Explicit Local Class Field Theory

An undergraduate thesis presented by

Dongryul Kim

Advisor : Professor Barry Mazur

Submitted in partial fulfillment of the honors requirements for
the degree of Bachelor of Arts in Mathematics

Department of Mathematics
Harvard University
Cambridge, Massachusetts

March 2019

List of symbols

$K, L, E, \ldots$	fields, mostly local fields
K^{sep}	the separable closure of a field K
K^{ab}	the abelian closure of a field K
$\mathrm{Gal}(L/K)$	the Galois group of L/K, for L/K a Galois extension
$N_{L/K}$	the norm map from $L^\times$ to $K^\times$ for L/K a field extension
$\lvert - \rvert_K$	the absolute value on a local field K
v_K	the valuation map on a local field K
$\mathcal{O}_K$	ring of integers of a local field K
$\mathfrak{m}_K$	the maximal ideal of $\mathcal{O}_K$, for K a local field
$k, l, \ldots$	the residue field of the local fields $K, L, \ldots$
π_K	a uniformizer for a local field K
K^{unr}	the maximal unramified extension of a local field K
$K^{\mathrm{unr},\wedge}$	the topological completion of the maximal unramified extension K^{unr}
$\mathrm{Frob}_{/K}$	the Frobenius automorphism of K^{unr}/K
$\widehat{K^\times}$	the profinite completion of the multiplicative group $K^\times$
$\theta_{/K}$	the absolute Artin reciprocity map from $\widehat{K^\times}$ to $\mathrm{Gal}(K^{\mathrm{sep}}/K)^{\mathrm{ab}}$
$\theta_{L/K}$	the relative Artin reciprocity map from $K^\times/NL^\times$ to $\mathrm{Gal}(L/K)^{\mathrm{ab}}$
A^G	the G-invariants of a G-module A
A_G	the G-coinvariants of a G-module A
$H^i(G, A)$	the ith group cohomology of a G-module A
$H_i(G, A)$	the ith group homology of a G-module A
$\hat{H}^i(G, A)$	the ith Tate cohomology of a G-module A, when G is finite
$F, G, \ldots$	formal groups laws or formal module laws
$[a]_F$	the power series representing multiplication by a in F
$\mathfrak{F}, \mathfrak{G}, \ldots$	formal groups or formal modules associated to the laws $F, G, \ldots$
$\mathfrak{LT}_{/K,\pi_K}$	the Lubin–Tate $\mathcal{O}_K$-module over $\mathcal{O}_K$ associated to π_K
$\mathfrak{LT}_{/K,\pi_K}[\mathfrak{m}_K^\infty]$	the torsion points of the Lubin–Tate $\mathcal{O}_K$-module $\mathfrak{LT}_{/K,\pi_K}$

Contents

Chapter 1

Introduction

An important object of interest in modern number theory are number fields, i.e., finite field extensions of $\mathbb{Q}$. By Galois theory, finite field extensions of $\mathbb{Q}$ are classified by open subgroups of the absolute Galois group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{sep}}/\mathbb{Q})$. Thus understanding finite extensions of $\mathbb{Q}$ can be translated to understanding the group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{sep}}/\mathbb{Q})$.

What class field theory attempts to do is to describe the abelianization of the absolute Galois group

$$\mathrm{Gal}(\mathbb{Q}^{\mathrm{sep}}/\mathbb{Q})^{\mathrm{ab}} \cong \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}),$$

or more generally,

$$\mathrm{Gal}(K^{\mathrm{sep}}/K)^{\mathrm{ab}} \cong \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

for certain fields K. Again by Galois theory, the abelianization of the absolute Galois group is the same as the Galois group of the maximal abelian extension, and therefore what class field theory deals with is the abelian extensions of K. There are two types of fields K we use in class field theory: local fields, $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ or their finite extensions, and global fields, $\mathbb{Q}$ or $\mathbb{F}_p(t)$ or their finite extensions. In this thesis, we are mainly interested the case when K is a finite extension of $\mathbb{Q}_p$, which we shall call a p-adic local field.

Local class field theory provides a surprising description of the abelianization of the absolute Galois group of a local field K of characteristic zero. The local Artin reciprocity map is an isomorphism

$$\theta_{/K} \colon \widehat{K^\times} \xrightarrow{\cong} \mathrm{Gal}(K^{\mathrm{sep}}/K)^{\mathrm{ab}},$$

and hence tells us that the abelianization of the absolute Galois group is in fact isomorphic to a slight modification of the multiplicative group $K^\times$.

The subject originated from the celebrated Kronecker–Weber theorem on abelian extensions of $\mathbb{Q}$. Proven in the late 19th century by Hilbert, building upon the works of Kronecker and Weber, the theorem gave a complete description of all abelian extensions of $\mathbb{Q}$, namely that all such extensions are contained in a cyclotomic field. This immediately gives a concrete description of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$, since we need only track where roots of unities are sent. However, this is not satisfactory as it does not contain information about abelian extensions over other number fields. As stated, it is not even apparent what the correct generalization it would be for other number fields.

In the late 1890s up until the 1920s, there was a lot of development on the generalization to arbitrary number fields. Weber formulated the notion of ray class groups and class fields, and Takagi showed that class fields are precisely abelian extensions of a given field. Artin conjectured the existence of the Artin reciprocity map for a number field and ultimately proved it in the 1920s, establishing global class field theory.

Curiously, the global case was dealt before local class field theory was introduced, despite the fact that modern treatments of global class field theory use local class field theory in constructing the Artin reciprocity map. Local fields such as the p-adic rational numbers were defined only in the late 1890s by Hensel, and local class field theory was developed by Hasse in the 1930s, after Artin reciprocity was proven.

In the modern literature, class field theory is usually stated in terms of the idele class group and proven using group cohomology. This is a formulation that was introduced after the main theorems were proved in the classical language. The language of adeles and ideles was developed and incorporated to class field theory by Chevalley in the 1930s. Group cohomology started to become a mathematical object of study only in the 1930s and 40s, and Hochschild and Nakayama reformulated class field theory in terms of group homology and cohomology in the 1950s. Tate introduced the Tate cohomology groups and simplified the cohomological arguments. At this point, the cohomological proof of class field theory was sufficiently optimized so that books such as Cassels–Fröhlich [CF67], written in the 1960s, is still used as the standard reference for class field theory.

There are multiple generalizations of class field theory, the most prominent one being the Langlands program. Introduced by Langlands in the late 1960s, the program attempts to relate representations of the absolute Galois group to automorphic objects. The GL_1 case precisely recovers class field theory, as 1-dimensional representations of a Galois group necessarily factors through the abelianization. The complete correspondence is far from being understood and is an active area of research.

The goal of this thesis is to provide two different explicit ways of understanding local class field theory—that is, two different constructions of the Artin reciprocity maps

$$\theta_{/K} \colon \widehat{K^\times} \cong \mathrm{Gal}(K^{\mathrm{sep}}/K)^{\mathrm{ab}}, \quad \theta_{L/K} \colon K^\times/NL^\times \cong \mathrm{Gal}(L/K)^{\mathrm{ab}}.$$

In Chapter 1, we review facts about local fields and provide a full statement of local class field theory. In Chapter 2, we construct the Artin reciprocity map using Tate cohomology, and use it to give an explicit description of the Artin reciprocity map. This description is explicit enough so that one can prove all the functoriality properties of the Artin map through that definition. In Chapter 3, we provide a second account of the same reciprocity map, but in an entirely different context. Here, the map is computed by constructing a certain 1-dimensional representation of the Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. Although the two descriptions are both very explicit, it is unclear to me how to connect the two approaches.

I would like to thank my thesis advisor Professor Barry Mazur for helpful and lengthy discussions in his office; without his advice and guidance, this thesis could not have existed. I am also indebted to Alison B. Miller, who first introduced me to the subject of class field theory in a year-long course. During my four years of study, the Harvard Mathematics Department provided wonderful opportunities and a great environment for cultivating myself as a mathematician. Finally, I would like to thank my family and friends, who sometimes endured my silly questions and sometimes provided moral support that kept me going.

1.1 Preliminaries

We provide a quick review on Galois theory and facts about local fields that we shall use in the following chapters.

1.1.1 Galois theory

Let K be a field.

Definition 1.1.1. A field extension $K \hookrightarrow L$ is called **Galois extension** if it is algebraic, normal, and separable. (We do not require finite degree.)

Definition 1.1.2. The **Galois group** $\mathrm{Gal}(L/K)$ of a Galois extension $K \hookrightarrow L$ is defined as the group of field automorphisms of L that fix K, where we consider K as a subfield of L. This is given the topology with base

$$\{U_S = \{g \in \mathrm{Gal}(L/K) : gx = x \text{ for all } x \in S\} : S \in L \text{ finite}\}.$$

This can also be described as the coarsest topology that makes the action $\mathrm{Gal}(L/K) \times L \to L$ continuous, where L is given the discrete topology.

Because every element $x \in L$ has finitely many Galois conjugates, we see that each U_x is a group of finite index. Using this observation, one proves that $\mathrm{Gal}(L/K)$ is a profinite group.

Definition 1.1.3. The **absolute Galois group** of K is defined as $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$, where K^{sep} is the separable closure of the field K.

Theorem 1.1.4 (fundamental theorem of Galois theory)**.** *There is an equivalence of categories*

$$F : \left\{ \begin{matrix} \textit{finite sets with a} \\ \textit{continuous left } G_K\textit{-action} \end{matrix} \right\} \xrightarrow{\simeq} \left\{ \begin{matrix} \textit{algebras over } K \textit{ that are isomorphic} \\ \textit{to } L_1 \times \cdots \times L_n \textit{ for } L_i \textit{ finite} \\ \textit{separable extensions of } K \end{matrix} \right\}^{\mathrm{op}}$$

that in particular sends

$$\coprod_i (G_K/H_i) \mapsto \prod_i (K^{\mathrm{sep}})^{H_i}$$

for open subgroups $H_i \subseteq G_K$. This functor moreover sends fiber products of G_K-sets to tensor products of algebras, i.e., for G_K-equivariant maps $S_1, S_2 \to T$ of sets there is an isomorphism

$$F(S_1 \times_T S_2) \cong F(S_1) \otimes_{F(T)} F(S_2).$$

Remark 1.1.5. The right hand side may also be considered as the category of finite étale schemes over $\operatorname{Spec} K$. Thus the theorem may be translated to the statement that the fundamental group of $\operatorname{Spec} K$ being well-defined and isomorphic to $\mathrm{Gal}(K^{\mathrm{sep}}/K)$.

This allows us to easily compute tensor product of fields. For instance, if L/K is a finite Galois extension, the K-algebra L corresponds to the finite set $\mathrm{Gal}(L/K)$ with the G_K-action given by left multiplication through the homomorphism $G_K \to \mathrm{Gal}(L/K)$. Then taking the product at the level of G_K-sets shows that there is an isomorphism

$$L \otimes_K L \cong \prod_{g \in \mathrm{Gal}(L/K)} L.$$

The map of algebras can be described more explicitly as

$$\sum_i l_{1,i} \otimes l_{2,i} \mapsto (\textstyle\sum_i l_{1,i} g(l_{2,i}))_{g \in \mathrm{Gal}(L/K)}.$$

If we want to work with infinite field extensions, we have the following correspondence.

Theorem 1.1.6 (fundamental theorem of infinitary Galois theory)**.** *There is a one-to-one correspondence*

$$\left\{ \textit{closed subgroups } H \subseteq G_K \right\} \quad \longleftrightarrow \quad \left\{ \begin{matrix} \textit{intermediate field} \\ \textit{extensions } K \subseteq L \subseteq K^{\mathrm{sep}} \end{matrix} \right\}$$

sending $H \mapsto (K^{\mathrm{sep}})^H$ and $L \mapsto \mathrm{Gal}(K^{\mathrm{sep}}/L)$.

1.1.2 Local fields

Definition 1.1.7. A **local field** is a field K equipped with an absolute value function $|-|_K : K \to \mathbb{R}_{\geq 0}$ satisfying the following properties:

(1) $|x| = 0$ if and only if $x = 0$,

(2) there is an element $x \in K$ such that $|x| \neq 0, 1$,

(3) $|xy| = |x||y|$ for all $x, y \in K$,

(4) $|x + y| \leq |x| + |y|$ for all $x, y \in K$,

(5) K is complete and locally compact with respect to the topology induced by the metric $d(x, y) = |x - y|$.

It is possible to classify all local fields up to isomorphism. A local field is either

- $\mathbb{R}$ or $\mathbb{C}$,
- a finite extension of $\mathbb{Q}_p$ for some prime p, or
- a finite extension of $\mathbb{F}_p((t))$ for some prime p.

In this thesis, we shall concern ourselves with only local fields K that fall in the second category. For conciseness, we define a **p-adic local field** as a field that is a finite extension of $\mathbb{Q}_p$. These satisfy a stronger triangle inequality; for every $x, y \in K$ we have

$$|x + y| \leq \max\{|x|, |y|\}.$$

This allows us to define another topology of K. If $c > 0$ is any real number, we may define a new absolute value $|-|'$ by $|x|' = |x|^c$. This absolute value also makes $(K, |-|'_K)$ into a local field. However, we shall regard the two local fields as isomorphic since the topologies agree. In fact, for any given discrete subgroup $\Gamma \subseteq \mathbb{R}^\times_{>0}$, there exists a unique normalization of the absolute value that makes

$$|K^\times| = \mathrm{im}(|-| : K^\times \to \mathbb{R}_{>0}) = \Gamma.$$

Definition 1.1.8. If an element $\pi \in K^\times$ satisfies $|\pi| < 1$ and has the property that $|\pi|$ generates the group $|K^\times| \subseteq \mathbb{R}^\times_{>0}$, then we say that π is a **uniformizer** for the field K.

If we denote the closed unit ball as

$$\mathcal{O}_K = \{x \in K : |x| \leq 1\} \subset K$$

this is a ring, and we call it the **ring of integers** in K. It is a discrete valuation ring with maximal ideal given by the open unit ball

$$\mathfrak{m}_K = \{x \in K : |x| < 1\} = (\pi) \subset \mathcal{O}_K.$$

We will prefer writing $\mathfrak{m}_K$ to (π) as the notation does not involve the choice of a uniformizer. The residue field $k = \mathcal{O}_K/\mathfrak{m}_K$ is a finite field of characteristic p.

Proposition 1.1.9 ([CF67], Section II.10)**.** *If K is a p-adic local field with absolute value $|-|_K$ and L/K is a finite field extension, then there exists a unique absolute value L that extends $|-|_K$. In particular, it is given by*

$$|x|_L = |N_{L/K}x|^{1/d}$$

where $d = [L : K]$ is the degree of the field extension.

If L/K is a finite extension of p-adic local fields, then it induces a field extension of residue fields ℓ/k and also a embedding of absolute value groups $|K^\times| \hookrightarrow |L^\times|$. We define the **ramification index** $e_{L/K}$ and the **inertia degree** $f_{L/K}$ as

$$e_{L/K} = [|L^\times| : |K^\times|], \quad f_{L/K} = [\ell : k].$$

Proposition 1.1.10 ([CF67], Proposition I.5.3)**.** *For any finite extension L/K of p-adic local fields, we have*

$$d = [L : K] = e_{L/K} f_{L/K}.$$

If L/K is finite Galois, we get a homomorphism of Galois groups, which turns out to be always surjective:

$$\mathrm{Gal}(L/K) \to \mathrm{Gal}(\ell/k) \to 1$$

If we denote by $K \subseteq L_0 \subseteq L$ the subextension corresponding to the kernel, then L_0/K is a Galois extension of degree f that has no ramification and L/L_0 is a Galois extension of degree e that has no inertia. Thus, in a sense, we may to study unramified extensions and totally ramified extensions (those with no inertia) separately.

Theorem 1.1.11 ([CF67], Theorem I.7.1)**.** *Let K be a p-adic local field. For every given integer $d \geq 1$, there exists a unique unramified extension L/K of degree f up to isomorphism. In particular, it is given by*

$$L = K(\zeta_{q^f-1})$$

where $q = |k|$ is the cardinality of the residue field.

Since it is a cyclotomic extension, it is Galois and moreover cyclic. Indeed, we have an isomorphism $\mathrm{Gal}(L/K) \cong \mathrm{Gal}(\ell/k) \cong \mathrm{Gal}(\mathbb{F}_{q^f}/\mathbb{F}_q) \cong \mathbb{Z}/f\mathbb{Z}$. We can even choose a canonical generator of this group, called the **Frobenius**,

$$\mathrm{Frob}_{L/K} \in \mathrm{Gal}(\ell/k) \cong \mathrm{Gal}(L/K); \quad x \mapsto x^q \text{ for } x \in \ell.$$

Of course, the $\mathrm{Frob}_{L/K}$-action on L is not given by $x \mapsto x^q$, but there is a unique lift of the Frobenius action on ℓ.

We can even take the union of all the unramified extensions. In this case, we obtain the **maximal unramified extension**

$$K^{\mathrm{unr}} = \bigcup_{f \geq 1} K(\zeta_{q^f-1}) \subseteq K^{\mathrm{sep}}.$$

We observe that the Galois group $\mathrm{Gal}(K^{\mathrm{unr}}/K)$ is computed by

$$\mathrm{Gal}(K^{\mathrm{unr}}/K) \cong \varprojlim_f \mathrm{Gal}(K(\zeta_{q^f-1})/K) \cong \varprojlim_f \mathbb{Z}/f\mathbb{Z} \cong \widehat{\mathbb{Z}},$$

where $\mathrm{Frob}_{K^{\mathrm{unr}}/K} \in \mathrm{Gal}(K^{\mathrm{unr}}/K)$ is a canonical element that topologically generates the group $\mathrm{Gal}(K^{\mathrm{unr}}/K)$.

Theorem 1.1.12 ([CF67], Theorem I.6.1)**.** *Let K be a p-adic local field.*

(1) If L/K is a totally ramified extension of degree e, then for any uniformizer $\pi_L \in \mathcal{O}_L$ we have $\mathcal{O}_L = \mathcal{O}_K[\pi_L]$. Moreover, the monic minimal polynomial of π_L over K is a degree e Eisenstein polynomial with coefficients in $\mathcal{O}_K$.

(2) If $f(x) \in \mathcal{O}_K$ is an Eisenstein polynomial of degree e, then the spitting field $K[x]/(f(x))$ is a totally ramified extension of degree e. Moreover, all roots of $f(x)$ in L are uniformizers.

This tells us that there are very many totally ramified extensions. Indeed, there is no "maximal totally ramified extension" that contains all and only totally ramified extensions.

1.2 Statement of local class field theory

Class field theory is a statement about the abelian part of the absolute Galois group. Using the infinite Galois correspondence, we easily see that

$$G_K^{\mathrm{ab}} = \mathrm{Gal}(K^{\mathrm{sep}}/K)^{\mathrm{ab}} \cong \mathrm{Gal}(K^{\mathrm{ab}}/K),$$

where the ab on the left hand side denotes the abelianization of a group and K^{ab} is the maximal abelian extension of K.

We note that with a choice of a uniformizer π_K of K, the multiplicative group splits topologically as a direct sum

$$K^\times \cong \mathbb{Z} \oplus \mathcal{O}_K^\times, \quad x \mapsto (\log_{|\pi_K|}|x|, x/\pi_K^{\log_{|\pi_K|}|x|}).$$

Here, we note that $\mathcal{O}_K^\times$ is a profinite group with the natural topology. Thus we may take the profinite completion and get an isomorphism

$$\widehat{K^\times} \cong \widehat{\mathbb{Z}} \oplus \mathcal{O}_K^\times$$

of profinite groups. Even if without the choice of the uniformizer, we still have a well-defined valuation map $v : K^\times \to \mathbb{Z}$ given by $x \mapsto \log_{|\pi_K|}|x|$ and it induces a homomorphism

$$v : \widehat{K^\times} \to \widehat{\mathbb{Z}}$$

of profinite groups.

Theorem 1.2.1 (local class field theory)**.** *Let K be a p-adic local field. There exists an isomorphism $\theta_{/K} : \widehat{K^\times} \to \mathrm{Gal}(K^{\mathrm{sep}}/K)^{\mathrm{ab}}$ of profinite groups satisfying the following:*

- *the valuation encodes the action on the maximal unramified,*

$$\begin{tikzcd} \widehat{K^\times} \arrow[r, "\theta_{/K}", "\cong"'] \arrow[d, two heads, "v"] & \mathrm{Gal}(K^{\mathrm{sep}}/K)^{\mathrm{ab}} \arrow[d, two heads, "\mathrm{res.}"] \\ \widehat{\mathbb{Z}} \arrow[r, "n \mapsto \mathrm{Frob}^n", "\cong"'] & \mathrm{Gal}(K^{\mathrm{unr}}/K) \end{tikzcd}$$

- *for a finite extension L/K, the inclusion of the Galois group corresponds to taking the norm,*

$$\begin{array}{ccc} \widehat{L^\times} & \xrightarrow[\cong]{\theta_{/L}} & \mathrm{Gal}(K^{\mathrm{sep}}/L)^{\mathrm{ab}} \\ \big\downarrow N & & \big\downarrow \text{inc.} \\ \widehat{K^\times} & \xrightarrow[\cong]{\theta_{/K}} & \mathrm{Gal}(K^{\mathrm{sep}}/K)^{\mathrm{ab}} \end{array}$$

  (here, note that the vertical map on the right side is not injective after abelianization even if $\mathrm{Gal}(K^{\mathrm{sep}}/L) \to \mathrm{Gal}(K^{\mathrm{sep}}/K)$ *is injective)*

- *if L/K is finite Galois, we may take the cokernel of the vertical maps in the previous diagram and obtain an isomorphism*

$$\theta_{L/K} : K^\times/NL^\times \to \mathrm{Gal}(L/K)^{\mathrm{ab}}$$

  (this is a consequence of the previous property).

The isomorphisms $\theta_{/K}$ and $\theta_{L/K}$ are called the **Artin reciprocity maps**. Oftentimes, one defines the **Weil group** of K as the inverse image of $\mathbb{Z}$ inside $\widehat{\mathbb{Z}}$,

$$W_K = [\mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Gal}(K^{\mathrm{unr}}/K) \cong \widehat{\mathbb{Z}}]^{-1}(\mathbb{Z}) \subseteq \mathrm{Gal}(K^{\mathrm{sep}}/K).$$

Then the Artin reciprocity map can be regarded as an isomorphism

$$\theta_{/K} : K^\times \to W_K^{\mathrm{ab}}.$$

Chapter 2

Explicit local class field theory via cohomology

In a modern treatment, class field theory is usually proven using the machinery of Galois cohomology. The goal of this section is to present a proof of class field theory using cohomology, and use it to produce an explicit description of the Artin reciprocity map.

