CAlGeN POW

June 7, 2015

Problems - 2014 Spring

Problem 1. Let $P(n)$ denote the greatest prime factor of n with $P(1) = 1$ for convenience. Prove that there exists infinitely many positive integers n satisfying $P(n) < P(n+1) < P(n+2)$.

Problem 2. $P_1, P_2, \cdots, P_n$ are n points inside a unit square. Let d_i be the distance from P_i to its nearest point. Prove that

$$d_1^2 + d_2^2 + \cdots + d_n^2 \leq 4.$$

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ such that f is differentiable and f, f' convex. For any real numbers $x_1, x_2, \cdots, x_n$, prove that

$$(n-2)\sum_{i=1}^{n} f(x_i) + nf\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \geq \sum_{i \neq j} f\left(\frac{(n-1)x_i + x_j}{n}\right).$$

Problem 4. Suppose a convex n-gon has a circumscribed circle Γ and a inscribed circle γ. Prove that for any point P on Γ, there exists a n-gon such that it has P as a vertex, Γ as the circumscribed circle, and γ as the inscribed circle.

Problem 5. A polynomial $P(x)$ with real coefficients satisfies $P(x) \geq 0$ for all $x \in \mathbb{R}$. Prove that there exist two polynomials Q and R, also with real coefficients, such that

$$P(x) = Q(x)^2 + R(x)^2.$$

Problem 6. Define a sequence $\{x_n\}$ of distinct natural numbers as follows. $x_1 = 1$ and x_n is determined as the least natural number, distinct from $x_1, \cdots, x_{n-1}$, such that $x_1 + x_2 \cdots + x_n$ is divisible by n. Prove that $x_{x_n} = n$ for all n.

Problem 7. Let P be a monic polynomial with complex coefficients. Prove that

$$diam\{z \in \mathbb{C} : |P(z)| \leq 1\} \geq 2.$$

Problem 8. Find all even, nonnegative, and differentiable functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the inequality

$$f(t) - (f(s) + f'(s)(t-s)) \geq f(t-s)$$

for every $t, s \in \mathbb{R}$.

Problem 9. Let A and B two convex polygons and let $\mathrm{vol}(P)$ denote the area of the polygon P. Prove that

$$\sqrt{\mathrm{vol}(A+B)} \geq \sqrt{\mathrm{vol}(A)} + \sqrt{\mathrm{vol}(B)}.$$

Problem 10. Let $I \subset \mathbb{R}$ be an open interval and f a function such that $f^{(n)}(x) \geq 0$ for all $x \in I$. Consider $n+1$ distinct real numbers $a_1, \cdots, a_{n+1} \in I$ and let P be the polynomial obtained by Lagrange's interpolation from the $n+1$ points $(a_1, f(a_1)), \cdots, (a_{n+1}, f(a_{n+1}))$. Prove that the coefficient of x^n in P is nonnegative.

Problem 11. Let G denote the set of all infinite sequences $(a_1, a_2, \cdots)$ of integers. We can add elements of G coordinate-wise, i.e.,

$$(a_1, a_2, \cdots) + (b_1, b_2, \cdots) = (a_1 + b_1, a_2 + b_2, \cdots).$$

Suppose $f : G \to \mathbb{Z}$ is a function satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in G$. Prove that

$$f(a_1, a_2, \cdots) = x_1 a_{k_1} + \cdots + x_n a_{k_n}$$

for some integers $x_1, \cdots, x_n, k_1, \cdots, k_n$.

Problem 12. Three squares of side length x_1, x_2 and x_3 are placed inside a 1×2 rectangle without any overlappings. Prove that $x_1 + x_2 + x_3 \leq 2$.

Problem 13. Let P be a polynomial with integer coefficients such that for every positive integer x, $P(x)$ is a perfect square. Prove that $P(x) = Q(x)^2$ for some polynomial $Q(x)$.

Problem 14. Two positive integers n and $k \geq 2$ are given, and a list of n integers is written in a row on a blackboard. Cinderella and her wicked Stepmother go through a sequence of rounds: Cinderella chooses a contiguous block of integers, and the Stepmother will either add 1 to all of them or subtract 1 from all of them. Cinderella can repeat this step as often as she likes, possibly adapting selections based on what her stepmother does. Prove that after a finite number of steps, Cinderella can reach a state where at least $n - k + 2$ of the numbers on the blackboard are all divisible by k.

Problems - 2014 Fall

Problem 15. For which n does there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that f takes each value exactly n times? ($|f^{-1}(x)| = n$ for all x)

Problem 16. Let $\mathcal{S}$ be a finite set of points in the plane (not all collinear), each colored red or blue. Show that there exists a monochromatic line passing through at least two points of $\mathcal{S}$.

Problem 17. Does there exist an infinite set M consisting of positive integers such that for any distinct $a, b \in M$, the sum $a + b$ is square-free?

Problem 18. Let Ω be the circumcircle and ω be the incircle of a triangle, and let P a arbitrary point inside ω. For a point X on Ω, let XY and XZ be chords of Ω which are tangent to ω. By CAlGeN 2014-4, we know that YZ is also tangent to ω. Find the locus of the isogonal conjugate of P respect to triangle XYZ where X moves around Ω.

Problem 19. Let $n \geq 2$ be a positive integer and let $A_1, \cdots, A_m$ be subsets of $\{1, 2, \cdots, n\}$ such that $|A_i| \geq 2$ for all i and $A_i \cap A_j \neq \emptyset$, $A_j \cap A_k \neq \emptyset$, $A_k \cap A_i \neq \emptyset$ imply $A_i \cap A_j \cap A_k \neq \emptyset$. Prove that $m \leq 2^{n-1} - 1$.

Problem 20. Let $S(n)$ be the sum of digits of n when represented by base 10. Show that

$$\lim_{n \to \infty} S(2^n) = \infty.$$

Problem 21. A positive integer n is given. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(a_1) + f(a_2) + \cdots + f(a_n) = f(b_1) + f(b_2) + \cdots + f(b_n)$$

for arbitrary real numbers $a_1, b_1, \cdots, a_n, b_n, c$ such that $P(x) = (x - a_1)(x - a_2) \cdots (x - a_n) = (x - b_1)(x - b_2) \cdots (x - b_n) + c$.

Problem 22. Let $k \geq 1$ and let $I_1, \cdots, I_k$ be non-degenerate open subintervals of $[0, 1]$. Prove

$$\sum \frac{1}{|I_i \cup I_j|} \geq k^2$$

where the summation is over all pairs (i, j) of indices such that I_i and I_j are not disjoint.

Problem 23. Let P be a convex polygon such that the distance between any two vertices does not exceed 1. Prove that the perimeter of P does not exceed π.

Problem 24. Let m, n be relatively prime positive integers, and $x > 0$ a real number. Prove that if $x^m + x^{-m}$ and $x^n + x^{-n}$ are both integers, then $x + x^{-1}$ is also an integer.

Problem 25. Let $A_1, A_2, A_3, A_4, A_5, A_6$ be six points on a circle, and $X_i = A_iA_{i-1} \cap A_{i+1}A_{i+2}$ where the indices are modulo 6. Denote by O_i the circumcenter of the triangle $\triangle X_iA_iA_{i+1}$. Prove that the three lines O_1O_4, O_2O_5, O_3O_6 are concurrent.

Problem 26. A real number is written in every unit square of an infinite square grid. Suppose that for each square which sides are parallel to the axes, the absolute value of the sum of values inside the square does not exceed 1. Prove that the absolute value of the sum of values inside any rectangle which sides are parallel to the axes does not exceed 4.

Problem 27. Prove that there exists a constant $k_n > 0$ dependent on n such that for all $x_1, \cdots, x_n \in [1 - k_n, 1 + k_n]$, we have

$$\frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_4} + \cdots + \frac{x_n}{x_1 + x_2} \geq \frac{n}{2}.$$

Problem 28. For every positive integer n, prove that there exists a set $S \subseteq \{n^2 + 1, n^2 + 2, \cdots, (n + 1)^2 - 1\}$ such that

$$\prod_{x \in S} x = 2m^2$$

for some integer m.

Problem 29. $\mathbb{N}$ is the set of nonnegative integers. For any subset S of $\mathbb{N}$, let $P(S)$ be the set of all *pairs* of members of S. (A *pair* is a unordered set of two distinct members) Partition $P(\mathbb{N})$ arbitrarily in to two sets P_1 and P_2. Prove that $\mathbb{N}$ must contain an infinite subset S such that either $P(S)$ is contained in P_1 or $P(S)$ is contained in P_2.

Problem 30. Let q be a rational number in the interval $(0, \frac{\pi^2}{6} - 1) \cup [1, \frac{\pi^2}{6})$. Prove that there exists integer $0 < x_1 < x_2 < \cdots < x_n$ such that

$$q = \frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_n^2}.$$

Problems - 2015 Spring

Problem 1. There are attached n^2 bulbs in an $n \times n$ board, some of them are with light on. There is a switch related to each row and column of the board. Turning a switch to its other position, it changes the lights of the bulbs in the appropriate row or column to their opposite. Show that with a suitable chain of switchings one can achieve that the difference between the number of shining bulbs and the number of dark bulbs is at least $\sqrt{n^3/2}$.

Problem 2. Let $a_1, a_2, \cdots, a_n$ be real numbers and

$$f(x) = \cos a_1 x + \cos a_2 x + \cdots + \cos a_n x.$$

Prove that there exists a positive integer $k \leq 2n$ for which

$$|f(k)| \geq \frac{1}{2}.$$

Problem 3. Alice and Bob play a game on a simple graph G with 2015 vertices. Alice first chooses a vertex and colors it red. After that, Bob and Alice alternatively picks an uncolored point adjacent to the last colored point, and color it red. The first who cannot pick an uncolored point loses. Prove that regardless of G, Alice always has the wining strategy.

Problem 4. Let α be a fixed positive number. Suppose that the set $\mathcal{A}$ consisting of positive integers satisfy

$$|\mathcal{A} \cap \{1, 2, \cdots, n\}| \geq \alpha n$$

for every positive integer n. Prove that there exists a constant c such that every positive integer is the sum of at most c elements of $\mathcal{A}$.

Problem 5. Three positive integers p, q, n and an injective function $f : \mathbb{Z}^2 \to \mathbb{R}$ are given. Prove that if $n > \binom{p+q-2}{p-1}$, either there exist integers $1 \leq x_0 < x_1 < \cdots < x_p \leq n$ such that

$$f(x_0, x_1) < f(x_1, x_2) < \cdots < f(x_{p-1}, x_p)$$

or there exist integers $1 \leq y_0 < \cdots < y_q \leq n$ such that

$$f(y_0, y_1) > f(y_1, y_2) > \cdots > f(y_{q-1}, y_q).$$

Problem 6. Suppose that for a function $f : \mathbb{R} \to \mathbb{R}$ and real numbers $a < b$ one has $f(x) = 0$ for all $x \in (a, b)$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$ if

$$\sum_{k=0}^{p-1} f\left(y + \frac{k}{p}\right) = 0$$

for every prime number p and every real number y.

Problem 7. Suppose that a $1 \times q$ rectangle can be partitioned into a finite number of squares. Prove that q is rational.

Problem 8. n points are placed in the three dimensional Euclidean space such that no four points are coplanar. Suppose that

$$\sum_{i=1}^{n}(\pi - \angle A_{i-1}A_iA_{i+1}) < 4\pi.$$

Prove that the closed curve $A_1A_2 \cdots A_nA_1$ is an unknot.

Problem 9. Let $A(x)$ denote the number of positive integers $n \leq x$ having at least one prime divisor greater than $\sqrt[3]{n}$. Prove that

$$\lim_{x\to\infty} \frac{A(x)}{x}$$

exists.

Problem 10. Let K be a finite field of p elements, where p is a prime. For every polynomial

$$f(x) = \sum_{i=0}^{n} a_i x^i \in K[x],$$

let

$$\overline{f(x)} = \sum_{i=0}^{n} a_i x^{p^i}.$$

Prove that for any pair of polynomials $f(x), g(x) \in K[x]$, the polynomial $f(x)$ divides $g(x)$ if and only if $\overline{f(x)}$ divides $\overline{g(x)}$.

Problem 11. Let $A \subset \{1, 2, \cdots, 1000\}$ be a set with 600 elements. Call a closed interval $[k, n]$ *good* if for every integer $1 \leq m \leq n - k$, there exist integers i and j in A such that $k \leq i < j \leq n$ and $j - i = m$. Prove that there exists integers $1 \leq a < b \leq 1000$ such that $[a, b]$ is good and $b - a \geq 199$.

Problem 12. Given a set X of points in the plane, let $f_X(n)$ be the largest possible area of a polygon with at most n vertices, all of which are points of X. Prove that if m, n are integers with $m \geq n > 2$ then $f_X(m) + f_X(n) \geq f_X(m+1) + f_X(n-1)$.

Problem 13. Let $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ be positive real numbers such that for all positive t there are at most $1/t$ pairs (i, j) satisfying $x_i + y_j \geq t$. Prove that

$$(x_1 + \cdots + x_n)(y_1 + \cdots + y_n) \leq \max_{1 \leq i \leq n} x_i + \max_{1 \leq i \leq n} y_i.$$

Deferred problems

Problem 14. Let S be a subset of $(\mathbb{Z}/3\mathbb{Z})^n$ having the property that for every distinct elements $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ of S, there exists $1 \le i \le n$ such that $a_i \equiv b_i + 1 \pmod 3$. Prove that $|S| \le 2^n$.

Problem 15. Call a sequence $\{x_n\}_{n\ge1}$ of positive integers 'careful' if $|x_n - x_{n+1}| \le 2014$ for all n. Suppose $\{a_n\}, \{b_n\}, \{c_n = a_n b_n\}$ are all careful sequences. Prove that there exists a constant C such that $\min\{a_n, b_n\} \le C$ for all n.

Problem 16. Consider a non-increasing infinite sequence $\{a_n\}$ with the first term $a_0 = 1$. Prove that there exists a positive integer n such that

$$\frac{a_0^2}{a_1} + \frac{a_1^2}{a_2} + \cdots + \frac{a_{n-1}^2}{a_n} \ge 3.99.$$

(Note that if $a_n = 2^{-n}$ for all n, then $\sum_{i=0}^{\infty} \frac{a_i^2}{a_{i+1}} = 4$)

Problem 17. Define a pair sequence $\{p_n\}$ and $\{q_n\}$ as

$$\frac{2^1}{1} + \frac{2^2}{2} + \cdots + \frac{2^n}{n} = \frac{p_n}{q_n}$$

where p_n and q_n are coprime. Prove that for all k, there exists a natural number n such that $p_n, p_{n+1}, p_{n+2}, \cdots$ are all divisible by 2^k.

Problem 18. Find all continuous functions $f : \mathbb{R}^+ \to \mathbb{R}$ such that

$$f(x^2 + xy) + f(xy + y^2) = f(2x^2) + f(2y^2)$$

for all $x, y > 0$.

Problem 19. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying

$$f(x, y) + f(x + y, z) = f(x, y + z) + f(y, z)$$

for all real numbers x, y and z. Prove that there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) + g(y) - g(x + y)$ for all real numbers x and y.

Problem 20. Find all continuous functions $f, g, h : \mathbb{R}^+ \to \mathbb{R}$ such that

$$f(x + y) + g(xy) = h(x) + h(y)$$

for any $x, y > 0$.

Problem 21. A constant $\epsilon > 0$ is given. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x + y) - f(x)(y)| \leq \epsilon$$

for arbitrary reals x and y.

Problem 22. Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be $2n$ positive real numbers. Prove that

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \geq \left(\sum_{i=1}^{n} \frac{a_i + b_i}{2}\right)\left(\sum_{i=1}^{n} \frac{2a_ib_i}{a_i + b_i}\right).$$

Problem 23. Does a real number $x > 0$ exist such that $\frac{2}{5} \leq \{x^n\} \leq \frac{3}{5}$ for all positive integers n? ($\{a\}$ denotes the decimal part of a)

Problem 24. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a set of distinct positive integers. Suppose that the 2^k possible sums of the elements of X are all different. Prove that

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} < 2.$$

Problem 25. Consider a tetrahedron and let r and R be the radius of its inscribed sphere and circumscribed sphere. Prove that $R \geq 3r$.

