New Advanced Problems in KöMaL

September 1993 – May 1997

N. 1. Two players play the following game. They move alternately, writing a real coefficient in polynomial

$$x^{10} + * \cdot x^9 + * \cdot x^8 + \cdots + * \cdot x + 1$$

to an empty $*$ place chosen by them. The first player wins if the polynomial obtained at the end of the game has no real roots. Which player has a winning strategy?

N. 2. Find the triangle of smallest area with which one can cover any triangle that has sides at most 1.

N. 3. Prove that for every real number $0 < \alpha < 1$ there exist integers $1 \le a_1 < a_2 < \cdots < a_n \le 2^{n-1}$ satisfying inequalities

$$[\alpha a_1] \le [\alpha^2 a_2] \le \cdots \le [\alpha^n a_n].$$

N. 4. Let P be an inner point of triangle ABC, and denote the distances of P from sides a, b and c by x, y and z, respectively. Find those points P for which the sum

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z}$$

is minimal.

N. 5. Let α be the greatest root of $x^3 - 3x^2 + 1 = 0$. Prove

$$\left[\alpha^{1993}\right] \equiv 4 \pmod{17},$$

i.e., that the integer part of α^{1993} leaves a remainder of 4 on division by 17.

N. 6. The black king is placed at the left upper corner of an $m \times n$ chessboard. Two players play a game as follows: they move alternately with the king, and they may move only such fields that have not been touched at an earlier stage of the game. The first one who cannot move, loses. Who has a winning strategy?

N. 7. The bisectors of the angles of triangle ABC meet the opposite sides at points P, Q and R, respectively. Prove that the perimeter of triangle PQR is at most the half of the perimeter of triangle ABC.

N. 8. Given a convex polygon, draw a straight line that halves the area of the polygon through each vertex of the polygon. Suppose that any such line cuts the polygon at a segment which is at most 1 unit long. Show that the area of the polygon is less than $\pi/4$.

N. 9. Let f_1 and f_2 denote the bisectors of two angle of a triangle, respectively. Let furthermore s_3 be the median starting at the third vertex of the triangle and s be the half of the perimeter. Prove that

$$f_1 + f_2 + s_3 \leq \sqrt{3}s.$$

N. 10. Let α be a fixed positive number. Suppose that the set $\mathcal{A}$ consisting of positive integers satisfy

$$|\mathcal{A} \cap \{1, 2, \cdots, n\}| \geq \alpha n$$

for every positive integer n. Prove that there exists a constant c dependent of α such that every positive integer is the sum of at most c elements of $\mathcal{A}$.

N. 11. Is it possible to cover the (3-dimensional) space by disjoint circles?

N. 12. Prove that for integers $n \geq 2$,

$$\left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) \log(n+1) > \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \log n$$

N. 13. Prove that an $n \times m$ board can be tiled by $k \times 1$ dominoes if and only if it can be tiled by translated copies of $k \times 1$ dominoes(i.e., if $k|m$ or $k|n$).

N. 14. Each element of the set $\{1, 2, \cdots, n\}$ is colored by one of three different colors so that each color class contains more than $n/4$ elements. Show that there exist x, y and z of three different color satisfying $x + y = z$. What does happen if we replace "more than" by "at least"?

N. 15. Let a and b be integers and suppose that

$$x^2 - ay^2 - bz^2 + abw^2 = 0$$

has a nontrivial (not-all-0) integer solution. Prove that

$$x^2 - ay^2 - bz^2 = 0$$

does so, too.

N. 16. Let $\mathcal{A}$ denote a set consisting of positive integers. Let $B(\mathcal{A}, n)$ denote the number of solutions of

$$a + a' = n \qquad (a, a' \in \mathcal{A}; a \leq a'),$$

for each positive integer n. Show that there exist instances of infinite sets $\mathcal{A}$ such that $B(\mathcal{A}, n)$ is even (resp. odd) only for finitely many positive integers n.

N. 17. The natural numbers are colored by finitely many colors. Show that there exist a color s and an integer m so that for every positive integer k there exist $a_1, a_2, \cdots, a_k$ of color s satisfying

$$0 < a_{j+1} - a_j \leq m \qquad (1 \leq j \leq k-1).$$

N. 18. A bishop is wandering on a $k \times n$ chessboard. It starts at a white corner of the board, moving diagonally. When it approaches an edge of the board, it is reflected. It stops wandering when it approaches a corner of the board again. Describe all the pairs (n, k) for which the bishop can wander all over the white fields of the board.

N. 19. There is a given set of n-tuples formed of positive integers. The n-tuple $(a_1, a_2, \cdots, a_n)$ is called a minimal element of the set if there does not exist an other element $(b_1, b_2, \cdots, b_n)$ in the set with $b_i \leq a_i (1 \leq i \leq n)$. Prove that the set has only finitely many minimal elements.

N. 20. There are attached n^2 bulbs in an $n \times n$ board, some of them are with light on. There is a switch related to each row and column of the board. Turning a switch to its other position, it changes the lights of the bulbs in the appropriate row or column to their opposite. Show that with a suitable chain of switchings one can achieve that the difference between the number of shining bulbs and the number of dark bulbs is at least $\sqrt{n^3/2}$.

N. 21. Show that every positive integer k has an integer multiple in the interval $[1, k^4]$ the decimal expression of which contains at most four different digits.

N. 22. Let there be given n points along a circle of unit radius so that the product of the distances of these points from any point on the circle is at most 2. Prove that the points form a regular n-gon.

