Diversity-Multiplexing Tradeoff in MIMO Channels

David Tse
Department of EECS, U.C. Berkeley

February 26, 2004

Intel Smart Antenna Workshop

Two objectives of the talk:

- Present a new performance metric for evaluating MIMO coding schemes.
- Give some examples of new coding schemes designed to optimize the metric.

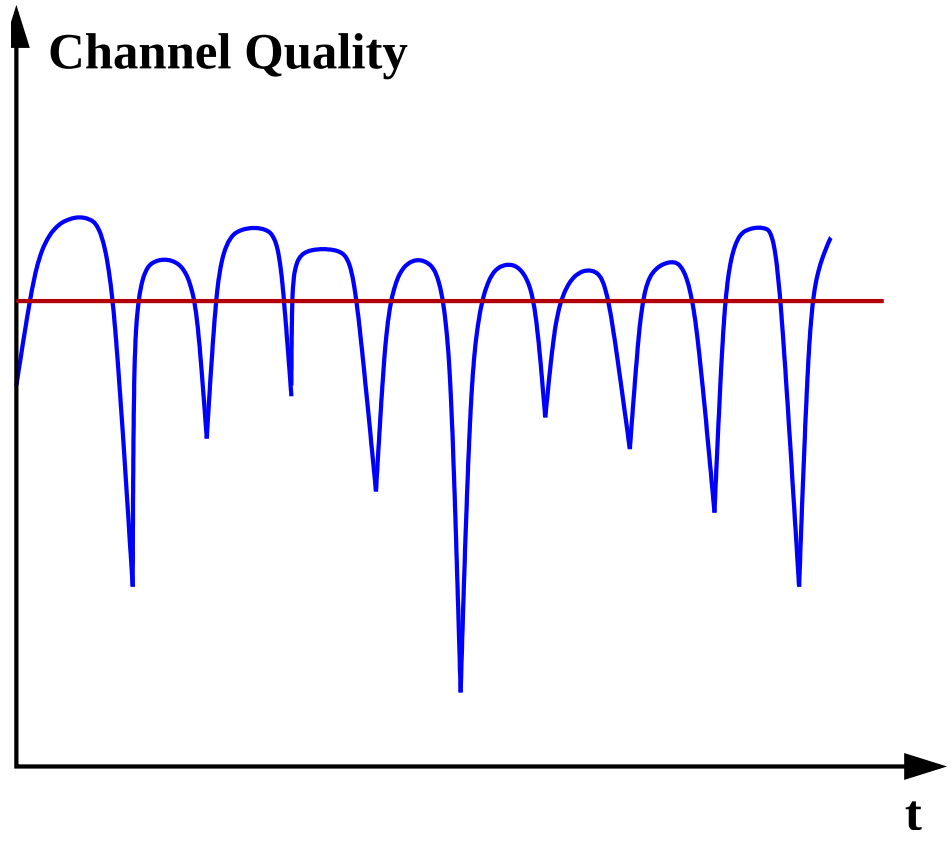
Diversity and Freedom

Two fundamental resources of a MIMO fading channel:

diversity

degrees of freedom

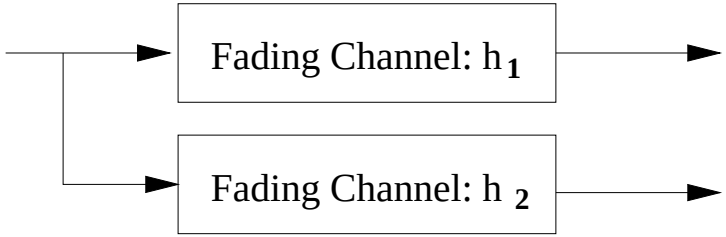
Diversity



A channel with more diversity has smaller probability in deep fades.

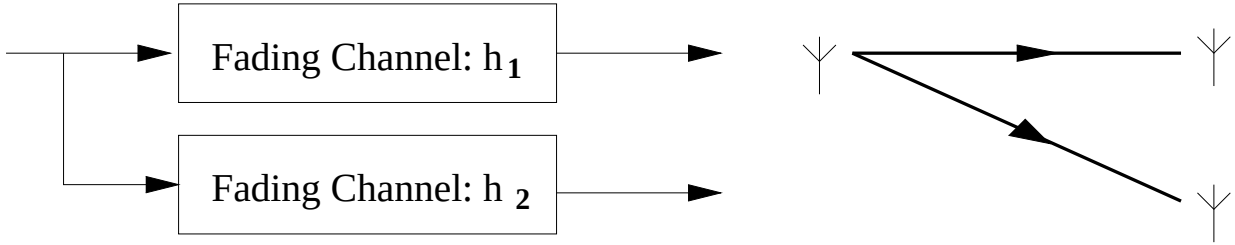
Diversity

Diversity



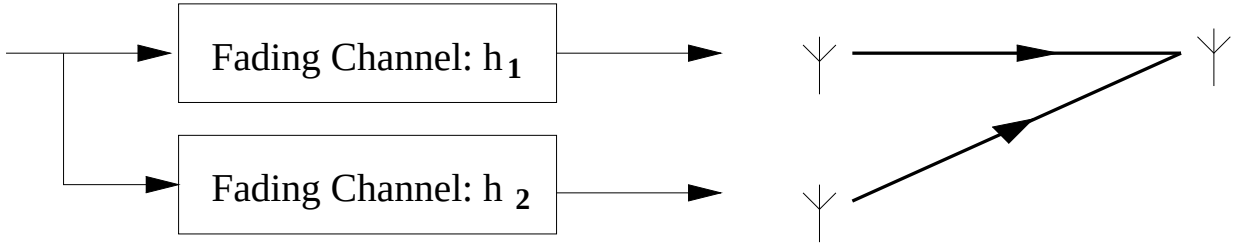
- Additional independent fading channels increase diversity.

Diversity



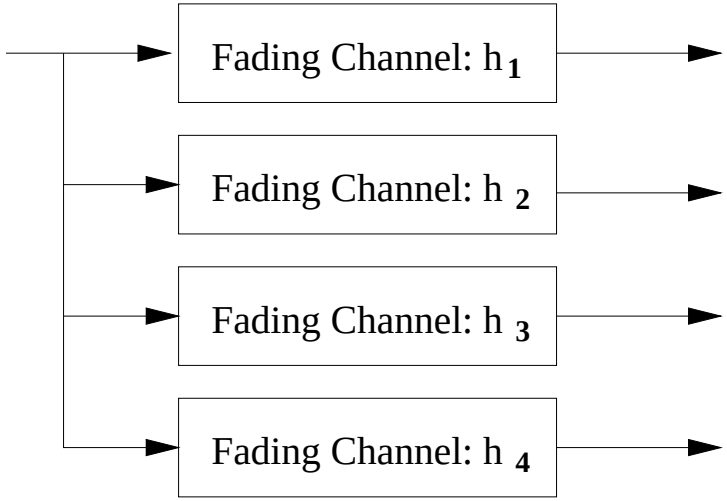
- Additional independent fading channels increase diversity.
- Spatial diversity

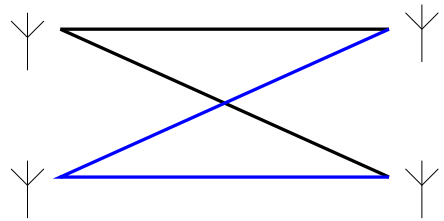
Diversity



- Additional independent fading channels increase diversity.
- Spatial diversity: receive, transmit

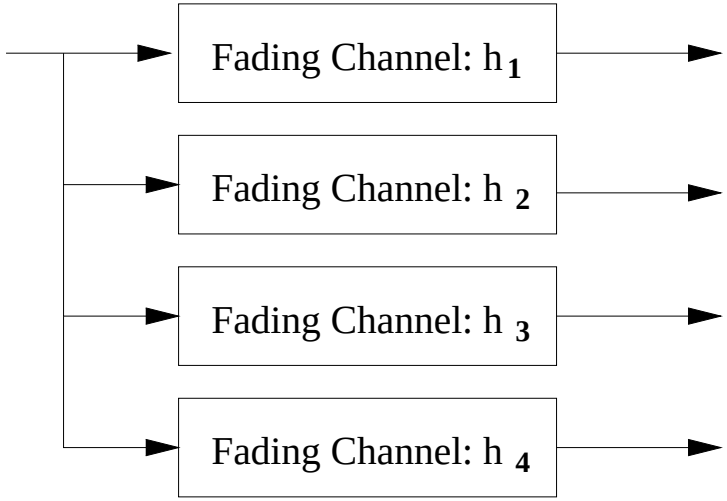
Diversity

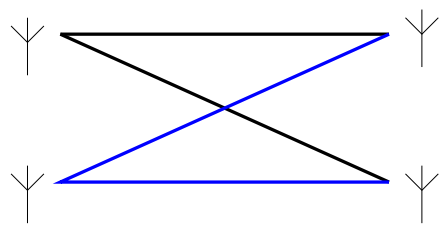




- Additional independent fading channels increase diversity.
- Spatial diversity: receive, transmit or both.

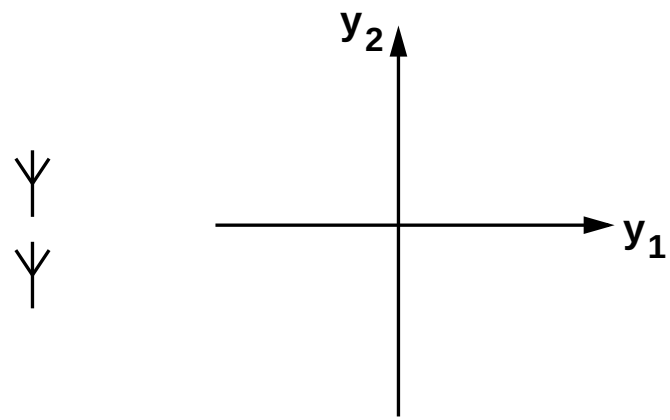
Diversity



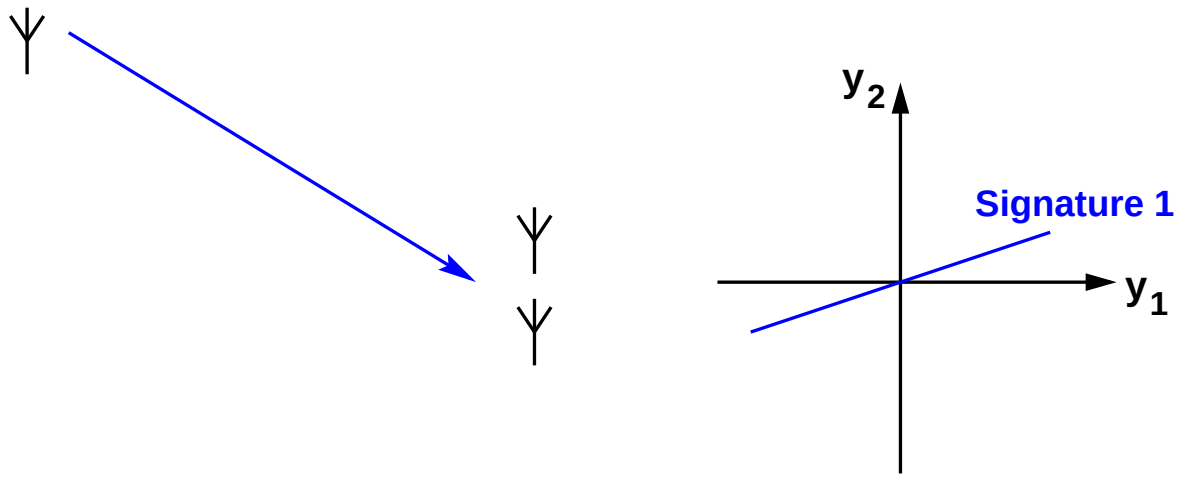


- Additional independent fading channels increase diversity.
- Spatial diversity: receive, transmit or both.
- For a m by n channel, diversity is mn.

