SKEIN LASAGNA MODULES FOR 2-HANDLEBODIES

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Abstract. Morrison, Walker, and Wedrich used the blob complex to construct a generalization of Khovanov-Rozansky homology to links in the boundary of a 4-manifold. The degree zero part of their theory, called the skein lasagna module, admits an elementary definition in terms of certain diagrams in the 4-manifold. We give a description of the skein lasagna module for 4-manifolds without 1- and 3-handles, and present some explicit calculations for disk bundles over S^2.

1. Introduction

Over the past twenty years, categorified knot invariants have been a central topic in low dimensional topology. The starting point was Khovanov's categorification of the Jones polynomial [13]. This was generalized by Khovanov and Rozansky in [18] to a sequence of link homology theories KhR_N for $N \geq 1$, where Khovanov homology corresponds to $N = 2$. Khovanov homology has been successfully used to give new, combinatorial proofs of deep results about smooth surfaces in 4-manifolds, such as the Milnor Conjecture [29] and the Thom Conjecture [22], for which the original proofs involved gauge theory [20, 21]. Furthermore, by now Khovanov homology has found its own novel topological applications, as for example in the work of Piccirillo [26, 27]. Still, compared to the invariants derived from gauge theory or Heegaard Floer homology, Khovanov homology has its limitations, due to the fact that its construction is *a priori* just for links in $\mathbb{R}^3$. In particular, a major open question is whether Khovanov or Khovanov-Rozansky homology can say something new about the classification of smooth 4-manifolds.

In [25], Morrison, Walker, and Wedrich proposed an extension of Khovanov-Rozansky homology to links in the boundaries of arbitrary oriented 4-manifolds. Specifically, they define an invariant

$$\mathcal{S}^N(W;L) = \bigoplus_{b\in\mathbb{Z}} \mathcal{S}^N_b(W;L) = \bigoplus_{b\in\mathbb{Z}} \Big(\bigoplus_{i,j\in\mathbb{Z}} \mathcal{S}^N_{b,i,j}(W;L) \Big),$$

which is a triply-graded Abelian group associated to a smooth, oriented 4-manifold W and a framed link $L \subset \partial W$. Two of the gradings (i and j) are the usual ones in Khovanov-Rozansky homology, and the third, the *blob grading* b, is new. The construction of $\mathcal{S}^N(W;L)$ starts by defining the part in blob degree zero, $\mathcal{S}^N_0(W;L)$, in a manner reminiscent to that of the skein modules of 3-manifolds. The group $\mathcal{S}^N_0(W;L)$, which we call the *skein lasagna module*, is generated by certain objects called *lasagna fillings* of W with boundary L, modulo an equivalence relation that captures the "local" cobordism relations in Khovanov-Rozansky homology. Once $\mathcal{S}^N_0(W;L)$ is defined, the higher degree groups $\mathcal{S}^N_b(W;L)$ for $b > 0$ are obtained from it using the machinery of blob homology from higher category theory [24].

It is shown in [25] that, when $W = B^4$, the invariant $\mathcal{S}^N_0(W;L)$ recovers the Khovanov-Rozansky homology $\mathrm{KhR}_N(L)$, and $\mathcal{S}^N_b(W;L) = 0$ for $b > 0$. While computing blob homology in general is rather daunting, the skein lasagna module $\mathcal{S}^N_0(W;L)$ has a relatively simple definition. Our goal here is to describe $\mathcal{S}^N_0(W;L)$ for a large class of non-trivial 4-manifolds W and links $L \subset \partial W$.

Precisely, we will be concerned with 2-*handlebodies*, that is, four-dimensional manifolds W obtained from B^4 by attaching n 2-handles. A Kirby diagram for such a manifold consists of a framed link $K \subset S^3$. For every homology class $\alpha \in H_2(W) \cong \mathbb{Z}^n$, we define the *cabled Khovanov-Rozansky*

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homology $\underline{\mathrm{KhR}}_{N,\alpha}(K)$ as the direct sum of the Khovanov-Rozansky homologies of an infinite collection of cables of K, divided by a certain equivalence relation. (The exact definition is given in Section 3.)

The skein lasagna module naturally decomposes according to the relative homology classes of lasagna fillings:

$$\mathcal{S}_0^N(W;L) = \bigoplus_{\alpha \in H_2(W,L)} \mathcal{S}_0^N(W;L,\alpha).$$

Our main result is the following.

Theorem 1.1. *Let W be the 4-manifold obtained from attaching 2-handles to B^4 along a framed n-component link K. For each $\alpha \in H_2(W) \cong \mathbb{Z}^n$, we have an isomorphism*

$$\Phi : \underline{\mathrm{KhR}}_{N,\alpha}(K) \xrightarrow{\cong} \mathcal{S}_0^N(W;\emptyset,\alpha).$$

In general, if we want to apply Theorem 1.1 to specific examples, we run into the difficulty of calculating Khovanov-Rozansky homology for an infinite family of cables. Nevertheless, we can do an explicit calculation when K is the 0-framed unknot, so that $W = S^2 \times D^2$.

Theorem 1.2. *For every $\alpha \in \mathbb{Z}$, the skein lasagna module $\mathcal{S}_0^N(S^2 \times D^2;\emptyset,\alpha)$ is a free abelian group supported in homological degree 0 and non-positive quantum degrees, with graded rank given by*

$$\sum_{j=0}^{\infty} \mathrm{rk}\, \mathcal{S}_{0,0,-j}^N(S^2 \times D^2;\emptyset,\alpha)x^j = \prod_{k=1}^{N-1} \frac{1}{1-x^{2k}}$$

In particular, for $N = 2$ (which corresponds to Khovanov homology), we have

$$\mathcal{S}_{0,0,j}^2(S^2 \times D^2;\emptyset,\alpha) = \begin{cases} \mathbb{Z} & \text{if } j=-2k,\ k \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

When $N = 2$, we also get some partial information for the p-framed unknot for $p \neq 0$. Then, W is the D^2-bundle over S^2 with Euler number p, which we denote by $D(p)$.

Theorem 1.3. *For $p > 0$ and $N = 2$, the part of the skein lasagna module of $D(p)$ that lies in class $\alpha = 0$ and homological degree 0 is*

$$\mathcal{S}_{0,0,*}^2(D(p);\emptyset,0) = 0.$$

On the other hand, for $p < 0$ we have

$$\mathcal{S}_{0,0,j}^2(D(p);\emptyset,0) = \begin{cases} \mathbb{Z} & \textit{if } \ j=0, \\ 0 & \textit{otherwise.} \end{cases}$$

While Theorem 1.1 was formulated for the case where the link $L \subset \partial W$ is empty, we can also handle the case of "local" links in ∂W, that is, those contained in a ball $B^3 \subset \partial W$. Indeed, we have the following tensor product formula for boundary connected sums. To state it, it is convenient to work with coefficients in a field $\Bbbk$, and write $\mathcal{S}^N(W;L;\Bbbk)$ for the corresponding skein lasagna module.

Theorem 1.4. *Let W_1 and W_2 be 4-manifolds with framed links $L_i \subset \partial W_i$ and let $W_1 \natural W_2$ denote their boundary connected sum along specified copies of $B^3 \subset \partial W_i$ away from the links L_i. Then,*

$$\mathcal{S}_0^N(W_1 \natural W_2; L_1 \cup L_2;\Bbbk) \cong \mathcal{S}_0^N(W_1;L_1;\Bbbk) \otimes \mathcal{S}_0^N(W_2;L_2;\Bbbk)$$

Applying Theorem 1.4 to $(W_1;L_1) = (W;\emptyset)$ and $(W_2;L_2) = (B^4;L)$, we obtain:

Corollary 1.5. *Let W be a 4-manifold and $L \subset B^3 \subset \partial W$ a framed link contained within a ball in the boundary of W. Then we have*

$$\mathcal{S}_0^N(W;L;\Bbbk) \cong \mathcal{S}_0^N(W;\emptyset;\Bbbk) \otimes \mathrm{KhR}_N(L;\Bbbk).$$

Furthermore, we can study skein lasagna modules associated to closed 4-manifolds. If the boundary of a 2-handlebody W is S^3, by attaching a ball we obtain a simply connected smooth 4-manifold X. We can associate to X the skein lasagna module $\mathcal{S}_0^N(X;\emptyset)$.

Proposition 1.6. *If X a closed smooth 4-manifold, then*

$$\mathcal{S}_0^N(X;\emptyset) \cong \mathcal{S}_0^N(X \setminus B^4;\emptyset).$$

In particular, we see that $\mathcal{S}_0^N(S^4;\emptyset) \cong \mathbb{Z}$.

It is an open question [19, Problem 4.18] whether every closed, smooth, simply connected four-manifold X admits a perfect Morse function or, equivalently, a Kirby diagram without 1-handles or 3-handles; i.e., whether $X \setminus B^4$ is a 2-handlebody. In practice, many four-manifolds are known to be of this form. The list of such manifolds include:

- $\mathbb{CP}^2$ and $S^2 \times S^2$;
- The elliptic surfaces $E(n)$, including the K3 surface $E(2)$; see for example [9, Figure 8.15];
- More generally, the log transforms $E(n)_p$; see [9, Corollary 8.3.17];
- The Dolgachev surface $E(1)_{2,3}$ and a few other elliptic surfaces of the form $E(n)_{p,q}$; see [1], [33];
- Smooth hypersurfaces in $\mathbb{CP}^3$; see for example [2, Section 12.3];
- The cyclic k-fold branched covers $V_k(d) \to \mathbb{CP}^2$ over curves of degree d, with $k|d$; cf. [2, Section 12.3];
- The Lefschetz fibrations $X(m,n)$ and $U(m,n)$ obtained as branched covers over curves in Hirzebruch surfaces; cf. [9, Figures 8.31 and 8.32].

Theorem 1.1, combined with Proposition 1.6, is a step towards understanding the skein lasagna modules for 4-manifolds from the above list. In particular, when $X = \mathbb{CP}^2$ or $\overline{\mathbb{CP}^2}$, we have a Kirby diagram with a single (± 1)-framed unknot, and Theorem 1.3 tells us its $N = 2$ skein lasagna module in class $\alpha = 0$ and homological degree zero:

$$\mathcal{S}^2_{0,0,*}(\mathbb{CP}^2;\emptyset,0) = 0, \qquad \mathcal{S}^2_{0,0,0}(\overline{\mathbb{CP}^2};\emptyset,0) \cong \mathbb{Z}. \tag{1}$$

For an invariant to be effective at detecting manifolds, one needs to be able to extract finite data from it. The calculations above indicate that the skein lasagna modules can have infinite rank overall, but may be finitely generated when we fix the bi-grading and the class α.

Question 1.7. *Is it true that for every 4-manifold W, framed link $L \subset W$, class $\alpha \in H_2(W, L)$, and values $i, j \in \mathbb{Z}$, the skein lasagna module $\mathcal{S}^N_{0,i,j}(W;L,\alpha)$ is finitely generated?*

Note that for the skein modules of closed 3-manifolds, a finite dimensionality result was recently proved by Gunningham, Jordan and Safronov [10].

A more ambitious problem is the following:

Question 1.8. *Can the invariant $\mathcal{S}_0^N(W;L)$ detect exotic smooth structures on the 4-manifold W?*

One indication that the answer might be positive is the behavior under orientation reversal. It is known that unitary TQFTs (which are symmetric under orientation reversal) cannot detect exotic smooth structures on simply connected 4-manifolds; see [8]. On the other hand, the Donaldson and Seiberg-Witten invariants (which can detect exotic smooth structures) are highly sensitive to the orientation. The skein lasagna modules $\mathcal{S}_0^N(W;L)$ are constructed from the cobordism maps on Khovanov-Rozansky homology, which are also sensitive to orientation. In fact, in the case $W = \mathbb{CP}^2$ and $N = 2$, one can see explicitly from (1) that the invariants of W and $\overline{W}$ are quite different.

Organization of the paper. In Section 2 we review the definition of skein lasagna modules and establish some simple properties, including Proposition 1.6. In Section 3 we give the definition of cabled Khovanov-Rozansky homology. In Section 4 we prove Theorem 1.1. In Sections 5 and 6 we do our explicit calculations from Theorems 1.2 and 1.3. In Section 7 we prove the connected sum formula, Theorem 1.4.

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2. Skein lasagna modules

2.1. **Conventions for Khovanov-Rozansky homology.** In this paper we follow [25] and, for a framed link $L \subset \mathbb{R}^3$, we write

$$\mathrm{KhR}_N(L) = \bigoplus_{i,j \in \mathbb{Z}} \mathrm{KhR}_N^{i,j}(L)$$

for the $\mathfrak{gl}_N$ version of Khovanov-Rozansky homology. Here, i denotes the homological grading and j denotes the quantum grading. For a bi-graded group W, we will denote by $W\{l\}$ the result of shifting its second grading (in our case, the quantum grading) by l:

$$W\{l\}^{i,j} = W^{i,j-l}.$$

For example, the invariant of the 0-framed unknot is the commutative Frobenius algebra

$$\mathcal{A} := \mathrm{KhR}_N(U, 0) = H^*(\mathbb{CP}^{N-1})\{1-N\} = (\mathbb{Z}[X]/\langle X^N \rangle)\{1-N\}, \tag{2}$$

with 1 in bidegree $(0, 1-N)$ and multiplication by X changing the bidegree by $(0, 2)$. The comultiplication on $\mathcal{A}$ (which corresponds to a pair-of-pants cobordism) is given by

$$\Delta(X^m) = \sum_{k=0}^{N-m-1} X^{k+m} \otimes X^{N-1-k} \tag{3}$$

and the counit on $\mathcal{A}$ is

$$\epsilon(X^m) = 0 \text{ for } 0 \le m \le N-2 \; ; \quad \epsilon(X^{N-1}) = 1. \tag{4}$$

Note that KhR_N is an invariant of *framed* links. We will distinguish it from the original $\mathfrak{sl}_N$ version of Khovanov-Rozansky homology from [18], which we denote by $\mathbf{KhR}_N$ and is independent of the framing. Further, $\mathbf{KhR}_N$ was defined only over $\mathbb{Q}$ whereas KhR_N has coefficients in $\mathbb{Z}$. The two theories differ by a shift in quantum grading:

$$\mathrm{KhR}_N(L) \otimes \mathbb{Q} \cong \mathbf{KhR}_N(L)\{Nw\},$$

where w is the writhe of a diagram in which the given framing of L is the blackboard framing.

