

# Shannon Meets Nash on the Interference Channel

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**Abstract**—The interference channel is the simplest communication scenario where multiple autonomous users compete for shared resources. We combine game theory and information theory to define the notion of a Nash equilibrium region of the interference channel. The notion is game theoretic: it captures the selfish behavior of each user as they compete. The notion is also information theoretic: it allows each user to use arbitrary communication strategies as it optimizes its own performance. We give an exact characterization of the Nash equilibrium region of the two-user linear deterministic interference channel and an approximate characterization of the Nash equilibrium region of the two-user Gaussian interference channel to within 1 bit/s/Hz.

**Index Terms**—Game theory, Han–Kobayashi scheme, interference channel, Nash equilibrium.

## I. INTRODUCTION

INFORMATION theory deals with the fundamental limits of communication. In network information theory, an object of central interest is the *capacity region* of the network: it is the set of all rate tuples of the users in the network that are simultaneously achievable by optimizing their communication strategies. Implicit in the definition is that users mutually agree on the optimized choice of their communication strategies. This may not be a realistic assumption if users are selfish and are only interested in maximizing their own benefit. Game theory provides a notion of a *Nash equilibrium* to characterize system operating points that are stable under such selfish behavior. In this paper, we define and explore an information theoretic Nash equilibrium region as the game theoretic counterpart of the capacity region of a network. While the Nash equilibrium region is naturally a subset of the (non-selfish) capacity region, in general not all points in the capacity region are Nash equilibria. The research question is then to characterize the Nash equilibrium region given a network and a model of the channels.

The two-user interference channel (IC) is perhaps the simplest communication scenario to study this problem. Here two point-to-point communication links interfere with each other

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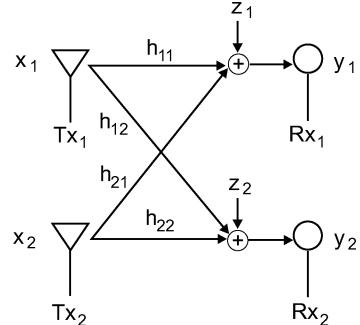


Fig. 1. Two-user Gaussian interference channel.

through cross-talk. Each transmitter has an independent message intended only for the corresponding receiver. The capacity region of this channel is the set of all simultaneously achievable rate pairs  $(R_1, R_2)$  in the two interfering links, and characterizes the fundamental tradeoff between the performance achievable on the two links in face of interference.

In the standard (non-selfish) setting, the users jointly choose encoding and decoding schemes to achieve a rate pair  $(R_1, R_2)$ . In the game theoretic setting, on the other hand, we study the case where each user individually chooses an encoding/decoding scheme in order to maximize his own transmission rate. The two users can be viewed as playing a *noncooperative* game, where a user's strategy is its encoding/decoding scheme and its payoff is its reliable rate. A Nash equilibrium (NE) is a pair of strategies for which there is no incentive for either user to unilaterally change its strategy to improve its own rate. These are incentive-compatible operating points. The Nash equilibrium region of the IC is the set of all reliable rate pairs each of which can be achieved at some NE. Our focus is on a "one-shot" game formulation in which each player has full information, i.e., both players know the channel statistics, the actions chosen by each player, as well as their pay-off function.

A particular IC we focus on in this paper is the two-user Gaussian IC shown in Fig. 1. This is a basic model in wireless and wireline channels (such as DSL). Game theoretic approaches for the Gaussian IC have been studied before, e.g., [5]–[8]. However, there are two key assumptions in these works: 1) the class of encoding strategies are constrained to random Gaussian codebooks; 2) the decoders are restricted to treat the interference as Gaussian noise and are hence sub-optimal. Because of these restrictions, the formulation in these works are not information-theoretic in nature. For example, a Nash equilibrium found under these assumptions may no longer be an equilibrium if users can adopt a different encoding or decoding strategy.

In this paper, we make three contributions. First, we give a precise formulation of an information theoretic NE region for general ICs, where the users are allowed to use *any* encoding

and decoding strategies. Second, we analyze the NE region of the two-user *linear deterministic* IC [3]. This type of deterministic channel model was first proposed by [4] in the analysis of Gaussian relay networks, and the deterministic IC has been shown to be a good approximation of the Gaussian IC in [3]. For this deterministic IC, we give a simple exact characterization of the NE region. At each of the rate pairs in the NE region, we provide explicit coding schemes that achieve the rate pair and such that no user has any incentive to deviate to improve his own rate. Somewhat surprisingly, we find that in all cases, there are always Nash equilibria which are *efficient*, i.e., they lie on the maximum sum-rate boundary of the capacity region. In particular, for channels with symmetrical channel gains, the symmetric rate point on the capacity region boundary is always a Nash equilibrium. Our third contribution is to use these insights to approximate the NE region of the Gaussian IC to within 1 bit/s/Hz. This result parallels the recent characterization of the (cooperative) capacity region of the same channel to within 1 bit [1].

## II. PROBLEM FORMULATION

Let us now formally define the communication situation for general interference channels. In subsequent sections, we will specialize to specific classes of interference channels.

Communication starts at time 0. User  $i$  communicates by coding over blocks of length  $N_i$  symbols,  $i = 1, 2$ . Transmitter  $i$  sends on block  $k$  information bits  $b_{i1}^{(k)}, \dots, b_{i,L_i}^{(k)}$  by transmitting a codeword denoted by  $\mathbf{x}_i^{(k)} = [\mathbf{x}_i^{(k)}(1), \dots, \mathbf{x}_i^{(k)}(N_i)]$ . All the information bits are equally probable and independent of each other. Receiver  $i$  observes on each block an output sequence through the interference channel, which specifies a stochastic mapping from the input sequences of user 1 and 2 to the output sequences of user 1 and 2. Given the observed sequence  $\{\mathbf{y}_i^{(k)} = [\mathbf{y}_i^{(k)}(1), \dots, \mathbf{y}_i^{(k)}(N_i)], k = 1, 2, \dots\}$ , receiver  $i$  generates guesses  $\hat{b}_{i\ell}^{(k)}$  for each of the information bits. Without loss of generality, we will assume that each receiver  $i$  performs maximum-likelihood decoding on each bit, i.e., it chooses the  $\hat{b}_i^{(k)}$  that maximizes the *a posteriori probability* of the observed sequence  $\mathbf{y}_i^{(1)}, \mathbf{y}_i^{(2)}, \dots$  given the transmitted bit  $b_{i\ell}^{(k)}$ .

Note that the communication scenario we defined here is more general than the one usually used in network information theory, as we allow the two users to code over different block lengths. However, such generality is necessary here, since even though the two users may agree *a priori* on a common block length, a selfish user may unilaterally decide to choose a different block length during the actual communication process.

A *strategy*  $s_i$  of user  $i$  is defined by its message encoding, which we assume to be the same on every block and involves:

- the number of information bits  $B_i$  and the block length  $N_i$  of the codewords;
- the codebook  $\mathcal{C}_i$ , i.e., the set of codewords employed by transmitter  $i$ ;
- the encoder  $f_i : \{1, \dots, 2^{B_i}\} \times \Omega_i \rightarrow \mathcal{C}_i$ , that maps on each block  $k$  the message  $m_i^{(k)} := (b_{i1}^{(k)}, \dots, b_{i,B_i}^{(k)})$  to a transmitted codeword  $\mathbf{x}_i^{(k)} = f_i(m_i^{(k)}, \omega_i^{(k)}) \in \mathcal{C}_i$ ;
- the rate of the code,  $R_i(s_i) = B_i/N_i$ .

A strategy  $s_1$  of user 1 and  $s_2$  of user 2 jointly determines the average bit error probabilities  $p_i^{(k)} := \frac{1}{B_i} \sum_{\ell=1}^{B_i} \mathcal{P}(\hat{b}_{i\ell}^{(k)} \neq b_{i\ell}^{(k)})$ ,  $i = 1, 2$ .<sup>1</sup> Note that if the two users use different block lengths, the error probability could vary from block to block even though each user uses the same encoding for all the blocks. However, if they use the same block length, then the error probability is the same across the blocks for a user, which we will denote by  $p_i$  for user  $i$ .

The encoder of each transmitter  $i$  may employ a stochastic mapping from the message to the transmitted codeword;  $\omega_i^{(k)} \in \Omega_i$  represents the randomness in that mapping. We assume that this randomness is independent between the two transmitters and across different blocks. Furthermore, we assume that each transmitter and its corresponding receiver have access to a source of *common randomness*, so that the realization  $\omega_i^{(k)}$  is known at both transmitter  $i$  and receiver  $i$ , but not at the other receiver or transmitter.<sup>2</sup>

For a given error probability threshold  $\epsilon > 0$ , we define an  $\epsilon$ -interference-channel-game as follows. Each user  $i$  chooses a strategy  $s_i$ ,  $i = 1, 2$ , and receives a pay-off of

$$\pi_i(s_1, s_2) = \begin{cases} R(s_i), & \text{if } p_i^{(k)}(s_1, s_2) \leq \epsilon, \forall k \\ 0, & \text{otherwise.} \end{cases}$$

In other words, a user's pay-off is equal to the rate of the code provided that the probability of error is no greater than  $\epsilon$ . A strategy pair  $(s_1, s_2)$  is defined to be  $(1 - \epsilon)$ -*reliable* provided that they result in an error probability  $p_i(s_1, s_2)$  of less than  $\epsilon$  for  $i = 1, 2$ . An  $(1 - \epsilon)$ -reliable pair of strategies is said to achieve the rate-pair  $(R(s_1), R(s_2))$ .

For an  $\epsilon$ -game, a strategy pair  $(s_1^*, s_2^*)$  is a Nash equilibrium (NE) if neither user can unilaterally deviate and improve their pay-off, i.e., if for each user  $i = 1, 2$ , there is no other strategy  $s_i$  such that<sup>3</sup>  $\pi_i(s_i, s_2^*) > \pi_i(s_1^*, s_2^*)$ . If user  $i$  attempts to transmit at a higher rate than what he is receiving in a Nash equilibrium and user  $j$  does not change her strategy, then user  $i$ 's error probability must be greater than  $\epsilon$ .

Similarly, a strategy pair  $(s_1^*, s_2^*)$  is an  $\eta$ -Nash equilibrium<sup>4</sup> ( $\eta$ -NE) of an  $\epsilon$ -game if neither user can unilaterally deviate and improve their pay-off by more than  $\eta$ , i.e., if for each user  $i$ , there is no other strategy  $s_i$  such that  $\pi_i(s_i, s_2^*) > \pi_i(s_1^*, s_2^*) + \eta$ . Note that when a user deviates, it does not care about the reliability of the other user but only its own reliability. So in the above definitions  $(s_i, s_j^*)$  is not necessarily  $(1 - \epsilon)$ -reliable.

Given any  $\bar{\epsilon} > 0$ , the capacity region  $\mathcal{C}$  of the interference channel is the closure of the set of all rate pairs  $(R_1, R_2)$  such that for every  $\epsilon \in (0, \bar{\epsilon})$ , there exists a  $(1 - \epsilon)$ -reliable strategy pair  $(s_1, s_2)$  which achieves the rate pair  $(R_1, R_2)$ . The *Nash equilibrium region*  $\mathcal{C}_{\text{NE}}$  of the interference channel is the closure of the set of rate pairs  $(R_1, R_2)$  such that for every  $\eta > 0$ , there

<sup>1</sup>Average bit error probabilities are more meaningful than codeword error probabilities in a setting, such as ours, where users can vary the blocklength they are using.

<sup>2</sup>Such common randomness is not needed for many of the results in the paper, but allowing for it simplifies our presentation.

<sup>3</sup>In this paper, we use the convention that  $j$  always denotes the other user from  $i$ .

<sup>4</sup>In the game theoretic literature, this is often referred to as an  $\epsilon$ -Nash equilibrium or simply an  $\epsilon$ -equilibrium for a game [9, page 143].

exists an  $\bar{\epsilon} > 0$  (dependent on  $\eta$ ) so that if  $\epsilon \in (0, \bar{\epsilon})$ , there exists a  $(1 - \epsilon)$ -reliable strategy pair  $(s_1, s_2)$  that achieves the rate-pair  $(R_1, R_2)$  and is a  $\eta$ -NE. Clearly,  $\mathcal{C}_{\text{NE}} \subset \mathcal{C}$ .

We make a few comments about the definition of  $\mathcal{C}_{\text{NE}}$ . In this definition, the parameter  $\bar{\epsilon}$  is introduced so that  $(1 - \epsilon)$ -reliable strategy pairs need only exist for “small enough” values of  $\epsilon$ . In the definition of the capacity region for the interference channel this constraint is not needed. The capacity region is equally well defined by requiring the given conditions to hold for any  $\epsilon > 0$  (since, clearly if a pair of strategies are  $(1 - \epsilon)$ -reliable, they are also  $(1 - \tilde{\epsilon})$ -reliable for all  $\tilde{\epsilon} > \epsilon$ ). However, when defining  $\mathcal{C}_{\text{NE}}$ , this condition is important. In particular a pair of strategies can be an  $\eta$ -NE for an  $\epsilon$ -game, but not an  $\eta$ -NE for an  $\tilde{\epsilon}$ -game with  $\tilde{\epsilon} > \epsilon$ , since increasing the error probability threshold enlarges the set of possible deviations an agent may make. As an extreme example, consider the case where  $\epsilon = 1$ , in which case each agent can achieve an arbitrarily high pay-off regardless of the action of the other user and so no  $\eta$ -NE exists. Thus, if we required our definition to hold for any  $\epsilon > 0$ ,  $\mathcal{C}_{\text{NE}}$  would be empty.

