\pm\infty$ (but such that the disks do not contain $\pm\infty$) and joining their boundaries by a handle, as shown in Figure 12. We call the resulting surface $\check{S}_\tau = \Sigma_m$.

Proposition 7.4. *$(\check{S}_\tau, \hat\alpha, \hat\beta, +\infty)$ is an admissible Heegaard diagram for the manifold $\mathcal{D}(L)$ # $(S^1 \times S^2)$, where $\mathcal{D}(L)$ indicates the double cover of S^3 branched over the link L and # denotes connected sum.*

Proof. The α curves form a crossingless matching of $2m$ points $\mu_1, \dots, \mu_{2m}$ in the complex plane $\mathbb{C}$. We add a point at infinity to $\mathbb{C}$ to obtain S^2 and think of that as the boundary of a ball B^3. We "lift" the α curves to form m segments $\alpha_1^\dagger, \dots, \alpha_m^\dagger$ in B^3 joining the

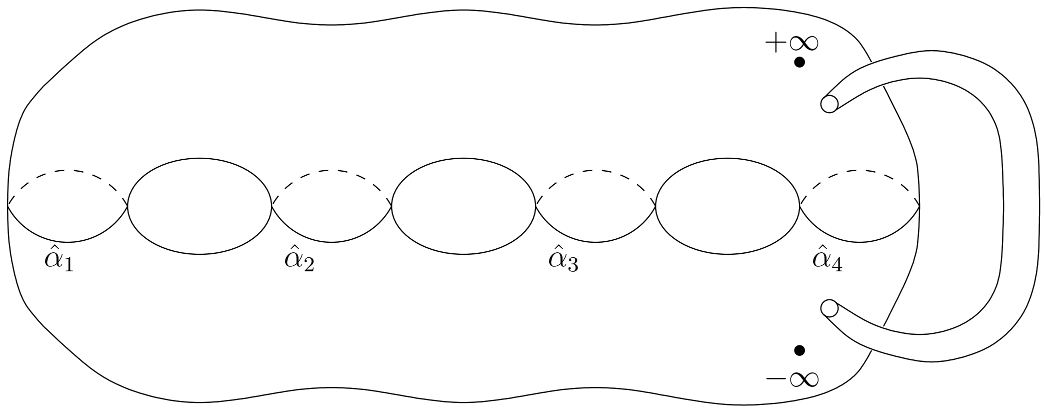

FIGURE 12. The surface $\check{S}_\tau$ for $m = 4$.

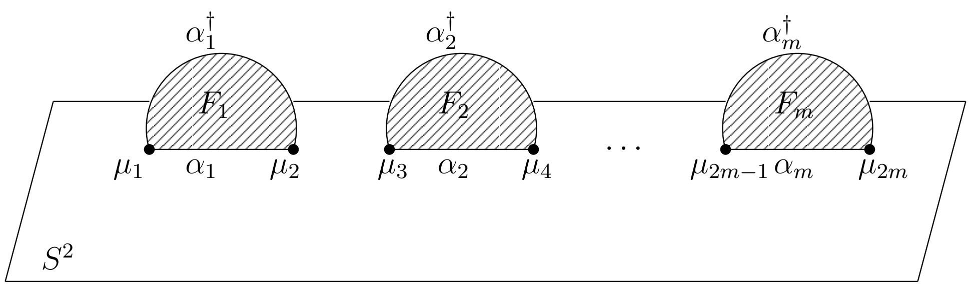

FIGURE 13. Lifting the α curves into the ball.

points μ_k in pairs just like the α's do, but such that their interiors do not intersect the boundary $\partial B^3 = S^2$. Then there are m small disks $F_1, \dots, F_m$ in B^3 with boundaries $\partial F_k = \alpha_k \cup (-\alpha_k^\dagger)$. This is shown in Figure 13.

We have a similar picture for the beta curves, with corresponding lifts $\beta_1^\dagger, \dots, \beta_m^\dagger$. We can form S^3 by joining the two balls B^3 along their common boundaries. One half contains the $\alpha^\dagger$ curves, and the other half the $\beta^\dagger$ curves. The union of the $\alpha^\dagger$ and $\beta^\dagger$ is exactly the link $L \subset S^3$.