The way most books on local class field theory prove it is through first computing the second cohomology group, and then using Tate's theorem to conclude that the cup product with the fundamental class $u_{L/K} \in H^2(L/K, L^\times)$ is produces an isomorphism $\hat{H}^{-2}(L/K, \mathbb{Z}) \to \hat{H}^0(L/K, L^\times)$. In particular, this is the approach taken by Cassels–Frölich [CF67], Serre [Ser79], and the cohomological parts of Milne [Mil13].

However, this proof is not so useful in producing an explicit characterization of the reciprocity. The approach we shall take in this chapter is the one by Dwork [Dwo58], which is also outlined in [Ser79] as a section and a series of exercises, and written out in detail in Snaith [Sna94]. The advantage of this proof of local class field theory is that it gives a more computable definition of the reciprocity map whose origin lies in computing the group cohomology. However, there is no counterpart for global class field theory to my knowledge.

2.1 Group homology and cohomology

Let G be a finite group, which will be the Galois group of a finite Galois extension.

Definition 2.1.1. A **G-module** is an abelian group A with a group homomorphism $G \to \mathrm{Aut}(A)$, i.e., an additive G-action. Alternatively, a G-module is a left $\mathbb{Z}[G]$-module.

Let us denote by $G\mathsf{-mod}$ the abelian category of G-modules, and by Ab the abelian category of abelian groups. The inclusion functor $\mathsf{Ab} \to G\mathsf{-mod}$ where a group acquires a trivial G-action has both a left and a right adjoint, called the **coinvariants** and the **invariants**,

$$A \mapsto A_G = A/\langle a - ga : a \in A, g \in G\rangle, \quad A \mapsto A^G = \{a \in A : ga = a \text{ for all } g \in G\}.$$

We may also think of this as

$$A_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} A, \quad A^G = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A).$$

Because $(-)_G : G\text{−}\mathsf{mod} \to \mathsf{Ab}$ is a left adjoint, it is right exact. Since the category $G\text{−}\mathsf{mod}$ has enough projectives, we may take its left derived functors,

$$H_i(G, A) = (L_i(-)_G)(A) \in \mathsf{Ab}.$$

This is called the **group homology**. Similarly, the invariants functor $(-)^G : G\text{−}\mathsf{mod} \to \mathsf{Ab}$ is a right adjoint and hence left exact, where $G\text{−}\mathsf{mod}$ has enough injectives, and thus we may consider its right derived functors

$$H^i(G, A) = (R^i(-)^G)(A) \in \mathsf{Ab},$$

which we call the **group cohomology** of A. By formal properties of the derived functor, any short exact sequence $0 \to A \to B \to C \to 0$ of G-modules induces a long exact sequence

$$\begin{gathered}\cdots \to H_1(G, A) \to H_1(G, B) \to H_1(G, C) \to A_G \to B_G \to C_G \to 0, \\ 0 \to A^G \to B^G \to C^G \to H^1(G, A) \to H^1(G, B) \to H^1(G, C) \to \cdots .\end{gathered}$$

Further assume that G is finite. For every G-module A, there is a norm map

$$N : A_G \to A^G; \quad [a] \mapsto \sum_{g \in G} ga.$$

This is a well-defined map, since $\sum_{g \in G} g(b - hb) = 0$ for any $b \in A$ and $g \in G$.

Definition 2.1.2. For G a finite group, we define **Tate cohomology** as the functor

$$\hat{H}^i(G, -) : G\text{−}\mathsf{mod} \to \mathsf{Ab}$$

given by

$$\hat{H}^i(G, A) = \begin{cases} H^i(G, A) & i \geq 1, \\ \mathrm{coker}(N : A_G \to A^G) & i = 0, \\ \ker(N : A_G \to A^G) & i = -1, \\ H_{-1-i}(G, A) & i \leq -2. \end{cases}$$

The advantage of making such a definition is that we can now string group homology and group cohomology together and regard it as a single cohomology theory. If we have a short exact sequence $0 \to A \to B \to C \to 0$ of G-modules, we can put the two long exact sequences coming from group homology and group cohomology in a single diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & \hat{H}^{-2}(G,C) \arrow[r] & A_G \arrow[r] \arrow[d, "N"] & B_G \arrow[r] \arrow[d, "N"] & C_G \arrow[r] \arrow[d, "N"] & 0 & \\
& 0 \arrow[r] & A^G \arrow[r] & B^G \arrow[r] & C^G \arrow[r] & \hat{H}^1(G,A) \arrow[r] & \cdots
\end{tikzcd}$$

The snake lemma applied to this diagram produces the following long exact sequence.

$$\begin{tikzcd}
\cdots \arrow[r] & \hat{H}^{-2}(G,B) \arrow[r] & \hat{H}^{-2}(G,C) \arrow[dll, rounded corners] \\
\hat{H}^{-1}(G,A) \arrow[r] & \hat{H}^{-1}(G,B) \arrow[r] & \hat{H}^{-1}(G,C) \arrow[dll, rounded corners] \\
\hat{H}^{0}(G,A) \arrow[r] & \hat{H}^{0}(G,B) \arrow[r] & \hat{H}^{0}(G,C) \arrow[dll, rounded corners] \\
\hat{H}^{1}(G,A) \arrow[r] & \hat{H}^{1}(G,B) \arrow[r] & \cdots
\end{tikzcd}$$

That is, we obtain connecting homomorphisms $\hat{H}^n(G,C) \to \hat{H}^{n+1}(G,A)$ for all integer n.

2.1.1 Computing group homology and cohomology

Recall that the invariants functor may be described as

$$G\text{−}\mathsf{mod} \to \mathsf{Ab}; \quad A \mapsto A_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} A.$$

Since this is tensor product with a certain module, the left derived functors may also be described as

$$H_i(G,A) = \operatorname{Tor}_i^{\mathbb{Z}[G]}(\mathbb{Z}, A).$$

This immediately tells us that instead of taking a resolution of A by projective left $\mathbb{Z}[G]$-modules, we may take a resolution of $\mathbb{Z}$ by projective right $\mathbb{Z}[G]$-modules to compute group homology. This is useful for computations, since finding one resolution for $\mathbb{Z}$ allows us to give concrete descriptions of all group homology $H_i(G,A)$.

Similarly, the coinvariants functor is

$$G\text{−}\mathsf{mod} \to \mathsf{Ab}; \quad A \mapsto A^G = \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A),$$

and it follows that

$$H^i(G,A) = \operatorname{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, A).$$

We then observe that the groups can be computed again by finding a resolution of $\mathbb{Z}$ by projective left $\mathbb{Z}[G]$-modules. Let us work out some examples.

Example 2.1.3. Let $G = \mathbb{Z}/n\mathbb{Z} = \langle t \rangle$ be a cyclic group, and let A be a G-module. Let us first compute group homology $H_i(G, A)$. There is an explicit projective resolution

$$\cdots \to \mathbb{Z}[G] \xrightarrow{1-t} \mathbb{Z}[G] \xrightarrow{1+\cdots+t^{n-1}} \mathbb{Z}[G] \xrightarrow{1-t} \mathbb{Z}[G] \to \mathbb{Z} \to 0$$

of the G-module $\mathbb{Z}$ with the trivial G-action.

If we want to compute group homology, we may take the tensor product of the chain complex with A over $\mathbb{Z}[G]$ and look at its homology groups. Because all the right $\mathbb{Z}[G]$-modules are free of rank 1, we end up with

$$\cdots \to A \xrightarrow{1+\cdots+t^{n-1}} A \xrightarrow{1-t} A \xrightarrow{1+\cdots+t^{n-1}} A \xrightarrow{1-t} A \to 0.$$

Therefore

$$H_i(G, A) = \begin{cases} A_G = \operatorname{coker}(1-t\colon A \to A) & i = 0, \\ \ker(1-t\colon A \to A)/\operatorname{im}(1+\cdots+t^{n-1}\colon A \to A) & i \geq 1 \text{ odd}, \\ \ker(1+\cdots+t^{n-1}\colon A \to A)/\operatorname{im}(1-t\colon A \to A) & i \geq 1 \text{ even}. \end{cases}$$

Similarly we can compute group cohomology by applying $\operatorname{Hom}_{\mathbb{Z}[G]}(-, A)$ to the sequence. The resulting dual chain complex is

$$0 \to A \xrightarrow{1-t} A \xrightarrow{1+\cdots+t^{n-1}} A \xrightarrow{1-t} A \xrightarrow{1+\cdots+t^{n-1}} A \to \cdots,$$

and therefore

$$H^i(G, A) = \begin{cases} A^G = \ker(1-t\colon A \to A) & i = 0, \\ \ker(1+\cdots+t^{n-1}\colon A \to A)/\operatorname{im}(1-t\colon A \to A) & i \geq 1 \text{ odd}, \\ \ker(1-t\colon A \to A)/\operatorname{im}(1+\cdots+t^{n-1}\colon A \to A) & i \geq 1 \text{ even}. \end{cases}$$

An interesting phenomenon occurs when we put the two together to form Tate cohomology. In the connecting part we get

$$\begin{aligned} \hat{H}^0 &= \operatorname{coker}(N\colon A_G \to A^G) = A^G/\operatorname{im}(N\colon A \to A), \\ \hat{H}^{-1} &= \ker(N\colon A_G \to A^G) = \ker(N\colon A \to A)/\operatorname{im}(1-t\colon A \to A). \end{aligned}$$

Therefore when we look at all the Tate cohomology groups, it is 2-periodic:

$$\hat{H}^i(G, A) = \begin{cases} \ker(1-t\colon A \to A)/\operatorname{im}(1+\cdots+t^{n-1}\colon A \to A) & i \text{ even}, \\ \ker(1+\cdots+t^{n-1}\colon A \to A)/\operatorname{im}(1-t\colon A \to A) & i \text{ odd}. \end{cases}$$

For groups that are not cyclic, we can use the **bar resolution**. This is an exact sequence of the form

$$\cdots \to \mathbb{Z}[G^4] \xrightarrow{d_3} \mathbb{Z}[G^3] \xrightarrow{d_2} \mathbb{Z}[G^2] \xrightarrow{d_1} \mathbb{Z}[G] \xrightarrow{d_0} \mathbb{Z} \to 0,$$

where each $\mathbb{Z}[G^n]$ has the structure of a left or right $\mathbb{Z}[G]$-module given by

$$g(g_1, \dots, g_n) = (gg_1, \dots, gg_n) \quad \text{or} \quad (g_1, \dots, g_n)g = (g_1 g, \dots, g_n g).$$

The maps d_i are defined by the formula

$$d_n((g_0, \dots, g_n)) = \sum_{i=0}^{n} (-1)^i (g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_n) \in \mathbb{Z}[G^n],$$

and it is readily verified that it is an exact sequence.[1] Therefore group homology can be computed as the homology of the chain complex

$$\cdots \to \mathbb{Z}[G^3] \otimes_{\mathbb{Z}[G]} A \xrightarrow{d_2} \mathbb{Z}[G^2] \otimes_{\mathbb{Z}[G]} A \xrightarrow{d_1} A \to 0$$

and similarly group cohomology can be computed as the cohomology of the chain complex

$$0 \to A \xrightarrow{d_1^\vee} \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^2], A) \xrightarrow{d_2^\vee} \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^3], A) \to \cdots.$$

Example 2.1.4. For low degree and simple enough G-modules A, the bar resolution is good enough to carry out computations. Let us look at the case when $A = \mathbb{Z}$, with the trivial group action. Then group homology is computed as the homology of the chain complex

$$\cdots \to \mathbb{Z}[G^3] \otimes_{\mathbb{Z}[G]} \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}[G^2] \otimes_{\mathbb{Z}[G]} \mathbb{Z} \to \mathbb{Z} \to 0.$$

Here, we may choose the basis

$$\{(1, g_1, g_2)\} \subseteq \mathbb{Z}[G^3], \quad \{(1, g_1)\} \subseteq \mathbb{Z}[G^2]$$

for the free right $\mathbb{Z}[G]$-modules, and use it to express the tensor products as

$$\mathbb{Z}[G^3] \otimes_{\mathbb{Z}[G]} \mathbb{Z} \cong \mathbb{Z}\langle (1, g_1, g_2) \rangle, \quad \mathbb{Z}[G^2] \otimes_{\mathbb{Z}[G]} \mathbb{Z} \cong \mathbb{Z}\langle (1, g_1) \rangle.$$

Then the maps d_2 and d_1 on these groups are expressed as

$$d_2((1, g_1, g_2)) = (1, g_2 g_1^{-1}) - (1, g_2) + (1, g_1), \quad d_2((1, g_1)) = 0.$$

It follows that the first group homology is

$$H_1(G, \mathbb{Z}) = \mathbb{Z}[G] / \langle g_2 g_1^{-1} - g_2 + g_1 \rangle = G^{\mathrm{ab}}$$

the abelianization of G. This is an algebraic incarnation of Hurewicz's theorem on $H_1(X; \mathbb{Z}) \cong \pi_1(X)^{\mathrm{ab}}$ for path-connected spaces X.

[1] One way of thinking about this sequence is as the simplicial chain complex associated to the nerve of the groupoid that has elements of G as objects and unique morphisms between any pair of objects. Exactness of the sequence follows from the fact that the category is equivalent to the trivial category with one object and one morphisms, and hence its classifying space is contractible.

2.1.2 The Hochschild–Serre spectral sequence

Suppose $H \subseteq G$ are finite groups, where H is normal inside G. Given a G-module A, we can take the H-invariants A^H and consider it as a G/H-module. Then further taking the G/H-invariants give the G-invariants. That is, the G-invariants functor

$$(-)^G : G\text{−}\mathsf{mod} \to \mathsf{Ab}$$

factors as a composition of functors

$$G\text{−}\mathsf{mod} \xrightarrow{(-)^H} G/H\text{−}\mathsf{mod} \xrightarrow{(-)^{G/H}} \mathsf{Ab}.$$

Hence, after verifying some technical conditions, we have the Grothendieck spectral sequence.

Proposition 2.1.5. *Let G be a finite group and $H \subseteq G$ be a normal subgroup. There is a cohomological* ***Hochschild–Serre spectral sequence***

$$E_2^{p,q} = H^p(G/H, H^q(H, A)) \Longrightarrow H^{p+q}(G, A).$$

Here, $H^q(H, A)$ has a natural G/H-action, since we may consider $(-)^H$ as a functor $G\text{−}\mathsf{mod} \to G/H\text{−}\mathsf{mod}$ instead of $G\text{−}\mathsf{mod} \to \mathsf{Ab}$.

Not surprisingly, there is an analogous statement for homology. This similarly follows from the decomposition of the functor $(-)_G$ into

$$G\text{−}\mathsf{mod} \xrightarrow{(-)_H} G/H\text{−}\mathsf{mod} \xrightarrow{(-)_{G/H}} \mathsf{Ab}.$$

Proposition 2.1.6. *Let G be a finite group and $H \subseteq G$ be a normal subgroup. There is a homological Hochschild–Serre spectral sequence*

$$E^2_{p,q} = H_p(G/H, H_q(H, A)) \Longrightarrow H_{p+q}(G, A).$$

From this we obtain a criterion for vanishing of Tate cohomology.

Lemma 2.1.7. *Let G be a finite group and $H \subseteq G$ be a normal subgroup. Assume that A is a G-module such that*

(i) $\hat{H}^i(H, A) = 0$ *for all integers* $i \in \mathbb{Z}$,

(ii) $\hat{H}^i(G/H, A^H) = 0$ *for all integers* $i \in \mathbb{Z}$.

Then we also have $\hat{H}^i(G, A) = 0$ for all integers $i \in \mathbb{Z}$.

Proof. We first note that (i) for $i = 0, -1$ implies that the norm map

$$N_H : A_H \to A^H$$

is an isomorphism of G/H-modules. If we look at the cohomological Hochschild–Serre spectral sequence, the E_2-page is

$$E_2^{p,q} = H^p(G/H, H^q(H, A)) = \begin{cases} H^p(G/H, A^H) & q = 0 \\ H^p(G/H, 0) & q \neq 0 \end{cases}$$
$$= \begin{cases} A^G & p = q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the spectral sequence degenerates at the E_2-page, hence

$$H^i(G, A) = \begin{cases} A^G & i = 0 \\ 0 & i \neq 0. \end{cases}$$

Similarly, we compute the E^2-page of the homological spectral sequence

$$E^2_{p,q} = H_p(G/H, H_q(H, A)) = \begin{cases} A_G & p = q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$H_i(G, A) = \begin{cases} A_G & i = 0 \\ 0 & i \neq 0. \end{cases}$$

This shows that $\hat{H}^i(G, A) = 0$ for $i \neq 0, -1$.

It now suffice to show that the norm map $N_G : A_G \to A^G$ is an isomorphism of abelian groups. To see this, we note that the norm map is the composition

$$A_G \cong (A_H)_{G/H} \xrightarrow{N_{G/H}} (A_H)^{G/H} \xrightarrow{N_H} (A^H)^{G/H} \cong A^G.$$

The map $N_{G/H}$ is an isomorphism by (ii) for $i = 0, -1$, and the map N_H is an isomorphism as we have seen above. This shows that $N_G : A_G \to A^G$ is also an isomorphism, and therefore $\hat{H}^i(G, A) = 0$ for $i = 0, -1$. □

2.2 The main computation

Let L/K be a finite Galois extension of p-adic local fields. Our goal in class field theory is to construct the Artin reciprocity map

$$\theta_{L/K} : K^\times/NL^\times \to \mathrm{Gal}(L/K)^{\mathrm{ab}}$$

and show that it is an isomorphism. The crucial observation is that the left hand side can be realized as the zeroth Tate cohomology

$$K^\times/NL^\times = (L^\times)^{\mathrm{Gal}(L/K)}/NL^\times = \hat{H}^0(\mathrm{Gal}(L/K), L^\times),$$

where the Galois group $\mathrm{Gal}(L/K)$ acts on the group $L^\times$ in the natural way. Similarly from Example 2.1.4, we see that

$$\mathrm{Gal}(L/K)^{\mathrm{ab}} = H_1(\mathrm{Gal}(L/K), \mathbb{Z}) = \hat{H}^{-2}(\mathrm{Gal}(L/K), \mathbb{Z})$$

where $\mathbb{Z}$ has a trivial $\mathrm{Gal}(L/K)$-action. Therefore, what we want is an isomorphism

$$\theta_{L/K} : \hat{H}^0(\mathrm{Gal}(L/K), L^\times) \to \hat{H}^{-2}(\mathrm{Gal}(L/K), \mathbb{Z}).$$

Our strategy for obtaining the isomorphism

$$\hat{H}^0(\mathrm{Gal}(L/K), L^\times) \xrightarrow{\theta_{L/K}} \hat{H}^{-2}(\mathrm{Gal}(L/K), \mathbb{Z})$$

is to find an exact sequence that looks like

$$0 \to L^\times \to A \xrightarrow{f} B \to \mathbb{Z} \to 0$$

of $\mathrm{Gal}(L/K)$-modules. If A and B further have the property that all Tate cohomology vanish, then the long exact sequence associated to the short exact sequences $0 \to L^\times \to A \to \mathrm{im}(f) \to 0$ and $0 \to \mathrm{im}(f) \to B \to \mathbb{Z} \to 0$ will give isomorphisms

$$\hat{H}^i(\mathrm{Gal}(L/K), L^\times) \cong \hat{H}^{i-1}(\mathrm{Gal}(L/K), \mathrm{im}(f)) \cong \hat{H}^{i-2}(\mathrm{Gal}(L/K), \mathbb{Z})$$

for all integers i. In particular, we will obtain local Artin reciprocity from the special case $i = 0$.

2.2.1 Base-changing to the maximal unramified extension

Let K be a p-adic local field. Recall that finite unramified extensions of K corresponds to finite extensions of the residue field k and moreover took the form of $K(\zeta_{q^f-1})$, where $q = |k|$ is the cardinality of the residue field. We defined the maximal unramified extension of K as

$$K^{\mathrm{unr}} = \bigcup_{f \geq 1} K(\zeta_{q^f-1}) \subseteq K^{\mathrm{sep}}.$$

However, there is one technical problem in working with this field; it has a discrete valuation induced from K, but the field is not complete with respect to the valuation. For example, if we take $K = \mathbb{Q}_p$ and the infinite sum

$$\sum_{n \geq 0} p^n \zeta_{p^n-1},$$

each finite truncation is in the field K^{unr} but the sum does not converge to any element of K^{unr}, by uniqueness of the Teichmüller expansion. Hence we can take the completion with respect to the valuation and define a new field $K^{\mathrm{unr},\wedge}$.

Example 2.2.1. If $K = \mathbb{Q}_p$, then the completed maximal unramified $K^{\mathrm{unr},\wedge}$ can also be constructed as the fraction field of $W(\mathbb{F}_p^{\mathrm{sep}})$, where W denotes the ring of Witt vectors.

On each finite unramified extension $K(\zeta_{q^f} - 1)/K$, there is a Frobenius action. Thus we have a Frobenius action on K^{unr} as well, and continuously extending the automorphism to the completion defines a canonical element

$$\mathrm{Frob}_{/K} \in \mathrm{Gal}(K^{\mathrm{unr},\wedge}/K)$$

of the Galois group. Let us prove a statement that will be used in the future. Denote by $\mathcal{O}_K^{\mathrm{unr},\wedge}$ the ring of integers in the field $K^{\mathrm{unr},\wedge}$.

Proposition 2.2.2. *The group homomorphisms*

$$\begin{aligned} \mathrm{La}_m \colon (\mathcal{O}_K^{\mathrm{unr},\wedge})^\times \to (\mathcal{O}_K^{\mathrm{unr},\wedge})^\times; \quad x \mapsto \mathrm{Frob}_{/K}(x)/x, \\ \mathrm{La}_a \colon \mathcal{O}_K^{\mathrm{unr},\wedge} \to \mathcal{O}_K^{\mathrm{unr},\wedge}; \quad x \mapsto \mathrm{Frob}_{/K}(x) - x \end{aligned}$$

are both surjective.

Proof. For $y \in (\mathcal{O}_K^{\mathrm{unr},\wedge})^\times$, we need to show that there exists a $x \in (\mathcal{O}_K^{\mathrm{unr},\wedge})^\times$ that solves the equation

$$\mathrm{Frob}_{/K}(x) = yx.$$

We find this x by inductively finding a sequence x_n such that

$$\mathrm{Frob}_{/K}(x_n) \equiv yx_n \pmod{\mathfrak{m}^n}$$

and $x_n \equiv x_{n-1} \pmod{\mathfrak{m}^{n-1}}$. Then the limit $x = \lim_{n\to\infty} x_n$ will be the solution we want. At the first step $n = 1$, the equation becomes

$$x_1^q \equiv \mathrm{Frob}_{/K}(x_1) \equiv yx_1 \pmod{\mathfrak{m}}$$

and so solving this equation is equivalent to solving the equation $\bar{x}_1^{q-1} = \bar{y}$ inside k^{sep}, where $\bar{y}$ is the image of y under $(\mathcal{O}_K^{\mathrm{unr},\wedge})^\times \to k^{\mathrm{sep},\times}$. This exists since the polynomial $f(x) = x^{q-1} - \bar{y}$ is separable over k^{sep} and hence has a root.