Next, we turn to the use of  $\eta$ -NE in the definition. A more natural approach would have been to instead simply use NE. In other words, define  $\mathcal{C}_{\text{NE}}$  to be the closure of the rate pairs  $(R_1, R_2)$  such that for any  $\epsilon$  small enough, that there exists a  $(1 - \epsilon)$ -reliable strategy pair  $(s_1, s_2)$  that achieves the rate-pair  $(R_1, R_2)$  and is a NE of an  $\epsilon$ -game. The difficulty with this definition is that to determine such a NE requires one to find a particular scheme that achieves the optimal rate for a given nonzero error probability. Finding such a scheme is extremely difficult and in general an open problem.<sup>5</sup> By introducing the slack  $\eta$ , these difficulties are removed. Moreover, since we require that this definition hold for all  $\eta > 0$ , this slack can be made arbitrarily small.

Finally, we would like to comment on the use of different block lengths in our definitions. First, it can be argued that if there is a  $(1 - \epsilon)$ -reliable strategy pair  $(s_1, s_2)$  that achieves a rate pair  $(R_1, R_2)$  using codes of block lengths  $N_1, N_2$ , then there exists a  $(1 - \epsilon)$  strategy pair that achieves the same rate pair but with each user using the same block length. This follows by considering using “super-blocks” of length  $N$ , where  $N$  is the least common multiple of  $N_1$  and  $N_2$ . Over these super-blocks the users can be viewed as using two equal-length codes. The error probabilities, being the average bit error probabilities now across longer blocks, remain less than  $\epsilon$ . This means that in computing the capacity region  $\mathcal{C}$ , we can without loss of generality consider only strategies in which both users use the same block lengths. Also, in the Nash equilibrium definitions, we can without loss of generality assume that in the nominal strategy, the two users use the same block length (although each user is allowed to deviate using another strategy of a different block length).

### III. LINEAR-DETERMINISTIC IC

#### A. Deterministic Channel Model

Let us now focus on a specific interference channel model: a linear deterministic IC analogous to the Gaussian IC. This

<sup>5</sup>Moreover, it is not even clear if there exists such a scheme, i.e., a scheme that achieves the supremum of the rates over all  $1 - \epsilon$  reliable schemes.

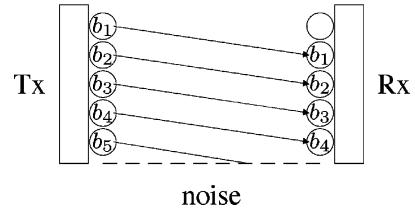


Fig. 2. Linear deterministic model for the point-to-point Gaussian channel. Each bit of the input occupies a signal level. Bits of lower significance are lost due to noise.

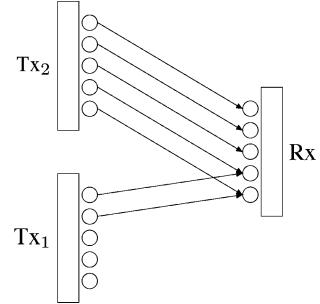


Fig. 3. Linear deterministic model for the Gaussian multiple-access channel. Incoming bits on the same level are added modulo two at the receiver. In this example, user 2 has the stronger channel and so compared to user 1, user 2 has more bits received above the noise level.

channel was first introduced in [4]. We begin by describing the linear deterministic channel model for the point-to-point AWGN channel, and then the two-user multiple-access channel. After understanding these examples, we present the linear deterministic IC.

Consider first the model for the point-to-point channel (see Fig. 2). The real-valued channel input is written in base 2; the signal—a vector of bits—is interpreted as occupying a succession of levels

$$x = 0.b_1 b_2 b_3 b_4 b_5 \dots$$

The most significant bit coincides with the highest level, the least significant bit with the lowest level. The levels attempt to capture the notion of *signal scale*; a level corresponds to a unit of power in the Gaussian channel, measured on the dB scale. Noise is modeled in the deterministic channel by truncation. Bits of smaller order than the noise are lost. Note that the number of bits above the noise floor correspond to  $\log_2 \text{SNR}$ , where SNR is the signal-to-noise ratio of the corresponding Gaussian channel.

The linear deterministic multiple-access channel is constructed similarly to the point-to-point channel (see Fig. 3). To model the super-position of signals at the receiver, the bits received on each level are added *modulo two*. Addition modulo two, rather than normal integer addition, is chosen to make the model more tractable. As a result, the levels do not interact with one another.

We proceed with the linear deterministic IC model (Fig. 4). There are two transmitter-receiver pairs (links), and as in the Gaussian case, each transmitter wants to communicate only with its corresponding receiver. The signal from transmitter  $i$ , as observed at receiver  $j$ , is scaled by a nonnegative integer gain  $a_{ji} = 2^{n_{ji}}$  (equivalently, the input column vector is shifted up by  $n_{ji}$ ). At each time  $t$ , the input and output, respectively, at link

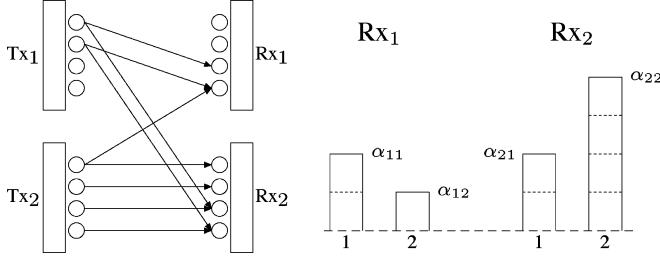


Fig. 4. At left is a linear deterministic IC. The more compact figure at right shows only the signals as observed at the receivers.

$i$  are  $\mathbf{x}_i(t), \mathbf{y}_i(t) \in \{0, 1\}^q$ , where  $q = \max_{ij} n_{ij}$ . Note that  $n_{ii}$  corresponds to  $\log_2 \text{SNR}_i$  and  $n_{ji}$  corresponds to  $\log_2 \text{INR}_{ji}$ , where  $\text{SNR}_i$  is the signal-to-noise ratio of link  $i$  and  $\text{INR}_{ji}$  is the interference-to-noise ratio at receiver  $j$  from transmitter  $i$  in the corresponding Gaussian IC.

The channel output at receiver  $i$  is given by

$$\mathbf{y}_i(t) = \mathbf{S}^{q-n_{i1}} \mathbf{x}_1(t) + \mathbf{S}^{q-n_{i2}} \mathbf{x}_2(t) \quad (1)$$

where summation and multiplication are in the binary field and  $\mathbf{S}$  is a  $q \times q$  shift matrix

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}. \quad (2)$$

If the inputs  $\mathbf{x}_i(t)$  are written as a binary number  $x_i$ , the channel can equivalently be written as

$$\begin{aligned} y_1 &= \lfloor a_{11}x_1 + a_{12}x_2 \rfloor \\ y_2 &= \lfloor a_{21}x_1 + a_{22}x_2 \rfloor \end{aligned}$$

where addition is performed on each bit (modulo two) and  $\lfloor \cdot \rfloor$  is the integer-part function.

In our analysis, it will be helpful to consult a different style of figure, as shown on the right-hand side of Fig. 4. This shows only the perspective of each receiver. Each incoming signal is shown as a column vector, with the highest element corresponding to the most significant bit and the portion below the noise floor truncated. The observed signal at each receiver is the modulo 2 sum of the elements on each level.

The linear deterministic IC is relatively simple, yet retains two essential features of the Gaussian IC: the loss of information due to noise, and the superposition of transmitted signals at each receiver. The modeling of noise can be understood through the point-to-point channel above. The superposition of transmitted signals at each receiver is captured by taking the modulo 2 sum of the incoming signals at each level, as in the model for the multiple-access channel.

## B. Main Results

To begin, we give the capacity region,  $\mathcal{C}$ , of the two-user linear deterministic IC. This region is given by Theorem 1 in

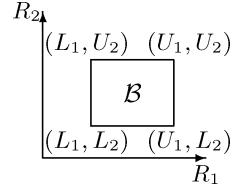


Fig. 5. Example of the box  $\mathcal{B}$ . The values of the four corner points are indicated in the figure.

[10], which applies to a larger class of deterministic interference channels. For our model, the resulting region becomes the set of non-negative rates satisfying:<sup>6</sup>

$$R_1 \leq n_{11} \quad (3)$$

$$R_2 \leq n_{22} \quad (4)$$

$$R_1 + R_2 \leq (n_{11} - n_{12})^+ + \max(n_{22}, n_{12}) \quad (5)$$

$$R_1 + R_2 \leq (n_{22} - n_{21})^+ + \max(n_{11}, n_{21}) \quad (6)$$

$$\begin{aligned} R_1 + R_2 &\leq \max(n_{21}, (n_{11} - n_{12})^+) \\ &\quad + \max(n_{12}, (n_{22} - n_{21})^+) \end{aligned} \quad (7)$$

$$\begin{aligned} 2R_1 + R_2 &\leq \max(n_{11}, n_{21}) + (n_{11} - n_{12})^+ \\ &\quad + \max(n_{12}, (n_{22} - n_{21})^+) \end{aligned} \quad (8)$$

$$\begin{aligned} R_1 + 2R_2 &\leq \max(n_{22}, n_{12}) + (n_{22} - n_{21})^+ \\ &\quad + \max(n_{21}, (n_{11} - n_{12})^+). \end{aligned} \quad (9)$$

Our first main result, stated in Theorem 1 below is to completely characterize  $\mathcal{C}_{\text{NE}}$  for the two-user linear deterministic IC. This characterization is in terms of  $\mathcal{C}$  and a “box”  $\mathcal{B}$  in  $\mathbb{R}_+^2$  given by (see Fig. 5)

$$\mathcal{B} = \{(R_1, R_2) : L_i \leq R_i \leq U_i, \quad \forall i = 1, 2\}$$

where for each user  $i = 1, 2$

$$L_i = (n_{ii} - n_{ij})^+, \quad (10)$$

and

$$U_i = \begin{cases} n_{ii} - \min(L_j, n_{ij}), & \text{if } n_{ij} \leq n_{ii}, \\ \min((n_{ij} - L_j)^+, n_{ii}), & \text{if } n_{ij} > n_{ii}. \end{cases} \quad (11)$$

We now state our main result for the linear deterministic model.

*Theorem 1:* For a two-user linear deterministic IC,  $\mathcal{C}_{\text{NE}} = \mathcal{C} \cap \mathcal{B}$ .

First let us interpret the bounds  $L_1, L_2, U_1, U_2$ . The number  $L_i$  is the number of levels that user  $i$  can transmit above the interference floor created by user  $j$ , i.e., the number of levels at receiver  $i$  that cannot see any interference from user  $j$ . These are always the most significant bits of user  $i$ 's transmitted signal. In the example channel in Fig. 4, these correspond to the top level for transmitter 1 ( $L_1 = 1$ ) and the top 2 levels for transmitter 2 ( $L_2 = 2$ ). The number  $U_i$  is the number of levels at receiver

<sup>6</sup>The boundaries of the region in [10] are given in terms of conditional entropies that must be maximized over any product distribution on the channel inputs. For our model the optimizing input distribution for each bound is always uniform over the input alphabet. The given bounds follow.

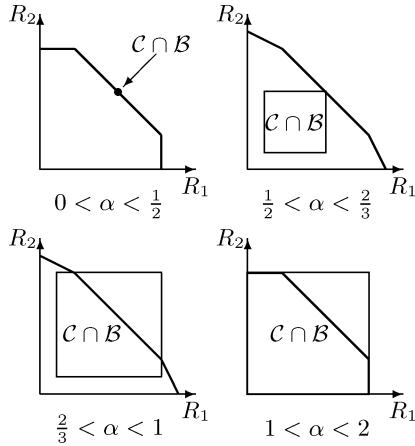


Fig. 6. Examples of  $\mathcal{C}_{\text{NE}} = \mathcal{C} \cap \mathcal{B}$  for a symmetric interference channel with normalized cross gain  $\alpha$ .

$i$  that receive signals from transmitter  $i$  but are free of interference from the top  $L_j$  levels from transmitter  $j$ . In the example channel in Fig. 4, at receiver 1 this is again given by the top level (i.e.,  $U_1 = L_1 = 1$ ), while at receiver 2, these correspond to the top two levels plus the bottom level so that  $U_2 = 3$ .

Intuitively, it is clear that at any  $\eta$ -NE, user  $i$  should have rate at least  $L_i$ : these levels are interference-free and user  $i$  can always send information at the maximum rate on these levels. This will create interference of maximum entropy at a certain subset of levels at receiver  $j$  and render them un-useable for user  $j$ . The rate for user  $j$  is bounded by the number of remaining levels that it can use. This is precisely the upper bound  $U_j$ . What Theorem 1 says is that any rate pair in the capacity region  $\mathcal{C}$  subject to these natural constraints is in  $\mathcal{C}_{\text{NE}}$ .

