

## CONTEXTUAL AREAS

# Fairness and Efficiency in Multiportfolio Optimization

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We deal with the problem faced by a portfolio manager in charge of multiple accounts. We argue that because of market impact costs, this setting differs in several subtle ways from the classical (single account) case, with the key distinction being that the performance of each individual account typically depends on the trading strategies of other accounts, as well. We propose a novel, tractable approach for jointly optimizing the trading activities of all accounts and also splitting the associated market impact costs between the accounts. Our approach allows the manager to balance the conflicting objectives of maximizing the aggregate gains from joint optimization and distributing them across the accounts in an equitable way. We perform numerical studies that suggest that our approach outperforms existing methods employed in the industry or discussed in the literature.

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## 1. Introduction

Since the seminal work of Markowitz (1952), multiple facets and extensions of the portfolio optimization problem have been studied in the literature of modern portfolio theory. A key realization in this context has been that maintaining an optimal portfolio for a client often involves considerable levels of trading, which incur transaction costs that can be substantial enough to erase true investment returns (see, e.g., Perold 1988, Kolm 2009, Johnson and Tabb 2007). From a regulatory angle, this has led to the Securities and Exchange Commission (SEC) adopting clear rules governing the behavior of investment advisers, commonly referred to as best execution rules: “As a fiduciary, an adviser has an obligation to obtain ‘best execution’ of clients’ transactions. In meeting this obligation, an adviser must execute securities transactions for clients in such a manner that the clients’ total cost or proceeds in each transaction is the most favorable under the circumstances” (Securities and Exchange Commission 2011). From an academic viewpoint, this has resulted in the development of several models that appropriately capture the many sources of transaction costs incurred when executing trades and mitigate their negative effects on net returns (see Fabozzi et al. Chap. 11, 2010 and references therein for a detailed discussion). In the vast majority of these studies, researchers have focused on a setting where a financial adviser is acting on behalf of a *single* client in order to optimally select, rebalance, or liquidate her portfolio.

In practice, however, financial service providers rarely manage a single portfolio (or account) because they typi-

cally offer their services to *multiple* clients simultaneously. Such providers range from wealth management firms serving few individual investors to large investment firms in charge of many hundreds of pension, mutual, and insurance funds. In fact, there is also a recent trend in the U.S. finance industry toward consolidation of asset management firms, the most notable examples being the acquisition of Barclays Global Investors by BlackRock and the Morgan Stanley/Smith Barney merger in 2009. As a result, the same manager can often end up advising multiple portfolios, with either similar or quite different sizes and compositions, reflecting potentially different objectives and requirements, levels of risk aversion, etc.; see Savelsbergh et al. (2010).

Some of the challenges faced by a financial adviser in charge of multiple portfolios are common with the classical (single) portfolio case, e.g., the uncertainty of the returns, the constraints on the positions that can be taken or on the risk involved, etc. Thus, a natural question to ask is whether the models and results developed in the literature for a single portfolio should be directly applied in the case of multiple portfolios. More precisely, is it optimal to simply treat the portfolios independently and simply apply the principles of (single) portfolio theory to manage each? Unfortunately, the answer to both questions is *no*: leading practitioners from Deutsche Bank, Goldman Sachs, and Axioma Inc. have recently recognized that as the number and/or size of portfolios under management grows, unique issues pertaining to the transaction costs arise, which—if *not properly accounted for*—can erase true investment gains (see O’Cinneide et al. 2006, Khodadadi et al. 2006, Stubbs and

Vandenbussche 2009). This calls for a new approach that extends existing single portfolio models by explicitly capturing all the relevant aspects that pertain specifically to a multiportfolio setting, while remaining well suited for use in practice. This is the main goal of the present paper.

The crux of the difference between the single and multiportfolio setting lies in *market impact costs*. These originate from price impact, as well as limited “at-the-money” liquidity and are the primary drivers of transaction costs as the trading volume increases.<sup>1</sup> A problematic interaction arises between the multiple portfolios because of market impact because the transaction costs incurred by each portfolio heavily depend on the trading activity of other portfolios as well. For instance, consider a situation where an adviser manages two portfolios, A and B. Portfolio A submits a large buy order for a particular asset. In case portfolio B now also wanted to submit a buy order for the same asset, the market impact costs that it would incur would be disproportionately high because of reduced liquidity and price impact from the trading activity of portfolio A.

This problem arises frequently in practice (O’Cinneide et al. 2006, Stubbs and Vandenbussche 2009) because managers often invest in similar or related assets coming from the same pool of available risky investments; this reflects a particular investment style, as well as issues of efficiency (e.g., managers becoming familiar with particular investment sectors or firms). The coupling between the accounts is furthermore exacerbated if the manager trades by aggregating the orders from several accounts together. For instance, in the example above, the manager would typically place a single (large) aggregate buy order for the asset, on behalf of both portfolios A and B. This latter practice is so common that it is explicitly mentioned in the SEC regulations (Securities and Exchange Commission 2011, see paragraph on “Duty of Best Execution”): “In directing orders for the purchase or sale of securities to a broker-dealer for execution, an adviser may aggregate or ‘bunch’ those orders on behalf of two or more of its accounts, so long as the bunching is done for purposes of achieving best execution, and no client is systematically advantaged or disadvantaged by the bunching. An adviser may include accounts in which it or its officers or employees have an interest in a bunched order. Advisers must have procedures in place that are designed to ensure that the trades are allocated in such a manner that all clients are treated fairly and equitably.”

In view of the remarks above, it can be seen that the problem faced by a manager advising multiple clients raises several unique challenges compared to the classical (single portfolio) setting. First, ignoring the problematic interactions between trading activities of multiple accounts can lead to inefficiencies that drastically reduce the benefits of rebalancing (Khodadadi et al. 2006, O’Cinneide et al. 2006). For a management scheme to be scalable, it is therefore a requirement to accurately reflect such interactions, and account for the *cumulative effect of trading*. This also entails the need to specify a fair way of *splitting the market impact costs* between the various accounts that are being

rebalanced. Second, since the accounts are coupled by market impact, there are potential gains from a *joint optimization* framework and the *coordination* of the rebalancing trades of individual portfolios. Since such benefits could be achieved by sharing information across the accounts, this raises a third issue, namely the question of *when and what information to make available*, so that the resulting gains are distributed in an equitable fashion among all the accounts.

To the best of our knowledge, the above problem, which we refer to as the multiportfolio optimization (MPO) problem, has been originally considered by practitioners. Several more or less ad hoc solution approaches have been recently documented (see Savelsbergh et al. 2010 for an account). The one that has become the industry standard and perhaps the simplest is to optimize each account independently (ignoring the presence of others), perform aggregated trades, and then split the resulting costs in a *pro rata fashion*, i.e., charging each account proportionally to its share of the aggregate trading activity. Such an approach of treating the accounts in isolation suffers from multiple weaknesses. For instance, it typically severely underestimates the (true) market impact costs incurred when trading and results in poor performance for all the clients; in fact, this approach may not yield Pareto optimal trades, which means that there may exist another set of trades for which each account performs at least as well and at least one account obtains strictly improved expected performance. We review this approach in §2 in more detail.

In the academic literature, the first paper to introduce the problem was O’Cinneide et al. (2006), which recognized the problematic interactions between the accounts and the resulting questions of fairness and potential biasing during simultaneous rebalancing. The authors propose a model for the MPO problem in which the objective to be maximized is the *social welfare*, i.e., the sum of the objective functions of the individual accounts, and argue that this is fair since the solution obtained is the same as if the clients directly competed against each other in a market for liquidity. The issue of splitting the market impact costs is not discussed, and the authors implicitly use in their model the de facto solution in the industry, namely, the pro rata scheme. Acknowledging that the social welfare scheme may result in severe inequalities in the distribution of the gains, Savelsbergh et al. (2010) propose solving the MPO problem by identifying the set of portfolios that form a *Cournot–Nash equilibrium*. In this model, each account optimizes its own objective assuming the trade decisions of all other accounts participating in the pooled trading are given by their best response. The resulting solution has the property that no account would have an incentive to unilaterally change its trades. For the case of quadratic utilities and quadratic trading costs, Savelsbergh et al. (2010) show how such a solution can be found by solving one instance of a multiportfolio optimization problem. A Cournot–Nash equilibrium solution, however, is neither necessarily Pareto

optimal nor fair because it forces clients to participate in an artificial game, a practice that might well violate the SEC best execution rules. Finally, it remains a stylized model that is intractable in the absence of strong assumptions. We discuss both the social welfare and Cournot–Nash approaches in §2 in more detail.

Our contributions in the present paper are as follows.

1. We introduce a model that explicitly acknowledges and addresses the three main challenges distinguishing the MPO problem from the single portfolio case. Our formulation is general and integrates well within the modern portfolio theory literature. It accommodates general market impact cost models and different trading schemes, and it can be utilized to extend a plethora of models dealing with a single portfolio in a multiportfolio setting (e.g., portfolio construction, optimal liquidation or execution, dynamic multiperiod models).

2. Our framework leads to a tractable convex optimization problem, which is scalable and can be routinely and reliably solved for large instances in practice.

3. Our framework allows the manager to jointly optimize the trades *and* the split of market impact costs. In contrast with existing approaches where the split is constrained (or determined *ex ante*) to have a specific form, our novel approach leverages that regulations offer the flexibility to managers to decide on the split in a fair and transparent way, under few constraints.

4. By maximizing a suitably modified objective function, our formulation always produces Pareto optimal solutions, while allowing the manager to explicitly trade off social welfare and fairness. In effect, by utilizing our scheme, one can virtually optimize efficiently over all prominent and tractable solution concepts in welfare economics, including utilitarianism, Nash bargaining solution (Nash 1950), generalized utilitarianism (Mas-Colell et al. 1995), maximin, etc.

## 2. The Multiportfolio Optimization Problem

The main goal of the present section is to formalize the multiportfolio optimization problem and discuss the main solution approaches used in practice and proposed in the literature for addressing it. For simplicity of exposition, we consider a stylized, one-period rebalancing problem; this allows us to better emphasize the key differences between the MPO and the single account setting as well as to compare our approach with the existing literature and practice, in §3. In §4, we discuss how the framework readily extends to more general settings.

A financial adviser is managing  $n$  distinct *portfolios* (or *accounts*), indexed by  $\mathcal{F} \stackrel{\text{def}}{=} \{1, \dots, n\}$ . For simplicity, we assume that the same pool of  $m$  risky *assets*, denoted by  $\mathcal{J} \stackrel{\text{def}}{=} \{1, \dots, m\}$ , is available for investment for all clients (for instance, this could be the entire universe of stocks in the Standard & Poor 500).

There is a single rebalancing period, and we use  $\mathbf{w}_i \in \mathbb{R}^m$  and  $\mathbf{x}_i \in \mathbb{R}^m$  to denote the initial wealth and the rebalancing trades of the  $i$ th account, respectively. More precisely,  $w_{ij}$  and  $x_{ij}$  denote the initial holding and traded amounts in the  $j$ th asset on behalf of the  $i$ th account, respectively, and we assume that both are expressed in units of currency. Let  $\mathbf{x} \stackrel{\text{def}}{=} (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) \in \mathbb{R}^{mn}$  be the vector containing all trades. Furthermore, the trades  $\mathbf{x}_i$  of the  $i$ th account are constrained to lie in a set of *feasible trades*  $\mathcal{C}_i$ ,<sup>2</sup> assumed to be a convex subset of  $\mathbb{R}^m$ . Constraints such as self-financing requirements, proximity to a target portfolio, sector exposure, and turnover can all be modeled in this framework (see, e.g., Bertsimas et al. 1999a, Fabozzi et al. 2007, references therein).

For each account, we introduce a function  $u_i: \mathbb{R}^m \rightarrow \mathbb{R}$  to quantify the expected utility derived by the account from its rebalancing trades;<sup>3</sup> that is, the utility derived by the  $i$ th account after rebalancing is  $u_i(\mathbf{x}_i)$ . The only requirements on functions  $\{u_i\}_{i \in \mathcal{F}}$  are that they are *concave* and *expressed in units of currency* for all accounts. The former requirement is standard in microeconomics and portfolio theory (Mas-Colell et al. 1995, Fabozzi et al. 2007), and the latter becomes relevant when discussing multiple portfolios since it allows comparing and aggregating the utilities of several accounts (O’Cinneide et al. 2006, Savelsbergh et al. 2010). We note that the most prominent examples of such utility functions are already expressed in units of currency, for instance, expected return  $\boldsymbol{\mu}^T(\mathbf{w}_i + \mathbf{x}_i)$  (where  $\boldsymbol{\mu} \in \mathbb{R}^m$  is a vector of expected returns), risk-adjusted expected return  $\boldsymbol{\mu}^T(\mathbf{w}_i + \mathbf{x}_i) - \lambda_i \sqrt{(\mathbf{w}_i + \mathbf{x}_i)^T \boldsymbol{\Sigma} (\mathbf{w}_i + \mathbf{x}_i)}$  (where  $\boldsymbol{\Sigma}$  is a covariance matrix, and  $\lambda_i \geq 0$  reflects risk aversion), etc.