N. 23. Show that every positive rational number is the sum of the cubes of three suitable positive rational numbers.

N. 24. Does there exist any triangle with the following property: whenever it is decomposed to similar triangles, these triangles should be similar to the original one, too?

N. 25. Find those obtuse angles γ for which every triangle that has an angle γ satisfies

$$\frac{T}{\sqrt{a^2b^2-4T^2}}+\frac{T}{\sqrt{b^2c^2-4T^2}}+\frac{T}{\sqrt{c^2a^2-4T^2}}\geq\frac{3\sqrt{3}}{2},$$

where a, b, c and T denotes the sides and the area of the triangle, respectively.

N. 26. Let p be a polynomial of degree at least two having rational coefficients. Let $q_1, q_2, \cdots$ be rational numbers so that $q_n = p(q_{n+1})$ for every integer $n \geq 1$. Show that the sequence (q_n) is a periodic one, i.e. there exists a positive integer k such that $q_{n+k} = q_n$ $(n \geq 1)$.

N. 27. An interval I is the union of not necessarily disjoint intervals $I_1, I_2, \cdots, I_n$. Prove that
a) the "left halves" of the intervals $I_1, I_2, \cdots, I_n$ (i.e. the intervals determined by the midpoints and the left endpoints of $I_1, I_2, \cdots, I_n$, respectively) cover at least half of the length of interval I;
b) omitting an arbitrary half (left or right) of each of the intervals $I_1, I_2, \cdots, I_n$, the remaining halves of the intervals cover at least the third part of I.

N. 28. Find those integers n for which a regular hexagon can be divided into n parallelograms that have equal areas.

N. 29. Prove that $\binom{n}{k}$ and $\binom{n}{l}$ have a common divisor greater than 1, whenever $1 \leq k, l < n$ are integers.

N. 30. Show that there exist infinitely many composite number n such that $n|2^n - 2$.

N. 31. The positive numbers $a_1, a_2, \cdots, a_n$ add up to 1. Prove that

$$\left(\frac{1}{a_1^2}-1\right)\left(\frac{1}{a_2^2}-1\right)\cdots\left(\frac{1}{a_n^2}-1\right)\geq\left(n^2-1\right)^n.$$

N. 32. Decompose a (planer) convex polygon M into finitely many convex polygons. The net obtained this way is called realizable if there exists a convex polytope such that M is a face of the polytope, and the orthogonal projection of the other faces to the plane of M gives the net at issue.
a) Show that if each polygon in the net is inscribed into a circle, containing the center of that circle, then the net is realizable.
b) Construct a net which is not realizable.

N. 33. Find the smallest number m, depending on n, with the following property. Given arbitrary positive integers $a_1 < a_2 < \cdots < a_n$, any number can be written as the sum of some consecutive a_i's at most m different ways.

N. 34. Assuming that $n > 1994$, show that the number of the pairwise non-isomorphic trees on n vertices lies between 2^n and 4^n. (Two graphs are isomorphic if there exists an edge preserving bijection between their vertex sets.)

N. 35. Finitely many switches and a bulb are attached to a table. The bulb can give red, blue and green light. Each switch has three different positions: red, blue and green. When all the switches are in the same positions, the bulb gives light of that color. In any combination of the positions of the switches, alterations of the positions of all the switches yield the change of the color of the bulb. Prove that there exists a switch which in itself determines the color of the bulb.

N. 36. Let a and n denote integers greater than 1. Show that the number of proper fractions having denominator $a^n - 1$ in their simplest form is divisible by n.

N. 37. Does there exist a sequence of natural number which contains each natural number infinitely many times and is periodic mod m for each positive integer m?

N. 38. Two ellipses have a common focus. Given their foci and major axes, construct their common tangents.

N. 39. Let $n > 3$ be an integer. Prove that $2^{\varphi(n)} - 1$ has a (proper) divisor which is prime to n. (For any positive integer n, $\varphi(n)$ denotes the number of positive integers prime to n and not greater than n; φ is the so-called Euler function.)

N. 40. Find those nonnegative integer valued functions f defined on the set of nonnegative integers which satisfy

$$\begin{aligned} &(i) \quad f(1) > 0; \\ &(ii) \quad f(m^2 + n^2) = f(m)^2 + f(n)^2 \end{aligned}$$

for arbitrary nonnegative integers m and n.

N. 41. The point P, Q and R divide the perimeter of triangle ABC to three equal parts. Prove that the area of triangle PQR is greater than $2/9$ the area of triangle ABC. Is constant $2/9$ best possible?

N. 42. Show that every positive integer can be expressed with the help of at most three 4's, applying the first four rules of arithmetic, extracting square roots and taking integer parts.

N. 43. Let k denote a positive integer. Describe the range of values of function $f(n) = [(n + n^{1/k})^{1/k}] + n$ defined on the set of natural numbers.

N. 44. Is it true that the Euler function φ takes the whole range of its values when restricted to the odd numbers?

N. 45. The nonzero sequence $a_1, a_2, \cdots$ satisfy the recurrence relations $a_{n+2} = |a_{n+1}| - a_n$. Show that after a certain point the sequence is periodic and its least period is of length 9.

N. 46. Let x and y be real numbers and suppose that $\frac{x^n - y^n}{x - y}$ has integer values for four consecutive positive integers n. Prove that it has integer values for all positive integers n.

N. 47. Show that there exist infinitely many integers n such that every prime divisor of $n^2 + 3$ has an integer multiple of the form $k^2 + 3$, where $0 \leq k \leq \sqrt{n}$ is an integer.