To illustrate this result, consider a symmetric interference channel in which  $n_{11} = n_{22}$  and  $n_{12} = n_{21}$ . Let  $\alpha = n_{ji}/n_{ii}$  be the normalized cross gain. Four examples of  $\mathcal{C}$  and  $\mathcal{B}$  corresponding to different ranges of  $\alpha$  are shown in Fig. 6. For  $0 < \alpha \leq \frac{1}{2}$ ,  $\mathcal{C}_{\text{NE}} = \mathcal{B}$  is a single point, which lies at the symmetric sum-rate point of  $\mathcal{C}$ . For  $\frac{1}{2} < \alpha \leq \frac{2}{3}$ , again  $\mathcal{C}_{\text{NE}} = \mathcal{B}$ .  $\mathcal{C}_{\text{NE}}$  contains a single efficient point (the symmetric sum-rate point in  $\mathcal{C}$ ), but now there are additional interior points of  $\mathcal{C}$ , which may be achieved as a Nash equilibrium.<sup>7</sup> For  $\frac{2}{3} < \alpha \leq 1$ ,  $\mathcal{C}_{\text{NE}}$  is the intersection of the simplex formed by the sum-rate constraint of  $\mathcal{C}$  and  $\mathcal{B}$ . In this case, there are multiple efficient points; in fact, the entire sum-rate face of  $\mathcal{C}$  is included in  $\mathcal{C}_{\text{NE}}$ . For  $1 \leq \alpha < 2$ ,  $\mathcal{C} \subset \mathcal{B}$  and so  $\mathcal{C}_{\text{NE}} = \mathcal{C}$ . For  $2 \leq \alpha$  (not shown)  $\mathcal{C} = \mathcal{B}$  and so again  $\mathcal{C}_{\text{NE}} = \mathcal{C}$ . Note that in all cases, the symmetric rate point is in  $\mathcal{C}_{\text{NE}}$ .

### C. Proofs

To prove Theorem 1, we first show that points outside of  $\mathcal{B}$  cannot be achievable as a Nash equilibrium, formalizing the intuition discussed earlier. We will then show that all points inside  $\mathcal{C} \cap \mathcal{B}$  can in fact be achieved.

<sup>7</sup>In a slight abuse of terminology, we say that points in  $\mathcal{C}_{\text{NE}}$  can be “achieved as a NE.”

#### 1) Nonequilibrium Points:

**Lemma 1:** If  $(R_1, R_2) \in \mathcal{C}_{\text{NE}}$ , then  $R_i \geq L_i = (n_{ii} - n_{ij})^+$  for  $i = 1, 2$ .

*Proof:* User  $i$ ’s  $L_i$  highest transmitted levels see no interference from user  $j$ ’s signal at receiver  $i$ . Hence, in any  $\epsilon$ -game, user  $i$  can always achieve a pay-off of  $R_i = L_i$  (with zero probability of error) by sending  $L_i$  uncoded bits, one on each of these levels, independent of user  $j$ ’s strategy. Thus, for  $(R_1, R_2)$  to be obtained as an  $\eta$ -NE of an  $\epsilon$ -game for any  $\epsilon > 0$ , it must be that  $R_i > L_i - \eta$  for all  $i$ ; otherwise, user  $i$  could deviate using the above strategy and improve his pay-off by  $\eta$ . If  $(R_1, R_2) \in \mathcal{C}_{\text{NE}}$ , then this inequality should hold for all  $\eta > 0$ . Taking the limit as  $\eta \rightarrow 0$ , the result follows. ■

**Lemma 2:** If  $(R_1, R_2) \in \mathcal{C}_{\text{NE}}$ , then for all  $i = 1, 2$ ,  $R_i \leq U_i$ , where  $U_i$  is given in (11).

*Proof:* Suppose  $(R_1, R_2) \in \mathcal{C}_{\text{NE}}$ . Without loss of generality, let us focus on user 1 to show that  $R_1 \leq U_1$ .

If  $n_{12} - L_2 \geq n_{11}$ , then  $U_1 = n_{11}$  and clearly  $R_1 \leq n_{11}$ , so there is nothing to prove. So in the following we can assume that

$$n_{12} - L_2 < n_{11}. \quad (12)$$

Fix an arbitrary  $\eta > 0$ . Given a sufficiently small  $\epsilon > 0$ , there exist a  $(1 - \epsilon)$ -reliable strategy pair  $(s_1^*, s_2^*)$  achieving the rate pair  $(R_1, R_2)$  that is also a  $\eta$ -NE. As remarked in Section II, we can assume that in this nominal strategy pair, both users use a common block length  $N$ . Applying Fano’s inequality to user 1 for the average bit error probability (see for example Theorem 4.3.2 in [11]), we get the bound, for any block  $k$

$$R_1 \leq \frac{I(m_1; \mathbf{y}_1 | \omega_1)}{N} + \delta \quad (13)$$

where  $\delta$  depends on  $\epsilon$  and goes to zero as  $\epsilon$  goes to zero. Here,  $m_1$  is the message of user 1 and  $\omega_1$  denotes any randomness in  $\mathbf{x}_1$ , which recall is known at receiver 1. Note that we drop the block indices of the message and the signals to simplify notation. Now

$$\begin{aligned} & \frac{1}{N} I(m_1; \mathbf{y}_1 | \omega_1) \\ & \leq \frac{1}{N} I(\mathbf{x}_1; \mathbf{y}_1 | \omega_1) \\ & \leq \frac{1}{N} I(\mathbf{x}_1; \mathbf{y}_1) \\ & = \frac{1}{N} [H(\mathbf{y}_1) - H(\mathbf{s}_2)] \\ & \leq \max(n_{11}, n_{12}) - \frac{H(\mathbf{s}_2)}{N} \end{aligned}$$

where  $\mathbf{s}_2$  is the signal from user 2 that is visible at receiver 1. Here, the second inequality follows since  $\omega_1 - \mathbf{x}_1 - \mathbf{y}_1$  forms a Markov chain. Combining this with the above inequality, we get:

$$R_1 \leq \max(n_{11}, n_{12}) - \frac{1}{N} H(\mathbf{s}_2) + \delta. \quad (14)$$

We now seek a bound on  $H(\mathbf{s}_2)$ . Applying Fano's inequality to user 2, we get

$$\begin{aligned} R_2 &\leq \frac{1}{N} I(m_2; \mathbf{y}_2 | \omega_2) + \delta \\ &\leq \frac{1}{N} I(\mathbf{x}_2; \mathbf{y}_2) + \delta \\ &= \frac{1}{N} I(\mathbf{u}_2, \mathbf{v}_2; \mathbf{y}_2) + \delta \end{aligned}$$

where  $\mathbf{u}_2$  is the part of user 2's transmitted signal  $\mathbf{x}_2$  which is on the top  $\min(L_2, n_{12})$  levels and  $\mathbf{v}_2$  is the rest. The significance of  $\mathbf{u}_2$  is that it is received without interference at receiver 2 and is visible at receiver 1 (i.e., part of  $\mathbf{s}_2$ ). Correspondingly, split user 2's received signal  $\mathbf{y}_2 = (\mathbf{u}_2, \tilde{\mathbf{y}}_2)$ . Now

$$\begin{aligned} I(\mathbf{u}_2, \mathbf{v}_2; \mathbf{y}_2) &= I(\mathbf{u}_2; \mathbf{y}_2) + I(\mathbf{v}_2; \mathbf{y}_2 | \mathbf{u}_2) \\ &= H(\mathbf{u}_2) + I(\mathbf{v}_2; \tilde{\mathbf{y}}_2 | \mathbf{u}_2) \\ &\leq H(\mathbf{u}_2) + I(\mathbf{v}_2; \tilde{\mathbf{y}}_2) \end{aligned}$$

since  $\mathbf{u}_2 - \mathbf{v}_2 - \tilde{\mathbf{y}}_2$  forms a Markov chain.

Combining this with the previous equation, we get

$$R_2 \leq \frac{1}{N} H(\mathbf{u}_2) + \frac{1}{N} I(\mathbf{v}_2; \tilde{\mathbf{y}}_2) + \delta. \quad (15)$$

Let us now consider an alternative strategy  $s'_2$  for user 2. This encoding strategy has two independent sub-codes. The first sub-code transmits uncoded bits on the top  $\min(L_2, n_{12})$  levels, achieving a rate of  $\min(L_2, n_{12})$  bits per symbol time with zero error. The second sub-code transmits on the remaining  $n_{22} - \min(L_2, n_{12})$  levels. It codes over  $K$  blocks of length  $N$  each. Each codeword in this code has  $K$  components, each spanning  $N$  symbol times, for a total length of  $KN$  symbol times. Each codeword is chosen randomly, with i.i.d.  $N$ -length components and each component chosen from the distribution of user 2's transmit signal  $\mathbf{v}_2$  under the original encoding strategy  $s_2^*$ . Note that since user 1's strategy  $s_1^*$  codes only within blocks of length  $N$  and sends independent message across different blocks, the interference from user 1 is i.i.d. across such blocks. User 2 thus faces a memoryless channel from block to block. Standard random coding arguments apply and one can show that for any  $\delta_1 > 0$ , there exists a large enough  $K$  such that strategy  $s'_2$  can achieve a rate of  $I(\mathbf{v}_2; \tilde{\mathbf{y}}_2) - \delta_1$  bits per block and with a probability of error of less than  $\epsilon$ . Thus, strategy  $s'_2$  achieves a total rate  $R'_2$  bits per symbol time reliably, where

$$R'_2 = \min(L_2, n_{12}) + \frac{1}{N} [I(\mathbf{v}_2; \tilde{\mathbf{y}}_2) - \delta_1]. \quad (16)$$

By definition of  $\eta$ -NE, strategy  $s'_2$  cannot perform much better than  $s_2^*$ , i.e.,  $R_2 + \eta \geq R'_2$ . Combining (16) and (15), we now have

$$\frac{1}{N} H(\mathbf{u}_2) \geq \min(L_2, n_{12}) - \delta_1/N - \delta - \eta. \quad (17)$$

Essentially, we have shown that user 2 under strategy  $s_2^*$  must be transmitting information at maximum entropy on these

$\min\{L_2, n_{12}\}$  levels by virtue of the fact that it forms a  $\eta$ -NE. Substituting this into (14) and observing that  $H(\mathbf{s}_2) \geq H(\mathbf{u}_2)$ , we get:

$$R_1 \leq \max(n_{11}, n_{12}) - \min(L_2, n_{12}) + \delta_1/N + 2\delta + \eta. \quad (18)$$

Under the condition (12), one can readily verify that  $U_1 = \max(n_{11}, n_{12}) - \min(L_2, n_{12})$ . Since  $\eta$ ,  $\delta$  and  $\delta_1$  can be chosen arbitrarily small, we have shown that  $R_1 \leq U_1$ . The proof is complete.  $\blacksquare$

**2) Achievable Nash Equilibria:** From Lemmas 1 and 2, it follows that  $\mathcal{C}_{\text{NE}} \subseteq \mathcal{C} \cap \mathcal{B}$ . In this section, we show that these two sets are in fact equal, proving Theorem 1. To do this we consider a modification of the class of Han–Kobayashi strategies presented in [2]. In these strategies, each user splits the transmitted information into two parts: private information to be decoded at only their own receiver and common information that can be decoded at both receivers. In [3] it is shown that a particular class of these strategies can achieve any point in the capacity region of the deterministic channel. The modification we make to these schemes is to allow each transmitter to include extra random bits in their common message. Next, we give some preliminary definitions related to Han–Kobayashi schemes and then formally define this class of strategies.

For a given linear deterministic interference channel, let  $\mathcal{X}_i$  denote the input alphabet of user  $i$ , i.e., this is the set of  $\max(n_{ii}, n_{ji})$  levels. We decompose this set as the direct product  $\mathcal{X}_{ip} \times \mathcal{X}_{ic}$ , so that for any  $\mathbf{x}_i \in \mathcal{X}_i$  can be written as  $\mathbf{x}_i = (\mathbf{x}_{ip}, \mathbf{x}_{ic})$ , where  $\mathbf{x}_{ip}$  denotes the  $(n_{ii} - n_{ji})^+$  least significant levels of  $\mathbf{x}_i$ , and  $\mathbf{x}_{ic}$  consists of the  $n_{ji}$  most significant levels. The significance of this decomposition is that the  $\mathbf{x}_{ip}$  consists of *private levels* for user  $i$  which are visible only at his receiver, while  $\mathbf{x}_{ic}$  consists of *common levels* that are visible at receiver  $j$ .

We define a *randomized Han–Kobayashi* scheme for a given block-length  $N$  to be a scheme in which each user  $i$  separates the message set  $\{1, \dots, 2^{B_i}\}$  into the direct product of a private message set  $\mathcal{M}_{ip}$  containing  $2^{NR_{ip}}$  messages and a common message set  $\mathcal{M}_{ic}$  containing  $2^{NR_{ic}}$  messages, where  $NR_{ip} + NR_{ic} = B_i$ . Additionally, each user  $i$  is allowed to have a *random common message* set  $\Omega_i$  consisting of  $2^{NR_{ir}}$  equally likely codewords; these can be thought of as  $NR_{ir}$  random bits that the transmitter generates using the common randomness, which is shared with the corresponding receiver. The message sets are then encoded using a superposition code as follows. First the transmitter encodes the common and common random message via a map  $f_{ic} : \mathcal{M}_{ic} \times \Omega_i \mapsto \mathcal{X}_{ic}^N$ , where the codebook is generated using an i.i.d. uniform distribution over the common levels. Next, the transmitter encodes the private message via a map  $f_{ip} : \mathcal{M}_{ip} \times \mathcal{M}_{ic} \times \Omega_i \mapsto \mathcal{X}_{ip}^N$ , where for each common codeword  $\mathbf{x}_{ic}$ , a different private codebook is generated using an i.i.d. uniform distribution over the private levels. Here, the common codeword  $\mathbf{x}_{ic}$  can be viewed as defining a cloud center and the private codeword can be viewed as defining the associated cloud points. Transmitter  $i$  then sends the superposition of these two codewords. In the special case

where  $R_{1r} = R_{2r} = 0$ , we refer to the resulting scheme as a *nonrandomized Han–Kobayashi* scheme.