As discussed in the introduction, we consider a case where trading is not frictionless, and as such, the transaction costs incurred by the manager when rebalancing the portfolios are nonzero. These transaction costs are commonly referred to as *implementation shortfall* in the literature. In practice, several sources of transaction costs can be encountered, some of which are explicit (e.g., brokerage commissions and fees, taxes, foreign exchange costs), whereas others are implicit (e.g., bid-ask spread, market impact, random price movement risk, opportunity cost). We direct the interested reader to Fabozzi et al. (2010, Chapter 11) and references therein for a detailed discussion. In the present paper, we focus on transaction costs due to market impact and use the terms transaction costs, implementation shortfall, and market impact costs interchangeably; i.e., we ignore all other sources of transaction costs. We initially assume that the market impact cost model used by the manager exactly corresponds to the model that governs the actual implementation shortfall incurred when executing trades.<sup>4</sup> After presenting our base model, we relax this assumption and show how our framework can be adapted to a more realistic setting, where the manager only has partial knowledge of the actual model.

Let the market impact costs resulting from the execution of trades  $\mathbf{x}$  be

$$t\left(\sum_{i \in \mathcal{J}} \mathbf{x}_i^+, \sum_{i \in \mathcal{J}} \mathbf{x}_i^-\right), \quad (1)$$

where  $\mathbf{x}_i^+ \stackrel{\text{def}}{=} \max(\mathbf{x}_i, \mathbf{0})$  and  $\mathbf{x}_i^- \stackrel{\text{def}}{=} \max(-\mathbf{x}_i, \mathbf{0})$  are the *buy* and *sell* orders for the  $i$ th account, respectively, and  $t: \mathbb{R}_+^m \times \mathbb{R}_+^m \rightarrow \mathbb{R}$  is a *market impact cost function*, expressed in currency units. That is, the function  $t$  takes as arguments the buy and sell orders submitted for execution and returns the market impact costs of the orders upon execution.

Several clarifications are in order. First, note that the arguments of  $t$  in (1) correspond to a trading mechanism whereby the manager first aggregates (or bunches) all the buy and sell orders from the accounts into a single buy and a single sell order, respectively. As discussed in the Introduction, this is common practice in multiportfolio management and is typically done for purposes of efficiency (Securities and Exchange Commission 2011, O’Cinneide et al. 2006, Fabozzi et al. 2007, Savelsbergh et al. 2010). Second, by taking the arguments of  $t$  to be separate buy and sell vectors, we are effectively forbidding the possibility of cross-trading, i.e., the netting of a buy and sell order for the same security “in-house,” followed by a market order for the remainder of the bigger trade. In §B.1 of the online appendix (available as supplemental material at <http://dx.doi.org/10.1287/opre.2014.1310>), we relax this assumption, and show how our model readily extends to settings where cross-trading is allowed.

There is a growing literature on market microstructure seeking to accurately model the functional form of  $t$  and the pricing and trading mechanisms resulting in such market impact costs (see, e.g., Roşu 2009, Obizhaeva and Wang 2013, Fabozzi et al. 2010, Tsoukalas et al. 2012). For the scope of our study, we do not adopt a particular pricing or market impact cost model. Instead, we only make the mild assumption that the function  $t$  is *jointly convex* in its arguments, and *componentwise increasing*. The former requirement reflects that the marginal market impact cost increases with the size of the trade, whereas the latter reflects the natural feature that more trading results in larger costs (for instance, due to reduced “at-the-money” liquidity). This is a commonly made assumption in the literature and is well aligned with empirical observations and practice (e.g., see Bertsimas and Lo 1998, Bertsimas et al. 1999a, Almgren and Chriss 2000, O’Cinneide et al. 2006, Brown et al. 2010, Lim and Wimonkittiwat 2011, Moallemi and Sağlam 2012).

For simplicity of exposition, we furthermore assume that the trading activity in a particular asset does not affect market impact costs associated with trading other assets (see, e.g., O’Cinneide et al. 2006, Brown et al. 2010, Fabozzi et al. 2010, Moallemi and Sağlam 2012). In other words,  $t$  is separable across assets and can be expressed as

$$t\left(\sum_{i \in \mathcal{J}} \mathbf{x}_i^+, \sum_{i \in \mathcal{J}} \mathbf{x}_i^-\right) = \sum_{j \in \mathcal{J}} t_j\left(\sum_{i \in \mathcal{J}} x_{ij}^+, \sum_{i \in \mathcal{J}} x_{ij}^-\right), \quad (2)$$

where  $t_j: \mathbb{R}_+^2 \rightarrow \mathbb{R}$  is the associated market impact cost function for the  $j$ th asset, and is jointly convex and componentwise increasing. In §4, we relax this assumption, and argue how our model readily extends to deal with cross-asset effects that may be encountered in practice (Savelsbergh et al. 2010, Tsoukalas et al. 2012).

Before formally introducing the MPO problem, let us first consider the standard setting, where there is a single account, e.g.,  $\mathcal{J} = \{i\}$ . In view of market impact costs, the *net utility* derived by the account, which we denote by  $U_i$ , is the utility from its holdings,  $u_i(\mathbf{w}_i + \mathbf{x}_i)$ , minus market impact costs,  $t(\mathbf{x}_i^+, \mathbf{x}_i^-)$ .<sup>5</sup> The portfolio selection problem can then be succinctly formulated as maximizing  $U_i$ , subject to trading constraints, i.e.,

$$\begin{aligned} & \text{maximize} \quad \{u_i(\mathbf{x}_i) - t(\mathbf{x}_i^+, \mathbf{x}_i^-)\} \\ & \text{subject to} \quad \mathbf{x}_i \in \mathcal{C}_i. \end{aligned} \quad (3)$$

This is a direct expression of the manager’s duty to obtain “best execution” for the clients’ transactions and has been studied extensively in the literature since Markowitz (1952).

Consider now the case where the manager is in charge of  $n$  portfolios, with  $n \geq 2$ . In contrast with the standard single portfolio optimization problem we just discussed, the MPO problem is more subtle. The three differentiating elements are the following:

1. *Splitting the market impact costs.* The net market impact costs incurred by the manager depend on the aggregate trades and thus on the activity of all the accounts, by (1). This immediately raises the question of how these costs should be split between the various participants. The SEC regulation is very strict on the matter, requiring a “fair and equitable” treatment of each client (Securities and Exchange Commission 2011), yet it does not specify a particular splitting mechanism.

2. *Optimizing over multiple objectives.* Because of market impact costs, the net utilities of the accounts are coupled. As such, a manager’s fiduciary duty requires solving a *multiobjective* optimization problem, whereby the net utilities  $\{U_i\}_{i \in \mathcal{J}}$  of all accounts are jointly optimized.

3. *Coordination benefits.* In a joint optimization framework, benefits are potentially achieved by coordination and sharing of information across the accounts. This raises the question of when and what information to make available, so that the resulting savings are distributed equitably, and all accounts are treated according to the “best execution” rules.

As a side remark, note that the reader might be tempted to conclude at this point that the coupling between the accounts has been artificially introduced in our model, as a result of the aggregate trading done by the manager. While aggregation is extremely common in practice, so that this reason alone should warrant the model, we note that, even if trading occurred separately (e.g., by deciding rebalancing trades and placing separate orders for each account), the transaction costs incurred by each client would still

depend on the activity of other clients because of *market impact*. Furthermore, this effect would exist no matter how the trades were executed, e.g., by splitting the order execution across larger periods of time, by placing separate simultaneous orders, etc.

We now review the most prominent solutions proposed in the industry and the academic literature for dealing with the three questions above.

### 2.1. Splitting the Market Impact Costs

**Industry.** To the best of our knowledge, the most common approach employed in practice is to split the market impact costs for a particular asset in a *pro rata* fashion, i.e., to charge each portfolio a cost proportional to its share of the total trade for that particular asset (O’Cinneide et al. 2006, Savelsbergh et al. 2010). The pro rata scheme is well defined for market impact costs that are separable across the assets, as in (2). In this context, when the trades for the  $j$ th asset are  $\{x_{ij}\}_{i \in \mathcal{J}}$ , the  $i$ th account is charged a cost of

$$\frac{x_{ij}}{\sum_{a \in \mathcal{J}} x_{aj}} t_j \left( \sum_{a \in \mathcal{J}} x_{aj}^+, \sum_{a \in \mathcal{J}} x_{aj}^- \right), \quad \forall i \in \mathcal{J}, j \in \mathcal{J}.$$

Hence, the total market impact cost charged to the  $i$ th account is

$$\sum_{j \in \mathcal{J}} \frac{x_{ij}}{\sum_{a \in \mathcal{J}} x_{aj}} t_j \left( \sum_{a \in \mathcal{J}} x_{aj}^+, \sum_{a \in \mathcal{J}} x_{aj}^- \right), \quad \forall i \in \mathcal{J}. \quad (4)$$

The pro rata scheme is easy to comprehend and apply and is often perceived as fair by portfolio managers (Fabozzi et al. 2007). However, it is not required by regulators, and it is inappropriate for market impact costs that are nonseparable across assets.<sup>6</sup> Moreover, it is also inadequate in case some of the accounts buy and some of the accounts sell a particular asset. In fact, in those cases expression (4) will not be well defined if the denominator were zero, i.e., if the net trade were zero. Also, according to (4), some accounts might end up being charged negative market impact costs. To overcome both these potential issues, managers typically resort to the otherwise unrealistic assumption of market impact costs that are separable for buy and sell orders; see Savelsbergh et al. (2010) and O’Cinneide et al. (2006) for more information. Furthermore, a pro rata scheme may lead to tractability issues, since the expression (4) is typically neither convex nor concave in  $\mathbf{x}$ . Finally, we argue in §3 that the pro rata scheme also fails to properly reflect all interactions between the accounts in an MPO setting, potentially resulting in an unfair split.

**Literature.** The question of how to split market impact costs has received little attention in the literature. Within the line of research focusing on the MPO problem, all papers that we are aware of either do not deal with that question or adopt the pro rata split without providing any

theoretical justification (Fabozzi et al. 2007, O’Cinneide et al. 2006, Savelsbergh et al. 2010).

A related body of work that studies fair and efficient cost sharing mechanisms is cooperative game theory. In a cooperative (cost) game, there are  $n$  players contemplating forming coalitions in undertaking particular projects. Typically, the collective costs incurred by the players are lower if they form a coalition, compared to the case where they act independently (as is also the case in the MPO problem). Cooperative game theory then suggests various solution concepts in sharing costs among the players in a fair way, for instance, the Shapley value and the nucleolus concepts (see Young 1995, Shapley 1953, Schmeidler 1969). All of these concepts critically rely on the existence of a *characteristic function*, which determines the collective costs incurred by any coalition of players. In the case of the MPO problem, however, the characteristic function cannot be defined: we can only determine the collective costs of *all* players, i.e., accounts in the MPO setting, through the aggregate market impact cost function  $t$ , but not the costs of any coalition formed as a strict subset of the players. The reason is that the costs of any coalition always depend on the trading activities of all the accounts because of market impact; thus, there are externalities between players involved in a coalition and players who are not, unlike the classical cooperative games. Finally, the solution concepts of cooperative game theory typically exhibit high computational complexity (the input characteristic function is already exponential in  $n$ ), which renders them impractical for large  $n$ . For more details, see Deng and Papadimitriou (1994).

### 2.2. Independent Solution

With regard to the second and third problems above, the simplest approach, which seems to be the industry standard, is to optimize each account in isolation, ignoring the presence of others (Savelsbergh et al. 2010, Fabozzi et al. 2007, Khodadadi et al. 2006). The resulting costs are then split pro rata. More precisely, a manager using the *independent scheme* would proceed as follows:

1. Solve problem (3) for each account  $i \in \mathcal{J}$ , and let  $\mathbf{x}_i^{\text{IND}}$  denote the optimal solution obtained.
2. Execute the aggregated buy and sell orders,

$$\sum_{i \in \mathcal{J}} (\mathbf{x}_i^{\text{IND}})^+ \quad \text{and} \quad \sum_{i \in \mathcal{J}} (\mathbf{x}_i^{\text{IND}})^-,$$

respectively, incurring a total cost according to (2).