N. 48. The circles $k_1, k_2, \cdots, k_6$ touch circle k at points $P_1, P_2, \cdots, P_6$, respectively in that order. Assuming furthermore that k_1 touches k_2, k_2 touches k_3, $\cdots$ and k_6 touches k_1, prove that the segments P_1P_4, P_2P_5 and P_3P_6 are concurrent.

N. 49. Find the positive integer solutions of the system of equations

$$\begin{aligned} a^2 + b^2 &= c^2 - d^2 \\ ab &= cd \end{aligned}$$

N. 50. How close can a number of form

$$\sqrt{2 \pm \sqrt{2 \pm \cdots \pm \sqrt{2}}}$$

be to 1?

N. 51. Show that there exists a polynomial $P(x, y, z)$ satisfying

$$P(t^{1993}, t^{1994}, t + t^{1995}) = t.$$

N. 52. Let $f_1, f_2, \cdots$ denote an arbitrary (infinite) sequence of real functions. Prove that there exist functions $\varphi_1, \varphi_2, \cdots, \varphi_{1994}$ such that each f_* is a composite function of some of them (the same function φ_i may be used several times.)

N. 53. Does there exist an arithmetic progression consisting of 1995 different terms, each of which is a proper power of a positive integer? What can we say about infinite sequences?

N. 54. There are n line segments in the plane, the sum of their lengths is 1. Prove that there exists a straight line so that the sum of the lengths of their projections to the line is less than $2/\pi$.

N. 55. Does there exist a polynomial with integer coefficients that does not have an integer root, but has a root modulo n for every positive integer n?

N. 56. There is a tiny invisible flea skipping on a square grid. Starting at the origin, it either skips away by one of the vectors u_1, u_2, u_3 or stays motionless at the twentieth second of every minute. We know vectors u_1, u_2, u_3, they do not lie in the same halfplane. We may poison two points of the grid at the fortieth seconds of each minute. If the flea is staying on one of those points, or visits such a point later, it groans "YUPP!", and perishes. Can we kill the flea for sure?

N. 57. We are given a quadrilateral inscribed in a circle, and an other circle that intersects every side of the quadrilateral at two inner points. Consider the four arcs of the circle lying inside the quadrilateral. Prove that the sum of the lengths of the opposites arcs are equal.

N. 58. We call a number to an almost perfect square if it is of the form pq where integers p and q satisfy $|p/q-1| < 1/1995$. Show that there exist infinitely many 6-tuples of consecutive integers that are almost perfect squares.

N. 59. On an electric circuit panel, chips P, Q, R and S are connected every possible way. For economical reasons, the wire connecting Q and S is on the back side of the panel, the others lie on the front side. Starting at chip P, an electron chooses among the three possible directions at every chip it reaches with equal probabilities. Find the probability of the even that the electron first gets to the back side of the panel at the 1995th step.

N. 60. Is it possible to choose 4 points in the plane so that all the distances they determine are odd integers?

N. 61. Find those numbers that can be written as the product of finitely many not necessarily different numbers of the form $\frac{n^2-1995^2}{n^2-1994^2}$, where $n > 1995$ is an integer.

N. 62. A lattice point is called visible (from the origin) if its coordinates are coprime numbers. Is there any lattice point the distance of which from each visible lattice point is at least 1995?

N. 63. A rook is moving on an $n \times n$ chessboard, it may move either down or to the right. It starts at the upper left corner of the board. Prove that the number of different ways it can get through to the lower right corner is at most 9^n. May the number 9 be replaced by any smaller one?

N. 64. A quadrilateral is inscribed in a circle, it has integer sides and its area is equal to its perimeter. Find all quadrilaterals that have these properties.

N. 65. The sum of the reciprocals of n positive integers is 1. Prove that each of them is less than n^{2^n}.

N. 66. Show that a triangle shaped sheet of paper can be folded "in half" such a way that the area covered by the folded sheet is at most 3/5-th of the area covered by the original sheet.

N. 67. Prove that there exist infinitely many pairs of positive integers (m, n) for which both divisibility relations $n|m^2 + 1$ and $m|n^2 + 1$ hold.

N. 68. Let H be a finite set consisting of integers. Let H_+ and H_- denote the sets formed by the sums resp. difference of pairs in H. (We pairs of identical numbers as well as negative differences.) Prove that

$$|H| \cdot |H_-| \leq |H_+|^2.$$

($|X|$ means the number of elements of set X.)

N. 69. In an inscribed quadrilateral $ABCD$ let P denote the intersection point of sides AB and CD and let Q denote the intersection point of BC and DA. Let E and F be those points of sides AB resp. CD for which PE resp. PF is the harmonic mean of PA and PB resp. PC and PD. Show that points Q, E and F are collinear.

N. 70. Let a, b, c and d denote distinct positive integers. Prove that the positive integers of the form $a \cdot c^n + b \cdot d^n$ have infinitely many prime factors altogether.

N. 71. Let $a_1, a_2, \cdots, a_n$ be real numbers and

$$f(x) = \cos a_1 x + \cos a_2 x + \cdots + \cos a_n x.$$

Prove that there exists a positive integer $k \leq 2n$ for which

$$|f(k)| \geq \frac{1}{2}.$$

N. 72. Show that the vertex set of a simple graph G can be divided into two distinct part so that deleting the edges between the two parts we obtain a graph G' every vertex of which has an even degree.