Problem 26. A is a finite subset of $\mathbb{N}$. Prove that there exists a finite set $A \subseteq B \subset \mathbb{N}$ such that each element of B is a divisor of the sum of all elements in B.

Problem 27. Let $\{a_n\}$ be a strictly increasing sequence of positive integers such that $a_{i+2} + a_i \geq 2a_{i+1}$ for all i. Also, every two terms of the sequence is relatively prime. Prove that

$$\sum_{i=1}^{\infty} \frac{1}{a_i}$$

converges.

Problem 28. A collection is a set allowing multiplicities. For a finite set $A \subset \mathbb{N}$, define the collection of natural numbers

$$A_2 = \{x + y | x, y \in A, x \neq y\}.$$

For example, $A_2 = \{2, 3, 3, 4, 4, 4, 5, 5, 6\}$ when $A = \{1, 2, 3\}$. If $X_2 = Y_2$ holds for two distinct finite subsets X, Y of $\mathbb{N}$, prove that $|X| = |Y|$ is a power of 2.

Problem 29. For a positive real number α, let

$$S_\alpha = \{ \lfloor n\alpha \rfloor \mid n = 1, 2, \cdots \}.$$

Prove that there are no three positive reals α, β, and γ such that S_α, S_β, and S_γ are pairwise disjoint.

Problem 30. Pooh Bear has $2n+1$ pots of honey. No matter which one of them he eats up, he can split the left $2n$ pots in to two groups of the same total weight, each consisting of n pots. Must all the $2n+1$ pots initially weigh the same?

Problem 31. Let P be a point on the interior of $\triangle ABC$ such that $\angle PBA + \angle PCB + \angle PAC = 90°$. If P' is the isogonal conjugate respect to $\triangle ABC$, prove that PP' passes through the circumcenter of $\triangle ABC$.

Problem 32. Let $\Gamma_a, \Gamma_b, \Gamma_c$ be three circles in triangle ABC which are externally tangent to each other. Suppose that Γ_a is tangent to AB and AC, Γ_b is tangent to BA and BC, and Γ_c is tangent to CA and CB. Let I be the incenter of triangle ABC. Prove that the common inscribed tangent of Γ_b and Γ_c is also an common inscribed tangent of the incircles of triangle IAB and IAC.

Problem 33. Let $x_1, x_2, \cdots, x_n$ be vectors in the d-dimensional Euclidean space $\mathbb{R}^d$, each with length at most 1. Suppose that $x_1 + x_2 + \cdots + x_n = 0$. Prove that there is a permutation π of $\{1, \cdots, n\}$ such that $\|x_{\pi(1)} + x_{\pi(2)} + \cdots + x_{\pi(i)}\| \leq d$ for all i.

Problem 34. Three integers a, b and c greater than 1 are given. Suppose there are infinitely many 3-tuples (x, y, z) of positive integers such that $a^x + b^y = c^z$. Prove that a, b, c are all powers of 2.

Problem 35. Let $P(x)$ be a nonzero polynomial with nonnegative integer coefficients. Suppose that for all natural numbers n, $\phi(n)$ divides $\phi(P(n))$. Is necessarily $P(0) = 0$?

Problem 36. A function $f : \mathbb{R} \to \mathbb{R}$ has the property that for every $x, y \in \mathbb{R}$, there exist a $0 < t < 1$ such that

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y).$$

Does this condition imply that

$$f(\frac{x+y}{2}) = \frac{f(x) + f(y)}{2}$$

for all $x, y \in \mathbb{R}$?

Problem 37. Suppose that for a function $f : \mathbb{R} \to \mathbb{R}$, its square f^2 and cube f^3 are infinitely differentiable. Is f necessarily differentiable too?

Problem 38. Define sequence $\{f_n\}_{n\geq 1}$ as follows: $f_1 = 1, f_2 = 1$ and $f_{i+2} = f_{i+1} + f_i$ for $i \geq 1$. Suppose that f_n^2 divides f_m where m, n are two positive integers. Show that f_n divides m.

Problem 39. Find the maximum number k such that

$$x^{kx} + y^{ky} \geq x^{ky} + y^{kx}$$

holds for all positive reals x and y.

Problem 40. Let f and g be polynomials with rational coefficients. Prove that $f(\mathbb{Q}) = g(\mathbb{Q})$ holds if and only if $f(x) = g(ax + b)$ for some rational numbers $a \neq 0$ and b.

Problem 41. Let Γ_1, Γ_2 and Γ_3 be three circles with no common interior. Let ℓ_i, m_i be the two common inscribed tangent of Γ_{i+1} and Γ_{i+2} where indices are modulo 3. Prove that ℓ_1, ℓ_2 and ℓ_3 concur if and only if m_1, m_2 and m_3 concur.

Solutions - 2014

Problem 1. *Let $P(n)$ denote the greatest prime factor of n, and let $P(1) = 1$ for convenience. Prove that there exist infinitely many positive integers n satisfying $P(n) < P(n+1) < P(n+2)$.*

Solution. Let p be an odd prime. We will observe the greatest prime factor of the numbers in the form of $p^{2^k} + 1$.

Let q a prime divisor of $p^{2^k} + 1$. Since $p^{2^k} \equiv -1 \pmod{q}$, the multiplicative order of p modulo q is divisible by 2^{k+1}, whereas it is not divisible by 2^k. This implies that it equals 2^k and that $q \equiv 1 \pmod{2^k}$, or $q > 2^k$. Hence $P(p^{2^k} + 1)$ is not bounded.

Now let us consider the least l such that $P(p^{2^l} + 1) > p$. By the factorization

$$p^{2^l} - 1 = (p-1)\prod_{i=0}^{l-1}(p^{2^i} + 1),$$

we obtain

$$P(p^{2^l} - 1) = \max\{P(p-1), P(p+1), \cdots, P(p^{2^{l-1}} + 1)\} < p.$$

In addition, it is clear that $P(p^{2^l}) = p$.

Therefore $n = p^{2^l} - 1$ satisfies the desired inequality, and since different p give different n, there exist infinitely many such n. $\square$

Comment. An analytic approach with the same arguments give us a lower bound of $\frac{c\log x}{(\log\log x)^2}$ for the number of $n \le x$ such that $P(n) < P(n+1) < P(n+2)$.

Problem 2. *$P_1, P_2, \cdots, P_n$ are n points inside a unit square. Let d_i be the distance from P_i to its nearest point. Prove that*

$$d_1^2 + d_2^2 + \cdots + d_n^2 \le 4.$$

Solution. We first prove the following lemma.

Lemma. Given a right triangle ABC with $\angle C = 90°$ and n points inside it. Then there exists a labeling $P_1, P_2, \cdots, P_n$ of these points such that

$$AP_1^2 + \sum_{k=1}^{n-1} P_kP_{k+1}^2 + P_nB^2 \le AB^2.$$

Proof. We prove it by induction on n. It is trivial when $n = 1$ since $\angle AP_1B \geq 90°$.

Now suppose that the statement holds for every smaller n. Let the projection of C on AB be H. If some of the points lie on the triangle ACH and some lie on BCH, we can use the induction hypothesis since the number of points in each triangle is strictly less than n. Therefore, there exist a labeling such that $AP_1^2 + P_1P_2^2 + \cdots + P_{k-1}P_k^2 + P_kC^2 \leq AC^2$ and $CP_{k+1}^2 + \cdots + P_nB^2 \leq CB^2$. If we add these two, we obtain $AP_1^2 + \cdots + P_kC^2 + CP_{k+1}^2 + \cdots + P_nB^2 \leq AC^2 + CB^2 = AB^2$. But since $\angle P_kCP_{k+1}$ is acute, $P_kC^2 + CP_{k+1}^2 \geq P_kP_{k+1}^2$. Therefore it follows that $AP_1^2 + \cdots + P_nB^2 \leq AB^2$.

If all the points lie on either triangle ACH or BCH, we cannot directly use the induction hypothesis. Assume that they all lie on triangle ACH, without the loss of generality. If the statement holds for ACH, we can label the points so that $AP_1^2 + \cdots + P_nC^2 \leq AC^2$. Also since $\angle P_nCB$ is acute, $AP_1^2 + \cdots + P_nB^2 \leq AP_1^2 + \cdots + P_nC^2 + BC^2 \leq AC^2 + BC^2 = AB^2$. Therefore, we can conclude that if the statement holds for triangle ACH, then it also holds for triangle ABC. Now, divide the triangle by its altitude in the same manner until the n points are separated. This ends in a finite number of times because the diameter of the triangle tends to zero. And when they are separated, we can finally use the induction hypothesis and prove that the statement is true for triangle ABC. $\square$

Let the unit square be $XYZW$. Using the lemma, label the points in $\triangle XZY$ by $Q_1, \cdots, Q_k$ and label the points in $\triangle ZXW$ by $R_1, \cdots, R_l$. k and l may be zero, but at least $k+l = n$ must be true. By the rule of the labeling, $XQ_1^2 + \cdots + Q_kZ^2 \leq XZ^2 = 2$, and also $ZR_1^2 + \cdots + R_lX^2 \leq XZ^2 = 2$. Now when we add these two up, we obtain $Q_1Q_2^2 + \cdots + Q_{k-1}Q_k^2 + Q_kR_1^2 + \cdots + R_lQ_1^2 \leq 4$ since $XQ_1^2 + XR_l^2 \leq Q_1R_l^2$ and $ZQ_k^2 + ZR_1^2 \leq Q_kR_1^2$. Note that each of the Q_iQ_{i+1} or R_iR_{i+1} is greater or equal to the distance to its nearest point of Q_i or R_i. By this, $Q_1Q_2^2 + \cdots + Q_kR_1^2 + \cdots + R_lQ_1^2 \leq 4$ implies $d_1^2 + \cdots + d_n^2 \leq 4$. $\square$

Problem 3. *Let $f : \mathbb{R} \to \mathbb{R}$ such that f is differentiable and f, f' convex. For any real numbers $x_1, x_2, \cdots, x_n$, prove that*

$$(n-2)\sum_{i=1}^{n} f(x_i) + nf\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \geq \sum_{i \neq j} f\left(\frac{(n-1)x_i + x_j}{n}\right).$$

Solution. With out the loss of generality, we may suppose that $x_1 \geq x_2 \geq \cdots \geq x_n$.

Considering the sequence $A_1 = \left(\frac{x_1+\cdots+x_k+(n-k)x_{k+1}}{n}, x_{k+1}, x_{k+1}, \cdots, x_{k+1}\right)$ with $k-1$ x_{k+1}'s and $A_2 = \left(\frac{x_1+(n-1)x_{k+1}}{n}, \frac{x_2+(n-1)x_{k+1}}{n}, \cdots, \frac{x_k+(n-1)x_{k+1}}{n}\right)$, one can easily check that A_1 majorizes A_2. Hence, by Karamata's inequality, it follows that

$$f\left(\frac{x_1+\cdots+x_k+(n-k)x_{k+1}}{n}\right)+(k-1)x_{k+1} \geq \sum_{j=1}^{k} f\left(\frac{x_j+(n-1)x_{k+1}}{n}\right).$$

If we sum these equations from $k=1$ to $k=n-1$, we then obtain

$$\sum_{k=2}^{n}(k-2)f(x_k)+\sum_{k=1}^{n-1} f\left(\frac{x_1+\cdots+x_k+(n-k)x_{k+1}}{n}\right) \geq \sum_{1\leq i<j\leq n} f\left(\frac{x_i+(n-1)x_j}{n}\right). \quad (1)$$

In order to prove the original inequality, we must now prove the complement inequality. If we let

$$\begin{aligned} &F(x_1, x_2, \cdots, x_n) \\ &\quad = \sum_{i=2}^{n}(n-i)f(x_i)+(n-2)f(x_1)+nf\left(\frac{x_1+x_2+\cdots x_n}{n}\right) \\ &\quad -\sum_{i=1}^{n-1} f\left(\frac{x_1+\cdots+x_i+(n-i)x_{i+1}}{n}\right)-\sum_{1\leq i<j\leq n} f\left(\frac{(n-1)x_i+x_j}{n}\right) \end{aligned}$$

the complement inequality will then be equivalent to $F(x_1, \cdots, x_n) \geq 0$. However, we will prove a more stronger claim that

$$\begin{aligned} F(x_1, \cdots, x_n) &\geq F(x_1, \cdots, x_{n-2}, x_{n-1}, x_{n-1}) \\ &\geq F(x_1, \cdots, x_{n-3}, x_{n-2}, x_{n-2}, x_{n-2}) \\ &\geq \cdots \\ &\geq F(x_1, x_1, \cdots, x_1) = 0. \end{aligned}$$

Consider the function $G(x) = F(x_1, \cdots, x_k, x, x, \cdots, x)$ with $k \geq 1$. It is sufficient to show that $G'(x) \leq 0$ whenever $x \leq x_k$. After some explicit calculations, we get

$$G'(x) = \frac{(n-k)}{n}\left\{kf'\left(\frac{x_1+\cdots+x_k+(n-k)x}{n}\right)-\sum_{j=1}^{k} f'\left(\frac{(n-1)x_j+x}{n}\right)\right\}.$$

Note that since f is a convex function, f' is increasing. Therefore,

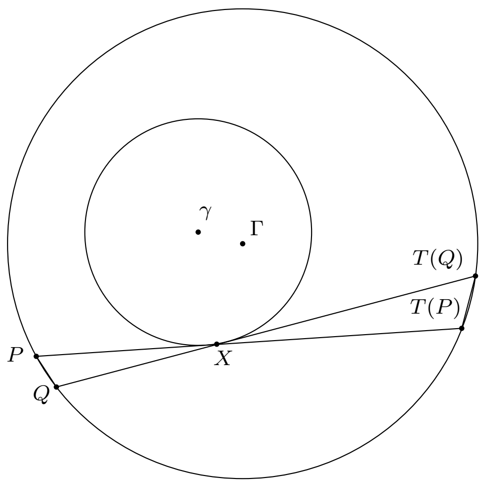
$$\begin{aligned}\sum_{j=1}^{k} f'\left(\frac{(n-1)x_j + x}{n}\right) &\geq kf'\left(\frac{x}{n} + \frac{n-1}{nk}(x_1 + \cdots + x_k)\right) \\ &\geq kf'\left(\frac{x_1 + \cdots + x_k + (n-k)x}{n}\right)\end{aligned}$$

follows from Jensen's inequality. Thus $F(x_1, \cdots, x_n) \geq 0$ for all $x_1 \geq \cdots \geq x_n$, and adding up with (1) gives us the desired inequality. □

Comment. This problem is from an article "About Surányi's Inequality" written by Mihály Bencze. By setting $f(x) = e^x$, we obtain Surányi's inequality as a direct corollary.

Problem 4. *Suppose a convex n-gon has a circumscribed circle Γ and a inscribed circle γ. Prove that for any point P on Γ, there exists a n-gon such that it has P as a vertex, Γ as the circumscribed circle, and γ as the inscribed circle.*

Solution.



Let $T : \Gamma \to \Gamma$ be a translation such that for every P on Γ, the line $PT(P)$ is tangent to γ in a counterclockwise manner. Also, let $f(P)$ be the length of the tangent drawn from P to γ.

Consider two close points P and Q on Γ and let X be the intersection of $PT(P)$ and $QT(Q)$. Now for the points on Γ, we can set up a coordinate system corresponding K to the central angle $\angle O\Gamma K$ where O(also on Γ) is the origin. We can also assume without the loss of generality that the radius of Γ is 1.

Since $P, Q, T(P)$ and $T(Q)$ are all points on Γ, the triangles PQX and $T(Q)T(P)X$ are similar. Therefore,

$$\begin{aligned}\lim_{Q\to P}\frac{T(Q)-T(P)}{Q-P} &= \lim_{Q\to P}\frac{T(Q)-T(P)}{T(P)T(Q)}\cdot\lim_{Q\to P}\frac{T(P)T(Q)}{PQ}\cdot\lim_{Q\to P}\frac{PQ}{P-Q}\\ &= 1\cdot\lim_{Q\to P}\frac{T(Q)X}{PX}\cdot 1 = \frac{\lim_{Q\to P}T(Q)X}{\lim_{Q\to P}PX}\\ &= \frac{f(T(P))}{f(P)}.\end{aligned}$$

In other words, $T'(\theta) = f(T(\theta))/f(\theta)$.