3. Charge the  $i$ th account in a pro rata fashion for the market impact costs, resulting in a *realized net utility* of

$$U_i^{\text{IND}} = u_i(\mathbf{x}_i^{\text{IND}}) - \sum_{j \in \mathcal{J}} \frac{x_{ij}^{\text{IND}}}{\sum_{a \in \mathcal{J}} x_{aj}^{\text{IND}}} \cdot t_j \left( \sum_{a \in \mathcal{J}} (x_{aj}^{\text{IND}})^+, \sum_{a \in \mathcal{J}} (x_{aj}^{\text{IND}})^- \right), \quad \forall i \in \mathcal{J}. \quad (5)$$

Note that for concave functions  $\{u_i\}_{i \in \mathcal{J}}$ , convex function  $t$ , and convex sets  $\{\mathcal{C}_i(\mathbf{w}_i)\}_{i \in \mathcal{J}}$ , Step 1 above requires solving a convex optimization problem in variables  $\mathbf{x}_i^+$ ,  $\mathbf{x}_i^-$  and can be efficiently solved to optimality via convex optimization techniques in many cases of practical interest (Boyd and Vandenberghe 2004). Therefore, this approach remains computationally tractable for a large number of accounts and assets. Since accounts are optimized independently and no information is shared across them, managers often regard the solution as being “fair” with respect to all clients (although this argument is sometimes challenged; Savelsbergh et al. 2010).

The approach is also known to have several serious weaknesses. First, by ignoring the presence of other accounts, it can significantly underestimate the true market impact costs incurred by each participant. In particular, in step 1, a client *anticipates* a utility of  $u_i(\mathbf{x}_i^{\text{IND}}) - t((\mathbf{x}_i^{\text{IND}})^+, (\mathbf{x}_i^{\text{IND}})^-)$ . However, the *realized* utility derived by the client is actually  $U_i^{\text{IND}}$ , in step 3, which is typically smaller than the anticipated utility (O’Cinneide et al. 2006, Savelsbergh et al. 2010). Second, based on the trades  $\mathbf{x}^{\text{IND}}$  and the pro rata split, the resulting utilities  $\{U_i^{\text{IND}}\}_{i \in \mathcal{J}}$  in (5) are not necessarily Pareto optimal for the MPO problem: one can find another set of portfolio trades such that the utility of *every* account is at least as large as  $U_i^{\text{IND}}$ , with some accounts further strictly improving.

### 2.3. Social Welfare Solution

A different approach, suggested by O’Cinneide et al. (2006), is the *social welfare scheme*, whereby the manager decides the trades so as to maximize the aggregate utility of all the accounts, i.e., the sum of the individual utilities derived from the holdings, minus the aggregate market impact costs. In other words, the manager would use the following scheme:

1. Solve the following optimization problem

$$\begin{aligned} & \text{maximize} \quad \left\{ \sum_{i \in \mathcal{J}} u_i(\mathbf{x}_i) - t \left( \sum_{i \in \mathcal{J}} \mathbf{x}_i^+, \sum_{i \in \mathcal{J}} \mathbf{x}_i^- \right) \right\} \\ & \text{subject to} \quad \mathbf{x}_i \in \mathcal{C}_i, \quad \forall i \in \mathcal{J}, \end{aligned} \quad (6)$$

and let  $\{\mathbf{x}_i^{\text{SOC}}\}_{i \in \mathcal{J}}$  denote the optimal solution obtained.

2. Execute the aggregate buy and sell orders.
3. Split the resulting market impact costs in a pro rata fashion, resulting in a realized utility of

$$\begin{aligned} U_i^{\text{SOC}} = & u_i(\mathbf{x}_i^{\text{SOC}}) - \sum_{j \in \mathcal{J}} \frac{x_{ij}^{\text{SOC}}}{\sum_{a \in \mathcal{J}} x_{aj}^{\text{SOC}}} t_j \\ & \cdot \left( \sum_{a \in \mathcal{J}} (x_{aj}^{\text{SOC}})^+, \sum_{a \in \mathcal{J}} (x_{aj}^{\text{SOC}})^- \right), \quad \forall i \in \mathcal{J}. \end{aligned} \quad (7)$$

As with the independent case, for concave functions  $\{u_i\}_{i \in \mathcal{J}}$ , convex function  $t$ , and convex sets  $\{\mathcal{C}_i(\mathbf{w}_i)\}_{i \in \mathcal{J}}$ , problem (6) above is convex and can be solved efficiently

for realistic sizes. The formulation is grounded in microeconomic theory (Mas-Colell et al. 1995), and the optimal solution is known to be Pareto optimal. Furthermore, the anticipated net utility exactly corresponds to the realized net utility for every account.

O’Cinneide et al. (2006) argue that the solution is also “fair” because it corresponds to the same trades obtained if clients were competing in an open market for liquidity. However, this notion of fairness is questionable, as one can construct simple examples to show that particular accounts can benefit disproportionately from the solution, at the expense of others (see, e.g., Savelsbergh et al. 2010). Moreover, accounts that derive a net utility strictly smaller than that obtained when they were optimized independently, i.e.,  $U_i^{\text{SOC}} < U_i^{\text{IND}}$ , could rightfully deem the social scheme as “unfair” since it coerces them to share their complete information with other accounts (through (6)) but results in worse outcomes (while increasing the utility of others).

### 2.4. Cournot–Nash Solution

Motivated by the shortcomings of the previous two approaches, Savelsbergh et al. (2010) suggest obtaining the trades for all the accounts by solving an equilibrium problem. More precisely, the manager would proceed as follows:

1. Compute the (best response) trades for the  $i$ th account by fixing  $\mathbf{x}_{-i} \stackrel{\text{def}}{=} (\mathbf{x}_a: a \neq i \in \mathcal{J})$  and solving the following optimization problem in variables  $\mathbf{x}_i$ :

$$\begin{aligned} & \text{maximize} \quad \left\{ u_i(\mathbf{x}_i) - \sum_{j \in \mathcal{J}} \frac{x_{ij}}{\sum_{a \in \mathcal{J}} x_{aj}} t_j \left( \sum_{a \in \mathcal{J}} (x_{aj})^+, \sum_{a \in \mathcal{J}} (x_{aj})^- \right) \right\} \\ & \text{subject to} \quad \mathbf{x}_i \in \mathcal{C}_i. \end{aligned} \quad (8)$$

Solve the *equilibrium problem*; i.e., let  $\mathbf{x}^{\text{CN}}$  be a solution with the property that every  $\mathbf{x}_i^{\text{CN}}$  is a best response to  $\mathbf{x}_{-i}^{\text{CN}}$ , for any  $i \in \mathcal{J}$ .

2. Execute the trades  $\mathbf{x}^{\text{CN}}$ .
3. Split the transaction costs in a pro rata fashion, yielding a realized net utility of

$$\begin{aligned} U_i^{\text{CN}} = & u_i(\mathbf{x}_i^{\text{CN}}) - \sum_{j \in \mathcal{J}} \frac{x_{ij}^{\text{CN}}}{\sum_{a \in \mathcal{J}} x_{aj}^{\text{CN}}} t_j \left( \sum_{a \in \mathcal{J}} (x_{aj}^{\text{CN}})^+, \sum_{a \in \mathcal{J}} (x_{aj}^{\text{CN}})^- \right), \\ & \forall i \in \mathcal{J}. \end{aligned} \quad (9)$$

The solution  $\mathbf{x}^{\text{CN}}$  is known as the *Cournot–Nash* solution and has solid foundations in microeconomics (Mas-Colell et al. 1995). It also has the property that the anticipated net utility corresponds to the realized net utility, for every account.

However, a pitfall with the approach is that the optimal solution is not necessarily Pareto optimal. Furthermore, just as with the social welfare scheme, it is possible to have  $U_i^{\text{CN}} < U_i^{\text{IND}}$ , raising the issue of fairness and willingness to share private information.

A third complication with the approach lies in the complexity of solving the overall equilibrium problem in Step 1. Although determining the best response in (8) can sometimes be done via convex optimization (e.g., when the utilities and market impact costs are quadratic; Savelsbergh et al. 2010), the overall problem is a mathematical program with equilibrium constraints, which is generally hard to solve (Luo et al. 1996). One approach to bypass intractability is to approximate the Cournot–Nash solution via an *iterative scheme*: the trades  $\mathbf{x}_i$  are determined for each account  $i$  under a guess for  $\mathbf{x}_{-i}$  and then the best responses are used as a guess in the next step (Fabozzi et al. 2007). Of course, if this process converges, it may do so very slowly and to a solution that is not necessarily Pareto optimal.

### 3. Our Model

In this section, we develop our framework to deal with the multiportfolio optimization (MPO) problem. We present our approach for the setting and assumptions introduced in §2; several relevant extensions are included in §4.

We first discuss our modeling choices for the three elements differentiating the MPO from the well-studied single portfolio optimization problem, namely,

(a) splitting the market impact costs of the aggregated trades between the individual accounts;

(b) optimizing over multiple objectives, i.e., the utilities of the individual accounts; and

(c) guaranteeing coordination benefits to every individual account in a joint optimization framework.

We then provide the formulation of our model, followed by a discussion. We also highlight how our formulation can be generalized for the case where the actual transaction cost model is unknown, with the manager having only partial knowledge.

#### 3.1. Splitting the Market Impact Costs

We allow the split of market impact costs to be a decision variable of the multiportfolio optimization problem. That is, we do not impose a particular functional form (e.g., pro rata, as in (4)) or any other mechanism for splitting the market impact costs *ex ante*. Instead, we introduce a minimal set of natural constraints on the split. We next describe our approach in more detail.

To introduce some notation, let  $\tau_{ij}$  be the amount charged to the  $i$ th account for trading the  $j$ th asset, for all  $i \in \mathcal{I}$  and  $j \in \mathcal{J}$ . Let  $\boldsymbol{\tau} \in \mathbb{R}^{mn}$  be the vector containing all those values. The net charges to the  $i$ th account, denoted by  $\tau_i$ , are in that case

$$\tau_i = \sum_{j \in \mathcal{J}} \tau_{ij}, \quad \forall i \in \mathcal{I}.$$

The utility that the  $i$ th account derives is then

$$U_i = u_i(\mathbf{x}_i) - \tau_i, \quad \forall i \in \mathcal{I}. \quad (10)$$

Note that the individual accounts are ultimately interested in a fair and equitable allocation of utilities  $\{U_i\}_{i \in \mathcal{I}}$  and, as such, in a fair decision concerning both the trades  $\mathbf{x}$  and the split of associated market impact costs  $\boldsymbol{\tau}$ .

From a regulatory perspective, there are few restrictions pertaining to the values that the allocated market impact costs  $\boldsymbol{\tau}$  can take, as discussed in §2.1. As such, we only impose the following natural constraints on  $\boldsymbol{\tau}$ :

(a) The amount charged to an account for trading a particular quantity of an asset is greater than or equal to the market impact cost of trading only that quantity; i.e.,

$$t_j(x_{ij}^+, x_{ij}^-) \leq \tau_{ij}, \quad \forall i \in \mathcal{I}, j \in \mathcal{J}. \quad (11)$$

(b) The amount charged to an account for trading a particular quantity of an asset is less than or equal to the externality it imposes on the aggregate market impact cost for that asset; i.e.,

$$\tau_{ij} \leq t_j\left(\sum_{a \in \mathcal{I}} x_{aj}^+, \sum_{a \in \mathcal{I}} x_{aj}^-\right) - t_j\left(\sum_{a \in \mathcal{I} \setminus \{i\}} x_{aj}^+, \sum_{a \in \mathcal{I} \setminus \{i\}} x_{aj}^-\right), \quad \forall i \in \mathcal{I}, j \in \mathcal{J}. \quad (12)$$

(c) The aggregate charge (to all the accounts) for trades in a particular asset equals the aggregate market impact cost for that asset; i.e.,

$$t_j\left(\sum_{a \in \mathcal{I}} x_{aj}^+, \sum_{a \in \mathcal{I}} x_{aj}^-\right) = \sum_{a \in \mathcal{I}} \tau_{aj}, \quad \forall j \in \mathcal{J}. \quad (13)$$

The constraints in (a) and (b) correspond to natural lower and upper bounds on the charges to each account. In particular, (11) ensures that the charge to an account is no less than the smallest possible it could incur, i.e., in a situation where the other accounts would not trade. Similarly, (12) requires that the charge is no larger than the additional market impact cost incurred because of the account's presence. As a technical remark, constraints (11) and (12) are always jointly feasible since functions  $t_j$  exhibit increasing differences because they are jointly convex and componentwise increasing.

Note that there are several advantages to using our approach instead of pro rata, the only other alternative in consideration. In fact, we now argue that a pro rata split may not even be an appropriate choice for the MPO setting. To this end, consider the following example.

