On the throughput capacity of random wireless networks

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Abstract

We consider the problem of how throughput in a wireless network with randomly located nodes scales as the number of users n grows. We show that randomly scattered nodes can achieve the same $1/\sqrt{n}$ per-node transmission rate of arbitrarily located nodes. This contrasts with previous achievable results suggesting that a $1/\sqrt{n \log n}$ reduced rate is the price to pay for the additional randomness introduced into the system.

Our results rely on percolation theory arguments. When the node density is too high the network is fully connected but generates excessive interference. In the low density regime the network looses connectivity. Percolation theory ensures that a connected backbone forms in the transition region between these two extreme scenarios. This backbone does not include all the nodes, nevertheless it is sufficiently rich in crossing paths so that it can transport the total amount of traffic. By operating the network in this transition region between order and disorder, we are able to prove our tight bound.

1 Introduction

A completely wireless network consists of n nodes that communicate over a common wireless channel. A natural question that arises in such systems is how the throughput scales with the number n. Typically, there are two ways of letting n tend to infinity. One can either keep the area on which the network is deployed constant, and make the node density λ tend to infinity (*dense* networks); or one can keep the node density λ constant, and increase the area to infinity (*extended* networks). In both of these settings, network theoretic lower bounds on achievable transmission rates can be obtained constructively, for given communication strategies and power attenuation laws; while information theoretic upper bounds must be obtained allowing arbitrary communication strategies and assuming only the power decay law in the propagation medium.

Consider now the following question originally due to Broadbent and Hammersley to introduce percolation theory [2]. Water is poured on one side of a large (ideally infinite) porous stone. What is the probability that the water finds a path to the opposite side? By modeling the stone as a square grid in which each edge can be open and hence traversed by water with probability p, and closed otherwise, independently of all other edges, one can

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show that for $p > 1/2$ water *percolates* trough the stone with probability one. One can then ask at what rate the water percolates and how it depends on p. In other words, how rich in disjoint paths is the connected component of open edges?

In this paper, we relate this latter question to throughput scaling in random wireless networks. We provide a mapping such that the open grid edges of a percolation model correspond to the presence of wireless transmitters in certain locations of the plane, and the open percolating paths represent a wireless backbone that is used to multi-hop packets across the network. Accordingly, to maximize throughput, we want to operate at $p > 1/2$, above the percolation threshold, so that we can guarantee the existence of many paths that traverse the network, but also have $p < 1$, so to avoid overcrowding and excessive interference. We show that controlling the parameter p corresponds in the wireless network to scaling transmission ranges at a given rate as $n \to \infty$, and we find the optimal scaling law corresponding to $1/2 < p < 1$. Percolation theory [9, 14] has been proposed in the past to study connectivity of wireless networks [1, 3, 6, 8, 15], but to the best of our knowledge it has never been used to show capacity results. We believe that the connection that we establish in this paper can be exploited in the future to solve other information and network theoretic questions where spatial randomness plays a key role.

Throughput scaling in wireless networks has received considerable attention in the past few years [4, 5, 10, 11, 12, 13, 16, 17]. The first paper [10] to address this problem considered the dense network case, and a traffic scenario where each node generates packets for a destination non-vanishingly far away. Using a network theoretic approach based on multi-hop communication, it showed a lower bound on the per-node rate of $\Omega(1/\sqrt{n})$ bit/sec, if nodes are arbitrarily located; and a lower bound of $\Omega(1/\sqrt{n \log n})$ bit/sec if nodes are randomly located [1]. These results rely on point to point connections delivering higher power as nodes tend to be closer to each other. In practice, this holds only as long as near field effects can be neglected. When the physical constraint of bounded power in the near field is enforced, computed bounds reduce to $1/n$ [4].

For extended networks, near field effects do not play a fundamental role and the power received by every node can be bounded without affecting the final result. In this case, the work in [17] presents an information theoretic bound of $\Theta(1/\sqrt{n})$ bit/sec per node, for arbitrarily located nodes satisfying a minimum distance constraint, and a power attenuation function that exhibits a power law behavior with exponent $\alpha > 6$, or an exponential attenuation. When nodes are randomly located, the work in [13] shows an upper bound of $O(1/n^{1/2-1/\alpha})$ that holds for $\alpha > 2$.

Constructive strategies proposed for networks of randomly located nodes [5, 10, 12, 16] achieve only $\Omega(1/\sqrt{n \log n})$ per-node bit rate, somehow suggesting that at least a $\sqrt{\log n}$ factor is the price to pay for randomness. Instead, we show the contrary, namely that it is possible to achieve a per-node throughput capacity of $\Omega(1/\sqrt{n})$ in random networks. Our result holds in the general case of extended networks with bounded transmitted and received power, and signal attenuation function whose tail exhibits a power law behavior with exponent $\alpha > 2$, or an exponential attenuation (which is typical if there is absorption

[1] We use the following notation throughout the paper: $f = O(g)$ if $\limsup_{n\to\infty} \frac{f(n)}{g(n)} < +\infty$; $f = \Omega(f)$ if $g = O(f)$; $f = \Theta(g)$ if $f = O(g)$ and $g = O(f)$. Thus all $O(\cdot)$ results are upper bounds, $\Omega(\cdot)$ results are lower bounds and $\Theta(\cdot)$ results are sharp scaling estimates.

in the medium [7]). The result also holds for dense networks, as long as near field effects are negligible, i.e., in the same setting of [10]. Finally, the proposed routing achieves the optimal average delay required for a packet to reach its destination, as defined in [5].