Let us now do the inductive step. Suppose we are given x_{n-1} satisfying the congruence equation. Pick an element $\alpha \in \mathfrak{m}_K^{n-1} - \mathfrak{m}_K^n$, and let us set $x_n = x_{n-1} + \alpha c$. Because we only care about x_n modulo $\mathfrak{m}^n$, we shall also care about c only modulo $\mathfrak{m}$. The equation $\mathrm{Frob}_{/K}(x_n) \equiv yx_n$ can be also written as

$$\mathrm{Frob}_{/K}(x_{n-1}) + \alpha c^q \equiv yx_{n-1} + y\alpha c \pmod{\mathfrak{m}^n},$$

and because x_{n-1} satisfies the congruence modulo $\mathfrak{m}^{n-1}$, this can also be written as

$$\frac{\mathrm{Frob}_{/K}(x_{n-1}) - yx_{n-1}}{\alpha} + c^q \equiv yc \pmod{\mathfrak{m}}.$$

Again, this has a solution $c \in k^{\mathrm{sep}}$ since the polynomial $c^q - yc + (\mathrm{const})$ is separable. This shows that we can choose c so that $\mathrm{Frob}_{/K}(x_n) \equiv yx_n \pmod{\mathfrak{m}^n}$.

For addition, we make a similar argument. This follows from the fact that the Frobenius on the residue field

$$k^{\mathrm{sep}} \to k^{\mathrm{sep}}; \quad x \mapsto x^q - x$$

is surjective. $\square$

This will play some role of a "universal cover" of K, as it unwraps the Frobenius inside the fundamental group attached to the finite residue field.

Lemma 2.2.3. *Let L/K be a finite extension of ramification index $e_{L/K} = e$. Then $L^{\mathrm{unr},\wedge}/K^{\mathrm{unr},\wedge}$ is a finite extension of degree e.*

Proof. Let L_0 be the intermediate subfield such that L_0/K is unramified and L/L_0 is totally ramified of degree e. (This can be done even if L/K is not Galois, since we may take L_0 as field generated by K and the Teichmüller lifts of elements of ℓ in L.) Then we observe that $K^{\mathrm{unr},\wedge} = L_0^{\mathrm{unr},\wedge}$ since any unramified extension of L_0 is also unramified over K. Hence we may reduce the problem to the case when L/K is totally ramified of degree e.

Consider all the fields as lying in $K^{\mathrm{sep},\wedge}$. Because unramified extensions of L are of the form $L(\zeta_{q^f-1}) = L \cdot K(\zeta_{q^f-1})$, we see that $L^{\mathrm{unr},\wedge}$ is the completion of the compositum of L and $K^{\mathrm{unr},\wedge}$. But because L/K is finite-dimensional, the compositum $L \cdot K^{\mathrm{unr},\wedge}$ is already complete. This shows that

$$L^{\mathrm{unr},\wedge} = L \cdot K^{\mathrm{unr},\wedge}$$

is simply the compositum. Now pick bases $\{1, \pi_L, \ldots, \pi_L^{e-1}\}$ of L over K. By looking at valuations, it follows that the elements are linearly independent over $K^{\mathrm{unr},\wedge}$ as well. Therefore they form a basis, and hence the degree $L^{\mathrm{unr},\wedge}/K^{\mathrm{unr},\wedge}$ is equal to e. $\square$

When L/K is a finite extension, we can tensor it with $K^{\mathrm{unr},\wedge}$ over K to get an algebra. Even though we cannot directly apply Theorem 1.1.4 as $K^{\mathrm{unr},\wedge}$, we still expect that the algebra splits as a product of fields. In fact, we have the following proposition.

Proposition 2.2.4. *Let L/K be a finite Galois extension. Assume that $K \subseteq L_0 \subseteq L$ where L_0/K is an unramified extension of degree f and L/L_0 is totally ramified of degree e. Then there exists an isomorphism of $K^{\mathrm{unr},\wedge}$-algebras*

$$\phi : L \otimes_K K^{\mathrm{unr},\wedge} \cong \prod_{i=0}^{f-1} L^{\mathrm{unr},\wedge}; \quad x \otimes y \mapsto (\sigma_i(x)y)$$

where $\{\sigma_0, \ldots, \sigma_{f-1}\} \subseteq \mathrm{Gal}(L/K)$ are lifts of elements of $\mathrm{Gal}(L_0/K)$ under the surjection $\mathrm{Gal}(L/K) \twoheadrightarrow \mathrm{Gal}(L_0/K)$.

Proof. Consider the maximal unramified subextension K_0/K of L/K. Then by finitary Galois theory, we have an isomorphism

$$\phi_0 \colon L \otimes_K K_0 \cong \prod_{i=0}^{f-1} L; \quad x \otimes y \mapsto (\sigma_i(x)y)$$

of K_0-algebras. When we base change both sides to $K^{\mathrm{unr},\wedge}$, we claim that we obtain the desired isomorphism ϕ. To check this, it suffices to show that the map

$$L \otimes_{K_0} K^{\mathrm{unr},\wedge} \to L^{\mathrm{unr},\wedge}; \quad x \otimes y \mapsto xy$$

is an isomorphism of $K^{\mathrm{unr},\wedge}$-algebras. This follows from the fact that the map is surjective and both sides have the same dimension over $K^{\mathrm{unr},\wedge}$. $\square$

2.2.2 The explicit resolution

Given L/K finite Galois, the K-algebra

$$L \otimes_K K^{\mathrm{unr},\wedge}$$

has two natural actions. There is first the $\mathrm{Gal}(L/K)$-action given by, for each $\sigma \in \mathrm{Gal}(L/K)$,

$$\sigma : L \otimes_K K^{\mathrm{unr},\wedge} \to L \otimes_K K^{\mathrm{unr},\wedge}; \quad x \otimes y \mapsto \sigma(x) \otimes y.$$

On the other hand, we note that there is a canonical generator $\mathrm{Frob}_{/K} \in \mathrm{Gal}(K^{\mathrm{unr}}/K)$, and it naturally extends to an automorphism of $K^{\mathrm{unr},\wedge}$. Then we can act on $L \otimes_K K^{\mathrm{unr},\wedge}$ as

$$\mathrm{Frob} : L \otimes_K K^{\mathrm{unr},\wedge} \to L \otimes_K K^{\mathrm{unr},\wedge}; \quad x \otimes y \mapsto x \otimes \mathrm{Frob}(y).$$

We note that the two actions naturally commute.

Consider the group of units in this K-algebra, $(L \otimes_K K^{\mathrm{unr},\wedge})^\times$. Let us define the homomorphism

$$\mathrm{La} : (L \otimes_K K^{\mathrm{unr},\wedge})^\times \to (L \otimes_K K^{\mathrm{unr},\wedge})^\times; \quad x \mapsto x^{-1}\,\mathrm{Frob}(x).$$

Because the Frobenius commutes with the Galois action, this map is $\mathrm{Gal}(L/K)$-equivariant.

From Proposition 2.2.4 we obtain an alternative description of this group of units and this homomorphism. First, we pick lifts

$$\{\sigma_0, \dots, \sigma_{f-1}\} \subseteq \mathrm{Gal}(L/K)$$

so that they restrict to corresponding powers of the Frobenius;

$$\sigma_i|_{L_0} = \mathrm{Frob}^i_{L_0/K} \in \mathrm{Gal}(L_0/K).$$

Then under the isomorphism $\phi : L \otimes_K K^{\mathrm{unr},\wedge} \to \prod_{i=1}^f L^{\mathrm{unr},\wedge}$, we Frobenius action can be identified with another map Frob on the right hand side that makes the following diagram commute.

$$\begin{array}{ccc} L \otimes_K K^{\mathrm{unr},\wedge} & \xrightarrow[\cong]{\phi} & \prod_{i=0}^{f-1} L^{\mathrm{unr},\wedge} \\ \downarrow{\scriptstyle \mathrm{Frob}} & & \downarrow{\scriptstyle \mathrm{Frob}} \\ L \otimes_K K^{\mathrm{unr},\wedge} & \xrightarrow[\cong]{\phi} & \prod_{i=0}^{f-1} L^{\mathrm{unr},\wedge} \end{array}$$

Lemma 2.2.5. *Under this setting, the map* Frob *on* $\prod_{i=1}^f L^{\mathrm{unr},\wedge}$ *is given by*

$$(x_0, x_1, \dots, x_{f-1}) \mapsto (\tau_{f-1}(x_{f-1}), \tau_0(x_0), \dots, \tau_{f-2}(x_{f-2}))$$

where $\tau_i \in \mathrm{Gal}(L^{\mathrm{unr},\wedge}/K)$ *are the unique elements satisfying*

$$\tau_i|_{K^{\mathrm{unr},\wedge}} = \mathrm{Frob}_{/K}, \quad \tau_i|_L = \sigma_{i+1}\sigma_i^{-1}.$$

Proof. First note that there indeed exist unique such τ_i, since $L \cap K^{\mathrm{unr},\wedge} = L_0$ and
$$\mathrm{Frob}_{/K}|_{L_0} = \mathrm{Frob}_{L_0/K} = (\sigma_{i+1}\sigma_i^{-1})|_{L_0}.$$

In the diagram, we see that there is a unique map making the diagram commute. The map described in the lemma is K-linear, and hence it suffices to check that the diagram commutes when we put this inside the diagram and evaluate on simple tensors. If we send $x \otimes y$ along the lower path, we obtain

$$x \otimes y \mapsto x \otimes \mathrm{Frob}_{/K}(y) \mapsto (\sigma_i(x)\,\mathrm{Frob}_{/K}(y))_{0\le i<f}.$$

If we send it along the upper path, we obtain

$$x \otimes y \mapsto (\sigma_i(x)y)_{0\le i<f} \mapsto (\tau_{i-1}(\sigma_{i-1}(x)y))_{0\le i<f}.$$

By the definition of τ, we see that

$$\tau_{i-1}(\sigma_{i-1}(x)y) = \tau_{i-1}(\sigma_{i-1}(x))\tau_{i-1}(y) = \sigma_i(x)\,\mathrm{Frob}_{/K}(y).$$

Therefore the two ways of composing morphisms agree. □

Corollary 2.2.6. *The map* La *on* $\prod_{i=1}^f (L^{\mathrm{unr},\wedge})^\times$ *is given by*

$$\mathrm{La} : (x_0, x_1, \ldots, x_{f-1}) \mapsto (\tfrac{\tau_{f-1}(x_{f-1})}{x_0}, \tfrac{\tau_0(x_0)}{x_1}, \ldots, \tfrac{\tau_{f-2}(x_{f-2})}{x_{f-1}}).$$

Our goal now is to compute the kernel and cokernel of this endomorphism La of $(L \otimes_K K^{\mathrm{unr},\wedge})^\times$.

Lemma 2.2.7. *The kernel of* La *is* $L^\times$*.*

Proof. The kernel is the set of x such that $\mathrm{Frob}(x) = x$. But we have that

$$(K^{\mathrm{unr},\wedge})^{\mathrm{Frob}=\mathrm{id}} = K,$$

since when we take Teichmüller expansions of an element fixed by the Frobenius, all the coefficients should lie in K. Because L is a vector space over K and Frob does not act on this part, we see that

$$(L \otimes_K K^{\mathrm{unr},\wedge})^{\mathrm{Frob}=\mathrm{id}} = L \otimes_K (K^{\mathrm{unr},\wedge})^{\mathrm{Frob}=\mathrm{id}} = L \otimes_K K = L.$$

If we take the group of units of both sides, we obtain the desired result. □

Lemma 2.2.8. *The cokernel of* La *is* $\mathbb{Z}$*, with the quotient map being*

$$(L \otimes_K K^{\mathrm{unr},\wedge})^\times \cong \prod_{i=1}^{f} (L^{\mathrm{unr},\wedge})^\times \to \mathbb{Z}; \quad (l_i) \mapsto \sum_{i=1}^{f} \log_{|\pi_L|}|l_i|.$$

Proof. It is clear that this map is surjective. It thus suffices to show that any element in the kernel is in the image of La. With the description of the map given in Corollary 2.2.6, it suffices to prove the following statement: if $y_0, \dots, y_{f-1} \in (L^{\mathrm{unr},\wedge})^\times$ satisfies $|y_0 \cdots y_{f-1}| = 1$, then there exist $x_0, \dots, x_{f-1} \in (L^{\mathrm{unr},\wedge})^\times$ satisfying

$$\tau_0(x_0) = y_1 x_1, \quad \tau_1(x_1) = y_2 x_2, \quad \dots, \quad \tau_{f-1}(x_{f-1}) = y_0 x_0.$$

Here, we note that choosing x_0 complete determines all other x_i by

$$x_1 = y_1^{-1}\tau_0(x_0), \quad x_2 = y_2^{-1}\tau_1(x_1) = y_2^{-1}\tau_1(y_1)^{-1}\tau_1\tau_0(x_0), \quad \dots,$$
$$x_{f-1} = y_{f-1}^{-1}(\tau_{f-2}(y_{f-2}))^{-1} \cdots (\tau_{f-2}\cdots\tau_1(y_1))^{-1}\tau_{f-2}\cdots\tau_0(x_0).$$

This in fact satisfies all the equations except for the last one $\tau_{f-1}(x_{f-1}) = y_0 x_0$. Therefore existence of the solution $(x_0, \dots, x_{f-1})$ is equivalent to the existence of a solution x_0 to the equation

$$x_0 = y_0^{-1}(\tau_{f-1}(y_{f-1}))^{-1} \cdots (\tau_{f-1}\cdots\tau_1(y_1))^{-1}(\tau_{f-1}\cdots\tau_0(x)).$$

Let us write $y = y_0^{-1} \cdots (\tau_{f-1}\cdots\tau_1(y_1))^{-1}$. Since any field automorphism fixes the absolute value, we see that

$$|y| = |y_0|^{-1}|y_{f-1}|^{-1} \cdots |y_1|^{-1} = 1^{-1} = 1$$

by the assumption on $y_0, \dots, y_{f-1}$. Moreover, we see that

$$(\tau_{f-1}\cdots\tau_0)|_{K^{\mathrm{unr},\wedge}} = \mathrm{Frob}_{/K}^f,$$
$$(\tau_{f-1}\cdots\tau_0)|_L = (\sigma_0\sigma_{f-1}^{-1})(\sigma_{f-1}\sigma_{f-2}^{-1}) \cdots (\sigma_1\sigma_0^{-1}) = \mathrm{id}_L.$$

It follows that $\tau_{f-1}\cdots\tau_0 = \mathrm{Frob}_{/L}$.

Therefore the statement we need to prove reduces to the following: if $y \in (L^{\mathrm{unr},\wedge})^\times$ satisfies $|y| = 1$ then there exists a $x_0 \in (L^{\mathrm{unr},\wedge})^\times$ such that

$$x_0 = y\,\mathrm{Frob}_{/L}(x_0).$$

This is precisely the statement of Proposition 2.2.2. ☐

Summing up, we have the following theorem.

Theorem 2.2.9. *For L/K a finite Galois extension, we have an exact sequence*

$$0 \to L^\times \to (L \otimes_K K^{\mathrm{unr},\wedge})^\times \xrightarrow{\mathrm{La}} (L \otimes_K K^{\mathrm{unr},\wedge})^\times \xrightarrow{\sum v} \mathbb{Z} \to 0$$

of $\mathrm{Gal}(L/K)$*-modules. (Here, $\mathbb{Z}$ has the trivial Galois action.)*

2.2.3 Vanishing of Tate cohomology

We will now prove that $(L \otimes_K K^{\mathrm{unr},\wedge})^\times$ is acyclic with respect to Tate cohomology. We will need the following theorem about the structure of the Galois group $\mathrm{Gal}(L/K)$.

Lemma 2.2.10. *If L/K is a finite Galois extension of p-adic local fields, then the Galois group $\mathrm{Gal}(L/K)$ is solvable.*

Proof. There is the ramification filtration

$$\mathrm{Gal}(L/K) \supseteq G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$$

defined by

$$G_i = \{g \in \mathrm{Gal}(L/K) : gx - x \in \mathfrak{m}_L^{i+1} \text{ for all } x \in \mathcal{O}_L\}.$$

The intersection of all of these groups is indeed trivial, since the extension L/K is generated by finitely many elements.

We first note that G_0 is the inertia subgroup sitting in the short exact sequence

$$1 \to G_0 \to \mathrm{Gal}(L/K) \to \mathrm{Gal}(l/k) \to 1.$$

So $\mathrm{Gal}(L/K)/G_0$ is a cyclic group. For the other quotients, we note that for $i \geq 0$ we may identify

$$G_i = \{g \in G_0 \subseteq \mathrm{Gal}(L/K) : g\pi_L - \pi_L \in \mathfrak{m}_L^{i+1}\}$$

if we pick a uniformizer $\pi_L \in L$. This is because being in G_0 implies that all Teichmüller lifts of k are fixed, then the ring $\mathcal{O}_L$ is generated over $\mathcal{O}_K$ by π_L and these Teichmüller lifts. This shows that we obtain a map

$$G_i \to (1 + \mathfrak{m}_L^i)^\times; \quad g \mapsto \frac{g\pi_L}{\pi_L}.$$

Taking the quotient by G_{i+1}, we may further define the map

$$G_i/G_{i+1} \to (1 + \mathfrak{m}_L^i)^\times/(1 + \mathfrak{m}_L^{i+1})^\times; \quad g \mapsto \left[\frac{g\pi_L}{\pi_L}\right].$$

This map is injective since G_{i+1} is the set of precisely those Galois automorphisms g for which $g\pi_L/\pi_L \in 1 + \mathfrak{m}_L^{i+1}$. We further claim that it is a group homomorphism. To see this, first note that the map is independent of the choice of uniformizer π_L. If we change π_L to $\varpi_L = u\pi_L$ for some unit $u \in \mathcal{O}_K^\times$, we see that

$$\frac{g\varpi_L/\varpi_L}{g\pi_L/\pi_L} = \frac{gu}{u} \in 1 + \mathfrak{m}_L^{i+1}$$

for all $g \in G_i$ since $gu - u \in \mathfrak{m}_L^{i+1}$ by definition. Therefore the map is independent of the choice of uniformizer. Then if we take $g_1, g_2 \in G_i$ we compute

$$g_1 g_2 \mapsto \left[\frac{g_1 g_2 \pi_L}{\pi_L}\right] = \left[\frac{g_1(g_2\pi_L)}{g_2\pi_L}\right]\left[\frac{g_2\pi_L}{\pi_L}\right] \in \frac{(1 + \mathfrak{m}_L^i)^\times}{(1 + \mathfrak{m}_L^{i+1})^\times}.$$

It follows that $G_i/G_{i+1} \to (1+\mathfrak{m}_L^i)^\times/(1+\mathfrak{m}_L^{i+1})^\times$ is indeed an injective group homomorphism. Since the target group is abelian, the quotient G_i/G_{i+1} is also abelian. $\square$

Lemma 2.2.11. *If* $\mathrm{Gal}(L/K)$ *is cyclic of prime order, then*

$$\hat{H}^i(\mathrm{Gal}(L/K), (L \otimes_K K^{\mathrm{unr},\wedge})^\times) = 0$$

for all $i \in \mathbb{Z}$.

Proof. From Example 2.1.3, we have isomorphisms

$$\hat{H}^{2i}(\mathrm{Gal}(L/K), (L \otimes_K K^{\mathrm{unr},\wedge})^\times) \cong \hat{H}^0(\mathrm{Gal}(L/K), (L \otimes_K K^{\mathrm{unr},\wedge})^\times),$$
$$\hat{H}^{2i-1}(\mathrm{Gal}(L/K), (L \otimes_K K^{\mathrm{unr},\wedge})^\times) \cong \hat{H}^{-1}(\mathrm{Gal}(L/K), (L \otimes_K K^{\mathrm{unr},\wedge})^\times).$$

Therefore it suffices to prove that

(1) the norm map $N_{L/K} \colon (L \otimes_K K^{\mathrm{unr},\wedge})^\times \to K^\times$ is surjective,
(2) the kernel of the norm map is generated as a group by elements of the form ga/a for $g \in \mathrm{Gal}(L/K)$ and $a \in (L \otimes_K K^{\mathrm{unr},\wedge})^\times$.

Since $\mathrm{Gal}(L/K)$ is a cyclic extension of prime degree, it is either unramified or totally ramified. If it is unramified, we note that Proposition 2.2.4 identifies

$$\phi \colon (L \otimes_K K^{\mathrm{unr},\wedge})^\times \cong \prod_{i=0}^{f-1} (K^{\mathrm{unr},\wedge})^\times; \quad x \otimes y \mapsto (\mathrm{Frob}_{L/K}^i(x)y)_{0 \le i < f}.$$

Under the isomorphism, the $\mathrm{Gal}(L/K)$-action is given by

$$\mathrm{Frob}_{L/K} \colon (x_0, \dots, x_{f-1}) \mapsto (x_1, \dots, x_{f-1}, x_0).$$

It immediately follows that the norm map is given by

$$(x_0, \dots, x_{f-1}) \mapsto (\textstyle\prod_{i=0}^{f-1} x_i, \dots, \prod_{i=0}^{f-1} x_i),$$

and hence surjects onto the $\mathrm{Gal}(L/K)$-invariants, while the kernel is generated by elements of the form $\mathrm{Frob}_{L/K}(x)/x$.[2]

Let us now consider the case when L/K is totally ramified. In this case, the same Proposition 2.2.4 identifies the module as the compositum

$$\phi \colon (L \otimes_K K^{\mathrm{unr},\wedge})^\times \cong (L^{\mathrm{unr},\wedge})^\times.$$

Furthermore, the Galois action is given acting through isomorphism $\mathrm{Gal}(L/K) \cong \mathrm{Gal}(L^{\mathrm{unr},\wedge}/K^{\mathrm{unr},\wedge})$. Hilbert's theorem 90 in its classical form precisely tells us that $Nx = 1$ for $x \in L^{\mathrm{unr},\wedge}$ implies $x = \sigma(y)/y$ for some $\sigma \in \mathrm{Gal}(L/K)$ a generator and $y \in L^{\mathrm{unr},\wedge}$. This implies that

$$\hat{H}^{-1}(\mathrm{Gal}(L/K), (L^{\mathrm{unr},\wedge})^\times) = 0.$$

[2] We may also argue that since this $\mathrm{Gal}(L/K)$-module is induced and coinduced, so all its group cohomology and homology vanish, and by 2-periodicity all Tate cohomology vanish.