N. 73. Is there any irrational number c in the interval $(0,1)$ such that neither c nor $\sqrt{c}$ has a digit 0 in its fractional part?

N. 74. A positive integer is called varied if multiplied by any positive integer m the product contains all the 10 positive digits. It is n-varied if it has the required property for every integer $1 \le m \le n$.
a) Prove that varied numbers do not exist.
b) Does there exist any 1995-varied number?

N. 75. Find the maximum ratio of the product of the sides and the product of the diagonals of an inscribed pentagon.

N. 76. For every positive integer n, prove that there exists a polynomial of degree n with integer coefficients of absolute value at most n, which admits 1 as a root with multiplicity of at least $[\sqrt{n}]$.

N. 77. Define a linear order $\prec$ on the set of positive integers so that $a \prec b \prec c$ implies $2b \neq a+c$. The relation $\prec$ is a linear order if a) every pair a, b satisfies one and only one of the relations $a \prec b$, $b \prec a$ and $a = b$ and b) $a \prec b$ and $b \prec c$ implies $a \prec c$.

N. 78. Given a circle, construct its center with the help of a single compass.

N. 79. A solution of the system of equations

$$x_1^3 + x_2^3 + x_3^3 = y_1^3 + y_2^3 + y_3^3,$$

$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3$$

is called trivial if the triplets x_1, x_2, x_3 and y_1, y_2, y_3 differ only in the order of the three numbers. Show that there exists an integer $n \ge 1995$ for which at least 99 percent of the solutions are trivial among the numbers $1, 2, \cdots, n$.

N. 80. Define sequences $(a_n), (b_n)$ and (c_n) in the following recursive manner. $a_1 = 1, b_1 = 2, c_1 = 4$; furthermore, for every integer $n > 1$, a_n is the smallest positive integer other than $a_1, \cdots, a_{n-1}$, $b_1, \cdots, b_{n-1}, c_1, \cdots, c_{n-1}$; b_n is the smallest positive integer other than $a_1, \cdots, a_n, b_1, \cdots, b_{n-1}$, $c_1, \cdots, c_{n-1}$; and $c_n = n + 2b_n - a_n$. Prove that $0 < (1+\sqrt{3})n - b_n < 2$.

N. 81. Let z denote a complex number of absolute value 1. Prove that there exists a polynomial of degree 1995, all the coefficients of which are $+1$ or -1, so that $|p(z)| \leq 4$.

N. 82. Show that there exists, for infinitely many n, a polynomial of degree n with the following properties: its coefficients are integers, its leading coefficient is less than 3^n, and it has n distinct roots in the interval $(0, 1)$.

N. 83. Is it true that every irrational number has an (integer) multiple among the decimals of which there are either infinitely many '0' digits or infinitely many '9' digits?

N. 84. In the interior of the unit square $ABCD$, two points P and Q are given. Prove that

$$13(PA + QC) + 14PQ + 15(PB + QD) > 38.$$

N. 85. We are given an $n \times n$ array of numbers $+1$ and -1 with the following property. Whenever we compare two rows of the array, the number of positions at which they coincide is the same as the number of positions at which they differ from each other. Prove that the sum of the entries of the array is not greater than $n^{3/2}$.

N. 86. We have two sand-glasses that can measure p and q minutes, respectively, where p and q are coprime positive integers. Our intention is to use them to measure n minutes. Initially, all the sand can be found in the lower parts of the glasses. When we start an experiment, and later, when one of our sand-glasses stops, we are allowed to turn upside down either any one or both of the glasses. Show that we can measure the required time, if $n \geq pq/2$.

N. 87. We are given a lattice rectangle the sides of which are not necessarily parallel to the coordinate axes. It is partitioned into lattice triangle of area $1/2$. Show that some of these triangles are right-angle ones, in fact, the number of right triangles in the decomposition is at least $2/\sqrt{5}$ times the length of the shorter side of the given rectangle.

N. 88. Is there any continuous function $f : [0, 1] \to \mathbb{R}$ that attains every element of its range an (finite) even number of times?

N. 89. Let, by definition a word be any finite sequence of letters. Given a word, we may repeatedly apply the following operations: a) We may erase either the first or the last letter of the word; b)We may 'double' the word, i.e. we may concatenate two identical copies of the word. Is it possible to alter the word $ABCD \cdots XYZ$ into the word $ZYX \cdots DCBA$ in this way?

N. 90. A convex polygon is orthogonally projected onto a plane. Show that its projection can be covered by a congruent copy of the polygon.

N. 91. Prove that for every positive integer n, there exists a polynomial p of degree at most $100n$, satisfying

$$|p(0)| > |p(1)| + |p(2)| + \cdots + |p(n^2)|.$$

N. 92. Let $a_1, a_2, \cdots, a_n$ be n different numbers. Derive

$$\sum_{i=1}^{n} \left(a_i \cdot \prod_{j \neq i} \frac{1}{a_i - a_j} \right) = 0.$$

N. 93. Define the Lucas numbers by

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n \qquad (n = 0, 1, \cdots).$$

Prove that if n is even, then every prime factor of $L_n - 2$ is a divisor of $L_{n+1} - 1$, too.

N. 94. Find a positive number z for which $[z^n] - n$ is an even number for every positive integer n.

N. 95. Let $n \geq 3$ be an integer. Show that one can partition a square into more than 2^n rectangles so that any line parallel to a side of the square intersects the interior of at most n rectangles. Prove furthermore, that no partition of the square into more than 3^n rectangles has the same property.