We call a randomized Han–Kobayashi scheme  $(1 - \epsilon)$ -*reliable* if each user  $i$  can decode their own private and common messages with an average probability of bit error no greater than  $\epsilon$ .<sup>8</sup>

Next we specify an achievable rate region for this class of schemes. This characterization is in terms of *modified MAC regions* for each of the two receivers. Specifically, the modified MAC region at receiver  $i$ ,  $\mathcal{R}_i^m$ , is the set of rates  $(R_{ic}, R_{ir}, R_{ip}, R_{jc}, R_{jr}, R_{jc})$  that satisfy

$$\begin{aligned} R_{ic} + R_{ip} + R_{jc} + R_{jr} &\leq \max(n_{ii}, n_{ij}) \\ R_{ip} + R_{jc} + R_{jr} &\leq \max(n_{ij}, (n_{ii} - n_{ji})^+) \\ R_{ip} &\leq (n_{ii} - n_{ji})^+ \\ R_{ic} + R_{ip} &\leq n_{ii}. \end{aligned} \quad (19)$$

The modified MAC region is derived by considering the MAC channel at receiver  $i$  consisting of three transmitters: one corresponding to user  $i$ 's own common message, one corresponding to user  $i$ 's own private message, and one corresponding to the combination user  $j$ 's common and common random messages. Here, user  $i$ 's own common random message can be ignored since it is known at receiver  $i$  and so can be removed. Likewise, user  $j$ 's private message can be ignored since it is not received at receiver  $i$ . The capacity region of this three user MAC will have seven constraints including the first four given in (19). The modification to this is that we drop the remaining three constraints by following similar arguments as in [12]. First, recall that a constraint for a MAC region that involves the rates  $\{R_i : i \in \mathcal{M}\}$  for some subset of the users  $\mathcal{M}$  corresponds to a bound on an error event where the message for each user in  $\mathcal{M}$  is in error and all other messages are correct [13]. The missing bounds correspond to error events that we ignore for one of the following two reasons. First, due to the use of superposition coding we can ignore constraints which correspond to making an error in the common message but not the private message. Second, from the point-of-view of user  $i$ , we can ignore the constraint that corresponds to an error in only user  $j$ 's signal.

From the above discussion and following similar arguments as in [3], we then have the following characterization of the rate-tuples that be achieved with this class of randomized Han–Kobayashi schemes.

**Lemma 3:**  $\mathcal{R}_{\text{RHK}} = \mathcal{R}_1^m \cap \mathcal{R}_2^m$  is an achievable region for randomized Han–Kobayashi schemes.

For some rate-tuples in  $\mathcal{R}_{\text{RHK}}$ , receiver  $i$  may not be able to reliably decode user  $j$ 's common and common random messages. In particular, this must be true if  $R_{jc} + R_{jr} > n_{ij}$ . However, if  $R_{ic} + R_{ip} \geq L_i$  for  $i = 1, 2$ , then the next lemma shows that this will be possible.

**Lemma 4:** Any rate-tuple  $(R_{1c}, R_{1r}, R_{1p}, R_{2c}, R_{2r}, R_{2p})$  in the interior of  $\mathcal{R}_{\text{RHK}}$  with  $R_{ic} + R_{ip} \geq L_i$  for  $i = 1, 2$  can be achieved by a randomized Han–Kobayashi scheme in which

<sup>8</sup>This is slight strengthening of the previous definition of a reliable strategy, which only required the overall average bit error probability to be no greater than  $\epsilon$ . Hence, an agent's pay-off in a  $\epsilon$ -game under a  $(1 - \epsilon)$ -reliable Han–Kobayashi scheme is again their rate.

each user  $i$  decodes user  $j$ 's common and common random message (with arbitrarily small probability of error).

*Proof:* First note that if a rate tuple is in the interior of  $\mathcal{R}_{\text{RHK}}$  with  $R_{ic} + R_{ip} \geq L_i = (n_{ii} - n_{ij})^+$ , then from the first constraint in (19) for receiver  $i$ , it follows that

$$R_{jc} + R_{jr} < n_{ij}. \quad (20)$$

Now consider a randomized Han–Kobayashi scheme which achieves this rate tuple. After user  $i$  decodes his own private and common messages, he will have a clean view of user  $j$ 's common message. Moreover, from the above constraint, the rate of this message is less than the capacity of the channel from user  $j$  to receiver  $i$  and so there must exist a randomized Han–Kobayashi scheme in which user  $i$  can also decode user  $j$ 's common messages with arbitrary reliability.<sup>9</sup> ■

A given rate tuple  $\mathbf{R} = (R_{1c}, R_{1r}, R_{1p}, R_{2c}, R_{2r}, R_{2p})$  is defined to be *self-saturated* at receiver  $i$  if  $\mathbf{R} \in \mathcal{R}_i^m$  at receiver  $i$ , but any other choice of  $R_{ic}$ ,  $R_{ir}$  and  $R_{ip}$  with a larger value of  $R_{ic} + R_{ip}$  (keeping all other rates fixed) will result in a rate-tuple that is not in  $\mathcal{R}_i^m$ . Clearly, a self-saturated rate-tuple must lie on the boundary of the modified MAC region at receiver  $i$ . Additionally, the constraints in (19) that are tight at this point must involve both  $R_{ip}$  and  $R_{ic}$ . If a rate-pair is self-saturated and  $R_{ip} + R_{ic} \geq L_i$ , then it will be useful to think about this in the context of a second MAC region at receiver  $i$  in which there are only two users, one corresponding to user  $i$ 's entire message (at rate  $R_i = R_{ic} + R_{ip}$ ) and a second that again corresponds to the common and common random messages sent by user  $j$  (at rate  $R_{jc} + R_{jr}$ ). This MAC region is given by

$$\begin{aligned} R_i + R_{jc} + R_{jr} &\leq \max(n_{ii}, n_{ij}) \\ R_i &\leq n_{ii} \\ R_{jc} + R_{jr} &\leq n_{ij}. \end{aligned} \quad (21)$$

It can be seen that if a rate tuple is self-saturated for user  $i$  and satisfies  $R_{ic} + R_{ip} \geq L_i$ , then the rates  $R_i$  and  $R_{jc} + R_{jr}$  must be in (21). Furthermore, if  $R_i$  is increased by any amount this will no longer be true. We use this to show that if a user is self-saturated and his rate is greater than  $L_i$ , then he can not deviate and improve his pay-off. The key idea here is that if a user could improve, then he will violate one of the MAC constraints in (21). This cannot be possible since after deviating and decoding his own message, he should still be able to decode the other user's message. The next lemma formalizes this argument.

**Lemma 5:** If a rate tuple  $(R_{1c}, R_{1r}, R_{1p}, R_{2c}, R_{2r}, R_{2p})$  is self-saturated with  $R_{ic} + R_{ip} \geq L_i$  for both both receivers  $i$ , then  $(R_{1p} + R_{1c}, R_{2p} + R_{2c}) \in \mathcal{C}_{\text{NE}}$ .

*Proof:* Given a rate tuple  $\mathbf{R} = (R_{1c}, R_{1r}, R_{1p}, R_{2c}, R_{2r}, R_{2p})$  that is self-saturated at both receivers and with  $R_{ic} + R_{ip} \geq L_i$ , it follows from Lemmas 3 and 4 that for any  $\eta > 0$ , and any  $\epsilon > 0$ , there exists a randomized Han–Kobayashi scheme achieving rates  $(R_{1c} - \eta/6, R_{1r} - \eta/6, R_{1p} - \eta/6, R_{2c} - \eta/6, R_{2r} - \eta/6, R_{2p} - \eta/6)$

<sup>9</sup>The Han–Kobayashi scheme under consideration may suffice, however, if this scheme does not provide a low enough probability of error, then a new scheme that achieves the same rate-tuple with the desired probability of error can be found (perhaps by using a longer blocklength).

for which each receiver can decode both his own common and private messages as well as the other user's common and common random messages with probability of error less than  $\epsilon$ .

Next we argue that for  $\epsilon$  small enough such a pair of strategies must be an  $\eta$ -NE of an  $\epsilon$ -game. First note that under these nominal strategies each user  $i$  will receive a pay-off of  $R_{ic} + R_{ip} - \eta/3$ . Assume that these strategies are not an equilibrium, and without loss of generality suppose that user 1 can deviate and improve his performance by at least  $\eta$ . After deviating, the rates for the MAC region at user 1 in (21) are given by  $\tilde{\mathbf{R}} = (\tilde{R}_1, R_{2c} + R_{2r} - \eta/3)$ , where  $\tilde{R}_1 \geq R_1 + 2\eta/3$ . Since the rate tuple  $\mathbf{R}$  is self-saturated, it follows that after this deviation, the rates  $\tilde{\mathbf{R}}$  must violate either the first or second constraint in (21) for  $i = 1$  by at least  $\eta/3$ .

Suppose that user 1 deviates to a blocklength  $N_1$  strategy. Then using Fano's inequality (as in (13)) for the average bit error probability and following the usual converse for a MAC channel, it must be that

$$\tilde{R}_1 \leq \frac{I(\mathbf{x}_1; \mathbf{y}_1 | \mathbf{x}_{2c})}{N_1} + \delta \quad (22)$$

where  $\delta$  goes to zero as the average bit error probability  $\epsilon$  does. In particular, choosing  $\epsilon$  small enough so that  $\delta$  is less than  $\eta/3$ , then (22) implies that

$$\tilde{R}_1 < n_{11} + \eta/3.$$

Hence, the second constraint in (21) can not be violated.

Likewise, since in the nominal strategy for user 1, he was able to decode user  $j$ 's common and common random signals, it follows that

$$R_{2c} + R_{2p} \leq \frac{I(\mathbf{x}_{2c}; \mathbf{y}_1)}{N'_i} + \delta' \quad (23)$$

where  $N'_i$  denotes the block length used in the nominal strategy. Combining (22) and (23) and choosing  $\epsilon$  small enough so that  $\delta + \delta' < \eta/3$ , we have<sup>10</sup>

$$\tilde{R}_1 + R_{2c} + R_{2p} \leq \max(n_{ii}, n_{ij}) + \eta/3 \quad (24)$$

which shows that the first constraint in (21) can not be violated.

Therefore, such a deviation cannot exist and the nominal strategy must be an  $\eta$ -NE for small enough  $\epsilon$ . Taking the limit as  $\eta \rightarrow 0$ , it follows that the desired rates must lie in  $\mathcal{C}_{\text{NE}}$ . ■

To prove that  $\mathcal{C}_{\text{NE}} = \mathcal{C} \cap \mathcal{B}$ , we will show for that any rate-point  $(R_1, R_2) \in \mathcal{C} \cap \mathcal{B}$ , there exists a feasible rate tuple in  $\mathcal{R}_{\text{RHK}}$  with  $R_i = R_{ic} + R_{ip}$  for  $i = 1, 2$  and that is self-saturated at both receivers. The desired result then follows directly from Lemma 5. As a first step toward doing this, we define a class of nonrandomized Han–Kobayashi rates  $(R_{1c}, R_{1p}, R_{2c}, R_{2p})$  at which each transmitter is fully utilizing its “interference-free” levels, i.e., the  $L_i = (n_{ii} - n_{ij})^+$  most significant levels at transmitter  $i$ . Depending on the channel

<sup>10</sup>Here, as discussed in Section II, we need to replace (22) and (23) with the corresponding expressions over super-blocks whose length is the least common multiple of  $N_i$  and  $N'_i$  so that that both equations are over the same block-length.

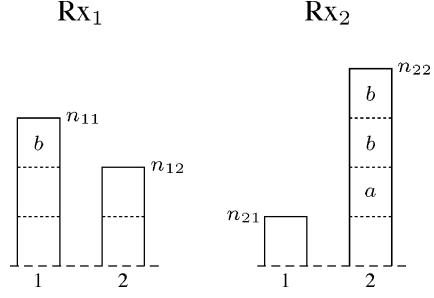


Fig. 7. This figure shows the interference free levels for each transmitter in the interference channel from Fig. 4. The levels for user  $i$  are indicated at that user's receiver by either a “a” or a “b”. The levels labeled with “a” correspond to “private levels,” which are accounted for in (25). The levels labeled with “b” correspond to common levels, which are accounted for in (26).

some of these levels may be common and some may be private. Specifically, there are

$$a_i = (n_{ii} - n_{ji} - n_{ij})^+$$

private interference free levels at user  $i$  and

$$b_i = (n_{ii} - \max(n_{ii} - n_{ji}, n_{ij}))^+$$

common interference free levels at user  $i$ . An example of these is shown in Fig. 7. We say that  $(R_{1c}, R_{1p}, R_{2c}, R_{2p})$  fully utilizes the interference free levels for user  $i$  if

$$R_{ip} \geq a_i \quad (25)$$

$$R_{ic} \geq b_i. \quad (26)$$

*Lemma 6:* If  $(R_{1c}, R_{1p}, R_{2c}, R_{2p})$  fully utilizes the interference free levels for each user  $i$ , then there exists random common rates  $R_{1r}, R_{2r} \geq 0$  such that  $(R_{1c}, R_{1r}, R_{1p}, R_{2c}, R_{2r}, R_{2p})$  is self-saturated at both receivers.

*Proof:* The intuition is as follows. If  $R_{ip} \geq a_i$  and  $R_{ic} \geq b_i$ , then all the interference-free levels are saturated at receiver  $i$  (i.e., at maximum entropy). The remaining  $n_{ij}$  levels are all reachable by the common signal from user  $j$ . By putting sufficient number of random bits on that common signal, these  $n_{ij}$  levels can be fully saturated as well.