For $\hat{H}^0$, we again use a proposition that we shall not prove. In Serre's *Local Fields* [Ser79], Section V.4, it is proven that the norm map

$$(L^{\mathrm{unr},\wedge})^\times \xrightarrow{N_{L/K}} (K^{\mathrm{unr},\wedge})^\times$$

is surjective. It immediately follows that

$$\hat{H}^0(\mathrm{Gal}(L/K), (L^{\mathrm{unr},\wedge})^\times) = 0.$$

Therefore we obtain the desired vanishing for L/K totally ramified as well. $\square$

Lemma 2.1.7 tells us that if we have a module that is acyclic over both H and G/H in an appropriate sense, then it will be acyclic over G. We can use this to arrive at the same conclusion for arbitrary finite Galois extensions, not necessarily cyclic of prime order.

Theorem 2.2.12. *If L/K is a finite Galois extension of p-adic local fields, then*

$$\hat{H}^i(\mathrm{Gal}(L/K), (L \otimes_K K^{\mathrm{unr},\wedge})^\times) = 0$$

for all $i \in \mathbb{Z}$.

Proof. Using Lemma 2.2.10 we can find a filtration

$$\mathrm{Gal}(L/K) = H_0 \supset H_1 \supset H_2 \supset \cdots \supset H_{t-1} \supset H_t = 1,$$

such that each quotient inclusion $H_{i+1} \subset H_i$ is a normal subgroup and H_i/H_{i+1} is a cyclic group of prime order. Denote $L_i = L^{H_i}$ so that we have a sequence of field extensions

$$K = L_0 \hookrightarrow L_1 \hookrightarrow L_2 \hookrightarrow \cdots \hookrightarrow L_{t-1} \hookrightarrow L_t = L.$$

We now show that

$$\hat{H}^i(\mathrm{Gal}(L/L_j), (L \otimes_{L_j} L_j^{\mathrm{unr},\wedge})^\times) = 0$$

for all $i \in \mathbb{Z}$, inductively in j starting from $j = t$ and descending to $j = 0$. In the base case $j = t$, this is trivially satisfied since $\mathrm{Gal}(L/L) = \{\mathrm{id}_L\}$ is the trivial group. Suppose that the statement holds for $j+1$ and let us try to prove it for j. We would like to apply Lemma 2.1.7 to the normal subgroup group $\mathrm{Gal}(L/L_{j+1}) \subseteq \mathrm{Gal}(L/L_j)$ and the Galois module $(L \otimes_{L_j} L_j^{\mathrm{unr},\wedge})^\times$.

By the induction hypothesis, we know that

$$\hat{H}^i(\mathrm{Gal}(L/L_{j+1}), (L \otimes_{L_{j+1}} L_{j+1}^{\mathrm{unr},\wedge})^\times) = 0.$$

Here, we observe that

$$L \otimes_{L_j} L_j^{\mathrm{unr},\wedge} \cong L \otimes_{L_{j+1}} (L_{j+1} \otimes_{L_j} L_j^{\mathrm{unr},\wedge}) \cong L \otimes_{L_{j+1}} (\textstyle\prod L_{j+1}^{\mathrm{unr},\wedge})$$

as $\mathrm{Gal}(L/L_{j+1})$-modules, and hence the $\mathrm{Gal}(L/L_{j+1})$-module $(L \otimes_{L_j} L_j^{\mathrm{unr},\wedge})^\times$ is simply a finite direct sum of copies of $(L \otimes_{L_{j+1}} L_{j+1}^{\mathrm{unr},\wedge})^\times$. It follows that

$$\hat{H}^i(\mathrm{Gal}(L/L_{j+1}), (L \otimes_{L_j} L_j^{\mathrm{unr},\wedge})^\times) = \bigoplus 0 = 0$$

for all $i \in \mathbb{Z}$.

To apply Lemma 2.1.7, it is now enough to verify that

$$\hat{H}^i(\mathrm{Gal}(L_{j+1}/L_j), ((L \otimes_{L_j} L_j^{\mathrm{unr},\wedge})^\times)^{\mathrm{Gal}(L/L_{j+1})}) = 0$$

for $i \in \mathbb{Z}$. Here, we simply note that the $\mathrm{Gal}(L/L_{j+1})$-fixed points of $L \otimes_{L_j} L_j^{\mathrm{unr},\wedge}$ is simply $L_{j+1} \otimes_{L_j} L_j^{\mathrm{unr},\wedge}$. Since L_{j+1}/L_j is cyclic of prime order by definition, vanishing of this Tate cohomology directly follows from Lemma 2.2.11. □

In the exact sequence

$$0 \to L^\times \to (L \otimes_K K^{\mathrm{unr},\wedge})^\times \xrightarrow{\mathrm{La}} (L \otimes_K K^{\mathrm{unr},\wedge})^\times \xrightarrow{\sum v} \mathbb{Z} \to 0$$

of Theorem 2.2.9, we now see that the middle two terms have zero Tate cohomology in all degrees.

Corollary 2.2.13. *The composition of the inverse of two coboundary maps gives an isomorphism*

$$\begin{aligned} -\theta_{L/K} : K^\times/NL^\times &\cong \hat{H}^0(\mathrm{Gal}(L/K), L^\times) \\ &\xrightarrow{\cong} \hat{H}^{-2}(\mathrm{Gal}(L/K), \mathbb{Z}) \cong \mathrm{Gal}(L/K)^{\mathrm{ab}}. \end{aligned}$$

(To agree with convention, we have to put in a minus sign; otherwise, the valuation on the left hand side recovers the inverse of the action on the maximal unramified on the right hand side.)

2.2.4 Explicit description of the Artin reciprocity map

We use the explicit resolution to give a more explicit description of the Artin reciprocity map $\theta_{L/K} : K^\times/NL^\times \to \mathrm{Gal}(L/K)^{\mathrm{ab}}$. The idea is that we can produce explicit resolutions of $\mathbb{Z}$ in other ways, for example, by the bar construction, and then compare the two resolutions we have.

For brevity let us denote $G = \mathrm{Gal}(L/K)$. The bar construction gives a resolution of $\mathbb{Z}$ by

$$0 \to \ker \delta_2 \to \mathbb{Z}[G^2] \xrightarrow{\delta_2} \mathbb{Z}[G] \xrightarrow{\delta_1} \mathbb{Z} \to 0.$$

We observe two facts:

- The G-modules $\mathbb{Z}[G^n]$ are free $\mathbb{Z}[G]$-modules, hence they are projective.
- Because $\mathbb{Z}[G]$ is both induced and coinduced, we see that all its group cohomology and homology vanish. Moreover, the 0th and -1th Tate cohomology vanish as well, and hence $\mathbb{Z}[G^n]$ are all acyclic with respect to Tate cohomology.

Because the modules are projective, we can lift maps to form the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \ker\delta_2 \arrow[r] \arrow[d, dashed, "\varphi"] & \mathbb{Z}[G^2] \arrow[r, "\delta_2"] \arrow[d, dashed] & \mathbb{Z}[G] \arrow[r, "\delta_1"] \arrow[d, dashed] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L^\times \arrow[r] & (L \otimes_K K^{\mathrm{unr},\wedge})^\times \arrow[r, "\mathrm{La}"] & (L \otimes_K K^{\mathrm{unr},\wedge})^\times \arrow[r, "\sum v"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Since the terms in the middle of the exact sequences are all acyclic with respect to Tate cohomology, the map φ will induces an isomorphism of cohomology groups

$$\hat{H}^0(\mathrm{Gal}(L/K), L^\times) \xleftarrow{\varphi_*} \hat{H}^0(\mathrm{Gal}(L/K), \ker\delta_2) \cong \mathrm{Gal}(L/K)^{\mathrm{ab}}.$$

To work this out precisely, we need a description of the map φ and the isomorphism $\hat{H}^0(\ker\delta_2) \cong \mathrm{Gal}(L/K)^{\mathrm{ab}}$.

First, let us explicitly figure out how the 0th cohomology of $\ker\delta_2$ is identified with the abelianization of the group G. The boundary map δ_2 takes the form of

$$\delta_2 : (g_1, g_2) \mapsto g_2 - g_1.$$

Then we see that if we take the G-invariants $\mathbb{Z}[G^2]$, which is the free abelian group generated by $\sum_{g\in G}(g, gg_0) \in \mathbb{Z}[G^2]$ for $g_0 \in G$, it is already contained in $\ker\delta_2$. Therefore

$$(\ker\delta_2)^G = \mathbb{Z}[G^2]^G.$$

To compute the norms, we see that the kernel $\ker\delta_2$ is the subgroup of $\mathbb{Z}[G^2]$ generated by elements of the form

$$(g_1, g_2) + (g_2, g_3) - (g_1, g_3) \in \mathbb{Z}[G^2].$$

It follows that the group of norms are generated by

$$N((g_1, g_2) + (g_2, g_3) - (g_1, g_3)) = N((1, g_1^{-1}g_2)) + N((1, g_2^{-1}g_3)) - N((1, g_1^{-1}g_3)).$$

Since the G-invariants is the free group generated by $N((1, g))$, we obtain an isomorphism

$$\hat{H}^0(\mathrm{Gal}(L/K), \ker\delta_2) \xrightarrow{\cong, N((1,g))\mapsto g} \mathbb{Z}[G]/\langle g_1 + g_2 - g_1g_2\rangle \cong G^{\mathrm{ab}}.$$

Here, we get the abelianization since $g_1g_2 = g_1 + g_2 = g_2 + g_1 = g_2g_1$ under the relation.

Let us now work out the map φ that induces the isomorphism on the 0th Tate cohomology. To obtain the first lift $\mathbb{Z}[G] \to (L \otimes_K K^{\mathrm{unr},\wedge})^\times$, we only need to choose where $1 \in \mathbb{Z}[G]$ maps to. Thus we pick an element $\alpha \in (L \otimes_K K^{\mathrm{urn}})^\times$ that has valuation equal to 1.

$$\mathbb{Z}[G] \to (L \otimes_K K^{\mathrm{unr},\wedge})^\times; \quad 1 \mapsto \alpha.$$

To obtain the next lift, we note that $\mathbb{Z}[G^2]$ as a $\mathbb{Z}[G]$-module is free with generators $\{(1,g) \in G\}$. Note that under the map

$$\mathbb{Z}[G^2] \xrightarrow{\delta_2} \mathbb{Z}[G] \xrightarrow{\alpha} (L \otimes_K K^{\mathrm{unr},\wedge})^\times,$$

the generators $(1,g)$ are sent to

$$(1,g) \mapsto g - 1 \mapsto \frac{g\alpha}{\alpha}.$$

Therefore we find lifts $\beta_g \in (L \otimes_K K^{\mathrm{unr},\wedge})^\times$ for each $g \in G$ satisfying

$$\mathrm{La}(\beta_g) = \frac{\mathrm{Frob}(\beta_g)}{\beta_g} = \frac{g\alpha}{\alpha} \in (L \otimes_K K^{\mathrm{unr},\wedge})^\times$$

and define the homomorphism

$$\beta : \mathbb{Z}[G^2] \to (L \otimes_K K^{\mathrm{unr},\wedge})^\times; \quad (g_1, g_2) = g_1(1, g_1^{-1} g_2) \mapsto g_1 \beta_{g_1^{-1} g_2}.$$

Now we can describe isomorphism on Tate cohomology. If we restrict the map β to $\ker \delta_2$, the isomorphism on the 0th Tate cohomology is given by

$$(\ker \delta_2)^{\mathrm{Gal}(L/K)} / N(\ker \delta_2) \xrightarrow{\beta_*} K^\times / NL^\times; \quad N((1,g)) \mapsto N\beta_g.$$

Here, we note that the norm $N\beta_g$ of β_g is indeed in $K^\times$ since first it is invariant under the action of $\mathrm{Gal}(L/K)$ and secondly

$$\frac{\mathrm{Frob}(N\beta_g)}{N\beta_g} = N\left(\frac{\mathrm{Frob}(\beta_g)}{\beta_g}\right) = N\left(\frac{g\alpha}{\alpha}\right) = 1$$

shows that it is also invariant under the action of Frob.

From the above discussion, we obtain the following theorem.

Theorem 2.2.14 (Dwork [Dwo58]). *Let L/K be a finite Galois extension of p-adic local fields. Fix a Galois automorphism $g \in \mathrm{Gal}(L/K)$. Choose*

(1) an element $\alpha \in (L \otimes_K K^{\mathrm{unr},\wedge})^\times$ such that $\sum_{i=1}^{f} v(\alpha_i) = 1$, and

(2) an element $\beta \in (L \otimes_K K^{\mathrm{unr},\wedge})^\times$ satisfying

$$\frac{\mathrm{Frob}(\beta)}{\beta} = \frac{g\alpha}{\alpha}.$$

Then $N\beta^{-1} \in K^\times$ and its equivalence class in $K^\times/NL^\times$ is independent of the choices of α and β. Moreover, this class is the image of $g \in \mathrm{Gal}(L/K)$ under the composition

$$\mathrm{Gal}(L/K) \twoheadrightarrow \mathrm{Gal}(L/K)^{\mathrm{ab}} \xrightarrow{\theta_{L/K}^{-1}} K^\times/NL^\times.$$

(We take the norm of β^{-1} instead of β to have the correct sign.)

In fact, the above description uniquely defines the Artin reciprocity map $\theta_{L/K}$. What do we do if we want to describe the absolute version of the Artin reciprocity map $\theta_{/K} : K^\times \to \mathrm{Gal}(K^{\mathrm{sep}}/K)^{\mathrm{ab}}$? We can compose with the natural restriction map

$$\mathrm{Gal}(K^{\mathrm{sep}}/K)^{\mathrm{ab}} \twoheadrightarrow \mathrm{Gal}(L/K)^{\mathrm{ab}} \xrightarrow{\theta_{L/K}^{-1}} K^\times/NL^\times$$

to obtain information about $\theta_{/K}$ up to $NL^\times$. As we use larger and larger L, we will get finer and finer descriptions of the reciprocity map $\theta_{/K}$.

2.2.5 Proof of local class field theory

We have constructed the isomorphisms between $K^\times/NL^\times$ and $\mathrm{Gal}(L/K)^{\mathrm{ab}}$ for finite Galois extensions L/K. We would now like to show that we may stack those relative isomorphisms together to obtain the absolute Artin reciprocity map

$$\theta_{/K} : \widehat{K^\times} \xrightarrow{\cong} \mathrm{Gal}(K^{\mathrm{sep}}/K)^{\mathrm{ab}}.$$

At a heuristic level, it makes sense that we have such an isomorphism since

$$\widehat{K^\times} = \varprojlim_U K^\times/U, \quad \mathrm{Gal}(K^{\mathrm{sep}}/K)^{\mathrm{ab}} = \varprojlim_L \mathrm{Gal}(L/K)^{\mathrm{ab}}$$

are both inverse limits of the objects appearing on either side of the relative Artin reciprocity map $\theta_{L/K}$. Thus, to achieve the goal we need two facts: firstly that the relative Artin map really behaves well when we change L to a larger field, and secondly that the groups $NL^\times$ can be arbitrary small open subgroups of $K^\times$.

Let us first prove that the Artin reciprocity map behaves well with respect to passing to large extensions.

Proposition 2.2.15. *The relative Artin reciprocity map $\theta_{L/K}$ constructed as in Corollary 2.2.13 satisfies the following property: if $E/L/K$ are finite extensions of a p-adic local field K, with both E/K and L/K Galois, the diagram*

$$\begin{tikzcd} K^\times/N_{E/K}E^\times \arrow[r, "\theta_{E/K}"] \arrow[d, two heads] & \mathrm{Gal}(E/K)^{\mathrm{ab}} \arrow[d, two heads] \\ K^\times/N_{L/K}L^\times \arrow[r, "\theta_{L/K}"] & \mathrm{Gal}(L/K)^{\mathrm{ab}} \end{tikzcd}$$

commutes. (Note that $N_{E/K}E^\times \subseteq N_{L/K}L^\times$ since $N_{E/K} = N_{L/K} \circ N_{E/L}$.)

Proof. We first note that the diagram

$$\begin{tikzcd} 0 \arrow[r] & E^\times \arrow[r] \arrow[d, "N_{E/L}"] & (E \otimes K^{\mathrm{unr},\wedge})^\times \arrow[r, "\mathrm{La}"] \arrow[d, "N_{E/L}"] & (E \otimes K^{\mathrm{unr},\wedge})^\times \arrow[r] \arrow[d, "N_{E/L}"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L^\times \arrow[r] & (L \otimes K^{\mathrm{unr},\wedge})^\times \arrow[r, "\mathrm{La}"] & (L \otimes K^{\mathrm{unr},\wedge})^\times \arrow[r] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

commutes. We use the explicit description of the Artin reciprocity map. Given a Galois automorphism $g \in \mathrm{Gal}(E/K)$, we need to show that first mapping under $\theta_{E/K}^{-1}$ and then projecting down is equal to first restricting to $\mathrm{Gal}(L/K)^{\mathrm{ab}}$ and then mapping through $\theta_{L/K}^{-1}$.

To see this, we first take an element $\alpha_E \in (E \otimes K^{\mathrm{unr},\wedge})^\times$ as in Theorem 2.2.14. Then $\alpha_L = N_{E/L}\alpha_E$ maps to $1 \in \mathbb{Z}$ and hence can be used to compute $\theta_{L/K}^{-1}(g)$ as well. We solve the equation

$$\frac{\mathrm{Frob}_{/K}(\beta_E)}{\beta_E} = \frac{g\alpha_E}{\alpha_E}$$

for $\beta_E \in (E \otimes K^{\mathrm{unr},\wedge})^\times$ and then we have

$$\theta_{E/K}^{-1}(g) = N_{E/K}\beta_E^{-1}.$$

On the other hand, applying $N_{E/L}$ on both sides give

$$\frac{\mathrm{Frob}_{/K}(N_{E/L}\beta_E)}{N_{E/L}\beta_E} = N_{E/L}\left(\frac{\mathrm{Frob}_{/K}(\beta_E)}{\beta_E}\right) = N_{E/L}\left(\frac{g\alpha_E}{\alpha_E}\right) = \frac{g\alpha_L}{\alpha_L}$$

since $\mathrm{Gal}(E/L) \subseteq \mathrm{Gal}(E/K)$ being normal implies $g \circ N_{E/L} = N_{E/L} \circ g$. Therefore we obtain

$$\theta_{L/K}^{-1}(g) = N_{L/K}(N_{E/L}\beta_E^{-1}) = N_{E/K}\beta_E^{-1} = N_{E/L}\theta_{E/K}^{-1}(g),$$

and this proves the proposition. □

Proposition 2.2.16. *Let K be a p-adic local field, and consider its multiplicative group $K^\times$ with the natural induced topology.*

- *(a) For any finite extension L/K, the subgroup of norms $NL^\times \subseteq K^\times$ is open.*
- *(b) For any finite index open subgroup $U \subseteq K^\times$, there exists a finite extension L/K for which $NL^\times \subseteq U$.*

Proof. (a) It suffices to show that $NL^\times$ contains $1 + \mathfrak{m}_K^t$ for large enough t. But we note that $NL^\times$ contains $(K^\times)^d$ where $d = [L : K]$ is the degree, and $(1 + \mathfrak{m}_K)^d$ contains some open neighborhood of 1 by Hensel's lemma.

(b) We first note that any finite index open subgroup $U \subseteq K^\times$ contains $(K^\times)^t$ for large enough t. This is because any index t subgroup necessarily contains $(K^\times)^t$. Hence it suffice to show that there exist finite L/K for which $NL^\times \subseteq (K^\times)^t$.

Before we construct this field extension L, we first observe that the Artin reciprocity map tells us that if $E, L/K$ are finite abelian extensions, then the norms of the compositum EL can be described as

$$N_{EL/K}(EL)^\times = (N_{E/K}E^\times) \cap (N_{L/K}L^\times) \subseteq K^\times.$$

This is because the kernel of $\mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Gal}(E/K) \times \mathrm{Gal}(L/K)$ corresponds to the kernel of $K^\times \to (K^\times/N_{E/K}E^\times) \times (K^\times/N_{L/K}L^\times)$.

Using this fact, we explicitly construct such a field L for which $NL^\times \subseteq (K^\times)^t$. We first adjoin the tth roots of unity to obtain $K_0 = K(\zeta_t)$ and then adjoin all tth roots of elements of K_0 to obtain $L = K_0(K_0^{1/t})$. This is indeed a finite extension because $(K_0^\times)^t \subseteq K_0^\times$ has finite index, and hence we need only adjoin finitely many $a^{1/t}$.

First, we note that

$$N_{L/K_0}L^\times = \bigcap_{a \in K_0^\times} N_{K_0(a^{1/t})/K_0}K_0(a^{1/t})^\times,$$

and because each $NK_0(a^{1/t})^\times$ has index t inside $K_0^\times$, each of the $NK_0(a^{1/t})^\times$ contains $(K_0^\times)^t$. It follows that

$$N_{L/K_0}L^\times \supseteq (K_0^\times)^t.$$

On the other hand, Kummer theory tells us that a degree t cyclic extension of K_0 uniquely looks like $K_0(a^{1/t})$ for $a \in K_0^\times/(K_0^\times)^t$, and hence

$$|\mathrm{Gal}(L/K_0)| = |\mathrm{Hom}(\mathrm{Gal}(L/K_0), \mu_t)| = [K_0^\times : (K_0^\times)^t].$$

Since the Artin reciprocity map gives an isomorphism

$$K_0^\times/N_{L/K_0}L^\times \cong \mathrm{Gal}(L/K_0),$$

it follows that

$$N_{L/K_0}L^\times = (K_0^\times)^t.$$

Then

$$N_{L/K}L^\times \subseteq N_{K_0/K}(N_{L/K_0}L^\times) = N_{K_0/K}((K_0^\times)^t) \subseteq (K^\times)^t,$$

and this finishes the proof. $\square$

Therefore we may construct and define the absolute Artin reciprocity map

$$\theta_{/K}^{-1}\colon\ \mathrm{Gal}(K^{\mathrm{sep}}/K)^{\mathrm{ab}} = \varprojlim_{L/K} \mathrm{Gal}(L/K)^{\mathrm{ab}} \xrightarrow{\theta_{L/K}^{-1}} \varprojlim_{L/K} K^\times/NL^\times = \widehat{K^\times},$$

which is an isomorphism since it is the projective limit of isomorphisms. To prove Theorem 1.2.1, we need to verify the listed properties.