In the case $N = 2$, we also have the ordinary Khovanov homology $\mathrm{Kh}(L)$ defined in [13]. As noted in [18], we have

$$\mathbf{KhR}_2^{i,j}(L) \cong \mathrm{Kh}^{i,-j}(\overline{L}) \otimes \mathbb{Q},$$

where $\overline{L}$ is the mirror of L. Moreover, from [13, Section 7.3], we know that the Khovanov complexes of L and $\overline{L}$ are related by duality, and therefore the Khovanov homologies are related by the universal coefficients theorem:

$$\mathrm{Kh}^{i,j}(\overline{L}) \cong \mathrm{Hom}(\mathrm{Kh}^{-i,-j}(L), \mathbb{Z}) \oplus \mathrm{Ext}(\mathrm{Kh}^{1-i,-j}(L), \mathbb{Z}). \tag{5}$$

In this paper we will not use $\mathbf{KhR}_N$. We will mostly work with KhR_N, but in Section 6 we will need to relate it to Kh because the relevant calculations in the literature are done in terms of Kh. The relation between the two theories is given by a (non-canonical) isomorphism:

$$\mathrm{KhR}_2^{i,j}(L) \cong \mathrm{Kh}^{i,-j-2w}(\overline{L}). \tag{6}$$

The usual Khovanov homology $\mathrm{Kh}(L)$ is functorial under cobordisms in $\mathbb{R}^3 \times [0,1]$, but only up to sign [17, 12]. On the other hand, the $\mathfrak{gl}_2$ version and, more generally, the $\mathfrak{gl}_N$ homology

$\mathrm{KhR}_N(L)$, are functorial over $\mathbb{Z}$ [4, 7]. Furthermore, it is shown in [25] that $\mathrm{KhR}_N(L)$ is a well-defined invariant of framed links in S^3, and is functorial under framed cobordisms in $S^3 \times [0,1]$. Given a framed cobordism $\Sigma \subset S^3 \times [0,1]$ from L_0 to L_1, the induced map

$$\mathrm{KhR}_N(\Sigma) : \mathrm{KhR}_N(L_0) \to \mathrm{KhR}_N(L_1)$$

is homogeneous of bidegree $(0, (1-N)\chi(\Sigma))$. For $N = 2$, this map agrees with the usual cobordism map

$$\mathrm{Kh}(\overline{\Sigma}) : \mathrm{Kh}(\overline{L}_0) \to \mathrm{Kh}(\overline{L}_1),$$

up to pre- and post-composition with the isomorphisms (6).

If we have an oriented manifold S diffeomorphic to the standard 3-sphere S^3, and a framed link $L \subset S$, we can define a canonical invariant $\mathrm{KhR}_N(S, L)$ as in [25, Definition 4.12]. When S is understood from the context, we will drop it from the notation and simply write $\mathrm{KhR}_N(L)$.

2.2. **Definition.** Let us review the construction of skein lasagna modules following [25, Section 5.2].

Let W be a smooth oriented 4-manifold and $L \subset \partial W$ a framed link. A *lasagna filling* $F = (\Sigma, \{(B_i, L_i, v_i)\})$ of W with boundary L consists of

- A finite collection of disjoint 4-balls B_i (called *input balls*) embedded in the interior or W;
- A framed oriented surface Σ properly embedded in $X \setminus \cup_i B_i$, meeting ∂W in L and meeting each ∂B_i in a link L_i; and
- for each i, a homogeneous label $v_i \in \mathrm{KhR}_N(\partial B_i, L_i)$.

The bidegree of a lasagna filling F is

$$\deg(F) := \sum_i \deg(v_i) + (0, (1-N)\chi(\Sigma)).$$

If W is a 4-ball, we can upgrade the functoriality of Kh to define a cobordism map

$$\mathrm{KhR}_N(\Sigma) : \bigotimes_i \mathrm{KhR}_N(\partial B_i, L_i) \to \mathrm{KhR}_N(\partial W, L)$$

and an evaluation

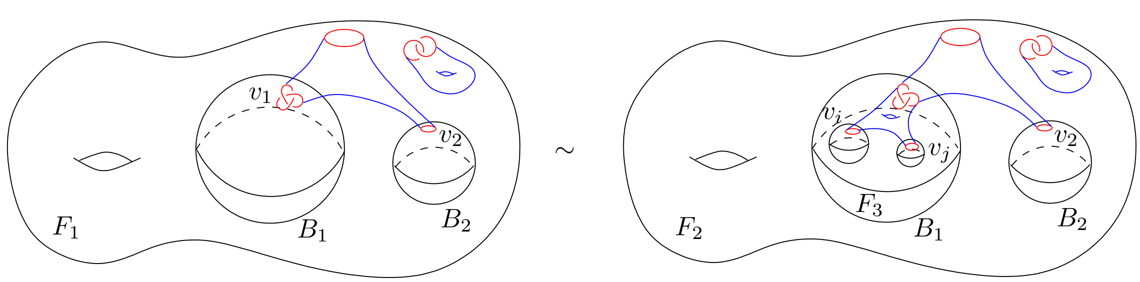
$$\mathrm{KhR}_N(F) := \mathrm{KhR}_N(\Sigma)(\otimes_i v_i) \in \mathrm{Kh}(\partial W, L).$$

We now define the skein lasagna module to be the bi-graded Abelian group

$$\mathcal{S}_0^N(W; L) := \mathbb{Z}\{\text{lasagna fillings } F \text{ of } W \text{ with boundary } L\}/\sim$$

where $\sim$ is the transitive and linear closure of the following relation:

- Linear combinations of lasagna fillings are set to be multilinear in the labels v_i;
- Furthermore, two lasagna fillings F_1 and F_2 are set to be equivalent if F_1 has an input ball B_1 with label v_1, and F_2 is obtained from F_1 by replacing B_1 with another lasagna filling F_3 of a 4-ball such that $v_1 = \mathrm{KhR}_N(F_3)$, followed by an isotopy rel boundary:



2.3. **Homology classes of lasagna fillings.** Given a lasagna filling F for $L \subset \partial W$ specified by the data $(\Sigma, \{(B_i, L_i, v_i)\})$, we denote by $[F]$ its equivalence class in $\mathcal{S}_0^N(W; L)$. We also define the homology class of F, denoted $[\![F]\!] \in H_2(W, L)$ by

$$[\![F]\!] = \big[\big(\Sigma, L \cup (\cup_i L_i)\big)\big] \in H_2(W, L \cup (\cup_i \partial B_i)) \cong H_2(W, L)$$

where $\big[\big(\Sigma, L \cup (\cup_i L_i)\big)\big]$ denotes the relative fundamental class of the surface Σ. If $F \sim F'$ are equivalent fillings in $\mathcal{S}_0^N(W; L)$, then F and F' agree up to isotopy outside of some disjoint balls. In particular, they are homologous relative to those balls, and thus homologous in $H_2(W, L)$. Thus, an equivalence class $[F]$ of lasagna fillings has a well-defined homology class $[\![F]\!]$.

Given $\alpha \in H_2(W, L)$, let $\mathcal{S}_0^N(W; L, \alpha)$ denote the subgroup of $\mathcal{S}_0^N(W; L)$ generated by fillings with homology class α. Since $\mathcal{S}_0^N(W; L)$ is generated by lasagna fillings and the fillings are partitioned according to their homology class, we obtain a decomposition

$$\mathcal{S}_0^N(W; L) = \bigoplus_{\alpha \in H_2(W,L)} \mathcal{S}_0^N(W; L, \alpha).$$

2.4. **Adding 3- and 4-handles.** Smooth 4-manifolds admit handle decompositions, which are represented pictorially by Kirby diagrams [9]. To compute $\mathcal{S}_0^N(W; \emptyset)$ for W an arbitrary smooth compact 4-manifold, we would need to understand how adding k-handles to W affects $\mathcal{S}_0^N(W; \emptyset)$. We will discuss 2-handles in detail in Section 4, and we cannot say much about 1-handles. For now, we present the following result about 3- and 4-handles.

Proposition 2.1. *Let $i : W \to W'$ be the inclusion of a 4-manifold W into W'. Then we have a natural map*

$$i_* : \mathcal{S}_0^N(W; \emptyset) \to \mathcal{S}_0^N(W'; \emptyset).$$

If W' is the result of a k-handle attachment to W, then i_* is a surjection for $k = 3$, and an isomorphism for $k = 4$.

Proof. Let $[F] \in \mathcal{S}_0^N(W; \emptyset)$ by the class of a lasagna filling F with surface Σ. Let $i(F)$ denote the filling F viewed inside of W'. If $F \sim \widetilde{F}$, then clearly $[i(F)] = [i(\widetilde{F})]$. Therefore, we have a well-defined map $i_* : \mathcal{S}_0^N(W; \emptyset) \to \mathcal{S}_0^N(W'; \emptyset)$ given by $i_*([F]) = [i(F)]$.

We observe that i_* is surjective if every surface $\Sigma' \subset W'$ can be isotoped to lie in W. Consider the case of W' being the result of attaching a k-handle h to W. Removing the cocore of h from W' produces a manifold that deformation retracts to W. In particular, if Σ can be isotoped to not intersect the cocore of h, then Σ can be isotoped to lie entirely in W. By transversality, this occurs when

$$\begin{aligned} \dim(\Sigma) + \dim(\text{cocore}(h)) &< 4 \\ 2 + (4 - k) &< 4 \\ 2 &< k \end{aligned}$$

When $k = 3$ or 4, the cocore is 1- or 0-dimensional, and hence embedded surfaces in W' can be isotoped off the handle. Thus, if W' is the result of attaching a 3-handle or 4-handle to W, then i_* is surjective.

If $i : W \to W'$ is a 4-handle addition, then surfaces in W' can be isotoped off the handle even in a one-parameter family. Therefore, if two lasagna fillings are equivalent in W', after we isotope them to lie in W they are still equivalent in W. It follows that i_* is injective, and therefore an isomorphism. ◻

Proof of Proposition 1.6. This is an immediate corollary of Proposition 2.1, because a closed 4-manifold X is obtained from $X \setminus B^4$ by attaching a 4-handle. ◻

3. The cabled Khovanov-Rozansky homology

Let $K \subset S^3$ be a framed oriented link with components $K_1, \dots, K_n$. Fix also two n-tuples of nonnegative integers, $k^- = (k_1^-, \dots, k_n^-)$ and $k^+ = (k_1^+, \dots, k_n^+)$. Let $K(k^-, k^+)$ denote the framed, oriented link obtained from k_i^- negatively oriented parallel strands to K_i (where the choice of parallel strand is determined by the framing) and k_i^+ positively oriented parallel strands. The framing on each of the parallel strands is the same as the framing on the corresponding knots K_i.

To be more precise, using the framing, we get a diffeomorphism f_i between a tubular neighborhood of K_i and $S^1 \times D^2$. For each i, we pick distinct points

$$x_1^-, \dots, x_{k_i^-}^-, x_1^+, \dots, x_{k_i^+}^+ \in D^2.$$

Then

$$K(k^-, k^+) = \bigcup_i f_i^{-1}(S^1 \times \{x_1^-, \dots, x_{k_i^-}^-, x_1^+, \dots, x_{k_i^+}^+\})$$

Remark 3.1. Suppose $n = 1$ so that K is a knot with some framing coefficient p. Then, as an unoriented link, $K(k^-, k^+)$ is the $(p(k^- + k^+), (k^- + k^+))$ cable of K.

Let B_n be the braid group on n strands, and $F : B_n \to S_n$ the natural homomorphism to the symmetric group. For $0 \le k \le n$, let $B_{k,n-k} = F^{-1}(S_k \times S_{n-k})$. Thus, if we view the braid group as the mapping class group of the punctured disk, then

$$B_{k_i^-, k_i^+} \subseteq B_{k_i^- + k_i^+}$$

consists of those self-diffeomorphisms that take the set of the first k_i^- punctures to itself. Observe that an element of $B_{k_i^-, k_i^+}$ produces a symmetry of the link $K(k^-, k^+)$. Therefore, there is a group action on Khovanov-Rozansky homology

$$\beta_i : B_{k_i^-, k_i^+} \to \mathrm{Aut}\big(\mathrm{KhR}_N(K(k^-, k^+))\big).$$

Let $e_i \in \mathbb{Z}^n$ denote the i^{th} basis vector. Observe that two strands parallel to K_i, if they have opposite orientations, co-bound a ribbon band R_i in S^3. By pushing R_i into $S^3 \times [0, 1]$ so that it is properly embedded there, removing a disk from R_i, and taking the disjoint union with the identity cobordisms on the other strands, we obtain an oriented cobordism Z_i from $K(k^-, k^+) \sqcup U$ to $K(k^- + e_i, k^+ + e_i)$. Here, U is the unknot and $\sqcup$ denotes split disjoint union. Observe that $\chi(Z_i) = -1$, and the framing on K_i induces a framing on Z_i. By the discussion in Section 2.1, there is a well-defined cobordism map

$$\mathrm{KhR}_N(Z_i) : \mathrm{KhR}_N(K(k^-, k^+) \sqcup U) \to \mathrm{KhR}_N(K(k^- + e_i, k^+ + e_i))$$

which changes the bi-grading by $(0, N - 1)$. Note that

$$\mathrm{KhR}_N(K(k^-, k^+) \sqcup U) \cong \mathrm{KhR}_N(K(k^-, k^+)) \otimes \mathrm{KhR}_N(U).$$

Recall from (2) that $\mathcal{A} = \mathrm{KhR}_N(U) \cong (\mathbb{Z}[X]/\langle X^N \rangle)\{1 - N\}$.

Thus, the information in the map $\mathrm{KhR}_N(Z_i)$ is encoded in N maps

$$\psi_i^{[m]} : \mathrm{KhR}_N(K(k^-, k^+)) \to \mathrm{KhR}_N(K(k^- + e_i, k^+ + e_i)),\ m = 0, \dots, N - 1$$

given by

$$\psi_i^{[m]}(v) = \mathrm{KhR}_N(Z_i)(v \otimes X^m). \tag{7}$$

Note that $\psi_i^{[m]}$ changes the bi-grading by $(0, 2m)$.

Remark 3.2. Let $\widehat{Z}_i$ be the cobordism from $K(k^-, k^+)$ to $K(k^- + e_i, k^+ + e_i)$ obtained from Z_i by reintroducing the disk that was removed from the ribbon R_i. Since the map associated to a disk in Khovanov-Rozansky homology takes $1 \mapsto 1$, we see that $\psi_i^{[0]} = \mathrm{KhR}_N(\widehat{Z}_i)$.

Remark 3.3. For conciseness, we did not include the link K and the values of k^- and k^+ in the notation β_i, R_i, Z_i, $\psi_i^{[m]}$.

Let W be the 2-handlebody obtained from B^4 by attaching handles along the link K. The homology $H_2(W)$ is freely generated by the cores of the handles, capped with Seifert surfaces for each K_i. We identify $H_2(W) \cong \mathbb{Z}^n$ by letting the capped core of the ith handle correspond to e_i.

For $\alpha = (\alpha_1, \dots, \alpha_n) \in H_2(W) \cong \mathbb{Z}^n$, let α^+ denote its positive part and α^- its negative part; i.e., $\alpha_i^+ = \max(\alpha_i, 0)$ and $\alpha_i^- = \min(\alpha_i, 0)$. We also let $|\alpha| = \sum_i |\alpha_i|$.

Definition 3.4. *The cabled Khovanov-Rozansky homology of K at level α is*

$$\underline{\mathrm{KhR}}_{N,\alpha}(K) = \Big(\bigoplus_{r \in \mathbb{N}^n} \mathrm{KhR}_N(K(r-\alpha^-, r+\alpha^+))\{(1-N)(2r+|\alpha|)\} \Big) / \sim$$

where the equivalence $\sim$ is the transitive and linear closure of the relations

$$\beta_i(b)v \sim v, \quad \psi_i^{[m]}(v) \sim 0 \text{ for } m < N-1, \quad \psi_i^{[N-1]}(v) \sim v \tag{8}$$

for all $i = 1, \dots, n$; $b \in B_{k_i^-, k_i^+}$, and $v \in \mathrm{KhR}_N(K(r-\alpha^-, r+\alpha^+))$.

Observe that the equivalence relation preserves the bi-grading, and hence there is an induced bi-grading on $\underline{\mathrm{KhR}}_{N,\alpha}(K)$.

Remark 3.5. In principle, there are several different maps of the form $\psi_i^{[m]}$ (with the same domain and target), corresponding to different choices of the pair of oppositely oriented strands that co-bound R_i. However, these maps differ by post-composition with some $\beta_i(b)$. Therefore, when we divide by the relations (8), which already include $\beta_i(b)v \sim v$, using one choice of $\psi_i^{[m]}$ is the same as using any other choice.