**EXAMPLE 1.** (a) There are  $n = 2$  accounts and  $m = 2$  assets. Account 1 invests a unit of currency in asset 1, whereas account 2 invests  $\theta$  units of currency in asset 1 and  $1 - \theta$  in asset 2, where  $0 \leq \theta \leq 1$ . That is,  $\mathcal{C}_1 = \{(1, 0)\}$  and  $\mathcal{C}_2 = \{(\theta, 1 - \theta)\}$ . Both assets are equally attractive to the accounts; i.e.,  $u_1$  and  $u_2$  are constant. The market impact cost functions for the two assets are

$$t_1(x^+, x^-) = (x^+)^2 + (x^-)^2,$$

$$t_2(x^+, x^-) = 3(x^+)^2 + 3(x^-)^2.$$

Suppose that account 2 is ignorant of the trading activity of account 1, e.g., as described in §2.2. In that case, it is easy to see that account 2 would trade  $\mathbf{x}_2 = (0.75, 0.25)$ , i.e., trade  $\theta = 0.75$ , in order to minimize its own anticipated market impact costs. The associated market impact costs of the pooled trades are

$$t_1(1 + 0.75, 0) = 1.75^2 = 3.0625,$$

$$t_2(0 + 0.25, 0) = 3(0.25)^2 = 0.1875.$$

A pro rata split of the market impact cost for asset 1 would charge account 1 with 1.75 and account 2 with 1.3125. Similarly, for asset 2, account 1 would be charged 0 and account 2 would be charged 0.1875. The net charges would be 1.75 for account 1 and 1.5 for account 2.

(b) Consider the same setting as in (a), but where the manager jointly optimizes the two accounts. In particular, in view of the high trading activity of account 1 in asset 1 (since account 1 trades  $\mathbf{x}_1 = (1, 0)$ ), account 2 lowers its target level  $\theta$  in Asset 1 from 0.75 to 0.5, in anticipation of high market impact costs. That is, account 2 now trades  $\mathbf{x}_2 = (0.5, 0.5)$ . In this scenario, the resulting market impact costs are

$$t_1(1 + 0.5, 0) = 1.5^2 = 2.25,$$

$$t_2(0 + 0.5, 0) = 3(0.5)^2 = 0.75,$$

and a pro rata split would charge both accounts with 1.5.

In comparing scenarios (a) and (b) in Example 1, note that account 2 is charged the same amount in both, whereas account 1 is charged 0.25 less in (B). That is, despite that account 2 adjusts its trading activity to lower aggregate market impact costs, it is unable to harvest any of those gains. On the contrary, account 1 is awarded all the benefits.

The example above illustrates that the pro rata split fails to account for adjustments in trading activities of individual accounts when a manager jointly optimizes. More generally, a pro rata split mechanism is based only on the actual trading activity and does not incorporate all interactions between the accounts in an MPO setting. In contrast, our approach allows the manager to account for such interactions.

Furthermore, as discussed in §2.1, (a) the pro rata split is inappropriate in cases where market impact costs are nonseparable across assets or when some accounts buy and others sell and (b) may lead to issues of tractability. Our approach of allowing the market impact cost split to be a decision variable of the optimization framework overcomes all those weaknesses.

Finally, compared to deciding the splitting mechanism *ex ante*, our approach provides more flexibility in optimizing over the utilities of the accounts, which is discussed next.

### 3.2. Optimizing Over Utilities

The MPO problem is a classical multiobjective optimization problem, where the manager needs to optimize performance by balancing  $n$  objectives, namely, the utilities  $\{U_i\}_{i \in \mathcal{J}}$  of the

accounts. By deciding on the trades  $\mathbf{x}$  and split of associated market impact costs  $\boldsymbol{\tau}$ , the manager decides, in essence, how utility (and gains) are allocated among the  $n$  accounts.

The aforementioned utility allocation problem has been well studied in welfare economics and bargaining (see Nash 1950, Mas-Colell et al. 1995). The standard solution approach in this line of literature is the introduction of a *welfare function*  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  of the allocation of utilities, which is used by the manager to rank allocations (see, e.g., Bergson 1938, Samuelson 1947). That is, if for a particular allocation of trades and split of costs, the accounts derive utilities  $\{U_i\}_{i \in \mathcal{J}}$ , the manager values this allocation according to  $f(U_1, \dots, U_n)$ . Consequently, the manager selects the trades and split of costs that maximize  $f$  over the set of feasible trades and splits.

Typically,  $f$  is assumed to be *componentwise increasing* and *concave*. Monotonicity is a natural requirement in view of the manager's fiduciary duty to the clients. Concavity allows  $f$  to exhibit diminishing marginal welfare increase as utilities increase and thus to possess *fairness properties*. To illustrate this, consider a situation where account A derives a lower utility than account B. A marginal increase in the utility of account A would then yield a higher welfare increase compared to a marginal increase in the utility of account B. As such, the former would be more desirable for the manager. This property of concave welfare functions typically leads to more even or fair distributions of utility; see also Bertsimas et al. (2012).

Some of the most prominent instances of welfare functions are the following:

- The utilitarian welfare function, also referred to as social welfare function, corresponds to the sum of the individual utilities (see §2.3). It is a natural choice in applications where the sum of the utilities corresponds to some measure of system efficiency. On the other hand, such an objective is neutral toward potential inequalities in the utility distribution among the players. It is therefore possible that the utilitarian solution is achieved at the expense of some players (see Young 1995, Savelsbergh et al. 2010 for an example). That is, the utilitarian objective puts no emphasis on the fairness properties of the allocation but rather on the net aggregate utility of all players.

Furthermore, note that the aforementioned shortcoming of the utilitarian function is exacerbated in our setting by differences in the sizes of the individual accounts. In particular, consider a situation where the size of one account is considerably larger than all others. Naturally, trades associated with the large account are more likely to be larger and thus to incur higher transaction costs. In optimizing aggregate utilities, the utilitarian function is then more likely to systematically focus on optimizing the trades of the large account at the expense of the smaller accounts. To alleviate this, a manager could use the *relative* utilitarian welfare function instead, which maximizes relative profits instead of absolute ones.<sup>7</sup>



• The *maximin* welfare function corresponds to the minimum utility derived by the players.<sup>8</sup> Unlike the utilitarian function, the maximin function has well established fairness properties, based on the Rawlsian concept of justice; see Rawls (1971) and Mas-Colell et al. (1995). However, the equitable allocation of utility under the maximin objective is often achieved at the expense of aggregate utility, i.e., social welfare; see Bertsimas et al. (2011).

• The *constant elasticity* (or  $\alpha$ -*fairness*) parameterized family of welfare functions, given by  $f(U_1, \dots, U_n) = \sum_{i=1}^n (U_i^{1-\alpha}) / (1-\alpha)$ , for  $\alpha \geq 0$ . This family subsumes utilitarianism ( $\alpha = 0$ ), the Nash bargaining solution ( $\alpha \rightarrow 1$ ), as well as maximin ( $\alpha \rightarrow \infty$ ). In fact, by varying  $\alpha$ , one adjusts attitudes toward inequity. For more details, we refer the reader to Bertsimas et al. (2012).

As the discussion above suggests, the manager needs to select a welfare function that trades off social welfare (sum of utilities) and fairness (equitable allocation of utilities). Such a selection problem is very relevant, but rather involved in practice. It entails the adoption of a particular fairness scheme, out of the multiple proposals in the abundant literature, ranging from welfare economics to philosophy. A discussion of such a choice is outside the scope of our paper, but we note that the framework introduced here is flexible enough to capture most sensible existing propositions. We direct the interested reader to Bertsimas et al. (2012) for a thorough overview of the literature and for guidelines of how the choice could be made in practice.

For our purposes, we assume that the manager has selected a concave, componentwise increasing welfare function  $f$  that is to be maximized in order to decide the trades  $\mathbf{x}$  and split of associated market impact costs  $\boldsymbol{\tau}$ . In §5, we discuss the utilitarian and maximin welfare functions, adapted to the MPO setting, in more detail.

### 3.3. Coordination Benefits

Unlike the independent solution, where each account is optimized separately and no private information is shared (see §2.2), in an MPO setting the manager jointly optimizes all accounts, and thus private information is being shared. In accordance with the best execution rules, the manager needs to ensure that such coordination yields gains to every individual account or at least does not inflict losses. Hence, we introduce the following constraints:

$$U_i = u_i(\mathbf{x}_i) - \tau_i \geq U_i^{\text{IND}}, \quad \forall i \in \mathcal{I}. \quad (14)$$

That is, the utility of every account  $i$  needs to be at least as large as its utility under the independent solution  $U_i^{\text{IND}}$ , where no information is shared. Otherwise, the  $i$ th account releases its information at a loss. Put differently, these constraints ensure that the MPO framework results in a Pareto improvement over the practice of treating accounts independently.

### 3.4. Main Formulation

We next provide the formulation of our solution approach for the MPO problem, based on our modeling choices

introduced above. The manager determines the trades  $\mathbf{x}$  and the split of market impact costs  $\boldsymbol{\tau}$  by solving the following convex optimization problem, in variables  $\mathbf{x}$ ,  $\mathbf{x}^+$ ,  $\mathbf{x}^-$  and  $\boldsymbol{\tau}$ :

$$\begin{aligned} & \text{maximize} \quad \{f(u_1(\mathbf{x}_1) - \tau_1, \dots, u_n(\mathbf{x}_n) - \tau_n)\} \\ & \text{subject to} \quad \mathbf{x}_i \in \mathcal{C}_i, \quad \forall i \in \mathcal{I}, \\ & \quad \mathbf{x}_i = \mathbf{x}_i^+ - \mathbf{x}_i^-, \quad \forall i \in \mathcal{I}, \\ & \quad \mathbf{x}_i^+, \mathbf{x}_i^- \geq 0, \quad \forall i \in \mathcal{I}, \\ & \quad \tau_i = \sum_{j \in \mathcal{J}} \tau_{ij}, \quad \forall i \in \mathcal{I}, \\ & \quad t_j(x_{ij}^+, x_{ij}^-) \leq \tau_{ij}, \quad \forall i \in \mathcal{I}, j \in \mathcal{J}, \end{aligned} \quad (15a)$$

$$t_j \left( \sum_{a \in \mathcal{J} \setminus \{i\}} x_{aj}^+, \sum_{a \in \mathcal{J} \setminus \{i\}} x_{aj}^- \right) \leq \sum_{a \in \mathcal{J} \setminus \{i\}} \tau_{aj}, \quad \forall i \in \mathcal{I}, j \in \mathcal{J}, \quad (15b)$$

$$t_j \left( \sum_{a \in \mathcal{J}} x_{aj}^+, \sum_{a \in \mathcal{J}} x_{aj}^- \right) \leq \sum_{a \in \mathcal{J}} \tau_{aj}, \quad \forall j \in \mathcal{J}, \quad (15c)$$

$$u_i(\mathbf{x}_i) - \tau_i \geq U_i^{\text{IND}}, \quad \forall i \in \mathcal{I}. \quad (15d)$$

Constraints (15d) correspond to the coordination benefits constraints (14). Constraints (15a)–(15c) correspond to (11)–(13), where (12)–(13) are reformulated equivalently so that the problem remains convex. To see this, note that by combining (12) and (13), we get (15b) by substitution. Finally, constraints (13) can be relaxed to inequalities, as in (15c), since they will be tight at optimality because  $f$  is componentwise increasing.

We now discuss the relative merits of our approach:

1. Our formulation allows the manager to jointly optimize the trades and the split of market impact costs in order to maximize the welfare objective. In contrast with existing approaches where the split is constrained (or determined *ex ante*) to have a specific form, our approach leverages that regulations offer the flexibility to managers to decide on the split in a fair and transparent way, under few constraints.

2. Our formulation leads to a tractable, convex optimization problem that is scalable and can be routinely and reliably solved for large instances in practice.

3. Our formulation produces utilities  $\{U_i\}_{i \in \mathcal{I}}$  that are Pareto optimal while also allowing the manager to trade off social welfare and fairness by selecting the welfare function  $f$ . By utilizing the convex feasible set of problem (15), one can virtually optimize efficiently over all prominent and tractable solution concepts in welfare economics, including utilitarianism, Nash bargaining solution (see Nash 1950), generalized utilitarianism (see Mas-Colell et al. 1995), maximin,  $\alpha$ -fairness, etc.

4. Our formulation is general enough to capture a multitude of interesting extensions that could be relevant in alternative settings. For instance, problem (15) accommodates any market impact cost function  $t$  (as long as it is

convex and increasing), including market impact cost functions that capture cross-asset effects. Problem (15) can also be generalized to multiperiod models, settings where cross-trading is allowed, etc. Finally, our formulation can also be extended to capture uncertainty in the market impact cost function and allow the manager to hedge against it.