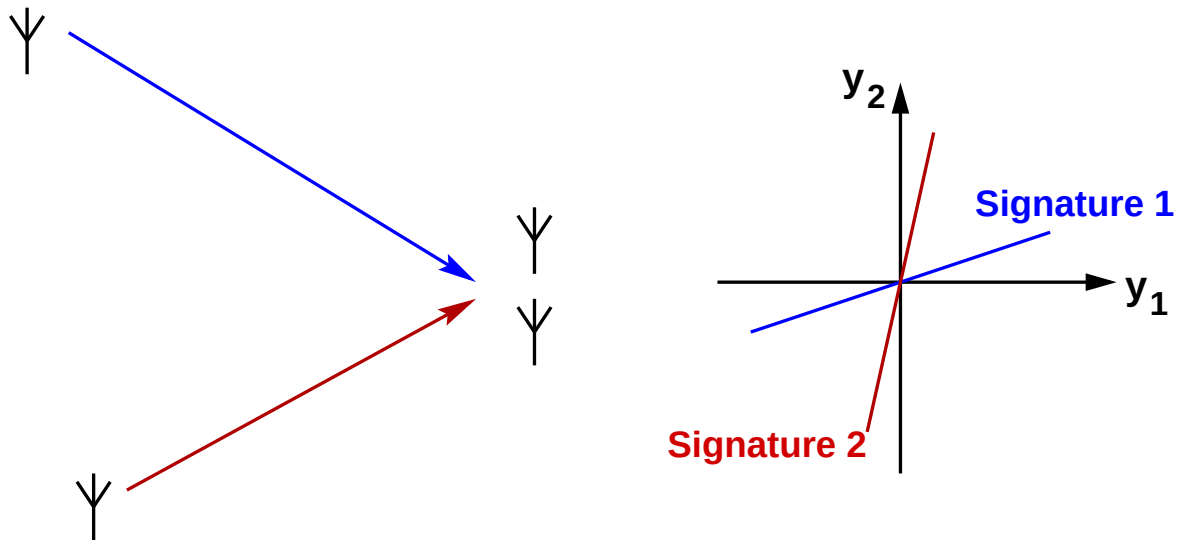
Degrees of Freedom



Degrees of Freedom

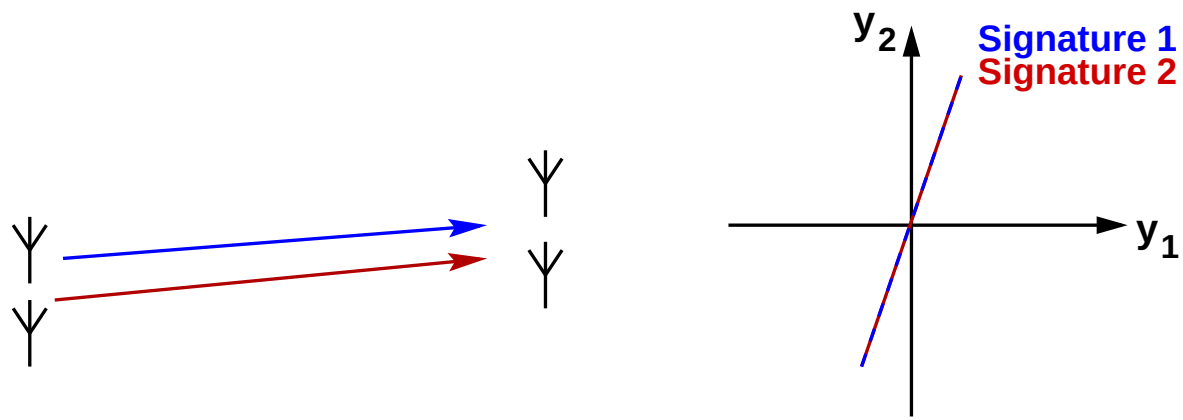


Degrees of Freedom



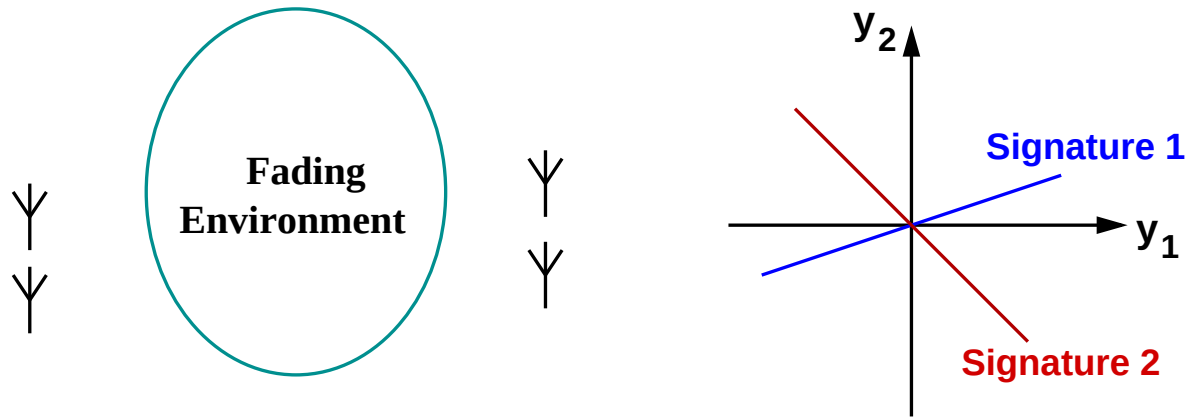
Signals arrive in multiple directions provide multiple degrees of freedom for communication.

Degrees of Freedom



Signals arrive in multiple directions provide multiple degrees of freedom for communication.

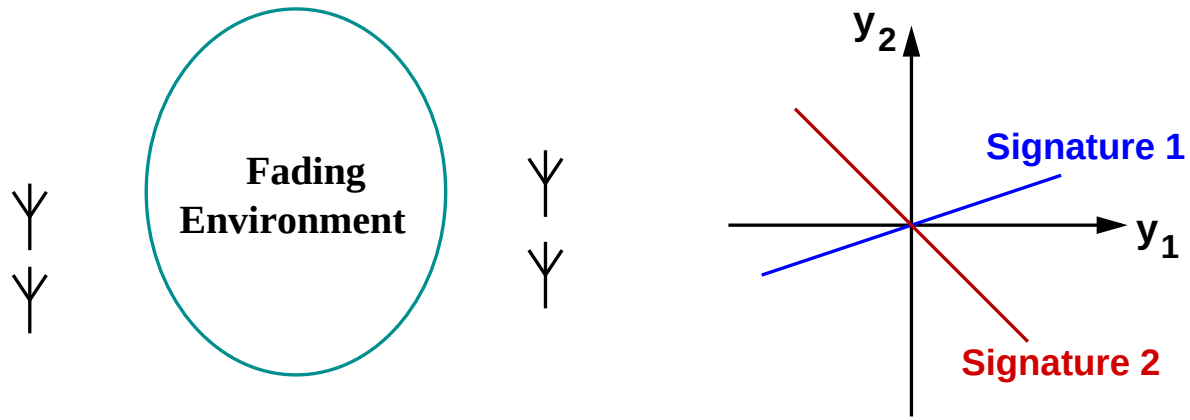
Degrees of Freedom



Signals arrive in multiple directions provide multiple degrees of freedom for communication.

Same effect can be obtained via scattering even when antennas are close together.

Degrees of Freedom



Signals arrive in multiple directions provide multiple degrees of freedom for communication.

Same effect can be obtained via scattering even when antennas are close together.

In a m by n channel with rich scattering, there are $\min\{m, n\}$ degrees of freedom.

Diversity and Freedom

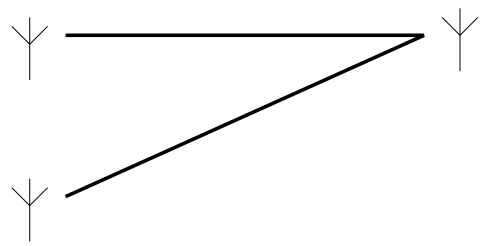
In a MIMO channel with rich scattering:

maximum diversity $= mn$

degrees of freedom $= \min\{m, n\}$

The name of the game in space-time coding is to design schemes which exploit as much of both these resources as possible.

Space-Time Code Examples: 2×1 Channel



Repetition Scheme:

$$\mathbf{X} = \begin{array}{c} \text{time} \\ \text{space} \end{array} \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix}$$

diversity: 2

data rate: 1/2 sym/s/Hz

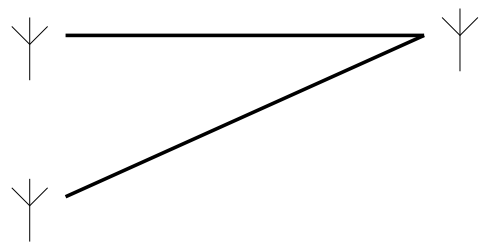
Alamouti Scheme:

$$\mathbf{X} = \begin{array}{c} \text{time} \\ \text{space} \end{array} \begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix}$$

diversity: 2

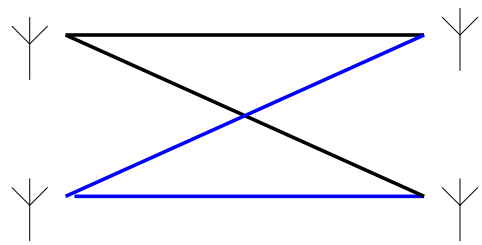
data rate: 1 sym/s/Hz

Performance Summary: 2×1 Channel



	Diversity gain	Degrees of freedom utilized /s/Hz
Repetition	2	$1/2$
Alamouti	2	1
channel itself	2	1

Space-Time Code Examples: 2×2 Channel



Repetition Scheme:

$$X = \begin{array}{c} \text{time} \\ \text{space} \end{array} \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix}$$

diversity gain : 4

data rate: 1/2 sym/s/Hz

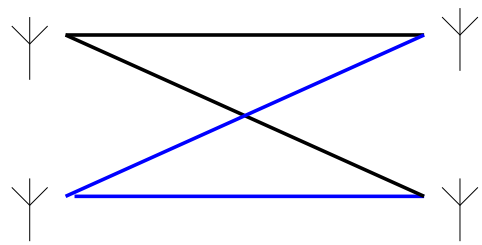
Alamouti Scheme:

$$X = \begin{array}{c} \text{time} \\ \text{space} \end{array} \begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix}$$

diversity gain : 4

data rate: 1 sym/s/Hz

Space-Time Code Examples: 2×2 Channel



Repetition Scheme:

$$X = \begin{array}{c} \text{time} \\ \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix} \\ \text{space} \end{array}$$

diversity: 4

data rate: 1/2 sym/s/Hz

Alamouti Scheme:

$$X = \begin{array}{c} \text{time} \\ \begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix} \\ \text{space} \end{array}$$

diversity: 4

data rate: 1 sym/s/Hz

But the 2×2 channel has 2 degrees of freedom!