Now let

$$I(\theta) = \int_{\theta}^{T(\theta)}\frac{d\phi}{f(\phi)}.$$

From the previous equation, it follows that

$$I'(\theta) = \frac{dT(\theta)}{d\theta}\frac{1}{f(T(\theta))} - \frac{d\theta}{d\theta}\frac{1}{f(\theta)} = 0,$$

or that $I(\theta)$ is constant for all θ. Let $I(\theta) = C$.

By the condition, there exists an point K such that $T^n(K) = K + 2\pi$. Thus the integral of $1/f(\phi)$ along the circumference of Γ once is equal to

$$\int_K^{K+2\pi}\frac{d\phi}{f(\phi)} = \int_K^{T(K)}\frac{d\phi}{f(\phi)} + \int_{T(K)}^{T^2(K)}\frac{d\phi}{f(\phi)} + \cdots + \int_{T^{n-1}(K)}^{T^n(K)}\frac{d\phi}{f(\phi)} = nC.$$

For any point P on Γ, one can analogously deduce that

$$\int_P^{T^n(P)}\frac{d\phi}{f(\phi)} = nC$$

whereas

$$\int_P^{P+2\pi}\frac{d\phi}{f(\phi)} = \int_K^{K+2\pi}\frac{d\phi}{f(\phi)} = nC.$$

Hence $T^n(P) = P + 2\pi$ for all P. $\square$

Comment. This result is commonly known as Poncelet's porism.

Problem 5. *A polynomial $P(x)$ with real coefficients satisfies $P(x) \geq 0$ for all $x \in \mathbb{R}$. Prove that there exist two polynomials Q and R, also with real coefficients, such that*

$$P(x) = Q(x)^2 + R(x)^2.$$

Solution. Consider all the complex roots of $P(x) = 0$. If there exists real roots, they must have even multiplicities. Furthermore, if z is a root, $\overline{z}$ must also be a root, and they must have equal multiplicities.

Now, let $z_1, \overline{z_1}, z_2, \overline{z_2}, \cdots, z_m, \overline{z_m}$ be all complex roots(including multiplicities) of P. Then

$$\begin{aligned}
P(x) &= c(x - z_1)\cdots(x - z_m)(x - \overline{z_1})\cdots(x - \overline{z_m}) \\
&= \sqrt{c}(x - z_1)\cdots(x - z_m)\overline{\sqrt{c}(x - z_1)\cdots(x - z_m)} \\
&= (Q(x) + iR(x))(Q(x) - iR(x)) \\
&= Q(x)^2 + R(x)^2
\end{aligned}$$

where $\sqrt{c}(x - z_1)\cdots(x - z_m) = Q(x) + iR(x)$.

Therefore, we are done. $\square$

Problem 6. *Define a sequence $\{x_n\}$ of distinct natural numbers as follows. $x_1 = 1$ and x_n is determined as the least natural number, distinct from $x_1, \cdots, x_{n-1}$, such that $x_1 + x_2 \cdots + x_n$ is divisible by n. Prove that $x_{x_n} = n$ for all n.*

Solution. Let $\varphi = \frac{1+\sqrt{5}}{2}$. We first note that the two sets $\{\lceil n\varphi \rceil : n \in \mathbb{N}\}$ and $\{\lceil n\varphi^2 \rceil : n \in \mathbb{N}\}$ partition the set of positive integers greater than 1. Define a new sequence as $a_1 = 1$, $a_{\lceil n\varphi \rceil} = \lceil n\varphi^2 \rceil$ and $a_{\lceil n\varphi^2 \rceil} = \lceil n\varphi \rceil$. Then it is clear that $a_{a_n} = n$ for all n.

We shall prove by induction that $\sum_{i=1}^{n} a_i = n\lceil \frac{n}{\varphi} \rceil$ and that a_n is the smallest integer except $1, a_1, \cdots, a_{n-1}$ such that $\sum_{i=1}^{n} a_i = n$ is divisible by n. For $n = 1$, it is trivial. Suppose that the hypothesis is true for $n = k$. There exist two cases: $k+1 = \lceil (m+1)\varphi \rceil$ for some m, or $k+1 = \lceil (m+1)\varphi^2 \rceil$ for some m.

Case 1. $k + 1 = \lceil m\varphi \rceil$ for some m.
By the definition, $a_{k+1} = \lceil m\varphi^2 \rceil = \lceil m(1 + \varphi) \rceil = m + \lceil m\varphi \rceil = m + k + 1$. Since $k + 1 = m\varphi + 1 - \{m\varphi\}$, we have

$$m - 1 < m - \frac{\{m\varphi\}}{\varphi} = \frac{k}{\varphi} < m < m + \frac{1 - \{m\varphi\}}{\varphi} = \frac{k+1}{\varphi} < m + 1$$

where $\{x\} = x - \lfloor x \rfloor$. It follows from this that $\lceil \frac{k}{\varphi} \rceil = m$ and $\lceil \frac{k+1}{\varphi} \rceil = m + 1$. Therefore by the induction hypothesis,

$$\sum_{i=1}^{k+1} a_i = k\lceil \frac{k}{\varphi} \rceil + a_{k+1} = km + m + k + 1 = (k+1)\lceil \frac{k+1}{\varphi} \rceil.$$

For the proof of minimality, note that $a_{k+1} = m + k + 1 < 2(k+1)$. It suffices to show that $a_i = m$ for some i with $1 \le i \le k$, or equivalently, $a_m \le k$. If $m = \lceil b\varphi^2 \rceil$ for some $b \in \mathbb{Z}_{>0}$, we are done since $a_m < m \le k$. Otherwise $m = \lceil b\varphi \rceil$ for some $b \in \mathbb{Z}_{>0}$, so $a_m = \lceil b\varphi^2 \rceil \le \lceil \lceil b\varphi \rceil \varphi \rceil = \lceil m\varphi \rceil = k+1$. Since $k+1 = \lceil m\varphi \rceil \ne \lceil b\varphi^2 \rceil = a_m$, we now have the desired inequality $a_m \le k$.

Case 2. $k+1 = \lceil m\varphi^2 \rceil = m + \lceil m\varphi \rceil$ for some m.
We proceed in the same manner. By the definition, $a_{k+1} = \lceil m\varphi \rceil = k+1-m$ and $k+1 = m\varphi^2 + 1 - \{m\varphi^2\}$.

As we have a partition, $k+1$ cannot be written in the form $\lceil i\varphi \rceil$. Thus there exists a positive integer c such that $\lceil c\varphi \rceil < k$ and $k+1 < \lceil (c+1)\varphi \rceil$ since $k > 1$. This gives $\lceil c\varphi \rceil = k$ and $\lceil (c+1)\varphi \rceil = k+2$.

Thus

$$c < c + \frac{1 - \{c\varphi\}}{\varphi} = \frac{k}{\varphi} = m\varphi - \frac{\{m\varphi^2\}}{\varphi} < m\varphi$$

and

$$m\varphi < m\varphi + \frac{1 - \{m\varphi^2\}}{\varphi} = \frac{k+1}{\varphi} = c + 1 - \frac{\{(c+1)\varphi\}}{\varphi} < c+1.$$

It then follows that $\lceil \frac{k}{\varphi} \rceil = \lceil m\varphi \rceil = \lceil \frac{k+1}{\varphi} \rceil = c+1$ and hence $k+1 = m + c + 1, a_{k+1} = c+1$. Therefore by the induction hypothesis,

$$\sum_{i=1}^{k+1} a_i = k\lceil \frac{k}{\varphi} \rceil + a_{k+1} = (k+1)(c+1) = (k+1)\lceil \frac{k+1}{\varphi} \rceil.$$

The minimality of $k+1-m$ is trivial since $a_{k+1} = k+1-m < k+1$.

Therefore in all cases the sequence $\{a_n\}$ satisfies all the definitions of $\{x_n\}$, and hence we can conclude that $a_n = x_n$ for all n. The fact $x_{x_n} = n$ is then obvious. $\square$

Comment. This solution was written by Ivan Loh.

Problem 7. *Let P be a monic polynomial with complex coefficients. Prove that*

$$diam\{z \in \mathbb{C} : |P(z)| \le 1\} \ge 2.$$

Solution. Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and $\Delta = \{z \in \hat{\mathbb{C}} : |z| > 1\}$. Let $E(P) = \{z \in \mathbb{C} : |P(z)| \le 1\}$ and E the convex hull of $E(P)$.

Note that $\hat{\mathbb{C}} \setminus E$ is a simply connected open set on the Riemann sphere. Hence by the Riemann mapping theorem, there exists an injective and analytic function $g : \Delta \to \hat{\mathbb{C}}$ such that $g(\Delta) = \hat{\mathbb{C}} \setminus E$. We can represent g as the form of

$$g(z) = bz + \sum_{j=0}^{\infty} b_j z^{-j}$$

where b and b_j are complex numbers. The higher order terms vanish since g is injective.

From the definition of E, for every $\epsilon > 0$ there exists a $\delta > 0$ such that $-\epsilon + n \log |1 + \delta| < 0$ or

$$-\epsilon \leq \log |z^{-n} p(g(z))|$$

for all $|z| = 1 + \delta$. Since $G(z) = \log |z^{-n} p(g(z))|$ is harmonic on Δ, we have

$$G(\infty) = \frac{1}{2\pi(1+\delta)} \int_0^{2\pi} G((1+\delta)e^{i\theta}) d\theta \geq \frac{-\epsilon}{2\pi(1+\delta)}$$

for every $\epsilon > 0$. Hence $n \log |b| = G(\infty) \geq 0$ and $|b| \geq 1$.

Now assume that $diam(E) < 2$. Then there will exist a $\delta > 0$ such that

$$|z^{-1}(g(z) - g(-z))| < 2$$

for all $|z| = 1 + \delta$. Since

$$F(z) = |z^{-1}(g(z) - g(-z))| = 2b + 2b_1 z^{-2} + 2b_2 z^{-4} + \cdots$$

is analytic on Δ, the maximum value of $|F(z)|$ will be attained at its boundary. Thus

$$2|b| = |F(\infty)| \leq \max_{|z|=1+\delta} |F(z)| = \max_{|z|=1+\delta} |z^{-1}(g(z) - g(-z))| < 2,$$

that is, $|b| < 1$, which is a contradiction.

Therefore $diam(E(P)) = diam(E) \geq 2$ and we are done. $\square$

Comment. The author apologizes for proposing a non-elementary problem without checking the solution. This problem comes from the book 'Polynomials and Polynomial Inequalities' by Borwein and Erdélyi.

Problem 8. *Find all even, nonnegative, and differentiable functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the inequality*

$$f(t) - (f(s) + f'(s)(t-s)) \geq f(t-s)$$

for every $t, s \in \mathbb{R}$.

Solution. Put $s-t$ for t and we get $f(s-t)-f(s)+f'(s)t \geq f(-t)$ and this is equivalent to $f(t-s)-f(s)+f'(s)t \geq f(t)$. Add this to the original equation, and we obtain

$$-2f(s)+sf'(s) \geq 0.$$

Now let $g(x)=f(x)x^{-2}$ for $x \neq 0$. Then obviously g is also an even function and $\frac{d}{dx}g=-2f(x)+xf'(x) \geq 0$. By the evenness of g, $\frac{d}{dx}g(x)=0$ for all $x \neq 0$. Therefore there exists an constant $k \geq 0$ such that $f(x)=kx^2$ for all $x \neq 0$.

However, plugging $s=t$ in the original equation gives $0 \geq f(0)$ and by the nonnegativity of f, $f(0)=0$. Thus $f(x)=kx^2$ for all x and it is easy to check that this function satisfies the original inequality. $\square$

Problem 9. *Let A and B two convex polygons and let* $\mathrm{vol}(P)$ *denote the area of the polygon P. Prove that*

$$\sqrt{\mathrm{vol}(A+B)} \geq \sqrt{\mathrm{vol}(A)}+\sqrt{\mathrm{vol}(B)}.$$

Solution. Let us set an arbitrary Cartesian coordinate on the plane. Let the projection of A on the x-axis be $[a_1,a_2]$ and for $a_1 \leq x_0 \leq a_2$, define $f(x_0)$ as the length of intersection of A and the line $x=x_0$. Define b_1,b_2 and $g(x)$ for B likewise. Then $\mathrm{vol}(A)=\int_{a_1}^{a_2} f(x)dx$ and $\mathrm{vol}(B)=\int_{b_1}^{b_2} g(x)dx$.

Now we lower bound the area of $A+B$. Let $a_2-a_1=\ell_1$ and $b_2-b_1=\ell_2$. The length of the intersection of $A+B$ and the line $x=x_0$ is at least $f(u)+g(v)$ for any $u \in [a_1,a_2]$, $v \in [b_1,b_2]$, $u+v=x_0$. Hence

$$\begin{aligned}
\mathrm{vol}(A+B) &\geq \int_{a_1+b_1}^{a_2+b_2} f\Big(\frac{\ell_1 x+a_1b_2-a_2b_1}{\ell_1+\ell_2}\Big)+g\Big(\frac{\ell_2 x+a_2b_1-a_1b_2}{\ell_1+\ell_2}\Big)dx\\
&= \frac{\ell_1+\ell_2}{\ell_1}\mathrm{vol}(A)+\frac{\ell_1+\ell_2}{\ell_2}\mathrm{vol}(B)\\
&= (\ell_1+\ell_2)\Big(\frac{\mathrm{vol}(A)}{\ell_1}+\frac{\mathrm{vol}(B)}{\ell_2}\Big)\\
&\geq (\sqrt{\mathrm{vol}(A)}+\sqrt{\mathrm{vol}(B)})^2.
\end{aligned}$$

Therefore $\sqrt{\mathrm{vol}(A+B)} \geq \sqrt{\mathrm{vol}(A)}+\sqrt{\mathrm{vol}(B)}$. $\square$

Problem 10. *Let $I \subset \mathbb{R}$ be an open interval and f a function such that $f^{(n)}(x) \geq 0$ for all $x \in I$. Consider $n+1$ distinct real numbers $a_1, \cdots, a_{n+1} \in$*

I and let P be the polynomial obtained by Lagrange's interpolation from the $n+1$ points $(a_1, f(a_1)), \cdots, (a_{n+1}, f(a_{n+1}))$. Prove that the coefficient of x^n in P is nonnegative.

Solution. Let $Q(x) = f(x) - P(x)$. Then obviously $Q(a_1) = Q(a_2) = \cdots = Q(a_{n+1}) = 0$.

Now by Rolle's theorem, there exists b_i for $i = 1, \cdots, n-1$ such that $a_i < b_i < a_{i+1}$ and $Q'(b_i) = 0$. Therefore, there exists at least $n-1$ distinct solutions for $Q'(x) = 0$ in I. Using the same argument, one can show that there exists at least $n-2$ distinct solutions for $Q''(x) = 0$ in I. Inductively, there exists a $c \in I$ such that $Q^{(n)}(c) = 0$. Therefore $P^{(n)}(c) = f^{(n)}(c) \geq 0$, and hence the coefficient of x^n in P is nonnegative. $\square$

Problem 11. *Let G denote the set of all infinite sequences $(a_1, a_2, \cdots)$ of integers. We can add elements of G coordinate-wise, i.e.,*

$$(a_1, a_2, \cdots) + (b_1, b_2, \cdots) = (a_1 + b_1, a_2 + b_2, \cdots).$$

Suppose $f : G \to \mathbb{Z}$ is a function satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in G$. Prove that

$$f(a_1, a_2, \cdots) = x_1 a_{k_1} + \cdots + x_n a_{k_n}$$

for some integers $x_1, \cdots, x_n, k_1, \cdots, k_n$.

Solution. Let $e_i = (0, \cdots, 0, 1, 0, \cdots)$ where 1 is in the ith entry. We first prove that $f(e_i) = 0$ for sufficiently large i.