Proposition 2.2.17. *The relative Artin reciprocity map $\theta_{L/K}$ constructed as in Corollary 2.2.13 satisfies the following property: if $E/L/K$ are finite extensions of a p-adic local field K, with E/K Galois, the diagram*

$$\begin{array}{ccc} L^\times/N_{E/L}E^\times & \xrightarrow{\theta_{E/L}} & \mathrm{Gal}(E/L)^{\mathrm{ab}} \\ \downarrow{\scriptstyle N_{L/K}} & & \downarrow{\scriptstyle \text{inc.}} \\ K^\times/N_{E/K}E^\times & \xrightarrow{\theta_{E/K}} & \mathrm{Gal}(E/K)^{\mathrm{ab}} \end{array}$$

commutes.

Proof. Note that there is a multiplication map

$$\mu\colon L \otimes_K K^{\mathrm{unr},\wedge} \to L^{\mathrm{unr},\wedge}; \quad x \otimes y \mapsto xy$$

of algebras, and it induces a map

$$E \otimes_K K^{\mathrm{unr},\wedge} = E \otimes_L (L \otimes_K K^{\mathrm{unr},\wedge}) \to E \otimes_L L^{\mathrm{unr},\wedge}.$$

Moreover, if we denote by $f = [l : k]$ the inertial degree of L/K, the diagram

$$\begin{tikzcd} L \otimes_K K^{\mathrm{unr},\wedge} \arrow[r, "\mathrm{id} \otimes \mathrm{Frob}^f_{/K}"] \arrow[d, "\mu"] & L \otimes_K K^{\mathrm{unr},\wedge} \arrow[d, "\mu"] \\ L^{\mathrm{unr},\wedge} \arrow[r, "\mathrm{Frob}_{/L}"] & L^{\mathrm{unr},\wedge} \end{tikzcd}$$

commutes. Hence tensoring with E over L shows that the diagram

$$\begin{tikzcd} E \otimes_K K^{\mathrm{unr},\wedge} \arrow[r, "\mathrm{Frob}^f_{/K}"] \arrow[d, "\mu"] & E \otimes_K K^{\mathrm{unr},\wedge} \arrow[d, "\mu"] \\ E \otimes_L L^{\mathrm{unr},\wedge} \arrow[r, "\mathrm{Frob}_{/L}"] & E \otimes_L L^{\mathrm{unr},\wedge} \end{tikzcd}$$

commutes.

We now use the description of the Artin reciprocity map as in Theorem 2.2.14. Fix a $g \in \mathrm{Gal}(E/L)$ which can also be thought of as an element of $\mathrm{Gal}(E/K)$. To compute $\theta_{E/K}^{-1}(g)$, we first choose an element $\alpha_K \in (E \otimes_K K^{\mathrm{unr},\wedge})^\times$ with valuation 1. If we solve the equation

$$\frac{\mathrm{Frob}_{/K}\,\beta}{\beta} = \frac{g\alpha_K}{\alpha_K}$$

for $\beta \in (E \otimes_K K^{\mathrm{unr},\wedge})^\times$, then

$$\theta_{E/K}^{-1}(g) = [N_{E/K}\beta^{-1}] \in K^\times / N_{E/K}E^\times.$$

On the other hand, we have

$$\frac{\mathrm{Frob}_{/L}(\mu\beta)}{\mu\beta} = \mu\Big(\frac{\mathrm{Frob}^f_{/K}\,\beta}{\beta}\Big) = \mu\Big(\frac{g(\alpha_K \,\mathrm{Frob}_{/K}\,\alpha_K \cdots \mathrm{Frob}^{f-1}_{/K}\,\alpha_K)}{\alpha_K \,\mathrm{Frob}_{/K}\,\alpha_K \cdots \mathrm{Frob}^{f-1}_{/K}\,\alpha_K}\Big).$$

Hence if we set

$$\alpha_L = \mu(\alpha_K \,\mathrm{Frob}_{/K}\,\alpha_K \cdots \mathrm{Frob}^{f-1}_{/K}\,\alpha_K)$$

then

$$\frac{\mathrm{Frob}_{/L}(\mu\beta)}{\mu\beta} = \frac{g\alpha_L}{\alpha_L}.$$

Here, we observe that the valuation of α_L in $(E \otimes_L L^{\mathrm{unr},\wedge})^\times$ is equal to the valuation of α_K inside $(E \otimes_K K^{\mathrm{unr},\wedge})^\times$ after expanding out the definition of the valuation. Therefore the valuation of α_L is 1, and hence can be used to compute the reciprocity map. In particular,

$$\theta_{E/L}^{-1}(g) = [N_{E/L}(\mu\beta)^{-1}] \in L^\times / N_{E/K} E^\times.$$

However, we note that

$$\begin{tikzcd}
E^\times = (E \otimes_K K)^\times \arrow[r, hook] \arrow[d, equal] & (E \otimes_K K^{\mathrm{unr},\wedge})^\times \arrow[d, "\mu"] \\
E^\times = (E \otimes_L L)^\times \arrow[r, hook] & (E \otimes_L L^{\mathrm{unr},\wedge})^\times
\end{tikzcd}$$

commutes, and hence when we restrict out attention to inside $E^\times$, the multiplication map μ is the identity. It follows that

$$\theta_{E/K}^{-1}(g) = [N_{E/K}\beta^{-1}] = N_{L/K}[N_{E/L}\beta^{-1}] = N_{L/K}\theta_{E/L}^{-1}(g),$$

and this proves the commutativity of the desired diagram. $\square$

Proposition 2.2.18. *The relative Artin reciprocity map $\theta_{L/K}$ constructed as in Corollary 2.2.13 satisfies the following property: if L/K is unramified of degree d, then the Artin map is given by*

$$\theta_{L/K} \colon K^\times / NL^\times \to \mathrm{Gal}(L/K) \cong \mathbb{Z}/n\mathbb{Z}; \quad x \mapsto \log_{|\pi_K|}|x|.$$

Proof. Again, we use the explicit description of the map from Theorem 2.2.14. Because L/K is unramified, Proposition 2.2.4 gives an alternative description

$$\phi \colon L \otimes_K K^{\mathrm{unr},\wedge} \cong (K^{\mathrm{unr},\wedge})^d; \quad x \otimes y \mapsto (\mathrm{Frob}_{L/K}^i(x)y),$$

with the Lang map given by

$$\mathrm{La} \colon (x_0, \dots, x_{d-1}) \mapsto \left(\tfrac{\mathrm{Frob}_{/K}(x_{d-1})}{x_0}, \tfrac{\mathrm{Frob}_{/K}(x_0)}{x_1}, \dots, \tfrac{\mathrm{Frob}_{/K}(x_{d-2})}{x_{d-1}}\right).$$

In our case, the $\mathrm{Gal}(L/K)$-action on $L \otimes_K K^{\mathrm{unr},\wedge}$ is simply given by shifting,

$$\mathrm{Frob}_{L/K} \colon (x_0, \dots, x_{d-1}) \mapsto (x_1, \dots, x_{d-1}, x_0).$$

To compute $\theta_{L/K}$, we first take

$$\alpha = (\pi_K, 1, 1, \dots, 1) \in ((K^{\mathrm{unr},\wedge})^\times)^d \cong (L \otimes_K K^{\mathrm{unr},\wedge})^\times,$$

and find a solution to $\mathrm{La}(\beta) = g\alpha/\alpha$. If $g = \mathrm{Frob}_{L/K}$ is the generator, we have

$$\frac{g\alpha}{\alpha} = (\pi_K^{-1}, 1, \dots, 1, \pi_K)$$

and hence

$$\beta = (1, \dots, 1, \pi_K^{-1}) \in (L \otimes_K K^{\mathrm{unr},\wedge})^\times$$

is a solution to $g\alpha/\alpha = \mathrm{La}(\beta)$. Therefore, when we take its norm we obtain

$$N\beta = (\pi_K^{-1}, \dots, \pi_K^{-1}) \in (L \otimes_K K^{\mathrm{unr},\wedge})^\times,$$

which corresponds to $\pi_K^{-1} \in K^\times$ under ϕ. Therefore $N\beta^{-1}$ defines the correct class in $K^\times/NL^\times$. $\square$

Combining all of the results in this subsection, we finally obtain a full proof of local class field theory, Theorem 1.2.1.

Chapter 3

Explicit local class field theory via Lubin–Tate groups

In the previous chapter, we used Galois cohomology to compute the abelianization of the Galois group. The way we carried out the computation was to use a certain acyclic resolution of $\mathbb{Z}$ or $L^\times$. The method of Lubin–Tate theory is to describe the local Artin map by explicitly constructing a representation.

The inverse of the Artin reciprocity map will gives us a group homomorphism

$$W_K \twoheadrightarrow W_K^{\mathrm{ab}} \xrightarrow{\theta_{/K}^{-1}} K^\times = \mathrm{GL}_1(K).$$

Thus this contains the same information as a 1-dimensional K-linear representation of the Weil group W_K.

Question. *Which* 1*-dimensional representation does this correspond to?*

The strategy is to find an algebro-geometric object defined over K that has a structure of a module. If X/K is a scheme for instance, the set of K^{sep}-points naturally has a $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-action. If X/K has an algebraic structure of an $\mathcal{O}_K$-module, for instance, then $X(K^{\mathrm{sep}})$ naturally is also an $\mathcal{O}_K$-module and the Galois action has to be $\mathcal{O}_K$-linear. In other words, the $\mathcal{O}_K$-module

$$X(K^{\mathrm{sep}})$$

is a $\mathcal{O}_K$-linear representation of the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$.

It turns out that schemes are too rigid to work with in this context. There are very few commutative group schemes (of finite type) out there, and it is almost impossible for them to have interesting $\mathcal{O}_K$-module structures. Thus we make a compromise by including "formal schemes" in our class of algebro-geometric objects.

This elegant approach to local class field theory was developed entirely by Lubin–Tate [LT65]. Interestingly, the approach was developed nearly a hundred years after Kronecker developed explicit global class field theory over $\mathbb{Q}$ and imaginary quadratic fields. The Lubin–Tate formal module is supposed to be the local analogue of an elliptic curve with complex multiplication, so that the ring of endomorphisms look like the ring of integers in the p-adic local field.

3.1 Formal schemes and groups

To define formal groups, we do not need the most general notion of a formal schemes. Thus instead of defining a formal group, we shall content with giving the one example we will be using.

Let A be a commutative ring, and consider the ring of formal power series

$$A[[x]] = \left\{ \sum_{i=0}^{\infty} a_i x^i : a_0, a_1, \ldots \in A \right\}.$$

This ring has a topology, with set of cosets of the ideals (x^n) being a topological basis. Instead of taking the set of prime ideals of the ring $A[[x]]$ and forming $\operatorname{Spec} A[[x]]$, we shall consider the ring as the inverse limit

$$A = A[x]/(x) \twoheadleftarrow A[x]/(x^2) \twoheadleftarrow A[x]/(x^3) \twoheadleftarrow A[x]/(x^4) \twoheadleftarrow \cdots$$

and then consider the scheme as the direct limit of the map on Spec, so

$$\operatorname{Spf} A[[x]] = \varinjlim(\operatorname{Spec} A[x]/(x) \hookrightarrow \operatorname{Spec} A[x]/(x^2) \hookrightarrow \operatorname{Spec} A[x]/(x^3) \hookrightarrow \cdots).$$

(The direct limit is taken inside the category of functors $\mathsf{Ring} \to \mathsf{Set}$, not in the category of schemes.) This process will somehow record the topological structure of $A[[x]]$ and produce a different object than $\operatorname{Spec} A[[x]]$.

We now contemplate on what it means for the formal scheme $\operatorname{Spf} A[[x]]$ to have a group structure. First, let us assume that the identity is at the point $x = 0$. A group multiplication map will be a map $\operatorname{Spf} A[[x]] \times \operatorname{Spf} A[[x]] \to \operatorname{Spf} A[[x]]$ and hence will correspond to a algebra homomorphism $A[[x]] \to A[[x]] \hat{\otimes} A[[x]] \cong A[[x_1, x_2]]$. Since $A[[x]]$ is topologically generated by the one element $x \in A[[x]]$, the homomorphism will be determined completely by this one element x. This motivates us to make the following definition.

Definition 3.1.1. Let A be a ring. An abelian group structure on $\operatorname{Spf} A[[x]]$, or a **formal group law** is a formal power series $F \in A[[x, y]]$ satisfying

(1) $F(x, y) \in x + y + xyA[[x, y]]$ (so that $F(x, 0) = x$ and $F(0, y) = y$),

(2) $F(x, y) = F(y, x)$ (so that the group is commutative),

(3) $F(x, F(y, z)) = F(F(x, y), z)$ (so that the group is associative),

(4) there exists a power series $G(x) \in A[[x]]$ such that $F(x, G(x)) = 0$ (so that there is an inverse).

The formal scheme $A[[x]]$ with this group structure given by F will be normally denoted in Fraktur, such as $\mathfrak{F}$.

Note that composition of formal power series such as $F(x, F(y, z))$ makes sense since (1) guarantees that only finitely many terms contribute to a given degree.

Lemma 3.1.2. *If a power series F satisfies (1), then it automatically satisfies (4) as well.*

Proof. We first set $G \equiv -x \pmod{x^2}$. Given G modulo x^n, we inductively lift it to G modulo x^{n+1}. If we are given G modulo x^n, (1) tells us that $F(x, G(x)) - x - G(x) \in xG(x)A[[x]]$ is determined modulo $n+1$. Thus to make $F(x, G(x)) = 0$, there exists a unique way to lift $G(x)$ to modulo x^{n+1}. □

We remark that if we take a functor-of-points point of view, the formal scheme $\operatorname{Spf} A[[x]]$ may be considered as a functor

$$A-\mathsf{alg} \to \mathsf{Set}; \quad B \mapsto \{\text{nilpotent elements of } B\}.$$

Then a formal group law may be thought of giving a factorization of this functor though the category of groups Grp. The group structure on the set of nilpotent elements is defined by F. Indeed $F(x, y) \in B$ makes sense if x and y are nilpotent, since terms with sufficiently high degree will be zero.

Example 3.1.3. There are two simple examples of formal group laws. The additive formal group law is defined by

$$F_a(x, y) = x + y \in A[[x, y]],$$

and the multiplicative formal group law is defined by

$$F_m(x, y) = x + y + xy \in A[[x, y]].$$

The multiplicative formal group law is called so since when we shift the identity element to 1, we have $F_m(x-1, y-1) = xy - 1$.

A homomorphism $f : \mathfrak{F} \to \mathfrak{G}$ of formal groups is a morphism of the underlying formal schemes that respect the additive structures. Since a map $\operatorname{Spf} A[[x]] \to \operatorname{Spf} A[[x]]$ is defined by the formal power series that x pulls back to, we can make the following definition.

Definition 3.1.4. Let $F, G \in A[[x, y]]$ be two formal group laws. A **homomorphism** $f : F \to G$ (or more precisely, $\mathfrak{F} \to \mathfrak{G}$) is a power series $f \in A[[x]]$ satisfying

(1) $f \in xA[[x]]$,
(2) $f(F(x, y)) = G(f(x), f(y))$.

Again, we may interpret this power series f to be giving a natural transformation between the functors that respect addition.

Example 3.1.5. If A contains $\mathbb{Q}$, then power series

$$f(x) = \exp(x) - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

gives a homomorphism $\mathfrak{F}_a \to \mathfrak{F}_m$. Similarly, the power series

$$g(x) = \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

gives a homomorphism $\mathfrak{F}_m \to \mathfrak{F}_a$ that is an inverse to f.

Similarly, we can define module structures on formal schemes. However, we shall only consider module structures that act by multiplication in the first order. Thus, a formal R-module over an R-algebra A will be a factorization $A-\mathsf{alg} \to R-\mathsf{mod} \to \mathsf{Set}$ satisfying this condition that its value on $A[x]/(x^2)$ is the ordinary R-module structure on A.

Definition 3.1.6. Let A be a commutative R-algebra. A **formal R-module structure** on $\operatorname{Spf} A[[x]]$ is a formal group law F over A with a collection of endomorphisms

$$[r]_F \in A[[x]], \quad [r]_F : F \to F$$

for each $r \in R$, satisfying

(1) $[0]_F(x) = 0$, $[1]_F(x) = x$,
(2) $F([r_1]_F(x), [r_2]_F(x)) = [r_1 + r_2]_F(x)$ for all $r_1, r_2 \in R$,
(3) $[r_1]_F([r_2]_F(x)) = [r_1 r_2]_F(x)$ for all $r_1, r_2 \in R$,
(4) $[r]_F(x) \in rx + x^2 A[[x]]$.

Because any ring is a commutative $\mathbb{Z}$-algebra, we see that a formal $\mathbb{Z}$-module over A is the same thing as a formal group over A.

Example 3.1.7. Consider the case $A = R$. The additive formal group $\mathfrak{F}_a$ is naturally a formal R-module with the action

$$[r]_{F_a}(x) = rx \in R[[x]].$$

However, there is no A-module structure on the multiplicative group $\mathfrak{F}_m$ except in certain cases. If A contains $\mathbb{Q}$ as before, we can define the action as

$$[r]_{F_m}(x) = \exp(r \log(1+x)) - 1 = rx + \binom{r}{2} x^2 + \binom{r}{3} x^3 + \cdots .$$

In fact, this is defined for some rings not containing $\mathbb{Q}$, such as $R = \mathbb{Z}$ or $R = \mathbb{Z}_p$ for instance.

Over general R, we can similarly define module homomorphisms of formal R-modules.

Definition 3.1.8. Let A be a commutative R-algebra, and let $\mathfrak{F}, \mathfrak{G}$ be two formal R-modules over A corresponding to the formal group laws F, G. A **module homomorphism** $f : \mathfrak{F} \to \mathfrak{G}$ is group homomorphism satisfying

$$f([r]_F(x)) = [r]_G(f(x))$$

for all $r \in R$.

We can do many of our normal algebra. The set of module homomorphisms $\mathfrak{F} \to \mathfrak{G}$ is going to naturally be an R-module. We are going to say that two formal R-modules over A are isomorphic if there are homomorphisms in both directions that compose to the identity homomorphism in both ways.

A useful point of view to take is to consider a formal R-module $\mathfrak{F}$ (or a formal group) as an abstract algebro-gometric object with the structure of a R-module over $\operatorname{Spec} A$. The identiification $\mathfrak{F} \cong \operatorname{Spf} A[[x]]$ should be considered as a choice of coordinates rather than a data provided in the definition.

Definition 3.1.9. Let A be a commutative R-algebra. A **formal R-module** $\mathfrak{F}$ over A is a formal scheme over A with the structrue of an addition map $\mathfrak{F} \times_{\operatorname{Spec} A} \mathfrak{F} \to \mathfrak{F}$ over A and a ring homomorphism $R \to \operatorname{End}_A(\mathfrak{F})$, such that there exists an isomorphism $\mathfrak{F} \cong \operatorname{Spf} A[[x]]$ of formal schemes over A that induces a formal R-module structure on $\operatorname{Spf} A[[x]]$.

Let K be a p-adic local field. At the end, we would like to consider only the case $A = R = \mathcal{O}_K$. In this case, we should think about what we would like to mean by $\mathfrak{F}(K^{\text{sep}})$, where $\mathfrak{F}$ is a formal $\mathcal{O}_K$-module over $\mathcal{O}_K$. One thing we may try is to think of $\mathfrak{F}$ as a functor taking $\mathcal{O}_K$-algebras to $\mathcal{O}_K$-modules, and evalaute it at the $\mathcal{O}_K$-algbera K^{sep}. However, the functor $\mathfrak{F}$ takes an algebra to the set of nilpotent elements, and this implies that the $\mathcal{O}_K$-module we get is trivial with this interpretation.

What makes the situation intereseting is the fact that the base ring $A = \mathcal{O}_K$ has a topology as well as the formal power series ring. Given any topologically inpotent element, i.e., an element of norm smaller than 1, we can put it inside a power series with coefficients in $\mathcal{O}_K$ and get a meaningful answer. This allows us to define a $\mathcal{O}_K$-module structure on the open unit disc

$$\mathfrak{m}_{K^{\text{sep}}} = \{x \in K^{\text{sep}} : |x| < 1\}.$$

It moreover has an $\operatorname{Gal}(K^{\text{sep}}/K)$-action, and also enough points to become an interseting Galois representation. If we want to make this construction rigorous, we can say that the formal $\mathcal{O}_K$-module in interest really is a map $\operatorname{Spf} \mathcal{O}_K[[x]] \to \operatorname{Spf} \mathcal{O}_K$, where $\mathcal{O}_K[[x]]$ is topologized with the ideal (π_K, x) and $\mathcal{O}_K$ is topoloigzed with (π_K). Then the ideal we are looking at is the $\operatorname{Spf} \mathcal{O}_{K^{\text{sep}}}$-points of $\operatorname{Spf} \mathcal{O}_K[[x]]$.

Another way of making sense of this is to view the formal $\mathcal{O}_K$-module as a p-divisible group. In this case, we will be taking the formal spectrum of $\mathcal{O}_K[[x]]$ with respect to yet another topology. The K^{sep}-points of this $\operatorname{Spf} \mathcal{O}_K[[x]]$ will be identified with the torsion points of $\mathfrak{m}_{K^{\text{sep}}}$ in the previous interpretation.

3.2 Classification of height 1 formal modules

Let K be a p-adic local field and take $R = A = \mathcal{O}_K$. We would first like to know what formal $\mathcal{O}_K$-modules over $\mathcal{O}_K$ there are.

To study formal $\mathcal{O}_K$-modules over $\mathcal{O}_K$, we first look at what happens when we reduce the coefficients of the formal group law modulo $\mathfrak{m}_K$. If we simply project down all the coefficients through $\mathcal{O}_K \twoheadrightarrow k$, then we obtain a formal $\mathcal{O}_K$-module over k. Our first goal is to classify those formal modules over k, and then study how they lift to formal modules over $\mathcal{O}_K$.

3.2.1 The Frobenius on a formal module

Fix $\mathfrak{F}$ a formal $\mathcal{O}_K$-module over k. Let $|k| = q$ be the cardinality of the residue field. Define the polynomial

$$\mathrm{Frob}_k(x) = x^q.$$

Then we have that

$$\mathrm{Frob}_k(f(x)) = f(\mathrm{Frob}_k(x)), \quad \mathrm{Frob}_k(F(x,y)) = F(\mathrm{Frob}_k(x), \mathrm{Frob}_k(y)).$$

It follows that

$$\mathrm{Frob}_k : \mathfrak{F} \to \mathfrak{F}$$

is an $\mathcal{O}_K$-linear endomorphism of $\mathfrak{F}$, and moreover it commutes with all endomorphisms of $\mathfrak{F}$.

A consequence of it is that the Frobenius is independent of the choice of coordinates on the formal module, i.e., behaves well with respect to isomorphisms. If $f \colon \mathfrak{F}_1 \to \mathfrak{F}_2$ is an isomorphism of formal $\mathcal{O}_K$ with formal group laws F_1, F_2, then

$$\mathrm{Frob}_k(f(x)) = f(\mathrm{Frob}_k(x)).$$

It follows that $\mathrm{Frob}_k(x)$ is a canonically defined endomorphism of the formal $\mathcal{O}_K$-module.