V-BLAST with Nulling

Send two independent uncoded streams over the two transmit antennas.

Demodulate each stream by nulling out the other stream.

Data rate: 2 sym/s/Hz

Diversity: 1

Winters, Salz and Gitlins 93:

Nulling out k interferers using n receive antennas yields a diversity gain of $n - k$.

Performance Summary: 2×2 Channel

	Diversity gain	d.o.f. utilized /s/Hz
Repetition	4	$1/2$
Alamouti	4	1
V-Blast with nulling	1	2
channel itself	4	2

Questions:

- Alaomuti is clearly better than repetition, but how can it be compared to V-Blast?

Performance Summary: 2×2 Channel

	Diversity gain	d.o.f. utilized /s/Hz
Repetition	4	$1/2$
Alamouti	4	1
V-Blast with nulling	1	2
channel itself	4	2

Questions:

- Alaomuti is clearly better than repetition, but how can it be compared to V-Blast?
- How does one quantify the "optimal" performance achievable by any scheme?

Performance Summary: 2×2 Channel

	Diversity gain	d.o.f. utilized /s/Hz
Repetition	4	$1/2$
Alamouti	4	1
V-Blast with nulling	1	2
channel itself	4	2

Questions:

- Alaomuti is clearly better than repetition, but how can it be compared to V-Blast?
- How does one quantify the "optimal" performance achievable by any scheme?
- We need to make the notions of "fiversity gain" and "d.o.f. utilized" precise and enrich them.

Classical Diversity Gain

Motivation: PAM

$$\mathbf{y} = \mathbf{h}\mathbf{x} + \mathbf{w} \qquad P_e \approx P(\|\mathbf{h}\| \text{ is small }) \propto \text{SNR}^{-1}$$

$$\left.\begin{aligned} \mathbf{y}_1 &= \mathbf{h}_1\mathbf{x} + \mathbf{w}_1 \\ \mathbf{y}_2 &= \mathbf{h}_2\mathbf{x} + \mathbf{w}_2 \end{aligned}\right\} \qquad \begin{aligned} P_e \quad &\approx P(\|\mathbf{h}_1\|, \|\mathbf{h}_2\| \text{ are both small}) \\ &\propto \text{SNR}^{-2} \end{aligned}$$

Classical Diversity Gain

Motivation: PAM

$$\mathbf{y} = \mathbf{h}\mathbf{x} + \mathbf{w} \qquad P_e \approx P(\|\mathbf{h}\| \text{ is small }) \propto \text{SNR}^{-1}$$

$$\left.\begin{aligned} \mathbf{y}_1 &= \mathbf{h}_1\mathbf{x} + \mathbf{w}_1 \\ \mathbf{y}_2 &= \mathbf{h}_2\mathbf{x} + \mathbf{w}_2 \end{aligned}\right\} \qquad \begin{aligned} P_e \quad &\approx P(\|\mathbf{h}_1\|, \|\mathbf{h}_2\| \text{ are both small}) \\ &\propto \text{SNR}^{-2} \end{aligned}$$

General Definition

A space-time coding scheme achieves (classical) diversity gain $d_{\max}$, if

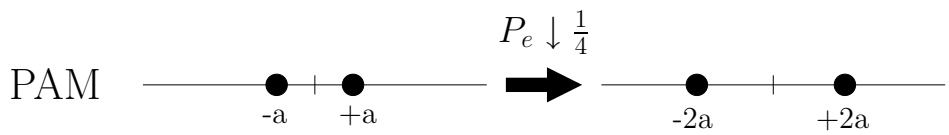
$$P_e(\text{SNR}) \sim \text{SNR}^{-d_{\max}}$$

for a fixed data rate.

i.e. error probability deceases by $2^{-d_{\max}}$ for every 3 dB increase in SNR, by $4^{-d_{\max}}$ for every 6dB increase, etc.

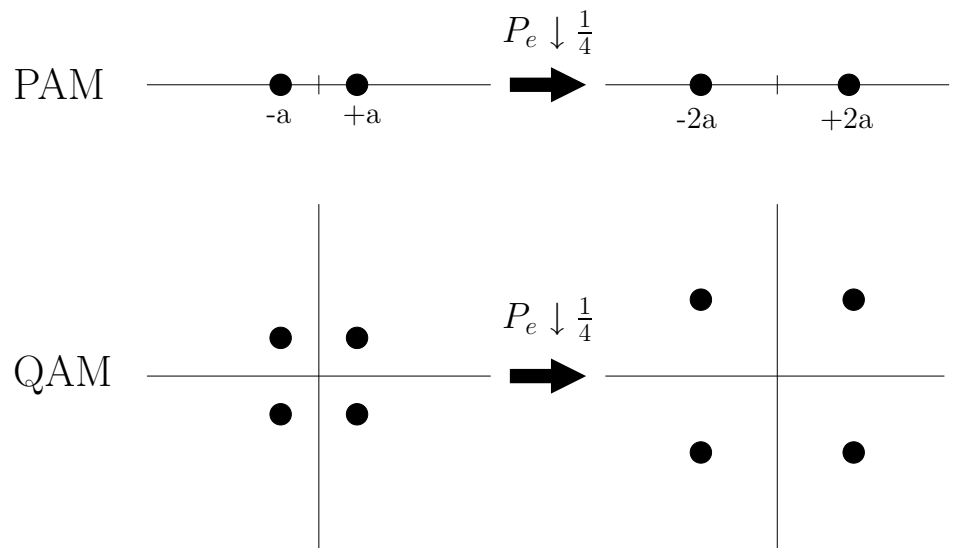
Example: PAM vs QAM in 1 by 1 Channel

Every 6 dB increase in SNR doubles the distance between constellation points for a given rate.



Example: PAM vs QAM in 1 by 1 Channel

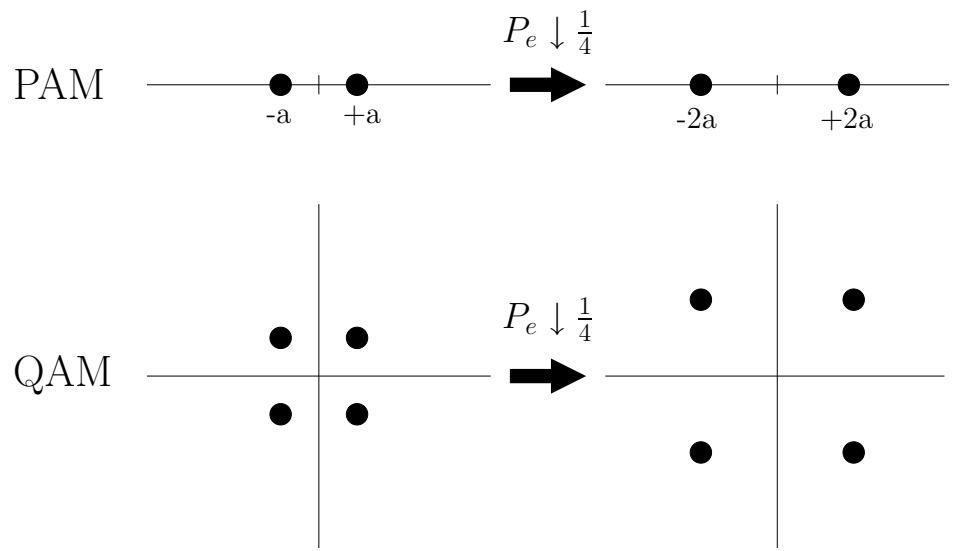
Every 6 dB increase in SNR doubles the distance between constellation points for a given rate.



Both PAM and QAM have the same (classical) diversity gain of 1.

Example: PAM vs QAM in 1 by 1 Channel

Every 6 dB increase in SNR doubles the distance between constellation points for a given rate.

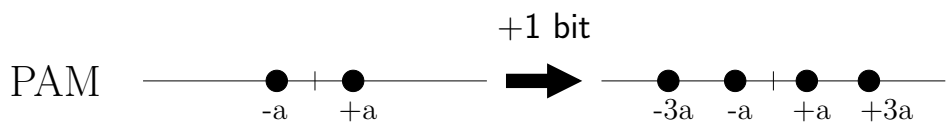


Both PAM and QAM have the same (classical) diversity gain of 1.

(classical) diversity gain does not say anything about the d.o.f. utilized by the scheme.

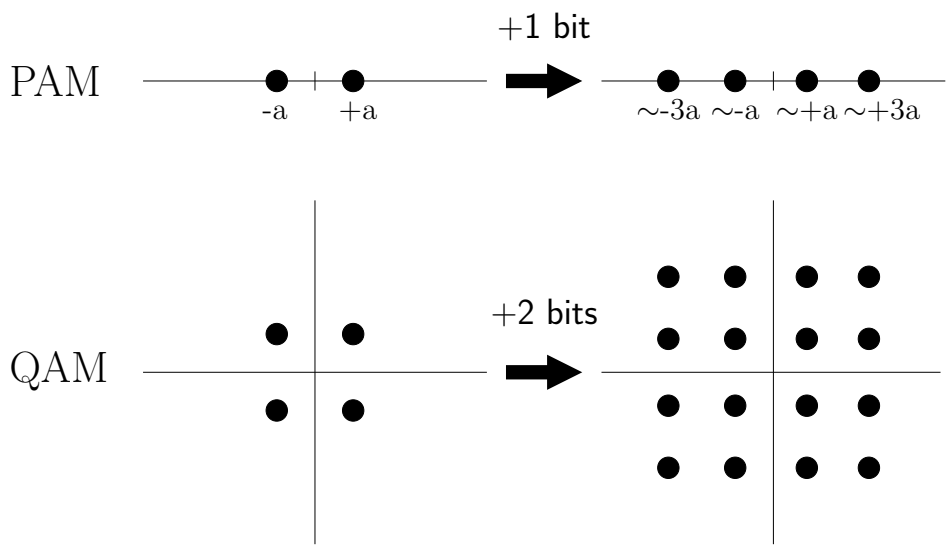
Ask a Dual Question

Every 6 dB doubles the constellation size for a given reliability, for PAM.



Ask a Dual Question

Every 6 dB doubles the constellation size for a given reliability, for PAM



But for QAM, every 6 dB quadruples the constellation size.

Degrees of Freedom Utilized

Definition:

A space-time coding scheme utilizes $r_{\max}$ degrees of freedom/s/Hz if the data rate scales like

$$R(\text{SNR}) \sim r_{\max} \log_2 \text{SNR} \quad \text{bits/s/Hz}$$

for a fixed error probability (reliability)

In a 1×1 channel, $r_{\max} = 1/2$ for PAM, $r_{\max} = 1$ for QAM.

Degrees of Freedom Utilized

Definition:

A space-time coding scheme utilizes $r_{\max}$ degrees of freedom/s/Hz if the data rate scales like

$$R(\text{SNR}) \sim r_{\max} \log_2 \text{SNR} \quad \text{bits/s/Hz}$$

for a fixed error probability (reliability)

In a 1×1 channel, $r_{\max} = 1/2$ for PAM, $r_{\max} = 1$ for QAM.

Note: A space-time coding scheme is a family of codes within a certain structure, with varying symbol alphabet as a function of SNR.

Diversity-Multiplexing Tradeoff

Every 3 dB increase in SNR yields

either

a $2^{-d_{\max}}$ decrease in error probability for a fixed rate;

or

$r_{\max}$ additional bits/s/Hz for a fixed reliability.