### 3.5. Unknown Transaction Costs

A standing assumption made so far was that the manager’s transaction cost model exactly corresponded to the model governing the actual costs incurred upon execution. In reality, however, the manager has access to models that are at best capable of estimating transaction costs (e.g., in expectation); realized costs are a priori unknown and potentially deviate from estimates.

We now relax this assumption. When costs are a priori unknown, it is no longer appropriate for a manager to decide the split using the static formulation (15); instead, a dynamic mechanism is required to determine how realized transaction costs should be divided among the multiple portfolios *ex post*.<sup>9</sup>

We first refine the model for transaction costs. We assume that the market impact costs associated with trades  $x^+$  and  $x^-$  in the  $j$ th asset are now random and are given by  $\tilde{t}_j(x^+, x^-, \xi)$ . Here,  $\xi$  is a random vector capturing all sources of noise that affect the market impact costs. In keeping with the assumptions in §2, we consider functions  $\tilde{t}_j$  that are jointly convex and componentwise increasing in their arguments, for any value of  $\xi$ . To avoid technical complications, we further assume that the distribution for  $\xi$  is such that all the expectation operators are finite.

We assume that the manager has access to an unbiased estimator of the true market impact costs, i.e., is able to reliably estimate the function

$$t_j(x^+, x^-) = \mathbb{E}[\tilde{t}_j(x^+, x^-, \xi)], \tag{16}$$

where  $\mathbb{E}[\cdot]$  denotes the expectation operator.

Using the predictive model above, the manager needs to decide on the trades  $\mathbf{x}$  for all the portfolios under management. After the trades are executed, the manager is only able to observe the realized transaction costs associated with the actual trades, namely,

$$\tilde{Z}_j \stackrel{\text{def}}{=} \tilde{t}_j\left(\sum_{a \in \mathcal{J}} x_{aj}^+, \sum_{a \in \mathcal{J}} x_{aj}^-, \xi\right),$$

for each asset  $j \in \mathcal{J}$ . She then needs to decide how to split these costs among the portfolios.

The key distinction between this setting and the one we have considered so far is that the manager is no longer able to choose a (static) split of the market impact costs, to be computed *ex ante*; instead, this split now depends on the realized market impact costs. More precisely, letting  $\tilde{\mathbf{Z}}$  denote the random vector with components  $\tilde{Z}_j$  and  $\mathbf{z}$  denote a realization of  $\tilde{\mathbf{Z}}$ , the manager now needs to select

a set of adjustable policies<sup>10</sup>  $\tau_{ij}$  such that  $\tau_{ij}(\mathbf{z})$  represents the *ex post* amount charged to the  $i$ th portfolio for trading in the  $j$ th asset. These policies, which are now part of the manager’s decision process, would then be required to obey certain constraints that are natural counterparts of our framework in §3.

In this context, the manager would solve the following stochastic optimization problem to determine the trades  $\mathbf{x}$  and the policies  $\tau_{ij}$ :

$$\text{(SP)} \quad \text{maximize} \quad \{f(u_1(\mathbf{x}_1) - \mathbb{E}[\tau_1(\tilde{\mathbf{Z}})], \dots, u_n(\mathbf{x}_n) - \mathbb{E}[\tau_n(\tilde{\mathbf{Z}})])\}$$

subject to

$$\mathbf{x}_i \in \mathcal{C}_i, \quad \forall i \in \mathcal{J},$$

$$\mathbf{x}_i = \mathbf{x}_i^+ - \mathbf{x}_i^-, \quad \forall i \in \mathcal{J},$$

$$\mathbf{x}_i^+, \mathbf{x}_i^- \geq 0, \quad \forall i \in \mathcal{J},$$

$$\tau_i(\tilde{\mathbf{Z}}) = \sum_{j \in \mathcal{J}} \tau_{ij}(\tilde{\mathbf{Z}}) \quad \text{a.s.}, \quad \forall i \in \mathcal{J},$$

$$\mathbb{E}[\tilde{t}_j(x_{ij}^+, x_{ij}^-, \xi)] \leq \mathbb{E}[\tau_{ij}(\tilde{\mathbf{Z}})], \quad \forall i \in \mathcal{J}, j \in \mathcal{J}, \tag{17a}$$

$$\mathbb{E}\left[\tilde{t}_j\left(\sum_{a \in \mathcal{J} \setminus \{i\}} x_{aj}^+, \sum_{a \in \mathcal{J} \setminus \{i\}} x_{aj}^-, \xi\right)\right] \leq \mathbb{E}\left[\sum_{a \in \mathcal{J} \setminus \{i\}} \tau_{aj}(\tilde{\mathbf{Z}})\right], \tag{17b}$$

$$\forall i \in \mathcal{J}, j \in \mathcal{J},$$

$$\tilde{t}_j\left(\sum_{a \in \mathcal{J}} x_{aj}^+, \sum_{a \in \mathcal{J}} x_{aj}^-, \xi\right) = \sum_{a \in \mathcal{J}} \tau_{aj}(\tilde{\mathbf{Z}}) \quad \text{a.s.}, \quad \forall j \in \mathcal{J}, \tag{17c}$$

$$u_i(\mathbf{x}_i) - \mathbb{E}[\tau_i(\tilde{\mathbf{Z}})] \geq U_i^{\text{IND}}, \quad \forall i \in \mathcal{J}, \tag{17d}$$

where “a.s.” denotes *almost surely*, i.e., for all realizations except perhaps on a set of measure zero.

Let us comment on the formulation. First, note that the objective function involves terms capturing the expected total charge to each account. This is consistent with our interpretation that  $u_i$  are *expected utilities* and is the standard approach in the literature (Fabozzi et al. 2010). Second, note that we require the constraints (17a), (17b), and (17d) to hold *in expectation*. The main reason behind this choice is pragmatic. Recall that the counterparts of these constraints in our original deterministic framework, i.e., Equations (11), (12), and (14), respectively, were all based on counterfactuals, i.e., hypothetical scenarios of what the market impact costs would have been under different sets of trades. In the current setting, the manager does not observe the true market impact costs under any set of trades except the executed ones. However, *ex ante*, counterfactual costs can still be inferred in expectation, using the unbiased estimator  $t_j$ . Therefore, imposing these constraints is still meaningful because they ensure that feasible cost allocation policies do not systematically favor a portfolio over another.

In contrast, note that we require constraints (17c) governing the distribution of the total transaction cost to hold

almost surely. Such a requirement is not based on any counterfactuals and is necessary since the realized transaction costs must always be covered, in every state of the world.

The main question remaining is how to solve problem (SP). Note that this is a two-stage stochastic program since  $\tau_{ij}$  are adjustable policies allowed to depend on the realization of the random vector  $\tilde{\mathbf{Z}}$ . The following result provides a solution to this problem.

**THEOREM 1.** Let  $\mathbf{x}_i^*$ ,  $(\mathbf{x}_i^*)^+$ ,  $(\mathbf{x}_i^*)^-$ ,  $\bar{\tau}_{ij}^*$ , and  $\bar{\tau}_i^*$  be an optimal solution to problem (15) with  $t_j$  given by (16). Then  $\mathbf{x}_i^*$ ,  $(\mathbf{x}_i^*)^+$ ,  $(\mathbf{x}_i^*)^-$ ,

$$\tau_{ij}^*(\mathbf{z}) \stackrel{\text{def}}{=} \frac{\bar{\tau}_{ij}^*}{\sum_{a \in \mathcal{J}} \bar{\tau}_{aj}^*} \cdot z_j, \quad \forall \mathbf{z}, \quad (18)$$

and

$$\tau_i^*(\mathbf{z}) \stackrel{\text{def}}{=} \sum_{j \in \mathcal{J}} \tau_{ij}^*(\mathbf{z})$$

are an optimal solution for problem (SP).

A proof of the theorem is included in §A of the appendix. The result above provides an implementable, optimal solution in case of unknown transaction costs: the manager first decides on the trades and plans to split the market impact costs “in expectation” by solving problem (15). The realized transaction costs are then split in a pro rata fashion, according to the values  $\bar{\tau}_{ij}^*$  obtained originally.

We believe that from a practical perspective, the scheme can be attractive to a manager. As an enhancement to the deterministic framework in (15), it is simple, intuitive, and relatively easy to justify to the account holders. To compute the optimal trades and policy for splitting the ex post costs, the same convex optimization problem (15) must be solved. Critically important, the manager only requires an unbiased estimator for the mean transaction costs, which should be considerably easier to obtain than an accurate model for  $\tilde{t}_j$  and the distribution of  $\xi$ —for instance, the manager could directly use one of the deterministic market impact cost models in the literature (effectively assuming that they provide unbiased estimates for the mean market impact costs) and then prorate the ex post realized costs according to (18).

## 4. Extensions

In this section, we present several important modeling extensions that can be readily embedded in our framework and conclude with a discussion.

### 4.1. Cross-Asset Price Impact

The base model we focused on in §§2 and 3 relied on the assumption that the trading activity in one asset does not affect the prices of other assets, which lead to the market impact cost function having the separable form in (2). In practice, however, that may not be true: for instance,

large trades in the stock of one company can often attract more trading (and hence price impact) in stocks of related companies; see Bertsimas et al. (1999b).

As we now argue, our framework readily extends to such a case. In fact, the only modification required pertains to the market impact cost split variables  $\tau$  and the associated constraints (11)–(13). More precisely, instead of deciding separate charges  $\tau_{ij}$  for each account  $i$  in each asset  $j$ , the manager would directly decide the total charge  $\tau_i$  for the  $i$ th account. This results in the following counterparts of our prior constraints:

$$(11): \quad t(\mathbf{x}_i^+, \mathbf{x}_i^-) \leq \tau_i, \quad \forall i \in \mathcal{J},$$

$$(12): \quad \tau_i \leq t\left(\sum_{a \in \mathcal{J}} \mathbf{x}_a^+, \sum_{a \in \mathcal{J}} \mathbf{x}_a^-\right) - t\left(\sum_{a \in \mathcal{J} \setminus \{i\}} \mathbf{x}_a^+, \sum_{a \in \mathcal{J} \setminus \{i\}} \mathbf{x}_a^-\right), \quad \forall i \in \mathcal{J},$$

$$(13): \quad \sum_{a \in \mathcal{J}} \tau_a = t\left(\sum_{a \in \mathcal{J}} \mathbf{x}_a^+, \sum_{a \in \mathcal{J}} \mathbf{x}_a^-\right).$$

The intuition is identical to that discussed in §3; the sole modification is that the constraints are now written cumulatively across assets. The manager can then solve the following MPO problem to determine  $\mathbf{x}$  and  $\tau$ :

$$\begin{aligned} & \text{maximize} \quad \{f(u_1(\mathbf{x}_1) - \tau_1, \dots, u_n(\mathbf{x}_n) - \tau_n)\} \\ & \text{subject to} \quad \mathbf{x}_i \in \mathcal{C}_i, \quad \forall i \in \mathcal{J}, \\ & \quad \mathbf{x}_i = \mathbf{x}_i^+ - \mathbf{x}_i^-, \quad \forall i \in \mathcal{J}, \\ & \quad \mathbf{x}_i^+, \mathbf{x}_i^- \geq 0, \quad \forall i \in \mathcal{J}, \\ & \quad t(\mathbf{x}_i^+, \mathbf{x}_i^-) \leq \tau_i, \quad \forall i \in \mathcal{J}, \\ & \quad t\left(\sum_{a \in \mathcal{J} \setminus \{i\}} \mathbf{x}_a^+, \sum_{a \in \mathcal{J} \setminus \{i\}} \mathbf{x}_a^-\right) \leq \sum_{a \in \mathcal{J} \setminus \{i\}} \tau_a, \quad \forall i \in \mathcal{J}, \\ & \quad t\left(\sum_{a \in \mathcal{J}} \mathbf{x}_a^+, \sum_{a \in \mathcal{J}} \mathbf{x}_a^-\right) \leq \sum_{a \in \mathcal{J}} \tau_a, \\ & \quad u_i(\mathbf{x}_i) - \tau_i \geq U_i^{\text{IND}}, \quad \forall i \in \mathcal{J}. \end{aligned} \quad (19)$$

We note that nonseparable market impact costs pose a challenge for standard schemes used in practice since the pro rata sharing fails to capture the cross-asset effects, in contrast to our model.

Note that formulation (19) subsumes our original model (15) since it deals with a more general case. In fact, the former problem is more compact, involving fewer variables than the latter, and  $\mathcal{O}(n)$  constraints instead of  $\mathcal{O}(mn)$ . A natural question to ask in this context is whether the two formulations are equivalent when the market impact cost function  $t$  is separable across assets. The following theorem formalizes this relationship.