If we take the double cover of B^3 branched over the curves $\alpha_k^\dagger$ we obtain a handlebody H_0 of genus $m-1$ with $\partial H_0 = \hat{S}_\tau$. The preimage of α_k is $\hat{\alpha}_k$, for $k = 1, \dots, m$. The homology classes $[\hat{\alpha}_k] \in H_1(\partial H_0; \mathbb{Z})$ add up to zero. Also, each curve $\hat{\alpha}_k$ bounds a disk in H_0 which is the preimage of F_k under the double covering map. Adding one extra handle to H_0 we obtain a genus m handlebody H. The picture is exactly the one in Figure 12, and the $\hat{\alpha}_k$ curves can serve as attaching circles for H.

The same construction works for the beta curves on the other half of B^3. The result is a handlebody H' with attaching circles $\hat{\beta}_k$. Gluing H and H' together along the boundary we obtain $\mathcal{D}(L)$ with one handle attached, which is $\mathcal{D}(L)\#(S^1 \times S^2)$, as desired.

We are left to check admissibility. Let $\mathcal{P} = \sum_{i=1}^s n_i D_i$ be a nontrivial periodic domain as in Definition 7.3, with $n_s = 0$ as required. The involution (35) induces an involution $i \to \bar{\imath}$ on the set $\{1, 2, \dots, s\}$, according to how a domain D_i on $\Sigma_m = \check{S}_\tau$ is taken to another domain $D_{\bar{\imath}}$. Because of our choice of the basepoint near infinity we have $\bar{s} = s$. We claim that

$$n_i + n_{\bar{i}} = 0 \tag{38}$$

for all $i = 1, \dots, s$.

Let D_i and D_j be two adjacent components, i.e. so that they have an edge E in common. The edge E could be either part of a $\hat{\alpha}$ curve or of a $\hat{\beta}$ curve. If we pick orientations on $\check{S}_\tau$ and on the $\hat{\alpha}$ and $\hat{\beta}$ curves such that the oriented boundary of D_i has an E term, then ∂D_j has a term $-E$. The involution (35) takes E to another edge $\bar{E}$, which lies on the same $\hat{\alpha}$ (or $\hat{\beta}$) curve, but comes with the opposite orientation. Note that $\partial D_{\bar{i}}$ has a term $\bar{E}$ and $\partial D_{\bar{j}}$ a term $-\bar{E}$. Since the boundary of $\mathcal{P} = \sum n_i D_i$ must be a sum of alpha and beta curves, E and $\bar{E}$ must appear with opposite signs as part of $\partial \mathcal{P}$. Therefore,

$$n_i - n_j = -n_{\bar{i}} + n_{\bar{j}}.$$

It follows that if i satisfies (38), then so does j. Since the surface $\check{S}_\tau$ is connected and we already know that $n_s = n_{\bar{s}} = 0$, we get recursively that (38) is true for all i. Since not all n_i's are zero by assumption, there must be at least one positive and one negative integer among them. This means that $(\check{S}_\tau, \hat{\alpha}, \hat{\beta}, +\infty)$ is admissible. □

A quick corollary of Proposition 7.4 is a description of a set of generators for the cochain complex $CF^*(\mathbb{T}_{\hat{\alpha}}, \mathbb{T}_{\hat{\beta}}) = \widehat{CF}^*(\mathcal{D}(L)\#(S^1 \times S^2))$. They are the intersection points in

$$\hat{\mathcal{G}} = \mathbb{T}_{\hat{\alpha}} \cap \mathbb{T}_{\hat{\beta}}.$$

In section 5.2 we denoted by $\bar{\mathcal{Z}}$ the union of all intersection points between the α and the β curves in the plane. Now let $\hat{\mathcal{Z}}$ be the union of all intersection points between the $\hat{\alpha}$ and the $\hat{\beta}$ curves in the surface $\hat{S}_\tau$, or equivalently in $\check{S}_\tau$. There is a natural map coming from the double cover

$$\hat{f} : \hat{\mathcal{Z}} \to \bar{\mathcal{Z}}. \tag{39}$$

An element $x \in \bar{\mathcal{Z}}$ has one preimage $\hat{e}_x \in \hat{\mathcal{Z}}$ under f in case it is one of the puncture points μ_k, and two preimages $\hat{e}_x, \hat{e}'_x$ otherwise. This situation is very similar to that of the map $f : \mathcal{Z} \to \bar{\mathcal{Z}}$ from (29). However, there is no canonical way of identifying the set $\{e_x, e'_x\}$ with $\{\hat{e}_x, \hat{e}'_x\}$. Hence $\mathcal{Z}$ and $\hat{\mathcal{Z}}$ have the same number of elements, but they cannot be identified in a natural way.