More rigorously, we will show that one can always increase  $R_{2r}$  such that the overall sum rate constraint (the first constraint in (19)) is tight, so that receiver 1 is saturated. Suppose no such choice of  $R_{2r}$  exists. Then, it must be that  $R_{2r}$  cannot be further increased because the second constraint in (19) for receiver  $i$  is tight. However, if this constraint is tight, then since  $R_{1c} \geq b_i$ , it can be seen that the first constraint at receiver  $i$  must also be tight, which is the desired contradiction. ■

As an example of the construction used in the proof of Lemma 6 consider a symmetric channel with  $n_{11} = n_{22} = 3$  and  $n_{12} = n_{21} = 2$  (see Fig. 8). Each user has 1 interference free level with  $a_i = 0$  and  $b_i = 1$ . Thus, the nonrandomized Han–Kobayashi rates given by  $R_{1c} = R_{2c} = 1$  and  $R_{1p} = R_{2p} = 0$  fully utilize the interference free levels at each receiver. These rates

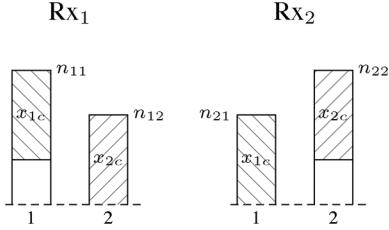


Fig. 8. Example of a nonrandomized Han-Kobayashi scheme for a symmetric channel with  $n_{ii} = 3$  and  $n_{ij} = 2$ . Here each user is using the rate split  $R_{ic} = 1$  and  $R_{ip} = 0$ , which fully utilizes the 1 interference free-level at each transmitter. This rate split is not self-saturating at either transmitter.

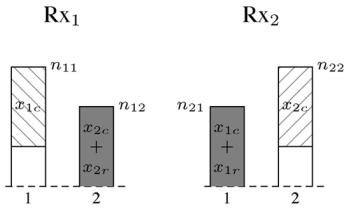


Fig. 9. Randomized Han-Kobayashi scheme which achieves the same rates as the nonrandomized scheme in Fig. 9 but is self-saturated. Here we do not show  $R_{ir}$  at receiver  $i$  since this signal can be removed using the assumed common randomness.

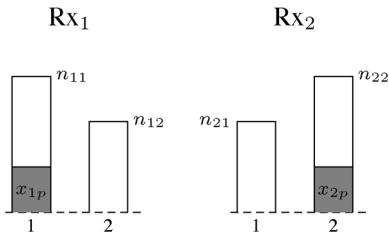


Fig. 10. Example of an alternative nonrandomized Han-Kobayashi rate-split that achieves the same rates as the scheme in Fig. 8 but does not fully utilize the interference-free levels.

are not self-saturating since each transmitter could increase  $R_{ip}$  by one. However, if each transmitter sets  $R_{ir} = 1$ , the resulting randomized rates will be self-saturated as shown in Fig. 9.

It follows from [3] that for any rate pair in  $\mathcal{C} \cap \mathcal{B}$ , there exists a nonrandomized Han-Kobayashi rate-split that lies in the achievable region in Lemma 3. If these Han-Kobayashi rates fully utilize the interference-free levels then we are done. Unfortunately, not all nonrandomized Han-Kobayashi rates fully utilize the interference-free levels. For example consider the symmetric channel in the previous paragraph. An alternative nonrandomized Han-Kobayashi rate-split is given by  $R_{1c} = R_{2c} = 0$  and  $R_{1p} = R_{2p} = 1$  (see Fig. 10). These rates do not fully utilize the interference-free levels and cannot be made into an equilibrium by simply increasing the users' common random rates. Though this set of rates does not fully utilize the interference-free levels, the previous example shows that there is another set of nonrandomized rates that do. The next lemma shows that this is always the case.

**Lemma 7:** Given any point  $(R_1, R_2) \in \mathcal{C} \cap \mathcal{B}$ , there exists a nonrandomized Han-Kobayashi rate split that fully utilizes the interference free levels at each transmitter.

*Proof:* To prove this lemma we begin with an arbitrary non-randomized Han-Kobayashi rate-split that satisfies Lemma 3 and show that this can always be transformed into one that fully utilizes the interference free levels at each transmitter. A key point here is that  $a_i + b_i = L_i$ , so that for any point in  $\mathcal{B}$  there will always be sufficient amount of "rate" available to meet the constraints in (25) and (26).

First we show that whenever  $R_{ip} < a_i$  for either user  $i$ , we can increase  $R_{ip}$  and decrease  $R_{ic}$  by the same amount until  $R_{ip} = a_i$ . The only way such a transformation could not be done is if the second constraint in (19) at receiver  $i$  prevented it. But by combining (20) at receiver  $j$  and the second constraint at receiver  $i$ , it can be seen that this will never happen for  $R_{ip} < a_i$ .

Thus, we can assume that  $R_{ip} \geq a_i$ . Given this, if a rate-pair does not fully utilize the interference free levels at transmitter  $i$ , it must be that  $R_{ic} < b_i$ . Suppose that this is true for receiver 1 and consider increasing  $R_{1c}$  and decreasing  $R_{1p}$  by the same amount until  $R_{1c} = b_1$ . Note that since  $a_1 + b_1 = L_1$  and  $R_{ic} + R_{ip} \geq L_1$ , when we decrease  $R_{1p}$  in this way it will never cause it to become less than  $a_1$ . Changing  $R_{1p}$  and  $R_{1c}$  in this manner will not violate any of the constraints in (19) at receiver 1, since every constraint involving  $R_{1c}$  also involves  $R_{1p}$ . If this can be done without violating the first or second constraints at receiver 2 then we are done. Otherwise, it must be that at least one of these constraints becomes tight when  $R_{1c}$  reaches the value  $R_{1c}^* = b_1 - \Delta$ , for some  $\Delta > 0$ . Note that

$$\begin{aligned} R_{2p} + R_{2c} + R_{1c}^* &= R_2 + R_{1c}^* \\ &\leq U_2 + b_1 - \Delta \\ &= \max(n_{22}, n_{21}) - \Delta \end{aligned} \quad (27)$$

and so the first constraint at receiver 2 can not be tight. This implies that the second constraint at receiver 2 must be tight, i.e.,

$$R_{2p} + R_{1c}^* = \max(n_{21}, n_{22} - n_{12}). \quad (28)$$

Combining this with (27), we have

$$\begin{aligned} R_{2c} &\leq \max(n_{22}, n_{21}) - \max(n_{21}, n_{22} - n_{12}) - \Delta \\ &= b_2 - \Delta. \end{aligned}$$

It then follows that  $R_{2p} \geq a_2 + \Delta$ . In other words, user 2's interference free levels must be "underutilized" by at least as much as user 1's.

Now consider increasing  $R_{1c}$  from  $R_{1c}^*$  by  $\Delta$  while simultaneously reducing  $R_{1p}$  and  $R_{2p}$  each by  $\Delta$  and also increasing  $R_{2c}$  by  $\Delta$ . The above calculations show that no constraint will be violated if we only changed  $R_{1c}$ ,  $R_{1p}$  and  $R_{2p}$  in this way. Likewise, by applying the same argument to  $R_{2c}$ ,  $R_{2p}$  and  $R_{1p}$ , changing these values will not violate any constraints. The only possible violation could occur in the first constraints at either MAC, which involve both  $R_{1c}$  and  $R_{2c}$ . However, this constraint also involves one of the users' private rates, and so cannot be violated by the same argument as in (27). After this transformation, the resulting rate-split will fully utilize the interference free levels at both receivers. ■

Combining Lemmas 5, 6, and 7, we have proven Theorem 1.

We also note that by direct calculation it can be shown that  $\mathcal{C}_{\text{NE}}$  always contains at least one efficient point, i.e., one point that is sum-rate optimal. Indeed it can be shown that for a symmetric channel, for  $\alpha \leq 2/3$ , the only efficient point in  $\mathcal{C}_{\text{NE}}$  is the symmetric sum-rate optimal point. For  $\alpha \geq 2/3$ , all sum-rate optimal points are in  $\mathcal{C}_{\text{NE}}$ .

#### IV. GAUSSIAN IC

In the previous section, we completely characterized the Nash equilibrium region for the two-user linear-deterministic IC. In this section, we show that an analogous result holds for the Gaussian IC within a one bit approximation.

##### A. Gaussian Channel Model

Here, our goal is to characterize rates in  $\mathcal{C}_{\text{NE}}$  for a two-user Gaussian IC represented by (see Fig. 1)

$$\begin{aligned} y_1 &= h_{11}x_1 + h_{12}x_2 + z_1 \\ y_2 &= h_{21}x_1 + h_{22}x_2 + z_2 \end{aligned} \quad (29)$$

where for  $i = 1, 2$ ,  $z_i \sim \mathcal{CN}(0, 1)$  and the input  $x_i \in \mathbb{C}$  is subject to the power constraint  $E[x_i^2] \leq P$ . Following [1], for  $i = 1, 2$ , we parameterize this channel by the signal-to-noise ratios  $\text{SNR}_i = P|h_{ii}|^2$  and the interference-to-noise ratios  $\text{INR}_{ij} = P|h_{ij}|^2$ .

The characterization of  $\mathcal{C}_{\text{NE}}$  in the linear deterministic case relied on knowing the exact capacity region  $\mathcal{C}$  for the deterministic IC and that any point in this region can be exactly achieved by a nonrandomized Han–Kobayashi scheme. For the Gaussian IC,  $\mathcal{C}$  is only known in the case of very weak [15]–[17] or very strong interference [2], [18]. Otherwise,  $\mathcal{C}$  is not known exactly but in [1] it is characterized to “within one bit” for all parameter ranges. This is done in part by showing that a class of (nonrandomized) Han–Kobayashi schemes can achieve any point with one bit of  $\mathcal{C}$ . This one-bit gap will effect how accurately we can characterize  $\mathcal{C}_{\text{NE}}$  in the Gaussian case. Namely, in general we will also be able to characterize this region only to within a one bit gap (though for particular channels this gap may be smaller).

##### B. Main Results

For the Gaussian IC our main result is to show an analogue to Theorem 1 that characterizes  $\mathcal{C}_{\text{NE}}$  to within one bit. In this case we will give both an inner bound and outer bound on  $\mathcal{C}_{\text{NE}}$ . Both of these bounds will be given in terms of a capacity region and a “box” as in the deterministic case. The true capacity region will be used for the outer bound, while an achievable “Han–Kobayashi” region,  $\mathcal{C}_{\text{HK}}$ , will be used for the inner bound. Here,  $\mathcal{C}_{\text{HK}}$  corresponds to the set of rates that are achievable using the specific class of Han–Kobayashi schemes in [1] (this will be defined more precisely in Section IV-C2). This region is within 1-bit of the capacity region  $\mathcal{C}$ , i.e., if  $(R_1, R_2) \in \mathcal{C}$ , then  $((R_1 - 1)^+, (R_2 - 1)^+) \in \mathcal{C}_{\text{HK}}$ .

The box  $\mathcal{B}$  used for the outer bound is given by

$$\mathcal{B} = \{(R_1, R_2) : L_i \leq R_i \leq U_i, \quad \forall i = 1, 2\}$$

where for each user  $i = 1, 2$

$$L_i := \log \left( 1 + \frac{\text{SNR}_i}{1 + \text{INR}_i} \right)$$

and

$$\begin{aligned} U_i &= \min \left\{ \log(1 + \text{SNR}_i + \text{INR}_{ij}) \right. \\ &\quad \left. - \log \left( 1 + \frac{[\text{SNR}_j - \max(\text{INR}_{ji}, \text{SNR}_j/\text{INR}_{ij})]^+}{1 + \text{INR}_{ji} + \max(\text{INR}_{ji}, \text{SNR}_j/\text{INR}_{ij})} \right) \right. \\ &\quad \left. \log(1 + \text{SNR}_i) \right\}. \end{aligned} \quad (30)$$

On the other hand, the inner bound is given in terms of the “box”

$$\mathcal{B}^- = \{(R_1, R_2) : L_i \leq R_i \leq \max(U_i - 1, L_i), \quad \forall i = 1, 2\}$$

which differs from  $\mathcal{B}$  by at most one bit.

We next state the analogous result to Theorem 1.

**Theorem 2:** For a two-user Gaussian IC,  $\mathcal{C}_{\text{HK}} \cap \mathcal{B}^- \subseteq \mathcal{C}_{\text{NE}} \subseteq \mathcal{C} \cap \mathcal{B}$ . Moreover, for a Gaussian IC with strong interference,  $\mathcal{C}_{\text{NE}} = \mathcal{C}_{\text{HK}} \cap \mathcal{B}$ .

In certain cases, when we know additional properties of the capacity region we can strengthen these results. For example for very weak interference, from the results in [15]–[17] it is known that the maximum sum-rate in  $\mathcal{C}$  is achieved by simply treating interference as noise, which is also in  $\mathcal{C}_{\text{HK}}$ . This corresponds exactly to the lower-left corner of  $\mathcal{B}$  and  $\mathcal{B}^-$ . Hence, Theorem 2 implies that  $\mathcal{C}_{\text{NE}}$  contains the single point  $(L_1, L_2)$ , and thus, in this case,  $\mathcal{C}_{\text{NE}}$  is characterized exactly.

We also note that the bounds in Theorem 2 can be shown to be within a constant gap of the bounds given in Theorem 1 for a related deterministic IC, which is obtained by the mapping  $n_{ii} = \lfloor \log(\text{SNR}_i) \rfloor$  and  $n_{ij} = \lfloor \log(\text{INR}_{ij}) \rfloor$ .

In the next section we will give a proof of Theorem 2 that is based on an analogy of each of the steps we used in the deterministic case.