4. 2-Handlebodies

In this section we prove our main result, about the skein lasagna modules of 2-handlebodies.

Proof of Theorem 1.1. We first define a map

$$\widetilde{\Phi} : \bigoplus_{r \in \mathbb{N}^n} \mathrm{KhR}_N(K(r-\alpha^-, r+\alpha^+))\{(1-N)(2r+|\alpha|)\} \to \mathcal{S}_0^N(W; \emptyset, \alpha), \quad \widetilde{\Phi}(v) = [F_v]$$

as follows. Let B be 4-dimensional ball slightly smaller than the 0-handle, contained in the interior of that handle. Given an element $v \in \mathrm{KhR}_N(K(r-\alpha^-, r+\alpha^+))$, we define $\widetilde{\Phi}(v)$ to be the class of the lasagna filling F_v with B as the only input ball, with B decorated with the framed, oriented link $K(r-\alpha^-, r+\alpha^+)$ and labeled by the element v, and with the surface given by the disjoint union of $r_i - \alpha_i^-$ negatively oriented discs parallel to the core of i^{th} 2-handle and $r_i + \alpha_i^+$ positively oriented such discs (union over all i). We will denote these disks by $C_j^{i,\pm}$, where $1 \leq j \leq r_i \pm \alpha_i^\pm$. Since the disks are contractible, they have unique framings. The homology class of this surface in $H_2(W, B) \cong H_2(W)$ is clearly α by construction. See Figure 1.

We claim that, under the equivalence relation from Definition 3.4, $\widetilde{\Phi}$ maps equivalent elements to the same class in $\mathcal{S}_0^N(W; \emptyset, \alpha)$. There are three types of relations to be checked.

First, consider the braid group action. Intuitively, this permutes the disks $C_j^{i,\pm}$ in the 2-handle. More precisely, a braid $b \in B_{k_i^-, k_i^+}$ gives a cobordism inside $D^2 \times [0,1]$. Taking the product of this cobordism with S^1, and using the identification between a neighborhood of $K_i \subset S^3$ and $S^1 \times D^2$, we get a cobordism

$$\Sigma_b \subset S^1 \times D^2 \times [0,1] \subset S^3 \times [0,1]$$

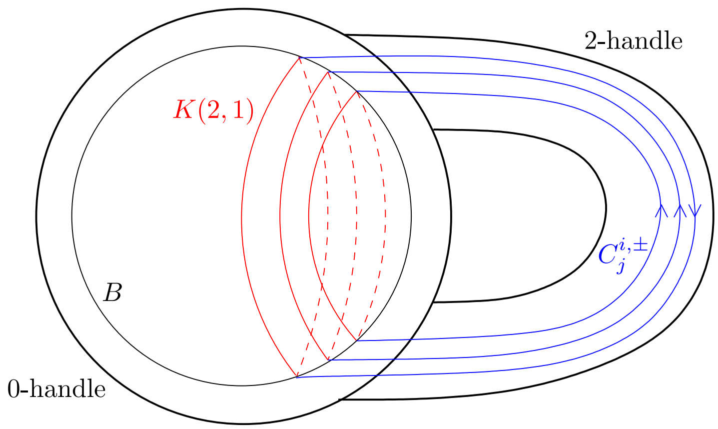


FIGURE 1. A generator for $\mathcal{S}_0^N(W;\emptyset,0)$.

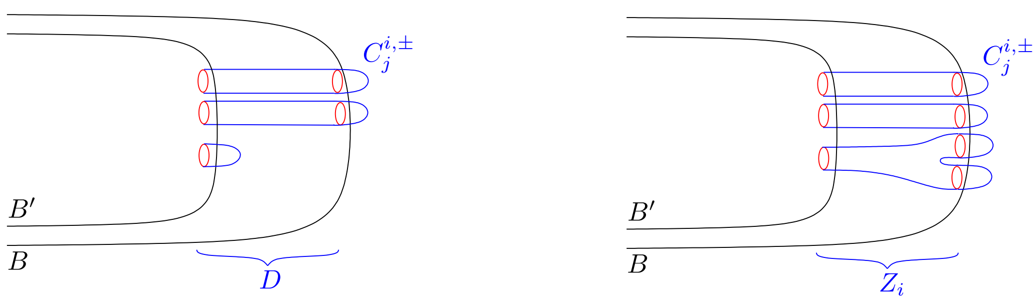


FIGURE 2. A schematic picture of the fillings E_w (left) and E'_w (right). For simplicity, we drew the components of the cable side by side, rather than nested.

from the cable $K(r-\alpha^-, r+\alpha^+)$ to itself. We can then view the filling F_v of W as obtained from $F_{\beta_i(b)v}$ by inserting into B a filling made of a smaller ball B' and the surface Σ_b, with the input labeled by v. Therefore, F_v and $F_{\beta_i(b)v}$ represent the same class in $\mathcal{S}_0^N(W;\emptyset,\alpha)$.

Second, consider a slightly smaller ball B' contained in B, so that the region between B' and B is a copy of $S^3\times[0,1]$. We put in that region the cobordism $D\subset S^3\times[0,1]$, from $K(r-\alpha^-,r+\alpha^+)\sqcup U$ to $K(r-\alpha^-,r+\alpha^+)$, which is simply the split disjoint union of the identity on $K(r-\alpha^-,r+\alpha^+)$ with a disk capping the unknot U. The cobordism map associated to the cap is the counit (4). Therefore,

$$\mathrm{KhR}_N(D)(v\otimes X^m)=0 \text{ for } m<N-1; \qquad \mathrm{KhR}_N(D)(v\otimes X^{N-1})=v. \tag{9}$$

For every $w\in \mathrm{KhR}_N(K(r-\alpha^-,r+\alpha^+)\sqcup U)$, we construct a lasagna filling E_w with input ball B', surface

$$D\cup\bigcup_{i,j}C_j^{i,-}\cup\bigcup_{i,j}C_j^{i,+}$$

and label w. Thus, E_w is obtained from the filling $F_{\mathrm{KhR}_N(D)(w)}$ by adjoining D. It follows that the fillings E_w and $F_{\mathrm{KhR}_N(D)(w)}$ are equivalent.

On the other hand, as in Section 3, we also have a cobordism $Z_i\subset S^3\times[0,1]$ from $K(r-\alpha^-,r+\alpha^+)\sqcup U$ to $K(r-\alpha^-+1,r+\alpha^++1)$. For every $w\in\mathrm{KhR}_N(K(r-\alpha^-,r+\alpha^+)\sqcup U)$, we construct a new lasagna filling E'_w from $F_{\mathrm{KhR}_N(Z_i)(w)}$ by adjoining Z_i in the region between B' and B. Then, the fillings E'_w and $F_{\mathrm{KhR}_N(Z_i)(w)}$ are equivalent.

Note that E'_w has the same input data as E_w (namely, the ball B' and the label w); see Figure 2. Moreover, the surface of E'_w is obtained from that of E_w by taking connected sum with the closed surface

$$C^{i,-}_{r_i-\alpha_i^-+1} \cup R_i \cup C^{i,+}_{r_i+\alpha_i^++1} \;\cong\; S^2, \tag{10}$$

where R_i is the ribbon between two oppositely oriented copies of K_i, as in Section 3. The copy of S^2 from (10) can be isotoped to lie entirely in the 2-handle. Taking a connected sum with such a sphere can be viewed as simply a surface isotopy. Therefore, E_w and E'_w are equivalent lasagna fillings. We conclude that

$$\widetilde{\Phi}(\mathrm{KhR}_N(D)(w)) = [F_{\mathrm{KhR}_N(D)(w)}] = [E_w] = [E'_w] = [F_{\mathrm{KhR}_N(Z_i)(w)}] = \widetilde{\Phi}(\mathrm{KhR}_N(Z_i)(w)),$$

for every $w \in \mathrm{KhR}_N(K(r-\alpha^-, r+\alpha^+) \sqcup U) \cong \mathrm{KhR}_N(K(r-\alpha^-, r+\alpha^+)) \otimes \mathcal{A}$.

Let $v \in \mathrm{KhR}_N(K(r-\alpha^-, r+\alpha^+))$. If we take $w = v \otimes X^m$, in view of Equations (7) and (9), we have

$$0 = \widetilde{\Phi}(\psi_i^{[m]}(v)) \text{ for } m < N-1; \qquad \widetilde{\Phi}(v) = \widetilde{\Phi}(\psi_i^{[N-1]}(v)).$$

We have now verified our claim that $\widetilde{\Phi}$ takes equivalent elements to the same equivalence class of lasagna fillings. This shows that $\widetilde{\Phi}$ descends to a map

$$\Phi : \underline{\mathrm{KhR}}_{N,\alpha}(K) \to \mathcal{S}_0^N(W; \emptyset, \alpha).$$

Next, we define an inverse Φ^{-1} to Φ. Let F be a lasagna filling with surface Σ. We can choose a ball containing all the input balls of F and find an equivalence from F to a filling with a single input ball. Thus, after isotopy, we can assume without loss of generality that F has a single input ball given by the 0-handle B, and intersects the cocores G_i of the 2-handles transversely in a number of points. After another isotopy, we can assume that Σ intersects the 2-handles only in core-parallel disks, one for each intersection with the cocores. (Equivalently, one can consider shrinking the 2-handles until all of the interesting topology of Σ is contained in the 0-handle.) These modifications show that F is equivalent to a filling of the form $\widetilde{\Phi}(v)$, and we let

$$\Phi^{-1}([F]) := [v].$$

Let us check that Φ^{-1} is well-defined.

Firstly, if we change a lasagna filling F by filling in an input ball in B^4, this does not change its its image under Φ^{-1}, because the intersection with the cocores of the 2-handles is unchanged.

Secondly, in the definition of Φ^{-1} we chose an isotopy of Σ that makes it transverse to the cocores G_i of the 2-handles. If we made a different choice, the isotopy relating the two choices is a 1-parameter family of surfaces $\Sigma_t, t \in [0,1]$. Let

$$Y = \bigcup_{t\in[0,1]} \big(\{t\} \times \Sigma_t\big) \subset [0,1] \times W.$$

We can assume that Y is a smooth 3-dimensional submanifold with boundary, and that Y intersects each $[0,1] \times G_i$ transversely in a 1-manifold P_i. Let

$$\pi_i : P_i \to [0,1]$$

be the composition of the inclusion into $[0,1] \times W$ with projection to the $[0,1]$ factor. After an isotopy of Y rel boundary, we arrange so that π_i are local diffeomorphisms away from finitely many critical points (caps and cups). The critical values $t_1 < \cdots < t_n$ of π_i, together with the endpoints $t_0 = 0$ and $t_{n+1} = 1$, split $[0,1]$ into finitely many intervals of the form $[t_k, t_{k+1}]$. After another isotopy of Y, we can assume that for some small $\epsilon > 0$, the collections of points $\pi_i^{-1}(t_k + \epsilon)$ and $\pi_i^{-1}(t_{k+1} - \epsilon)$ coincide, as oriented submanifolds of the cocore G_i. Thus, on the interval $[t_k + \epsilon, t_{k+1} - \epsilon]$, the surfaces Σ_t stay transverse to the cocores, and the only effect of varying the lasagna filling is to replace the element v with $\beta_i(b)(v)$ for some braid $b \in B_{k_i^-,k_i+}$. However, we have $\beta_i(b)(v) \sim v$, so the value of $\Phi^{-1}([F])$ is unchanged.

At the critical values of π_i, the surfaces Σ_t are no longer transverse to the cocores of the 2-handles. Rather, what happens is that we introduce or remove two intersections of opposite signs. Introducing such points corresponds to "pushing a disk" from one lasagna filling into the cocore; that is, replacing a lasagna filling of the form E_w with one of the form E'_w, for some $w \in \mathrm{KhR}_N(K(r-\alpha^-, r+\alpha^+) \sqcup U)$. The corresponding values of Φ^{-1} for these fillings are $F_{\mathrm{KhR}_N(D)(w)}$ and $F_{\mathrm{KhR}_N(Z_i)(w)}$. When $w = v \otimes X^m$ for $m < N-1$, these give 0 and $\psi_i^{[m]}(v)$, and when $w = v \otimes X^{N-1}$, they give v and $\psi_i^{[N-1]}(v)$. Since the equivalence that gives the cabled Khovanov-Rozansky homology includes the relations

$$\psi_i^{[m]}(v) \sim 0 \text{ for } m < N-1, \quad \psi_i^{[N-1]}(v) \sim v,$$

we see that the value of $\Phi^{-1}([F])$ is the same for isotopic lasagna fillings.

This completes the proof that Φ^{-1} is well-defined. It is straightforward to check that Φ and Φ^{-1}, as defined above, are inverse to each other. $\square$

Remark 4.1. Theorem 1.1 implies that $\underline{\mathrm{KhR}}_{N,\alpha}(K)$ is invariant under handleslides among the components of the link K. One can also give a more direct proof of this fact, using just the functoriality of Khovanov-Rozansky homology, and without reference to the skein algebra. We leave this as an exercise for the reader.

5. The 0-framed unknot

Note that $S^2 \times D^2$ is the result of attaching a 2-handle along a 0-framed unknot. Thus, Theorem 1.2 is a direct consequence of Theorem 1.1 and the following proposition.

Proposition 5.1. *Let $(U, 0)$ be the unknot with framing 0. The cabled Khovanov-Rozansky homology of $(U, 0)$ at level $\alpha \in \mathbb{Z}$ is a free abelian group supported in homological degree 0 and non-positive quantum degrees, with graded rank given by*

$$\sum_{j=0}^{\infty} \mathrm{rk}\, \underline{\mathrm{KhR}}^{0,-j}_{N,\alpha}(U,0) x^j = \prod_{k=1}^{N-1} \frac{1}{1-x^{2k}}$$

Proof. From Definition 3.4 we have

$$\underline{\mathrm{KhR}}_{N,\alpha}(U,0) = \Big(\bigoplus_{r\in\mathbb{N}} \mathrm{KhR}_N(U(r-\alpha^-, r+\alpha^+)\{(1-N)(2r+|\alpha|)\}\Big)/\sim$$

where $U(r-\alpha^-, r+\alpha^+)$ is the $(2r+|\alpha|)$-component unlink. The Khovanov homology of the n-component unlink is $\mathcal{A}^{\otimes n}$, where $\mathcal{A} = (\mathbb{Z}[X]/\langle X^N \rangle)\{1-N\}$. To study the equivalence relation $\sim$ we need to distinguish between two types of components of the unlink: $r-\alpha^-$ negatively oriented and $r+\alpha^+$ positively oriented (although, of course, changing the orientation of a component of the unlink still yields the same oriented link, up to isotopy). We will use $\mathcal{B} = (\mathbb{Z}[Y]/\langle Y^N \rangle)\{1-N\}$ to denote the copies of $\mathcal{A}$ associated to positively oriented components, and reserve $\mathcal{A}$ for those associated to negatively oriented components. Thus,

$$\underline{\mathrm{KhR}}_{N,\alpha}(U,0) = \Big(\bigoplus_{r\in\mathbb{N}} \mathcal{A}^{\otimes(r-\alpha^-)} \otimes \mathcal{B}^{\otimes(r+\alpha^+)}\{(1-N)(2r+|\alpha|)\}\Big)/\sim$$

It is clear that this group is supported in homological degree 0. We first consider the braid group action, which in this case is simply the action of $S_{r-\alpha^-} \times S_{r+\alpha^+}$ permuting the tensor factors. A basis of $\mathcal{A}^{\otimes(r-\alpha^-)} \otimes \mathcal{B}^{(r+\alpha^+)}$ is given by the monomials

$$\left\{ \prod_{i=1}^{r-\alpha^-} \prod_{j=1}^{r+\alpha^+} X_i^{d_i} Y_j^{e_j} \right\}_{0 \le d_i, e_j \le N-1}$$

These basis elements are permuted by the symmetric group action. Reducing modulo this action corresponds to regarding the multidegree $(d_1, \dots, d_{r-\alpha^-})$ as an unordered partition $d_1 + \cdots + d_{r-\alpha^-}$. Let us introduce some notation for partitions. We let $\mathcal{P}_{N-1}(q;r)$ denote the set of unordered partitions of q into at most r (nonempty) parts, each of which is less than or equal to $N-1$. We let $\mathcal{P}_{N-1}(q;\infty)$ be the set of partitions of q into parts of size at most $N-1$. The function $P_{N-1}(q) = |\mathcal{P}_{N-1}(q;\infty)|$ is the cardinality of this set. Finally, we let let $\mathcal{P}_{N-1}(-;r)$ be the set of partitions of any number into at most r parts no greater than $N-1$.