The set $\hat{\mathcal{G}}$ can be recovered from $\hat{\mathcal{Z}}$ similarly to how $\mathcal{G}$ and $\bar{\mathcal{G}}$ were constructed from $\mathcal{Z}$ and $\bar{\mathcal{Z}}$ in section 5.2. It follows that if we fix identifications of $\{e_x, e'_x\}$ with $\{\hat{e}_x, \hat{e}'_x\}$ for all the $x \in \mathcal{Z}$ that are not punctures, we get an identification of $\hat{\mathcal{G}}$ with the set of Bigelow generators $\mathcal{G}$.

7.4. **Grading.** The elements of $\hat{\mathcal{G}}$ form generators for $\widehat{CF}(\mathcal{D}(L)\#(S^1 \times S^2))$, and we would like to understand their cohomological grading. As explained in section 7.2, this grading is well-defined only modulo 2. However, it can be improved to a $\mathbb{Z}$ grading in the following way.

First, note that because our identification of $\hat{\mathcal{G}}$ with the set $\mathcal{G}$ of Bigelow generators was not canonical, we cannot talk about all the gradings defined in sections 5 and 6 on $\mathcal{G}$ as gradings on $\hat{\mathcal{G}}$. Nevertheless, if a grading $F : \mathcal{G} \to \mathbb{Z}$ is stable in the sense of Definition 5.4, then it comes from a grading $\bar{F}$ on $\bar{\mathcal{G}}$, and by composing with (39) we can think of F as a well-defined grading on $\hat{\mathcal{G}}$. This is the case of the gradings T and $P - Q$ (or $\tilde{P} - Q$) as seen in section 5.2 and in Lemma 6.4.

Now, instead of considering the tori $\mathbb{T}_{\hat\alpha}, \mathbb{T}_{\hat\beta}$ as embedded in $\mathrm{Sym}^m(\check{S}_\tau)$, we view them as totally real submanifolds of $W = \mathrm{Sym}^m(\hat{S}_\tau) - \nabla$ as in Proposition 7.2. Recall from the proof of Proposition 7.1 that W is an affine algebraic variety given by the equations (36) in the coefficients of A, B and D. Let us equip $W = \mathrm{Fix}(\sigma|_{\mathcal{Y}_{m,\tau}})$ with the restriction of the Kähler form $\tilde\Omega$ on $\mathcal{Y}_{m,\tau}$. We apply the formalism in section 3.6 (the totally real case) to $Y = W, \mathcal{T} = \mathbb{T}_{\hat\alpha}$ and $\mathcal{T}' = \mathbb{T}_{\hat\beta}$.

Proposition 7.5. *There exists a complex volume form Θ on W so that we can endow $\mathbb{T}_{\hat\alpha}$ and $\mathbb{T}_{\hat\beta}$ with gradings in the sense of Definition 3.4. The resulting absolute Maslov grading on the elements of $\hat{\mathcal{G}} = \mathbb{T}_{\hat\alpha} \cap \mathbb{T}_{\hat\beta}$ is $\tilde{P} - Q + T$.*

For the proof we need the following:

Lemma 7.6. *Let $w_1, \dots, w_n$ be n formal variables, and set $Q(t) = \prod(t - w_i) = t^n - s_1 t^{n-1} + \cdots + (-1)^n s_n$, so that $s_1, s_2, \dots, s_n$ are the symmetric polynomials in $w_1, \dots, w_n$. Then:*

$$ds_1 \wedge \cdots \wedge ds_n = \prod_{i<j}(w_i - w_j) \cdot dw_1 \wedge \cdots \wedge dw_n.$$

Proof. Differentiating the relation $Q(w_i) = 0$ we get $\sum_{j=1}^n (-1)^j w_i^{n-j} ds_j = -Q'(w_i) dw_i$ for all $i = 1, \dots, n$. The formula for the Vandermonde determinant gives

$$(-1)^{n(n+1)/2} \prod_{i<j}(w_i - w_j) \cdot ds_1 \wedge \cdots \wedge ds_n = (-1)^n \prod_i Q'(w_i) \cdot dw_1 \wedge \cdots \wedge dw_n. \tag{40}$$