**THEOREM 2.** Suppose that the market impact cost function  $t$  is separable across assets; i.e., it satisfies (2), and  $t(\mathbf{0}) = 0$ . Then the feasible set of (19) is identical to the feasible set of (15) projected on the variables  $\{\mathbf{x}_i, \mathbf{x}_i^+, \mathbf{x}_i^-, \tau_i\}_{i \in \mathcal{J}}$ . In particular, the two formulations have the same optimal values.

A proof of the theorem is included in §A of the appendix. The additional requirement  $t(\mathbf{0}) = 0$  is trivially true in practice. In this sense, the result in Theorem 2 becomes very relevant from a pragmatic viewpoint: even when the market impact costs are separable, the manager can find the optimal trades  $\mathbf{x}_i$  and optimal charges  $\tau_i$  for each account by solving the more compact formulation (19).

## 4.2. Multiperiod Models

The models discussed thus far have been primarily focused on single period portfolio selection. In practice, however, most portfolio optimization problems are dynamic, involving (investment) decisions taken at multiple points in time. In this section, we demonstrate how our framework can be extended to such a setting. The literature on multiperiod models for single portfolio optimization is already vast, covering various facets of the problem (see, e.g., Fabozzi et al. 2010, Brown and Smith 2011 for more references). For illustration purposes, we focus on one particular application, which has attracted considerable interest, particularly in the context of market impact costs.

More precisely, we consider the optimal execution problem faced by a manager liquidating a large portfolio and seeking to minimize trading costs by splitting the trades across several periods in time. Because of market impact, short-term return predictability, and/or potential constraints on trading, the problem of finding an optimal execution schedule is nontrivial and has received considerable attention in the literature (see, e.g., Bertsimas and Lo 1998, Almgren and Chriss 2000, Moazeni et al. 2010, Tsoukalas et al. 2012, and references therein). The setting that we adopt here is most closely aligned with that in Moallemi and Sağlam (2012), to which we direct the interested reader for details and discussions of underlying assumptions. We first describe the single portfolio model in the former paper and then extend it to the MPO setting.

We consider a manager who would like to liquidate a single portfolio with initial holdings  $\mathbf{w}(0) \in \mathbb{R}_+^n$  before a final time  $T$ . We assume that trades occur at discrete times  $k = 1, 2, \dots, T$  and define the *execution schedule* as the collection  $(\mathbf{x}(1), \mathbf{x}(2), \dots, \mathbf{x}(T))$ , where  $\mathbf{x}(k) \in \mathbb{R}^m$  denotes the trades executed in time period  $k$  (positive and negative components denote buy and sell orders, respectively). As such, the holdings of the portfolio at the beginning of period  $k$  are given by  $\mathbf{w}(k) = \mathbf{w}(0) + \sum_{s=1}^k \mathbf{x}(s)$ .

The portfolio holdings and the trading schedule must typically also satisfy certain constraints. In an execution problem, natural requirements are  $\mathbf{w}(T) = \mathbf{0}$ , i.e., the entire portfolio should be liquidated, and  $\mathbf{x}(k) \leq \mathbf{0}$ , i.e., only selling should occur during the trading horizon.

Let  $\mathbf{r}(k+1) \in \mathbb{R}^m$  denote the price changes in the  $m$  assets from period  $k$  to  $k+1$ . We assume that  $\mathbf{r}(k+1)$  are driven by  $K$  factors  $\mathbf{F}(k) \in \mathbb{R}^K$ , which follow a mean-reverting process. More formally, we consider the following

dynamics for the price changes and factor realizations (see Moallemi and Sağlam 2012 for details):

$$\mathbf{F}(k+1) = (I - \Phi)\mathbf{F}(k) + \boldsymbol{\epsilon}^{(1)}(k+1),$$

$$\mathbf{r}(k+1) = \boldsymbol{\mu} + B\mathbf{F}(k) + \boldsymbol{\epsilon}^{(2)}(k+1),$$

where  $B \in \mathbb{R}^{m \times K}$  is a constant matrix of factor loadings;  $\Phi \in \mathbb{R}^{K \times K}$  is a diagonal matrix of mean reversion coefficients;  $\boldsymbol{\mu} \in \mathbb{R}^n$  is the mean return; and the noise terms are independent (across time and returns/factors), normally distributed, with zero means and covariances

$$\text{cov}(\boldsymbol{\epsilon}^{(1)}(k+1)) = \Psi \in \mathbb{R}^{K \times K} \quad \text{and}$$

$$\text{cov}(\boldsymbol{\epsilon}^{(2)}(k+1)) = \Sigma \in \mathbb{R}^{m \times m}.$$

When executing a trade  $\mathbf{x}$  in any period  $k$ , the manager incurs transaction costs (primarily due to market impact), modeled as  $t(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \Lambda \mathbf{x}$ , where  $\Lambda \in \mathbb{R}^{m \times m}$  is a positive semidefinite matrix. As in Moallemi and Sağlam (2012), we assume that the manager is risk neutral, and his objective is to maximize the total expected excess profits from trading, net of transaction costs.

Since the returns are stochastic, the decisions taken by the manager do not have to be fixed, i.e., static; instead, the manager can choose trading schedules that consist of *nonanticipative dynamic policies*.<sup>11</sup> Finding the optimal such policy is generally computationally intractable because of the high dimensionality of the problem (one must keep track of the portfolio weights in each asset). As such, a natural approach is to look for suboptimal policies with good performance. For our subsequent analysis, we focus on only one such policy, namely, model predictive control (MPC), which is well established and often delivers good performance in practice (we direct the interested reader to Moallemi and Sağlam 2012, who compare this with several other alternatives).

In the MPC heuristic, at each trading time  $k$ , the manager would solve a problem over the remaining periods  $k, k+1, \dots, T$ , to determine a deterministic execution schedule  $(\mathbf{x}(k), \mathbf{x}(k+1), \dots, \mathbf{x}(T))$  conditional on the available information but would only implement the first trade  $\mathbf{x}(k)$ . More formally, the manager would solve the following quadratic program:

$$\begin{aligned} & \underset{\mathbf{x}(k), \dots, \mathbf{x}(T)}{\text{maximize}} \quad \left\{ \sum_{s=k}^T (\mathbf{w}(s)^T B(I - \Phi)^{s-k} \mathbf{F}(k) - \frac{1}{2} \mathbf{x}(s)^T \Lambda \mathbf{x}(s)) \right\} \\ & \text{subject to} \quad \mathbf{x}(s) = \mathbf{w}(s) - \mathbf{w}(s-1), \quad \forall s \in \{k, \dots, T\}, \\ & \quad \mathbf{x}(s) \leq \mathbf{0}, \mathbf{w}(s) \geq \mathbf{0}, \quad \forall s \in \{k, \dots, T\}, \\ & \quad \mathbf{w}(T) = \mathbf{0}. \end{aligned} \tag{20}$$

Let us now consider an MPO setting, where the manager is in charge of liquidating  $n$  accounts, indexed by  $i \in \mathcal{I}$ . Just as with the setting in §2, the presence of market impact costs would again result in questions concerning an appropriate split of trading costs as well as designing the optimal

execution schedules that appropriately take this subsequent split into account.

We adopt the framework introduced in §3, suitably extended. In particular, at any stage  $k$  in time, the manager would solve the following MPO equivalent of the MPC formulation, with variables  $\{\mathbf{x}_i(s), \tau_i(s)\}_{s \in \{k, \dots, T\}, i \in \mathcal{I}}$ :

$$\begin{aligned} & \text{maximize } f\left(\sum_{s=k}^T (\mathbf{w}_1(s)^T B(I - \Phi)^{s-k} - \tau_1(s)), \dots, \right. \\ & \quad \left. \sum_{s=k}^T (\mathbf{w}_n(s)^T B(I - \Phi)^{s-k} - \tau_n(s))\right) \\ & \text{subject to } \mathbf{x}_i(s) = \mathbf{w}_i(s) - \mathbf{w}_i(s-1), \\ & \quad \forall s \in \{k, \dots, T\}, \quad \forall i \in \mathcal{I}, \\ & \mathbf{x}_i(s) \leq \mathbf{0}, \mathbf{w}_i(s) \geq \mathbf{0}, \quad \forall s \in \{k, \dots, T\}, \quad \forall i \in \mathcal{I}, \\ & \mathbf{w}_i(T) = \mathbf{0}, \quad \forall i \in \mathcal{I}, \\ & \frac{1}{2} \mathbf{x}_i(s)^T \Lambda \mathbf{x}_i(s) \leq \tau_i(s), \quad \forall s \in \{k, \dots, T\}, \\ & \quad \forall i \in \mathcal{I}, \\ & \frac{1}{2} \left( \sum_{a \in \mathcal{I} \setminus \{i\}} \mathbf{x}_a(s) \right)^T \Lambda \left( \sum_{a \in \mathcal{I} \setminus \{i\}} \mathbf{x}_a(s) \right) \leq \sum_{a \in \mathcal{I} \setminus \{i\}} \tau_a(s), \\ & \quad \forall s \in \{k, \dots, T\}, \quad \forall i \in \mathcal{I}, \\ & \frac{1}{2} \left( \sum_{a \in \mathcal{I}} \mathbf{x}_a(s) \right)^T \Lambda \left( \sum_{a \in \mathcal{I}} \mathbf{x}_a(s) \right) \leq \sum_{a \in \mathcal{I}} \tau_a(s), \\ & \quad \forall s \in \{k, \dots, T\}, \\ & \sum_{s=k}^T (\mathbf{w}_i(s)^T B(I - \Phi)^{s-k} - \tau_i(s)) \geq U_i^{\text{IND}}(k), \\ & \quad \forall i \in \mathcal{I}. \quad (21) \end{aligned}$$

Few remarks are in order. First,  $U_i^{\text{IND}}(k)$  has the same interpretation as in §2—it reflects the realized net utility that would be obtained by the  $i$ th account, when problem (20) is solved for each account in isolation to determine the optimal execution schedule conditional on the available information, but then the trades are actually aggregated and the resulting market impact costs are split in a pro rata fashion.

Second, cost allocation in the above formulation is performed on a per-period basis, since in practice managers often face accounting requirements and need to report trades and cost allocations at every period. In the absence of such requirements, however, problem (21) can be reformulated using a single cost allocation, thus providing more flexibility.

Third, note that extending the MPC scheme to an MPO setting essentially entails solving the same class of problems, but with suitably enlarged sizes. As is typical in an MPC scheme, a manager would only implement the decisions for period  $k$  resulting from the solution of problem (21); i.e., he would effectively execute the first set of trades  $\{\mathbf{x}_i(k)\}_{i \in \mathcal{I}}$  and split the resulting transaction costs according to  $\{\tau_i(k)\}_{i \in \mathcal{I}}$ . In period  $k + 1$ , a similar model

would then be solved to determine trades and cost splits for  $k + 1, \dots, T$ . Therefore, conceptually, the model is as straightforward to implement and test as the single portfolio setup in (20). However, it does require solving larger problems; for instance, problem (21) has  $\mathcal{O}(mnT)$  variables and constraints. We further explore this issue in §5.

Finally, note that Theorem 2 is readily applicable in this setting as well; as such, one could equivalently reformulate problem (21).

### 4.3. Discussion

The extensions discussed above highlight that our framework is *general* and adapts readily to important settings other than the one considered in §2. In fact, we argue that our framework can be leveraged to extend many models proposed in modern portfolio theory that deal with managing a single portfolio in frictional markets, to the case of multiple portfolios. Our claim is because for piecewise linear  $f$ ,<sup>12</sup> our framework does not change the underlying complexity of an optimization model for a single portfolio. That is, the addition of the extra variable  $\tau$  for the trading costs split, the split constraints (11)–(13), and the coordination benefits constraints (14) do not change the complexity of an optimization model for a single portfolio that already accounts for transaction costs. For instance, consider the formulation for asset allocation proposed by Bertsimas et al. (1999a) that involved solving a mixed-integer linear optimization problem. Their model can be readily extended to a multiportfolio setting where one similarly needs to solve a mixed-integer linear optimization problem. Similarly, the deleveraging problem considered by Brown et al. (2010) involves a quadratically constrained quadratic optimization problem in the single portfolio case; the same holds true if one were to extend that model for a multiportfolio setting. In case the selected welfare function  $f$  is not piecewise linear, our formulation still leads to a convex optimization problem as long as the original single portfolio problem is also convex.

As per the discussion above, our framework leads to scalable and tractable extensions of many single portfolio optimization problems studied in the literature. Other relevant extensions include situations where cross-trading of assets is allowed between accounts, transaction cost models that capture permanent price impact effects, and cases where other types of nonseparable transaction costs are present (e.g., fixed costs or fees, etc.) The first two aforementioned extensions are discussed in detail in §B of the appendix.

## 5. Numerical Studies

We present studies that illustrate the performance of our framework in practice.