For a partition $\mathbf{d} = (d_1 + \cdots + d_k) \in \mathcal{P}_{N-1}(-; r-\alpha^-)$ with $k \le r - \alpha^-$, let

$$\mathbf{X}^{\mathbf{d}} := \left[\prod_{i=0}^{k} X_i^{d_i} \right]$$

be the class of a monomial in the quotient $\mathcal{A}^{\otimes(r-\alpha^-)}/S^r$. Observe that if we rearrange the parts, the class of the monomial is unchanged. Then, a basis for the quotient $\mathcal{A}^{\otimes(r-\alpha^-)} \otimes \mathcal{B}^{(r+\alpha^+)}/(S^r \times S^r)$ is given by the pairs of equivalence classes of monomials

$$(\mathbf{X}^{\mathbf{d}}, \mathbf{Y}^{\mathbf{e}}) \ \text{ for } \ (\mathbf{d}, \mathbf{e}) \in \mathcal{P}_{N-1}(-; r-\alpha^-) \times \mathcal{P}_{N-1}(-; r+\alpha^+).$$

After applying the shift by $(1-N)(2r + |\alpha|)$ in the definition of the cabled Khovanov-Rozansky homology, we find that a basis element labelled by $(\mathbf{d}, \mathbf{e}) \in \mathcal{P}_{N-1}(q_-; r-\alpha^-) \times \mathcal{P}_{N-1}(q_+; r+\alpha^+)$ has quantum degree $(1-N)(4r + 2|\alpha|) + 2(q_- + q_+)$.

We now consider the maps $\psi^{[m]}$. It will be more convenient to use a slightly different basis than the one we have described. We let $V^a = X^{N-1-a}$ and $W^b = Y^{N-1-b}$ be new bases of $\mathcal{A}$ and $\mathcal{B}$ respectively. As before, we have that the pairs

$$(\mathbf{V}^{\mathbf{d}}, \mathbf{W}^{\mathbf{e}}) = \left(\left[\prod_i V_i^{d_i} \right], \left[\prod_j W_j^{e_j} \right] \right)$$

give a basis for $\mathcal{A}^{\otimes(r-\alpha^-)} \otimes \mathcal{B}^{(r+\alpha^+)}/(S^r \times S^r)$. The degree of such an element is $-2(q_- + q_+)$, where $\mathbf{d} \in \mathcal{P}_{N-1}(q_-; r-\alpha^-)$ and $\mathbf{e} \in \mathcal{P}_{N-1}(q_+; r+\alpha^+)$.

For the 0-framed unknot, the maps $\psi^{[m]}$ are given by multiplication by $\Delta(X^m)$, which is defined in Equation (3). In our new basis, the map $\psi^{[N-1]}$ is simply multiplication by 1. If we use the partition labels as shorthand for the basis elements themselves, then $\psi^{[N-1]}$ corresponds to the natural identification of partitions

$$\mathcal{P}_{N-1}(-; r-\alpha^-) \times \mathcal{P}_{N-1}(-; r+\alpha^+) \hookrightarrow \mathcal{P}_{N-1}(-; r+1-\alpha^-) \times \mathcal{P}_{N-1}(-; r+1+\alpha^+)$$

The identification $v \sim \psi^{[N-1]}(v)$ thereby allows us to regard the pair $(\mathbf{d}, \mathbf{e})$ in the r^{th} summand as equivalent to the same pair in the $(r+1)^{\text{st}}$ summand. Thus, we can drop the restriction on the number of parts in our partitions and obtain

$$\underline{\mathrm{KhR}}_{N,\alpha}(U, 0) \cong \mathbb{Z}\langle \mathcal{P}_{N-1}(-;\infty) \times \mathcal{P}_{N-1}(-;\infty) \rangle / (\psi^{[m]} \sim 0, m \le N-2)$$

where we use the notation $\mathbb{Z}\langle S \rangle$ for the free abelian group on a set S.

We now consider the final equivalence relation setting the image of the $\psi^{[m]}$ equal to zero. Under $\psi^{[m]}$, a generator $(\mathbf{d}, \mathbf{e})$ is sent to

$$\psi^{[m]}((\mathbf{d}, \mathbf{e})) = \sum_{k=0}^{N-m-1} (\mathbf{d} + (N-1-k-m), \mathbf{e} + k)$$

Setting this quantity equal to zero and solving for the last term in the sum (with $k = N - m - 1$), we have

$$(\mathbf{d}, \mathbf{e} + (N - m - 1)) = - \sum_{k=0}^{N-m-2} (\mathbf{d} + (N - 1 - k - m), \mathbf{e} + k) \tag{11}$$

for $m = 0, \ldots, N-2$. Equation 11 lets us remove a part of size $N-m-1$ from the second partition in the pair and replace it with (a sum over terms with) smaller parts. Since $m = 0, \ldots, N-2$, we can actually remove parts of any size. Thus, repeatedly applying Equation 11 allows one to reduce an element $(\mathbf{d}, \mathbf{e})$ to one of the form $(\mathbf{d}', \emptyset)$. Hence, we have

$$\underline{\mathrm{KhR}}_{N,\alpha}(U, 0) \cong \mathbb{Z}\langle \mathcal{P}_{N-1}(-; \infty)\rangle$$

where a partition $\mathbf{d} \in \mathcal{P}_{N-1}(q; \infty)$ has degree $-2q$. Therefore,

$$\underline{\mathrm{KhR}}^{0,-2q}_{N,\alpha}(U, 0) \cong \mathbb{Z}\langle \mathcal{P}_{N-1}(q; \infty)\rangle \cong \mathbb{Z}^{P_{N-1}(q)}$$

and the invariant is zero in all other bidegrees. The generating function for $P_{N-1}(q)$ is

$$\sum_{q=0}^{\infty} P_{N-1}(q) x^q = \prod_{k=0}^{N-1} \frac{1}{1 - x^k}$$

Applying this formula gives the advertised answer for the graded dimension of $\underline{\mathrm{KhR}}^{0,j}_{N,\alpha}(U, 0)$. □

Remark 5.2. For low values of N, one can write explicit formulae for $P_{N-1}(q)$. For example, for $N = 2$ we have $P_1(q) = 1$ and for $N = 3$, $P_2(q) = 1 + \lfloor \frac{q}{2} \rfloor$.

6. The unknot with non-zero framing

The goal of this section is to prove Theorem 1.3. This will be a direct consequence of Theorem 1.1 and the following:

Proposition 6.1. *Let (U, p) be the unknot with framing p.*

(*a*) *For $N = 2$ and $p > 0$, the cabled Khovanov-Rozansky homology of (U, p) in homological degree 0 and at level $0 \in \mathbb{Z}$ is given by*

$$\underline{\mathrm{KhR}}^{0,j}_{2,0}(U, p) = 0, \quad \forall j \in \mathbb{Z}.$$

(*b*) *On the other hand, for $N = 2$ and $p < 0$ we have*

$$\underline{\mathrm{KhR}}^{0,j}_{2,0}(U, p) = \begin{cases} \mathbb{Z} & \text{if } j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Before proving Proposition 6.1, we need some preliminaries.

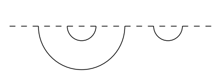
6.1. The ring H^n and tangle invariants.

In [14], Khovanov extended his theory Kh to tangles. We briefly sketch his construction here.

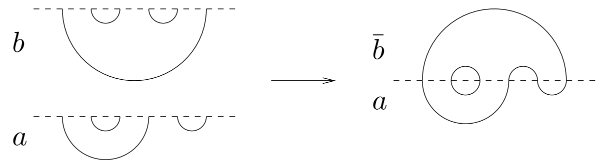
Throughout this section $N = 2$, and the invariant of the unknot is

$$\mathcal{A} = (\mathbb{Z}[X]/\langle X^2 \rangle)\{1\} = \mathrm{Span}\{1, X\}.$$

Let $\mathfrak{C}_n$ be the set of crossingless matchings between $2n$ points on a line. For example, the following is an element of $\mathfrak{C}_3$:



For $a \in \mathfrak{C}_n$, we let $\bar{a}$ denote its reflection in the dashed line. Then, for every $a, b \in \mathfrak{C}_n$, the composition $\bar{b}a$ is a collection of k circles in the plane:



To $\bar{b}a$ we associate the tensor product of copies of $\mathcal{A}$, one for each circle:

$$\mathcal{F}(\bar{b}a) = \mathcal{A}^{\otimes k}.$$

Khovanov constructs a finite-dimensional graded ring

$$H^n := \bigoplus_{a,b \in \mathfrak{C}_n} {}_a(H^n)_b$$

where

$${}_a(H^n)_b = \mathcal{F}(\bar{b}a)\{n\}.$$

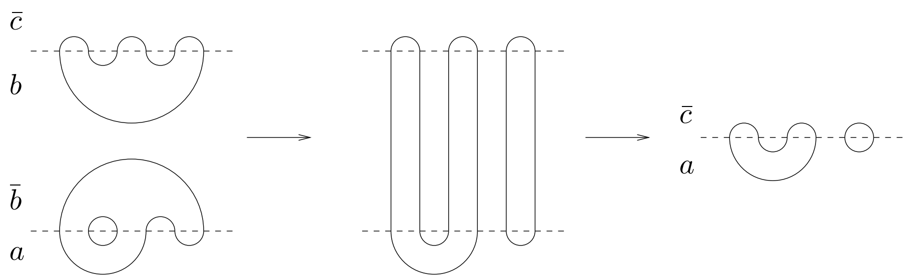
The multiplication on H^n is

$${}_a(H^n)_b \otimes {}_d(H^n)_c \to 0 \text{ if } b \neq d$$

and

$${}_a(H^n)_b \otimes {}_b(H^n)_c \to {}_a(H^n)_c$$

is given by "compressing" $\bar{b}b$ into the identity tangle, using the multiplication and comultiplication maps on the Frobenius algebra $\mathcal{A}$:



Consider now a tangle T represented by a diagram inside a rectangle, connecting $2n$ points at the bottom to $2m$ points at the top. This is called an (m, n)-tangle in the terminology of [14]. To T Khovanov associates a complex of (H^n, H^m)-bimodules

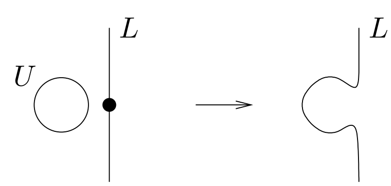
$$\mathcal{F}(T) = \bigoplus_{a \in \mathfrak{C}_n} \bigoplus_{b \in \mathfrak{C}_m} {}_a(\mathcal{F}(T))_b \tag{12}$$

where ${}_a(\mathcal{F}(T))_b$ is the usual Khovanov complex associated to the link obtained from the composition $\bar{b}Ta$. Different diagrams for the same tangle produce homotopy equivalent complexes.

Finally, let us discuss a module action on the rings H^n. This is the analogue of the action of

$$R = \mathbb{Z}[X]/\langle X^2 \rangle = \mathcal{A}\{-1\}$$

on the Khovanov homology of a link L, which was constructed in [15]. The action of $r \in R$ is given by introducing a small unknot near a basepoint on the knot, marking it with r, and applying the multiplication map induced by the saddle cobordism from $L \sqcup U$ to L:



More generally, if L has m components, there is an action of $R^{\otimes m}$ by using unknots near basepoints on each component; cf. [11, Section 2] and [3, Section 2.2].

In our case, we view the ring H^n as an algebra over

$$R^{\otimes 2n} = \mathbb{Z}[X_1, \dots, X_{2n}]/\langle X_1^2, \dots, X_{2n}^2\rangle \tag{13}$$

by using unknots near each of the $2n$ points on the line. We set X_i to be $(-1)^i$ times the X action from the unknot near the ith point.

If T is an (m, n)-tangle, then the complex of bimodules $\mathcal{F}(T)$ gets induced actions of $R^{\otimes 2n}$ (from the points at the bottom) and of $R^{\otimes 2m}$ (from the points at the top).

Remark 6.2. The analogues of the H^n rings for the $\mathfrak{gl}_2$ theory KhR_2 were constructed by Ehrig, Stroppel and Tubbenhauer in [6], [5]. It would be more natural to work with them, because the skein lasagna algebras are defined in terms of KhR_2. However, we chose to use the original H^n in order to be able to use various results from the literature that were proved in that context.

6.2. **The center of H^n.** We will be interested in the center of the ring H^n, which was computed in [16].

Theorem 6.3 (Khovanov [16])**.** *The center $Z(H^n)$ is isomorphic to the polynomial ring $\mathbb{Z}[X_1, \dots, X_{2n}]$ modulo the ideal generated by the elements*

$$X_i^2, \quad i = 1, \dots, 2n$$

and

$$\sum_{|I|=k} X_I, \quad k = 1, \dots, 2n,$$

where $X_I = X_{i_1} \dots X_{i_k}$ for $I = \{i_1, \dots, i_k\}$, and the sum is over all the cardinality k subsets of $\{1, \dots, 2n\}$.

We grade $Z(H^n)$ so that each X_i is in degree 2. This corresponds to the convention for KhR_2, and is opposite the convention for Kh; see Section 2.1.

Starting from the description in Theorem 6.3, we see that $Z(H^n)_{2k}$ is generated by X_I over all cardinality k subsets $I \subseteq \{1, \dots, 2n\}$, subject to the linear relations:

$$\sum_{|I|=k,\ I \supset J} X_I = 0, \tag{14}$$

for every subset $J \subseteq \{1, \dots, 2n\}$ of cardinality $|J| < k$.

Stošić [31, Proposition 1] showed that

$$Z(H^n)_j \cong \begin{cases} \mathbb{Z}^{\binom{2n}{k} - \binom{2n}{k-1}} & \text{if } j = 2k,\ k = 0, \dots, n, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

The total rank of $Z(H^n)$ is $\binom{2n}{n}$. We can give a concrete basis for $Z(H^n)$ as follows.

Definition 6.4. *A subset $I \subseteq \{1, 2, \dots, 2n\}$ is called* admissible *if*

$$|I \cap \{1, 2, \dots, m\}| \leq \frac{m}{2}, \tag{16}$$

for all $m = 1, \dots, 2k$. We let

$$A_k^n = \{I \subseteq \{1, 2, \dots, 2n\} \mid I \textit{ is admissible}, |I| = k\}.$$

Proposition 6.5 (Lemma 8 in [16])**.** *A basis for $Z(H^n)_{2k}$ consists of the elements X_I for $I \in A_k^n$.*

Example 6.6. When $n = 2$, the ranks of $Z(H^2)$ in degrees $0, 2$ and 4 are 1, 3 and 2, respectively. A basis is given by 1, X_2, X_3, X_4, X_2X_4 and X_3X_4.