Diversity-Multiplexing Tradeoff

Every 3 dB increase in SNR yields

either

a $2^{-d_{\max}}$ decrease in error probability for a fixed rate;

or

$r_{\max}$ additional bits/s/Hz for a fixed reliability.

But these are two extremes of a rate-reliability tradeoff.

Diversity-Multiplexing Tradeoff

Every 3 dB increase in SNR yields

either

a $2^{-d_{\max}}$ decrease in error probability for a fixed rate;

or

$r_{\max}$ additional bits/s/Hz for a fixed reliability.

But these are two extremes of a rate-reliability tradeoff.

More generally, one can increase reliability and the data rate at the same time.

Diversity-Multiplexing Tradeoff of A Scheme

(Zheng and Tse 03)

Definition

A space-time coding scheme achieves a diversity-multiplexing tradeoff curve $d(r)$ if for each multiplexing gain r, simultaneously

$$R(\text{SNR}) \sim r \log_2 \text{SNR} \quad \text{bits/s/Hz}$$

and

$$P_e(\text{SNR}) \sim \text{SNR}^{-d(r)}.$$

Diversity-Multiplexing Tradeoff of A Scheme

(Zheng and Tse 03)

Definition

A space-time coding scheme achieves a diversity-multiplexing tradeoff curve $d(r)$ if for each multiplexing gain r, simultaneously

$$R(\text{SNR}) \sim r \log_2 \text{SNR} \quad \text{bits/s/Hz}$$

and

$$P_e(\text{SNR}) \sim \text{SNR}^{-d(r)}.$$

The largest multiplexing gain is $r_{\max}$, the d.o.f. utilized by the scheme.

The largest diversity gain is $d_{\max} = d(0)$, the classical diversity gain.

Diversity-Multiplexing Tradeoff of the Channel

Definition

The diversity-multiplexing tradeoff $d^*(r)$ of a MIMO channel is the best possible diversity-multiplexing tradeoff achievable by any scheme.

$r^*_{\max}$ is the largest multiplexing gain achievable in the channel.

$d^*_{\max} = d^*(0)$ is the largest diversity gain achievable.

Diversity-Multiplexing Tradeoff of the Channel

Definition

The diversity-multiplexing tradeoff $d^*(r)$ of a MIMO channel is the best possible diversity-multiplexing tradeoff achievable by any scheme.

$r^*_{\max}$ is the largest multiplexing gain achievable in the channel.

$d^*_{\max} = d^*(0)$ is the largest diversity gain achievable.

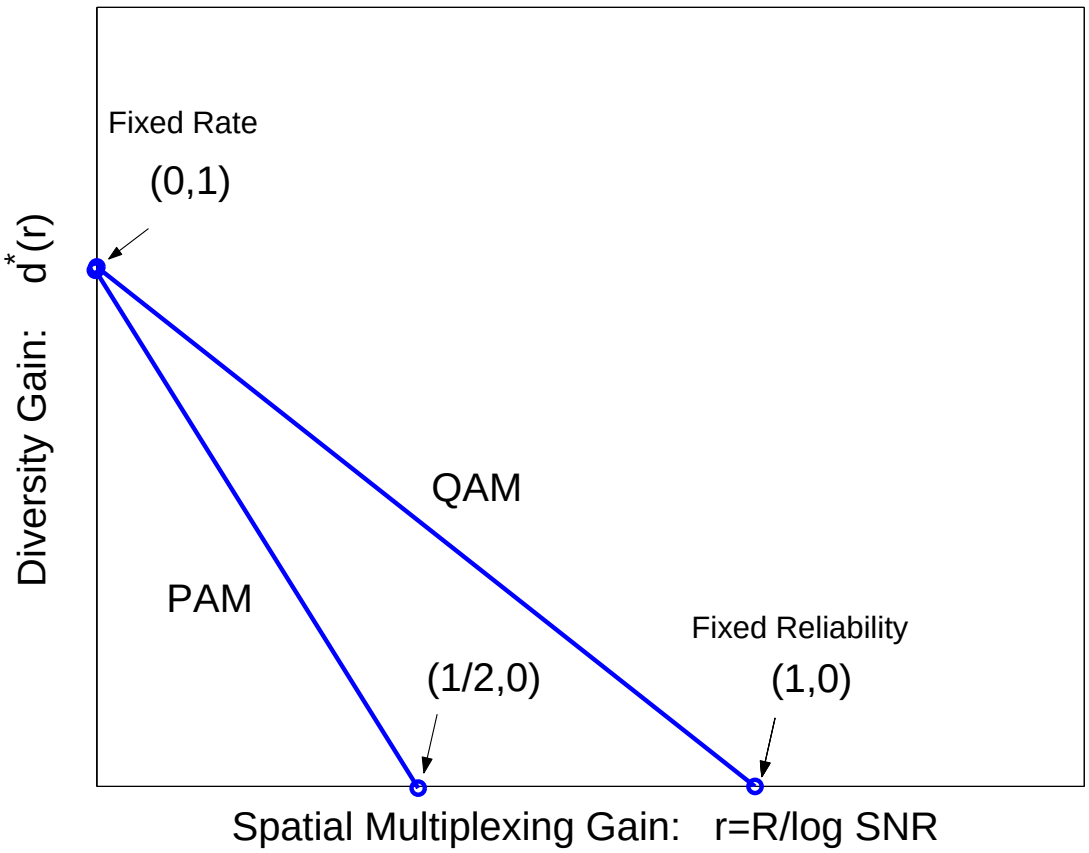
For a $m \times n$ MIMO channel, it is not difficult to show:

$$r^*_{\max} = \min\{m, n\}$$

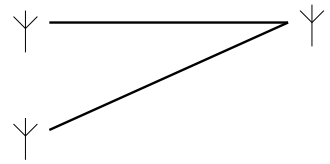
$$d^*_{\max} = mn$$

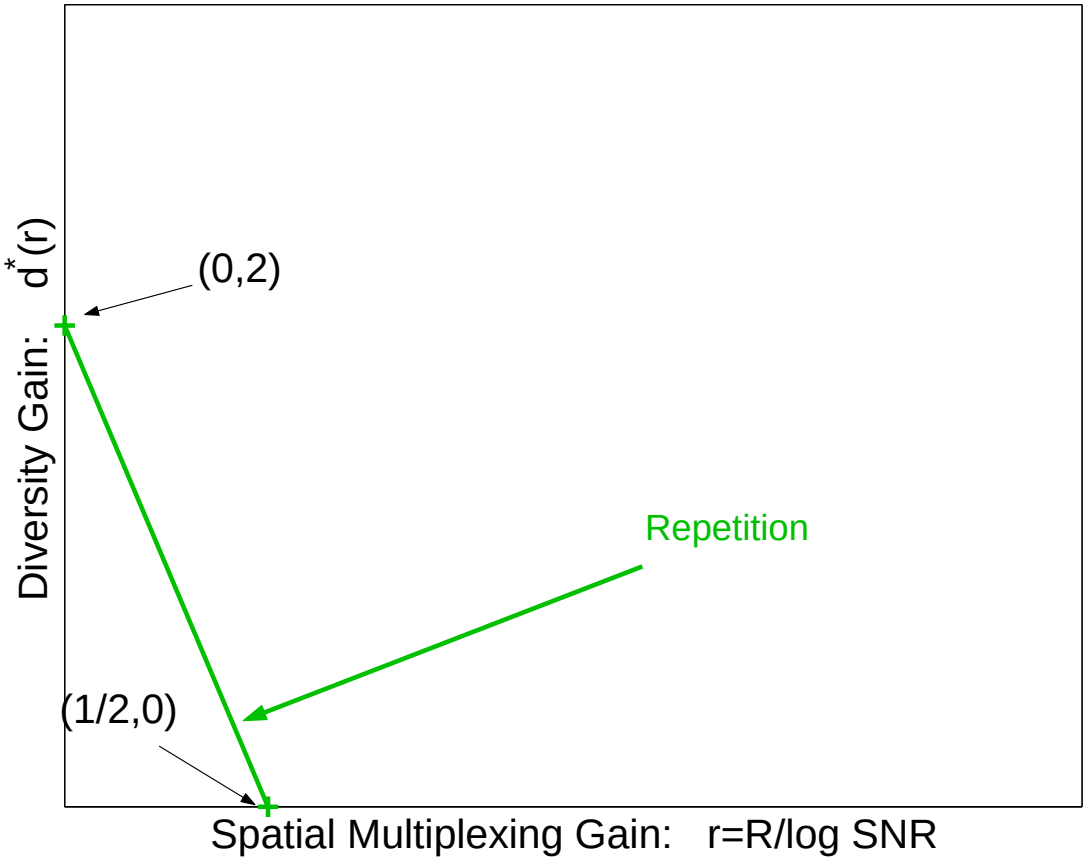
What is more interesting is how the entire curve looks like.