The rest of the paper is organized as follows. In the next section we summarize our model and state our main results. In Section 3 we give an overview of our protocol and provide some intuition on why it achieves the optimal transmission rate. In Section 4 we formalize the correspondence with percolation theory and show how to construct the wireless backbone. Section 5 describes the three phases of the protocol and proves our main results. Finally, Section 7 concludes the paper.

2 Main results

We construct a random extended network by placing nodes according to a Poisson point process of unit intensity on the plane and focus our attention to the square $[0, \sqrt{n}] \times [0, \sqrt{n}]$. Similarly, we construct a dense network by placing nodes according to a Poisson point process of intensity n over a square of unit area. We are mainly concerned with events that occur inside these squares with high probability (w.h.p.), that is, with probability tending to one as $n \to \infty$. We denote the Euclidean distance between two nodes i and j by d_{ij}. We pick uniformly at random a matching of source-destination pairs, so that each node is the destination of exactly one source. We assume all nodes transmit at constant power P, and that node j receives the transmitted signal from node i with power $P\ell(i,j)$, where $\ell(i,j)$ indicates the path loss between i and j. In this paper we are concerned with lower bounds on achievable rates, hence, we assume a model of multi-hop communication such that two nodes can establish a direct wireless link of capacity

$$C(i \to j) = \frac{1}{2} \log \left(1 + \frac{P\ell(i,j)}{N_0 + \sum_{k \neq i} P\ell(k,j)} \right) \text{ bps/Hz},$$

where N_0 is the ambient noise power at the receiver.

The per-node throughput capacity $T(n)$ of the network is defined as the number of bits per second that every node can transmit w.h.p. to its destination.

The per-packet delay $D(n)$ is the sum of the times the packet spends at each relay node. When we measure this, we scale the packet size by $T(n)$. In this way the transmission delay at each node is constant and the per-packet delay corresponds to the number of hops needed to reach its destination. In practice, thanks to scaling, $D(n)$ captures the dynamics of network and not of the transmission delay, see also [5].

Our results are the following.

Theorem 1 *Assuming a power attenuation function of the type* $\ell(i,j) = \min\{1, e^{-\gamma d_{ij}}/d_{ij}^{\alpha}\}$ *with* $\alpha > 0, \gamma > 0$ *or* $\alpha > 2, \gamma = 0$*, a per-node throughput capacity of*

$$T(n) = \Omega(1/\sqrt{n}) \text{ bit/sec}$$

is achievable in a random extended network, with a corresponding average packet delay at most

$$E(D(n)) = O(\sqrt{n}).$$

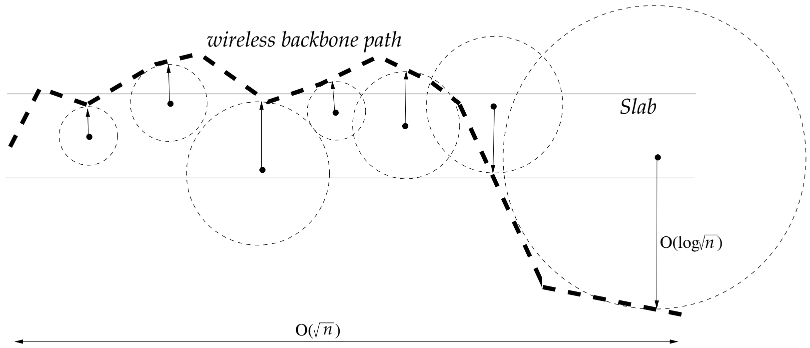

Figure 1: Nodes inside a slab of constant width access a path of the wireless backbone in single hops of length at most proportional to $\log \sqrt{n}$. Packets are carried across a distance $\sqrt{n}$ along the wireless backbone in constant-length hops.

Theorem 2 *Assuming a power attenuation function of the type* $\ell(i,j) = e^{-\gamma d_{ij}}/d_{ij}^{\alpha}$ *with* $\alpha > 0, \gamma > 0$, *or* $\alpha > 2, \gamma = 0$, *a per-node throughput capacity of*

$$T(n) = \Omega(1/\sqrt{n}) \textit{ bit/sec}$$

is achievable in a random dense network, with a corresponding average packet delay at most

$$E(D(n)) = O(\sqrt{n}).$$

Note that the result of Theorem 2 relies on an ideal power attenuation function that is singular at the origin. In practice, this result holds as long as near field effects are neglected and it is presented for completeness, as it matches the dense network model of [10]. The per-node throughput in dense networks and in presence of bounded attenuation function cannot scale better than $1/n$, [4].

3 Overview of the solution

The main idea of our proposed solution is to have a wireless backbone that carries packets across the network at constant rate, using short hops, and to drain the rest of the traffic to the wireless backbone using single hops of longer length. See Fig. 1 for a schematic representation.