N. 96. Prove that there is no arithmetic progression consisting of four different perfect squares.

N. 97. Define a sequence (a_n) by

$$a_0 = 3, \quad a_1 = 0, \quad a_2 = 2, \quad a_{n+3} = a_{n+1} + a_n \qquad (n = 0, 1, 2, \cdots).$$

Prove that, for any prime number p, $p | a_p$.

N. 98. Let p denote a polynomial with integer coefficients. Prove that for arbitrary integers m and n, n divides $p(m+n) - p(m)$. Is there any function $p : \mathbb{Z} \to \mathbb{Z}$ that is not a polynomial with integer coefficients, yet has the previous property?

N. 99. Show that, for infinitely many positive integers L, every term of the sequence

$$a_0 = 0, \quad a_{n+1} = \frac{1}{L - a_n} \qquad (n = 0, 1, 2, \cdots)$$

can be written as the ratio of two successive Fibonacci numbers.

N. 100. There are given $3k+2$ points in the plane, no three of them are collinear. Prove that there is one point which determines at least $k+1$ different distances with the other points.

N. 101. Prove

$$\sum_{j=0}^{r}\binom{d}{j}\binom{d-r-j-1}{r-j}=\frac{2^r}{r!}\prod_{k=0}^{r-1}(d-(1+2k))$$

N. 102. Choose 1996 straight lines in the plane such a way that any two of them are intersecting and no three of them are concurrent. The lines divide the planes into regions. Find the minimum resp. the maximum number of angular regions that may arise in this way.

N. 103. Call a (simple, connected) graph randomly Eulerian if whichever way we walk along its (consecutive) edges, taking care that we visit every edge only once, we visit every edge of the graph sooner or later. Find all randomly Eulerian graphs.

N. 104. The terms of a sequence (a_n) satisfy

$$\sum_{d|n} a_d = 2^n \qquad (n=1,2,\cdots).$$

Prove that $n|a_n$ for every positive integer n.

N. 105. We are given lot of different sets, their number is large enough (i.e. greater than a certain sufficiently large number.) Prove that among the given sets one can find 1996 sets so that no one of them is obtained as the union of two other sets.

N. 106. In a graph of 1996 vertices that does not contain an isolated vertex(i.e. a vertex with degree zero) we select a subset of vertices at random. Prove that they form a covering system with probability less than 0.51, that is, the probability having an edge in the graph without an endpoint belonging to the selected set of vertices is greater than 0.49.

N. 107. Given a graph, we may walk on the vertices of the graph according to the following rules. In each step we may move from a vertex to one of its neighbors (if there is any) that we have not visited before. Call a graph randomly Hamiltonian if whichever way we walk on its vertices (according to the above described rule), we visit every vertex of the graph sooner or later. Find all randomly Hamiltonian graphs.

N. 108. On the surface of a sphere there is a given closed curve that intersects with every great circle of the sphere. Prove that the length of the curve is not less than the circumference of a great circle of the sphere.

N. 109. A positive integer is called "almost perfect" if the sum of its divisors exceeds twice the number by 1. Prove that every almost perfect number is odd.

N. 110. Find those positive integer parameter λ for which

$$x^2 + y^2 + z^2 + v^2 = \lambda xyzv$$

has a positive integer solution.

N. 111. Find the minimum number of edges a graph on 10 vertices must have if any 5 of its vertices induce at least 2 edges.

N. 112. We are given a complete graph of $n \geq 3$ vertices. Find the smallest possible value of k for which the edges of the graph can be colored by k given colors so that
1) there is no color which is assigned to each and every edge of the graph,
2) there is no edge which is colored by all the k given colors, and
3) in every triangle, each color is assigned to an odd number of edges.

N. 113. A certain polynomial p of degree n assigns $\frac{k}{k+1}$ to k for $k = 0, 1, \cdots, n$. Find $p(n+1)$.

N. 114. Let p denote a prime number of the form $3k + 1$. Prove that there exist positive integers $a < b < \sqrt{p}$ such that $a^3 \equiv b^2 \pmod{p}$.

N. 115. Given two nonnegative fractions, there median is a nonnegative fraction whose numerator and denominator are obtained as the sum of the numerators resp. the sum of denominators of the given fractions, assuming that they are written in their simplified form, respectively. Consider of sequence of three fractions so that the middle one is the median of the other two fractions. Delete one of the outer ones and write their median in between the remaining ones to form a sequence similar to the original one. We may apply this procedure repeatedly. Prove that if $0 < q < 1$ is an arbitrary fraction then, starting with the sequence $(\frac{0}{1}, \frac{1}{2}, \frac{1}{1})$, there is a unique way to use the above described method in order to obtain a sequence whose middle term is q.

N. 116. The ground-plan of a gallery is shaped like a concave n-gon. Guards should be placed in the room so that each point of each wall would be visible from at least one of the posts. Fine the maximum number of guards we may need to satisfy this requirement.

N. 117. The positive integers a_k ($1 \leq k \leq 2^n$) satisfy $a_k \leq k$ Prove that the sequence $a_1, a_2, \cdots, a_{2^n}$ contains a monotone increasing subsequence of length $n+1$.

N. 118. Find those positive integers d for which there exists a positive integer divisible by $d \cdot 2^{1996}$ that can be written in decimal system using only the digits 1 and 2.

N. 119. Find a triangle-free graph whose vertices cannot be colored properly using 4 colors. A coloring of the vertices of a graph is called proper if adjacent (i.e., neighboring) vertices have different colors.