Observe that $Q'(w_i) = \prod_{j \neq i}(w_j - w_i)$. The result now follows after dividing both sides of (40) by $(-1)^{n(n+1)/2} \prod_{i<j}(w_i - w_j)$. □

Proof of Proposition 7.5. We start with the construction of the complex volume form Θ on W. Recall that $\hat{S}_\tau$ is given by the equation $u^2 + P_\tau(z) = 0$ and this gives a set of coordinates u_j, z_j $(j = 1, \dots, m)$ on its symmetric product. Set

$$\Theta = \prod_{1 \leq i < j \leq m} (z_i - z_j) \cdot \prod_{j=1}^m \frac{dz_j}{u_j}. \tag{41}$$

Note that dz_j/u_j is a well-defined form on $\hat{S}_\tau$, and hence the expression (41) is well-defined as a form on the Cartesian product $(\hat{S}_\tau)^m$. It is invariant under the action of the symmetric group on m elements, because if we switch z_i and z_j this produces a minus sign in both of the products appearing in (41). Thus it descends to a complex m-form on $\mathrm{Sym}^m(\hat{S}_\tau)$ which clearly has no zeros or poles outside $\Delta \cup \nabla$, and therefore gives a good volume form there.

We claim that Θ extends to a volume form on all of $W = \mathrm{Sym}^m(\hat{S}_\tau) - \nabla$. We need to check this for points $x \in \Delta - \nabla$. A point of this type is an m-tuple of pairs (u_j, z_j) with $u_j^2 = -P_\tau(z_j) \neq 0$ for all $j = 1, \dots, m$, and $(u_i, z_i) = (u_j, z_j)$ for some $i \neq j$. The point x determines a partition of $\{1, \dots, m\}$ into blocks of the form $\{j_1, \dots, j_n\}$ such that $z_{j_1} = \cdots = z_{j_n}$ and the z_j's are different for j in different blocks. Putting together the symmetric functions $s_1, \dots, s_n$ in $z_{j_1}, \dots, z_{j_n}$ for each block we form a set of local coordinates on W around x. Using Lemma 7.6 and the fact that $u_j \neq 0$ we get that Θ is a nonzero multiple of the product of all $ds_1 \wedge \cdots \wedge ds_n$, taken over all blocks and in a neighborhood of x. Thus Θ extends to a well-defined complex volume form near any $x \in W$.

Now we can do a computation similar to that in section 6.2. A point $x \in \mathbb{T}_{\hat{\beta}}$ has coordinates (u_j, z_j), with $z_j = \beta_j(t_j)$ for some $t_j \in [0,1]$ and $u_j = \pm\sqrt{-P_\tau(\beta_j(t_j))}$. The resulting square phase map (17) on $\mathbb{T}_{\hat{\beta}}$ is

$$\theta_{\mathbb{T}_{\hat{\beta}}} : \mathbb{T}_{\hat{\beta}} \to \mathbb{C}^*/\mathbb{R}_+, \ \theta_{\mathbb{T}_{\hat{\beta}}}(x) = \prod_{1 \leq i < j \leq m} \big(\beta_i(t_i) - \beta_j(t_j)\big)^2 \cdot \prod_{j=1}^{m} \frac{\beta_j'(t_j)^2}{-P_\tau(\beta_j(t_j))}.$$

Since $H^1(\mathbb{T}_{\hat{\beta}})$ is nontrivial, we did not know *a priori* that the square phase map lifted to a real-valued function. Now we know that this is actually the case, because it factors through the projection to the contractible space $\beta_1 \times \cdots \times \beta_m$.