Let $s_i = f(e_i)$. Define a sequence of integers $0 < n_1 < n_2 < \cdots$ so that for all $k \geq 1$,

$$\sum_{i=1}^{k} |s_i| 2^{n_i} < 2^{n_{k+1}-1}.$$

Now consider $x = (2^{n_1}, 2^{n_2}, \cdots)$. Then

$$\begin{aligned} f(x) &= f(2^{n_1} e_1 + \cdots + 2^{n_k} e^k + 2^{n_{k+1}} (e_{k+1} + \cdots)) \\ &= \sum_{i=1}^{k} s_i 2^{n_i} + 2^{n_{k+1}} f(t_k) \end{aligned}$$

where $t_k = (e_{k+1} + 2^{n_{k+2} - n_{k+1}} e_{k+2} + \cdots)$.

By the triangle inequality, we have

$$2^{n_{k+1}} |f(t_k)| \leq \sum_{i=1}^{k} |s_i| 2^{n_i} + |f(x)| < 2^{n_{k+1}-1} + |f(x)|$$

and therefore $|f(t_k)| < \frac{1}{2} + |f(x)|2^{-n_{k+1}}$. Thus, we have $f(t_k) = 0$ for sufficiently large k. Also, since $t_j - 2^{n_{j+2}-n_{j+1}}t_{j+1} = e^{j+1}$, we have $s_j = f(e_j) = 0$ for sufficiently large j.

Now suppose that $s_i = 0$ for all $i > N$. Let $g : G \to \mathbb{Z}$ be a function which maps

$$g(a_1, a_2, \cdots) = f(a_1, a_2, \cdots) - s_1a_1 - \cdots - s_na_n.$$

Then obviously $g(e_i) = 0$ for all i. We shall now prove that $g(x) = 0$ for all $x \in G$.

Let $x = (x_1, x_2, \cdots)$. Since 2^n and 3^n are relatively prime there exist integers y_n and z_n such that $x_n = y_n2^n + z_n3^n$. Then $x = y + z$ where $y = (2y_1, 4y_2, \cdots)$ and $z = (3z_1, 9z_2, \cdots)$.

For every $k \geq 1$, we have

$$\begin{aligned} f(y) &= f(2a_1, 4a_2, \cdots, 2^{k-1}a_{k-1}, 0, 0, \cdots) \\ &\qquad + f(0, \cdots, 0, 2^ka_k, 2^{k+1}a_{k+1}, \cdots) \\ &= 0 + 2^kf(0, \cdots, 0, a_k, 2a_{k+1}, \cdots). \end{aligned}$$

Hence, $f(y)$ is divisible by 2^k for all $k \geq 1$, and therefore $f(y) = 0$. Similarly, $f(z) = 0$ and $f(x) = f(y) - f(z) = 0$.

Thus for all $x \in G$, we have $f(a_1, a_2, \cdots) = s_1a_1 + \cdots + s_na_n$. □

Problem 12. *Three squares of side length x_1, x_2 and x_3 are placed inside a 1×2 rectangle without any overlaps. Prove that $x_1 + x_2 + x_3 \leq 2$.*

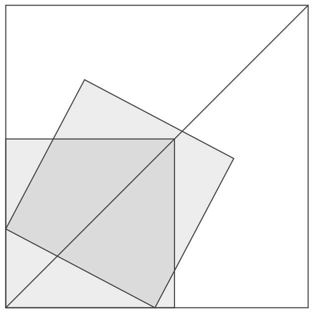
Solution. For two non-intersecting convex polygons A and B, say that A is on the left side of B or $A < B$ if there is a horizontal line ℓ intersecting with both A and B such that the intersection of A with ℓ is on the left of B with ℓ. Then we see $A < B$ and $B < A$ are incompatible. In fact, $A_1 < A_2 < \cdots < A_n < A_1$ cannot hold for any distinct polygons. (We leave the proof for the reader)

Let X_i be the square with side length x_i. Of the three squares, there exists a minimal element of this order relation. Let it be X_1. Then X_1 can slide freely in the left direction without colliding with any other square. Thus, we may assume that at least one vertex of X_1 is on the left side of the 1×2 rectangle. Likewise, let X_3 be the maximal element except X_1 and we may assume that one vertex of X_3 lie on the right side of the rectangle.

Let the line the separates the rectangle to two unit squares be ℓ. Then ℓ cannot intersect X_1 as well as X_3. Therefore if we define another relation for the vertical direction, neither $X_1 < X_3$ nor $X_1 > X_3$ does not hold. Hence X_1 and X_3 are both either a maximal element or a minimal element. Sliding the squares again, we arrive at a state where both X_1 and X_3 have a vertex on at least two sides of the 1×2 rectangle.

At this point, we prove the following lemma.

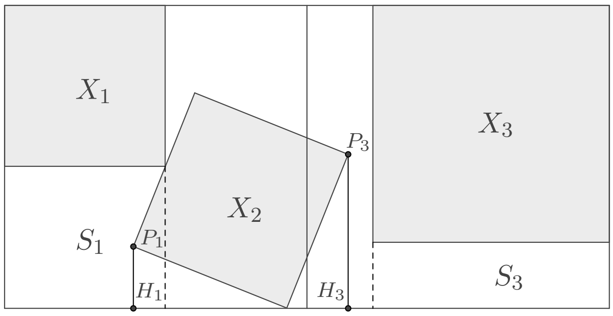
Lemma. Three squares of side length x_1 and x_2 are placed inside a unit square without any overlaps. Then $x_1 + x_2 \leq 1$.



Proof. Let the square with side length x_1 be at the left side and at the bottom side between the two squares. We may slide this square to the left bottom corner and the other one to the right top corner. After some calculations, it can be verified that the length of the diagonal occupied by the square with side length x_1 is at least $\sqrt{2}x_1$, where 'occupation' includes the unusable space at the corner. (Again, we leave it to the reader) Hence we may rotate the square to share the left bottom corner with the unit square without causing any overlaps. Doing the same thing to the other square, we arrive at a state where two squares occupy opposite corners of the unit square. Then it is obvious that $x_1 + x_2 \leq 1$. $\square$

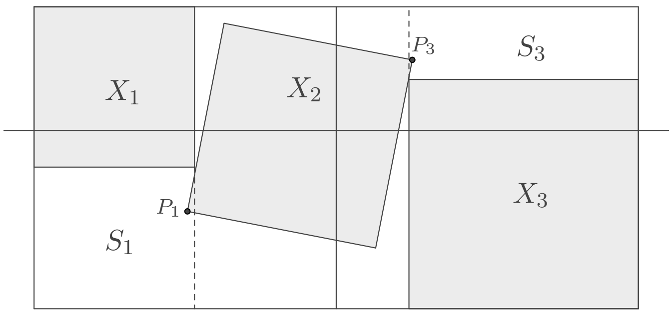
In our case, since X_1 is the square on the most left side and on the most top/bottom side, we may do the same thing to X_1 and for the same reason, also to X_3. Let the vertex on the most left side among the vertices of X_2 be P_1 and let P_3 be the vertex opposite of P_1.

Case 1. X_1 and X_3 occupies the top corner



Let the region below X_1 be S_1 and the region below X_3 be S_3. Let the perpendiculars from P_1 and P_3 to the bottom side be H_1 and H_3 respectively. Since $P_1H_1 + P_3H_3 \geq H_1H_3$, the point P_1 is not in S_1 or P_3 is not in S_3. Without loss of generality suppose that P_3 is not in S_3 as in the figure. Then both X_1 and X_2 are contained in a rectangle of size $1 \times (2 - x_3)$, and by our lemma, $x_1 + x_2 \leq 2 - x_3$. Hence $x_1 + x_2 + x_3 \leq 2$.

Case 2. X_1 occupies the top corner and X_3 occupies the bottom corner



Let S_1 be the region below X_1 and S_3 be the region above X_3. If $x_1+x_3 \leq 1$, then $x_1 + x_2 + x_3 \leq 1 + x_2 \leq 2$. Therefore we assume that $x_1 + x_3 > 1$. If P_1 is not in S_1, then X_2 and X_3 lie in a square of side length $2 - x_1$, and by our lemma, $x_2 + x_3 \leq 2 - x_1$. Therefore we assume that P_1 is in S_1 and likewise, that P_3 is in S_3.

Now we draw a horizontal line ℓ which passes through both X_1 and X_3. (This is possible since $x_1 + x_3 > 1$) Then the sum of the lengths of the intersection of ℓ with the squares will not exceed 2. On the other hand the three lengths are x_1, $\geq x_2$ and x_3. Hence $x_1 + x_2 + x_3 \leq 2$. $\square$

Problem 13. *Let P be a polynomial with integer coefficients such that for every positive integer x, $P(x)$ is a perfect square. Prove that $P(x) = Q(x)^2$ for some polynomial $Q(x)$.*

Solution. Let $P(x) = P_1(x)P_2(x)\cdots P_k(x)Q(x)^2$ where $P_i(x)$ are all different and irreducible. Suppose that $k \geq 1$. Since distinct irreducible polynomials are relatively prime, there exist polynomials A and B in $\mathbb{Z}[x]$ such that $A_{ij}P_i + B_{ij}P_j = c_{ij}$ for an nonzero integer c_{ij}. Let $M = \max_{1\leq i<j\leq k} c_{i,j} + 1$ and $M = 1$ if $k = 1$. Then for any prime $p > M$ and an arbitrary integer x, the prime p cannot divide both $P_i(x)$ and $P_j(x)$. Hence p divides at most one of $P_i(x)$. But by the condition, $P_1(x)P_2(x)\cdots P_k(x)$ is a square. Thus the p-adic order of $P_i(x)$ must always be even.

Since $P_1(x)$ is irreducible, P_1 and P_1' is relatively prime, where P_1' is the derivative of P_1. Therefore there exists polynomials X and Y in $\mathbb{Z}[x]$ such that $XP_1 + YP_1' = z$ is a non-zero integer. Since $P_1(x)$ is not constant, there exists infinitely many prime divisors of $P_1(x)$. (If the only possible prime divisors are $p_1, \cdots, p_t$, then one can show that $P(n + xP(n))$ is bounded for every n) Suppose for some prime $p > M$, prime p is divides $P_1(x)$. Since P_1 has integer coefficients, $P_1(x+p)$ is also divisible by p and hence divisible by p^2. Since $0 \equiv P_1(x+tp) \equiv P_1(x) + pP_1'(x) \bmod p^2$, we get that p divides $P_1'(x)$. Thus p also divides z because p divides both $P_1(x)$ and $P_1'(x)$. But this holds for infinitely many primes p since $P_1(\mathbb{Z})$ have infinitely many prime divisors. Thus we arrive at a contradiction and $k = 0$. $\square$

Problem 14. *Two positive integers n and $k \geq 2$ are given, and a list of n integers is written in a row on a blackboard. Cinderella and her wicked Stepmother go through a sequence of rounds: Cinderella chooses a contiguous block of integers, and the Stepmother will either add 1 to all of them or subtract 1 from all of them. Cinderella can repeat this step as often as she likes, possibly adapting selections based on what her stepmother does. Prove that after a finite number of steps, Cinderella can reach a state where at least $n - k + 2$ of the numbers on the blackboard are all divisible by k.*

Solution. Before we start the proof, we prove the following lemma.

Lemma. Let n, k, t be three given positive integers such that $n \geq k-2 \geq 0$. Initially, a list of n integers $a_1, a_2, \cdots, a_n$ is written in a row on a blackboard where $0 < a_1 < k$. Whatever the Stepmother does, Cinderella can always make in finite time a state where $a_1 = 0$ or $a_1 = k$ or there exist at least $n - k + 2$ multiples of t among $a_2, \cdots, a_n$.

Proof. We prove that if the Stepmother never let $a_1 = 0$ or $a_1 = k$, then we can make at least $n - k + 2$ multiples of t by induction on (k, n) in lexicographical order.

(i) $k = 2$

Since $a_1 = 1$, Cinderella may choose only a_1 to force the Stepmother to make $a_1 = 0$ or $a_1 = 2$.

(ii) $n = k - 2$

There exists at lest $n - k + 2 = 0$ multiples of t among $a_2, \cdots, a_n$.

(iii) Assume that the statement is true for $k - 1$ or for $(k, n - 1)$ where $k \geq 3$ and $n \geq k - 1$.

Cinderella will use the following strategy. She first assumes that the Stepmother will never let $a_1 = 0$ or $a_1 = k - 1$ and apply the strategy used for $(k - 1, n - 1)$ on $a_1, \cdots, a_{n-1}$ without touching a_n. If the Stepmother actually does let $a_1 = k - 1$, then Cinderella choses the whole block from a_1 to a_n. After the Stepmother subtracts 1 from every number without any other choice, she forgets everything and again assumes that that the Stepmother will never let $a_1 = 0$ or $a_1 = k - 1$.

First suppose that the Stepmother will let $a_1 = k - 1$ for over t times. Since a_n is altered only when the Stepmother subtracts 1, among the t times, a_n would have been a multiple of t. At this state, Cinderella can dispose a_n and use the strategy for $(k, n - 1)$ on $a_1, \cdots, a_{n-1}$.

Suppose the contrary, that the Stepmother will not let $a_1 = k - 1$ after some time. Then by the induction hypothesis, Cinderella can make at least $(n - 1) - (k - 1) + 2 = n - k + 2$ multiples of t among $a_2, \cdots, a_{n-1}$. $\square$

Now we return to the original problem. We use an induction on n with k fixed.

(i) $n = k - 2$

It is obvious since $n - k + 2 = 0$.

(ii) Assume that it is true for $n - 1$ when $n \geq k - 1$.

Since we are interested only on whether a number is a multiple of k, we may replace each number to the remainder when divided by k. Since $0 \leq a_1 < k$, we may apply our lemma to prove that Cinderella can make either a_1 be a multiple of k or there exist at least $n-k+2$ multiples of k among $a_2, \cdots, a_n$. If a_1 becomes of multiple of k then Cinderella may dispose of a_1 and apply the induction hypothesis on $a_2, \cdots, a_n$. Also if $n - k + 2$ multiples of k are made, then we are done. $\square$

Comment. This problem is from the 2014 Canadian Mathematical Olympiad.

Problem 15. *For which n does there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that f takes each value exactly n times? ($|f^{-1}(x)| = n$ for all x)*

Solution. We prove that such f exists if and only if n is an odd integer. We first show that there exists an f if n is odd. Let

$$f(x) = -\cos\left(x - \pi n \left\lfloor \frac{x}{\pi n} \right\rfloor\right) + 2\left\lfloor \frac{x}{\pi n} \right\rfloor .$$

One can check that f is continuous since it is connected at $x = \pi k n$ where k is an integer.

Now suppose that n is even. Let $f^{-1}(0) = \{x_1, x_2, \cdots, x_n\}$ where $x_1 < x_2 < \cdots < x_n$. Let $a_0 = x_1 - 1$, $a_n = x_n + 1$ and $a_i = (x_i + x_{i+1})/2$ for $i = 1, \cdots, n-1$. Since every $a_0, \cdots, a_n, x_1, \cdots, x_n$ are distinct, $f(b_i) \neq 0$ for every i. Therefore we can find an $\epsilon > 0$ such that $\epsilon < f(b_i)$ for all i.

Let us consider the solutions of $f(x) = \pm\epsilon$. By the condition $|f^{-1}(x)| = n$, there exist exactly $2n$ solutions. On the other hand, by the intermediate value theorem, there exists a solution in each of the $2n$ non-intersecting intervals $(a_0, x_1), (x_1, a_1), (a_1, x_2), \cdots, (x_n, a_n)$. Hence there should exist a unique solution in each of those intervals. Let $\xi_1 \in (a_0, x_1)$, $\xi_1 \in (x_1, a_1)$, $\cdots$, $\xi_{2n} \in (x_n, a_n)$ be those solutions. It is obvious that $f(\xi_1), f(\xi_2), \cdots, f(\xi_{2n})$ consists of n ϵ's and n $-\epsilon$'s.