N. 120. Given a positive integer n, let $f(n)$ denote the number of ways n can be represented as the sum of terms $2^a 5^b$ (a, b are nonnegative integers) so that no term divides any other term in the summation. Two representations are considered to be the same if they differ only in the order of the addends. Prove that $f(n)$ is not bounded.

N. 121. A disc of radius 2000 is divided into smaller parts by 1996 straight lines. Prove that a disc of 1 can be placed into one of these parts.

N. 122. Let a, b and c denote integers, not all of which are 0. Prove

$$\left|\sqrt[3]{4}a + \sqrt[3]{2}b + c\right| \geq \frac{1}{4a^2 + 3b^2 + 2c^2}.$$

N. 123. A certain simple graph has the following property: given any nonempty subset H of its vertex set, there is a vertex x of the graph (x may belong to H) so that the number of edges connecting x with the points in H is odd. Prove that the graph has an even number of vertices.

N. 124. Prove

$$\frac{\binom{2n}{0}}{x} + \frac{\binom{2n}{2}}{x+2} + \cdots + \frac{\binom{2n}{2n}}{x+2n} > \frac{\binom{2n}{1}}{x+1} + \frac{\binom{2n}{3}}{x+3} + \cdots + \frac{\binom{2n}{2n-1}}{x+2n-1}$$

where $x > 0$ and n is a positive integer.

N. 125. Given that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and the sequence $f(a), f(2a), f(3a), \cdots$ converges to 0 for any positive real number a, prove that $\lim_{x\to\infty} f(x) = 0$.

N. 126. A cuboid, or a rectangular box, is partitioned into smaller cuboids. Given that each of the smaller cuboids has an edge of integer length, prove that the original cuboid has an edge of integer length, too.

N. 127. Each entry of a given $n \times n$ array is either $+1, -1$ or 0. If we consider the sum of the entries in each row resp. column of the array, we find that the $2n$ results obtained this way are all different. Prove that n is an even number.

N. 128. There are given n points in the plane, no three of which are collinear. Each line segment determined by them is colored with one of two given colors. Prove that the graph obtained this way contains a monochromatic spanning tree no two edges of which cross each other. (A spanning tree of a graph is a connected cycle-free graph, i.e., a tree, the vertex set of which coincides with that of the original graph, and the edges of which belong to the edge set of the original graph.)

N. 129. The sequence $a_1, a_2, \cdots$ consists of positive integers. Suppose that $a_n \le n$ for every n. Prove that there exists a non-constant arithmetic progression every element of which can be written as the signed sum of a few initial terms of the sequence (a_n), i.e., in the form $\pm a_1 \pm a_2 \pm \cdots \pm a_k$. Show that the statement does not remain valid under the weaker assumption that the sequence (a_n/n) is bounded.

N. 130. Let a_n and b_n denote positive integers defined by

$$a_n = \binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots + \binom{n}{3\left\lfloor \frac{n}{3} \right\rfloor}$$

$$b_n = \binom{n}{0} + \binom{n}{5} + \binom{n}{10} + \cdots + \binom{n}{5\left\lfloor \frac{n}{5} \right\rfloor}$$

respectively. Prove that the sequence $(a_n - 2^n/3)$ is bounded, but the sequence $(b_n - 2^n/5)$ is note bounded.

N. 131. Prove that if $k > 1$ and $0 < x < \pi/K$, then

$$\frac{\sin Kx}{\sin x} < Ke^{-\frac{K^2-1}{6}x^2}.$$

N. 132. The point (a_1, a_2, a_3) is said to be above (resp. below) the point (b_1, b_2, b_3) if $a_1 = b_1, a_2 = b_2$ and $a_3 > b_3$ (resp. $a_3 < b_3$). Suppose that none of the pairwise skew straight lines $e_1, e_2, \cdots, e_{2k}$ $(k \ge 2)$ is parallel to the z-axis. Suppose furthermore that among their orthogonal projections to the xy-plane there are neither two parallel lines, nor three concurrent ones. May it happen that moving along one of the given lines, those points, above resp. below which a point of another line e_i can be found, appear in an alternating order?

N. 133. Prove that for all prime numbers $p > 3$, $\binom{2p-1}{p-1} - 1$ is divisible by p^3.

N. 134. Let a_n denote the number of occurrences of digit 1 in their binary expression of 3^{2^n}. Prove that $\lim_{n\to\infty} a_n = \infty$.

N. 135. Let $A \subset (0,1)$ be the union of a finite number of intervals whose lengths add up to $1/3$. Let $B = \{\{x - y\} : x, y \in A\}$, that is, the set consisting of the fractional parts of the pairwise differences of the elements of A. Prove that B is also a finite union of intervals, and the sum of the lengths of these intervals is at least $2/3$.

N. 136. Let $a < b$ be arbitrary positive integers. Prove that there exists a prime after the division by which the remainder of a is greater than that of b.

N. 137. Prove that for every positive integer k there exist positive integers $a_1 < a_2 < \cdots < a_k$ such that $a_i - a_j | a_i^{1997}$ for $1 \leq i, j \leq k$, $i \neq j$. Prove, furthermore, that for all such sequences $a_k > 2^{ck}$ holds with a positive constant c.