##### C. Proofs

1) *Nonequilibrium Points:* We begin by showing that certain rate-pairs can not be in  $\mathcal{C}_{\text{NE}}$ .

**Lemma 8:** If  $(R_1, R_2) \in \mathcal{C}_{\text{NE}}$ , then  $R_i \geq L_i := \log(1 + \frac{\text{SNR}_i}{1 + \text{INR}_i})$  for  $i = 1, 2$ .

*Proof:* Regardless of user  $j$ ’s strategy, user  $i$  can always achieve at least rate  $\log(1 + \frac{\text{SNR}_i}{1 + \text{INR}_i})$  (with arbitrarily small probability of error) by treating user  $j$ ’s signal as Gaussian noise. Hence, this is always a possible deviation for user  $i$  in any  $\epsilon$ -game. Thus, user  $i$ ’s rate in any  $\eta$ -NE must be at least  $L_i - \eta$ . ■

The bound in Lemma 8 is a direct analog to the bound in Lemma 1 for the linear deterministic channel, which characterizes the lower bounds of the box  $\mathcal{B}$ . The next lemma gives an upper bound corresponding to the bound in Lemma 2.

**Lemma 9:** If  $(R_1, R_2) \in \mathcal{C}_{\text{NE}}$ , then  $R_i \leq U_i$ , where  $U_i$  is given in (30).

*Proof:* Suppose  $(R_1, R_2) \in \mathcal{C}_{\text{NE}}$ . Without loss of generality, let us focus on user 1 to show the upper bound on  $R_1$ .

We define a parameter

$$\sigma_v^2 := \max \left( \frac{\text{INR}_{21}}{\text{SNR}_2}, \frac{1}{\text{INR}_{12}} \right). \quad (31)$$

Consider first the case that  $\sigma_v^2 > 1$ : this corresponds to the case in the deterministic channel when the interference from the signal user 2 transmits at its interference-free levels appears below noise level at receiver 1. In this case, there will be no minimum amount of interference that user 2 will cause to user 1 at a NE, and we simply bound  $R_1$  by its point-to-point capacity

$$R_1 \leq \log(1 + \text{SNR}_1). \quad (32)$$

The case when  $\sigma_v \leq 1$  is more interesting and we need a tighter bound on  $R_1$ . Fix  $\eta > 0$  and arbitrary. Given a sufficiently small  $\epsilon > 0$ , there exist a  $(1 - \epsilon)$ -reliable strategy pair  $(s_1^*, s_2^*)$  achieving the rate pair  $(R_1, R_2)$  that is also an  $\eta$ -NE. As remarked in Section II, we can assume that in this nominal strategy pair, both users use a common block length  $N$ . Applying Fano's inequality to user 1 for average bit error probability, we get the bound, for any block  $k$

$$R_1 \leq \frac{I(m_1; \mathbf{y}_1 | \omega_1)}{N} + \delta \quad (33)$$

where  $m_1$  is the message of user 1,  $\mathbf{y}_1$  is user 1's received signal over the block,  $\delta$  depends on  $\epsilon$  and goes to zero as  $\epsilon$  goes to zero, and  $\omega_1$  denotes any common randomness in  $\mathbf{x}_1$ . Note that we drop the block indices of the message and the signals to simplify notation. Now

$$\begin{aligned} & \frac{1}{N} I(m_1; \mathbf{y}_1 | \omega_1) \\ & \leq \frac{1}{N} I(\mathbf{x}_1; \mathbf{y}_1 | \omega_1) \\ & \leq \frac{1}{N} I(\mathbf{x}_1; \mathbf{y}_1) \\ & = \frac{1}{N} [h(\mathbf{y}_1) - h(\mathbf{y}_1 | \mathbf{x}_1)] \\ & = \frac{1}{N} [h(\mathbf{y}_1) - h(\mathbf{z}_1) - h(\mathbf{y}_1 | \mathbf{x}_1) + h(\mathbf{z}_1)] \\ & \leq \log(1 + \text{SNR}_1 + \text{INR}_{12}) - \frac{I(\mathbf{x}_2; \tilde{\mathbf{y}}_1)}{N} \end{aligned}$$

where

$$\tilde{\mathbf{y}}_1 := h_{12} \mathbf{x}_2 + \mathbf{z}_1.$$

Combining this with the above inequality, we get

$$R_1 \leq \log(1 + \text{SNR}_1 + \text{INR}_{12}) - \frac{I(\mathbf{x}_2; \tilde{\mathbf{y}}_1)}{N}. \quad (34)$$

The term  $I(\mathbf{x}_2; \tilde{\mathbf{y}}_1)$  plays the role of  $H(\mathbf{s}_2)$  in the linear deterministic case. We now seek a lower bound on  $I(\mathbf{x}_2; \tilde{\mathbf{y}}_1)$ . Applying Fano's inequality to user 2, we get

$$\begin{aligned} R_2 & \leq \frac{1}{N} I(m_2; \mathbf{y}_2 | \omega_2) + \delta \\ & \leq \frac{1}{N} I(\mathbf{x}_2; \mathbf{y}_2) + \delta \\ & = \frac{1}{N} I(\mathbf{u}_2, \mathbf{v}_2; \mathbf{y}_2) + \delta \end{aligned}$$

where

$$\mathbf{u}_2 = \mathbf{x}_2 + \mathbf{v}_2$$

and  $\mathbf{v}_2 \sim \mathcal{CN}(0, \sigma_v^2 I_N)$  independent of everything else, with  $\sigma_v^2$  defined as in (31).

Now,

$$\begin{aligned} I(\mathbf{u}_2, \mathbf{v}_2; \mathbf{y}_2) & = I(\mathbf{u}_2; \mathbf{y}_2) + I(\mathbf{v}_2; \mathbf{y}_2 | \mathbf{u}_2) \\ & = I(\mathbf{u}_2; \mathbf{y}_2) + I(\mathbf{v}_2; \tilde{\mathbf{y}}_2 | \mathbf{u}_2) \\ & \leq I(\mathbf{u}_2; \mathbf{y}_2) + I(\mathbf{v}_2; \tilde{\mathbf{y}}_2) \end{aligned}$$

where

$$\tilde{\mathbf{y}}_2 = \mathbf{y}_2 - h_{22} \mathbf{u}_2 = h_{22} \mathbf{v}_2 + h_{21} \mathbf{x}_1 + \mathbf{z}_2$$

and the last inequality above follows from the Markov chain  $\mathbf{u}_2 - \mathbf{v}_2 - \tilde{\mathbf{y}}_2$ .

Combining this with the previous equation, we get

$$R_2 \leq \frac{1}{N} I(\mathbf{u}_2; \mathbf{y}_2) + \frac{1}{N} I(\mathbf{v}_2; \tilde{\mathbf{y}}_2) + \delta. \quad (35)$$

Let us now consider an alternative strategy  $s_2'$  for user 2: a superposition of two i.i.d. Gaussian codebooks, one with each component of each codeword having variance  $1 - \sigma_v^2$ , and one with each component of each codeword having variance  $\sigma_v^2$ . The codes have block length  $NK$ . If we let  $K \rightarrow \infty$ , then standard random coding arguments and the chain rule of mutual information implies that this scheme can achieve a rate of

$$\frac{1}{N} I(\tilde{\mathbf{u}}_2, \mathbf{v}_2; \mathbf{y}'_2) = \frac{1}{N} I(\tilde{\mathbf{u}}_2; \mathbf{y}'_2) + \frac{1}{N} I(\mathbf{v}_2; \tilde{\mathbf{y}}_2) \quad (36)$$

where  $\tilde{\mathbf{u}}_2 \sim \mathcal{CN}(0, 1 - \sigma_v^2 I_N)$  and  $\tilde{\mathbf{u}}_2$  and  $\mathbf{v}_2$  are independent, and

$$\mathbf{y}'_2 = h_{22}(\tilde{\mathbf{u}}_2 + \mathbf{v}_2) + h_{21} \mathbf{x}_1 + \mathbf{z}_2.$$

We have

$$\frac{1}{N} I(\tilde{\mathbf{u}}_2; \mathbf{y}'_2) \geq \log \left( 1 + \frac{\text{SNR}_2 (1 - \sigma_v^2)}{1 + \text{SNR}_2 \sigma_v^2 + \text{INR}_{21}} \right)$$

using a worst-case Gaussian noise argument [20] on  $\mathbf{v}_2$  and  $\mathbf{x}_1$ . Substituting this into (36) gives a lower bound on the rate this alternative strategy can obtain. Comparing this with (35) and using the assumption that we are operating at an *eta*-NE implies that

$$\frac{1}{N} I(\mathbf{u}_2; \mathbf{y}_2) \geq \log \left( 1 + \frac{\text{SNR}_2 (1 - \sigma_v^2)}{1 + \text{SNR}_2 \sigma_v^2 + \text{INR}_{21}} \right) - \eta \quad (37)$$

Next we relate  $I(\mathbf{x}_2; \tilde{\mathbf{y}}_1)$  to  $I(\mathbf{u}_2; \mathbf{y}_2)$  and complete the argument. First note that

$$\begin{aligned} & I(\mathbf{x}_2; \tilde{\mathbf{y}}_1) \\ & = I(\mathbf{x}_2; h_{12} \mathbf{x}_2 + \mathbf{z}_1) \\ & = I(\mathbf{x}_2; \mathbf{x}_2 + \tilde{\mathbf{z}}_1), \quad \tilde{\mathbf{z}}_1 \sim \mathcal{CN} \left( 0, \frac{1}{\sqrt{\text{INR}_{12}}} I_N \right) \\ & \stackrel{(a)}{\geq} I(\mathbf{x}_2; \mathbf{x}_2 + \mathbf{v}_2) \\ & = I(\mathbf{u}_2, \mathbf{x}_2) \\ & \geq I(\mathbf{u}_2; \mathbf{y}_2) \\ & \stackrel{(b)}{\geq} N \left[ \log \left( 1 + \frac{\text{SNR}_2 (1 - \sigma_v^2)}{1 + \text{SNR}_2 \sigma_v^2 + \text{INR}_{21}} \right) - \eta \right] \end{aligned}$$

where (a) follows since  $\sigma_v^2 = \max(\frac{\text{INR}_{21}}{\text{SNR}_2}, \frac{1}{\text{INR}_{12}}) \geq \frac{1}{\text{INR}_{12}}$  and (b) follows from (37).

Substituting this into (34), we get the final result

$$\begin{aligned} R_1 &\leq \log(1 + \text{SNR}_1 + \text{INR}_{12}) \\ &\quad - \log \left( 1 + \frac{\text{SNR}_2(1 - \sigma_v^2)}{1 + \text{SNR}_2\sigma_v^2 + \text{INR}_{21}} \right) + \eta \\ &= \log(1 + \text{SNR}_1 + \text{INR}_{12}) \\ &\quad - \log \left( 1 + \frac{\text{SNR}_2 - \max(\text{INR}_{21}, \text{SNR}_2/\text{INR}_{12})}{1 + \text{INR}_{21} + \max(\text{INR}_{21}, \text{SNR}_2/\text{INR}_{12})} \right) \\ &\quad + \eta. \end{aligned}$$

Combining this with inequality (32) and letting  $\eta \rightarrow 0$  yields the desired result.  $\blacksquare$

2) *Achievable Nash Equilibrium:* The lemmas in the previous section provide an outer bound on  $\mathcal{C}_{\text{NE}}$ . In this section we give an inner bound on  $\mathcal{C}_{\text{NE}}$  by showing that this set contains  $\mathcal{C}_{\text{HK}} \cap \mathcal{B}^-$ . Motivated by the deterministic analysis, we again consider a modified Han–Kobayashi scheme in which each user may send a private message, a common message, and also a common random message that is generated using the common randomness they share with their own receiver. Additionally, in the Gaussian case, we allow a user to also send a private random message using their common randomness. In the deterministic case, sending such a message would not serve any purpose since a user’s private signal does not appear at all at the other receiver. In the Gaussian case, a user’s private signal is present at the other user and the private random message is used to ensure that the effect of this signal is essentially the same as “noise.” All of these messages are again encoded using a superposition code, which we define formally next.

For a given Gaussian IC, let  $P_{ip}$  and  $P_{ic}$  denote a user’s private and common power respectively, where  $P_{ip} + P_{ic} = P$ . As in [1], we assume that  $P_{ip}$  is set as follows:

$$|h_{ji}|^2 P_{ip} = \begin{cases} \min(1, \text{INR}_{ji}), & \text{if } \text{INR}_{ji} < \text{SNR}_j \\ 0, & \text{otherwise.} \end{cases}$$

Let  $\text{INR}_{ji}^p = |h_{ji}|^2 P_{ip}$  denote the INR at receiver  $j$  due to this choice of  $P_{ip}$  and let  $\text{SNR}_i^p = |h_{ii}|^2 P_{ip}$  denote the corresponding SNR at receiver  $i$ . Note that when  $\text{SNR}_j > \text{INR}_{ji} > 1$ , the received interference power at receiver  $j$  due to user  $i$ ’s private power is at the noise level.