There is a similar square phase map for $\mathbb{T}_{\hat{\alpha}}$, and that turns out to be null-homotopic because the α curves are horizontal. Thus, we can endow both $\mathbb{T}_{\hat{\alpha}}$ and $\mathbb{T}_{\hat{\beta}}$ with gradings, i.e. with lifts of the square phase maps to $\tilde{\theta}_{\mathbb{T}_{\hat{\alpha}}} : \mathbb{T}_{\hat{\alpha}} \to \mathbb{R}, \tilde{\theta}_{\mathbb{T}_{\hat{\beta}}} : \mathbb{T}_{\hat{\beta}} \to \mathbb{R}$. We choose them just like in the Seidel-Smith picture (section 3.6), i.e. we obtain the grading on $\mathbb{T}_{\hat{\beta}}$ from that on $\mathbb{T}_{\hat{\alpha}}$ by following continuously the family of crossingless matchings in the plane determined by the braid $b \times 1^m$.

As explained in section 3.6, this induces an absolute Maslov grading $\tilde{R}(x) \in \mathbb{Z}$ on the points $x \in \mathbb{T}_{\hat{\alpha}} \cap \mathbb{T}_{\hat{\beta}}$. To make it more explicit, recall that in the standard picture from section 3.5 all alpha curves are subsets of the real line. We can assume that they always intersect the beta curves at 90 degree angles. Also, at the endpoints μ_j the beta curves need to be parametrized so that their derivatives vanish. Then $\theta_{\mathbb{T}_{\hat{\beta}}}(x) = -1 \in S^1$ for every $x \in \mathbb{T}_{\hat{\alpha}} \cap \mathbb{T}_{\hat{\beta}}$. Hence $\tilde{\theta}_{\mathbb{T}_{\hat{\beta}}}(x)$ is an odd integer. If the lift $\tilde{\theta}_{\mathbb{T}_{\hat{\beta}}}$ is chosen so that the special point $(\mu_2, \ldots, \mu_{2m})$ is mapped to 1, then the Maslov index $\tilde{R}(x)$ is equal to $k \in \mathbb{Z}$ at the points where $\tilde{\theta}_{\mathbb{T}_{\hat{\beta}}}(x) = 2k+1$.

Just like in section 6.2, we get that the resulting Maslov grading can be computed from a flattened braid diagram. We use the standard additivity properties of the index. The factor $\prod\big(\beta_i(t_i) - \beta_j(t_j)\big)^2$ counts the twisting of the points around each other, which is the T grading.

The second factor produces a grading which is clearly both additive and stable. We decompose it into contributions coming from

$$f_j(t_j) = \frac{\beta_j'(t_j)^2}{-P_\tau(\beta_j(t_j))} \tag{42}$$

for each j. The value of the respective contribution $C^*(p)$ at a point $p = \beta_j(t_j) \in \alpha_i \cap \beta_j$ can be computed as follows. At the endpoint μ_{2j} of β_j we have $C^*(\mu_{2j}) = 0$. We then follow the curve β_j in reverse until we hit p, and look at the lift $\tilde{f}_j$ of f_j to a real-valued function. Then $C^*(p)$ describes on which sheet we end up, i.e. it equals half the difference $f_j(t_j) - f_j(1)$. It is a bit hard to see the result because at the endpoints of β both the numerator and the denominator in (42) are zero. However, the Maslov index is invariant under deformation, so we can replace β_j with one half H_j of the corresponding figure-eight E_j. (It does not matter which of the two halves we choose.) H_j is an arc ending at ν_j near μ_{2j} and starting at the intersection point near the other endpoint of β_j. Now C^* decomposes into a contribution from the numerator $H_j'(t_j)^2$, which is the projective grading $\tilde{P}^*$, and one from the denominator $(-1)^m \prod P_\tau(H_j(t_j))$, which records the twisting around the puncture points and hence gives the Q^* grading. Summing up all these contributions we get $\tilde{R} = \tilde{P} - Q + T$, as desired. $\square$

Proof of Theorem 1.4. The identification claimed in the first part of the theorem was explained at the end of section 7.3. The second part of the theorem is basically Proposition 7.5.

Proof of Corollary 1.5. Recall that in the introduction we denoted

$$R = \tilde{R} - (m+w)/2 = P + (J/2).$$

The expression for the Jones polynomial in terms of P and R follows readily from (26), Bigelow's formula (28), and Remark 6.3. □

7.5. **Reduced theories.** Seidel and Smith suggested the involution τ as a key to a geometric interpretation of the spectral sequence in [24], which relates Khovanov homology to $\widehat{HF}(\mathcal{D}(L)\#(S^1 \times S^2))$.