N. 138. We are given a function $f : \mathbb{R} \to \mathbb{R}$. If a, b, c are different real numbers, then $2c \neq a + b$ implies $2f(c) \neq f(a) + f(b)$ and then

$$f\left(\frac{2ab - ac - bc}{a + b - 2c}\right) = \frac{2f(a)f(b) - f(a)f(c) - f(b)f(c)}{f(a) + f(b) - 2f(c)}$$

holds. Prove that f is a linear function.

N. 139. Find the number of those permutations of the numbers $1, 2, \cdots, n$ which do not preserve the original order of any three elements.

N. 140. The positive integers $a_1 < a_2 < \cdots < a_9$ have the following property: all sums (of at least one but at most nine different summands) that can be formed of them are different. Prove that $a_9 > 100$.

N. 141. There are given $n \geq 6$ points on a circle such that any possible distance among them occurs at most twice. Prove that at least $\lfloor n/2 \rfloor - 2$ of these distances occur only once.

N. 142. Let n denote any positive integer. Prove that every polynomial can be expressed as a signed sum of the nth powers of n appropriate polynomials. (By a polynomial we mean a polynomial with real coefficients.)

N. 143. Is there any positive integer n for which all prime divisors of $2^n - 1$ are less than $2^{n/1997}$?

Hints

1. *The second player wins. The second player first uses up all the even degree coefficients.*

2. *The answer is* $\frac{1}{2}\cos 10°$, *when* $AC = 1$, $BC = \frac{2}{\sqrt{3}}\cos 10°$ *and* $\angle C = 60°$.

3. *Consider two cases:* $\alpha \geq 1/2$ *and* $\alpha < 1/2$.

4. *Let* S *denote the area of triangle* ABC. *It is obvious that* $ax + by + cz = 2S$.

5. *Let* $\alpha > \beta > \gamma$ *be the three real roots of* $x^3 - 3x^2 + 1 = 0$. *Observe the sequence* $a_n = \alpha^n + \beta^n + \gamma^n$. *Also note that* $x^3 - 3x^2 + 1 \equiv (x-4)(x-5)(x+6) \pmod{17}$.

6. *Try to partition the whole board into dominoes. If one places the king on an unused domino, the other places the king to the second field of the domino. The first player wins if and only if* mn *is even.*

7. *It can be proved that* $QR \leq (BR + CQ)/2$ *both geometrically and algebraically. For the geometrical solution, consider the intersections of line* BC *with the circumcircles of triangles* BRQ *and* CRQ.

8. *Consider the butterfly shaped region cut by two 'adjacent' 'midlines' of the polygon. Let* θ *be the angle of the butterfly and prove that the area is less than* $\frac{1}{4}\sin\theta < \theta/4$.

9. *Let the length of the sides be* a, b *and* c. *Since* $f_1^2 = ((b+c)^2 - a^2)bc/(b+c)^2$, *we have* $f_1 \leq 1/2\sqrt{(b+c)^2 - a^2}$.

10. *The greatest such* α *in this problem is called the Schnirelmann density of the set* $\mathcal{A}$. *For a set* $\mathcal{A}$, *denote its Schnirelmann density by* $\sigma\mathcal{A}$. *It is sufficient to prove* $\sigma(\mathcal{A} \cup \mathcal{B} \cup (\mathcal{A} + \mathcal{B})) \geq \sigma\mathcal{A} + \sigma\mathcal{B} - \sigma\mathcal{A}\sigma\mathcal{B}$.

11. *It is possible. First prove that if* u, v *are distinct points on* S^2, *then* $S^2 \setminus \{u, v\}$ *is coverable by disjoint circles and use this fact to cover the whole space.*

12. *Note that* $\log(1 + \frac{1}{n}) = \int_n^{n+1} 1/x\, dx > \frac{1}{n+1}$ *and that* $1 + \cdots + \frac{1}{n-1} > \log n + 1/4$.

13. *At the* i*th row,* j *column square, assign a weight of* ζ^{i+j} *where* $\zeta = e^{2\pi i/k}$.

14. *Suppose the contrary and let* $1, 2, \cdots, k$ *be colored in blue while* $k+1$ *is colored in red. If* $k \geq 2$, *then there cannot be a lot of green numbers.*

15. *The condition states that there exist two integers of the form* $x^2 - ay^2$ *whose ratio is* b. *We need to prove that there exists an integer of the form* $x^2 - ay^2$ *which is a product of* b *and a perfect square. Note that the form* $x^2 - ay^2$ *is closed under multiplication.*

16. *Just inductively construct such a set, being careful not to make it finite.*

17. *Call a set* S good *if there exists an* m *such that there exists* $a_1, \cdots, a_k \in S$ *such that the distance between adjacent two numbers are at most* m *for every* k. *Show that if a good set is partitioned into two sets, one of the two is also good.*

18. *The pairs for which* $\gcd(n-1, k-1) = 1$. *Extend the board to twice its original size and make the bishop wander around a torus.*

19. *Use induction on the dimension.*

20. *Let* r_i *be the difference of shining bulbs and dark bulbs in the* i*th column. By a column switchings, we can obtain a total difference of* $r_1 + \cdots + r_n$. *Thus it is sufficient to prove that a suitable chain of row switching results in* $r_1 + \cdots + r_n \geq \sqrt{n^3/2}$.

21. *The difference of two numbers whose digits consist of* 0 *and* 1 *has only* 0, 1, 8 *and* 9 *as digits.*

22. *Interpret the condition as a inequality on a complex polynomial. Add some of the values and prove that the average is at least* 2.

23. *An old result of Ryley. Let* $u = x + y + z$, $v = y + z$ *and write* $x^3 + y^3 + z^3$ *in terms of* u, v, *and* z.