We now wish to provide some intuition on how this strategy works. The wireless backbone consists of paths of constant length hops. Each hop generates an interference footprint proportional to the square of its length, and there are a constant number of nodes that fall within this footprint. This allows to have a constant transmission rate along the path. Each path, however, needs to relay packets coming from other nodes that access the backbone in single hops. This traffic is at most proportional to $\sqrt{n}$, if we associate to each path only nodes that are within a slab of constant width that crosses the network area, see Fig. 1. Hence, the per-node throughput on the backbone can be of order $1/\sqrt{n}$. Now, let us look at the throughput of the nodes that access the wireless backbone in single hops. We will show that these single hops are of length at most proportional to $\log \sqrt{n}$ and can sustain a rate higher than $1/\sqrt{n}$. Furthermore, there is no relay burden for nodes accessing the backbone in single hops. It follows that the bottleneck is represented by the nodes on the backbone that transmit at a rate of $1/\sqrt{n}$.

There are three key points in this reasoning: i) there exist paths of constant hop length that cross the entire network forming the wireless backbone, ii) these paths can be put into a one to one correspondence with $\sqrt{n}$ slabs of constant width, each containing at most a constant times $\sqrt{n}$ number of nodes, and iii) these paths are somehow regularly spaced so that there is always one within a $\log \sqrt{n}$ distance factor from any node in the network.

In the following, Theorem 3 ensures the existence of many paths using percolation theory arguments. Theorem 4 shows that each path in the wireless backbone can transport packets at a constant rate, and that packets can be drained to the backbone at a rate higher than $\sqrt{n}$. Finally, Lemmas 3 and 4 are needed to bound the number of nodes that access any given path.

4 Percolation results

In this section we establish the percolation results that are needed to show existence of a cluster of nodes forming the wireless backbone. The objective is to formally construct a mesh of paths that can simultaneously carry information across the network at a constant rate, independent of the number of nodes n. We call this mesh the *highway system*, and will use it to carry packets over most of the distance.

To begin our construction, we divide the area into squares of constant side length c, as depicted in the left-hand of Fig. 2. By adjusting c, we can adjust the probability that a square contains at least one point:

$$P[\text{a square contains at least one point}] = 1 - e^{-c^2} := p. \tag{1}$$

We say that a square is *open* if it contains at least one point, and *closed* otherwise; note that the status of the squares is i.i.d.

We now map our construction to a bond percolation model. We draw an horizontal edge across half of the squares, and a vertical edge across the others, as shown on the right-hand side of Fig. 2. In this way we obtain a grid of horizontal and vertical edges, each edge being open, independently of all other edges, with probability p. We call a path *open* (resp. *closed*) if it contains only open (resp. closed) edges. Note that, for c large enough, our construction produces winding open paths that cross the network area, see Fig. 3. Next, we turn to the question of how many of these paths there are.

4.1 Number of disjoint crossing paths

Let us divide the network area into horizontal rectangles $\overline{R}_n$, of size $\sqrt{n} \times \sqrt{2}c\kappa \log \frac{\sqrt{n}}{\sqrt{2}c}$, for some constant $\kappa > 0$, see Fig. 4. Each of these rectangles has thus lattice size $m \times \kappa \log m$ in the bond percolation model, with $m = \frac{\sqrt{n}}{\sqrt{2}c}$ (as the edges have length $\sqrt{2}c$). We want to show that there exist many disjoint open paths from left to right inside such rectangles.

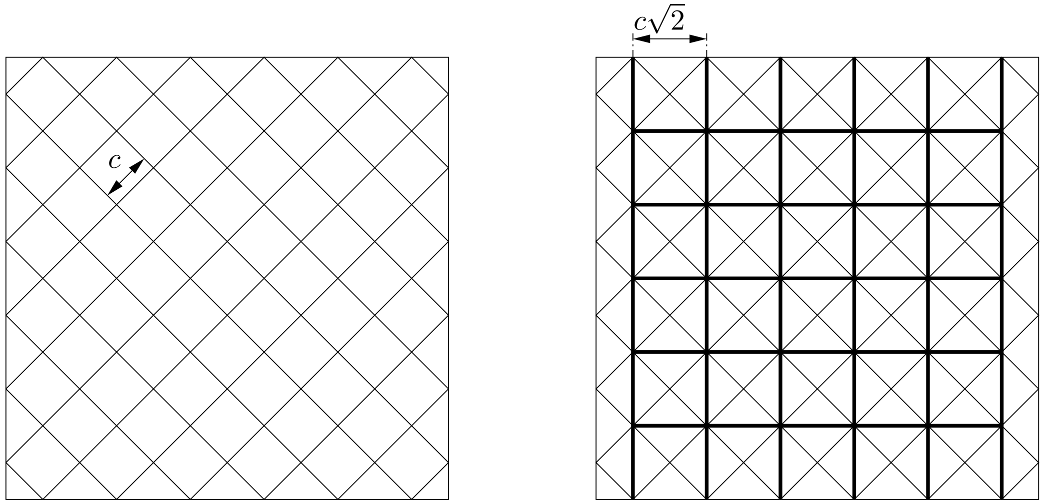

Figure 2: Construction of the bond percolation model. We declare each square on the left-hand side of the picture open, if there is at least a Poisson point inside it, closed otherwise. This corresponds to associate an edge to each square, traversing it diagonally, as depicted on the right-hand side of the figure, and declare the edge either open or closed according to the state of the corresponding square.