One can read off from [16] an explicit description of the elements X_i. Let $p_1, \dots, p_{2n}$ be the $2n$ points on the line (in this order) that we connect by crossingless matchings. Then, we have

$$X_i = \sum_{a \in \mathfrak{C}_n} {}_a(X_i)_a, \quad {}_a(X_i)_a \in {}_a(H^n)_a \cong \mathcal{A}^{\otimes k}\{n\}$$

where ${}_a(X_i)_a$ is the tensor product of $1 \in \mathcal{A}$ for each circle not going through p_i, and of $(-1)^i X \in \mathcal{A}$ for the circle going through p_i. In other words, X_i exactly correspond to the variables in the ring $R^{\otimes 2n}$ from (13), applied to the identity element $1 \in Z(H^n) \subset H^n$. Thus, we can improve Theorem 6.3 to a statement about $R^{\otimes 2n}$-algebras:

Proposition 6.7. *As a $R^{\otimes 2n}$-algebra, the center $Z(H^n)$ is isomorphic to*

$$R^{\otimes 2n} / \big\langle \sum_{|I|=k} X_I, \quad k = 1, \dots, 2n \big\rangle$$

where the sum is over all the cardinality k subsets of $\{1, \dots, 2n\}$.

6.3. **The dual of the center.** For an Abelian group V, we will denote by $V^\vee = \mathrm{Hom}(V, \mathbb{Z})$ its dual. For example, $Z(H^n)^\vee_{2k}$ is the dual of $Z(H^n)_{2k}$. In view of (15), this is a free Abelian group of rank $\binom{2n}{k} - \binom{2n}{k-1}$. We will describe a set of interesting elements in $Z(H^n)_{2k}$.

Let Z^n_k be the Abelian group freely generated by elements X_I for $I \subset \{1, \dots, 2n\}$ with $|I| = k$. The dual $(Z^n_k)^\vee$ has a dual basis given by $X_I^\vee$, where $X_I^\vee(X_J)$ is Kronecker's δ_{IJ}. We introduce a multiplication on $\oplus_k (Z^n_k)^\vee$ by setting

$$X_I^\vee \cdot X_J^\vee = \begin{cases} X^\vee_{I \cup J} & \text{if } I \cap J = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $Z(H^n)_{2k}$ is the quotient of Z^n_k by the relations (14). It follows that $Z(H^n)^\vee_{2k}$ is a subgroup of $(Z^n_k)^\vee$ consisting of those functions $f : Z^n_k \to \mathbb{Z}$ such that

$$f\Big(\sum_{|I|=k,\ I \supseteq J} X_I \Big) = 0, \tag{17}$$

for every subset $J \subseteq \{1, \dots, 2n\}$ with $|J| < k$.

Definition 6.8. *A* partial matching *of $\{1, 2, \dots, 2n\}$ is a set*

$$\mathbf{m} = \{p_1, \dots, p_k\}$$

consisting of k disjoint pairs of elements from $\{1, 2, \dots, 2n\}$, for some $k \le m$. A pair $p = (i, j)$ is called balanced *if it consists of an odd number and an even number, and a partial matching $\mathbf{m}$ is called* balanced *if all the pairs p_i in $\mathbf{m}$ are balanced.*

Example 6.9. The following is a balanced partial matching of $\{1, \dots, 8\}$:

$$\{(2, 5), (3, 8), (6, 7)\}.$$

Given a partial matching $\mathbf{m} = \{(i_1, j_1), \dots, (i_k, j_k)\}$, we define an element $f_{\mathbf{m}} \in (Z^n_k)^\vee$ by

$$f_{\mathbf{m}} = \prod_{s=1}^{k} (X^\vee_{i_s} - X^\vee_{j_s}).$$

Lemma 6.10. *For every partial matching $\mathbf{m}$, the element $f_{\mathbf{m}}$ satisfies the relations* (17)*, and therefore can be viewed as an element of $Z(H^n)^\vee_{2k}$.*

Proof. When we expand

$$\Big(\prod_{s=1}^{k}(X_{i_s}^\vee - X_{j_s}^\vee)\Big)\big(\sum_{|I|=k,\ I\supseteq J} X_I\big) \tag{18}$$

we get nonzero contributions only if J is made of elements that all appear in $\mathbf{m}$, but such that each pair in $\mathbf{m}$ contains at most one element from J. Since $|J| < k$, there is at least one pair in $\mathbf{m}$, say (i_s, j_s), that does not contain elements of J. Thus, in the expansion of (18) we get $2^{k-|J|}$ terms of ± 1, and in fact half of them are $+1$ and half are -1: the terms coming from $X_{i_s}^\vee$ cancel out with those from $X_{j_s}^\vee$. □

We now exhibit a set of generators for $Z(H^n)_{2k}^\vee$. (Note that it will not usually be a basis.)

Proposition 6.11. *The elements $f_{\mathbf{m}}$, over all balanced partial matchings $\mathbf{m}$ of cardinality k, generate the group $Z(H^n)_{2k}^\vee$.*

Proof. Let us first show that $f_{\mathbf{m}}$, over all (not necessarily balanced) partial matchings $\mathbf{m}$ of cardinality k, generate $Z(H^n)_{2k}^\vee$. Let V be the Abelian group freely generated by partial matchings of cardinality k. We need to show that the linear homomorphism

$$V \to Z(H^n)_{2k}^\vee, \quad \mathbf{m} \mapsto f_{\mathbf{m}}$$

is surjective. This is equivalent to showing that its dual $Z(H^n)_{2k} \to V^\vee$ is injective. Proposition 6.5 tells us that a basis of $Z(H^n)_{2k}$ is given by X_I with $I \in A_k^n$. Therefore, what we need to check is that, if we have numbers $a_I \in \mathbb{Z}$ such that

$$f_{\mathbf{m}}\big(\sum_{I\in A_k^n} a_I X_I\big) = 0, \tag{19}$$

for all partial matchings $\mathbf{m}$ of cardinality k, then $a_I = 0$ for all $I \in A_k^n$.

We will prove this claim by induction on n. The base case $n = 0$ is clear, because $f_\emptyset = 1$.

For the inductive step, assume the corresponding statement is true for $n-1$ and all possible k. Suppose we have numbers a_I satisfying (19). Consider first the partial matchings $\mathbf{m}$ that consist only of pairs not involving the last two elements $2n-1$ and $2n$. For such $\mathbf{m}$, we have $f_{\mathbf{m}}(X_I) = 0$ when $I \cap \{2n-1, 2n\} \neq \emptyset$. If $I \in A_k^n$ has $I \cap \{2n-1, 2n\} = \emptyset$, then I is an admissible subset of $\{1, \dots, 2n-2\}$, and we can also view $\mathbf{m}$ as a partial matching of $\{1, \dots, 2n-2\}$. Applying the inductive hypothesis for $n-1$ and k, we deduce that

$$a_I = 0,\ \forall I \in A_k^n \text{ with } I \cap \{2n-1, 2n\} = \emptyset. \tag{20}$$

Next, consider an arbitrary partial matching $\mathbf{m}$ of $\{1, \dots, 2n-2\}$ of cardinality $k-1$. If $k < n$, there exists some $i \in \{1, \dots, 2n-2\}$ that does not appear in any of the pairs in $\mathbf{m}$. Define the matching

$$\mathbf{m}' = \mathbf{m} \cup \{(2n-1, i)\}$$

so that

$$f_{\mathbf{m}'} = (X_{2n-1}^\vee - X_i^\vee) \cdot f_{\mathbf{m}}.$$

Applying (19) for $\mathbf{m}'$, and using (20), we get

$$0 = f_{\mathbf{m}'}\big(\sum_{I\in A_k^n} a_I X_I\big) = f_{\mathbf{m}}\Big(\sum_{\substack{I=J\cup\{2n-1\}\\ J\in A_{k-1}^{n-1}}} a_I X_J\Big).$$

Since this is true for all possible $\mathbf{m}$, from the inductive hypothesis for $n-1$ and $k-1$, we deduce that

$$a_I = 0,\ \forall I \in A_k^n \text{ with } I \cap \{2n-1, 2n\} = \{2n-1\}. \tag{21}$$

Observe that if $I \cap \{2n-1, 2n\} = \{2n-1\}$, the admissibility condition (16) for I applied to $m = 2n-1$ shows that our hypothesis $k < n$ must be satisfied.

Let $\mathbf{m}$ still be a partial matching of $\{1, \dots, 2n-2\}$ of cardinality $k-1$, and set

$$\mathbf{m}'' = \mathbf{m} \cup \{(2n, 2n-1)\}$$

so that

$$f_{\mathbf{m}''} = (X^\vee_{2n} - X^\vee_{2n-1}) \cdot f_{\mathbf{m}}.$$

Applying (19) for $\mathbf{m}'$, and using (20) and (21), we find that

$$0 = f_{\mathbf{m}''}\Big(\sum_{I \in A^n_k} a_I X_I\Big) = f_{\mathbf{m}}\Big(\sum_{\substack{I = J \cup \{2n\} \\ J \in A^{n-1}_{k-1}}} a_I X_J\Big).$$

Applying the inductive hypothesis for $n-1$ and $k-1$, we deduce that

$$a_I = 0, \ \forall I \in A^n_k \text{ with } I \cap \{2n-1, 2n\} = \{2n\}. \tag{22}$$

Finally, consider an arbitrary partial matching $\mathbf{m}$ of $\{1, \dots, 2n-2\}$ of cardinality $k-2$. Since $k \leq n$, we can find $i, j \in \{1, \dots, 2n-2\}$ that are not in any pair in $\mathbf{m}$. Let

$$\mathbf{m}' = \mathbf{m} \cup \{(2n-1, i), (2n, j)\}$$

so that

$$f_{\mathbf{m}'} = (X^\vee_{2n-1} - X^\vee_i)(X^\vee_{2n} - X^\vee_j) \cdot f_{\mathbf{m}}.$$

Applying (19) for $\mathbf{m}'$, and using (20), (21) and (22), we get

$$0 = f_{\mathbf{m}'}\Big(\sum_{I \in A^n_k} a_I X_I\Big) = f_{\mathbf{m}}\Big(\sum_{\substack{I = J \cup \{2n-1, 2n\} \\ J \in A^{n-1}_{k-2}}} a_I X_J\Big).$$

From the inductive hypothesis for $n-1$ and $k-2$, we conclude that

$$a_I = 0, \ \forall I \in A^n_k \text{ with } \{2n-1, 2n\} \subset I. \tag{23}$$

This shows that all a_I vanish, and therefore $f_{\mathbf{m}}$ generate $Z(H^n)^\vee_{2k}$.

To see that $f_{\mathbf{m}}$ for balanced $\mathbf{m}$ also suffice to generate $Z(H^n)^\vee_{2k}$, we will prove that every $f_{\mathbf{m}}$ is a linear combination of the balanced ones. We will do this inductively: If $\mathbf{m}$ is not balanced, we will express $f_{\mathbf{m}}$ as a linear combination of elements corresponding to matchings that have fewer unbalanced pairs than $\mathbf{m}$.

An unbalanced partial matching $\mathbf{m}$ must contain a pair of odd elements, or a pair of even elements. Suppose it contains both: a pair (a, b) of odd elements, and a pair (c, d) of even elements. Using the relation

$$(X^\vee_a - X^\vee_b)(X^\vee_c - X^\vee_d) = (X^\vee_a - X^\vee_c)(X^\vee_b - X^\vee_d) + (X^\vee_a - X^\vee_d)(X^\vee_c - X^\vee_b)$$

we can turn $f_{\mathbf{m}}$ into a sum $f_{\mathbf{m}'} + f_{\mathbf{m}''}$, where $\mathbf{m}'$ and $\mathbf{m}''$ have two fewer unbalanced pairs than $\mathbf{m}$.

If $\mathbf{m}$ does not contain both types of unbalanced pairs, without loss of generality let us suppose it only contains pairs made of even elements, in addition to possibly some balanced pairs. In total, there are more even than odd elements in the pairs in $\mathbf{m}$, so there must be an odd number $a \in \{1, \dots, 2n\}$ that is not contained in any pair in $\mathbf{m}$. Let (b, c) be a pair in $\mathbf{m}$ with b and c both even. Using the relation

$$X^\vee_b - X^\vee_c = (X^\vee_b - X^\vee_a) + (X^\vee_a - X^\vee_c)$$

we can express $f_{\mathbf{m}}$ as $f_{\mathbf{m}'} + f_{\mathbf{m}''}$, where $\mathbf{m}'$ and $\mathbf{m}''$ have one fewer unbalanced pair compared to $\mathbf{m}$. This completes the proof. □

6.4. **Hochschild homology and cohomology.** To study the cabled Khovanov-Rozansky homology of the p-framed unknot, we need to first understand the homologies of the cables of (U, p). Since we restrict ourselves to level $\alpha = 0$, these cables are the $(2n, 2np)$-torus links $T'_{2n,2np}$, with n strands positively oriented and n strands negatively oriented, going through p full twists. The Khovanov homology of these links was studied by Stošić in [31]. There, for $p > 0$, he showed that

$$\mathrm{Kh}^{i,j}(T'_{2n,2np}) = 0 \ \text{ if } i > 0 \text{ or } j > 0$$

and, furthermore, in the maximal homological degree 0 we have

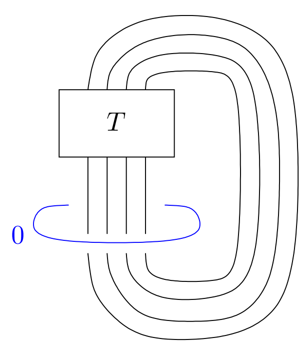
$$\mathrm{Kh}^{0,j}(T'_{2n,2np}) = \begin{cases} \mathbb{Z}^{\binom{2n}{n-k} - \binom{2n}{n-k-1}} & \text{if } j = -2k,\ k = 0, \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$

See [31, Corollaries 2 and 4]. For $p < 0$, the formula (5) for the Khovanov homology of the mirror gives

$$\mathrm{Kh}^{0,j}(T'_{2n,2np}) = \begin{cases} \mathbb{Z}^{\binom{2n}{n-k} - \binom{2n}{n-k-1}} & \text{if } j = 2k,\ k = 0, \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Note the similarity between these answers and (15). (When $n = 1$, this was first observed by Przytycki in [28].) In fact, a more direct relation between $\mathrm{Kh}^{0,j}(T'_{2n,2np})$ and $Z(H^n)$ comes from [30]. There, Rozansky constructs a Khovanov homology for framed links in $S^1 \times S^2$. We will only need the case of null-homologous links in $S^1 \times S^2$, in which case the framing dependence can be cancelled by a suitable shift in gradings, as shown by Willis in [32]. The Khovanov homology of a null-homologous link $L \subset S^1 \times S^2$ is a well-defined bi-graded group, defined as follows.