Example: 1×1 Channel

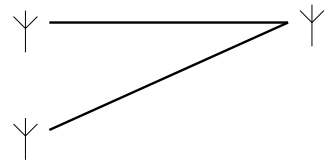


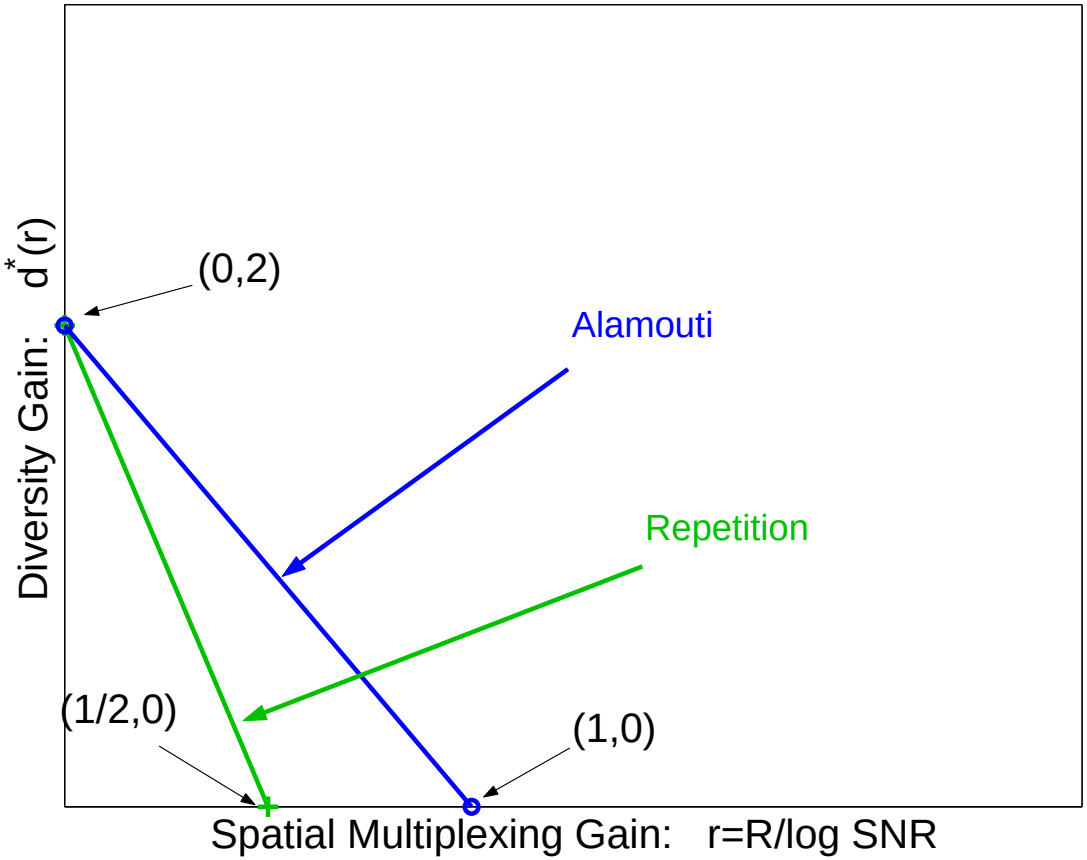
Example: 2×1 Channel



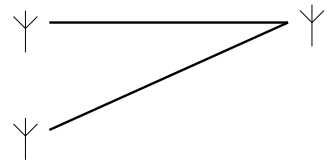


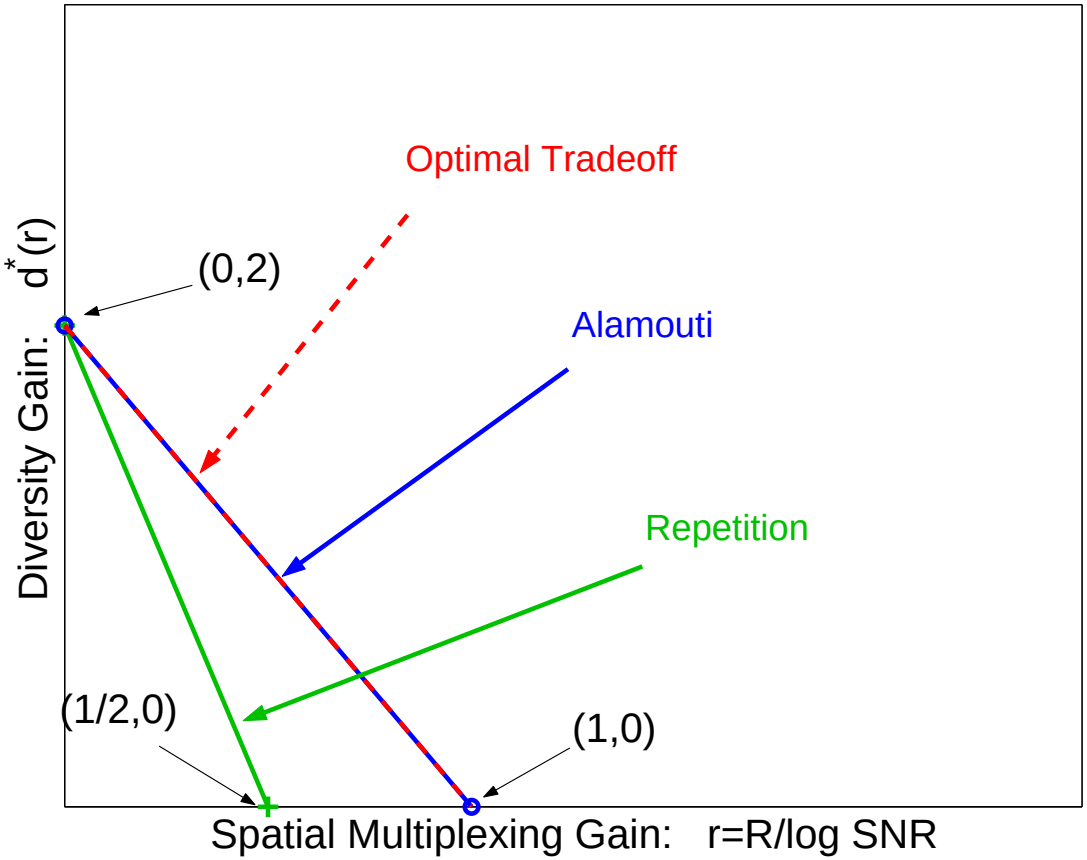
Example: 2×1 Channel



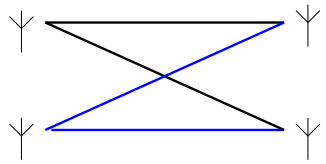


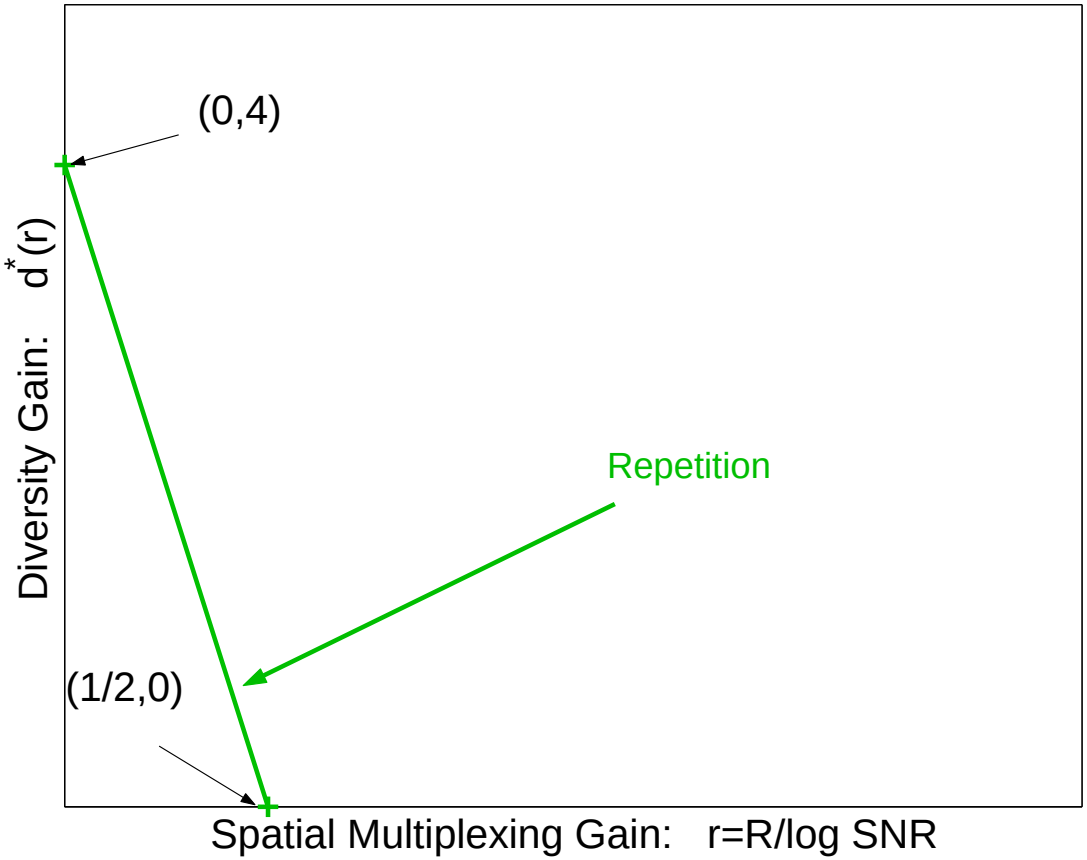
Example: 2×1 Channel



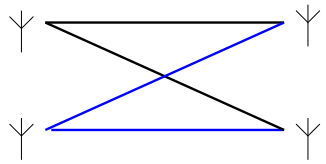


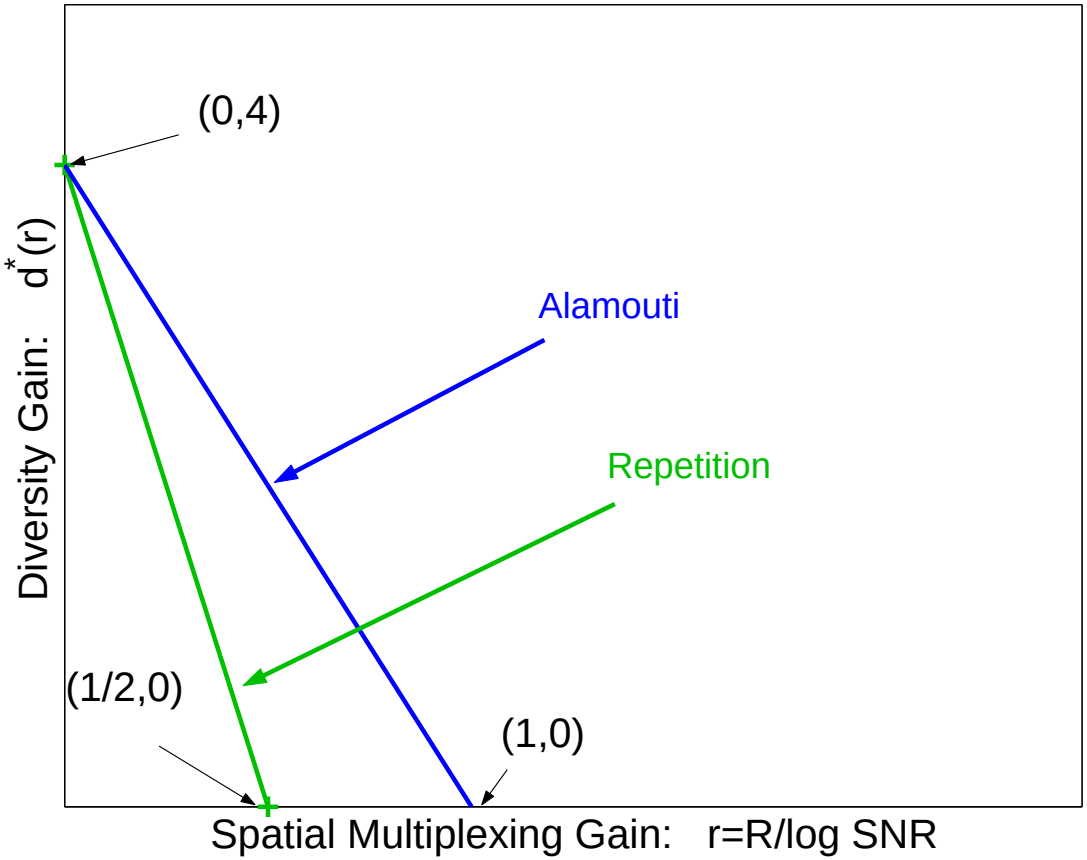
Example: 2×2 Channel



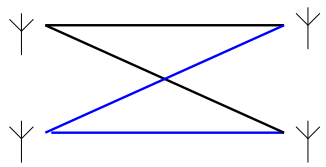


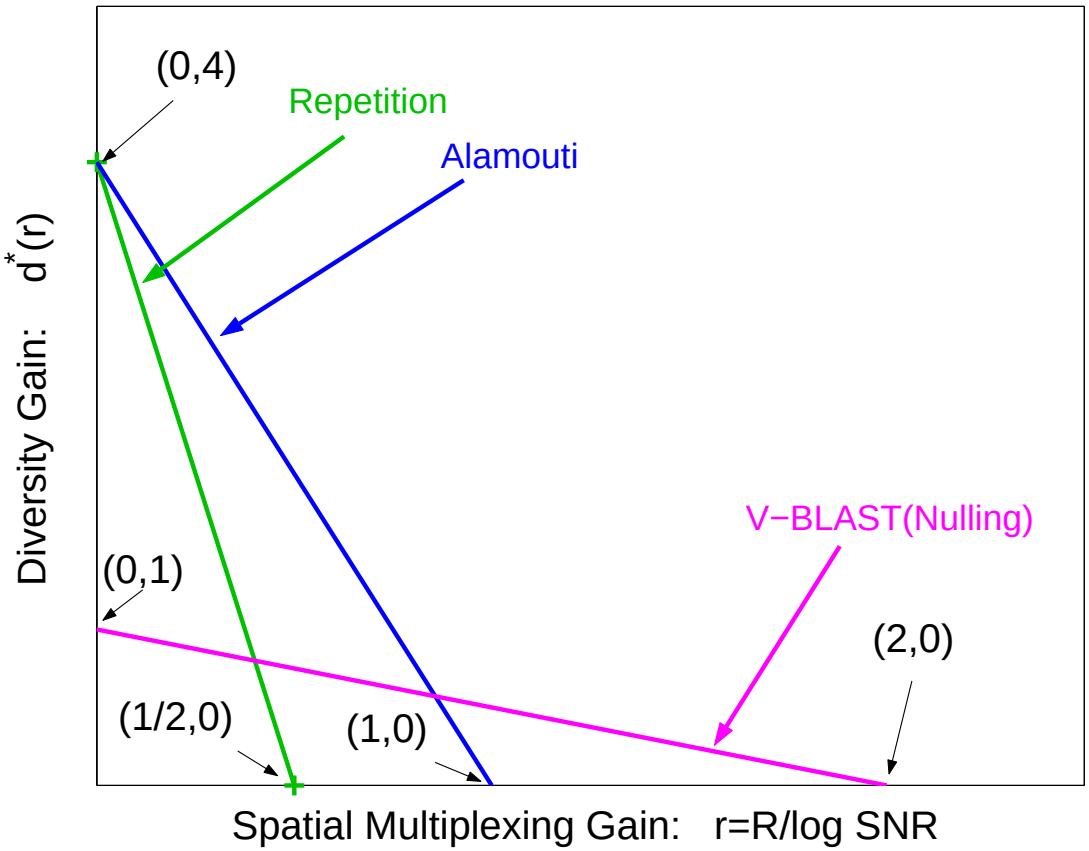