Lemma 3.2.1 ([HG94], Lemma 4.1)**.** *Let $\mathfrak{F}$ be a formal $\mathcal{O}_K$-module over k. Then for any endomorphism $f \in \mathrm{End}_{\mathcal{O}_K}(\mathfrak{F})$, either $f = 0 \in k[[x]]$ or there uniquely exists an integer $h \geq 0$ such that*

$$f(x) = g(x^{q^h}), \quad g'(0) \neq 0.$$

Proof. We may suppose that f is not 0. As we can set $h = 0$ if $f'(0) \neq 0$, we also assume that $f'(0) = 0$. We take the identity

$$f(F(x,y)) = F(f(x), f(y)),$$

differentiate with respect to y to obtain

$$\partial_2 F(x,y) f'(F(x,y)) = f'(y) \partial_2 F(f(x), f(y)).$$

We first note that $\partial_2 F(x,y) \in 1 + xk[[x,y]]$. Hence when we plug in $y = 0$, this becomes a polynomial in $1 + xk[[x]]$ and hence a unit in $k[[x]]$. Because

$$\partial_2 F(x,0) f'(x) = f'(0) \partial_2 F(f(x),0) = 0$$

by the condition $f'(0) = 0$, we obtain $f'(x)$. This shows that $f(x) = f_1(x^p)$ for some polynomial $f_1 \in k[[x]]$.

We now claim that f_1 is again an endomorphism of $\mathfrak{F}$. This is because even though $\mathrm{Frob}_{\mathbb{F}_p}$ is not an $\mathcal{O}_K$-linear endomorphism of $\mathfrak{F}$, it commutes with all endomorphisms. We have

$$\begin{aligned}\mathrm{Frob}_{\mathbb{F}_p}(f_1(F(x,y)) &= f(F(x,y)) = F(f(x), f(y)) \\ &= F(\mathrm{Frob}_{\mathbb{F}_p}(f_1(x)), \mathrm{Frob}_{\mathbb{F}_p}(f_1(y))) = \mathrm{Frob}_{\mathbb{F}_p}(F(f_1(x), f_1(y)))\end{aligned}$$

and thus $f_1(F(x,y)) = F(f_1(x), f_1(y))$, and similarly

$$\begin{aligned}\mathrm{Frob}_{\mathbb{F}_p}(f_1([a]_F(x))) &= f([a]_F(x)) = [a]_F(f(x)) \\ &= [a]_F(\mathrm{Frob}_{\mathbb{F}_p}(f_1(x))) = \mathrm{Frob}_{\mathbb{F}_p}([a]_F(f_1(x))\end{aligned}$$

implies $f_1([a]_F(x)) = [a]_F(f_1(x))$.

Since f_1 is an endomorphism of F, we may apply the same argument to f_1. Either $f_1'(0) \neq 0$ or we can find a power series f_2 satisfying $f_2(x^p) = f_1$. Iterating the process, we arrive at a polynomial g with

$$f(x) = g(x^{p^t}), \quad g'(0) \neq 0.$$

Note that the process ends since we assumed $f \neq 0$.

Finally, we show that p^t is actually a power of q. To see this, we return to the identity

$$[a]_F(f(x)) = f([a]_F(x)).$$

If the lowest order term in $f(x)$ is of the form cx^{p^t} where $c \neq 0 \in k$, then in the expansion of both sides, we will get

$$\bar{a}cx^{p^t} + O(x^{p^t+1}) = c(\bar{a}x)^{p^t} + O(x^{p^t+1})$$

since $[a]_F(x) = \bar{a}x + O(x^2)$. It follows that $\bar{a} = \bar{a}^{p^t}$ for all $\bar{a} \in k$, and because $k = \mathbb{F}_q$ this is true if and only if p^t is a power of q. $\square$

Definition 3.2.2. This unique integer $h = \mathrm{ht}(h) \geq 0$ is called the **height** of the $\mathcal{O}_K$-linear endomorphism $f\colon \mathfrak{F} \to \mathfrak{F}$. If $f = 0$, we say that the height of f is infinity.

Note that the height of an endomorphism is independent of the choice of basis. If we have another presentation of the formal module $\mathfrak{F}$ by a different choice of coordinates, the transition function is given by a power series $g(x) = a_1 x + \cdots \in k[[x]]^\times$. Then f corresponds to the power series $g(f(g^{-1}(x)))$. Because $g(x)$ has a nonzero linear term, we see that $g(f(g^{-1}(x)))$ is a power series of the same height. This shows that for an abstract $\mathcal{O}_K$-linear endomorphism of an abstract formal $\mathcal{O}_K$-module over k, the height independent of choice of coordinates and hence well-defined.

This can be used to moreover define an invariant of a formal $\mathcal{O}_K$-module $\mathfrak{F}$ over k. Let π_K be an arbitrary uniformizer of K. The map $[\pi_K]_F$ is an $\mathcal{O}_K$-linear endomorphism of $\mathfrak{F}$ and hence we may consider its height h and write

$$[\pi_K]_F = g(\mathrm{Frob}_k^h(x)), \quad g'(0) \neq 0.$$

If we change the uniformizer to $\pi'_K = \pi_K u$, where $u \in \mathcal{O}_K^\times$ is a unit, then

$$[\pi'_K]_F = [u]_F[\pi_K]_F = [u]_F(g(\mathrm{Frob}_k^h(x))), \quad ([u]_F \circ g)'(0) = \bar{u}g'(0) \neq 0.$$

This shows that the height of $[\pi'_K]_F$ is also h. Thus the height of multiplication by a uniformizer is an invariant of the formal $\mathcal{O}_K$-module.

Definition 3.2.3. The **height** $\mathrm{ht}(\mathfrak{F})$of a formal $\mathcal{O}_K$-module $\mathfrak{F}$ over k is the height of $[\pi_K]$ for any choice of uniformizer $\pi_K \in \mathcal{O}_K$. The **height** of a formal $\mathcal{O}_K$-module over $\mathcal{O}_K$ is the height of the reduced formal module defined over k.

Example 3.2.4. Let us take the additive formal group law $F_a(x, y) = x + y$, considered as a formal $\mathcal{O}_K$-module over $\mathcal{O}_K$. This has

$$[\pi_K]_F(x) = \pi_K x \equiv 0 \pmod{\mathfrak{m}_K},$$

so this formal group has infinite height.

Example 3.2.5. Consider the multiplication formal group law $F_m(x, y) = x + y + xy$, considered as a formal $\mathbb{Z}_p$-module over $\mathbb{Z}_p$. This has

$$[p]_F(x) = (1 + x)^p - 1 = px + \tfrac{p(p-1)}{2}x + \cdots + px^{p-1} + x^p$$

and hence after reducing modulo p, we obtain $[p]_F(x) = x^p \in \mathbb{F}_p[[x]]$. This shows that $\mathfrak{F}_m$ has height 1.

3.2.2 Classifying formal modules over the residue field

Our strategy for classifying for classifying formal $\mathcal{O}_K$-modules $\mathfrak{F}$ over k is to look at ring of endomorphisms

$$\mathrm{End}_{\mathcal{O}_K}(\mathfrak{F}).$$

There are many elements in this endomorphism ring; it contains all $[a]_F$ for $a \in \mathcal{O}_K$, and it contains Frob_k. Here, note that all $[a]_F$ commute with each other, since the ring $\mathcal{O}_K$ is commutative, and also commute with Frob_k since the Frobenius commutes with everything. Later, we are going to see that $\mathrm{End}_{\mathcal{O}_K}(\mathfrak{F})$ is topologically generated by Frob_k and $\mathcal{O}_K$ and hence the endomorphism ring is also commutative.

Ignoring the infinite height case, let us suppose that $\mathfrak{F}$ is a formal $\mathcal{O}_K$-module over k of finite height h. By definition, multiplication by π_K takes the form of

$$[\pi_K]_F(x) = c_1 x^{q^h} + c_2 x^{2q^h} + \cdots .$$

Because $\mathrm{Frob}_k(x) = x^q$, we further see that

$$\mathrm{Frob}_k^\alpha [\pi_K^\beta]_F(x) = (\text{nonzero constant}) x^{q^{\alpha+h\beta}} + (\text{constant}) x^{2q^{\alpha+h\beta}} + \cdots$$

has height exactly $\alpha + h\beta$.

Lemma 3.2.6. *Let $f, g \in \mathfrak{F}$ be two $\mathcal{O}_K$-linear endomorphisms of a formal $\mathcal{O}_K$-module $\mathfrak{F}$ over k. Then*

$$\mathrm{ht}(fg) = \mathrm{ht}(f) + \mathrm{ht}(g), \quad \mathrm{ht}(f+g) \geq \min(\mathrm{ht}(f), \mathrm{ht}(g)),$$

where equality on the second equation holds if $\mathrm{ht}(f) \neq \mathrm{ht}(g)$*.*

Proof. This follows from expanding corresponding power series. Because $f = c_f x^{q^{\mathrm{ht}(f)}} + (\text{higher})$ and similarly $g = c_g x^{q^{\mathrm{ht}(g)}} + (\text{higher})$, we have

$$\begin{aligned} fg &= c_f c_g x^{q^{\mathrm{ht}(f)+\mathrm{ht}(g)}} + (\text{higher}), \\ F(f,g) &= c_f x^{q^{\mathrm{ht}(f)}} + c_g x^{q^{\mathrm{ht}(g)}} + O(x^{\min(q^{\mathrm{ht}(f)}, q^{\mathrm{ht}(g)})+1}). \end{aligned}$$

The lemma follows. □

So the height function almost works like a valuation function. We can use this formal property to get a better understanding of the subring generated by $\mathcal{O}_K$ and Frob_k.

Lemma 3.2.7. *Suppose that $\mathfrak{F}$ has finite height $h = \mathrm{ht}(\mathfrak{F}) < \infty$. Then for any endomorphism $f \in \mathrm{End}_{\mathcal{O}_k}(\mathfrak{F})$, there uniquely exist $a_0, \ldots, a_{h-1} \in \mathcal{O}_K$ satisfying*

$$f = a_0 + a_1 \mathrm{Frob}_k + a_2 \mathrm{Frob}_k^2 + \cdots + a_{h-1} \mathrm{Frob}_k^{h-1} \in \mathrm{End}_{\mathcal{O}_K}(\mathfrak{F}).$$

In particular, $\mathrm{End}_{\mathcal{O}_K}(\mathfrak{F})$ is a free $\mathcal{O}_K$-module of rank h with $1, \mathrm{Frob}_k, \ldots, \mathrm{Frob}_k^{h-1}$ being a basis.

Proof. Note that

$$\begin{aligned} &\mathrm{ht}(1) = 0, \quad \mathrm{ht}(\mathrm{Frob}_k) = 1, \quad \ldots, \quad \mathrm{ht}(\mathrm{Frob}_k^{h-1}) = h-1, \\ &\mathrm{ht}(\pi_K) = h, \quad \mathrm{ht}(\pi_K \mathrm{Frob}_k) = h+1, \quad \ldots, \quad \mathrm{ht}(\pi_K \mathrm{Frob}_k^{h-1}) = 2h-1, \\ &\mathrm{ht}(\pi_K^2) = 2h, \quad \cdots . \end{aligned}$$

Therefore by correcting terms inductively, we see that there are constants $a_0, \ldots, a_{h-1} \in \mathcal{O}_K$, unique up to $\mathfrak{m}_K^t$, such that

$$\mathrm{ht}(f - a_0 - a_1 \mathrm{Frob}_k - \cdots - a_{h-1} \mathrm{Frob}_k^{h-1}) \geq th.$$

Therefore as we pass to the limit, completeness of $\mathcal{O}_K$ shows that there uniquely exist $a_0, \ldots, a_{h-1} \in \mathcal{O}_K$ that satisfies the equation. □

Corollary 3.2.8. *Suppose a formal $\mathcal{O}_K$-module $\mathfrak{F}$ over k has finite height $h = \mathrm{ht}(\mathfrak{F}) < \infty$. Then the endomorphism ring is*

$$\mathrm{End}_{\mathcal{O}_K}(\mathfrak{F}) = \mathcal{O}_K[\mathrm{Frob}_k]/p(\mathrm{Frob}_k)$$

for a uniquely defined Eisenstein polynomial $p(x) \in \mathcal{O}_K[x]$ of degree h.

Proof. If we apply Lemma 3.2.7 to the endomorphism

$$\mathrm{Frob}_k^h \in \mathrm{End}_{\mathcal{O}_K}(\mathfrak{F}),$$

then we obtain a unique monic degree h polynomial $p(x) \in \mathcal{O}_K$ for which $p(\mathrm{Frob}_k) = 0$. To show that it is an Eisenstein polynomial, note that

$$h = \mathrm{ht}(\mathrm{Frob}_k^h) = \mathrm{ht}(a_0 + a_1 \mathrm{Frob}_k + \cdots + a_{h-1} \mathrm{Frob}_k^{h-1}).$$

The heights of all $a_0, a_1 \mathrm{Frob}_k, \ldots, a_{h-1} \mathrm{Frob}_k^{h-1}$ are distinct since they have different remainders when divided by h, and hence

$$h = \min(\mathrm{ht}(a_0), \mathrm{ht}(a_1 \mathrm{Frob}_k), \ldots, \mathrm{ht}(a_{h-1} \mathrm{Frob}_k^{h-1}))$$

This shows that $|a_0| = |\pi_K|$ and $|a_1|, \ldots, |a_{h-1}| \leq |\pi_K|$. That is, $p(x)$ is indeed an Eisenstein polynomial. $\square$

Therefore, we can associate to each formal $\mathcal{O}_K$-module over k of height h, an Eisenstein polynomial of degree h.

$$\left\{\begin{matrix}\text{formal } \mathcal{O}_K\text{-module} \\ \text{over } k \text{ of height } h\end{matrix}\right\}/\text{iso.} \quad \longrightarrow \quad \left\{\begin{matrix}\text{Eisenstein polynomials} \\ p(x) \in \mathcal{O}_K[x] \text{ of degree } h\end{matrix}\right\}$$

In particular, for $h = 1$, we have an invariant

$$\begin{aligned}\left\{\begin{matrix}\text{formal } \mathcal{O}_K\text{-module} \\ \text{over } \mathcal{O}_K \text{ of height } 1\end{matrix}\right\}/\text{iso.} \quad &\xrightarrow{\text{reduce}} \quad \left\{\begin{matrix}\text{formal } \mathcal{O}_K\text{-module} \\ \text{over } k \text{ of height } 1\end{matrix}\right\}/\text{iso.} \\ &\longrightarrow \quad \left\{\text{uniformizers } \pi_K \in \mathcal{O}_K\right\}.\end{aligned}$$

We claim that this is a complete invariant formal $\mathcal{O}_K$-modules over both k and over $\mathcal{O}_K$. That is, given a uniformizer $\pi_K \in \mathcal{O}_K$ there exists a unique formal $\mathcal{O}_K$-module over k on which Frob_k acts as π_K, and a unique formal $\mathcal{O}_K$-module over $\mathcal{O}_K$ that reduces this module over k. To prove this, we should construct formal $\mathcal{O}_K$-modules.

Remark 3.2.9. Let $K = \mathbb{Q}_p$ and consider a formal group $\mathfrak{F}$ over $\mathbb{F}_p$ on which $\mathrm{Frob}_{\mathbb{F}_p} = [\pi_{\mathbb{Q}_p}]_F$. Dieudonné theory associates to $\mathfrak{F}$ a Diedonné module $\mathbf{D}(\mathfrak{F})$, which turns to be a free rank 1 module over $\mathbb{Z}_p$ on which Frobenius is multiplication by $\pi_{\mathbb{Q}_p}$.

3.2.3 Lubin–Tate formal modules

Let us first fix a uniformizer $\pi_K \in \mathcal{O}_K$. We would like to construct a formal $\mathcal{O}_K$-module over $\mathcal{O}_K$, such that when we reduce the coefficients to k the Frobenius Frob_k acts as multiplication by π_K. This means that

$$[\pi_K]_F(x) \in \pi_K x + x^q + \mathfrak{m}_K x^2 \mathcal{O}_K[[x]].$$

Definition 3.2.10. A **Lubin–Tate series** for the uniformizer π_K is a power series $f(x) \in \mathcal{O}_K[[x]]$ satisfying the property that

$$f(x) \equiv \pi_K x + x^q \pmod{\mathfrak{m}_K x^2}.$$

We claim that for any Lubin–Tate series $f(x)$, there exists a unique formal $\mathcal{O}_K$-module law F over $\mathcal{O}_K$ satisfying

$$[\pi_K]_F(x) = f(x).$$

Moreover, we will show that if the F_1 and F_2 are formal group laws for two Lubin–Tate series f_1 and f_2, then the two are uniquely isomorphic through a change of coordinates that is the identity at first order. This will show that there is a canonical formal $\mathcal{O}_K$-module $\mathfrak{LT}_{/K,\pi_K}$ over $\mathcal{O}_K$ that depends only on the choice of uniformizer π_K, on which the choice of coordinates corresponds to the choice of the Lubin–Tate series $[\pi_K]$.

We shall prove our claims by a useful lemma.

Lemma 3.2.11 ([CF67], Proposition VI.3.5)**.** *Let K be a p-adic local field and fix π_K a uniformizer. Suppose f, g are Lubin–Tate series for π_K, and let $\phi_1(x_1, \dots, x_n)$ be a linear polynomial with coefficients in $\mathcal{O}_K$. Then there exists a unique $\phi \in \mathcal{O}_K[[x_1, \dots, x_n]]$ satisfying*

- *(i)* $\phi \equiv \phi_1 \bmod (x_1, \dots, x_n)^2$,
- *(ii)* $f(\phi(x_1, \dots, x_n)) = \phi(g(x_1), \dots, g(x_n))$.

Proof. As usual, we inductively construct power series ϕ_m that are correct up to degree m. That is, if we are given ϕ_{m-1} such that

$$f(\phi_{m-1}(x_1, \dots, x_n)) \equiv \phi_{m-1}(g(x_1), \dots, g(x_n)) \pmod{(x_1, \dots, x_n)^m},$$

it suffices to show that there exists a ϕ_m, unique up to $(x_1, \dots, x_n)^{m+1}$, such that $\phi_m \equiv \phi_{m+1}$ modulo $(x_1, \dots, x_n)^{m-1}$ and

$$f(\phi_m(x_1, \dots, x_n)) \equiv \phi_m(g(x_1), \dots, g(x_n)) \pmod{(x_1, \dots, x_n)^{m+1}}.$$

Given ϕ_{m-1}, we modify it to $\phi_m = \phi_{m-1} + \psi_m$, where ψ_m is a degree m homogeneous polynomial. Then we have

$$\begin{aligned} f(\phi_m(x_1, \dots, x_n)) &\equiv f(\phi_{m-1}(x_1, \dots, x_n)) + \pi_K \psi_m(x_1, \dots, x_n) \\ \phi_m(g(x_1), \dots, g(x_n)) &\equiv \phi_{m-1}(g(x_1), \dots, g(x_n)) + \pi_K^m \psi_m(x_1, \dots, x_n) \\ &\qquad\qquad \pmod{(x_1, \dots, x_n)^{m+1}}. \end{aligned}$$

This in fact shows that there exists a unique ψ_m satisfying $f(\phi_m) \equiv \phi_m(g)$ modulo degree $m+1$, if and only if the degree m homogeneous part of

$$f(\phi_{m-1}(x_1, \dots, x_n)) - \phi_{m-1}(g(x_1), \dots, g(x_n))$$

is divisible by $\pi_K - \pi_K^m$. Since $m \geq 2$, being divisible by $\pi_K - \pi_K^m$ is equivalent to being divisible by π_K.

However, if we reduce all the coefficients modulo $\mathfrak{m}_K$, we see that we automatically have

$$\begin{aligned} f(\phi_{m-1}(x_1, \dots, x_n)) &\equiv \phi_{m-1}(x_1, \dots, x_n)^q \\ &\equiv \phi_{m-1}(x_1^q, \dots, x_n^q) \equiv \phi_{m-1}(g(x_1), \dots, g(x_n)) \pmod{\mathfrak{m}_K}, \end{aligned}$$

since $\mathrm{Frob}_k(x) = x^q$ commutes with all polynomials over the residue field k. This shows that $f(\phi_{m-1}(x_1, \dots, x_n)) - \phi_{m-1}(g(x_1), \dots, g(x_n))$ is divisible by π_K, and hence there is a unique correction of ϕ_{m-1} to ϕ_m for $m \geq 2$. □

We can now apply this lemma to construct the Lubin–Tate formal module law associated to a Lubin–Tate series, prove uniqueness, and show that there is a canonical formal $\mathcal{O}_K$-module over $\mathcal{O}_K$ not depending on the choice of the Lubin–Tate series.

Proposition 3.2.12. *Fix a Lubin–Tate series $f(x)$ for a uniformizer π_K. Then*

(a) There exists a unique formal group law F over $\mathcal{O}_K$ for which $f(x)$ is a group endomorphism.

(b) This formal group law F can be uniquely promoted to define a formal $\mathcal{O}_K$-module law over $\mathcal{O}_K$ satisfying the property that $f(x) = [\pi_K]_F(x)$.

Proof. (a) Let us first apply Lemma 3.2.11 to find a $F(x, y) \in \mathcal{O}_K[[x, y]]$ such that

$$F(x, y) \equiv x + y \pmod{(x, y)^2}, \quad F(f(x), f(y)) = f(F(x, y)).$$

We now need to verify that $F(x, y)$ is indeed a formal group law. This will follow from the uniqueness part of Lemma 3.2.11. First, if we define another polynomial $F_{\mathrm{swap}}(x, y) = F(y, x)$ by switching the two variables, then we see that F_{swap} satisfies the identities

$$\begin{aligned} F_{\mathrm{swap}}(x, y) &\equiv x + y \pmod{(x, y)^2}, \\ F_{\mathrm{swap}}(f(x), f(y)) &= F(f(y), f(x)) = f(F(y, x)) = f(F_{\mathrm{swap}}(x, y)). \end{aligned}$$

This shows that $F = F_{\mathrm{swap}}$ and hence F is symmetric. Similarly, we can show that F satisfies $F(x, 0) = x$, by applying the uniqueness part lemma to the polynomials $F(x, 0)$ and x in one variables. For associativity, we apply it to the polynomials $F(F(x, y), z)$ and $F(x, F(y, z))$ in three variables.