24. *There does exist. Focus on the angles of the triangle, and see what happens if the angles of original triangle are linearly independent over the rationals.*

25. *The left hand side is* $\frac{1}{2}(\tan A + \tan B + \tan C)$.

26. *Let* $q_i = a_i/b_i$ *where* a_i *and* b_i *are relatively prime. Prove that* b_i *must be bounded and that* q_i *is also bounded. Then show that there are only finitely many possibilities for* q_i.

27. *Let the union of the left(resp. arbitrary) halves be* $J_1, \cdots, J_k$. *Suppose that the sum of the lengths do not exceed one half(resp. one third) of the length of* I *and prove that* $I_1, \cdots, I_n$ *cannot cover* I.

28. *Prove that the sum of areas of the parallelograms of specific orientation is one third the area of the hexagon.*

29. *Suppose that* $k, l \le n/2$. *Then we have* $\binom{n}{k}\binom{n-k}{l} = \binom{n}{l}\binom{n-l}{k}$.

30. *If* n *satisfies the condition, then* $2^n - 1$ *also satisfies the condition.*

31. *Multiply* $a_1^2 a_2^2 \cdots a_n^2$ *to both sides and use AM-GM.*

32. *Incomprehensible.*

33. *The maximum value among the consecutive numbers should decrease if the number of numbers increase. The answer is* $\lceil m/2 \rceil$.

34. *Use Cayley's formula to prove that the number exceeds* 2^n. *For the* 4^n *side, just make a root and read from top to bottom, left to right, making notes when the parent node changes. Then the notes determine a tree while at most* $2n$ *bits are used in the notes.*

36. *In the multiplicative group* $(\mathbb{Z}/(a^n - 1)\mathbb{Z})^\times$, *the order of* a *is* n.

42. *Show that every integer can be represented in the form of* $[(4/(4^{2^{-n}} - 1))^{2^{-m}}]$.

43. *Since* $f(n+1) \ge f(n) + 1$, *we only need to check when* $f(n+1) > f(n) + 1$. *The answer is* $3\mathbb{N}$ *for* $k = 1$, *and* $\mathbb{N} \setminus \{m^k : m \in \mathbb{N}\}$ *for* $k > 1$.

50. *Always choose the '$-$'.*

71. *Let* $(x^2 - 2\cos a_1 x + 1) \cdots (x^2 - 2\cos a_n x + 1) = c_{2n}x^2n + c_{2n-1}x^{2n-1} + \cdots + c_1 x + c_0$. *Then* $\sum_{i=0}^{2n} c_i f(x+i) = 0$ *for all* x. *Let* $|c_t|$ *be the maximum absolute value among the numbers. Then let* $x = -t$ *and prove that* $|f(k)| \ge \frac{1}{2}$ *for some* k, *noting* $f(x) = f(-x)$.

74. *a) Since every integer has a multiple of the form* $99 \cdots 90 \cdots 0$. *b) Yes. Show that* 1000000 $990000 \cdots 02000001$ *is a 1999-varied number.*

76. *Since* f *admits* 1 *as a root with multiplicity* m *if and only if* $f(1) = f'(1) = \cdots = f^{(m-1)}(1) = 0$, *the problem is equivalent to proving that* $a_0\vec{v_0} + \cdots + a_n\vec{v_n} = 0$ *for some non-trivial* $a_0, \cdots, a_n$, *where* $\vec{v_i} = (\binom{0}{i}, \binom{1}{i}, \cdots, \binom{n}{i})$. *Now proceed using the pigeonhole principle.*

77. *For* $a = k_t k_{t-1} \cdots k_1 k_0$ *in base* 2, *let* $f(a) = 0.k_0 k_1 \cdots k_t$ *in base* 2. *Let* $a \prec b$ *if and only if* $f(a) < f(b)$.

81. *Use the fact that* $|1 + z + \cdots + z^k| \le 2/|1 - z|$. *One can even attain the bounds* $|p(z)| \le \sqrt{2}$.

83. *Yes. It is not difficult to prove that if* α *is a irrational number, any digit* $k\alpha$ *becomes* 0 *or* 9 *for a suitable* $k \le C$.

84. *It is not difficult to show that by translating P and Q so that the center of the square is the midpoint of PQ, we may decrease the left hand side. Then, the problem reduces to finding the least value of $13PA + 14PO + 15PB$, where O is the center of the square. Let K be a point which satisfies the conditions $KA = 15/14, KB = 13/14$. Using Ptolemy's theorem on $KAPB$, we obtain the value $2\sqrt{19^2 + 2^2}$ as the minimum.*

85. *Sum up the inner product of the rows. Let r_i be the i row. Then $|\sum r_i|^2 = n^2 + \sum r_i \cdot r_j$. Use Cauchy's inequality to obtain the result.*

88. *Yes. Let $f(x) \geq 0$ for all x and $f(x) = -x$ if $x \leq 0$. It is not difficult to construct one.*

89. *If α, β, γ are words, show that one can transform $\alpha\beta\gamma$ to $\alpha\gamma\beta$.*

92. *Consider the Lagrange interpolation of $f(a_i) = a_i^2$. Obviously $f(x) = x^2$ and $f(0) = 0$.*

93. *With a bit of algebraic number theory, the problem becomes trivial. Since $L_n = \alpha^n + \beta^n$ where $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, we can easily show $p|\alpha^n - 1$ when $p|\alpha^n + \beta^n - 2$.*