Example: 2×2 Channel



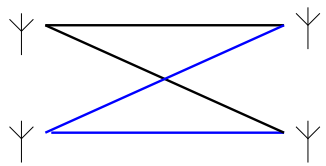


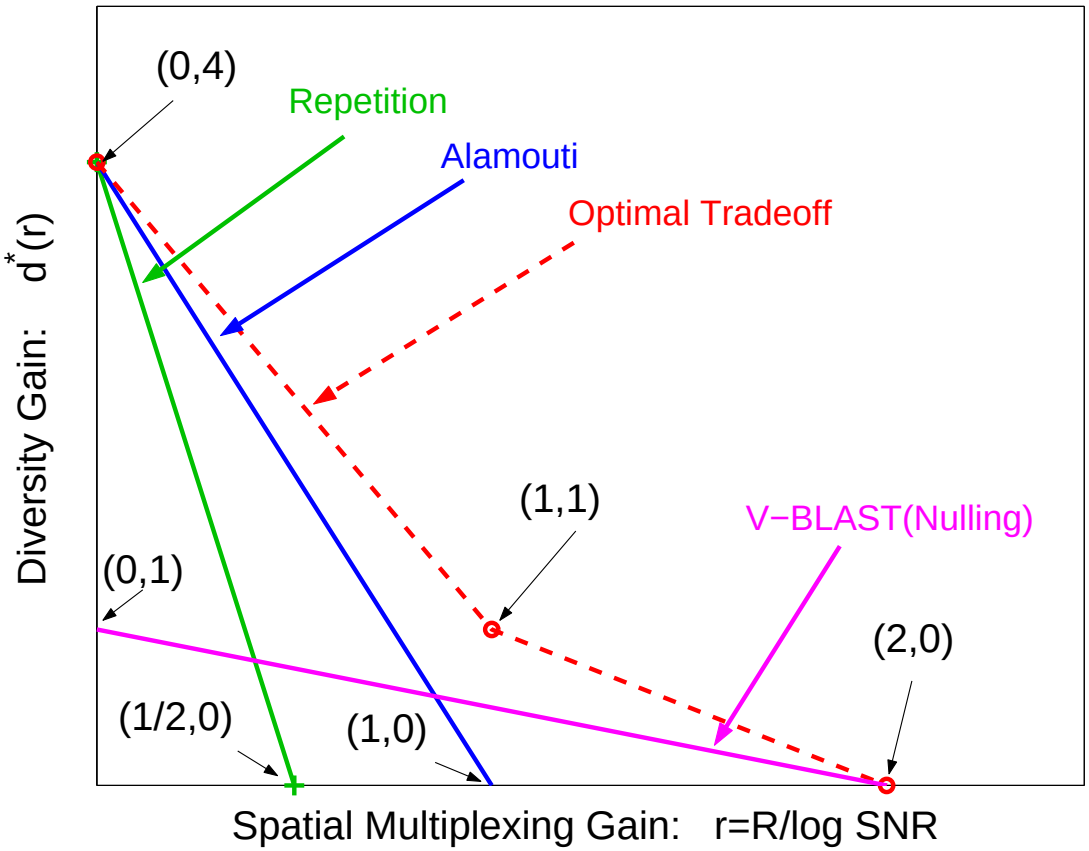
Example: 2×2 Channel



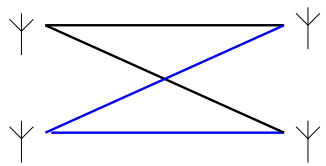


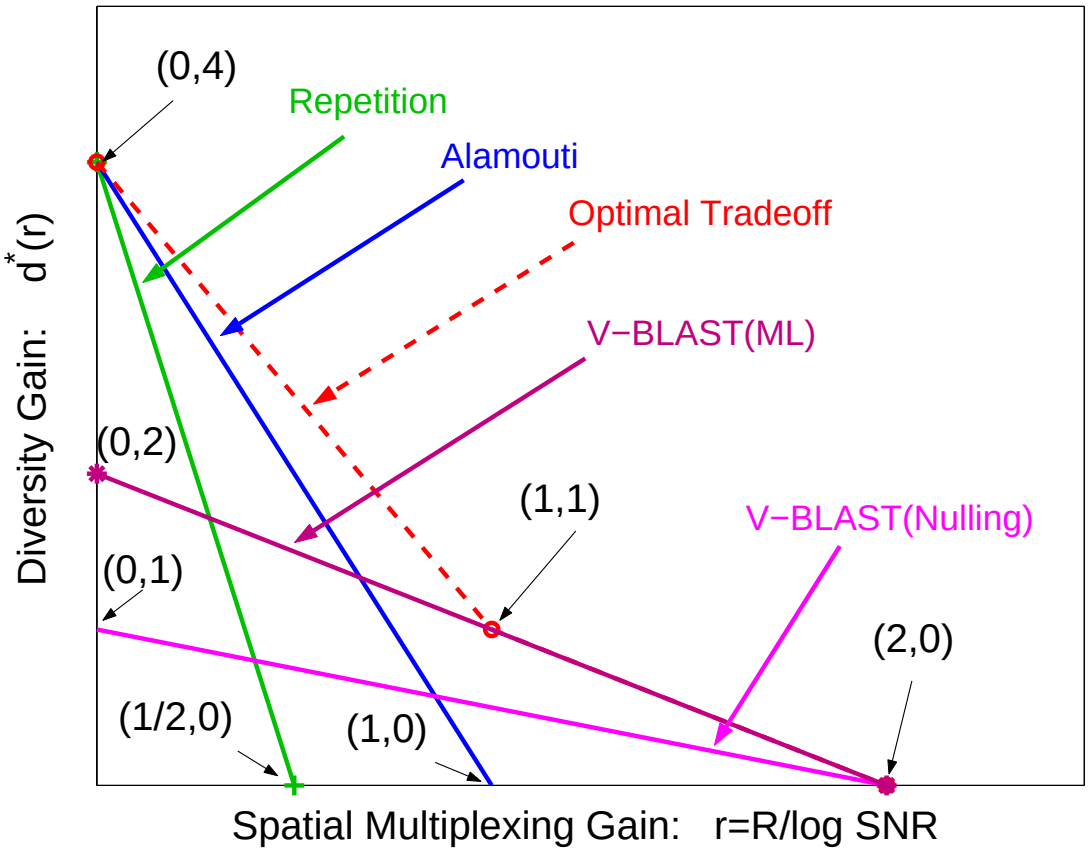
Example: 2×2 Channel



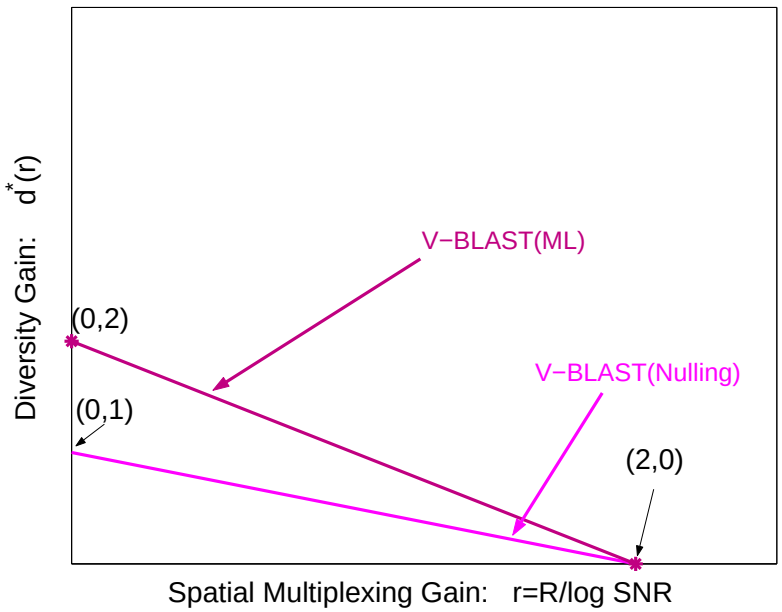


Example: 2×2 Channel





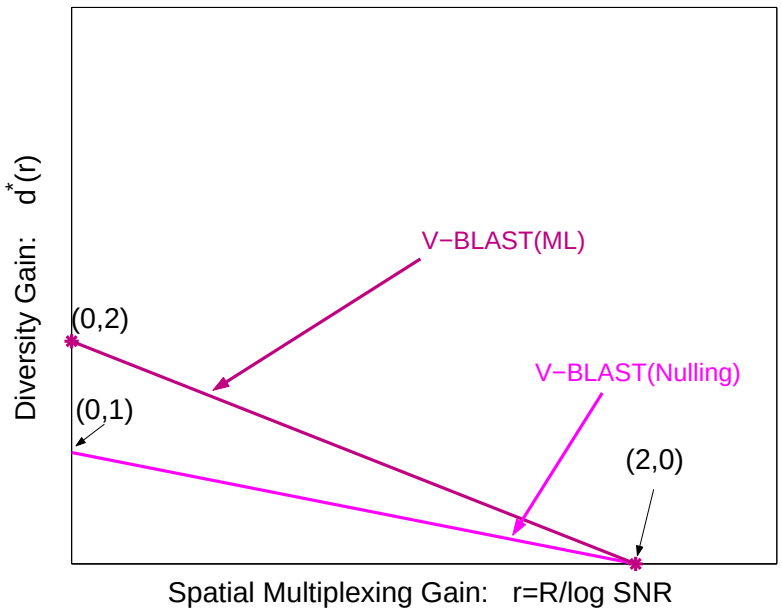
ML vs Nulling in V-Blast



Winters, Salz and Gitlins 93:

Nulling out k interferers using n receive antennas provides a diversity gain of $n - k$.