Suppose that L is given as the circular closure of a tangle T from $2n$ to $2n$ points. In the standard picture of $S^1 \times S^2$ as 0-surgery on the unknot, this corresponds to connecting the $2n$ pairs of points by arcs going through the unknot:



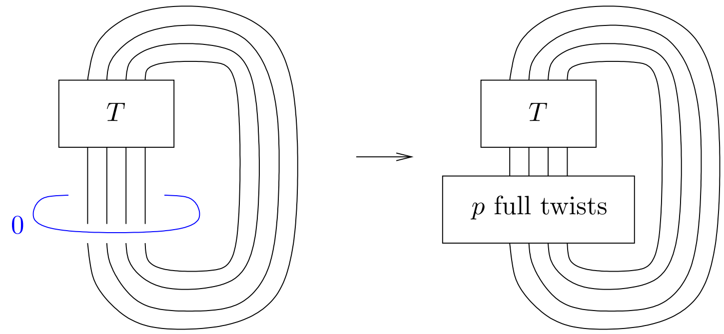
Recall from (12) that to the tangle T, Khovanov associated an (H^n, H^n)-bimodule $\mathcal{F}(T)$. The Khovanov homology of L is set to be the Hochschild homology of $\mathcal{F}(T)$:

$$\mathrm{Kh}^{i,*}(L) := \mathrm{HH}_i(\mathcal{F}(T)). \tag{24}$$

Moreover, in [30, Theorem 6.8], Rozansky shows that there is a canonical isomorphism

$$\mathrm{Kh}^{i,*}(L) \cong \mathrm{Kh}^{i,*}(L(p)) \quad \text{for} \quad i \geq n_+ - 2p + 2, \tag{25}$$

where n_+ is the number of positive crossings in a diagram for T, and $L(p) \subset S^3$ is the link obtained by inserting p full twists in the corresponding diagram for L at the place where the $2n$ arcs went through the 0-framed unknot:



Let us specialize to the case when $i = 0$ and T is the identity tangle $\mathrm{Id}_{n,n}$ on $2n$ alternately oriented strands. As in [23], we denote the corresponding link $L \subset S^1 \times S^2$ by $F_{n,n}$. We have

$$F_{n,n}(p) = T'_{2n,2np}.$$

We find that, for every $p > 0$,

$$\mathrm{Kh}^{0,*}(T'_{2n,2np}) \cong \mathrm{HH}_0(H^n).$$

Using (5), we get from here description of $\mathrm{Kh}^{0,*}(T'_{2n,2np})$ for $p < 0$. Then, the dual of the Hochschild homology of H^n is the Hochschild cohomology of H^n (up to a degree shift):

$$\mathrm{Kh}^{0,*}(T'_{2n,2np}) \cong \mathrm{Kh}^{0,*}(T'_{2n,-2np})^\vee \cong \mathrm{HH}_0(H^n)^\vee \cong \mathrm{HH}^0(H^n)\{-2n\} \quad \text{for } p < 0.$$

See [30, Theorem 6.9]. (Some care has to be taken with respect to grading conventions: Rozansky puts X in degree 2, so its quantum grading is the negative of the usual quantum grading in Kh.)

The zeroth Hochschild cohomology of a ring equals its center. Therefore, we have canonical isomorphisms

$$\mathrm{Kh}^{0,j}(T'_{2n,2np}) \cong Z(H^n)^\vee_{2n+j} \quad \text{for} \quad p > 0 \tag{26}$$

and

$$\mathrm{Kh}^{0,j}(T'_{2n,2np}) \cong Z(H^n)_{2n-j} \quad \text{for} \quad p < 0. \tag{27}$$

Let us re-write (26) and (27) in terms of the Khovanov-Rozansky homology KhR_2, which is related to Kh by the formula (6). We are interested in $T'_{2n,2np}$ as a framed link, in which every component has framing p. A diagram for this framed link is obtained from the standard diagram of the torus link $T'_{2n,2np}$ (which has writhe $-2np$) by adding p kinks in each component. The writhe of the resulting diagram is then $w = 0$. Therefore, in view of (6), we obtain:

$$\mathrm{KhR}_2^{0,j}(T'_{2n,2np}) \cong Z(H^n)_{2n+j} \quad \text{for} \quad p > 0 \tag{28}$$

and

$$\mathrm{KhR}_2^{0,j}(T'_{2n,2np}) \cong Z(H^n)^\vee_{2n-j} \quad \text{for} \quad p < 0. \tag{29}$$

Note that Theorem 6.3 gives an explicit description of the center $Z(H^n)$.

6.5. **Cobordism maps and the braid action.** In the definition of the cabled Khovanov-Rozansky homology we had the cobordism maps $\psi_i^{[m]}$. In our case, there is a single knot component, so we will drop the subscript $i = 1$ from the notation. Further, since $N = 2$, the values of m can be 0 or 1. We will simply write ψ for $\psi^{[0]}$ and ϕ for $\psi^{[1]}$.

Thus, we are interested in the maps:

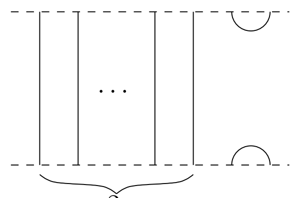
$$\psi = \mathrm{KhR}_2(Z)(\cdot \otimes 1) : \mathrm{KhR}_2^{0,j}(T'_{2n,2np}) \to \mathrm{KhR}_2^{0,j}(T'_{2n+2,(2n+2)p}) \tag{30}$$

and

$$\phi = \mathrm{KhR}_2(Z)(\cdot \otimes X) : \mathrm{KhR}_2^{0,j}(T'_{2n,2np}) \to \mathrm{KhR}_2^{0,i+2}(T'_{2n+2,(2n+2)p}), \tag{31}$$

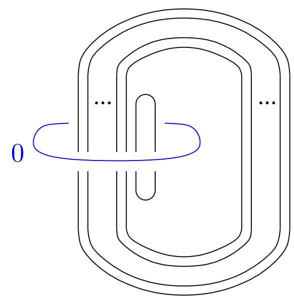
where $Z = Z_1$ is the saddle cobordism from $T'_{2n,2np} \sqcup U$ to $T'_{2n+2,(2n+2)p}$, which introduces two new strands in the cable; cf. Section 3. We are interested in computing ψ and ϕ as maps relating $Z(H^n)$ and $Z(H^{n+1})$, under the identifications (28) and (29).

For this, we introduce the $(n+1, n+1)$-tangle

$J :=$ 

We denote by $M = \mathcal{F}(J)$ the (H^{n+1}, H^{n+1})-bimodule associated to J.

Observe that the circular closure of J in $S^1 \times S^2$ is the following link:



This is the split disjoint union $F_{n,n} \sqcup U$, which can also be represented as the circular closure of the (n,n)-tangle $\mathrm{Id}_{n,n} \sqcup U$. Therefore, we have two different ways of expressing the Khovanov homology of $F_{n,n} \sqcup U$ in terms of Hochschild homology:

$$\mathrm{HH}_i(M) \cong \mathrm{Kh}^{i,*}(F_{n,n} \sqcup U) \cong \mathrm{Kh}^{i,*}(F_{n,n}) \otimes \mathcal{A} \cong \mathrm{HH}_i(H^n) \otimes \mathcal{A}. \tag{32}$$

To be in line with the conventions in this paper, we will work with KhR_2 instead of Kh; compare (6). Using the duality coming from the Frobenius trace on H^n as in [30, Section 6.3], we can turn statements about Hochschild homology into ones about Hochschild cohomology. Thus, from Equation (32) for $i = 0$, we deduce the existence of an isomorphism

$$\mathrm{HH}^0(H^n) \otimes \mathcal{A} \xrightarrow{\cong} \mathrm{HH}^0(M). \tag{33}$$

We have $\mathrm{HH}^0(H^n) = Z(H^n)$, whereas $\mathrm{HH}^0(M)$ is given by

$$\mathrm{HH}^0(M) = \{m \in M \mid mh = hm, \ \forall h \in H^{n+1}\}. \tag{34}$$

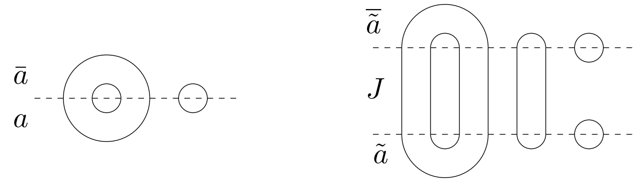
Proposition 6.12 below will give an explicit formula for the isomorphism in (33), up to sign. Before stating it, let us introduce some notation for certain elements of the bimodule M. Recall that

$$M = \bigoplus_{a,b \in \mathfrak{C}_{n+1}} {}_aM_b,$$

where ${}_aM_b$ is the complex associated to the link $\bar{b}Ja$.

Suppose we have a crossingless matching $a \in \mathfrak{C}_n$. From a we can construct crossingless matchings in $\mathfrak{C}_{n+1}$ in two ways. First, we get a matching $\tilde{a} \in \mathfrak{C}_{n+1}$ by connecting the last two endpoints p_{2n+1}

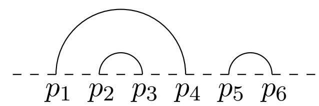
and p_{2n+2}. Then, the link $\bar{\tilde{a}} J \tilde{a}$ is the split disjoint union of $\bar{a}a$ and two unknots:



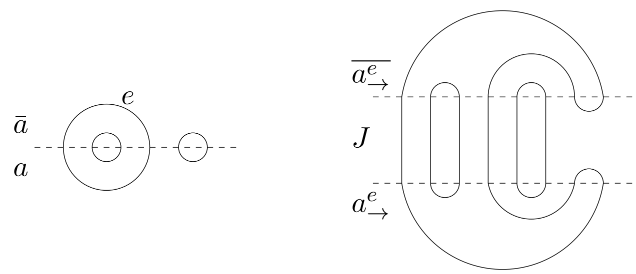
Given an element $x \in {}_a(H^n)_a$ and $w \in \mathrm{KhR}_2(U \sqcup U) \cong \mathcal{A} \otimes \mathcal{A}$, we get an element

$$x \otimes w \in {}_{\tilde{a}}M_{\tilde{a}} \subset M.$$

Second, given $a \in \mathfrak{C}_n$, let $\mathrm{Out}(a)$ denote the set of "outer" arcs in a, that is, those connecting points p_i and p_j (for $i < j$) such that no points p_k and p_l with $k < i < j < l$ are matched in a. For example, when a is the matching



the outer arcs are those from p_1 to p_4, and from p_5 to p_6. For $e \in \mathrm{Out}(a)$ connecting p_i to p_j with $i < j$, we define a crossingless matching $a^e_{\rightarrow} \in \mathfrak{C}_{n+1}$ by connecting p_i to p_{2n+2} and p_j to p_{2n+1}. Notice that the link $\overline{a^e_{\rightarrow}} J a^e_{\rightarrow}$ is diffeomorphic to $\bar{a}a$:



Under this diffeomorphism, an element $x \in {}_a(H^n)_a$ produces a corresponding element

$$x^e_{\rightarrow} \in {}_{a^e_{\rightarrow}}M_{a^e_{\rightarrow}} \subset M$$

We let

$$x_{\rightarrow} := \sum_{e \in \mathrm{Out}(a)} x^e_{\rightarrow} \in M.$$

The assignment $x \mapsto x_{\rightarrow}$ extends linearly to a map

$$\bigoplus_{a \in \mathfrak{C}_n} {}_a(H^n)_a \rightarrow \bigoplus_{a \in \mathfrak{C}_{n+1}} {}_aM_a, \quad x \mapsto x_{\rightarrow}$$

Furthermore, given $v \in \mathcal{A}$, we will denote by $x_{\rightarrow} \cdot v$ the result of acting by v using the multiplication coming from a small unknot near p_{2n}. In other words, in terms of the module action described at the end of Section 6.1, we identify the variable $X \in \mathcal{A}$ with X_{2n} and act accordingly.

Let us define a map

$$\Theta : Z(H^n) \otimes \mathcal{A} \rightarrow M, \quad \Theta(x \otimes v) = x \otimes \Delta(v) + x_{\rightarrow} \cdot v.$$

Observe that the map Θ is injective (because so is Δ).

Proposition 6.12. *The image of the map* Θ *is* $\mathrm{HH}^0(M) \subset M$*, and the isomorphism from* (33) *is given by* $\pm\Theta$*.*

Proof. We first determine $\mathrm{HH}^0(M)$ as a subset of M. Given $m \in \mathrm{HH}^0(M)$, let us write

$$m = \sum_{a,b \in \mathfrak{C}_{n+1}} {}_a m_b \ , \quad {}_a m_b \in {}_a M_b.$$

The defining property of elements $m \in \mathrm{HH}^0(M)$ is that they commute with all elements of H^{n+1}; cf. (34). In particular, they commute with the idempotents ${}_a 1_a$ corresponding to each $a \in \mathfrak{C}_{n+1}$. It follows that

$${}_a m_b = 0 \ \text{ if } a \neq b.$$

Note that every crossingless matching $b \in \mathfrak{C}_{n+1}$ is either of the form $\tilde{a}$ or $a^e_{\rightarrow}$, for some $a \in \mathfrak{C}_n$ and $e \in \mathrm{Out}(a)$. Let us write

$$m = m' + m'' \tag{35}$$

where

$$m' = \sum_{a \in \mathfrak{C}_n} {}_{\tilde{a}} m_{\tilde{a}} \quad \text{and} \quad m'' = \sum_{a \in \mathfrak{C}_n} \sum_{e \in \mathrm{Out}(a)} {}_{a^e_{\rightarrow}} m_{a^e_{\rightarrow}}. \tag{36}$$

Let us analyze the commutation relations between m and elements

$$h \in \bigoplus_{a \in \mathfrak{C}_n} {}_{\tilde{a}}(H^{n+1})_{\tilde{a}} \cong \bigoplus_{a \in \mathfrak{C}_n} {}_a(H^n)_a \otimes \mathcal{A} \cong H^n \otimes \mathcal{A}.$$

We have $m''h = hm'' = 0$, and therefore $m'h = hm'$. Notice that

$$m' \in \bigoplus_{a \in \mathfrak{C}_n} {}_{\tilde{a}} M_{\tilde{a}} \cong \bigoplus_{a \in \mathfrak{C}_n} {}_a(H^n)_a \otimes \mathcal{A}^{\otimes 2} \cong H^n \otimes \mathcal{A}^{\otimes 2}.$$

The relations $m'h = hm'$ (over all possible h) imply that $m' \in Z(H^n) \otimes \mathcal{A}^{\dagger}$, where

$$\mathcal{A}^{\dagger} = \{ w \in \mathcal{A}^{\otimes 2} \mid w \otimes x = x \otimes w \in \mathcal{A}^{\otimes 3}, \text{ for all } \ x \in \mathcal{A} \}.$$