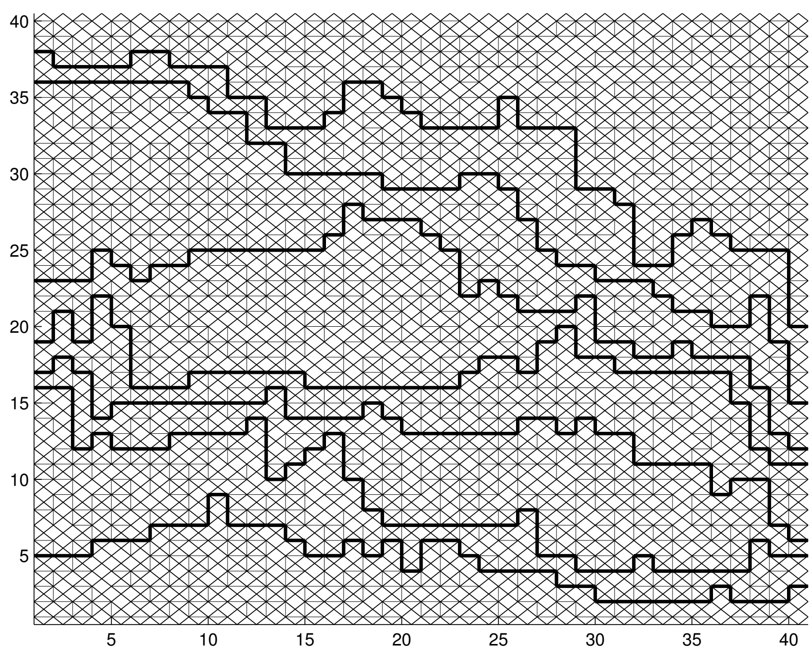

Figure 3: Horizontal paths in a 40×40 bond percolation model obtained by computer simulation. Each square is traversed by an open edge with probability p ($p = 0.7$ here). Closed edges are not depicted. We find seven disjoint open path crossing the area from left to right.

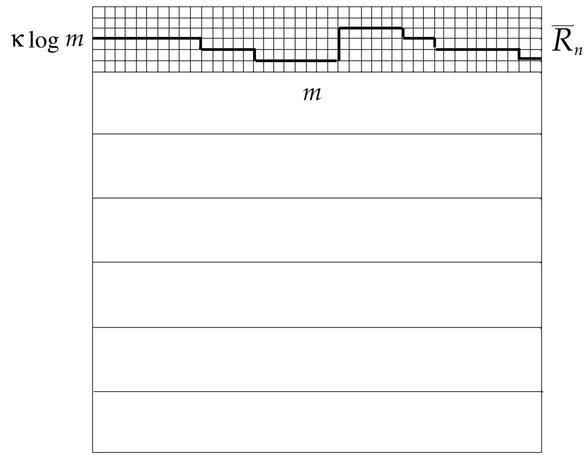

Figure 4: The network area is divided into $m/\kappa \log m$ horizontal rectangles of lattice size $m \times \kappa \log m$. A left to right crossing of rectangle $\overline{R}_n$ is shown.

Theorem 3 *For any constant $\kappa > 0$ and if c is so large that*

$$c^2 > \log 6 + \frac{2}{\kappa}, \tag{2}$$

then there exists a strictly positive constant $\beta = \beta(c, \kappa)$ such that w.h.p. there exist $\beta\kappa \log m = \beta\kappa \log \frac{\sqrt{n}}{\sqrt{2}c}$ disjoint open paths inside each rectangle $\overline{R}_n$, that cross it from left to right.

In order to prove this theorem we need some preliminary results expressed by the lemmas below. We denote by P_p the product measure with open edge density p.

Lemma 1 *Let S_n be a square lattice of size $n \times n$ and $0 < p \leq 1$. The probability that there exists an open path from the center 0 of S_n to its boundary ∂S_n is upper bounded by*

$$P_p(0 \leftrightarrow \partial S_n) \leq \frac{4}{3}(3p)^n.$$

Proof: If a path starts at 0 and touches ∂S_n, its length is at least n. Thus, denoting by $N(n)$ the number of open paths of length n starting at the origin:

$$P_p(0 \leftrightarrow \partial S_n) \leq P_p(N(n) \geq 1).$$

As paths of length n are open with probability p^n, we can bound this probability by

$$P_p(N(n) \geq 1) \leq p^n \sigma(n),$$

where $\sigma(n)$ denotes the number of paths of length n starting at the origin. This number is obviously not larger than

$$\sigma(n) \leq 4 \cdot 3^{n-1}.$$

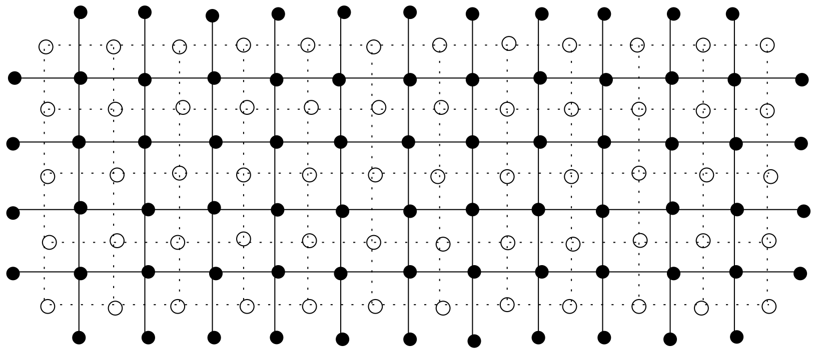

Figure 5: A picture of $\overline{R}_n$ (solid line) and its dual graph (dotted line) . Note that if there is no open path traversing the rectangle from left to right, then there is a closed path in the dual graph traversing it from top to bottom.