*Numerical Study 1.* A manager is in charge of  $n = 3$  portfolios, investing in a market consisting of  $m = 100$  assets.

There are 20 factors that drive the returns of the assets, assumed to be independently and identically distributed following a standard normal distribution. The return of the  $j$ th

asset is  $r_j = \mu_j + \mathbf{a}_j^T \mathbf{f} + \epsilon_j$ , where  $\mu_j$  is the expected (annualized) return,  $\mathbf{f}$  is the vector of factors,  $\mathbf{a}_j$  is the vector of exposure coefficients to the factors and  $\epsilon_j$  is an idiosyncratic noise term. The noise terms are independently and identically distributed, following a zero-mean normal distribution. Let  $\Sigma$  be the covariance matrix of the returns  $\{r_j\}_{j \in \mathcal{J}}$ . The exposure coefficients and the volatilities of the noise terms are randomly selected, subject to the (annualized) volatilities of the returns being between 15% and 45%. Similarly, the expected returns  $\boldsymbol{\mu}$  are randomly selected between  $-20\%$  and  $40\%$ .

The market impact cost function is quadratic and separable across assets as well as separable and symmetric across buys and sells. That is, the market impact cost for trading the  $j$ th asset is given by

$$t_j(x^+, x^-) = \alpha_j((x^+)^2 + (x^-)^2).$$

The coefficients  $\{\alpha_j\}_{j \in \mathcal{J}}$  are randomly selected between 2 and 10. We use such a simplistic and stylized model for impact costs (a) for simplicity of the exposition and (b) in order to be able to compare the performance of our framework with all the solution concepts that have been proposed so far.<sup>13</sup>

The initial holdings  $\{\mathbf{w}_i\}_{i \in \mathcal{J}}$  are assumed to be three different market indices that the portfolios are tracking. Their compositions are randomly generated, subject to their (annualized) volatilities being  $\sigma_1 = 5\%$ ,  $\sigma_2 = 10\%$ , and  $\sigma_3 = 20\%$ , respectively. Accounts 1 and 3 are of the same size, whereas account 2 is twice as large; i.e.,  $\mathbf{1}^T \mathbf{w}_1 = \frac{1}{2} \mathbf{1}^T \mathbf{w}_2 = \mathbf{1}^T \mathbf{w}_3$ , where  $\mathbf{1}$  is the vector of all ones.

The manager needs to perform self-financing rebalancing trades  $\mathbf{x}$ , such that the turnover for each account is at most 10% and its risk exposure does not increase; i.e.,

$$\mathcal{C}_i = \{\mathbf{x}_i \in \mathbb{R}^m \mid \mathbf{1}^T \mathbf{x}_i = 0, \|\mathbf{x}_i\|_1 \leq 10\% \cdot \mathbf{1}^T \mathbf{w}_i, (\mathbf{w}_i + \mathbf{x}_i)^T \Sigma (\mathbf{w}_i + \mathbf{x}_i) \leq (\sigma_i \mathbf{1}^T \mathbf{w}_i)^2\}, \quad \forall i \in \mathcal{J}.$$

For simplicity, we assume that the  $i$ th account derives utility equal to the (expected monetary) profits it makes because of trading; i.e.,

$$u_i(\mathbf{x}_i) = \boldsymbol{\mu}^T \mathbf{x}_i, \quad \forall i \in \mathcal{J}.$$

One could equivalently consider normalizing the utility of each portfolio by its wealth. That is, one could consider the quantities  $\boldsymbol{\mu}^T \mathbf{x}_i / \mathbf{1}^T \mathbf{w}_i$ ,  $\forall i \in \mathcal{J}$ , that correspond to the *active returns* of the accounts and the quantities  $U_i / \mathbf{1}^T \mathbf{w}_i$ ,  $\forall i \in \mathcal{J}$ , that correspond to the *net active returns* of the accounts (adjusted for transaction costs). In fact, these are the values we report in our numerical results since they yield a (normalized) performance measure that is more readily interpretable and compared across the accounts.

We first consider the independent solution for deciding the rebalancing trades (see §2.2) that we view as the baseline

case. We then consider the MPO solutions of social welfare and Cournot–Nash (see §§2.3–2.4). We contrast them with the maximin solution, which we obtain by utilizing our framework (see §3). That is, we make a particular selection for the welfare function  $f$  in our framework (15) that specifies how we trade off efficiency and fairness. Note that as we discussed in §3, such a selection depends on what one considers as “fair,” details of which are outside the scope of this paper; the selection of the maximin function is made here for illustration purposes and is discussed next.

For the maximin solution, we have

$$f(U_1, U_2, \dots, U_n) = \min_{i \in \mathcal{J}} \left\{ \frac{U_i - U_i^{\text{IND}}}{U_i^{\text{IND}}} \right\}. \quad (22)$$

Recall that  $U_i$  is the utility of the  $i$ th account, adjusted for transaction costs, for the maximin solution under consideration; see (10). Similarly,  $U_i^{\text{IND}}$  is the utility under the independent solution; see (5). The welfare function  $f$  evaluated at  $\{U_i\}_{i \in \mathcal{J}}$  equals to the minimum increase in utility, relative to the utility under the independent solution, across all accounts.<sup>14</sup> One can think about the maximin solution as follows. The independent solution is the case where the accounts do not “cooperate” and are optimized independently. Under the MPO maximin approach, the accounts do “cooperate” and are jointly optimized. Moreover, the gains from joint optimization are split in a way that maximizes the minimum (relative) benefit of each account from this “cooperation.”

The outcomes of the numerical study are included in Tables 1 and 2. We report the active returns, transaction costs, and net active returns of the portfolios under the different schemes we consider. We also report as *Total* the corresponding values for the portfolios in aggregation.<sup>15</sup> Figure 1 depicts the increase of the net active return of each portfolio under the MPO schemes, relative to the independent scheme.

The results confirm our earlier claims. In particular,

- By optimizing the accounts in isolation, the independent scheme generates trades with significant overlap. This translates in *realized* market impact costs that are significantly larger than *anticipated* ones, which, in turn, imply realized net utilities considerably smaller than anticipated ones (as reflected in Table 1, the latter are by 15%–30% smaller than the former).

- All MPO schemes (social, Cournot–Nash, and maximin) result in lower market impact costs for the accounts (compare the results of Table 2 with those *realized* under Table 1). In this particular study, this also translates into strictly improved net utilities for all accounts compared to the independent scheme, as reflected in Figure 1.

- The three schemes discussed have very different fairness properties. As shown in Figure 1, both social and Cournot–Nash tend to result in widely different improvement levels for the accounts. Under social, the first two accounts improve their net active returns (as compared to

**Table 1.** Anticipated and realized active returns (i.e., normalized utilities  $u_i$ ); transactions costs; and net active returns (i.e., normalized utilities  $U_i$  adjusted for transaction costs), for the three portfolios under the independent scheme in numerical study 1.

	Independent								
	Anticipated				Realized				
	1	2	3	Total	1	2	3	Total	
Active return (in %)	2.37	2.24	2.41	2.32	2.37	2.24	2.41	2.32	2.32
Transaction cost (in %)	0.39	0.84	0.64	0.67	0.98	1.05	1.23	1.08	1.08
Net active return (in %)	1.99	1.41	1.77	1.64	1.39	1.19	1.18	1.24	1.24

**Table 2.** Realized active returns (i.e., normalized utilities  $u_i$ ); transactions costs; and net active returns (i.e., normalized utilities  $U_i$  adjusted for transaction costs), for the three portfolios under the social, Cournot–Nash, and maximin schemes in numerical study 1.

	Social				Cournot–Nash				Maximin			
	1	2	3	Total	1	2	3	Total	1	2	3	Total
Active return (in %)	2.06	2.18	2.24	2.17	2.24	2.17	2.35	2.24	2.06	2.18	2.24	2.17
Transaction cost (as %)	0.59	0.91	0.77	0.80	0.70	0.94	0.98	0.89	0.52	0.86	0.94	0.80
Net active return (in %)	1.48	1.27	1.47	1.37	1.54	1.24	1.37	1.35	1.54	1.32	1.30	1.37

the independent scheme) by roughly 6%, whereas the third account achieves a staggering improvement of 25%. Under Cournot–Nash, the first account improves by more than 10%, the second by less than 4%, and the third by more than 16%. As expected, under the maximin scheme, all accounts improve by exactly the same amount (namely 10.7%).

• By definition, the social scheme maximizes the aggregate performance of all portfolios (recorded under Total) and achieves an active return that is 10.7% higher compared to the independent scheme. This is strictly larger than the increase achieved under Cournot–Nash but exactly equal to that under maximin! This reflects an attractive feature of maximin and our framework in this case, namely, that by taking an approach that optimizes jointly over the trades and

split of transaction costs, one can achieve a considerably fairer split of the improvements without sacrificing aggregate performance.

*Numerical Study 2.* We consider a similar setup to the one discussed in study 1, where the manager is in charge of  $n = 6$  accounts. The  $i$ th account derives utility equal to the (expected monetary) profits it makes because of trading, adjusted for risk, i.e.,

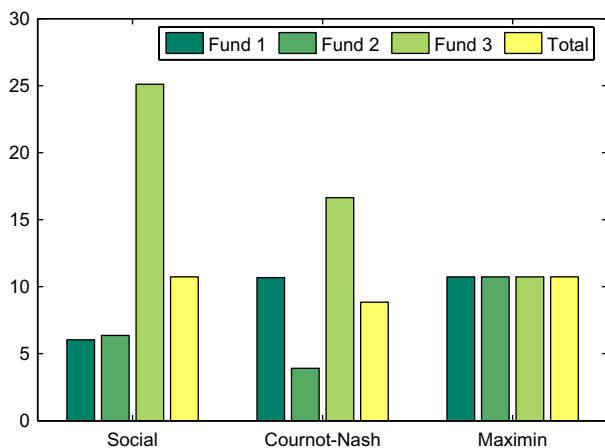
$$u_i(\mathbf{x}_i) = \boldsymbol{\mu}^T \mathbf{x}_i - \lambda_i (\mathbf{w}_i + \mathbf{x}_i)^T \boldsymbol{\Sigma} (\mathbf{w}_i + \mathbf{x}_i), \quad \forall i \in \mathcal{J},$$

where  $\lambda_i$  is a parameter that measures risk aversion. The values  $\{\lambda_i\}_{i \in \mathcal{J}}$  are randomly selected between  $10^{-4}$  and  $2.5 \times 10^{-4}$ .

The results mimic those of the previous study. In particular, the independent scheme again considerably underestimates the market impact costs, resulting in lower realized net utilities. MPO approaches partially correct for the effect by resulting in improvements in net active returns. Figure 2 depicts the increase of the net active return of each portfolio under the MPO schemes, relative to the independent scheme.

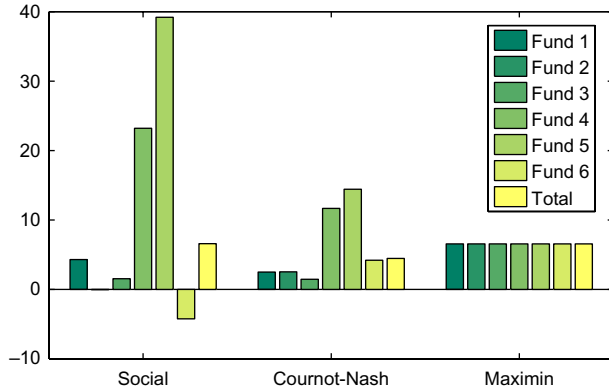
Note that the social scheme again results in severe inequalities in the distribution of gains: as can be seen from Figure 2, portfolio 5 achieves a 40% improvement over the independent scheme, portfolio 2 achieves almost no improvement, and portfolio 6 suffers a 5% decline in its active return. On the other hand, the maximin approach provides a constant improvement of 6.5% to each account. In terms of aggregate performance, the aggregate improvement under the social scheme is 6.6%, compared to 6.5% under the maximin scheme. That is, the maximin scheme again provides an equitable distribution of gains over the portfolios, without sacrificing aggregate performance. Finally, the Cournot–Nash scheme provides both an unequal distribution of gains as well as inferior aggregate performance improvement of 4.5%.

**Figure 1.** (Color online) Relative increase (in %) of the net active returns, i.e., net utilities  $U_i$ , of the three portfolios in numerical study 1 under the social, Cournot–Nash, and maximin schemes, compared to the independent scheme.





**Figure 2.** (Color online) Relative increase (in %) of the net active returns, i.e., net utilities  $U_i$ , of the six portfolios in numerical study 2 under the social, Cournot–Nash, and maximin schemes, compared to the independent scheme.



*Numerical Study 3.* In this study, we focus on the impact of our proposal to allow the transaction cost splits to be decision variables, rather than pro rata. While this choice clearly results in a more flexible model (and hence improved performance), it is relevant to also examine the extent to which trading patterns resulting from the two approaches differ.