As in the deterministic case, we define a randomized Han–Kobayashi scheme to be one in which each user  $i$  separates their message set into  $\mathcal{M}_{ip} \times \mathcal{M}_{ic}$  with rates  $R_{ip}$  and  $R_{ic}$ , respectively, and also generates a random common message set  $\Omega_{ir}$  with rate  $R_{ir}$ . Additionally, we allow each transmitter to generate a random private message set,  $\Omega_{is}$  with rate  $R_{is}$ . These messages are then encoded using a superposition code as follows. First, the transmitter encodes the common message  $m_{ic} \in \mathcal{M}_{ic}$  and common random message  $\omega_{ir} \in \Omega_{ir}$  into a codeword  $\mathbf{x}_{ic}(m_{ic}, \omega_{ir})$  from a codebook that satisfies the average power constraint of  $P_{ic}$ . Given this codeword, it encodes the private message  $m_{ip}$  and private random message  $\omega_{is} \in \Omega_{is}$  into a codeword  $\mathbf{x}_{ip}(m_{ip}, \omega_{is}, \mathbf{x}_{ic})$  from a codebook that is indexed by the common codeword  $\mathbf{x}_{ic}$  and satisfies the average power constraint of  $P_{ip}$ . It then transmits the superposition  $\mathbf{x}_i = \mathbf{x}_{ic} + \mathbf{x}_{ip}$ . As in the deterministic case, we call such a

scheme  $(1 - \epsilon)$ -reliable if a user is able to decode both his common and private messages with reliability of  $(1 - \epsilon)$ .

We again introduce a modified MAC region  $\mathcal{R}_i^m$  for each receiver  $i$  that we will use to characterize the rates achievable by such a scheme. This is the set of rate tuples that satisfy

$$\begin{aligned} R_{ic} + R_{ip} + R_{jc} + R_{jr} &\leq \log \left( 1 + \frac{\text{SNR}_i + \text{INR}_{ij} - \text{INR}_{ij}^p}{\text{INR}_{ij}^p + 1} \right) \\ R_{ip} + R_{jc} + R_{jr} &\leq \log \left( 1 + \frac{\text{SNR}_i^p + \text{INR}_{ij} - \text{INR}_{ij}^p}{\text{INR}_{ij}^p + 1} \right) \\ R_{ip} &\leq \log \left( 1 + \frac{\text{SNR}_i^p}{\text{INR}_{ij}^p + 1} \right) \\ R_{ic} + R_{ip} &\leq \log \left( 1 + \frac{\text{SNR}_i}{\text{INR}_{ij}^p + 1} \right). \end{aligned} \quad (38)$$

As in the deterministic case, these constraints arise from considering the three user MAC region at receiver  $i$  corresponding to the rates  $R_{ic}$ ,  $R_{ip}$  and  $R_{jc} + R_{jr}$ . Using these regions we then have the following characterization of rate-splits that can be achieved with this class of schemes.

*Lemma 10:*  $\mathcal{R}_{\text{RHK}} = \mathcal{R}_1^m \cap \mathcal{R}_2^m$  is an achievable region for randomized Han–Kobayashi schemes.

We define the Han–Kobayashi region  $\mathcal{C}_{\text{HK}}$  in Theorem 2 as the set of rates  $(R_1, R_2)$  for which there exists a rate split  $(R_{1c}, R_{1r}, R_{1p}, R_{1s}, R_{2c}, R_{2r}, R_{2p}, R_{1s}) \in \mathcal{R}_{\text{RHK}}$  with  $R_1 = R_{1c} + R_{1p}$  and  $R_2 = R_{2c} + R_{2p}$ . Using [1], it follows that  $\mathcal{C}_{\text{HK}}$  is within one-bit of  $\mathcal{C}$ .<sup>11</sup> Next, we give an analog to Lemma 4.

*Lemma 11:* Any rate-tuple  $(R_{1c}, R_{1r}, R_{1p}, R_{1s}, R_{2c}, R_{2r}, R_{2p}, R_{2s})$  in the interior of  $\mathcal{R}_{\text{RHK}}$  with  $R_{ic} + R_{ip} \geq L_i$  for  $i = 1, 2$  can be achieved by a randomized Han–Kobayashi scheme in which each user  $i$  decodes user  $j$ ’s common and common random message (with arbitrarily small probability of error).

The proof here follows exactly the same steps as in the deterministic case. In particular for  $R_{ic} + R_{ip} \geq L_i$ , note that

$$R_{jc} + R_{jr} < \log \left( 1 + \frac{\text{INR}_{ij} - \text{INR}_{ij}^p}{1 + \text{INR}_{ij}^p} \right) \quad (39)$$

which implies that if user  $i$  can decode his own common and private messages, he will be receiving user  $j$ ’s common messages at a rate less than the capacity of the channel over which these messages are sent.

Note also that the private random rate  $R_{is}$  does not show up in any of the constraints for  $\mathcal{R}_{\text{RHK}}$ . This is because in these constraints each user  $i$  is treating the other user’s private message as worst-case noise with power  $\text{INR}_{ij}^p$  and can remove their own random private message. The next lemma shows that the random

<sup>11</sup>The rate region studied in [1] corresponds to the rates  $R_1$  and  $R_2$  that can be achieved with rate-splits in  $\mathcal{R}_{\text{RHK}}$  where the common random and private random rates of both users are zero. It can be seen that the resulting region is equivalent to  $\mathcal{C}_{\text{HK}}$  as defined here.

private rate can always be chosen so that there is essentially no loss in this assumption.

*Lemma 12:* Given any  $\delta > 0$  and any rate tuple in  $\mathcal{R}_{\text{RHK}}$  with

$$R_{js} = (\log(1 + \text{INR}_{ij}^p) - R_{jp} - \delta/2)^+$$

then for a large enough block-length  $N$ , this rate tuple can be achieved by a randomized Han–Kobayashi scheme such that

$$\frac{1}{N}h(h_{ij}\mathbf{x}_j^p + \mathbf{z}_1) \geq \log(\pi e(1 + \text{INR}_{ij}^p)) - \delta.$$

*Proof:* Given a rate tuple that satisfies the conditions of this lemma and a constant  $\delta > 0$ , we next describe a specific encoding of user  $j$ 's private messages to ensure the conditions for this lemma hold true. Let  $w_j = (m_{jp}, \omega_{js})$  denote the total private message to be encoded by user  $j$  (chosen from a total private codebook with rate  $R_{jp} + R_{js}$ ). To encode this message<sup>12</sup> we consider the following two cases: (i)  $R_{jp} \leq \log(1 + \text{INR}_{ij}^p) - \delta/2$  and (ii)  $R_{jp} > \log(1 + \text{INR}_{ij}^p) - \delta/2$ . In each case we will separate  $w_j$  into two messages so that  $w_j = (w_j^1, w_j^2)$ .

*Case 1:*  $R_{jp} \leq \log(1 + \text{INR}_{ij}^p) - \delta/2$ . In this case we set  $w_j^1 = m_{jp}$ , i.e., this is the private message which is to be decoded at receiver  $j$ . Since the rate-tuple is in  $\mathcal{R}_{\text{RHK}}$ , this message must be decodable over a Gaussian channel with a capacity of  $\log(1 + \text{SNR}_j^p)$ . We then set  $w_j^2 = \omega_{js}$ , i.e., this is the private random message sent by user  $j$ . By assumption this message will have a rate of  $R_{js} = \log(1 + \text{INR}_{ij}^p) - R_{jp} - \delta/2$ .

By choosing  $N$  large enough, there will exist a Gaussian broadcast codebook for these messages so that for a given reliability,  $w_j^1$  and  $w_j^2$  can be received reliably over the Gaussian channel given by

$$y = h_{ij}x_j^p + z_i$$

where  $x_j^p$  has average power  $P_{jp}$  and the noise variance is 1, and  $w_j^1$  can be received at user  $j$ 's receiver given  $w_j^2$  (equivalently given the private random message  $\omega_{js}$ ). Note that the first channel has a capacity of  $\log(1 + \text{INR}_{ij}^p)$ .

By applying Fano's inequality to the first receiver, we have that for a large enough reliability we can find a block-length  $N$  so that

$$N(\log(1 + \text{INR}_{ij}^p) - \delta/2) \leq I(w_j; \mathbf{y}) + N\delta/2$$

where  $\mathbf{y} = h_{ij}\mathbf{x}_j^p + \mathbf{z}_1$  denotes the received signal over a block of length  $N$ .

Now

$$I(w_j; \mathbf{y}) = h(\mathbf{y}) - N \log(\pi e).$$

Hence, we have

$$h(\mathbf{y}) \geq N(\log(\pi e(1 + \text{INR}_{ij}^p)) - \delta)$$

as desired.

<sup>12</sup>To keep the overall superposition code structure for our class of Han–Kobayashi schemes, we need to construct such a private codebook for each codeword  $\mathbf{x}_{jc}$  in user  $j$ 's common codebook. Here we focus on one such codebook.

*Case 2:*  $R_{jp} > \log(1 + \text{INR}_{ij}^p) - \delta/2$ . In this case we set  $R_{js} = 0$  and choose  $w_j^1$  and  $w_j^2$  so that  $m_{jp} = (w_j^1, w_j^2)$  where  $w_j^1$  is chosen from a code book with rate  $\log(1 + \text{INR}_{ij}^p) - \delta/2$  and  $w_j^2$  is chosen from a codebook with rate  $R_{jp} - \log(1 + \text{INR}_{ij}^p) + \delta/2$ .

By choosing  $N$  large enough, there will exist a Gaussian broadcast codebook for these messages so that for a given reliability,  $w_j^1$  and  $w_j^2$  can be received at user  $j$ 's receiver, and given  $w_j^2$ ,  $w_j^1$  can be received reliably over the Gaussian channel

$$y = h_{ij}x_j^p + z_i.$$

Applying Fano's inequality at the second receiver we have

$$NR_{jp} \leq I(w_j; \mathbf{y}, w_j^2) + N\delta/2.$$

Now

$$\begin{aligned} I(w_j; \mathbf{y}, w_j^2) &= I(w; w_j^2) + I(w; \mathbf{y} | w_j^2) \\ &= N(R_{jp} - \log(1 + \text{INR}_{ij}^p) + \delta/2) \\ &\quad + h(\mathbf{y} | w_j^2) - N \log(\pi e). \end{aligned}$$

Hence, we have

$$h(\mathbf{y} | w_j^2) \geq N(\log(\pi e(1 + \text{INR}_{ij}^p)) - \delta).$$

Dropping the conditioning and dividing by  $N$ , it follows that

$$\frac{1}{N}h(\mathbf{y}) \geq \log(\pi e(1 + \text{INR}_{ij}^p)) - \delta$$

as desired.  $\blacksquare$

Note that under the given power constraints,  $N \log(\pi e(1 + \text{INR}_{ij}^p))$  is the maximum possible value for  $h(\mathbf{y})$ , which is achieved when  $\mathbf{y}$  is a sequence of i.i.d. Gaussian random variables. Hence, this lemma can be viewed as showing that when the private rates are sufficiently high,  $h(\mathbf{y})$  is well approximated by simply viewing  $\mathbf{y}$  as i.i.d. Gaussian.

We say that a rate-split  $\mathbf{R}$  is *self saturated* at receiver  $i$  if  $\mathbf{R} \in \mathcal{R}_i^m$  and any other choice of  $R_{ic}$  and  $R_{ip}$  in which  $R_{ip} + R_{ic}$  is increased (keeping all other rates fixed) will result in a rate-split that is not in  $\mathcal{R}_i^m$ . Similar to the deterministic case, it can be shown that if receiver  $i$  is self-saturated and  $R_{ic} + R_{ip} \geq L_i$  then  $R_i = R_{ip} + R_{ic}$  and  $R_{jc} + R_{jr}$  must be inside the following two user MAC region:

$$\begin{aligned} R_i + R_{jc} + R_{jr} &\leq \log\left(1 + \frac{\text{SNR}_i + \text{INR}_{ij} - \text{INR}_{ij}^p}{\text{INR}_{ij}^p + 1}\right) \\ R_i &\leq \log\left(1 + \frac{\text{SNR}_i}{\text{INR}_{ij}^p + 1}\right) \\ R_{jc} + R_{jr} &\leq \log\left(1 + \frac{\text{INR}_{ij} - \text{INR}_{ij}^p}{1 + \text{INR}_{ij}^p}\right). \end{aligned} \quad (40)$$

Moreover, if  $R_i$  is increased then this rate pair will no longer be in this region. Using this we have the next lemma which gives an analogous result to Lemma 5 for the deterministic channel.

*Lemma 13:* If there exists a rate tuple  $\mathbf{R}$  that is self-saturated with  $R_{ic} + R_{ip} \geq L_i$  for both both receivers  $i$ , then  $(R_{1p} + R_{1c}, R_{2p} + R_{2c}) \in \mathcal{C}_{\text{NE}}$ .

*Proof:* The proof follows a similar argument as that for Lemma 5. Given a rate-tuple  $\mathbf{R} = (R_{1c}, R_{1r}, R_{1p}, R_{1s}, R_{2c}, R_{2r}, R_{2p}, R_{2s})$  that satisfies the conditions in the lemma, then it follows from Lemmas 10, 11, and 12 that for any  $\eta > 0$  and  $\epsilon > 0$ , there exists a randomized Han–Kobayashi scheme achieving rates  $(R_{1c} - \eta/6, R_{1r} - \eta/6, R_{1p} - \eta/6, \tilde{R}_{1s}, R_{2c} - \eta/6, R_{2r} - \eta/6, R_{2p} - \eta/6, \tilde{R}_{2s})$  for which each receiver can decode his own common and private messages as well as the other user's common and common random messages with probability of error less than  $\epsilon$ . Moreover, by possibly changing the private random rates  $\tilde{R}_{1s}$  and  $\tilde{R}_{2s}$  to satisfy Lemma 12 such a scheme can be found for which

$$\frac{1}{N}h(h_{ij}\mathbf{x}_j^p + \mathbf{z}_1) \geq \log(\pi e(1 + \text{INR}_{ij}^p)) - \eta/6 \quad (41)$$

for each user  $i$ .