Combining these three inequalities, we obtain

$$P_p(0 \leftrightarrow \partial S_n) \leq 4 \cdot 3^{n-1} p^n = \frac{4}{3}(3p)^n.$$

□

The next well known results follows directly from Theorem 2.45 in [9] and the remarks thereafter.

Lemma 2 *Let R_n be a rectangle embedded in the square lattice. Let A_n be the event that there exists an open path between the left and right sides of R_n and $I_r(A_n)$ the event that there exist r edge-disjoint such crossings. We have*

$$1 - P_p(I_r(A_n)) \leq \left(\frac{p}{p-p'}\right)^r [1 - P_{p'}(A_n)]$$

for any $0 \leq p' < p \leq 1$.

Proof of Theorem 3: We consider bond percolation in the rectangle $\overline{R}_n$, with each edge having probability p' to be open, independently of all other edges. Let the *dual graph* of $\overline{R}_n$ be obtained by placing a vertex in each square of the percolation lattice, and joining two such vertices by an edge whenever the corresponding squares share a side, see Fig. 5. An edge of the dual is open if it crosses an open edge of the original lattice, it is closed otherwise. Let A_n be the event of having at least one open path inside $\overline{R}_n$ that crosses it from left to right, and let B_n be the event that a closed path crosses the rectangle vertically in the dual lattice. We have $A_n \bigcap B_n = \emptyset$, because if both A_n and B_n occur, then there must be an intersection between an open edge of $\overline{R}_n$ and a closed edge of its dual, which is impossible. Moreover, whenever A_n does not occur then B_n occurs (one can be convinced of this again by looking at Fig. 5). It follows that A_n and B_n are disjoint events which partition the sample space, and hence $P_{p'}(A_n) + P_{p'}(B_n) = 1$.

We index the nodes at the base of the dual graph by index i, and denote by $i \leftrightarrow \Box\overline{R}_n$ the existence of a closed path in the dual from node i to the opposite side $\Box\overline{R}_n$ at the top of the dual graph. As $\overline{R}_n$ has width $\kappa \log m$, we have for any $0 \le q \le 1$:

$$P_q(i \leftrightarrow \Box\overline{R}_n) \le P_q(0 \leftrightarrow \partial S_{\kappa \log m}).$$

Therefore, as edges are closed with probability $1-p'$,

$$\begin{aligned} P_{p'}(B_n) &\le \sum_{i=1}^{m} P_{1-p'}(i \leftrightarrow \Box\overline{R}_n) \\ &\le \sum_{i=1}^{m} P_{1-p'}(0 \leftrightarrow \partial S_{\kappa \log m}) \\ &\le \frac{4m}{3}(3(1-p'))^{\kappa \log m} \end{aligned}$$

where the first inequality is a union bound, and the third inequality follows from Lemma 1.

Now we look at the event that $\beta\kappa \log m$ disjoint paths exist. We apply Lemma 2 to rectangle $\overline{R}_n$ with $r = \beta\kappa \log m$, obtaining

$$1 - P_p(I_{\beta\kappa \log m}(A_n)) \le \left(\frac{p}{p-p'}\right)^{\beta\kappa \log m} P_{p'}(B_n), \tag{3}$$

for any $p' < p$. Let us choose $p' = 2p - 1$. We have thus

$$\frac{p}{p-p'} = e^{c^2} - 1 < e^{c^2},$$

and

$$1 - p' = 2(1-p) = 2e^{-c^2}.$$

Hence, Equation (3) becomes

$$\begin{aligned} 1 - P_p(I_{\beta\kappa \log m}(A_n)) &\le \left(e^{c^2}\right)^{\beta\kappa \log m} P_{p'}(B_n) \\ &\le m^{\beta\kappa c^2} \frac{4m}{3}(6e^{-c^2})^{\kappa \log m} \\ &= m^{\beta\kappa c^2} \frac{4m}{3} m^{-\kappa c^2 + \kappa \log 6} \\ &= \frac{4}{3} m^{(\beta-1)\kappa c^2 + \kappa \log 6 + 1}. \end{aligned}$$

The probability to find at least $\beta \log m$ paths in $\overline{R}_n$ is thus

$$P_p(I_{\beta\kappa \log m}(A_n)) \ge 1 - \frac{4}{3} m^{(\beta-1)\kappa c^2 + \kappa \log 6 + 1}.$$

As this happens independently in each of the $\frac{m}{\kappa \log m}$ rectangles, the probability of having $\beta\kappa \log m$ disjoint paths in each rectangle is

$$P_p(I_{\beta\kappa \log m}(A_n))^{\frac{m}{\kappa \log m}} \ge \left(1 - \frac{4}{3} m^{(\beta-1)\kappa c^2 + \kappa \log 6 + 1}\right)^{\frac{m}{\kappa \log m}}.$$

Finally note that if $(\beta - 1)\kappa c^2 + \kappa \log 6 + 1 \le -1$, the above expression tends to 1 when m goes to infinity. Thus, if $c^2 > \log 6 + 2/\kappa$, one can choose

$$\beta(c, \kappa) = 1 - \frac{\kappa \log 6 + 2}{\kappa c^2} > 0$$

such that the above condition is fulfilled. □

Joining all the rectangles together, we obtain $\beta\sqrt{n}$ paths in the whole network. The same is true of course if we divide the area into vertical rectangles and look for paths crossing the area from bottom to top. Using a simple union bound argument, we conclude that there exist $\beta\sqrt{n}$ horizontal and $\beta\sqrt{n}$ vertical disjoint paths simultaneously with high probability. These paths form a grid, that we call the *highway system*.