(b) From the lemma, we see that for any $a \in \mathcal{O}_K$ there exists a unique polynomial $[a]_F(x) \in ax + x^2\mathcal{O}[[x]]$ such that $f([a]_F(x)) = [a]_F(f(x))$. We claim that this is indeed an endomorphism of the group law F. To see this, we look at the polynomials $F([a]_F(x), [a]_F(y))$ and $[a]_F(F(x, y))$ which are polynomials in two variables, and observe that

$$f(F([a]_F(x), [a]_F(y))) = F(f([a]_F(x)), f([a]_F(y))) = F([a]_F(f(x)), [a]_F(f(y))),$$
$$f([a]_F(F(x, y))) = [a]_F(f(F(x, y))) = [a]_F(F(f(x), f(y))).$$

Therefore by uniqueness of the lemma, we see that the polynomials $F([a]_F(x), [a]_F(y))$ and $[a]_F(F(x, y))$ agree. That is, $[a]_F$ is indeed an endomorphism of F.

Properties such as $[a+b]_F(x) = F([a]_F(x), [b]_F(x))$ or $[ab]_F(x) = [a]_F([b]_F(x))$ can be verified by checking that both sides commute with f and have the same linear term. Moreover, $[\pi_K]_F(x) = f(x)$ since $f(f(x)) = f(f(x))$ and $f(x) \in \pi_K x + x\mathcal{O}_K[[x]]$. □

Therefore we have constructed a formal $\mathcal{O}_K$-module law over $\mathcal{O}_K$, out of a choice of uniformizer π_K and a Lubin–Tate series $f(x)$ for π_K. We now see what happens when we change the Lubin–Tate series.

Proposition 3.2.13. *Fix a uniformizer π_K, and choose two Lubin–Tate series f_1, f_2. Consider two formal $\mathcal{O}_K$-module laws F_1, F_2 over $\mathcal{O}_K$ satisfying $[\pi_K]_{F_1} = f_1$ and $[\pi_K]_{F_2} = f_2$. Then for each $c \in \mathcal{O}_K$, there exists a unique power series $g(x) \in cx + x^2\mathcal{O}_K[[x]]$ that defines a $\mathcal{O}_K$-linear homomorphism $g\colon F_1 \to F_2$, i.e.,*

$$g(F_1(x, y)) = F_2(g(x), g(y)), \quad g([a]_{F_1}(x)) = [a]_{F_2}(g(x))$$

for all $a \in \mathcal{O}_K$.

Proof. We first note that if g is indeed a homomorphism then $g(f_1(x)) = f_2(g(x))$ by setting $a = \pi_K$. By Lemma 3.2.11 there exists a unique such $g \in cx + x^2\mathcal{O}_K[[x]]$.

We now show that this unique g is indeed a $\mathcal{O}_K$-linear homomorphism. First we note that

$$g(F_1(f_1(x), f_1(y))) = g(f_1(F_1(x, y))) = f_2(g(F_1(x, y))),$$
$$F_2(g(f_1(x)), g(f_1(y))) = F_2(f_2(g(x)), f_2(g(y))) = f_2(F_2(g(x), g(y))).$$

Thus uniqueness shows that $g(F_1(x, y)) = F_2(g(x), g(y))$. Similarly, we have

$$g([a]_{F_1}(f_1(x))) = g(f_1([a]_{F_1}(x))) = f_2(g([a]_{F_1}(x))),$$
$$[a]_{F_2}(g(f_1(x))) = [a]_{F_2}(f_2(g(x))) = f_2([a]_{F_2}(g(x))),$$

and therefore $g([a]_{F_1}(x)) = [a]_{F_2}(g(x))$. □

This tells us that once we choose the uniformizer $\pi_K \in \mathcal{O}_K$, the choice of the Lubin–Tate series does not matter when constructing the formal module. Summing up, we obtain the following theorem.

Theorem 3.2.14. *Let K be a p-adic local field and π_K be a uniformizer. Then there exists a formal $\mathcal{O}_K$-module $\mathfrak{LT}_{/K,\pi_K}$ over $\mathcal{O}_K$, satisfying the property that* *(i) for any choice of coordinates on $\mathfrak{LT}_{/K,\pi_K}$, the endomorphism $[\pi_K]$ is a Lubin–Tate series for π_K, and*

(ii) conversely for any Lubin–Tate polynomial f, there exists a choice of coordinates, unique up to $\mathcal{O}_K^\times$, such that $[\pi_K] = f$.

We have previously stated that we can define an invariant of a formal $\mathcal{O}_K$-module over $\mathcal{O}_K$ of height 1. This shows that the invariant is complete, in the sense that every uniformizer comes from a unique formal $\mathcal{O}_K$-module over $\mathcal{O}_K$ of height 1, namely the Lubin–Tate module.

Corollary 3.2.15. *There is a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{formal } \mathcal{O}_K\textit{-modules} \\ \textit{over } \mathcal{O}_K \textit{ of height } 1\end{matrix}\right\}/\textit{iso.} \quad \longleftrightarrow \quad \left\{\textit{uniformizers } \pi_K \in K\right\};$$
$$\mathfrak{LT}_{/K,\pi_K} \quad \longleftrightarrow \quad \pi_K.$$

That is, every formal $\mathcal{O}_K$-module over $\mathcal{O}_K$ of height 1 is a Lubin–Tate module for some uniformizer.

3.3 $\mathfrak{m}_K$-divisible modules

It is not necessary to define p-divisible groups or π-divisible modules in developing Lubin–Tate theory. However, any formal module may be considered as a π-divisible module, and this gives a different perspective on Lubin–Tate groups. Even better, there are π-divisible modules that do not come from formal modules, which makes the theory of π-divisible modules much richer than that of formal modules.

We start with the observation that if we take an arbitrary formal $\mathcal{O}_K$-module and look at the $\mathfrak{m}_K$-torsion, we get a finite group scheme that is moreover a $\mathcal{O}_K$-module. For $\mathfrak{F}$ a formal $\mathcal{O}_K$-module over $\mathcal{O}_K$, let us look at the kernel of the $\pi_K^n \colon \mathfrak{F} \to \mathfrak{F}$. (For the moment, let us take the perspective of identifying a formal module with $\operatorname{Spf} \mathcal{O}_K[[x]]$, where the topology on $\mathcal{O}_K[[x]]$ is generated by the ideal (p, x).) This may be considered as the fiber product

$$\begin{tikzcd} \mathfrak{F}[\mathfrak{m}_K^n] \arrow[r, hook] \arrow[d] & \mathfrak{F} \arrow[d, "\pi_K^n"] \\ \operatorname{Spf} \mathcal{O}_K \arrow[r, hook, "0"] & \mathfrak{F}, \end{tikzcd}$$

and therefore can be given an algebro-geometric structure. Note that this is independent of the choice of uniformizer π_K^n.

We can figure out what $\mathfrak{F}[\mathfrak{m}_K^n]$ looks like after choosing coordinates. If the formal module $\mathfrak{F}$ has height h, then the kernel is

$$\operatorname{Spf}(\mathcal{O}_K[[x]]/([\pi_K^n]_F(x)))$$

where $[\pi_K^n]_F(x)$ is a power series of height nh. From this we see that this is a flat finite group scheme of order q^{nh} over $\operatorname{Spf} \mathcal{O}_K$, heuristically.

Definition 3.3.1. Let A be an integral domain, and let $\mathcal{O}_K$ be a ring of integers of a p-adic local field K. A $\mathfrak{m}_K$**-divisible module** (or a **Barsotti–Tate module**) of height h is an sequence G_n of finite flat $\mathcal{O}_K$-module schemes over A with closed embeddings

$$\operatorname{Spec} A = G_0 \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow \cdots,$$

such that

(i) each G_n has order q^{nh} over A, and

(ii) the inclusion $G_n \hookrightarrow G_{n+1}$ identifies G_n as the $\mathfrak{m}_K^n$-torsion of the group scheme G_{n+1}.

Example 3.3.2. Let $\mathfrak{F}$ be a formal $\mathcal{O}_K$-module over $\mathcal{O}_K$ of height h. We claim that $\mathfrak{F}$ may be considered as a $\mathfrak{m}_K$-divisible module, if we set G_n to be the $\mathfrak{m}_K^n$-torsion points of $\mathfrak{F}$. More precisely, we choose coordinates to identify $\mathfrak{F}$ with the formal $\mathcal{O}_K$-module law F, and then set

$$G_n = \operatorname{Spec}(\mathcal{O}_K[[x]]/([\pi_K^n]_F(x))).$$

Because $[\pi_K^n]_F$ is a power series with height nh, it follows that the $\mathcal{O}_K$-algebra $\mathcal{O}_K[[x]]/([\pi_K^n](x))$ is a free module over $\mathcal{O}_K$ of rank q^{nh}. Moreover, it is clear from the construction that each G_n indeed has a structure of an $\mathcal{O}_K$-module, there are closed embeddings $G_n \hookrightarrow G_{n+1}$, and that G_n is identified with the $\mathfrak{m}_K^n$-torsion of G_{n+1}.

Let us now only focus on those $\mathfrak{m}_K$-divisible modules that have height $h = 1$. The two conditions in the definition imply that G_n look like $\mathcal{O}_K/\mathfrak{m}_K^n$ as $\mathcal{O}_K$-modules. To be precise, let us take B an arbitrary A-algebra and take the B-points

$$G_0(B) \hookrightarrow G_1(B) \hookrightarrow G_2(B) \hookrightarrow \cdots.$$

In reasonable situations, such as when B is an integral domain and flat over A, the group G_n having order q^{nh} implies that there are at most q^{nh} ways of lifting the morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ to $\operatorname{Spec} B \to G_n$. Hence each $G_n(B)$ is a finite $\mathcal{O}_K$-module with at most q^{nh} elements.

By the classification of finitely generated modules over principal ideal domains, each $\mathcal{O}_K$-module $G_n(B)$ is isomorphic to

$$G_n(B) \cong \mathcal{O}_K/\mathfrak{m}_K^{e_{n,1}} \oplus \cdots \oplus \mathcal{O}_K/\mathfrak{m}_K^{e_{n,t_n}},$$

where $e_{n,i} \geq 1$ and $\sum_i e_{n,i} \leq n$ by the cardinality condition. But condition (ii) implies that $G_1(B)$ is the $\mathfrak{m}_K$-torsion points of $G_n(B)$, and from our identification of $G_n(B)$ we see that

$$G_1(B) \cong G_n(B)[\mathfrak{m}_K] \cong (\mathcal{O}_K/\mathfrak{m}_K)^{\oplus t_n}.$$

This implies that $t_n = 1$, and $e_{n,1} \leq n$. Let us write $e_{n,1} = e_n$ for convenience, so that

$$G_n(B) \cong \mathcal{O}_K/\mathfrak{m}_K^{e_n}$$

as $\mathcal{O}_K$-modules. For $m \leq n$, condition (ii) also implies that $G_m(B) \cong G_n(B)[\mathfrak{m}_K^m]$, and thus $e_m = \min(m, e_n)$. Therefore one of the two happens:

- $e_n = n$ for all n, and hence $G_n(B) \cong \mathcal{O}_K/\mathfrak{m}_K^n$.
- there exists an integer N such that $e_n = n$ for $n \leq N$ and $e_n = N$ for $n \geq N$.

If B is an algebraically closed field of characteristic zero, for instance, the cardinality of $G_n(B)$ is exactly q^{nh}. It follows that $G_n(B) \cong \mathcal{O}_K/\mathfrak{m}_K^n$ for all n.

There are examples of $\mathfrak{m}_K$-divisible modules of height 1 that do not arise from formal $\mathcal{O}_K$-modules.

Example 3.3.3. Let K be a field totally ramified over $\mathbb{Q}_p$ of degree d. We can take the group scheme $\mathbb{G}_m$ (which also is a formal $\mathbb{Z}_p$-module) and tensor it over $\mathbb{Z}_p$ with $\mathcal{O}_K$ to get a $\mathfrak{m}_K$-divisible module

$$G = \mathbb{G}_m \otimes_{\mathbb{Z}_p} \mathcal{O}_K.$$

More precisely, we first take a uniformizer $\pi_K \in \mathbb{Q}^{\mathrm{sep}}$ of K so that $\mathcal{O}_K = \mathbb{Z}_p\langle 1, \pi_K, \ldots, \pi_K^{d-1}\rangle$, and let π_K satisfy the Eisenstein polynomial

$$\pi_K^d = a_0 + a_1\pi_K + \cdots + a_{d-1}\pi_K^{d-1}, \quad a_0, \ldots, a_{d-1} \in p\mathbb{Z}.$$

We now define the finite group schemes as

$$\mu_{p^n} = \operatorname{Spec}(\mathcal{O}_K[x]/(x^{p^n} - 1)), \quad G_{nd} = \mu_{p^n}^d = \mu_{p^n} \times \cdots \times \mu_{p^n}$$

and endow it with a $\mathcal{O}_K$-module structure so that π_K acts as

$$[\pi_K] = \begin{pmatrix} 0 & \cdots & 0 & a_0 \\ 1 & \cdots & 0 & a_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & a_d \end{pmatrix}.$$

Then it readily follows that $G_{md} \cong G_{nd}[p^m]$ for $m \leq n$, with the natural inclusions $G_{md} \hookrightarrow G_{nd}$ induced by $\mu_{p^m} \hookrightarrow \mu_{p^n}$. Therefore, if we set $G_n = G_{Nd}[\mathfrak{m}_K^n]$ for large enough N, it will be independent of the choice of N and give G the structure of a $\mathfrak{m}_K$-divisible group of height 1.

To see that this $\mathfrak{m}_K$-divisible module does not arise from a formal $\mathcal{O}_K$-module (if $d \geq 2$), we base change to the residue field $\mathbb{F}_p$. We note that

$$\begin{aligned} G_d \times_{\operatorname{Spec}\mathcal{O}_K} \operatorname{Spec}\mathbb{F}_p &\cong \operatorname{Spec}\mathbb{F}_p[x_1, \ldots, x_d]/(x_1^p, \ldots, x_d^p), \\ \mathfrak{LT}_{/K,\pi_K}[\mathfrak{m}_K^d] \times_{\operatorname{Spec}\mathcal{O}_K} \operatorname{Spec}\mathbb{F}_p &\cong \operatorname{Spec}\mathbb{F}_p[x]/(x^{pd}). \end{aligned}$$

The two rings on the right side are distinct, because every p-th power is zero in $\mathbb{F}_p[x_1, \ldots, x_d]/(x_1^p, \ldots, x_d^p)$, while in $\operatorname{Spec}\mathbb{F}_p[x]$ we have $x^p \neq 0$.

Example 3.3.4. We can easily find and even classify étale π_K-divisible modules of height 1 over $\mathcal{O}_K$, i.e., those with each G_n étale over $\operatorname{Spec}\mathcal{O}_K$. Once we fix a geometric point ξ of $\operatorname{Spec}\mathcal{O}_K$, we get an equivalence

$$\left\{\begin{matrix}\text{étale } \mathcal{O}_K\text{-module schemes} \\ \text{of order } q^n \text{ over } \mathcal{O}_K\end{matrix}\right\} \quad \longleftrightarrow \quad \left\{\begin{matrix}\mathcal{O}_K\text{-linear representations} \\ \pi_1^{\mathrm{et}}(\operatorname{Spec}\mathcal{O}_K, \xi) \to \operatorname{Aut}_{\mathcal{O}_K}(M) \\ \text{where } M \text{ has cardinality } q^n\end{matrix}\right\}.$$

We note that the étale fundamental group of $\operatorname{Spec}\mathcal{O}_K$ is $\operatorname{Gal}(k^{\mathrm{sep}}/k)$, since $\mathcal{O}_K$ is a henselian local ring. Because k is a finite field, we thus obtain $\pi_1^{\mathrm{et}}(\operatorname{Spec}\mathcal{O}_K,\xi)\cong\hat{\mathbb{Z}}$ with the Frobenius being a topological generator. On the other hand, because $M_n\cong\mathcal{O}_K/\mathfrak{m}_K^n$ for each n, we have a canonical isomorphism $\operatorname{Aut}_{\mathcal{O}_K}(M_n)\cong(\mathcal{O}_K/\mathfrak{m}_K^n)^\times$ for each n. By looking at all n at the same time, we see that there is a correspondence

$$\left\{\begin{matrix}\text{étale }\mathfrak{m}_K\text{-divisible modules}\\ \text{of height 1 over }\mathcal{O}_K\end{matrix}\right\}/\text{iso.}\quad\longleftrightarrow\quad\varprojlim_n(\mathcal{O}_K/\mathfrak{m}_K^n)^\times\cong\mathcal{O}_K^\times$$

where the right hand side holds because $\mathcal{O}_K^\times$ is already complete. Therefore étale $\mathfrak{m}_K$-divisible modules are classified by $\{x\in K:|x|=1\}$ whereas formal $\mathcal{O}_K$-modules are classified by $\{x\in K:|x|=|\pi_K|\}$.

In this perspective of considering a formal $\mathcal{O}_K$-module as particular type of a $\mathfrak{m}_K$-divisible group, the formal module $\mathfrak{F}$ is really built out of finite $\mathcal{O}_K$-module schemes over $\mathcal{O}_K$ that may be called

$$\operatorname{Spec}\mathcal{O}_K\hookrightarrow\mathfrak{F}[\mathfrak{m}_K]\hookrightarrow\mathfrak{F}[\mathfrak{m}_K^2]\hookrightarrow\mathfrak{F}[\mathfrak{m}_K^3]\hookrightarrow\cdots$$

with a bit of abuse of notation. Then we may define the $\mathcal{O}_K$-module

$$\mathfrak{F}[\mathfrak{m}_K^\infty](K^{\mathrm{sep}})=\varinjlim(\mathfrak{F}[\mathfrak{m}_K](K^{\mathrm{sep}})\hookrightarrow\mathfrak{F}[\mathfrak{m}_K^2](K^{\mathrm{sep}})\hookrightarrow\mathfrak{F}[\mathfrak{m}_K^3](K^{\mathrm{sep}})\hookrightarrow\cdots).$$

Remark 3.3.5. In a sense, this construction may be taken as given a functor-of-points description of the algebro-geometric object $\mathfrak{F}[\mathfrak{m}_K^\infty]$. At the least, since it is a filtered colimit of representable functors, it is going to be a sheaf in the flat topology.

We are mostly interested in formal $\mathcal{O}_K$-modules over $\mathcal{O}_K$ of height 1, i.e., Lubin–Tate modules in view of the classification given in Corollary 3.2.15.

Proposition 3.3.6. *Let $\mathfrak{F}$ be a formal $\mathcal{O}_K$-module over $\mathcal{O}_K$ of height 1. As a $\mathcal{O}_K$-module, the torsion points of $\mathfrak{F}$ is noncanonically isomorphic to*

$$\mathfrak{F}[\mathfrak{m}_K^\infty](K^{\mathrm{sep}})\cong K/\mathcal{O}_K.$$

Proof. Note that K^{sep} is algebraically closed with characteristic zero. Because each $\mathfrak{F}[\mathfrak{m}_K^n]$ is a flat finite scheme over $\mathcal{O}_K$ of order q^n, the number of K^{sep}-points is precisely

$$\#\mathfrak{F}[\mathfrak{m}_K^n](K^{\mathrm{sep}})=q^n.$$

Then from the previous discussion, we see that we have noncanonical isomorphisms

$$\mathfrak{F}[\mathfrak{m}_K^n](K^{\mathrm{sep}})\cong\mathcal{O}_K/\mathfrak{m}_K^n$$

of $\mathcal{O}_K$-modules, for all integers $n\geq 0$.

Pick a uniformizer $\pi_K\in\mathcal{O}_K$. Then each multiplication map

$$\mathfrak{F}[\mathfrak{m}_K^n](K^{\mathrm{sep}})\xrightarrow{\pi_K}\mathfrak{F}[\mathfrak{m}_K^{n-1}](K^{\mathrm{sep}})$$

is surjective since it looks like $\pi_K\colon \mathcal{O}_K/\mathfrak{m}_K^n \to \mathcal{O}_K/\mathfrak{m}_K^{n-1}$. Therefore we may pick a sequence of elements

$$a_n \in \mathfrak{F}[\mathfrak{m}_K^n](K^{\mathrm{sep}}), \quad \pi_K a_n = a_{n-1}.$$

We define the map

$$K/\mathcal{O}_K \to \mathfrak{F}[\mathfrak{m}_K^\infty](K^{\mathrm{sep}}); \quad \frac{c}{\pi_K^n} + \mathcal{O}_K \mapsto c a_n$$

and observe that this is well-defined and an isomorphism. $\square$

3.4 Reconstruction of the reciprocity map

Let us now try to reconstruct the Artin reciprocity map

$$\theta_{/K} : \widehat{K^\times} \to \mathrm{Gal}(K^{\mathrm{sep}}/K)$$

using the Lubin–Tate modules.

For K a p-adic local field, fix π_K a uniformizer and consider $\mathfrak{LT}_{/K,\pi_K}$ the Lubin–Tate formal $\mathcal{O}_K$-module over $\mathcal{O}_K$ associated to π_K. If we look at all the torsion points, there is a natural Galois action

$$\mathrm{Gal}(K^{\mathrm{sep}}/K) \circlearrowright \mathfrak{LT}_{/K,\pi_K}[\mathfrak{m}_K^\infty](K^{\mathrm{sep}})$$

given by composing the point $\operatorname{Spec} K^{\mathrm{sep}} \to \mathfrak{LT}_{/K,\pi_K}[\mathfrak{m}_K^n]$ with an automorphism of $\operatorname{Spec} K^{\mathrm{sep}}$. Moreover, this action respects the $\mathcal{O}_K$-module structure, and hence we obtain a group homomorphism

$$\mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Aut}_{\mathcal{O}_K}(\mathfrak{LT}_{/K,\pi_K}[\mathfrak{m}_K^\infty](K^{\mathrm{sep}})).$$

Here, we can compute this group of $\mathcal{O}_K$-linear automorphism precisely, as we have identified the $\mathcal{O}_K$-module structure on torsions of $\mathfrak{LT}_{/K,\pi_K}$. There exists a natural map

$$\mathcal{O}_K^\times \to \mathrm{Aut}_{\mathcal{O}_K}(\mathfrak{LT}_{/K,\pi_K}[\mathfrak{m}_K^\infty](K^{\mathrm{sep}})); \quad a \mapsto [x \mapsto ax],$$

and we claim that this is indeed an isomorphism. To show this, it suffices to note that $\mathfrak{LT}_{/K,\pi_K}[\mathfrak{m}_K^\infty](K^{\mathrm{sep}}) \cong K/\mathcal{O}_K$, and then

$$\mathrm{Aut}_{\mathcal{O}_K}(K/\mathcal{O}_K) \cong \varprojlim_n \mathrm{Aut}_{\mathcal{O}_K}(\mathfrak{m}_K^{-n}/\mathcal{O}_K) \cong \varprojlim_n (\mathcal{O}_K/\mathfrak{m}_K^n)^\times \cong \mathcal{O}_K^\times.$$

Therefore we canonically obtain a group homomorphism

$$\mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Aut}_{\mathcal{O}_K}(\mathfrak{LT}_{/K,\pi_K}[\mathfrak{m}_k^\infty](K^{\mathrm{sep}})) \cong \mathcal{O}_K^\times.$$

This map turns out to be almost equal to the inverse of the Artin reciprocity map $\theta_{/K} : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \widehat{K^\times}$. To show this, we first determine the representations for different π_K interact with each other.