If $f(a_1) > 0$, then (x_1, a_1) and (a_1, x_2) each must contain a solution of $f(x) = \epsilon$ since $f(x_1) = f(x_2) = 0 < \epsilon < f(a_1)$. Hence $f(\xi_2) = f(\xi_3) = \epsilon$. On the contrary, $f(\xi_2) = f(\xi_3) = -\epsilon$ if $f(a_1) < 0$. Therefore in either cases, the number of ϵ's in $\{f(\xi_2), f(\xi_3)\}$ must be even. Likewise, the number of ϵ's in $\{f(\xi_{2i}), f(\xi_{2i+1})\}$ should also be even for $i = 1, \cdots, n-1$. But since the total number of ϵ's among $f(\xi_i)$s is n, which is even, $\{f(\xi_1, f(\xi_{2n})\}$ also must contain even number of ϵ's. Thus $f(\xi_1) = f(\xi_{2n}) = \pm\epsilon$.

If $f(\xi_1) = f(\xi_{2n}) = \epsilon$, it means that $f(x) > 0$ for all $x \in (-\infty, x_1) \cup (x_n, \infty)$. Then f would obviously be bounded below since f which is a continuous function attains a negative value only when $x \in (x_1, x_n)$. Hence it contradicts with the condition that $|f^{-1}(x)| = n$ for all x. If $f(\xi_1) = f(\xi_{2n}) = -\epsilon$, then f would be bounded above and it will also lead to a contradiction. Hence n must be odd. $\square$

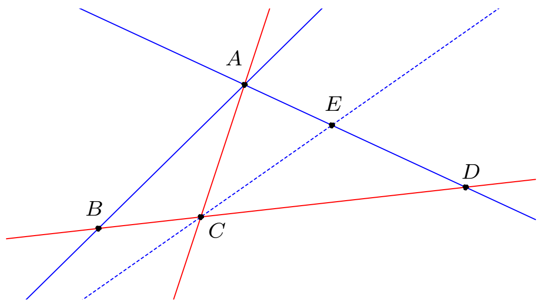
Problem 16. *Let $\mathcal{S}$ be a finite set of points in the plane (not all collinear), each colored red or blue. Show that there exists a monochromatic line passing through at least two points of $\mathcal{S}$.*

Solution. Suppose the contrary, and consider the projective dual of this situation. Then the lines of $\mathcal{L}$ are not all concurrent at one point, and there doesn't exist a point which has only one color of at least two lines of $\mathcal{L}$ passing through it.

Since the lines of $\mathcal{L}$ do not pass through one point, there exist at least three lines in $\mathcal{L}$. Suppose that there are at least two blue lines ℓ_1 and ℓ_2, and let $A = \ell_1 \cap \ell_2$. By the condition, there exists a red line ℓ_3 passing A_0. If all red lines pass through A_0, then for every blue $\ell \in \mathcal{L}$, the intersection $\ell \cap \ell_1$ must be A_0 since a red line must pass it. Hence every line passes through A_0 and we arrive at a contradiction. Therefore some red line ℓ_4 must not pass through A_0. Let $B_0 = \ell_1 \cap \ell_4$, $D_0 = \ell_2 \cap \ell_4$, and $C_0 = \ell_3 \cap \ell_4$.

Now let $\omega \notin \mathcal{L}$ be a line such that exactly one of B_0 or D_0 lies between C_0 and $\omega \cap \ell_4$. Set ω at the line at infinity, and delete ω to make a real Euclidean plane. Then by our setting, C_0 must lie between B_0 and D_0.

For a triangle XYW and a point Z on the segment YW, call such (X, Y, Z, W) a *good configuration* if XY and XW has the same color and XZ and YW has the opposite same color. By our setting of the line at infinity, there must exist at least one good configuration. Let (A, B, C, D) be the good configuration with the smallest area of ABD.



Without loss of generality, let AB, AD blue and AC, BD red. By our condition, there should exist a blue line ℓ passing through C. Since B and D are on either side, ℓ must intersect either segment AB or AD. Suppose ℓ intersects AD at E. Then we have a smaller good configuration (C, A, E, D) and hence a contradiction. □

Comment. This statement is commonly known as the Motzkin-Rabin Theorem.

Problem 17. *Does there exist an infinite set M consisting of positive integers such that for any distinct $a, b \in M$, the sum $a + b$ is square-free?*

Solution. We shall prove that it is possible to construct such M inductively.

Let $S \subset \mathbb{Z}^+$ be a finite set such that all the elements of $S+S$ are square-free. It is sufficient to prove that there exists an integer k such that $(S \cup \{k\}) + (S \cup \{k\})$ contains only square-free elements.

Let p be an arbitrary prime. Since $S+S$ does not contain a multiple of p^2, either $x \equiv 1 \pmod{p^2}$ or $x \equiv p^2-1 \pmod{p^2}$ does not have a solution in S. Let $n_p \in \{1, p^2-1\}$ be the number which makes $x \equiv n_p \pmod{p^2}$ have no solution in S.

Since the sum of the inverse square of primes converges, there exists a sufficiently large constant N such that

$$\sum_{p>N} \frac{1}{p^2} < \frac{1}{2(|S|+1)}.$$

Let $A = \prod_{p \le N} p^2$. By the Chinese remainder theorem, there exists a number B such that $x \equiv B \pmod{A}$ becomes a common solution of $x \equiv -n_p \pmod{p^2}$ for all primes p not exceeding N. Then for any integer $k = At+B$, the set $(S \cup \{k\}) + (S \cup \{k\})$ will not contain any multiples of p^2 where $p \le N$ is a prime number.

Now fix a sufficiently large integer M. Let $I = \{1, 2, \cdots, M\}$ and define

$$A_p = \{t \le M : p^2 | x + At + B \text{ for some } x \in S \cup \{0\}\}$$

for any prime $p > N$. If $t \notin A_p$ for any p, then $k = At+B$ will be the number such that $(S \cup \{k\}) + (S \cup \{k\})$ contains only square-free elements, which we are looking for. Since A is relatively prime with p, there exists a unique solution to $x + At + B \equiv 0 \pmod{p^2}$ where t is the variable. Therefore,

$$|A_p| \le (|S|+1) \left\lceil \frac{M}{p^2} \right\rceil \le \frac{M(|S|+1)}{p^2} + |S| + 1.$$

Also note that $|A_p| = 0$ if $p > \sqrt{\max\{S\} + AM + B}$.

Now we shall give a lower bound to the number of elements of $I - \bigcup_{p>N}$.

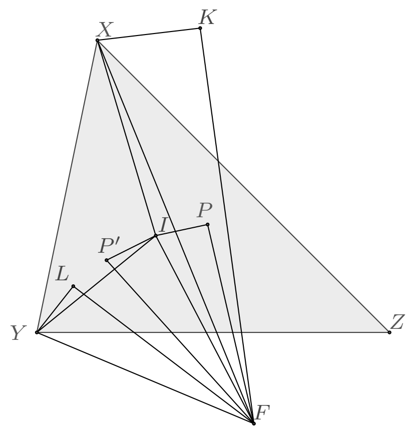
$$\begin{aligned}
\left| I - \bigcup_{p>N} A_p \right| &= \left| I - \bigcup_{N<p\le\sqrt{\max\{S\}+AM+B}} A_p \right| \\
&\le |I| - \sum_{N<p\le\sqrt{\max\{S\}+AM+B}} |A_p| \\
&\le M - M(|S|+1) \sum_{N<p} \frac{1}{p^2} - (|S|+1)\sqrt{\max\{S\}+AM+B} \\
&> \frac{M}{2} - (|S|+1)\sqrt{\max\{S\}+AM+B}
\end{aligned}$$

Hence if M is sufficiently large, $I - \bigcup_{p>N} \neq \emptyset$. Therefore there exists a number t which is in non of the A_ps, and for such t, every element of $(S \cup \{k\}) + (S \cup \{k\})$ is square-free where $k = At + B$. $\square$

Comment. This problem is from the 2011 International Mathematics Competition for University Students.

Problem 18. *Let Ω be the circumcircle and ω be the incircle of a triangle, and let P a arbitrary point inside ω. For a point X on Ω, let XY and XZ be chords of Ω which are tangent to ω. By CAlGeN 2014-4, we know that YZ is also tangent to ω. Find the locus of the isogonal conjugate of P respect to triangle XYZ where X moves around Ω.*

Solution.



Let P' be the isogonal conjugate of P respect to $\triangle XYZ$. Let F be a point which makes triangles $\triangle FPI$ and $\triangle FIP'$ similar, and let K, L be two points which make $\triangle FKX$, $\triangle FPI$, $\triangle FLY$ all similar. We shall first prove that F lies on the circumcircle of $\triangle XYZ$.

Since $\triangle FKX \sim \triangle FPI \sim \triangle FIP'$, the point F is the center of the spiral similarity of $\triangle KPI \sim \triangle XIP'$. Hence $\angle IKP = \angle P'XI = \angle PXI$ and

X, K, P, I are concyclic. Likewise, Y, L, I, P are also concyclic. Therefore,

$$\begin{aligned}
\angle FXZ &= \angle IXZ + \angle FXK - \angle IXK \\
&= \tfrac{1}{2}\angle X + \angle FIP - (180^\circ - \angle KPI) \\
&= \tfrac{1}{2}\angle X + \angle FIP + \angle XIP' - 180^\circ \\
&= \tfrac{1}{2}\angle X + \angle FIP + \angle XIY - \angle YIP' - 180^\circ \\
&= -\tfrac{1}{2}\angle Y + \angle FIP - \angle LPI \\
&= -\angle IYZ + \angle FYL - \angle LYI = \angle FYZ
\end{aligned}$$

and thus F lies on Ω.

Now set a complex coordinate which makes Ω the unit circle. Let $c \in \mathbb{C}$ be the center of ω and let p, p' be P, P' respectively. If f denotes the coordinate of F, then $f\bar{f} = 1$ since F is on the unit circle. Also, $\triangle FPI \sim \triangle FIP'$ implies

$$\frac{c-f}{p-f} = \frac{p'-f}{c-f}$$

and some calculations shows that

$$p' = f\frac{\bar{p}-\bar{f}}{p-f}\frac{(p-c)^2}{|p|^2-1} + \frac{\bar{p}c^2+p-2c}{|p|^2-1}.$$

Since the $|f| = 1$, it is obvious that p' lies on the circle of radius $\frac{|p-c|^2}{1-|p|^2}$ around $\frac{\bar{p}c^2+p-2c}{|p|^2-1}$. Hence the locus of P' is a circle. $\square$

Comment. This proof was given by Alexander Skutin in the article "On Rotation of a Isogonal Point," *Jour. of Classical Geom.* **2**(2013), 66-67.

Problem 19. *Let $n \geq 2$ be a positive integer and let $A_1, \cdots, A_m$ be subsets of $\{1, 2, \cdots, n\}$ such that $|A_i| \geq 2$ for all i and $A_i \cap A_j \neq \emptyset$, $A_j \cap A_k \neq \emptyset$, $A_k \cap A_i \neq \emptyset$ imply $A_i \cap A_j \cap A_k \neq \emptyset$. Prove that $m \leq 2^{n-1} - 1$.*

Solution. We prove by induction on n. Since the case $n = 2$ is obvious, we assume that our statement holds for $n = 2, \cdots, n-1$ and that $n \geq 3$. We shall consider two cases.

Case 1. There do not exist two sets A_i, A_j such that $|A_i \cup A_j| = n$ and $|A_i \cap A_j| = 1$.

For any set $S \subset \{1, \cdots, n-1\}$, the sets $S \cup \{n\}$ and $\{1, \cdots, n\} - S$ cannot coexist. Hence there are at most 2^{n-2} sets among $A_1, \cdots, A_m$ which contain n. On the other hand, there are at most $2^{n-2} - 1$ sets that does not contain

n by the induction hypothesis. Hence $m \le 2^{n-2} + 2^{n-2} - 1 = 2^{n-1} - 1$ in total.

Case 2. There exist two sets A_i, A_j such that $|A_i \cup A_j| = n$ and $|A_i \cap A_j| = 1$.

Let $A_i \cap A_j = \{x\}$. Let $|A_i| = r + 1$ and $|A_j| = s + 1$. It is obvious that $r + s = n - 1$ by the condition. Note that for any k, both $A_k \subseteq A_i$ and $A_k \subseteq A_j$ cannot both be true. Hence there are three possibilities: $A_k \subseteq A_i$, $A_k \subseteq A_j$, or $A_k \not\subset A_i, A_j$.

There are at most $2^r - 1$ sets A_k such that $A_k \subseteq A_i$ by the induction hypothesis. Likewise, there are at most $2^s - 1$ sets contained in A_j. If $A_k \not\subset A_i, A_j$, then $A_k \cap A_i \neq \emptyset$ and $A_k \cap A_j \neq \emptyset$ since $A_i \cup A_j = \{1, \cdots, n\}$. Hence by the condition of the problem, we have $A_i \cap A_j \cap A_k \neq \emptyset$ or $x \in A_k$. Now $A_k = \{x\} \cup (A_k - A_i) \cup (A_k - A_j)$, and these three sets are nonempty and disjoint. There are $2^s - 1$ possibilities for $A_k - A_i$ since $n - |A_i| = s$, and there are $2^r - 1$ possibilities for $A_k - A_j$. Hence there are at most $(2^r - 1)(2^s - 1)$ possible k's such that $A_k \not\subset A_i, A_j$. By summing up these upper bounds, we obtain $m \le (2^r - 1) + (2^s - 1) + (2^r - 1)(2^s - 1) = 2^{n-1} - 1$. □

Problem 20. *Let $S(n)$ be the sum of digits of n when represented by base 10. Show that*

$$\lim_{n\to\infty} S(2^n) = \infty.$$

Solution. Suppose that 2^n when represented by base 10 is $\overline{a_k a_{k-1} \cdots a_1 a_0}$. Suppose that $k \ge m$ but $a_{m-1}, \cdots, a_{t+1}, a_t$ are all 0. Since $\overline{a_{m-1} a_{m-2} \cdots a_0} = \overline{a_{t-1} \cdots a_0}$ is a multiple of 2^m and is definitely not 0, we have $\overline{a_{t-1} \cdots a_0} \ge 2^m$. Therefore $10^t > 2^m$ and hence

$$\frac{m}{t} < \log_2 10 < 4.$$

Now for each $t \ge 1$, at least one of $a_t, a_{t+1}, \cdots, a_{4t-1}$ is not zero. Thus

$$S(2^n) \ge \lfloor \log_4 k \rfloor \ge \log_4 k - 1 > \log_4(n \log_{10} 2 - 1) - 1$$

and hence

$$\lim_{n\to\infty} S(2^n) = \infty.$$

□

Problem 21. *A positive integer n is given. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy*

$$f(a_1) + f(a_2) + \cdots + f(a_n) = f(b_1) + f(b_2) + \cdots + f(b_n)$$

for arbitrary real numbers $a_1, b_1, \cdots, a_n, b_n, c$ such that $P(x) = (x - a_1)(x - a_2)\cdots(x - a_n) = (x - b_1)(x - b_2)\cdots(x - b_n) + c$.

Solution. If there is a real number c such that

$$P(x) = (x - a_1)(x - a_2)\cdots(x - a_n) = (x - b_1)(x - b_2)\cdots(x - b_n) + c,$$

let us say for convenience that a_i, b_i satisfies the condition $\mathcal{C}(n)$. Also, if a function f satisfies the condition in the problem for n, let us say that f has property $\mathcal{P}(n)$. We first prove the following lemma and then the two claims.

Lemma. Let $f : (\alpha, \beta) \to \mathbb{R}$ be a function with property $\mathcal{P}(n)$ and suppose that $f(x) = p(x)$ if $x \in (\gamma, \delta)$ for a polynomial p with degree at most $n - 1$ and a subinterval $(\gamma, \delta) \subset (\alpha, \beta)$. Then $f(x) = p(x)$ for all $x \in (\alpha, \beta)$.

Proof. Since $p(x)$ is a polynomial of degree less than n, it obviously has the property $\mathcal{P}(n)$. Therefore we can define a new function g on (α, β) such that $g(x) = f(x) - p(x)$. Then g will also have property $\mathcal{P}(n)$ and $g(x) = 0$ for all $x \in (\gamma, \delta)$.