4.2 Capacity of the percolation cluster

Along the paths of the highway system, we choose one node per edge, that relays the packets. This is possible as the paths are formed by open edges, which are associated to non-empty squares. The paths are thus made of a chain of nodes such that the distance between two consecutive nodes is at most $2\sqrt{2}c$.

To actually transport packets along the paths, we set up a TDMA scheme. When a node transmits, other nodes that are sufficiently far away can simultaneously transmit, without causing excessive interference. Theorem 4 makes this precise, ensuring that a constant rate R, independent of n, can be achieved on all the paths simultaneously as $n \to \infty$. Note that this theorem gives a more general result, that will be useful also in Section 5.

Theorem 4 *For any given integer $d > 0$ there exists a TDMA scheduling, such that one node per square can transmit to any destination located within a radius of d squares (in Manhattan distance) with fixed rate $R(d)$ independent of n.*

When d goes to infinity, the asymptotic behavior of the rate is given by

$$R(d) = \Omega\left(d^{-\alpha-2}e^{-\gamma c d}\right).$$

Proof: We take a coordinate system, and label each square with two integer coordinates (in our construction, the axis of the coordinate system are diagonal). Then we take an integer k, and consider the subset of squares whose two coordinates are a multiple of k (see left-hand side of Figure 6). By translation, we can construct k^2 disjoint equivalent subsets. This allows us to build the following TDMA scheme: we define k^2 time slots, during which only nodes from a particular subset are allowed to emit. We assume also that at most one node per square emits at the same time, and that they all emit with the same power P.

Let us consider one particular square. We suppose that the emitter in this square emits towards a destination located in a square at distance at most d. We compute the signal-to-interference ratio at the receiver. First, we choose the number of time slots k^2 as follows:

$$k = 2(d + 1)$$

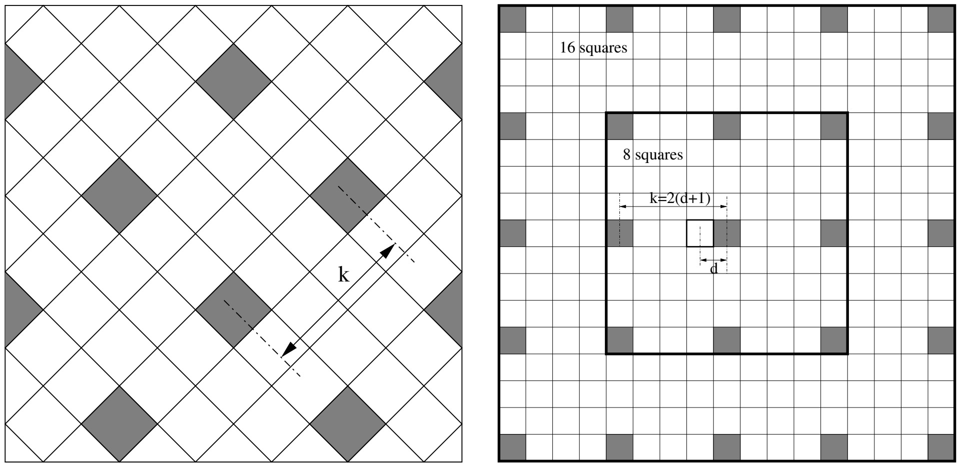

Figure 6: Left hand-side: A subset of squares. Right-hand side: Construction of the lower bound on the interference term.

To find an upper bound to the interferences, we observe that with this choice, the emitters in the 8 first closest squares are located at a distance at least $d+2$ (in squares) from the receiver (see right-hand side of Figure 6). This means that the Euclidean distance between the receiver and the 8 closest interferers is at least $c(d+1)$. The 16 next closest squares are at distance at least $3d+4$ (in squares), and the Euclidean distance between the receiver and the 16 next interferers is therefore at least $c(3d+3)$, and so on. The sum of the interferences $I(d)$ can be bounded as follows:

$$\begin{aligned} I(d) &\leq \sum_{i=1}^{\infty} 8i\, Pl(c(2i-1)(d+1)) \\ &= \sum_{i=1}^{\infty} 8i\, P \min\{1, [c(2i-1)(d+1)]^{-\alpha}\} e^{-\gamma c(2i-1)(d+1)} \\ &\leq \sum_{i=1}^{\infty} 8i\, P [c(2i-1)(d+1)]^{-\alpha} e^{-\gamma c(2i-1)(d+1)} \\ &= P[c(d+1)]^{-\alpha} e^{-\gamma c(d+1)} \sum_{i=1}^{\infty} 8i\, (2i-1)^{-\alpha} e^{-\gamma c(d+1)(2i-2)}. \end{aligned}$$

The sum in $I(d)$ clearly converges if $\alpha > 2$ or $\gamma > 0$.