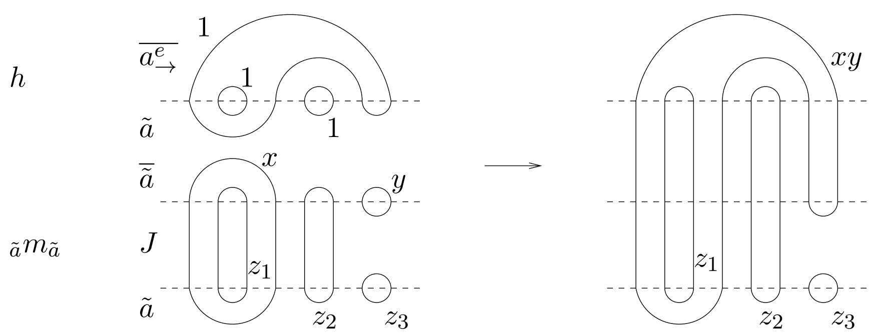
It is easy to see that $\mathcal{A}^{\dagger}$ is spanned by $1 \otimes X + X \otimes 1$ and $X \otimes X$, that is, it coincides with the image of the comultiplication Δ. Therefore,

$$m' \in Z(H^n) \otimes \Delta(\mathcal{A}). \tag{37}$$

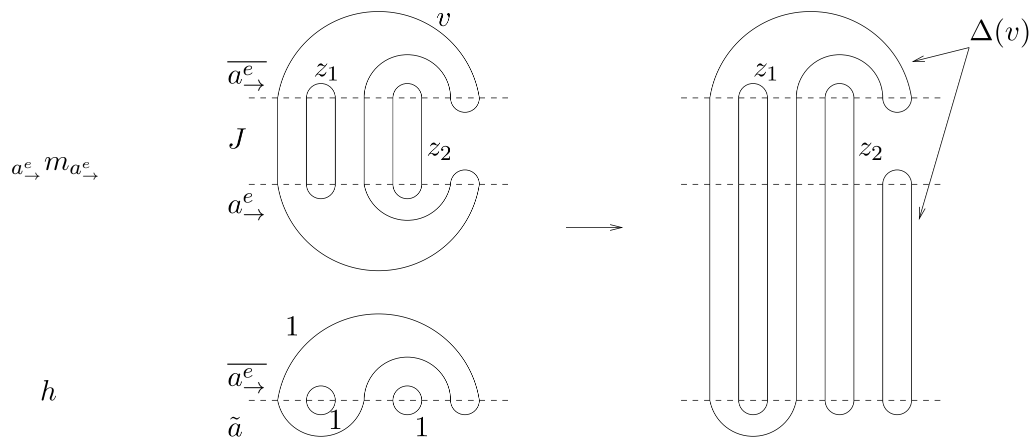
Next, let $a \in \mathfrak{C}_n$ and $e \in \mathrm{Out}(a)$. Consider the element $h \in {}_{\tilde{a}}(H^n)_{a^e_{\rightarrow}}$ obtained by marking with 1 all the circles in the tangle $\overline{a^e_{\rightarrow}}\tilde{a}$. The commutation $mh = hm$ reduces to the relation:

$$({}_{\tilde{a}} m_{\tilde{a}}) \cdot h = h \cdot ({}_{a^e_{\rightarrow}} m_{a^e_{\rightarrow}}). \tag{38}$$

Observe that the left hand side of (38) is given by multiplying the values from two circles in $\overline{a^e_{\rightarrow}} J \tilde{a}$ (the one containing the arc e and the one at the top through p_{2n+1} and p_{2n+2}), while keeping the values on the other circles of $\overline{a^e_{\rightarrow}} J \tilde{a}$ the same, as in the following example:



On the other hand, the right hand side of (38) is given by the comultiplication Δ applied to the copy of $\mathcal{A}$ coming from the circle going through the last two points p_{2n+1} and p_{2n+2}:



Therefore, if we write

$$_{\tilde{a}}m_{\tilde{a}} = \sum_i x_i \otimes \Delta(v_i) \in {}_a(H^n)_a \otimes \Delta(\mathcal{A}),$$

from (38) we deduce that

$$_{a^e_{\rightarrow}}m_{a^e_{\rightarrow}} = \sum_i (x_i)^e_{\rightarrow} \cdot v_i. \tag{39}$$

It follows from (35), (36), (37) and (39) that m is in the image of Θ. Conversely, it can be checked that all the elements in the image of Θ commute with every $h \in H^{n+1}$. Therefore,

$$\mathrm{HH}^0(M) = \mathrm{Im}(\Theta) \subset M.$$

To pin down the isomorphism (33), we use the module action by $R^{\otimes(2n+1)}$. Recall from Proposition 6.7 the description of $\mathrm{HH}^0(H^n) = Z(H^n)$ as an $R^{\otimes 2n}$-algebra. It follows that

$$Z(H^n) \otimes \mathcal{A} \cong R^{\otimes(2n+1)} / \big\langle \sum_{|I|=k} X_I, \quad k = 1, \dots, 2n \big\rangle.$$

On the other hand, as noted at the end of Section 6.1, an H^{n+1}-module such as M admits an action of R^{2n+2} (say, from the points at the bottom of the tangle). Since the last two points are connected in the tangle J, it follows that X_{2n+2} acts on M by $-X_{2n+1}$. We can thus focus on the action of $R^{\otimes(2n+1)}$ on M, using the first $2n+1$ variables. This descends to an action of $R^{\otimes(2n+1)}$ on $\mathrm{HH}^0(M) \subset M$. The constructions in [30] preserve the $R^{\otimes(2n+1)}$ actions, and therefore the isomorphism (33) is one of (graded) $R^{\otimes(2n+1)}$-modules.

Observe also that the map Θ preserves the module actions. Since the only graded automorphisms of

$$R^{\otimes(2n+1)} / \big\langle \sum_{|I|=k} X_I, \quad k = 1, \dots, 2n \big\rangle \tag{40}$$

as $R^{\otimes(2n+1)}$-modules are $\pm\,\mathrm{Id}$, it follows that given two modules isomorphic to (40), the isomorphism between them is uniquely determined up to sign. Therefore, (33) must be given by $\pm\Theta$. (We conjecture that it is Θ.) □

We can now compute the maps from (30) and (31).

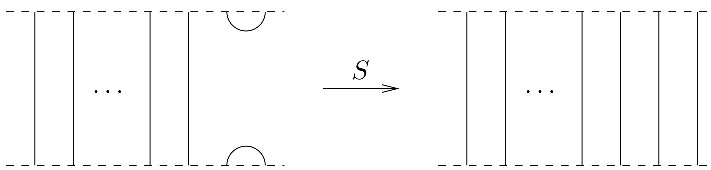
Proposition 6.13. *Let $p > 0$. Under the identification* (28)*, and using the description of $Z(H^n)$ from Theorem 6.3, the maps ψ and ϕ from* (30), (31) *are given by*

$$\psi : Z(H^n)_{2n+j} \to Z(H^{n+1})_{2n+2+j}, \quad \psi(X_I) = \pm X_I \cdot (X_{2n+2} - X_{2n+1}) \tag{41}$$

and

$$\phi : Z(H^n)_{2n+j} \to Z(H^{n+1})_{2n+4+j}, \quad \phi(X_I) = \pm X_I \cdot (-X_{2n+1}X_{2n+2}). \tag{42}$$

Proof. Consider the saddle cobordism S from the $(n+1, n+1)$-tangle J the identity $\mathrm{Id}_{n+1,n+1}$:



This induces a cobordism map between the associated (H^{n+1}, H^{n+1})-bimodules:

$$M = \mathcal{F}(J) \xrightarrow{\mathcal{F}(S)} \mathcal{F}(\mathrm{Id}_{n+1,n+1}) = H^{n+1}.$$

By restriction, we get a map on Hochschild cohomology

$$\mathcal{F}(S) : \mathrm{HH}^0(M) \to \mathrm{HH}^0(H^{n+1}).$$

By taking circular closures in $S^1 \times S^2$, the cobordism S produces a cobordism in $[0,1] \times S^1 \times S^2$ between the links $F_{n,n} \sqcup U$ and $F_{n+1,n+1}$. Furthermore, by introducing p full twists in place of the 0-framed unknot, we get the saddle cobordism Z from $T'_{2n,2np} \sqcup U$ to $T'_{2n+2,(2n+2)p}$ which produces the maps ψ and ϕ.

Equation (25) relates the Khovanov homology of a link $L \subset S^1 \times S^2$ to its counterpart $L(p) \subset S^3$ obtained by introducing $p > 0$ full twists. We get an isomorphism in homological degree $i = 0$ provided that L has no positive crossings. The arguments given in [30] apply equally well to saddle cobordisms such as S, showing that there is a commutative diagram

$$\begin{array}{ccccc}
\mathrm{KhR}_2^{0,*}(T'_{2n,2np} \sqcup U) & & \xrightarrow{\quad \mathrm{KhR}_2^{0,*}(Z) \quad} & & \mathrm{KhR}_2^{0,*}(T'_{2n+2,(2n+2)p}) \\
\downarrow & & & & \downarrow \\
Z(H^n) \otimes \mathcal{A} & \xrightarrow{\cong} & \mathrm{HH}^0(M) & \xrightarrow{\mathcal{F}(S)} & \mathrm{HH}^0(H^{n+1}).
\end{array}$$

The first isomorphism in the bottom row is (33), which is $\pm\Theta$ according to Proposition 6.12. Therefore,

$$\psi(x) = \pm\mathcal{F}(S)(\Theta(x \otimes 1)), \quad \phi(x) = \pm\mathcal{F}(S)(\Theta(x \otimes X).$$

The maps ψ and ϕ preserve the $R^{\otimes 2n}$-module action, so to describe them it suffices to evaluate them on 1. We have

$$\begin{aligned}
\psi(1) &= \pm\mathcal{F}(S)(\Theta(x \otimes 1)) \\
&= \pm\mathcal{F}(S)(1 \otimes \Delta(1) + 1_{\rightarrow}) \\
&= \pm\mathcal{F}(S)\Big(\sum_{a \in \mathfrak{C}_n} ({}_{\tilde{a}}1_{\tilde{a}}) \otimes (1 \otimes X + X \otimes 1) + \sum_{a \in \mathfrak{C}_n} \sum_{e \in \mathrm{Out}(a)} 1^e_{\rightarrow}\Big) \\
&= \pm\Big(\sum_{a \in \mathfrak{C}_n} 2({}_{\tilde{a}}(X_{2n+2})_{\tilde{a}}) + \sum_{a \in \mathfrak{C}_n} \sum_{e \in \mathrm{Out}(a)} \big({}_{a^e_{\rightarrow}}(X_{2n+2})_{a^e_{\rightarrow}} - {}_{a^e_{\rightarrow}}(X_{2n+1})_{a^e_{\rightarrow}}\big)\Big) \\
&= \pm(X_{2n+2} - X_{2n+1}),
\end{aligned}$$

where in the last equation we used the fact that $X_{2n+1} = -X_{2n+2}$ on summands of H^{n+1} of the form ${}_{\tilde{a}}(H^{n+1})_{\tilde{a}}$. This proves (41).

Moreover,

$$\begin{aligned}
\phi(1) &= \pm\mathcal{F}(S)(\Theta(x\otimes X))\\
&= \pm\mathcal{F}(S)(1\otimes\Delta(X)+1_{\to}\cdot X)\\
&= \pm\mathcal{F}(S)\Big(\sum_{a\in\mathfrak{C}_n}({}_{\tilde a}1_{\tilde a})\otimes X\otimes X+\sum_{a\in\mathfrak{C}_n}\sum_{e\in\mathrm{Out}(a)}1^{e}_{\to}\cdot X_{2n+2}\Big)\\
&= \pm\Big(0+\sum_{a\in\mathfrak{C}_n}\sum_{e\in\mathrm{Out}(a)}\big({}_{a^{e}_{\to}}(-X_{2n+1}X_{2n+2})_{a^{e}_{\to}}\big)\Big)\\
&= \mp(X_{2n+1}X_{2n+2}).
\end{aligned}$$

This proves (42). □

Proposition 6.14. *Let $p<0$. Under the identification* (29)*, and using the description of $Z(H^n)$ from Theorem 6.3, the maps ψ and ϕ from* (30)*,* (31) *are*

$$\psi : Z(H^n)^{\vee}_{2n-j}\to Z(H^{n+1})^{\vee}_{2n+2-j}$$

given by

$$\psi(f)(X_I)=\pm\begin{cases} f(X_J) & \text{if } I=J\cup\{2n+2\},\ J\subset\{1,\dots,2n\}\\ -f(X_J) & \text{if } I=J\cup\{2n+1\},\ J\subset\{1,\dots,2n\}\\ 0 & \text{otherwise,}\end{cases}\tag{43}$$

and

$$\phi : Z(H^n)^{\vee}_{2n-j}\to Z(H^{n+1})^{\vee}_{2n-j}$$

given by

$$\phi(f)(X_I)=\pm\begin{cases} f(X_I) & \text{if } I\subset\{1,\dots,2n\}\\ 0 & \text{otherwise.}\end{cases}\tag{44}$$

Proof. This is similar to Proposition 6.13, except we should consider the reverse saddle cobordism S^r from the identity tangle $\mathrm{Id}_{n+1,n+1}$ to J. This gives a map on Hochschild cohomology

$$\mathcal{F}(S^r):\mathrm{HH}^0(H^{n+1})\to\mathrm{HH}^0(M).$$

We get a commutative diagram

$$\begin{array}{ccccc}
\big(\mathrm{KhR}_2^{0,*}(T'_{2n+2,(2n+2)p})\big)^{\vee} & & \xrightarrow{(\mathrm{KhR}_2^{0,*}(Z))^{\vee}} & & \big(\mathrm{KhR}_2^{0,*}(T'_{2n,2np}\sqcup U)\big)^{\vee}\\
\downarrow & & & & \downarrow\\
\mathrm{HH}^0(H^{n+1}) & \xrightarrow{\mathcal{F}(S^r)} & \mathrm{HH}^0(M) & \xrightarrow{\cong} & Z(H^n)\otimes\mathcal{A}
\end{array}$$

where the isomorphism in the last arrow at the bottom is $\pm\Theta^{-1}$. To compute $\Theta^{-1}\circ\mathcal{F}(S^r)$, observe that this preserves the $R^{\otimes 2n}$-module structure, and therefore it suffices to evaluate it on 1, X_{2n+1}, X_{2n+2} and $X_{2n+1}X_{2n+2}$. A straightforward calculation gives

$$\begin{aligned}
&(\Theta^{-1}\circ\mathcal{F}(S^r))(1)=1\otimes 1,\\
&(\Theta^{-1}\circ\mathcal{F}(S^r))(X_{2n+2})=(\Theta^{-1}\circ\mathcal{F}(S^r))(-X_{2n+1})=1\otimes X,\\
&(\Theta^{-1}\circ\mathcal{F}(S^r))(X_{2n+1}X_{2n+2})=0.
\end{aligned}$$

This describes (up to sign) the dual of the map $\mathrm{KhR}_2^{0,*}(Z)$. By taking duals, we obtain the desired description of the maps ψ and ϕ. □

Remark 6.15. In Propositions 6.13 and 6.14 we only specified the maps ϕ and ψ up to a sign. We conjecture that all symbols $\pm$ should be $+$, and $\mp$ should be $-$.

One last ingredient in the definition of cabled Khovanov-Rozansky homology is the braid group action. In [16, Section 5], Khovanov proves that the braid group action on H^n induces an action of the symmetric group S_{2n} on the center $Z(H^n)$, which permutes the variables X_i. In our case, we are interested in the subgroup $B_{n,n}$, which acts on $Z(H^n)$ via the product $S_n \times S_n$. The first factor permutes the odd variables X_{2i+1}, and the second the even variables X_{2i}.

6.6. **Proof of Proposition 6.1.** From the definition of cabled Khovanov-Rozansky homology, we have

$$\underline{\mathrm{KhR}}_{2,0}^{0,j}(U,p) = \Big(\bigoplus_{n\in\mathbb{N}} \mathrm{KhR}_2^{0,2n+j}(T'_{2n,2np})\Big)/\sim$$

where we divide by the linear and transitive closure of the relations of the form

$$\beta_i(b)v \sim v, \quad \psi(v) \sim 0, \quad \phi(v) \sim v. \tag{45}$$

In the case $p > 0$, using (28) we get

$$\underline{\mathrm{KhR}}_{2,0}^{0,j}(U,p) \cong \Big(\bigoplus_{n\in\mathbb{N}} Z(H^n)_{4n+j}\Big)/\sim$$

Since we divide by the relations $\psi(v) \sim 0$, and ψ is given in (41) by multiplication with $\pm(X_{2n+2} - X_{2n+1})$, we find that the variables X_{2n+1} and X_{2n+2} are identified in the quotient. Using (42), we get that, up to a sign, ϕ is given by multiplication with

$$X_{2n+1}X_{2n+2} = X_{2n+1}^2 = 0.$$

Therefore, after dividing by the relations $\phi(v) \sim v$, everything collapses to zero:

$$\underline{\mathrm{KhR}}_{2,0}^{0,j}(U,p) = 0.$$

Let us now consider the case $p < 0$. Using (29), we get

$$\underline{\mathrm{KhR}}_{2,0}^{0,j}(U,p) = \Big(\bigoplus_{n\in\mathbb{N}} Z(H^n)^\vee_{-j}\Big)/\sim \tag{46}$$

It follows that $\underline{\mathrm{KhR}}_{2,0}^{0,*}(U,p)$ is supported in quantum gradings of the form $j = -2k$ for $k \geq 0$.