Now suppose that there exists a value x such that $g(x) \neq 0$. Without loss of generality, assume that $x \leq \gamma$ and consider the supremum

$$\gamma_0 = \sup\{x \leq \gamma : g(x) \neq 0\}.$$

Obviously, there exist a_i, b_i such that $\gamma_0 < a_i, b_i < \delta - \epsilon_0$ except for $a_1 = \gamma_0 - \epsilon_0/2$ where $\epsilon_0 > 0$ is sufficiently small. Since $a_i + \epsilon, b_i + \epsilon$ also satisfies $\mathcal{C}(n)$ for $0 \leq \epsilon \leq \epsilon_0/2$, we have $\sum g(a_i + \epsilon) = \sum g(b_i + \epsilon)$ and hence $g(a_1 + \epsilon) = 0$ since $a_2 + \epsilon, \cdots, a_n + \epsilon, b_i + \epsilon \in (\gamma_0, \delta)$. Therefore $g(x) = 0$ for all $x \in [\gamma_0 - \epsilon_0/2, \gamma_0]$, and we arrive at a contradiction because of the definition of γ_0. $\square$

Claim 1. *If a continuous function $f : (a, b) \to \mathbb{R}$ has property $\mathcal{P}(2^n)$, then f must be a polynomial with degree at most $2^n - 1$.*

Proof. We shall prove by induction on n. It is trivial for $n = 0$. Assume the statement for $n - 1$ where $n \geq 1$. Let $m = (a + b)/2$ and define a new function $g : [0, (b - a)^2/4) \to \mathbb{R}$ as

$$g(x^2) = f(m - x) + f(m + x).$$

Suppose that $a_i, b_{i\,(i=1,\cdots,2^{n-1}}$ are sequences satisfying $\mathcal{C}(2^{n-1})$. Then it is easy to check that $m\pm\sqrt{a_i}, m\pm\sqrt{b_i}$ satisfy $\mathcal{C}(2^n)$. Since that implies $\sum f(m\pm\sqrt{a_i}) = \sum f(m\pm\sqrt{b_i})$ which can be expressed in terms of g as $\sum g(a_i) = \sum g(b_i)$, it is verified that g has property $\mathcal{P}(2^{n-1})$. Thus by our inductive hypothesis, we get that g is a polynomial with degree at most $2^{n-1}-1$. ($g(0)$ follows naturally since g is continuous)

Let $\epsilon > 0$ a positive real sufficiently smaller than $(b-a)/2^{n+3}$. By considering a similar function $h : [0, (b-a-2\epsilon)^2/4) \to \mathbb{R}$ defined by $h(x^2) = f(m+\epsilon-x)+f(m+\epsilon+x)$, we can prove by the same arguments that h is also a polynomial with degree at most $2^{n-1}-1$. Since $f(x+2\epsilon)-f(x) = h((x-m+\epsilon)^2) - g((x-m)^2)$ becomes a polynomial of degree at most 2^n-2, there exists a polynomial $p_1(x)$ with degree at most 2^n-1 such that $f(m+2\epsilon k) = p(m+2\epsilon k)$ for all integers k whose absolute value is smaller than about $(b-a)/(4\epsilon) \gg 2^n$.

Now let l be any positive integer. Using ϵ/l instead of ϵ, we deduce that there exists another polynomial p_l with degree at most 2^n-1 such that $f(m+2\epsilon k/l) = p_l(m+2\epsilon k/l)$ for not too big k. Since

$$p(m+2\epsilon k) = f(m+2\epsilon k) = f(m+2\epsilon kl/l) = p_l(m+2\epsilon k)$$

for all integers with absolute value smaller than 2^n, the polynomial $p-p_l$ has more than 2^n zeros while having a degree at most 2^n-1. Hence p_l is identical to p and we have $f(m+2\epsilon k/l) = p(m+2\epsilon k/l)$ for all integers k, l such that $|k/l| < c$ for some constant $c > 0$. Since f is continuous, the fact that $f(m+q) = p(m+q)$ for every $q \in \mathbb{Q}\cap(-c,c)$ implies that $f(x) = p(x)$ for every $x \in (m-c, m+c)$. Now our lemma implies that f is a polynomial with degree at most 2^n-1 in (a,b). $\square$

Claim 2. *Property $\mathcal{P}(n)$ implies property $\mathcal{P}(n+1)$.*

Proof. Let f be a function with property $\mathcal{P}(n)$. We shall prove that f also possesses $\mathcal{P}(n+1)$.

Consider two sequences a_i, b_i satisfying $\mathcal{C}(n+1)$ and suppose without loss of generality that $a_1 \geq a_2 \geq \cdots \geq a_{n+1}$ and $b_1 \geq \cdots \geq b_{n+1}$. Also, we may assume that $a_1 > b_1$. Then by the condition of $\mathcal{C}(n+1)$, we have

$$\cdots \leq a_4 < b_4 \leq b_3 < a_3 \leq a_2 < b_2 \leq b_1 < a_1.$$

Let $p(x) = (x-a_1)(x-a_2)\cdots(x-a_{n+1}) = (x-b_1)\cdots(x-b_{n+1}) - c$ and consider the equation $p(x) = c(x-a_1)/(a_1-b_{n+1})$. By the intermediate value theorem, there exists a solution between a_i and b_i exclusive for $i =$

$2, 3, \cdots, n$. Let c_i be the solution. Then $a_1, c_2, c_3, \cdots, c_n, b_{n+1}$ are $n+1$ distinct roots for $p(x) - c(x-a_1)/(a_1 - b_{n+1})$. Hence

$$p(x) = (x-a_1)(x-c_2)\cdots(x-c_n)(x-b_{n+1}) + c\,\frac{x-a_1}{a_1-b_{n+1}}$$

since $p(x)$ is monic.

We now consider the sum $\sum f(a_i)$. Since

$$\begin{aligned}\frac{p(x)}{x-a_1} &= (x-a_2)\cdots(x-a_{n+1}) \\ &= (x-c_2)\cdots(x-c_n)(x-b_{n+1}) + \frac{c}{a_1-b_{n+1}}\end{aligned}$$

the sequences $a_2, \cdots, a_{n+1}$ and $c_2, \cdots, c_n, b_{n+1}$ satisfies $\mathcal{C}(n)$. And by the property $\mathcal{P}(n)$, we obtain

$$\sum_{i=1}^{n+1} f(a_i) = f(a_1) + \sum_{i=2}^{n} f(c_i) + f(b_{n+1}).$$

Similarly,

$$\begin{aligned}\frac{p(x)+c}{x-b_{n+1}} &= (x-b_1)\cdots(x-b_n) \\ &= (x-a_1)(x-c_2)\cdots(x-c_n) + \frac{c}{a_1-b_{n+1}}\end{aligned}$$

and we obtain

$$f(a_1) + \sum_{i=2}^{n} f(c_i) + f(b_{n+1}) = \sum_{i=1}^{n+1}.$$

Therefore $\sum f(a_i) = \sum f(b_i)$ for all sequences a_i, b_i satisfying $\mathcal{C}(n)$ and thus f also has property $\mathcal{P}(n+1)$. $\square$

We now turn to the original problem. Since f has property $\mathcal{P}(n)$ and $n < 2^n$, one can prove that it also has property $\mathcal{P}(2^n)$ by applying *Claim 2* multiple times. Then *Claim 1* tells us that f must be a polynomial with degree at most $2^n - 1$. A quick check shows that the coefficients of $x^n, x^{n+1}, \cdots$ must be zero, and that f satisfies $\mathcal{P}(n)$ if f is a polynomial with degree at most $n-1$. Hence, f has property $\mathcal{P}(n)$ if and only if f is a polynomial with degree at most $n-1$. $\square$

Problem 22. *Let $k \geq 1$ and let $I_1, \cdots, I_k$ be non-degenerate open subintervals of* $[0,1]$. *Prove*

$$\sum \frac{1}{|I_i \cup I_j|} \geq k^2$$

where the summation is over all pairs (i, j) of indices such that I_i and I_j are not disjoint.

Solution. For convenience, let

$$e(i,j) = \begin{cases} 0 & \text{if } I_i \cap I_j = \emptyset \\ 1 & \text{if } I_i \cap I_j \neq \emptyset \end{cases}$$

Also, for every interval I_i, define a function $f_i : [0,1] \to \mathbb{R}$ as

$$f_i(x) = \begin{cases} 1 & \text{if } x \notin I_i \\ 1 - \frac{1}{|I_i|} & \text{if } x \in I_i \end{cases}$$

We now consider the value of $\int_0^1 f_i(x) f_j(x)\, dx$.

Suppose that $|I_i \cap I_j| = c$. Since the value of $f_i(x)f_j(x)$ is 1 for length $1 - |I_i| - |I_j| + c$, $1 - \frac{1}{|I_i|}$ for length $|I_i| - c$, $1 - \frac{1}{|I_j|}$ for length $|I_j| - c$, and $(1 - \frac{1}{|I_i|})(1 - \frac{1}{|I_j|})$ for length c, we have

$$\begin{aligned} \int_0^1 f_i(x) f_j(x)\, dx &= (1 - |I_i| - |I_j| + c) + (|I_i| - c)\left(1 - \frac{1}{|I_i|}\right) \\ &\quad + (|I_j| - c)\left(1 - \frac{1}{|I_j|}\right) + c\left(1 - \frac{1}{|I_i|}\right)\left(1 - \frac{1}{|I_j|}\right) \\ &= -1 + \frac{c}{|I_i||I_j|} \end{aligned}$$

We first prove that

$$\int_0^1 f_i(x) f_j(x)\, dx \le -1 + \frac{e(i,j)}{|I_i \cup I_j|}.$$

Case 1. I_i and I_j are disjoint.
Since $c = 0$, we have

$$\int_0^1 f_i(x) f_j(x)\, dx = -1 \le -1.$$

Case 2. I_i and I_j are not disjoint.

Since $|I_i| \ge c$ and $|I_j| \ge c$, we have $(|I_i| - c)(|I_j| - c) \ge 0$. Hence $|I_i||I_j| \ge c(|I_i| + |I_j| - c)$ and

$$\int_0^1 f_i(x) f_j(x)\, dx = -1 + \frac{c}{|I_i||I_j|} \le -1 + \frac{1}{|I_i| + |I_j| - c} = -1 + \frac{1}{|I_i| \cup |I_j|}.$$

Now summing up these inequalities implies

$$0 \leq \int_0^1 \left(\sum_{i=1}^k f_i(x) \right)^2 dx = \sum_{i,j=1}^k \int_0^1 f_i(x) f_j(x)\, dx$$
$$\leq \sum_{i,j=1}^k \left(-1 + \frac{e(i,j)}{|I_i \cup I_j|} \right) = -k^2 + \sum_{i,j=1}^k \frac{e(i,j)}{|I_i \cup I_j|}$$

and hence

$$\sum_{I_i \cap I_j \neq \emptyset} \frac{1}{|I_i \cup I_j|} \geq k^2$$

□

Problem 23. *Let P be a convex polygon such that the distance between any two vertices does not exceed 1. Prove that the perimeter of P does not exceed π.*

Solution. Let $\ell(\theta)$ be the line obtained by rotating the x-axis through θ about the origin, and for an object $\mathcal{O}$, let $\ell(\mathcal{O}, \theta)$ be the length of the projection of $\mathcal{O}$ on $\ell(\theta)$. It is then obvious that $\ell(\mathcal{O}, \theta) = \ell(\mathcal{O}, \theta + \pi)$.

Let $s_1, \cdots, s_n$ be the sides of P. First note that

$$\int_0^\pi \ell(s_i, \theta)\, d\theta$$

is twice the length of s_i since $\int_0^\pi \cos x\, dx = 2$. Hence

$$\int_0^\pi (\ell(s_1, \theta) + \cdots + \ell(s_n, \theta))\, d\theta$$

is the twice of the perimeter of P.

On the other hand, $\ell(s_1, \theta) + \cdots + \ell(s_n, \theta) = 2\ell(P, \theta) \leq 2$ since the diameter of P less or equal to 1. Therefore

$$\int_0^\pi (\ell(s_1, \theta) + \cdots + \ell(s_n, \theta))\, d\theta \leq \int_0^\pi 2\, d\theta = 2\pi$$

and thus the perimeter of P does not exceed π. □

Problem 24. *Let m, n be relatively prime positive integers, and $x > 0$ a real number. Prove that if $x^m + x^{-m}$ and $x^n + x^{-n}$ are both integers, then $x + x^{-1}$ is also an integer.*

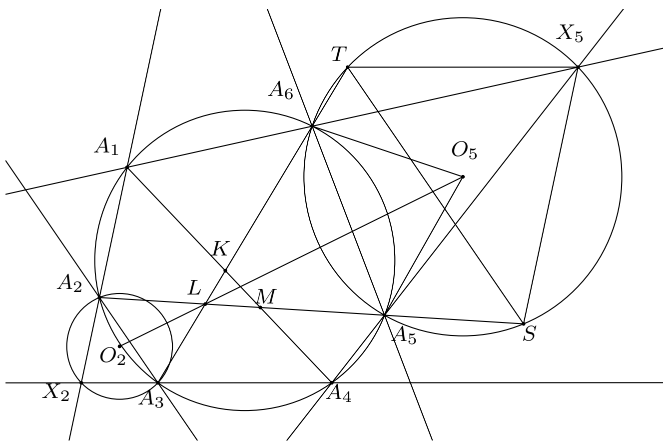
Solution. We shall first prove that $x+x^{-1}$ is an rational number. Consider the minimal polynomial $p(x)$ of x over the field of rationals. Since $x^m + x^{-m} = a$ and $x^n + x^{-n} = b$ are both integers, $p(x)$ must divide both $x^{2m} - ax^m + 1$ and $x^{2n} - bx^n + 1$.

The zeros of $x^{2m} - ax^m + 1 = 0$ in $\mathbb{C}$ are $\zeta_m^k((a \pm \sqrt{a^2-4})/2)^{1/m}$, where $\zeta_m = e^{2\pi i/m}$. Likewise, the zeros of $x^{2n} - bx^n + 1$ are $\zeta_n^k((b \pm \sqrt{b^2-4})/2)^{1/n}$ where $\zeta = e^{2\pi i/n}$. Since $p(x)$ divides both polynomials, the zeros of $p(x)$ must be zeros of both polynomials. However, looking at the argument of the complexes, we obtain that only the real numbers can be the zeros of both, since m and n are relatively prime. Therefore, the degree of $p(x)$ is either 1 or 2. In both cases, $x + x^{-1}$ becomes a rational.

Suppose $x+x^1 = p/q$, where p and q are relatively prime positive integers. Since $x^2+x^{-2} = (p/q)^2-2 = (p^2-2q^2)/q^2$, the denominator of x^2+x^{-2} in its simplest form is q^2. In a similar manner, using $(x^n+x^{-n})(x+x^{-1}) = (x^{n+1}+ x{-}n-1) + (x^{n-1} + x^{-n+1})$, one can easily show that the denominator of x^n+x^{-n} is q^n. Since x^m+x^{-m} is an integer, q must be 1 and $x+x^{-1} \in \mathbb{Z}$. $\square$

Problem 25. *Let $A_1, A_2, A_3, A_4, A_5, A_6$ be six points on a circle, and $X_i = A_iA_{i-1} \cap A_{i+1}A_{i+2}$ where the indices are modulo 6. Denote by O_i the circumcenter of the triangle $\triangle X_iA_iA_{i+1}$. Prove that the three lines O_1O_4, O_2O_5, O_3O_6 are concurrent.*

Solution.



Let K, L, M be the intersections of the diagonals A_1A_4, A_2A_5, A_3A_6, and let T, S be the second intersections of lines A_3A_6, A_2A_5 and circle O_5. Since $\angle A_3A_2A_5 = \angle A_3A_6A_5 = \angle A_5ST$, the lines A_2A_3 and ST are parallel. X_5T and X_2A_3 are also parallel since $\angle A_3TX_5 = \angle A_6A_5A_4 = \angle X_2A_3A_6$, and likewise, SX_5 and A_2X_2 too are parallel. Hence $\triangle STX_5$ and $\triangle A_2A_3X_2$ have a center of similarity, which must be L, and thus O_2, O_5, L are collinear.