Now we want to bound from below the signal received from the emitter. We observe first that the distance between the emitter and the receiver is at most

$$\sqrt{(cd)^2 + c^2} \leq c(d+1).$$

The strength $S(d)$ of the signal at the receiver can be thus bounded by

$$\begin{aligned} S(d) &\geq Pl(c(d+1)) \\ &= P \min\{1, [c(d+1)]^{-\alpha}\} e^{-\gamma c(d+1)}. \end{aligned}$$

Finally, we obtain a bound on the signal-to-interference ratio

$$\begin{aligned} SNIR(d) &= \frac{S(d)}{N_0 + I(d)} \\ &\geq \frac{P \min\{1, [c(d+1)]^{-\alpha}\} e^{-\gamma c(d+1)}}{N_0 + P[c(d+1)]^{-\alpha} e^{-\gamma c(d+1)} \sum_{i=1}^{\infty} 8i\,(2i-1)^{-\alpha} e^{-\gamma c(d+1)(2i-2)}} \end{aligned}$$

As the above expression does not depend on n, the first part of the theorem is proven.

We now look at the asymptotic behavior of the SNIR for large d. If $c(d+1) \geq 1$, we can remove the minimum and write

$$\begin{aligned} SNIR(d) &\geq \frac{P[c(d+1)]^{-\alpha} e^{-\gamma c(d+1)}}{N_0 + P[c(d+1)]^{-\alpha} e^{-\gamma c(d+1)} \sum_{i=1}^{\infty} 8i\,(2i-1)^{-\alpha} e^{-\gamma c(d+1)(2i-2)}} \\ &= \frac{1}{N_0[c(d+1)]^{\alpha} e^{\gamma c(d+1)}/P + \sum_{i=1}^{\infty} 8i\,(2i-1)^{-\alpha} e^{-\gamma c(d+1)(2i-2)}} \end{aligned}$$

The second term in the denominator clearly decreases when d goes to infinity. The first term grows like $d^{\alpha} e^{\gamma cd}$. The whole fraction therefore decreases like $1/d^{\alpha} e^{\gamma cd}$. The throughput on each link is given by $\log(1 + SNIR(d))$, and therefore also decreases like $1/d^{\alpha} e^{\gamma cd}$.

Now we have to divide this throughput by the number of time slots k^2 used in the TDMA scheme. As $k = 2(d+1)$, the number of time slots increases like d^2. So, finally, the actual throughput available in each square decreases like $d^{-\alpha-2} e^{-\gamma cd}$. □

Corollary 1 *For any given integer $d > 0$ there exists a TDMA scheduling, such that one node per square can receive from any emitter located within a radius of d squares with fixed rate $R(d)$ independent of n.*

When d goes to infinity, the asymptotic behavior of the rate is given by

$$R(d) = \Omega\left(d^{-\alpha-2} e^{-\gamma cd}\right).$$

Proof: This result is obtained by switching the role of emitters and receivers in the above proof. Distances remain the same, and all equations still hold. □

5 Protocol

In this section, we describe the actual routing protocol and show that it achieves the desired $\Omega(\sqrt{n})$ throughput capacity.

The protocol uses 4 separate time slots: a first one for draining packets to the highway, a second one to transport packets on the "horizontal" highways connecting the left and right edges of the domain, a third one to transport packets on the "vertical" highways connecting the top and bottom edges of the domain, and a fourth one to deliver packets to the destination. The draining and delivery phases use direct transmission, while the highway phases use multiple hops. We show that the throughput bottleneck is in the highway phase that can sustain a rate per node proportional to $1/\sqrt{n}$ bit per second.

We start by proving two simple lemmas that will be useful for the capacity calculation. They are followed by three propositions that prove our main result.

Lemma 3 *Divide the network area in $\sqrt{n}/c$ boxes of side length c. The probability that there are less than $c^2 \log n$ nodes in each box tends to one when n goes to infinity.*

Proof: The number of nodes in each square is a Poisson random variable of parameter c^2. Let us denote one of these variables by X. Chernoff's inequality implies that

$$P(X > c^2 \log n) \leq e^{-sc^2 \log n} E[e^{sX}],$$

for any $s > 0$. We choose here $s = 2/c^2$ and obtain

$$P(X \leq c^2 \log n) \geq 1 - n^{-2} e^{c^2(e^{2/c^2}-1)}.$$

As the numbers of nodes in all of the n/c^2 squares are i.i.d, we have

$$P(X \leq c^2 \log n)^{\frac{n}{c^2}} \geq \left(1 - \frac{e^{c^2(e^{2/c^2}-1)}}{n^2}\right)^{\frac{n}{c^2}}.$$

The latter expression tends to one when n goes to infinity. $\square$

Lemma 4 *Divide the network into horizontal slabs of constant width $\sqrt{2}c/\beta$. The probability that each slab contains less than $2c\sqrt{2n}/\beta$ nodes tends to one when n goes to infinity.*

Proof: The number of nodes in the i-th slab is a Poisson random variable of parameter $c\sqrt{2n}/\beta$, that we denote by N_i here. We apply Chernoff's inequality:

$$\begin{aligned} P(N_i > 2\frac{c\sqrt{2n}}{\beta}) &\leq e^{-2\frac{c\sqrt{2n}}{\beta}s} E[e^{sN_i}] \\ &= e^{-2\frac{c\sqrt{2n}}{\beta}s} e^{c\sqrt{2n}/\beta(e^s-1)} \\ &= e^{\frac{c\sqrt{2n}}{\beta}(e^s-2s-1)}. \end{aligned}$$