To address this, we consider an identical setup to that of study 1 and solve the following problem for determining the optimal trades and market impact cost split under a maximin objective:

$$\begin{aligned}
 &\text{maximize} \quad \min_{i \in \mathcal{F}} \left\{ \frac{(\boldsymbol{\mu}^T \mathbf{x}_i - \tau_i) - U_i^{\text{IND}}}{U_i^{\text{IND}}} \right\} \\
 &\text{subject to} \quad \mathbf{x}_i \in \mathcal{C}_i, \quad \forall i \in \mathcal{F}, \\
 &\quad \mathbf{x}_i = \mathbf{x}_i^+ - \mathbf{x}_i^-, \quad \forall i \in \mathcal{F}, \\
 &\quad \mathbf{x}_i^+, \mathbf{x}_i^- \geq 0, \quad \forall i \in \mathcal{F}, \\
 &\quad \tau_i = \sum_{j \in \mathcal{F}} \tau_{ij}, \quad \forall i \in \mathcal{F}, \\
 &\quad \boldsymbol{\mu}^T \mathbf{x}_i - \tau_i \geq U_i^{\text{IND}}, \quad \forall i \in \mathcal{F}, \\
 &\quad \tau_{ij} \geq \frac{x_{ij}^+}{\sum_{a \in \mathcal{F}} x_{aj}^+} \cdot \underbrace{\alpha_j \left( \sum_{a \in \mathcal{F}} x_{aj}^+ \right)^2}_{\text{market impact cost for buying the } j\text{-th asset}} + \frac{x_{ij}^-}{\sum_{a \in \mathcal{F}} x_{aj}^-} \\
 &\quad \quad \cdot \underbrace{\alpha_j \left( \sum_{a \in \mathcal{F}} x_{aj}^- \right)^2}_{\text{market impact cost for selling the } j\text{-th asset}}, \quad \forall i \in \mathcal{F}, j \in \mathcal{F}.
 \end{aligned}$$

The MPO model above restricts the split of costs to be pro rata. Note that the formulation is a nonconvex, quadratically constrained quadratic program (QCQP), which is generally an NP-hard problem (Boyd and Vandenberghe 2004)—this is consistent with our claim in §2.1 that the pro rata split readily results in intractable models. Several approaches

have been studied for obtaining approximate solutions to this problem—we resort to the well-established linearization technique, which entails solving a sequence of convex QCQPs (see, e.g., d’Aspremont and Boyd 2003 for details).

In the approximate solution we obtained, each account derived a relative increase of 10.57% in net active return—slightly lower than the value of 10.7% achieved by the corresponding maximin MPO model in study 1, where cost splits were not constrained to be pro rata. Figure 3 also depicts the associated trades generated by these two approaches—note that the trading activity of each portfolio is not substantially different (the largest weight change is 0.0038).

*Numerical Study 4.* We present an application of our approach to a multiperiod setting in order to further evaluate its performance as well as its computational burden.

We consider the execution problem analyzed in §4.2, where a manager is in charge of liquidating  $n$  portfolios of  $m$  assets, over a trading horizon split into  $T$  periods. We study two problems of different sizes: (a)  $n = 6, m = 30, T = 10$ , and (b)  $n = 10, m = 100, T = 10$ .

The remaining problem parameters are generated as follows. Initial portfolio weights for each portfolio are randomly, uniformly sampled, and normalized so as to sum up to one. We let  $K = 5$ . Factor loadings  $\mathbf{B}$  and initial factor values  $\mathbf{F}(0)$  are sampled according to a standard normal distribution. Mean reversion parameters  $\Phi$  are randomly selected between zero and one. The volatilities  $\Psi$  of the noise terms affecting the factors are randomly selected between 0% and 10%. Transaction costs parameters  $\Lambda$  and the welfare function  $f$  are chosen as in studies 1 and 2.

As described in §4.2, we use the MPC heuristic to solve the multiperiod execution problem. We first optimize the execution schedules of the accounts independently and allocate the resulting market impact costs pro rata; i.e., we use the independent solution concept. We then use our MPO solution approach. In order to compare the two approaches, we use the same set of 1,000 simulation factor and return paths. We solve the resulting second-order cone programs using CPLEX. Table 3 reports the average relative increase of excess return (per account) under the MPO approach compared to the independent approach. We also report CPU times for the two approaches for the first trading period, which is the most computationally intensive one.

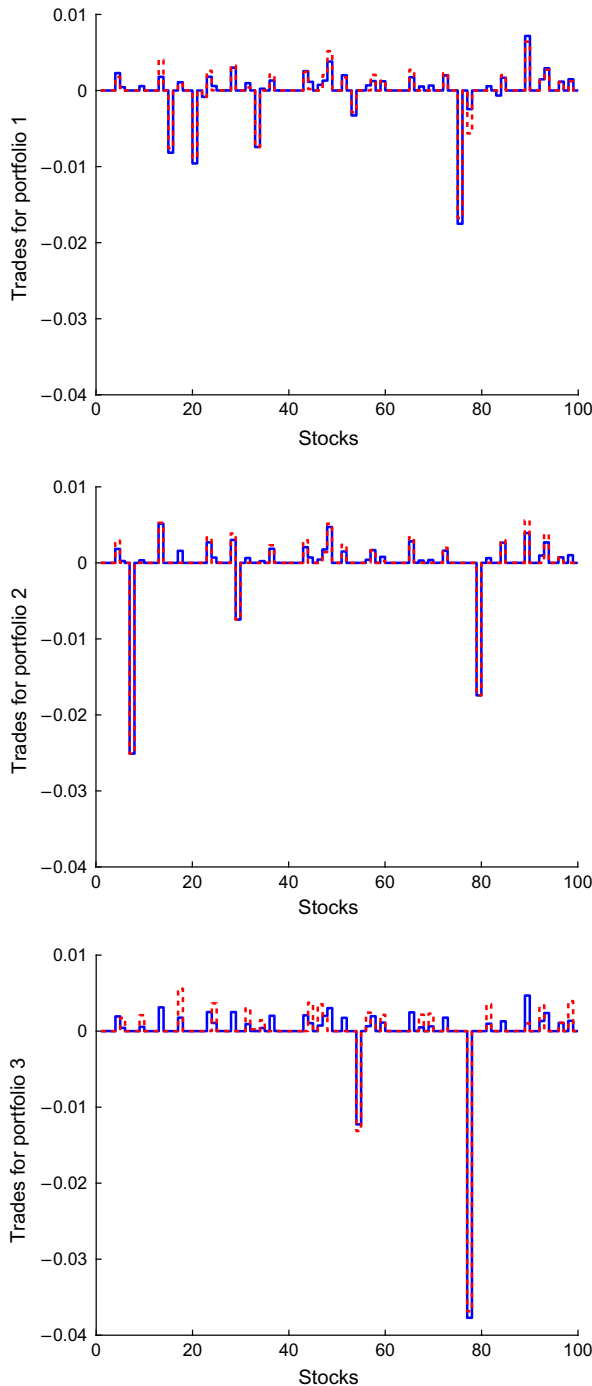
The MPO approach delivers considerable improvements in excess return. The additional computational burden is manageable from a practical perspective. Nevertheless, for larger scale instances, decomposition techniques can be deployed to enable parallelization, which would drastically reduce computational time requirements. An example of such technique would be the cutting plane Dantzig-Wolfe decomposition method, where constraints (12) and (13) are treated as the coupling constraints; we refer the reader to Bertsekas (1999, §6.4.1) for more details.

## 6. Conclusions

Modern portfolio theory encompasses a variety of powerful tools and methods useful for investment management. The



**Figure 3.** (Color online) Trades for the three portfolios in numerical studies 1 and 2 under the maximin MPO scheme, when the split of costs is constrained to be pro rata (dashed line) or not (solid line).



vast majority of those methods address different problems faced by a single investor. In practice however, portfolio managers or asset management firms are in charge of the investments of numerous clients, with an ongoing trend toward further consolidation in the U.S. finance industry.

**Table 3.** Average relative increase of excess return (per account) under the MPO approach compared to the independent approach, as well as CPU times for the two approaches for the first trading period.

	$n = 6, m = 30,$ $T = 10$	$n = 10, m = 100,$ $T = 10$
Rel. return increase under MPO (in %) (S.E.)	18.2 (2.8)	28.9 (4.7)
CPU time for independent approach	17 sec	18 sec
CPU time for MPO approach	40 sec	122 sec

In this paper, we discussed the unique challenges that arise in portfolio management in case one is in charge of multiple accounts. We argued that the problematic interactions that arise between multiple accounts in a frictional market call for a different approach that jointly optimizes/manages the accounts, rather than independently according to the classical portfolio theory paradigm. In the context of a joint management framework, however, one needs to ensure that the different portfolios are treated equitably; in fact, the SEC requires joint management of portfolios to be carried out in a transparent and fair way but without, however, providing further precise regulations or requirements.

We proposed a novel framework that allows a manager to jointly optimize multiple portfolios, subject to the SEC regulations. Our framework offers the manager the flexibility of selecting her preferred notion of fairness in balancing the performance of all portfolios she is in charge of. Incorporated in the framework is also a novel method of splitting market impact costs (incurred by the trading activity of the jointly managed accounts), in a way that is fair and also captures the aforementioned problematic interactions between them.

We compared our framework with the few of existing solution concepts proposed in the literature and used in practice. We established that our framework outperforms them by discussing both their theoretical properties and their performance in the numerical studies we conducted.

Finally, we illustrated another unique feature of our approach, namely, its generality: we demonstrated how it can be utilized to extend tractable single portfolio management methods to multiportfolio settings without sacrificing tractability or increasing the underlying computational complexity of the original method.

### Supplemental Material

Supplemental material to this paper is available at <http://dx.doi.org/10.1287/http://dx.doi.org/10.1287/opre.2014.1310>.

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## Endnotes

1. For more information on market impact costs, see Obizhaeva and Wang (2013), Almgren and Chriss (2000), and Bertsimas and Lo (1998).
2. The feasible set  $\mathcal{C}_i$  can also depend on the initial holdings  $\mathbf{w}_i$ . We omit writing this explicitly in order to simplify notation.
3. As with the feasible sets, the utility  $u_i$  can also depend on the initial holdings  $\mathbf{w}_i$ , which we also omit writing explicitly.
4. This assumption is standard in all treatments of the MPO problem in the literature, as well as in papers dealing with market impact costs, which implicitly assume that the models are sufficiently reliable for the purposes of assessing costs and deciding trades (see, e.g., Almgren and Chriss 2000, Brown et al. 2010, Moazeni et al. 2010, Moallemi and Sağlam 2012, and references therein).
5. We tacitly assumed here that the net utility of the  $i$ th account is quasilinear. Our framework readily extends to the more general case where the net utility  $U_i$  is a concave function of the utility  $u_i$  and the associated market impact costs  $t$ .
6. In this case, a manager could still apply the scheme, provided that individual cost components for each asset are available; this would effectively amount to ignoring all cross-asset interactions, which may be inappropriate.
7. For more information on utilitarianism, including relative utilitarianism and a comparison of the two principles, we refer the interested reader to Young (1995), Pivato (2008), and Dhillon and Mertens (1999).
8. As a technical remark, note that the maximin welfare function is not componentwise increasing but rather nondecreasing. Since this can lead to inefficiencies, one can instead consider a lexicographic maximization counterpart, known as max-min fairness; see Bertsimas et al. (2011).
9. Note that this challenge is absent in the single portfolio setting because the manager unequivocally allocates realized costs to the single portfolio under management.
10. More formally, the policies  $\tau_{ij}$  are adapted to the filtration generated by the random vector  $\tilde{\mathbf{Z}}$ .
11. More formally, the trades  $\mathbf{x}_k$  can be functions that are adapted to the filtration induced by the stochastic processes  $\varepsilon_k^{(1)}, \varepsilon_k^{(2)}$  (see Moallemi and Sağlam 2012 for details).
12. As we argued in §3, a piecewise linear functional form for  $f$  already captures the two most useful welfare functions, the utilitarian, i.e., sum of utilities, and maximin, i.e., min of utilities.
13. Recall that unlike our approach, the Cournot–Nash approach leads to intractable equilibrium problems for other (more realistic) impact cost models, such as the ones in Kolm (2009), Moallemi and Sağlam (2012), or Tsoukalas et al. (2012).
14. As a technical remark, note that in case the maximization of  $f$  in (22) does not produce a Pareto optimal point, one can always use a lexicographic maximin form for  $f$ ; see Bertsimas et al. (2011) for details.
15. For instance, the total active return reported is  $\boldsymbol{\mu}^T(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3) / \mathbf{1}^T(\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3)$ .

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