ML vs Nulling in V-Blast



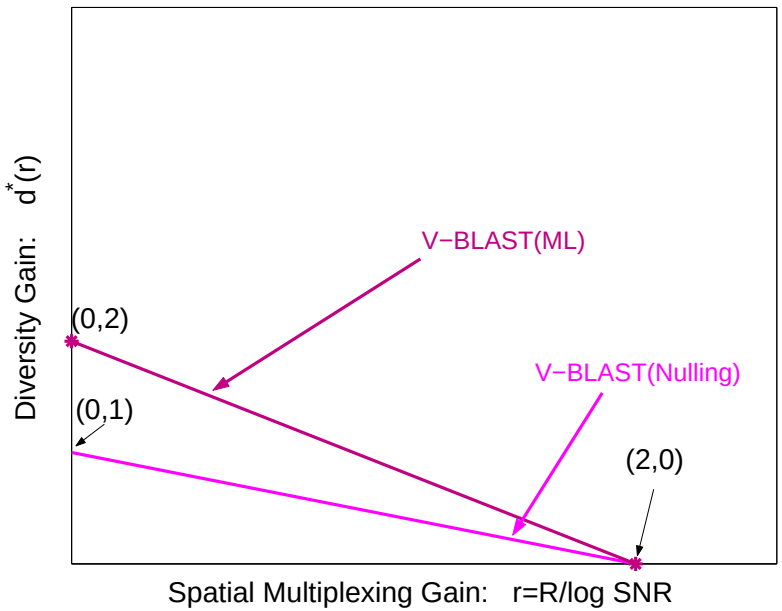
Winters, Salz and Gitlins 93:

Nulling out k interferers using n receive antennas provides a diversity gain of $n - k$.

Tse,Viswanath and Zheng 03:

Jointly detecting all users provides a diversity gain of n to each.

ML vs Nulling in V-Blast



Winters, Salz and Gitlins 93:

Nulling out k interferers using n receive antennas provides a diversity gain of $n - k$.

Tse,Viswanath and Zheng 03:

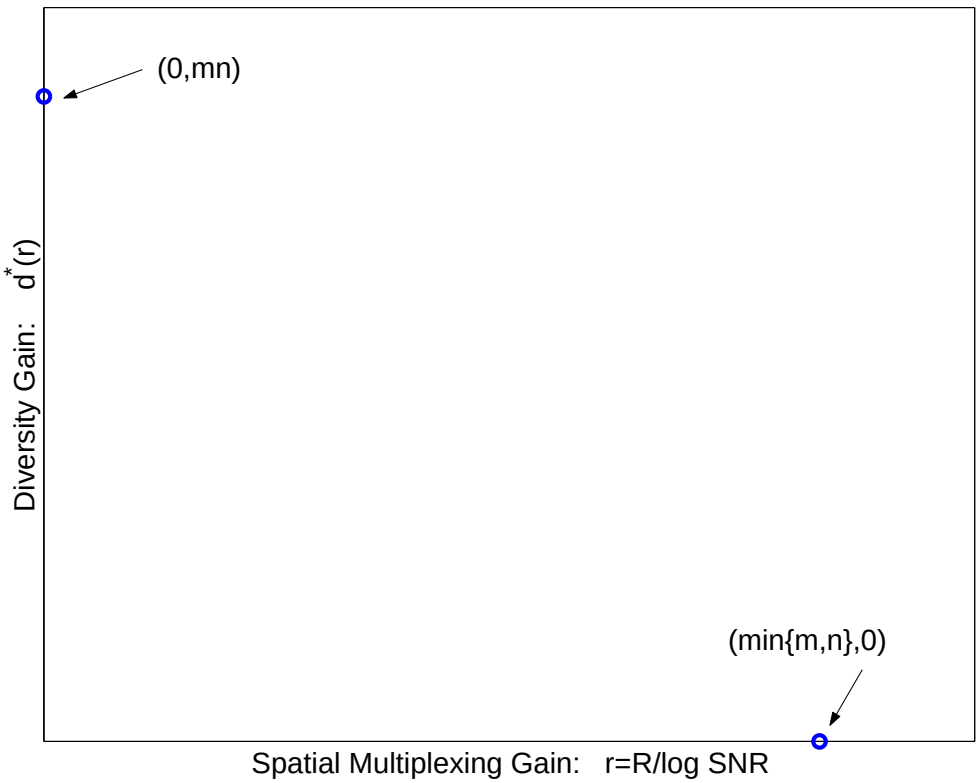
Jointly detecting all users provides a diversity gain of n to each.

There is free lunch. (?)

Optimal D-M Tradeoff for General $m \times n$ Channel

(Zheng and Tse 03)

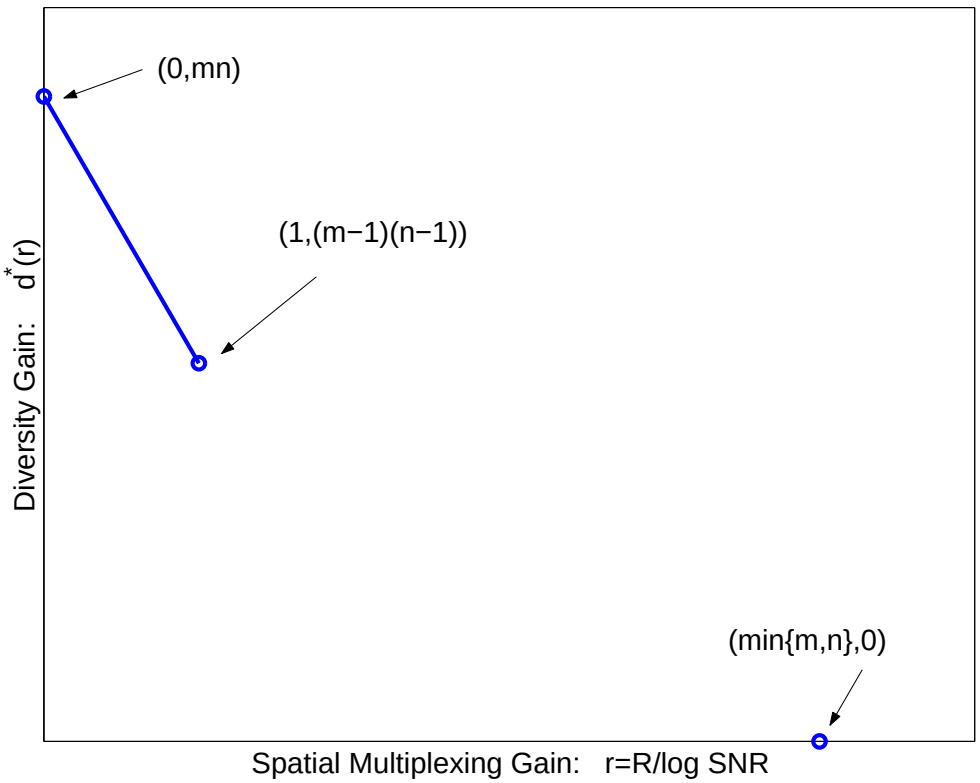
As long as block length $l \geq m + n - 1$:



Optimal D-M Tradeoff for General $m \times n$ Channel

(Zheng and Tse 03)

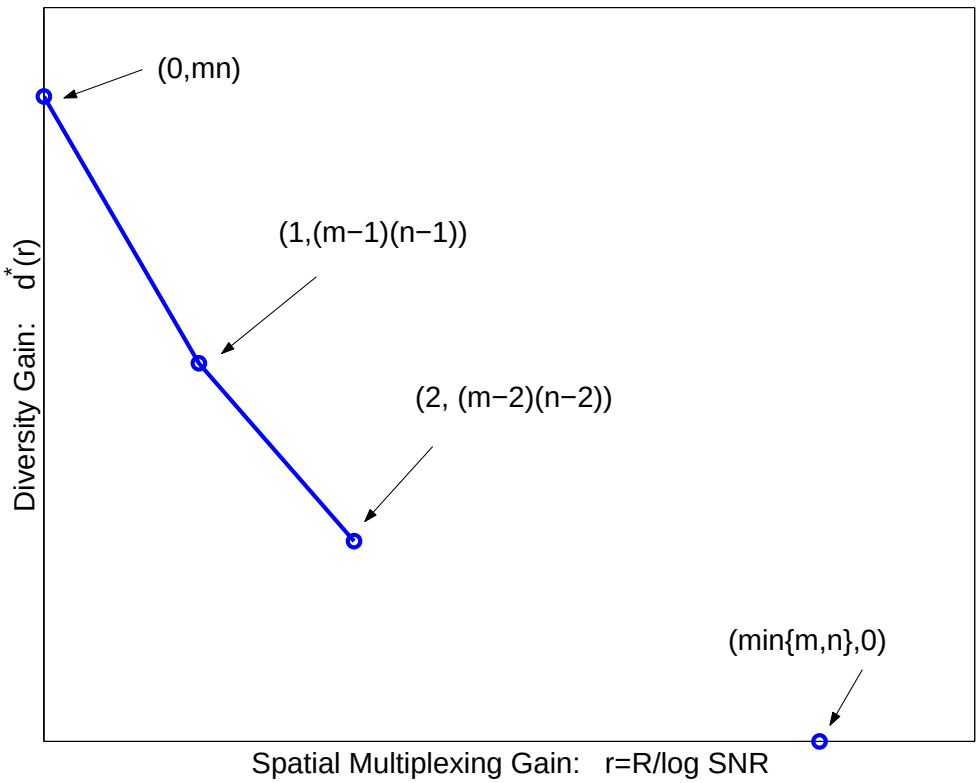
As long as block length $l \geq m + n - 1$:



Optimal D-M Tradeoff for General $m \times n$ Channel

(Zheng and Tse 03)

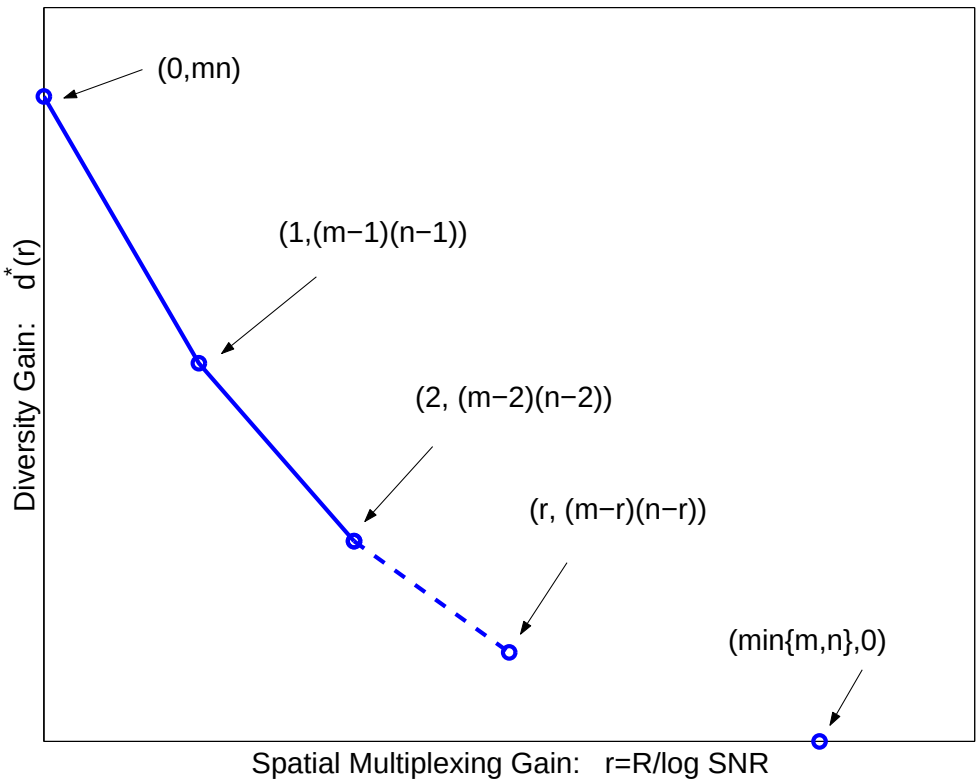
As long as block length $l \geq m + n - 1$:



Optimal D-M Tradeoff for General $m \times n$ Channel

(Zheng and Tse 03)

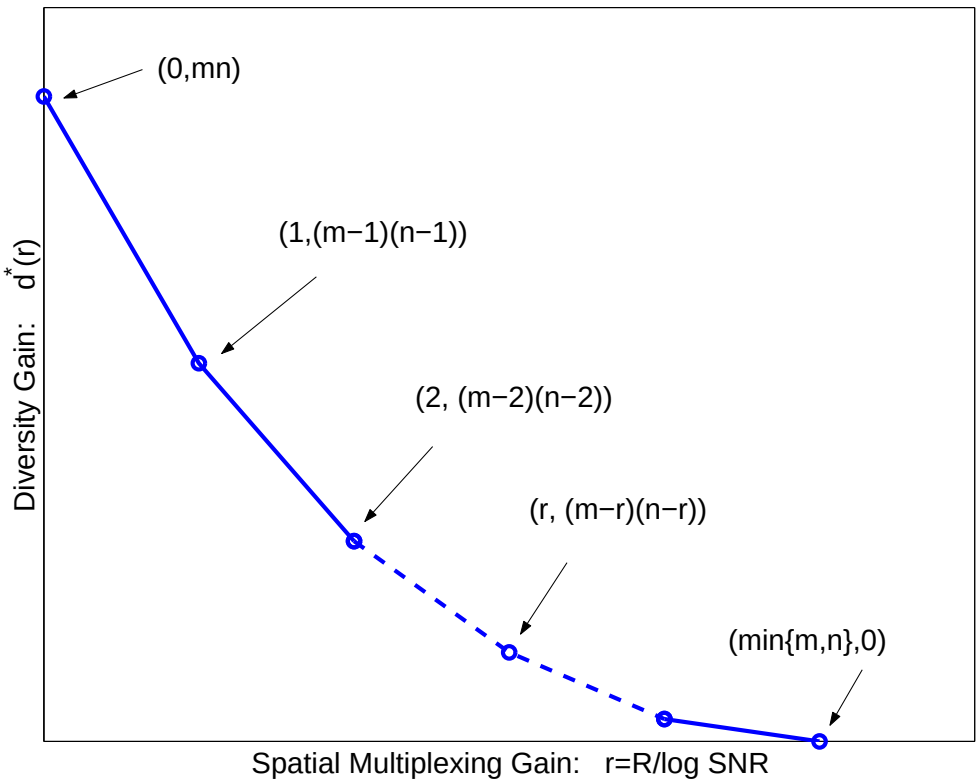
As long as block length $l \geq m + n - 1$:



Optimal D-M Tradeoff for General $m \times n$ Channel

(Zheng and Tse 03)

As long as block length $l \geq m + n - 1$:



For integer r, it is *as though* r transmit and r receive antennas were dedicated for multiplexing and the rest provide diversity.

Achieving Optimal Diversity-Multiplexing Tradeoff

- Hao and Wornell 03: MIMO rotation code (2×2 channel only).
- Tavildar and Viswanath 04: D-Blast plus permutation code.
- El Gamal, Caire and Damen 03: Lattice codes.

Hao and Wornell 03

Alamouti scheme:

$$\begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix}$$

Hao and Wornell's scheme:

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

where

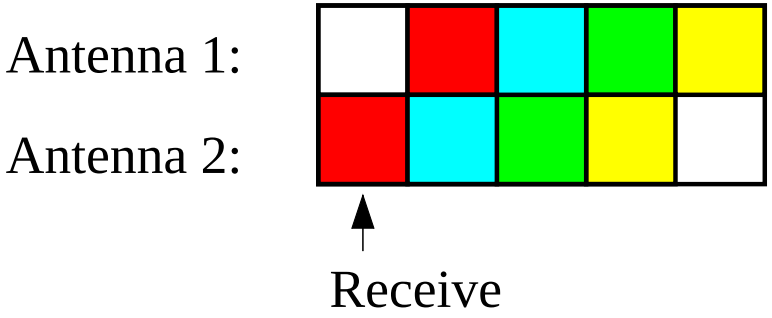
$$\begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \mathrm{Rotate}(\theta_1^*) \begin{bmatrix} u_1 \\ u_4 \end{bmatrix} \qquad \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \mathrm{Rotate}(\theta_2^*) \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}$$

and u_1, u_2, u_3, u_4 are independent QAM symbols.

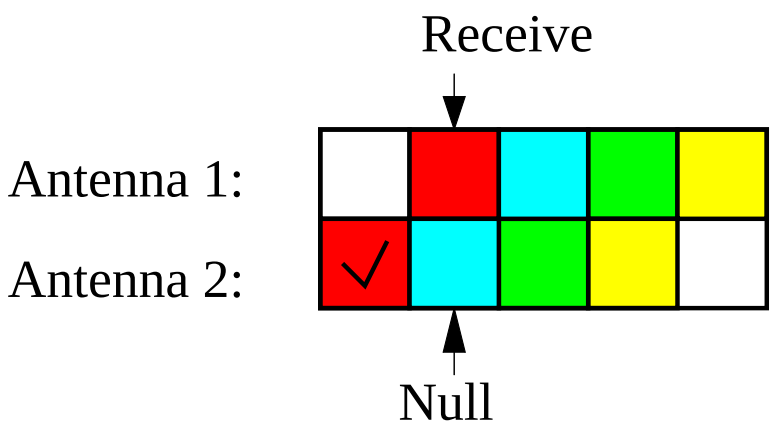
Tavildar and Viswanth 04

- First use D-Blast to convert the MIMO channel into a parallel channel.
- Then design permutation codes to achieve the optimal diversity-multiplexing tradeoff on the parallel channel.

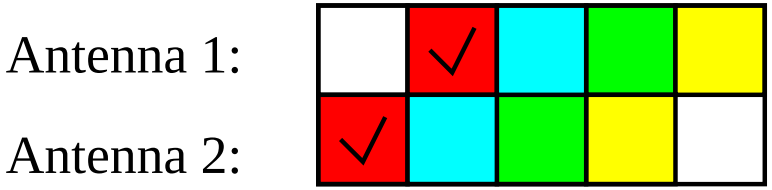
D-BLAST



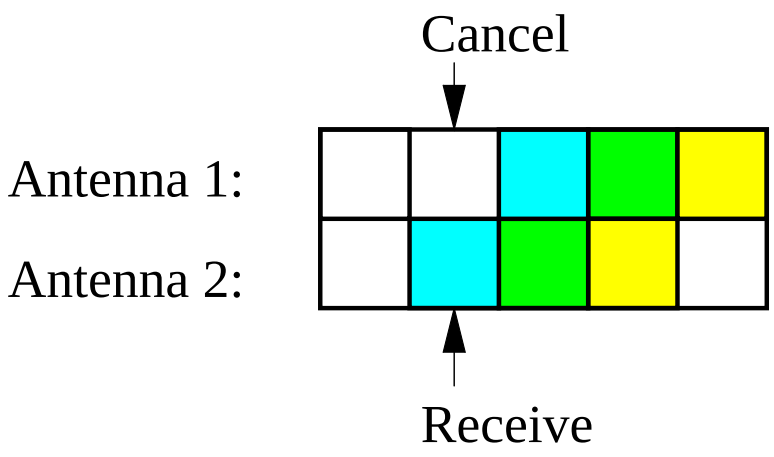
D-BLAST



D-BLAST

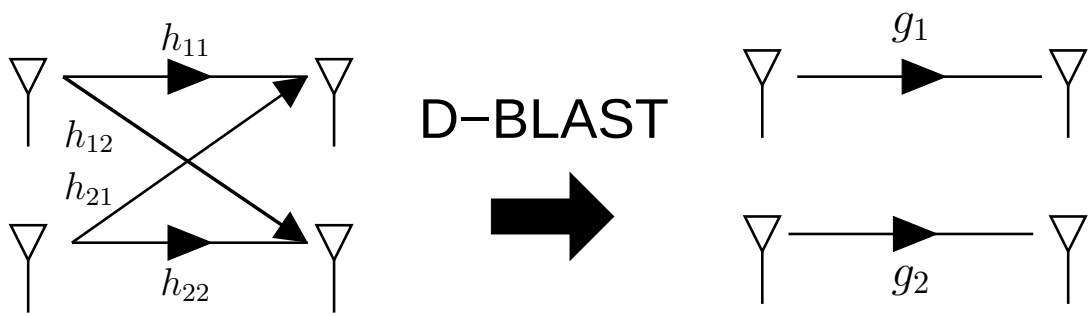


D-BLAST



Original D-Blast is sub-optimal.

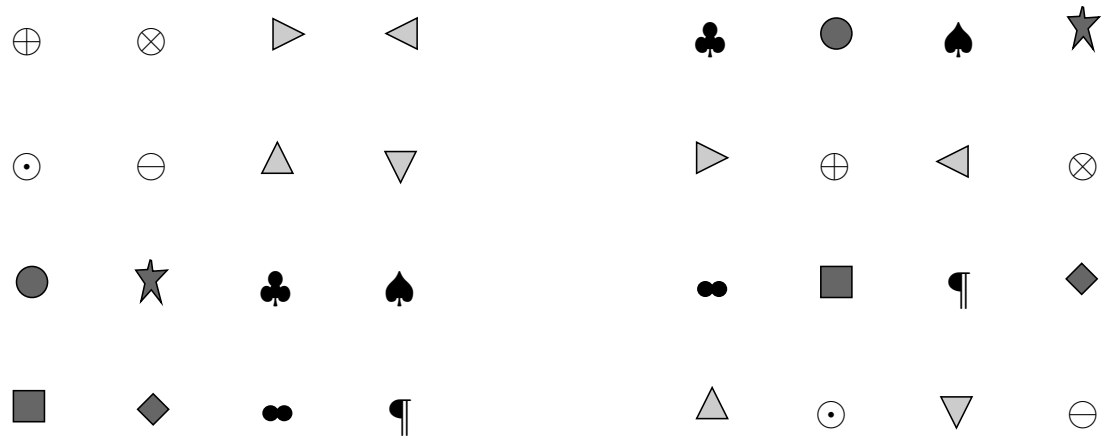
D-Blast with MMSE suppression is information lossless



Permutation Coding for Parallel Channel

The channel is parallel but the fading at the different sub-channels are correlated.

Nevertheless it is shown that the permutation codes can achieve the optimal diversity-multiplexing tradeoff of the parallel channel.



Conclusion

Diversity-multiplexing tradeoff is a unified way to look at space-time code design for MIMO channels.

It puts diversity and multiplexing on an equal footing.

It provides a framework to compare existing schemes as well as stimulates the design of new schemes.