Now we calculate the ratio of the sine values in order to apply Ceva's theorem. Since

$$\frac{\sin\angle A_6LO_5}{\sin\angle O_5LA_5}\cdot\frac{\sin\angle LA_5O_5}{\sin\angle O_5A_5A_6}\cdot\frac{\sin\angle A_5A_6O_5}{\sin\angle O_5A_6L}=1,$$

we have

$$\frac{\sin\angle A_6LO_5}{\sin\angle O_5LA_5}=\frac{\sin\angle O_5A_6L}{\sin\angle LA_5O_5}=\frac{\cos|A_4A_6-A_1A_3|}{\cos|A_1A_5-A_2A_4|},$$

where the A_iA_j represents $\angle A_iOA_j/2$. The product of the three values equals 1 because the expression is cyclic, and thus O_1O_4, O_2O_5, O_3O_6 are concurrent. □

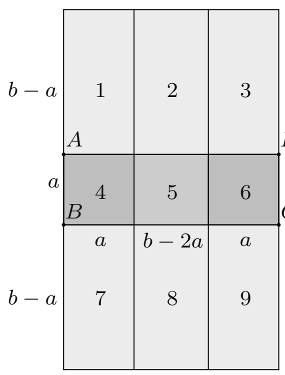
Problem 26. *A real number is written in every unit square of an infinite square grid. Suppose that for each square which sides are parallel to the axes, the absolute value of the sum of values inside the square does not exceed* 1*. Prove that the absolute value of the sum of values inside any rectangle which sides are parallel to the axes does not exceed* 4.

Solution. For convenience, let $P(a, b, x)$ be the statement that for all $a \times b$ rectangle(where $a \leq b$), the absolute value of the sum of the numbers in the rectangle does not exceed x.

We first prove the following lemma.

Lemma. If $P(|b-2a|, 2b-a, x)$ holds, then $P(a, b, (x+8)/3)$ holds.

Proof.



For a $a \times b$ rectangle, $ABCD$,

$$\begin{aligned}3(456) =& (4) + (6) + (123456) + (456789) + (258) \\ & - (12) - (23) - (78) - (89)\end{aligned}$$

and hence the absolute value in $ABCD$ does not exceed $(x + 8)/3$. $\square$

If $b = a$ or $b = 2a$, then the problem is obvious. We consider two cases: $b > 2a$ and $a < b < 2a$.

First suppose that $b > 2a$. Construct a sequence $\{x_n\}$ as $x_0 = ab$ and $x_{n+1} = (x_n + 32)/9$. Obviously, the sequence always converge to 4.

Since a $a \times b$ consists of ab unit squares, $P(a, b, x_0)$ holds. Also, if $P(a, b, x_n)$ holds, then $P(3a, 3b, x_n)$ holds(since we may construct a new square grid by merging every 3×3 blocks), and by our lemma, $P(b-2a, 2b-a, (x_n + 8)/3)$ holds, and again by our lemma, $P(a, b, x_{n+1})$ holds. Hence, $P(a, b, x_n)$ holds for every n, and since the sequence converges to 4, $P(a, b, 4)$ must also hold.

If $a < b < 2a$, then construct two sequence $\{a_n\}, \{b_n\}$ such that $a_0 = a, b_0 = b$, and $a_{n+1} = 2a_n - b_n, b_{n+1} = 2b_n - a_n$. Since $a_n + b_n = a_{n+1} + b_{n+1}$ and $3(a_n - b_n) = a_{n+1} - b_{n+1}$, there must be a k such that $a_{k+1} < 0 < a_k$, and for that, b_k must be greater than $2a_k$.

Now we use our lemma again. As shown above, $P(a_k, b_k, 4)$ holds. Since $a_k = |2a_{k-1} - b_{k-1}|$ and $b_k = 2b_{k-1} - a_{k-1}$, we have $P(a_{k-1}, b_{k-1}, 4)$, and so on. Therefore, $P(a, b, 4) = P(a_0, b_0, 4)$ holds. $\square$

Problem 27. *Prove that there exists a constant $k_n > 0$ dependent on n such that for all $x_1, \cdots, x_n \in [1 - k_n, 1 + k_n]$, we have*

$$\frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_4} + \cdots + \frac{x_n}{x_1 + x_2} \geq \frac{n}{2}.$$

Solution. Let $a < 1$ be a constant which will be determined afterwards. If k_n is small enough, $x_i - \frac{a}{2}(x_{i+1} + x_{i+2}) > 0$ for all i. Suppose for now that this is the case.

Subtracting $na/2$ from each side, we easily see that the inequality is equivalent to

$$\sum_{i=1}^{n} \frac{x_i - \frac{a}{2}(x_{i+1} + x_{i+2})}{x_{i+1} + x_{i+2}} \geq \frac{n}{2}(1 - a).$$

Since Cauchy's inequality states that

$$\sum_{i=1}^{n} \frac{x_i - \frac{a}{2}(x_{i+1} + x_{i+2})}{x_{i+1} + x_{i+2}} \geq \frac{(1 - a)^2 (\sum x_i)^2}{\sum x_i(x_{i+1} + x_{i+2}) - \frac{a}{2} \sum (x_i + x_{i+1})^2},$$

it is enough to prove that

$$\frac{n}{2}(\sum x_i(x_{i+1}+x_{i+2}) - \frac{a}{2}\sum(x_i+x_{i+1})^2) \le (1-a)(\sum x_i)^2.$$

Noting that $\sum x_i(x_{i+1}+x_{i+2}) = \sum(x_i+x_{i+1})(x_{i+1}+x_{i+2}) - \frac{1}{2}\sum(x_i + x_{i+1})^2$, let us substitute $x_i + x_{i+1} = 2y_i$. Then the inequality we are to show becomes

$$n(2\sum y_i y_{i+1} - (a+1)\sum y_i^2) \le (1-a)(\sum y_i)^2.$$

Subtracting $n(1-a)\sum y_i^2$, we may simplify both sides to

$$-n\sum(y_i - y_{i+1})^2 \le -(1-a)\sum_{i<j}(y_i - y_j)^2.$$

Since for any $i<j$

$$n\left(\sum_{i=1}^{n}(y_i - y_{i+1})^2\right) \ge \left(\sum_{i=1}^{n}|y_i - y_{i+1}|\right)^2 \ge (y_i - y_j)^2,$$

adding these inequalities, we obtain

$$n\sum(y_i - y_{i+1})^2 \ge \frac{2}{n(n-1)}\sum_{i<j}(y_i - y_j)^2.$$

Hence $a = 1 - \frac{2}{n(n-1)}$ works, and in order for $x_i - \frac{a}{2}(x_{i+1}+x_{i+2})$ to always be positive, $k_n < (1-a)/(1+a)$ is sufficient. $\square$

Comment. This problem was proposed by Vasile Cîrtoaje and the presented solution was given by 'harazi'.

Problem 28. *For every positive integer n, prove that there exists a set $S \subseteq \{n^2+1, n^2+2, \cdots, (n+1)^2-1\}$ such that*

$$\prod_{x\in S} x = 2m^2$$

for some integer m.

Solution. Let $x = n+1-\sqrt{2n+1}$ and $y = n+1+\sqrt{2n+1}$. Then it is easily verifiable that $x+y = 2n+2$ and $xy = n^2$, and hence $\sqrt{x} - \sqrt{y} = \sqrt{2}$.

For any integer k such that $x \le k < k+1 \le y$, we first prove that there exists an integer c such that $n^2 \le ck < c(k+1) \le (n+1)^2$. Let $k = n-t$, where t is an integer. By the restriction on k, we have $-\sqrt{2n+1} \le t \le \sqrt{2n+1} - 1$. Hence we obtain $(t^2+2t)/2 \le n$.

We divide into two cases: when $(t^2+2t)/2 \le n \le t^2+t-1$ and when $t^2+t \le n$. If $(t^2+2t)/2 \le n \le t^2+t-1$ set $c = n+t+2$. Then $ck = (n+t+2)(n-t) = n^2+2n-t(t+2) \ge n^2$ and $c(k+1) = (n+t+2)(n-t+1) = n^2+3n-t^2-t+2 \le (n+1)^2$. Otherwise, if $t^2+t \le n$, let $c = n+t+1$. Then $ck = (n+t+1)(n-t) = n^2+n-t(t+1) \ge n^2$ and $c(k+1) = (n+t+1)(n-t+1) = (n+1)^2-t^2 \le (n+1)^2$. Thus in both cases there exists such c.

Since $\sqrt{x/2} - \sqrt{y/2} = \sqrt{2/2} = 1$, there exists an integer b such that $\sqrt{x/2} \le b \le \sqrt{y/2}$, or $x \le 2b^2 \le y$. Likewise, $x \le a^2 \le y$ for some integer a. Denote $m_1 = \min\{a^2, 2b^2\}$ and $m_2 = \max\{a^2, 2b^2\}$. For every $m_1 \le x \le m_2-1$, we have proved that there exists a c_x such that $c_x x, c_x(x+1) \in [n^2, (n+1)^2]$. Hence

$$\prod_{x=m_1}^{m_2-1} (c_x x)(c_x(x+1)) = a^2 \cdot 2b^2 \cdot (c_{m_1} \cdots c_{m_2-1} \cdot (m_1+1) \cdots (m_2-1))^2$$

is twice a square, and excluding the duplicates of $c_x x$ and $c_x(x+1)$ by pairs, we are done. □

Problem 29. $\mathbb{N}$ *is the set of nonnegative integers. For any subset S of $\mathbb{N}$, let $P(S)$ be the set of all* pairs *of members of S. (A* pair *is a unordered set of two distinct members) Partition $P(\mathbb{N})$ arbitrarily in to two sets P_1 and P_2. Prove that $\mathbb{N}$ must contain an infinite subset S such that either $P(S)$ is contained in P_1 or $P(S)$ is contained in P_2.*

Solution. Suppose that for any infinite subset of S, $P(S)$ has an element that is contained in P_1 and another element that is contained in P_2.

Let $A_1 = A_2 = \{1\}$. We will expand A_1 and A_2 so that $P(A_1)$ is contained in P_1 and $P(A_2)$ is contained in P_2. If there exists an element of $\mathbb{N}$ such that the pair of x with any member of A_1 is contained in P_1, then we put x in A_1. By this algorithm, A_1 will expand until every element of $\mathbb{N}$ has a pair with sum member of A_1 which is contained in P_2. This is guaranteed to happen in finite time by the hypothesis above. Likewise, A_2 will also expand until every element of $\mathbb{N}$ has a pair with sum member of A_2 which is contained in P_1.

Now A_1 and A_2 are finite subsets of $\mathbb{N}$, and for every $x \in \mathbb{N} \setminus (A_1 \cup A_2)$ there exist $a_1 \in A_1$ and $a_2 \in A_2$ such that $\{x, a_1\} \in P_2$ and $\{x, a_2\} \in P_1$. a_1 and a_2 is dependent of x. Since A_1 and A_2 are both finite, the number of possible tuples (a_1, a_2) is also finite. Thus there exists a specific tuple

(a_1, a_2) that appear infinitely many times. Let X denote the infinite set of xs that $\{x, a_1\} \in P_2$ and $\{x, a_2\} \in P_1$.

Since X is infinite, we can make this same argument again. Then we have two elements b_1 and b_2 in X such that the set of $y \in X$s such that $\{y, b_1\} \in P_2$ and $\{y, b_2\} \in P_1$ is infinite. Denote this set Y and use the argument over and over again. This constructs two infinite sets $S_1 = \{a_1, b_1, \cdots\}$ and $S_2 = \{a_2, b_2, \cdots\}$ such that $P(S_1) \in P_2$ and $P(S_2) \in P_1$. Hence we reach a contradiction. $\square$

Problem 30. *Let q be a rational number in the interval $(0, \frac{\pi^2}{6} - 1) \cup [1, \frac{\pi^2}{6})$. Prove that there exists integer $0 < x_1 < x_2 < \cdots < x_n$ such that*

$$q = \frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_n^2}.$$

Solution. We start with two lemmas.

Lemma 1. *Every integer $200 \leq x \leq 2000$ can be represented as a sum of distinct squares strictly less than 20^2.*

Proof. It is a matter of computation. $\square$

Lemma 2. *Let $n > 20$ be a positive integer and let $P = \{(s_1 s_2 \cdots s_k)^2 : 1 \leq s_1 < \cdots < s_k \leq n\}$. Then every integer $200 \leq x < (n!/6)^2$ can be represented as a sum of distinct elements of P.*

Proof. We first prove that there exists a sequence $(m_1, \cdots, m_s)$ consisting of elements of P such that $20^2 = m_1 < \cdots < m_t = (n!/6)^2$ and $m_{i+1}/m_i < 2$ for all $1 \leq i < t$.

Construct a finite sequence $L = (L_1, \cdots, L_s)$ of subsets of $\{3, 4, \cdots, n\}$ as follows.

(1) $L_1 = \{20\}$.

(2) Suppose that $L_k = \{v, v+1, \cdots, n\}$ for some $v \geq 5$. Then $L_{k+1} = \{3, 4, v+1, v+2, \cdots, n-1\}$.

(3) Suppose that $L_k = \{i_1, i_2, \cdots, i_h\}$ in increasing order and there exists a largest $m \leq h$ such that $i_m < n$ and $i_m + 1 \notin L_k$. Then $L_{k+1} = (L_k \setminus \{i_m\}) \cup \{i_m + 1\}$.

The formation of L_{k+1} from L_k is continued until the term $\{4, 5, \cdots, n\}$ is reached.

Now replace every term X by $(\prod X)^2$ to obtain a new sequence $L' = (L'_1, \cdots, L'_s)$. First of all, $L'_1 = 20^2$ and $L'_s = (n!/6)^2$. Also, for every $1 \le i < s$, the ratio L'_{i+1}/L'_i is either $(12/n)^2$ or $((v+1)/v)^2$ for some $v \ge 3$. Hence $L'_{i+1}/L'_i < 2$ for all i. Thus rearranging the sequence in increasing order and deleting duplicates, we obtain a strictly increasing sequence $(m_1, \cdots, m_t)$ whose ratio between consecutive terms does not exceed 2. Also $m_1 = 20^2$ and $m_t = (n!/6)^2$.

For any integer $200 \le x < m_t = (n!/6)^2$, let $y = x - 200$ and consider the maximum m_{a_1} not exceeding y. Since $m_{a_1+1} > y$ and $m_{a_1+1} < 2m_{a_1}$, it follows that $y - m_{a_1} < m_{a_1}$. Now consider the maximum m_{a_2} not exceeding $y - m_{a_1}$. Clearly $a_1 > a_2$, and repeating the previous argument, we have $y - m_{a_1} - m_{a_2} < m_{a_2}$. This procedure continues until $y - m_{a_1} - \cdots - m_{a_k} < m_1$ for some $a_1, \cdots, a_k$. Then $x - m_{a_1} - \cdots - m_{a_k} < m_1 + 200$. Hence it follows from Lemma 1 that there exists $1 \le b_1 < \cdots < b_l < 20$ such that

$$x - m_{a_1} - \cdots - m_{a_k} = b_1^2 + \cdots + b_l^2.$$

Since $b_i < 20$, they are all elements of P, and are distinct from $m_{a_1}, \cdots, m_{a_k}$. Thus x can be represented as a sum of distinct elements of P. $\square$

We now return to the original problem. It is sufficient to prove that for any rational $0 < q < \pi^2/6 - 1$, there exists $2 \le x_1 < \cdots < x_n$ such that $q = x_1^{-2} + \cdots + x_n^{-2}$. We again use the greedy algorithm. Let n_1^{-2} be the greatest square reciprocal not exceeding q, and let n_2^{-2} be the greatest square reciprocal aside from n_1^{-2}, and so on forth. Since $q < \zeta(2) - 1$, there exists a point where $n_t > n_{t-1} + 1$ and $n_t > 20$. Then

$$0 < q - \frac{1}{n_1^2} - \cdots - \frac{1}{n_t^2} < \frac{1}{(n_t-1)^2} - \frac{1}{n_t^2} = \frac{2n_t - 1}{n_t^2(n_t-1)^2} < \frac{1}{n_t^2}.$$

On the other hand, there exists a sufficiently large integer N such that $(N!)^2(q - n_1^{-2} - \cdots - n_t^{-2})$ is an integer at least 200. Let $q - n_1^{-2} - \cdots - n_t^{-2} = R/(N!)^2$. Note that $R/(N!)^2 < 1/n_t^2 < 1/36$ or $200 < R < (N!/6)^2$. By Lemma 2, there exist distinct divisors $p_1, \cdots, p_s$ of $N!$ such that $R = p_1^2 + \cdots + p_s^2$. Then $R/(N!)^2 = (N!/p_1)^{-2} + \cdots + (N!/p_s)^{-2}$. Because $R/(N!)^2 < 1/n_t^2$, each of $N!/p_i$ must be greater that n_t and hence $n_1, \cdots, n_t, N!/p_1, \cdots, N!/p_s$ are all distinct. Thus

$$q = \frac{1}{n_1^2} + \cdots + \frac{1}{n_t^2} + \frac{1}{(N!/p_1)^2} + \cdots + \frac{1}{(N!/p_s)^2}$$

is a representation of q as a sum of distinct squares. $\square$

Comment. This is a result of Ronald Graham, "On Finite Sums of Unit Fractions", *Proc. London Math. Soc.* **14** (1964), 193-207.