We start by looking at the quantum grading $j = 0$. From Theorem 6.3 we see that each $Z(H^n)_0$ is a copy of $\mathbb{Z}$ (generated by 1), and hence the same is true for $Z(H^n)^\vee_0$. In the equivalence relation we have no relations of the form $\psi(v) \sim 0$, because the targets of the maps ψ are in degrees $j < 0$. The braid group action is the identity, and from (44) we see that the maps

$$\phi : Z(H^n)^\vee_0 \to Z(H^{n+1})^\vee_0$$

are isomorphisms. Hence, the relations $\phi(v) \sim v$ identify together all the different $Z(H^n)_0 \cong \mathbb{Z}$, and we have

$$\underline{\mathrm{KhR}}_{2,0}^{0,0}(U,p) \cong \mathbb{Z}.$$

Next, we look at quantum gradings $j = -2k$ with $k > 0$. Using the notation $X_I^\vee$ from Section 6.3, the formula (43) for the map ψ can be re-written as

$$\psi(f) = \pm f \cdot (X^\vee_{2n+2} - X^\vee_{2n+1}).$$

Consider the elements $f_{\mathbf{m}} \in Z(H^n)^\vee_{2k}$, where $\mathbf{m}$ is a partial matching of $\{1, \ldots, 2n\}$; cf. Lemma 6.10. We have

$$\psi(f_{\mathbf{m}}) = \pm f_{\mathbf{m}\,\cup\{(2n+2,2n+1)\}}.$$

Therefore, after dividing by the relations $\psi(v) \sim 0$, all elements of the form $f_{\mathbf{m}}$ are set to zero, provided that $\mathbf{m}$ is a matching containing the last pair $(2n + 2, 2n + 1)$.

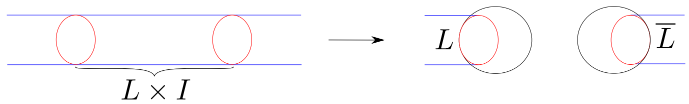


FIGURE 3. The neck-cutting lemma.

If $\mathbf{m}$ is a nonempty balanced partial matching of $\{1, 2, \dots, 2n+2\}$, the action of a suitable element in the braid group $B_{n+1,n+1}$ (factoring through $S_{n+1} \times S_{n+1}$) can take $f_{\mathbf{m}}$ to $\pm f_{\mathbf{m}'}$ where $\mathbf{m}'$ is a balanced matching containing the pair $(2n+2, 2n+1)$. It follows that all $f_{\mathbf{m}}$ are set to zero, for nonempty balanced partial matchings $\mathbf{m}$. Proposition 6.11 says that these elements $f_{\mathbf{m}}$ generate $Z(H^{n+1})^{\vee}_{2k}$, and therefore the whole group collapses to zero after we divide by the equivalence relation.

This concludes the proof of Proposition 6.1 and hence of Theorem 1.3.

7. Connected Sums

In this section, we prove Theorem 1.4. We will work with coefficients in a field $\Bbbk$. We write $\mathrm{KhR}_N(L;\Bbbk)$ for the $\mathfrak{gl}_N$ Khovanov-Rozansky homology of the framed link L with coefficients in $\Bbbk$. We write $\mathcal{S}_0^N(W;L;\Bbbk)$ for the skein lasagna module obtained using $\mathrm{KhR}_N(L;\Bbbk)$ instead of $\mathrm{KhR}_N(L)$.

Remark 7.1. If $\mathrm{char}(\Bbbk)=0$, then $\mathrm{KhR}_N(L;\Bbbk) \cong \mathrm{KhR}_N(L)\otimes_{\mathbb{Z}}\Bbbk$ and $\mathcal{S}_0^N(W;L;\Bbbk) \cong \mathcal{S}_0^N(W;L)\otimes_{\mathbb{Z}}\Bbbk$. In general, this is not true, because of the presence of Tor terms in the universal coefficients theorem.

Recall that, over a field, the Khovanov-Rozansky homology of a link L is isomorphic to the dual of the Khovanov-Rozansky homology of the mirror $\overline{L}$:

$$\mathrm{KhR}_N(L;\Bbbk) \cong \mathrm{KhR}_N(\overline{L};\Bbbk)^{\vee}. \tag{47}$$

Given an element $v \in \mathrm{KhR}_N(L;\Bbbk)$, we will write $v^{\vee}$ for its dual in $\mathrm{KhR}_N(\overline{L};\Bbbk)$. In terms of a basis $\{u_i\}$ for $\mathrm{KhR}_N(L;\Bbbk)$, the isomorphism (47) comes from the element

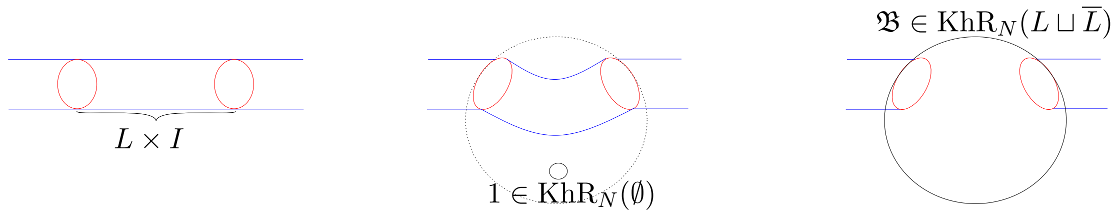
$$\mathfrak{B} := \sum_i u_i \otimes u_i^{\vee} \in \mathrm{KhR}_N(L \sqcup \overline{L};\Bbbk) \cong \mathrm{KhR}_N(L;\Bbbk) \otimes \mathrm{KhR}_N(\overline{L};\Bbbk).$$

We will need a lemma for cutting necks of surfaces in lasagna fillings.

Lemma 7.2. *Let F be a lasagna filling of W with boundary L and surface Σ, which we decompose as $\Sigma = \Sigma^{\circ} \cup (L \times I)$ for some link $L \subset \mathbb{R}^3$, and such that $L \times I$ is contained in a ball. Given $v \in \mathrm{KhR}_N(L;\Bbbk)$, let $\widetilde{F}(v)$ be the lasagna filling specified by the same data as F, except we replace the neck $L \times I$ with two input balls: one decorated with the link L and labelled by v and the other decorated with $\overline{L}$ and labelled by $v^{\vee}$. (See Figure 3.) If $\{u_i\}$ is a basis for $\mathrm{KhR}_N(L;\Bbbk)$, then:*

$$[F] = \sum_i [\widetilde{F}(u_i)] \in \mathcal{S}_0^N(W;L;\Bbbk).$$

Proof. First observe that we can always add an input ball decorated by the empty set and labelled by $1 \in \mathrm{KhR}_N(\emptyset)$ to any filling without affecting its class in $\mathcal{S}_0^N$. Add such an input ball near the neck $L \times I$. Enclose this new input ball together with the neck inside of a larger ball, as in Figure 4. We can thereby view the cylindrical neck $L \times I$ as a cobordism from the empty set to $L \sqcup \overline{L}$. The image of $1 \in \mathrm{KhR}_N(\emptyset)$ under this cobordism map is $\mathfrak{B}$. Evaluating this cobordism gives a sum of fillings with the new ball decorated with $L \sqcup \overline{L}$ and labelled by the $u_i \otimes u_i^{\vee}$. The claim follows by splitting this input ball into two balls, one with L and the other with $\overline{L}$. ∎



The neck $L \times I$ inside of the surface Σ

Add a new input ball decorated with $\emptyset$ and labelled by 1.

Evaluate the cobordism in the enclosed region to obtain an equivalent diagram.

FIGURE 4. Proof of the neck-cutting lemma.

We now provide the proof of the tensor product formula for boundary connected sums.

Proof of Theorem 1.4. We define the isomorphism

$$\Psi : \mathcal{S}_0^N(W_1; L_1; \Bbbk) \otimes \mathcal{S}_0^N(W_2; L_2) \to \mathcal{S}_0^N(W_1 \natural W_2; L_1 \cup L_2; \Bbbk)$$

on simple tensors by setting $\Psi([F_1] \otimes [F_2])$ to be the lasagna filling represented by $F_1 \cup F_2$.

We define an inverse to Ψ as follows. The boundary connected sum is obtained from W_1 and W_2 by identifying 3-dimensional balls $B_1 \subset \partial W_1$ and $B_2 \subset \partial W_2$; we write B for $B_1 = B_2$ as a subset of $W_1 \natural W_2$. Let F be a lasagna filling of $W_1 \natural W_2$ with boundary $L_1 \cup L_2$ and surface Σ. After an isotopy, we can arrange that:

(a) The input balls for F are disjoint from B;
(b) The surface Σ intersects B transversely in a link J.

Decompose $\Sigma = \Sigma_1 \cup_L \Sigma_2$ where $\Sigma_i \subset W_i$. We can apply Lemma 7.2 to cut along J and obtain

$$[F] = \sum_i [\widetilde{F}(u_i)] \in \mathcal{S}_0^N(W_1 \natural W_2; L_1 \cup L_2; \Bbbk)$$

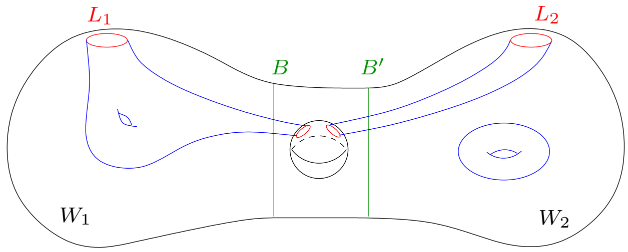
where each $\widetilde{F}(u_i)$ is of the form $F_i^1 \cup F_i^2$, with fillings F_i^j of W_j with boundary L_j, $j = 1, 2$. Then $\Psi(\sum_i [F_i^1] \otimes [F_i^2]) = [F]$, and we set

$$\Psi^{-1}([F]) = \sum_i [F_i^1] \otimes [F_i^2].$$

We need to make sure that Ψ^{-1} is well-defined. Filling in one of the input balls of F (in either W_1 or W_2) with another lasagna filling does not change the equivalence classes $[F_i^1]$ and $[F_i^2]$, so the value of Ψ^{-1} is unchanged.

What is left to show is that $\Psi^{-1}([F])$ does not depend on the choice of isotopy used to ensure the conditions (a) and (b) above. Consider an isotopy that moves the lasagna filling $F = F_{(0)}$ in a family $F_{(t)}, t \in [0, 1]$, such that the final filling $F(1)$ also satisfies (a) and (b).

With regard to (a), we can imagine the input balls of the fillings to be small (i.e., neighborhoods of points). Generically, in a one-parameter family such as $F_{(t)}$, there can be finite many times t where an input ball passes from one side of B to the other. Moving the input ball to the other side is equivalent to replacing B with an isotopic ball B', such that the region between B and B' is a cylinder $B^3 \times [0,1]$:



We obtain $\Psi^{-1}([F])$ in one case by cutting the filling F along B, and in the other case by cutting it along B'. By Lemma 7.2, both of these are equivalent to cutting along both B and B', and therefore equivalent to each other.

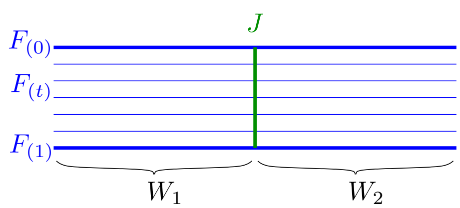
To deal with (b), without loss of generality, we can now assume that throughout the isotopy $F_{(t)}$, the input balls do not intersect B. Let $\Sigma_{(t)}$ be the surfaces for $F_{(t)}$. The intersections

$$J_{(t)} := \Sigma_{(t)} \cap B$$

may fail to be transverse at various points in $(0,1)$, but generically we can assume that the union

$$J = \bigcup_{t\in[0,1]} \big(\{t\} \times J_{(t)}\big) \subset [0,1] \times B$$

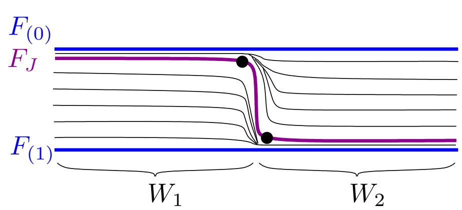
is a smooth link cobordism between the links $J_{(0)}$ and $J_{(1)}$. Schematically, we draw this as:



We identify $[0,1] \times B$ with a neighborhood of B in $W_1 \natural W_2$, and insert J in there. By smoothing the corners of

$$\big(F_{(0)}|_{W_1}\big) \cup J \cup \big(F_{(1)}|_{W_2}\big)$$

we obtain a new lasagna filling F_J. This is isotopic to $F_{(0)}$ by an isotopy supported in W_2 and is isotopic to $F_{(1)}$ by an isotopy supported in W_1:



For example, the isotopy between $F_{(0)}$ and F_J is given at time t by smoothing the corners of

$$\big(F_{(0)}|_{W_1}\big) \cup \bigcup_{s\in[0,t]} \big(\{s\} \times J_{(s)}\big) \cup \big(F_{(t)}|_{W_2}\big).$$

Applying Ψ^{-1} to $F_{(0)}$ consists in cutting its neck at $\{0\} \times B$, which is equivalent to cutting the neck of F_J at $\{0\}\times B$ (because they are related by an isotopy supported in W_2). Similarly, applying Ψ^{-1} to $F_{(1)}$ is equivalent to cutting the neck of F_J at $\{1\} \times B$. From Lemma 7.2 we see that the results of cutting F_J at $\{0\} \times B$ and $\{1\} \times B$ are equivalent, because they are each equivalent to cutting the neck in both places.

This completes the proof of well-definedness for Ψ^{-1}. The fact that Ψ and Ψ^{-1} are inverse to each other is immediate from the construction. □

We can also deduce the same result for interior connected sums.

Corollary 7.3. *Let $(W_1; L_1)$ and $(W_2; L_2)$ be a pair of 4-manifolds with links in the boundaries. Let $W_1 \# W_2$ denote their interior connected sum. Then,*

$$\mathcal{S}_0^N(W_1 \# W_2; L_1 \cup L_2; \Bbbk) \cong \mathcal{S}_0^N(W_1; L_1; \Bbbk) \otimes \mathcal{S}_0^N(W_2; L_2; \Bbbk)$$

Proof. By Proposition 2.1, we can add and remove small 4-balls without affecting $\mathcal{S}_0^N$. We remove a small 4-ball from each of the W_i, then perform the boundary connect sum along 3-balls in the new 3-sphere boundary components. The two boundaries glue together to give a 3-sphere boundary component in the connected sum, which we then fill in with a 4-ball to obtain $W_1 \# W_2$. Applications of Theorem 1.4 and Proposition 2.1 give the result. □

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