3.4.1 Isomorphism over the maximal unramified

Although the formal $\mathcal{O}_K$-module schemes $\mathfrak{LT}_{/K,\pi_K}$ are not isomorphic over $\mathcal{O}_K$, they all become isomorphic once we base change to the maximal unramified $\mathcal{O}_K^{\mathrm{unr},\wedge}$. To show this, we need a lemma that takes a very similar form to Lemma 3.2.11.

Recall that there is a canonical Frobenius automorphism

$$\mathrm{Frob}_{/K} \in \mathrm{Gal}(K^{\mathrm{unr},\wedge}/K),$$

and this acts on the ring of power series $\mathcal{O}_K^{\mathrm{unr},\wedge}[[x]]$ by acting on each coefficient. Denote this action by f^σ, so that $f(x) = \sum_i a_i x^i$ gives

$$f^\sigma(x) = \mathrm{Frob}_{/K}(a_0) + \mathrm{Frob}_{/K}(a_1)x + \mathrm{Frob}_{/K}(a_2)x^2 + \cdots .$$

Lemma 3.4.1. *Let K be a p-adic local field and let $f \in x\mathcal{O}_K[[x]]^\times$ be a power series with no constant term and invertible degree 1 coefficient. Then there exists a power series $\phi \in x\mathcal{O}_K^{\mathrm{unr},\wedge}[[x]]^\times$ satisfying*

$$\phi^\sigma(x) = \phi(f(x)).$$

Proof. Again, we try to determine each coefficient inductively. Let us write

$$f(x) = a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

If we want to determine the degree 1 coefficient, $\phi(x) = c_1 x + O(x^2)$, we need to solve the equation

$$\mathrm{Frob}_{/K}(c_1) = a_1 c_1.$$

This is solvable by Proposition 2.2.2. If we have a ϕ_{m-1} that satisfies $\phi_{m-1}^\sigma = \phi_{m-1}(f)$ up to degree $m-1$, then we define $\phi_m = \phi_{m-1} + c_m x^m$ for ψ some homogeneous degree m polynomial, where c_m needs to satisfy

$$\mathrm{Frob}_{/K}(c_m) \equiv c_m + (\phi_{m-1}(f) - \phi_{m-1}^\sigma) \pmod{x^m}.$$

This is possible by the additive part of Proposition 2.2.2. ∎

Proposition 3.4.2. *Let $\pi_{K,1}, \pi_{K,2}$ be two uniformizers of K and consider the corresponding Lubin–Tate modules $\mathfrak{LT}_{/K,\pi_{K,1}}$ and $\mathfrak{LT}_{/K,\pi_{K,2}}$. If we base change to $\mathcal{O}_K^{\mathrm{unr},\wedge}$, then there exists an isomorphism*

$$\mathfrak{LT}_{/K,\pi_{K,1}} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\,\mathcal{O}_K^{\mathrm{unr},\wedge} \cong \mathfrak{LT}_{/K,\pi_{K,2}} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\,\mathcal{O}_K^{\mathrm{unr},\wedge}$$

of formal $\mathcal{O}_K$-modules over $\mathcal{O}_K^{\mathrm{unr},\wedge}$. In other words, if we choose coordinates and identify the formal Lubin–Tate modules with the formal $\mathcal{O}_K$-module laws F_1, F_2, then there exists a power series $f(x) \in x\mathcal{O}_K^{\mathrm{unr},\wedge}[[x]]^\times$ for which

$$f(F_1(x,y)) = F_2(f(x), f(y)), \quad f([a]_{F_1}(x)) = [a]_{F_2}(f(x))$$

for all $a \in \mathcal{O}_K$. Furthermore, we may choose the homomorphism f to satisfy

$$f([u]_{F_1}(x)) = [u]_{F_2}(f(x)) = f^\sigma(x).$$

where we define $u \in \mathcal{O}_K^\times$ so that $\pi_{K,2} = u\pi_{K,1}$.

Proof. Let us first choose a power series $f_1 \in x\mathcal{O}_K[[x]]^\times$ satisfying

$$f_1([u]_{F_1}(x)) = f_1^\sigma(x),$$

using Lemma 3.4.1. Then we can conjugate F_1 by f and consider the formal group law

$$F_3(x,y) = f_1(F_1(f_1^{-1}(x), f_1^{-1}(y))), \quad [a]_{F_3} = f_1([a]_{F_1}(f_1^{-1}(x))).$$

Then when we act by the Frobenius, we see that

$$\begin{aligned} F_3^\sigma(x,y) &= f_1([u]_{F_1}(F_1([u^{-1}]_{F_1}(f_1^{-1}(x)), [u^{-1}]_{F_1}(f_1^{-1}(x))))) \\ &= f_1(F_1(f_1^{-1}(x), f_1^{-1}(y))) = F_3(x,y), \\ [a]_{F_3}^\sigma(x) &= f_1([u]_{F_1}([a]_{F_1}([u^{-1}]_{F_1}(f_1^{-1}(x))))) = f_1([a]_{F_1}(f_1^{-1}(x))) = [a]_{F_3}(x). \end{aligned}$$

Therefore, the formal $\mathcal{O}_K$-module law F_3 actually has all coefficients in $\mathcal{O}_K$.

Moreover, the series $[\pi_{K,2}]_{F_3}$ can be described as

$$[\pi_{K,2}]_{F_3}(x) = f_1([u]_{F_1}([\pi_{K,1}]_{F_1}(f_1^{-1}(x)))) = f_1^\sigma([\pi_{K,1}]_{F_1}(f_1^{-1}(x))),$$

and thus when we reduce this modulo $\mathfrak{m}_K$,

$$\begin{aligned} f_1^\sigma([\pi_{K,1}]_{F_1}(f_1^{-1}(x))) &\equiv f_1^\sigma(\mathrm{Frob}_k(f_1^{-1}(x))) \\ &\equiv f_1^\sigma((f_1^\sigma)^{-1}(\mathrm{Frob}_k(x))) \equiv x^q \pmod{\mathfrak{m}_K} \end{aligned}$$

is a Lubin–Tate series. Therefore the formal $\mathcal{O}_K$-module law F_3 defines the Lubin–Tate module for $\pi_{K,2}$, hence is equivalent to F_2. Therefore

$$\mathfrak{LT}_{/K,\pi_{K,1}} \cong \mathfrak{F}_1 \xrightarrow{f_1} \mathfrak{F}_3 \cong \mathfrak{LT}_{/K,\pi_{K,2}} \cong \mathfrak{F}_2.$$

To obtain the isomorphism between F_1 and F_2, we can simply compose f_1 with the isomorphism between F_3 and F_2. $\square$

This allows us to compare Lubin–Tate modules coming from different uniformizers. Let $\pi_{K,1}$ and $\pi_{K,2} = u\pi_{K,1}$ be uniformizers of K, and consider Lubin–Tate $\mathcal{O}_K$-modules laws F_1, F_2 over $\mathcal{O}_K$ corresponding to $\pi_{K,1}$ and $\pi_{K,2}$. We have seen that there exists a power series $f \in x\mathcal{O}_K^{\mathrm{unr},\wedge}[[x]]^\times$ satisfying

$$f^\sigma(x) = f([u]_{F_1}(x)), \quad f([a]_{F_1}(x)) = [a]_{F_2}(f(x)).$$

Then it induces an isomorphism

$$\mathfrak{LT}_{/K,\pi_{K,1}}[\mathfrak{m}_K^\infty](K^{\mathrm{sep}}) \xrightarrow{f} \mathfrak{LT}_{/K,\pi_{K,2}}[\mathfrak{m}_K^\infty](K^{\mathrm{sep}}); \quad x \mapsto f(x)$$

of $\mathcal{O}_K$-modules.

Let us see how the Galois action on the two sides compare with each other. For any element $g \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$, we note that

$$g^{-1}(f(g(x))) = f^g(x),$$

where $f^g \in \mathcal{O}_K^{\mathrm{unr},\wedge}[[x]]$ is the power series obtained by applying g to all the coefficients of f. By definition, the coefficients of f are all in $\mathcal{O}_K^{\mathrm{unr},\wedge}$, and hence f^g is only sensible to the restriction of g to K^{unr}. Let us suppose that

$$g|_{K^{\mathrm{unr}}} = \mathrm{Frob}_{/K}^{\alpha} \in \mathrm{Gal}(K^{\mathrm{unr}}/K) \quad \text{for} \quad \alpha \in \widehat{\mathbb{Z}}.$$

Then

$$g^{-1}(f(g(x))) = f^g(x) = f^{(\sigma^\alpha)}(x) = f([u^\alpha]_{F_1}(x)) \in \mathcal{O}_K^{\mathrm{unr},\wedge}[[x]],$$

where both the α-powers make sense by choosing a sequence $\alpha_i \to \alpha$ for $\alpha_i \in \mathbb{Z}$ and taking the limit after evaluating.

Let us denote by

$$\rho_1, \rho_2 \colon \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathcal{O}_K^\times$$

the group homomorphisms coming from $\mathcal{O}_K$-linear representations of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ associated to the Lubin–Tate modules $\mathfrak{LT}_{/K,\pi_{K,1}}$ and $\mathfrak{LT}_{/K,\pi_{K,2}}$. These maps are characterized by the property that with an arbitrary choice of coordinates,

$$g(x) = [\rho_i(g)]_{F_i}(x)$$

for all $x \in \mathfrak{LT}_{/K,\pi_{K,i}}[\mathfrak{m}_K^\infty](K^{\mathrm{sep}})$. Therefore

$$\begin{aligned} f([\rho_1(g)]_{F_1}(x)) &= f(g(x)) = g(f([u^\alpha]_{F_1}(x))) \\ &= [\rho_2(g)]_{F_2}(f([u^\alpha]_{F_1}(x))) = f([u^\alpha \rho_2(g)]_{F_1}(x)) \end{aligned}$$

implies that $[\rho_1(g)]_{F_1} = [u^\alpha \rho_2(g)]_{F_1}$. Looking at the linear term, we obtain

$$\rho_1(g) = u^\alpha \rho_2(g)$$

if $g \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ restricts to $\mathrm{Frob}_{/K}^\alpha \in \mathrm{Gal}(K^{\mathrm{unr}}/K)$. To make the formula nicer, we can further write this as

$$\pi_{K,1}^\alpha \rho_1(g) = \pi_{K,2}^\alpha \rho_2(g) \in \widehat{K^\times},$$

since $u \in \mathcal{O}_K^\times$ was defined as $\pi_{K,2} = u\pi_{K,1}$.

Proposition 3.4.3. *Choose π_K a uniformizer of K and define a map*

$$\rho_{LT} \colon \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \widehat{K^\times}; \quad g \mapsto \beta\pi_K^\alpha$$

where $\beta \in \mathcal{O}_K^\times$ is defined to satisfy

$$g(x) = [\beta]_{\mathfrak{LT}_{/K,\pi_K}}(x)$$

for all $x \in \mathfrak{LT}_{/K,\pi_K}[\mathfrak{m}_K^\infty](K^{\mathrm{sep}})$ and $\alpha \in \widehat{\mathbb{Z}}$ is defined to satisfy

$$g|_{K^{\mathrm{unr}}} = \mathrm{Frob}_{/K}^\alpha \in \mathrm{Gal}(K^{\mathrm{unr}}/K).$$

Then map ρ_{LT} does not depend on the choice of uniformizer π_K. In particular, ρ_{LT} restricts to a map

$$\rho_{LT}|_{W_K} \colon W_K \to K^\times$$

that does not depend on the choice of uniformizer π_K, which can also be thought of as a 1-dimensional representation of the Weil group W_K.

We have thus obtained a canonical map that only depends on the p-adic local field K and not on any auxiliary choices. Our next goal is to prove that this indeed is the inverse of the Artin reciprocity map.

3.4.2 Explicit description of the Artin reciprocity map again

In Proposition 3.4.3, we constructed a continuous homomorphism

$$\rho_{LT}\colon \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \widehat{K^\times}.$$

We want to show that this map is equal to the composition

$$\rho_\theta\colon \mathrm{Gal}(K^{\mathrm{sep}}/K) \twoheadrightarrow \mathrm{Gal}(K^{\mathrm{sep}}/K)^{\mathrm{ab}} \xrightarrow{\theta_{/K}^{-1}} \widehat{K^\times}.$$

To prove the theorem, we need to use the existence of the Artin reciprocity map $\theta_{/K}$ and all the properties it has. Hence this chapter does not by itself give a proof of local class field theory; it produces the Artin reciprocity map, but to show that it is an isomorphism we need the results from the previous chapter.

Lemma 3.4.4. *Let $\pi_K \in K$ be a uniformizer. Then the field extension*

$$K_{\pi,n} = K(\mathfrak{LT}_{/K,\pi_K}[\mathfrak{m}_K^n](K^{\mathrm{sep}}))$$

obtained by adjoining all the $\mathfrak{m}_K^n$-torsion points of the Lubin–Tate module $\mathfrak{LT}_{/K,\pi_K}$ to K is abelian extension of degree $q^{n-1}(q-1)$. Moreover, $K_{\pi,n}$ is totally ramified over K and

$$\pi_K \in N_{K_{\pi,n}/K} K_{\pi,n}^\times.$$

Proof. For computational simplicity, consider the polynomial $f(x) = \pi_K x + x^q$. Then there exists a choice of coordinates on $\mathfrak{LT}_{/K,\pi_K}$ that makes $[\pi_K](x) = f(x)$. By definition, $K_{\pi,n}$ is obtained from K by adjoining all the zeros of

$$f^{(n)}(x) = \frac{f^{(n)}(x)}{f^{(n-1)}(x)} \cdots \frac{f(f(x))}{f(x)} \frac{f(x)}{x} x.$$

Here, we note that each $f^{(k)}(x)/f^{(k-1)}(x)$ is

$$\frac{f^{(k)}(x)}{f^{(k-1)}(x)} = \pi_K + (f^{(k-1)}(x))^{q-1}$$

and hence is an Eisenstein polynomial of degree $q^{k-1}(q-1)$ and constant term π_K.

Pick a root α of $f^{(n)}(x)/f^{(n-1)}(x)$ so that $f^{(n-1)}(\alpha) \neq 0$ but $f^{(n)}(\alpha) = 0$. Because α is a $\mathfrak{m}_K^n$-torsion point of the Lubin–Tate module that is not a $\mathfrak{m}_K^{n-1}$-torsion, all other $\mathfrak{m}_K^n$-torsion points can by obtained by considering $[a](\alpha)$ for $a \in \mathcal{O}_K$. It follows that all other $\mathfrak{m}_K^n$-torsion are in generated by α over K, and therefore

$$K_{\pi,n} = K(\alpha).$$

Because α is a root of a degree $q^{(n-1)}(q-1)$ Eisenstein polynomial, it immediately follows that $K_{\pi,n}$ is totally ramified over K with degree $q^{(n-1)}(q-1)$. To see that this is an abelian extension, we note that the Galois conjugates of α are

$$\{[a](\alpha) : a \in (\mathcal{O}_K/\mathfrak{m}_K^n)^\times\},$$

since these are $q^{(n-1)}(q-1)$ distinct zeros of $f^{(n)}/f^{(n-1)}$. Thus a Galois automorphism of $K_{\pi,n}$ sends $\alpha \mapsto [a](\alpha)$ for some $a \in (\mathcal{O}_K/\mathfrak{m}_K^n)^\times$, and this automatically sends

$$[b](\alpha) \mapsto [b]([a](\alpha)) = [ab](\alpha)$$

for all $b \in (\mathcal{O}_K/\mathfrak{m}_K^n)^\times$. It follows that we obtain a natural identification

$$\mathrm{Gal}(K_{\pi,n}/K) \cong (\mathcal{O}_K/\mathfrak{m}_K^n)^\times; \quad (\alpha \mapsto [a](\alpha)) \leftrightarrow a.$$

(This identification does not even depend on the choice of the root α.) It follows that $K_{\pi,n}$ is an abelian extension. □

First note that both maps ρ_{LT}, ρ_θ recovers the valuation and restricted to the maximal unramified K^{unr}, i.e., the diagram

$$\begin{tikzcd}
\mathrm{Gal}(K^{\mathrm{sep}}/K) \arrow[r, "\rho"] \arrow[d, "\mathrm{res}"] & \widehat{K^\times} \arrow[d, "v"] \\
\mathrm{Gal}(K^{\mathrm{unr}}/K) & \widehat{\mathbb{Z}} \arrow[l, "\alpha \mapsto \mathrm{Frob}_{/K}^\alpha"', "\cong"]
\end{tikzcd}$$

commutes. The statement for ρ_{LT} holds by construction, as the Lubin–Tate module only sees the $\mathcal{O}_K^\times$ part and we multiplied it with a power of a uniformizer according to how it acts K^{unr}. On the other hand, the statement for ρ_θ holds since it was one of the desired properties of the Artin reciprocity map.

We observe two facts.

- Suppose L/K is a finite abelian extension. If $\rho_\theta(g) \in NL^\times$, then $g|_L = \mathrm{id}_L \in \mathrm{Gal}(L/K)$. This is because the $\theta_{L/K}^{-1} \colon \mathrm{Gal}(L/K) \to K^\times/NL^\times$ is an isomorphism.

$$\begin{tikzcd}
\mathrm{Gal}(K^{\mathrm{sep}}/K) \arrow[r, two heads] \arrow[d, two heads, "\mathrm{res}"] & \mathrm{Gal}(K^{\mathrm{sep}}/K)^{\mathrm{ab}} \arrow[r, "\theta_{/K}^{-1}", "\cong"'] \arrow[d, two heads, "\mathrm{res}^{\mathrm{ab}}"] & \widehat{K^\times} \arrow[d, two heads] \\
\mathrm{Gal}(L/K) \arrow[r, equal] & \mathrm{Gal}(L/K)^{\mathrm{ab}} \arrow[r, "\theta_{L/K}^{-1}", "\cong"'] & K^\times/NL^\times \cong \widehat{K^\times}/N\widehat{L^\times}
\end{tikzcd}$$

- Let $\pi_K \in K$ be a uniformizer. Suppose $g \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ satisfies that $g|_{K^{\mathrm{unr}}} = \mathrm{Frob}_{/K}$ and $g|_{K_{\pi_K,n}} = \mathrm{id}_{K_{\pi_K,n}}$ for all $n \geq 0$. Then $\rho_{LT}(g) = \pi_K$. This follows directly from the definition of ρ_{LT}.

Theorem 3.4.5. *The two homomorphisms*

$$\rho_\theta, \rho_{LT} \colon \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \widehat{K^\times}$$

are identical.

Proof. Since the completion of $\mathbb{Z}$ is topologically generated by 1, it is enough to show that

$$\rho_\theta(g) = \rho_{LT}(g) \in K$$

is the same uniformizer, for all $g \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ that restricts to $\mathrm{Frob}_{/K} \in \mathrm{Gal}(K^{\mathrm{unr}}/K)$.

Fix $g \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ that restricts to the Frobenius. Then $\pi_K = \rho_\theta(g)$ is a uniformizer. Moreover, we see from Lemma 3.4.4 that

$$\rho_\theta(g) = \pi_K \in N_{K_{\pi_K,n}/K} K^\times_{\pi_K,n}$$

for all $n \geq 0$. It follows that

$$g|_{K_{\pi_K,n}} = \mathrm{id}_{K_{\pi_K,n}} \in \mathrm{Gal}(K_{\pi_K,n}/K).$$

Since we picked g to restrict to $\mathrm{Frob}_{/K}$ on K^{unr}, it moreover follows that

$$\rho_{LT}(g) = \pi_K.$$

Therefore $\rho_{LT}(g) = \rho_\theta(g)$. □

Since we have an explicit description of the map ρ_{LT} and it is essentially the inverse of $\theta_{/K}$, we obtain a second explicit description of the Artin reciprocity map.

Corollary 3.4.6. *Let K be a p-adic local field, and fix a uniformizer $\pi_K \in K$. For a Galois automorphism $g \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$, consider*

(1) the element $\alpha \in \widehat{\mathbb{Z}}$ such that $g|_{K^{\mathrm{unr}}} = \mathrm{Frob}^\alpha_{/K} \in \mathrm{Gal}(K^{\mathrm{unr}}/K)$, and

(2) the element $\beta \in \mathcal{O}_K^\times$ that satisfies $g(x) = [\beta](x)$ for all torsion points

$$x \in \mathfrak{LT}_{/K,\pi_K}[\mathfrak{m}_K^\infty](K^{\mathrm{sep}}).$$

Then $\beta\pi_K^\alpha \in \widehat{K^\times}$ is the image of g under the composition

$$\mathrm{Gal}(K^{\mathrm{sep}}/K) \twoheadrightarrow \mathrm{Gal}(K^{\mathrm{sep}}/K)^{\mathrm{ab}} \xrightarrow{\theta_{/K}^{-1}} \widehat{K^\times},$$

and hence is independent of the choice of the uniformier π_K.

Bibliography

[CF67] J. W. S. Cassels and A. Fröhlich, *Algebraic number theory*, Proceedings of an instructional conference organized by the London Mathematical Society (a NATO Advanced Study Institute) with the support of the International Mathematical Union. Edited by J. W. S. Cassels and A. Fröhlich, Academic Press, London; Thompson Book Co., Inc., Washington, D.C., 1967. MR 0215665

[Dwo58] Bernard Dwork, *Norm residue symbol in local number fields*, Abh. Math. Sem. Univ. Hamburg **22** (1958), 180–190. MR 0098079

[HG94] M. J. Hopkins and B. H. Gross, *Equivariant vector bundles on the Lubin-Tate moduli space*, Topology and representation theory (Evanston, IL, 1992), Contemp. Math., vol. 158, Amer. Math. Soc., Providence, RI, 1994, pp. 23–88. MR 1263712

[LT65] Jonathan Lubin and John Tate, *Formal complex multiplication in local fields*, Ann. of Math. (2) **81** (1965), 380–387. MR 0172878

[Mil13] J.S. Milne, *Class field theory (v4.02)*, 2013, Available at www.jmilne.org/math/, pp. 281+viii.

[Ser79] Jean-Pierre Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York-Berlin, 1979, Translated from the French by Marvin Jay Greenberg. MR 554237

[Sna94] Victor P. Snaith, *Galois module structure*, Fields Institute Monographs, vol. 2, American Mathematical Society, Providence, RI, 1994. MR 1307386