Solutions - 2015

Problem 1. *There are attached n^2 bulbs in an $n \times n$ board, some of them are with light on. There is a switch related to each row and column of the board. Turning a switch to its other position, it changes the lights of the bulbs in the appropriate row or column to their opposite. Show that with a suitable chain of switchings one can achieve that the difference between the number of shining bulbs and the number of dark bulbs is at least $\sqrt{n^3/2}$.*

Solution. Let c_i be the difference between the number of 2 and dark bulbs in the ith column. Turning any switch related to a column does not affect the value of c_i for any i, and moreover, after a certain sequence of turning switches related to columns, it is possible to attain a state in which the difference between the total number of shining and dark bulbs is $c_1 + \cdots + c_n$. Hence it is sufficient to show that after a suitable sequence of turning switches, it is possible to make $c_1 + \cdots + c_n \geq \sqrt{n^3/2}$.

Now for every switch associated to a row, turn it independently with probability $1/2$. Then for each column, the 2^n possibilities will occur with the same probability $1/2^n$. Hence the expected value of c_1 is

$$\mathrm{E}[c_1] = \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} |n - 2i| = \frac{n}{4^m} \binom{2m}{m}$$

where $m = \lfloor n/2 \rfloor$. It is easily verifiable that $\binom{2m}{m} \geq 4^m/\sqrt{4m}$ for all $m \geq 1$, and hence we have $\mathrm{E}[c_1] \geq n/\sqrt{4m} \geq \sqrt{n/2}$.

Adding it for all i, it follows that

$$\mathrm{E}(c_1 + \cdots + c_n) = \mathrm{E}(c_1) + \cdots + \mathrm{E}(c_n) \geq \sqrt{n^3/2}.$$

Thus there exists a sequence of turning switches related with rows such that the result satisfies $c_1 + \cdots + c_n \geq \sqrt{n^3/2}$. $\square$

Problem 2. *Let $a_1, a_2, \cdots, a_n$ be real numbers and*

$$f(x) = \cos a_1 x + \cos a_2 x + \cdots + \cos a_n x.$$

Prove that there exists a positive integer $k \leq 2n$ for which

$$|f(k)| \geq \frac{1}{2}.$$

Solution. First note that $\cos a(x-1) - 2\cos a \cos ax + \cos a(x+1) = 0$ for all a and x. Let

$$P(u) = \prod_{j=1}^{n}(u^2 - 2(\cos a_j)u + 1) = b_0 + b_1 u + \cdots + b_{2n}u^{2n}.$$

Since $(u^2 - 2(\cos a_j)u + 1)$ is divides $P(u)$ for every j, the sum

$$b_0 f_j(x) + b_1 f_j(x+1) + \cdots + b_{2n} f(x+2n) = 0$$

where $f_j(x) = \cos a_j x$. Hence adding it for all j, we obtain the equality

$$b_0 f(x) + b_1 f(x+1) + \cdots + b_{2n} f(x+2n) = 0$$

which holds for every number x.

Now let b_m be the number with maximum absolute value among the numbers $b_0, \cdots, b_{2n}$. Letting $x = -m$ in the equality above, we obtain

$$b_m f(0) = \sum_{\substack{0 \le j \le 2n \\ j \ne m}} -b_j f(j-m).$$

Because b_m is the one with largest absolute value and because $f(x) = f(-x)$ for all x, we have

$$n = f(0) = \sum_{\substack{0 \le j \le 2n \\ j \ne m}} -\frac{b_j}{b_m} f(j-m) \le \sum_{\substack{0 \le j \le 2n \\ j \ne m}} |f(j-m)| \le 2n \max_{1 \le k \le 2n} |f(k)|.$$

Hence there exists a $1 \le k \le 2n$ for which $|f(k)| \ge 1/2$. □

Problem 3. *Alice and Bob play a game on a simple graph G with* 2015 *vertices. Alice first chooses a vertex and colors it red. After that, Bob and Alice alternatively picks an uncolored point adjacent to the last colored point, and color it red. The first who cannot pick an uncolored point loses. Prove that regardless of G, Alice always has the wining strategy.*

Solution. Consider the maximum matching of G and let it be (a_i, b_i) for $1 \le i \le m$. For convenience, let $M = \{(a_1, b_1), \cdots, (a_m, b_m)\}$ and $S = \{a_1, b_1, \cdots, a_m, b_m\}$. Since the number of vertices is odd, there exists a vertex v_0 which is not in S.

Let Alice first pick v_0. Then Bob shall pick a vertex u_0 connected to v_0 if such exists. Since (a_i, b_i) is the maximum matching, there is no edge connecting two points outside of S. Thus the vertex u_0, which Bob had picked, should be an element of S. If Bob has picked a_i for some i, then let Alice pick b_i, and if Bob has picked b_i for some i, let Alice pick a_i. Let this vertex Alice has picked be v_1.

Now, Bob should pick some vertex u_1 adjacent to v_1 which is still unoccupied. If u_1 is not in S, then (v_0, u_0), (v_1, u_1), and $M \setminus \{(u_0, v_1)\}$ becomes a bigger matching, contradicting our assumption. Thus u_1 must be a element of S, and this again allows Alice to pick v_2 so that (u_1, v_2) is an edge of M.

Repeating this argument, if Bob can pick an unoccupied vertex, then it must always be an element of S, and Alice always can pick the next vertex so that those two form an edge of M. Thus it must be Bob who first run out of choices and lose. $\square$

Problem 4. *Let α be a fixed positive number. Suppose that the set $\mathcal{A}$ consisting of positive integers satisfy*

$$|\mathcal{A} \cap \{1, 2, \cdots, n\}| \geq \alpha n$$

for every positive integer n. Prove that there exists a constant c such that every positive integer is the sum of at most c elements of $\mathcal{A}$.

Solution. For convenience, say that A is α-dense if $|A \cap \{1, 2, \cdots, n\}| \geq \alpha n$ for any positive integer n. Also, denote $A \oplus B = A \cup B \cup (A + B)$ where $A + B = \{a + b : a \in A,\, b \in B\}$.

Lemma. Let S be a σ-dense set and T a τ-dense set where $\sigma, \tau > 0$. Then $A \oplus B$ is a $\sigma + \tau - \sigma\tau$-dense set.

Proof. We shall prove that $|(S \oplus T) \cap \{1, \cdots, n\}| \geq (\sigma + \tau - \sigma\tau)n$ for all n. Let $1 = s_1 < \cdots < s_k$ be the all positive integer elements S not exceeding n. By the condition, $k \geq \sigma n$.

Since T is τ-dense, there is at least $\tau(s_{j+1} - s_j - 1)$ elements of T at most $s_{j+1} - s_j - 1$, and hence $S \oplus T$ has at least $\tau(s_{j+1} - s_j - 1) + 1$ elements in $[s_j, s_{j+1})$. Likewise, there are at least $\tau(n - s_k) + 1$ elements in $[s_k, n]$. Adding those up, the total number of elements of $S \oplus T$ not exceeding n is

$$\tau(n - s_1 - (k - 1)) + k = \tau n + (1 - \tau)k \geq \tau n + (1 - \tau)\sigma n = (\tau + \sigma - \tau\sigma)n.$$

$\square$

Denote $\mathcal{A}^{[k]}$ by the sum $\mathcal{A} \oplus \mathcal{A} \oplus \cdots \oplus \mathcal{A}$, where $\mathcal{A}$ is added k times. It is sufficient to prove that for some c, the set $\mathcal{A}^{[k]}$ is equal to the set of positive integers. Applying our lemma multiple times, we get that $\mathcal{A}^{[k]}$ is a $1-(1-\alpha)^k$-dense set. Thus for some k_0, the set $\mathcal{A}^{[k_0]}$ becomes an 1/2-dense set.

Now let n be an arbitrary positive integer which is not an element of $\mathcal{A}^{[k_0]}$. Since the set is 1/2-dense, there exists at least $n/2$ elements below n. Hence by the pigeonhole principle, there exists an i such that both i and $n-i$ are elements of $\mathcal{A}^{[k_0]}$. Thus every positive integer can be represented as a sum of at most 2 elements of $\mathcal{A}^{[k_0]}$, or as a sum of at most $2k_0$ elements of $\mathcal{A}$. $\square$

Comment. For a set $\mathcal{A}$, the Schnirelmann density of $\mathcal{A}$ is defined as

$$\sigma\mathcal{A} = \inf \frac{|\mathcal{A} \cap \{1, \cdots, n\}|}{n}.$$

Schnirelmann himself proved the lemma used above, and Mann(1948) first proved the stronger result stating that $\sigma(A \oplus B) \geq \min\{1, \sigma A + \sigma B\}$.

Problem 5. *Three positive integers p, q, n and an injective function $f : \mathbb{Z}^2 \to \mathbb{R}$ are given. Prove that if $n > \binom{p+q-2}{p-1}$, either there exist integers $1 \leq x_0 < x_1 < \cdots < x_p \leq n$ such that*

$$f(x_0, x_1) < f(x_1, x_2) < \cdots < f(x_{p-1}, x_p)$$

or there exist integers $1 \leq y_0 < \cdots < y_q \leq n$ such that

$$f(y_0, y_1) > f(y_1, y_2) > \cdots > f(y_{q-1}, y_q).$$

Solution. We use induction on $p+q$. Since the statement is obvious when either p or q is 1, we shall assume that both p and q are at least 2.

Partition the set $\{1, 2, \cdots, n\}$ into the following sets.

$$P = \{1 < a < n : \max_{1 \leq x < a} f(x, a) < \max_{a < y \leq n} f(a, y)\} \cup \{1\}$$

$$Q = \{1 < a < n : \max_{1 \leq x < a} f(x, a) > \max_{a < y \leq n} f(a, y)\} \cup \{n\}$$

Since each of $n > \binom{p+q-2}{p-1} = \binom{p+q-3}{p-2} + \binom{p+q-3}{p-1}$, either $|P| > \binom{p+q-3}{p-2}$ or $|Q| > \binom{p+q-3}{p-1}$.

First suppose that $|P| > \binom{p+q-3}{p-2}$. By the induction hypothesis, there exists either integers $1 \leq x_0 < \cdots < x_{p-1} < n$ in P such that

$$f(x_0, x_1) < \cdots < f(x_{p-2}, x_{p-1})$$

or integers $1 \le y_0 < \cdots y_q < n$ in P such that

$$f(y_0, y_1) > \cdots > f(y_{q-1}, y_q).$$

If there exist such $y_0, \cdots y_q$, then we are done. If there exist such $x_0, \cdots, x_{p-1}$, then we can find a $x_{p-1} < x \le n$ such that $f(x_{p-2}, x_{p-1}) < f(x_{p-1}, x_p)$ since the set P is defined in such way.

If $|Q| > \binom{p+q-3}{p-1}$, we may find such $x_0, \cdots, x_p$ or $y_0, \cdots, y_q$ in a similar manner using the induction hypothesis for p and $q-1$. $\square$

Comment. The number $\binom{p+q-2}{p-1}$ is tight; there might not exist $x_0, \cdots, x_p$ and $y_0, \cdots, y_q$ if $n = \binom{p+q-2}{p-1}$.

Problem 6. *Suppose that for a function $f : \mathbb{R} \to \mathbb{R}$ and real numbers $a < b$ one has $f(x) = 0$ for all $x \in (a, b)$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$ if*

$$\sum_{k=0}^{p-1} f\left(y + \frac{k}{p}\right) = 0$$

for every prime number p and every real number y.

Solution. Let $p_1, \cdots, p_n$ be the first n primes and let $N = p_1 \cdots p_n$. For $1 \le i \le n$, let

$$f_i(x) = \frac{x^N - 1}{x^{N/p_i} - 1} = 1 + x^{N/p_i} + \cdots + x^{N(p_i-1)/p_i}.$$

Since the common roots of all the polynomials $f_1, \cdots, f_n$ are of the form $e^{2\pi i a/N}$ where a and N are relatively prime, the greatest common divisor of $f_1, \cdots, f_n$ is the Nth cyclotomic polynomial $\Phi_N(x)$. Hence there exists polynomials $g_1, \cdots, g_n$ such that

$$f_1 g_1 + f_2 g_2 + \cdots + f_n g_n = \Phi_N(x).$$

Now consider a linear operator $T : \mathbb{R}[x] \to \mathbb{R}^{\mathbb{R}}$ such that $T(x^t)$ equals $y \mapsto f(y + t/N)$. Then T maps the polynomial $x^t f_i$ to a functions that maps y to

$$\sum_{k=0}^{p-1} f\left(y + \frac{t}{N} + \frac{k}{p_i}\right) = 0$$

Thus $T(x^t f_i) = 0$ for all t and i.

Since T is linear $T(g_i f_i) = 0$ for all i and hence $T(\Phi_N(x)) = 0$. Let $\Phi_N(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$ where $d = \phi(N) = (p_1 - 1)\cdots(p_n - 1)$. Then

$$T(\Phi_N) = f\Big(y + \frac{d}{N}\Big) + a_{d-1} f\Big(y + \frac{d-1}{N}\Big) + \cdots + a_0 f(y) = 0$$

for all y.

Let n be a sufficiently large integer such that

$$\prod_{i \le n} \frac{p_i - 1}{p_i} < b - a.$$

For any $y \in [b - d/N, b - (d-1)/N)$, we have

$$f(y + d/N) = a_{d-1} f\Big(y + \frac{d-1}{N}\Big) + \cdots + a_0 f(y) = 0,$$

and hence $f(x) = 0$ for all $x \in (a, b + 1/N)$. Iterating this procedure, we get that $f(x) = 0$ for all $x > a$, and since $f(x) + f(x + 1/2) = 0$ for all x, the function f is identical to 0. $\square$

Problem 7. *Suppose that a $1 \times q$ rectangle can be partitioned into a finite number of squares. Prove that q is rational.*

Solution. Suppose that q is irrational. Since 1 and q are linearly independent on the field $\mathbb{Q}$, we may consider a $\mathbb{Q}$-linear function $f : \mathbb{R} \to \mathbb{R}$ such that $f(1) = 1$ and $f(q) = -1$.

For a rectangle with side-lengths a and b, assign to it a value of $f(a)f(b)$. By the linearity of f, when a $x \times y$ rectangle is partitioned into $x_i \times y_j$ rectangles, the sum of the assigned values is preserved since

$$f(x_1 + \cdots + x_k) f(y_1 + \cdots + y_l) = \sum_{i=1}^{k} \sum_{j=1}^{l} f(x_i) f(y_j).$$

Now if the $1 \times q$ rectangle is partitioned into squares, cut the squares by all the lines passing through each point vertex of the square and parallel to the axes. Then sum of the assigned values is equal to $f(1)f(q) = -1$, but is the sum of the assigned value of the squares. Since for each square of side-length l, the assigned value is $f(l)^2 \ge 0$, the two sum cannot be the same. $\square$