The work of Ozsváth and Szabó in [24] involves the reduced Khovanov homology $\widetilde{Kh}(L)$. This was defined in [11]. It is a homology theory combinatorially defined starting with a plane diagram for the link L, just like $Kh(L)$. It is an invariant of L only when considered with $\mathbb{Z}/2$ coefficients. With $\mathbb{Z}$ coefficients $\widetilde{Kh}(L)$ depends on a distinguished component of L; in particular it still gives a well-defined invariant of knots, for example. The Euler characteristic of $\widetilde{Kh}$ is the usual Jones polynomial $V_L(t)$.

The spectral sequence in [24] is defined with $\mathbb{Z}/2$ coefficients only, has as E^2 term the reduced Khovanov homology of the mirror of L, and converges to $E^\infty = \widehat{HF}(\mathcal{D}(L);\mathbb{Z}/2)$. (Taking the mirror only has the effect of changing the signs in the bidegree of Kh and $\widetilde{Kh}$ in a well-understood manner [9], so we will not worry about it.) If we add an unlinked unknot O to L, we have $\widetilde{Kh}(L \cup O) = Kh(L)$ and, according to [23, Proposition 6.4]:

$$\widehat{HF}^*(\mathcal{D}(L\cup O)) = \widehat{HF}^*(\mathcal{D}(L)\#(S^1\times S^2)) = \widehat{HF}^*(\mathcal{D}(L)) \otimes H^*(S^1).$$

Thus we have a spectral sequence from Kh of the mirror of L to $\widehat{HF}(\mathcal{D}(L)\#(S^1 \times S^2))$, which is what we mentioned before.

It is worthwhile seeing how many of the constructions in this paper admit a "reduced" version as well. First of all, if in Proposition 7.4 we eliminate a pair of corresponding $\hat{\alpha}$ and $\hat{\beta}$ curves, say $\hat{\alpha}_m$ and $\hat{\beta}_m$, and use $\Sigma_{m-1} = \hat{S}_\tau \cup \{\pm\infty\}$ without the handle instead of $\check{S}_\tau$, then the same proof applies and we get an admissible Heegaard diagram for $\mathcal{D}(L)$. Thus we can describe a set of generators for $\widehat{HF}(\mathcal{D}(L))$.

Jacob Rasmussen pointed out to the author that there is a reduced variant of Bigelow's picture. We need a mild condition on the flattened braid diagram, namely that there is a path from μ_{2m} to infinity that does not intersect any of the α and β curves in any other point. Under this condition one can eliminate α_m and β_m from the picture, get a set of reduced Bigelow generators, compute their Q, T and J gradings as before, and obtain the Jones polynomial V_L instead of the unnormalized one J_L. (For the Q grading, one still needs to consider the total winding number around all the μ_k's, including μ_{2m-1} and μ_{2m}.)

Finally, on the Seidel-Smith side a reduced version is yet to be developed. We expect that the construction from [30] works also for the slice $\mathcal{S}_{m-1}$ instead of $\mathcal{S}_m$, and that one can define a similar Lagrangian Floer cohomology theory on $\mathcal{Y}_{m-1,\tau}$ for $\tau \in \mathrm{Conf}_0^{2m}(\mathbb{C})$. The result is likely to be a well-defined invariant for knots with $\mathbb{Z}$ coefficients, and for links with $\mathbb{Z}/2$ coefficients, as is the case for reduced Khovanov homology. The reduced Bigelow generators should then give a set of generators for the reduced Seidel-Smith cochain complex.

References

[1] S. Bigelow, *A homological definition of the Jones polynomial,* in Invariants of knots and 3-manifolds (Kyoto, 2001), 29-41, Geom. Topol. Monogr. **4**, Geom. Topol. Publ., Coventry, 2002.

[2] A. Beauville, *Variétés kähleriennes dont la première classe de Chern est nulle,* J. Diff. Geom. **18** (1983), 755-782.

[3] A. Floer, *Morse theory for Lagrangian intersections,* J. Diff. Geom. **28** (1988), 513-547.

[4] J. Fogarty, *Algebraic families on an algebraic surface,* Amer. J. Math. **90** (1968), 511-521.

[5] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Lagrangian intersection Floer theory - anomaly and obstruction,* preprint (2000), available at `www.math.kyoto-u.ac.jp/~fukaya/fukaya.html`

[6] A. Grothendieck, *Techniques de construction et theéoremes d'existence en géométrie algebrique, IV: Les schémas de Hilbert,* Sém. Bourbaki **221** (1960/61).