Next we argue that for  $\epsilon$  small enough such a pair of strategies must be an  $\eta$ -NE of an  $\epsilon$ -game. Assume that these strategies are not an  $\eta$ -NE, and without loss of generality suppose that user 1 can deviate and improve his performance by at least  $\eta$ . After deviating, the rates for the MAC region at user 1 in (40) are given by  $\tilde{\mathbf{R}} = (\tilde{R}_1, R_{2c} + R_{2r} - \eta/3)$ , where  $\tilde{R}_1 \geq R_1 + 2\eta/3$ . After this deviation, the rates  $\tilde{\mathbf{R}}$  must violate either the first or second constraint in (40) for  $i = 1$  by at least  $\eta/3$ .

Suppose that user 1 deviates to a blocklength  $N$  strategy, which without loss of generality we can assume is the same as the original strategy. Then from Fano's inequality for the average bit error probability it must be that

$$\tilde{R}_1 \leq \frac{I(\mathbf{x}_1; \mathbf{y}_1 | \mathbf{x}_{2c})}{N} + \delta \quad (42)$$

where  $\delta$  goes to zero as the average bit error probability  $\epsilon$  does. Note that

$$\begin{aligned} \frac{I(\mathbf{x}_1; \mathbf{y}_1 | \mathbf{x}_{2c})}{N} &= \frac{1}{N}(h(\mathbf{y}_1 | \mathbf{x}_{2c}) - h(\mathbf{y}_1 | \mathbf{x}_1, \mathbf{x}_{2c})) \\ &\leq \log(\pi e(1 + \text{SNR}_1 + \text{INR}_{12}^p)) \\ &\quad - \log(\pi e(1 + \text{INR}_{12}^p)) + \eta/6 \\ &= \log\left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_{12}^p}\right) + \eta/6, \end{aligned}$$

where the second line follows from (41). Choosing  $\epsilon$  small enough and combining this with (42) implies that

$$\tilde{R}_1 < \log\left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_{12}^p}\right) + \eta/3.$$

Hence, the second constraint in (21) can not be violated.

Likewise, since in the nominal strategy user 1 was able to decode user  $j$ 's common and common random signals, it follows that

$$R_{2c} + R_{2p} \leq \frac{I(\mathbf{x}_{2c}; \mathbf{y}_1)}{N} + \delta'. \quad (43)$$

Combining (42) and (43) and choosing  $\epsilon$  small enough so that  $\delta + \delta' < \eta/6$ , we have

$$\begin{aligned} &\tilde{R}_1 + R_{2c} + R_{2p} \\ &\leq \frac{1}{N}I(\mathbf{x}_{2c}, \mathbf{x}_1; \mathbf{y}_1) + \eta/6 \\ &\leq \log(\pi e(1 + \text{SNR}_1 + \text{INR}_{12})) \\ &\quad - \log(\pi e(1 + \text{INR}_{12}^p)) + \eta/3 \\ &= \log\left(1 + \frac{\text{SNR}_1 + \text{INR}_{12} + \text{INR}_{12}^p}{1 + \text{INR}_{12}^p}\right) + \eta/3 \end{aligned}$$

which shows that the first constraint in (21) can not be violated.

Therefore, such a deviation can not exist and the nominal strategy must be a  $\eta$ -NE for small enough  $\epsilon$ . Taking the limit as  $\eta \rightarrow 0$ , it follows that the desired rates must lie in  $\mathcal{C}_{\text{NE}}$ . ■

Next, we turn to proving an analogue of Lemma 6 for the Gaussian model. To do this we need to define a parallel notion to the interference-free levels in the deterministic channel. In a Gaussian channel, this again corresponds to the rate  $L_i$ , which can be achieved by treating interference as Gaussian noise. We still want to constrain both the common and private rates of each transmitter so that this rate is utilized with as much “common rate” as possible. Specifically, let

$$a_i = \log\left(1 + \frac{\text{SNR}_i^p}{1 + \text{INR}_{ij}}\right)$$

be the required private rate at user  $i$  and

$$b_i = \log\left(1 + \frac{\text{SNR}_i - \text{SNR}_i^p}{1 + \text{SNR}_i^p + \text{INR}_{ij}}\right)$$

be the required common rate at user  $i$ . The rate  $a_i$  is the rate achieved by user  $i$ 's private signal when treating the aggregate interference plus noise as Gaussian noise, while the rate  $b_i$  is the rate achieved by user  $i$ 's common signal when treating its own private signal plus interference plus noise as Gaussian noise.

We say that  $(R_{ic}, R_{ip})$  fully utilizes the interference free rate for user  $i$  if

$$R_{ip} \geq a_i \quad (44)$$

$$R_{ic} \geq b_i. \quad (45)$$

With this definition we have the following analogue of Lemma 6.

*Lemma 14:* If  $(R_{1c}, R_{1p}, R_{2c}, R_{2p})$  fully utilizes the interference free rate for each user  $i$ , then there exists random common rates  $R_{1r}, R_{2r} \geq 0$  such that  $(R_{1c}, R_{1r}, R_{1p}, R_{1s}, R_{2c}, R_{2r}, R_{2p}, R_{2s})$  is self-saturated at both receivers for any choice of  $R_{1s}, R_{2s}$ .

*Proof:* This proof parallels exactly the proof of Lemma 6. We will show that one can always increase  $R_{2r}$  such that the overall sum rate constraint (the first constraint in (38)) is tight, so that receiver 1 is self-saturated. Suppose no such choice of  $R_{2r}$  exists. Then it must be that  $R_{2r}$  cannot be further increased because the second constraint in (38) is tight. However, if this constraint is tight, then since  $R_{1c} \geq b_1$ , it can be seen that the first constraint must also be tight. ■

To complete the analogue of the deterministic analysis, we need a parallel result to Lemma 7, which we state next.

**Lemma 15:** For any point  $(R_1, R_2) \in \mathcal{C}_{\text{HK}} \cap \mathcal{B}^-$  there exists a nonrandomized Han–Kobayashi rate-split that fully utilizes the interference-free levels at each transmitter.

*Proof:* As in the deterministic case, we begin with an arbitrary nonrandomized Han–Kobayashi rate-split and show that this can always be transformed into one that fully utilizes the interference free levels at each receiver. Note that again since  $a_i + b_i = L_i$ , for any point in  $\mathcal{B}^-$  there will always be a sufficient amount of “rate” available to meet the constraints in (44) and (45).

First, we show that if  $R_{ip} < a_i$  for either user  $i$ , then we can always increase  $R_{ip}$  and decrease  $R_{ic}$  by the same amount until  $R_{ip} = a_i$ . The only way such a transformation could not be done is if the second constraint in (38) prevented it. But by combining (39) with this constraint, it can be seen that this will never happen for  $R_{ip} < a_i$ . Moreover, since  $R_{ip}$  does not appear in any of the constraints in (38) at receiver  $j$ , such a change will never result in any of those constraints being violated.

Thus, we can assume that  $R_{ip} \geq a_i$ . Given this, if a rate-pair does not fully utilize the interference free levels at transmitter  $i$ , it must be that  $R_{ic} < b_i$ . Suppose that this is true for receiver 1 and consider increasing  $R_{1c}$  and decreasing  $R_{1p}$  by the same amount until  $R_{1c} = b_1$ . Note that since  $a_1 + b_1 = L_1$  and  $R_{ic} + R_{ip} \geq L_1$ , when we decrease  $R_{1p}$  in this way it will never cause it to become less than  $a_1$ . Changing  $R_{1p}$  and  $R_{1c}$  in this manner will not violate any of the generalized MAC constraints at receiver 1, since every constraint in (38) involving  $R_{1c}$  also involves  $R_{1p}$ . If this can be done without violating any constraints at receiver 2 then we are done. Otherwise, it must be that at least one of the constraints at receiver 2 involving  $R_{1c}$  becomes tight when  $R_{1c}$  reaches the value  $R_{1c}^* = b_1 - \Delta$ , for some  $\Delta > 0$ . The possible constraints here are the first and second. By the definition of  $\mathcal{B}^-$  we have that  $R_{2p} + R_{2c} \leq U_2 - 1$  and so

$$\begin{aligned} R_{2p} + R_{2c} + R_{1c}^* &= R_2 + R_{1c}^* \\ &\leq U_2 - 1 + b_1 - \Delta \\ &\leq \log(1 + \text{SNR}_2 + \text{INR}_{21}) - 1 - \Delta \\ &\leq \log\left(1 + \frac{\text{SNR}_2 + \text{INR}_{21} - \text{INR}_{12}^p}{\text{INR}_{12}^p + 1}\right) - \Delta. \end{aligned} \quad (46)$$

This shows that the first constraint can not be tight. Note that the last inequality followed since  $\text{INR}_{12}^p \leq 1$ . This implies that the second constraint must be tight, i.e.,

$$R_{2p} + R_{1c}^* = \log\left(1 + \frac{\text{SNR}_2^p + \text{INR}_{21} - \text{INR}_{21}^p}{\text{INR}_{21}^p + 1}\right). \quad (47)$$

Proceeding as in (46) we have

$$\begin{aligned} R_{2p} + R_{2c} + R_{1c}^* &\leq \log(1 + \text{SNR}_2 + \text{INR}_{21}) - 1 - \Delta \\ &\leq \log\left(1 + \frac{\text{SNR}_2 + \text{INR}_{21} - \text{INR}_{12}^p}{\text{INR}_{21}^p + 1}\right) - \Delta. \end{aligned}$$

Combining this with (47) we have

$$\begin{aligned} R_{2c} &\leq \log\left(1 + \frac{\text{SNR}_2 - \text{SNR}_2^p - \text{INR}_{12}^p}{1 + \text{SNR}_2^p + \text{INR}_{21}}\right) - \Delta \\ &\leq b_2 - \Delta. \end{aligned}$$

And so it must also be that  $R_{2p} \geq a_2 + \Delta$ . Now consider increasing  $R_{1c}$  from  $R_{1c}^*$  by  $\Delta$ , while simultaneously reducing  $R_{1p}$  and  $R_{2p}$  each by  $\Delta$  and also increasing  $R_{2c}$  by  $\Delta$ . As in the deterministic analysis, the above calculations show that we will not violate any of the constraints in the modified MAC regions at either receiver when doing this. After this transformation, the resulting rate-split will fully utilize the interference free levels at both receivers. ■

Combining the previous lemmas, we have shown that all points in  $\mathcal{C}_{\text{HK}} \cap \mathcal{B}^-$  are in  $\mathcal{C}_{\text{NE}}$ , proving the first part of Theorem 2. Applying the next lemma will complete this proof by generalizing Lemma 15 for strong IC and showing that in that case the conclusions apply for all points in  $\mathcal{C} \cap \mathcal{B}$ .

**Lemma 16:** For a strong IC, any point  $(R_1, R_2) \in \mathcal{C} \cap \mathcal{B}$  can be achieved by a nonrandomized Han–Kobayashi rate split that fully utilizes the interference-free levels at each transmitter.

*Proof:* Recall, the for a strong interference channel, we set  $\text{INR}_{ij}^p = 0$  for each user  $i$  and so we have  $a_i = 0$  and  $b_i = L_i$ . It follows that for any point  $(R_1, R_2) \in \mathcal{C}_{\text{HK}} \cap \mathcal{B}$  there will be exactly one nonrandomized Han–Kobayashi rate split that achieves this rate, namely the one with  $R_{1c} = R_1$ ,  $R_{2c} = R_2$  and  $R_{1p} = R_{2p} = 0$ . This will trivially fully utilize the interference-free levels at each transmitter. Furthermore, for such a channel  $\mathcal{C}_{\text{HK}} = \mathcal{C}$ , completing the proof. ■

As in the deterministic analysis, it can also be shown by direct calculation that  $\mathcal{C}_{\text{NE}}$  will always contain at least one point that is sum-rate optimal to within 1 bit.

## V. CONCLUSION

We have formulated the new information theoretic notion of a Nash equilibrium region for interference channels. Moreover, we have used this notion to characterize the equilibria in both deterministic and Gaussian ICs. In the deterministic case we are able to exactly specify the Nash equilibrium region, while in the Gaussian case we characterize the Nash equilibrium region to within one bit. The analysis for the Gaussian case directly parallels our analysis for the deterministic case, and thus serves as another illustration of the utility of deterministic models in providing useful insights for the more complicated Gaussian setting.

Our approach here is based on assuming that a given transmitter and the intended receiver share a source of common randomness. However, in the case of deterministic channels, it is shown in [14] that this is not needed. Specifically, it is possible to achieve all points in  $\mathcal{C} \cap \mathcal{B}$  by time-sharing among structured schemes that do no coding over time and use no common randomness (though proving this appears to require more detailed combinatorial arguments than we have used here). The key property of these structured schemes is that the common signals of the two users are *segregated* into separate levels at each of the receivers (in contrast to the random coding schemes

considered in this paper, where the common signals of the two users are all mixed up.) Each transmitter may still use randomness to send a jamming signal on specified levels, but by aligning the jamming signal with the interfering common signal at the node's own receiver, the receiver does not need to decode it. Hence, the receiver needs not know the random bits generating the jamming signal. Such schemes can likely also be translated to the Gaussian settings by using structured codes instead of the Gaussian Han–Kobayashi schemes considered here.

The games we were considering here were games with full information, i.e., each user has perfect knowledge of all channel gains as well as the code-books of the other user. One possible future direction for this work would be to relax this assumption and consider games with incomplete information. Another natural direction would be to consider interference networks with more than 2 users. Some preliminary work in this direction for deterministic channels is given in [21] where it is shown that with more than two user efficient equilibria may no longer exist.

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