[7] V. Jones, *A polynomial invariant for knots via von Neumann algebras,* Bull. Amer. Math. Soc. **12** (1985), 103-111.

[8] S. Katz and C.-C. M. Liu, *Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc,* Adv. Theor. Math. Phys. **5** (2001) no.1, 1-49.

[9] M. Khovanov, *A categorification of the Jones polynomial,* Duke Math. J. **101** (2000) no. 3, 359-426.

[10] M. Khovanov, *A functor-valued invariant of tangles,* Algebr. Geom. Topol. **2** (2002), 665-741.

[11] M. Khovanov, *Patterns in knot cohomology I,* Experiment. Math. **12** (2003), no. 3, 365-374.

[12] M. Khovanov and P. Seidel, *Quivers, Floer cohomology, and braid group actions,* J. Amer. Math. Soc. **15** (2002), 203-271.

[13] M. Kontsevich, *Homological algebra of mirror symmetry,* in Proceedings of the International Congress of Mathematicians (Zürich, 1994), p. 120-139, Birkhäuser, 1995.

[14] P. B. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients,* J. Diff. Geom. **29** (1989), 665-683.

[15] P. B. Kronheimer, *Instantons and the geometry of the nilpotent variety,* J. Diff. Geom. **32** (1990), 473-490.

[16] P. B. Kronheimer and H. Nakajima, *Yang-Mills instantons on ALE gravitational instantons,* Math. Ann. **288** (1990), 263-307.

[17] R. J. Lawrence, *Homological representations of the Hecke algebra,* Comm. Math. Phys. **135** (1990), no. 1, 141-191.

[18] H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras,* Duke Math. J. **76** (1994) no. 2, 365-416.

[19] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces,* Univ. Lect. Series **18,** Amer. Math. Soc, 1999.

[20] Y.-G. Oh, *Riemann-Hilbert problem and application to the perturbation theory of analytic discs,* Kyungpook Math. J. **35** (1995) no. 1, 39-75.

[21] Y.-G. Oh, *On the structure of pseudo-holomorphic discs with totally real boundary conditions,* J. Geom. Anal. **7** (1997) no. 2, 305-327.

[22] P. Ozsváth and Z. Szabó, *Holomorphic disks and topological invariants for closed three-manifolds,* Annals of Math. **159** (2004), no. 3, 1027-1158.

[23] P. Ozsváth and Z. Szabó, *Holomorphic disks and three-manifold invariants: properties and applications,* Annals of Math. **159** (2004), no. 3, 1159-1245.

[24] P. Ozsváth and Z. Szabó, *On the Heegaard Floer homology of branched double-covers,* Adv. Math. **194** (2005), 1-33.

[25] M. Pozniak, *Floer homology, Novikov rings and clean intersections,* in Northern California Symplectic Geometry Seminar, p. 119-181, Amer. Math. Soc., 1999.

[26] J. Robbin and D. Salamon, *The Maslov index for paths,* Topology **32** (1993) no. 4, 827-844.

[27] Z. Qin and W. Wang, *Hilbert schemes of points on the minimal resolution and soliton equations,* preprint, math.QA/0404540.

[28] J. Rasmussen, *Khovanov homology and the slice genus,* preprint, math.GT/0402131.

[29] P. Seidel, *Graded Lagrangian submanifolds,* Bull. Soc. Math. France **128** (2000), 103-146.

[30] P. Seidel and I. Smith, *A link invariant from the symplectic geometry of nilpotent slices,* preprint, math.SG/0405089.

[31] I. Smith, *Symplectic geometry of the adjoint quotient I (joint with Paul Seidel),* lectures at MSRI in Spring 2004, `www.msri.org/publications/ln/msri/2004/symplecticgeom/smith/1/index.html`

[32] W. Wang, *Hilbert schemes, wreath products, and the McKay correspondence,* preprint, math.AG/9912104.

[33] A. Weinstein, *Lagrangian submanifolds and hamiltonian systems,* Annals of Math. **98** (1973), 377-410.

Department of Mathematics, Princeton University, Princeton, NJ 08540
E-mail address: cmanoles@math.princeton.edu