Thus, if we take $s = 1$,

$$P(N_i > 2\frac{c\sqrt{2n}}{\beta}) \leq e^{\frac{c\sqrt{2n}}{\beta}(e-3)}.$$

The probability that each of the $\beta\sqrt{n}/c\sqrt{2}$ slabs contains less than $2c\sqrt{2n}/\beta$ nodes is thus

$$P(N_i \leq 2c\sqrt{2n}/\beta,\ \forall i) \geq \left(1 - e^{\frac{c\sqrt{2n}}{\beta}(e-3)}\right)^{\frac{\beta\sqrt{n}}{c\sqrt{2}}}.$$

The latter expression tends to one when n goes to infinity. $\square$

Proposition 1 (Draining Phase) *Each node can transmit packets to the highway system with rate of order*

$$\Omega\left((\log \sqrt{n})^{-\alpha-2}(\log n)^{-1} n^{-c\kappa\gamma/2}\right),$$

for any constant $\kappa > 0$ and if c is so large that (2) holds.

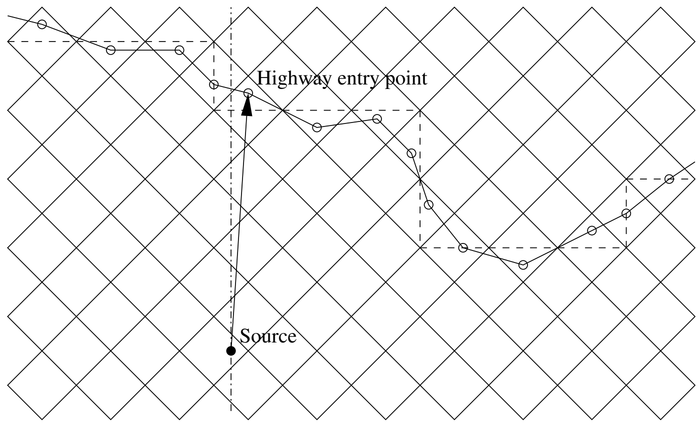

Figure 7: Draining of packets to the highways

Proof: We consider carrying packets from sources to the highways. We start by dividing the network area into $\beta\sqrt{n}/c\sqrt{2}$ horizontal slabs. As there are exactly as many slabs as horizontal edge-disjoint paths, we can impose that nodes from the i-th slab send their packets to the i-th horizontal path. Note that each path may not be contained in its corresponding slab, but it may deviate from it (recall Fig. 1). However, Theorem 3 bounds the amount of deviation.

More precisely, to each source in the i-th slab, we assign an *entry point* on the i-th horizontal path. The entry point is defined as the node on the horizontal path closest to the vertical line drawn from the source point, see Fig. 7. The source then transmits its packet to the entry point in a single hop. Theorem 3 ensures that the distance between sources and entry points is never larger than $\kappa \log m = \kappa \log \frac{\sqrt{n}}{\sqrt{2}c}$ squares. This is because each rectangle $\overline{R}_n$ of width $\kappa \log m$ contains $\beta\kappa \log m$ paths, and therefore each source finds its highway within the same rectangle.

To compute the rate at which nodes can send their packets to entry points, we let $d = \kappa \log \frac{\sqrt{n}}{\sqrt{2}c}$ in Theorem 4. We obtain that one node per square can send packets to its entry point with rate

$$\begin{aligned} R(\kappa \log \frac{\sqrt{n}}{\sqrt{2}c}) &= \Omega\left((\kappa \log \frac{\sqrt{n}}{\sqrt{2}c})^{-\alpha-2} e^{-\gamma c \kappa \log \frac{\sqrt{n}}{\sqrt{2}c}} \right) \\ &= \Omega\left((\log \sqrt{n})^{-\alpha-2} e^{-c\kappa\gamma \log \sqrt{n}} \right) \\ &= \Omega\left((\log \sqrt{n})^{-\alpha-2} n^{-c\kappa\gamma/2} \right). \end{aligned}$$

But as there are possibly many nodes in the squares, they have to share this bandwidth.

Using Lemma 3, we can conclude that the transmit rate of each node in the draining phase of our protocol is at least $R(d)/c^2 \log n$, which concludes the proof. □

Proposition 2 (Highway Phase) *Along the highway packets can be relayed at rate at least $\beta/2k^2\sqrt{n}$ bits/sec per-node.*

Proof: We now compute the rate that can be sustained during each highway phase of our protocol. Each node generates packets at constant rate W and we must carry these packets towards the point on the vertical highway appropriate for delivery.

We divide horizontal and vertical traffic, adopting the following simple routing policy: packets are carried along horizontal highways until they reach the crossing with their target vertical highway. Then, they are carried along vertical highways until they reach the appropriate point for delivery.

We start considering the horizontal traffic. We consider a node sitting on the i-th horizontal highway, and compute the traffic that goes through it. Actually, a packet will travel through this node if it was generated in the i-th slab, and has a destination on the other side of the node. So, at most, our node will relay all the traffic generated in the i-th slab.

According to Lemma 4, a node on a horizontal highway must therefore relay at most $2Wc\sqrt{2n}/\beta$ bits per second. As the maximal distance between hops is constant ($2\sqrt{2}c$), the throughput along highways is independent of n (see Section 4.2), one can set the rate per node to $W = \Omega(1/\sqrt{n})$ without overloading links, with high probability.

The problem for vertical traffic is the dual of the previous one. We can use the same arguments, except that W now describes the *receiving rate* of the nodes. Since each node is the destination of exactly one source, the throughput per node becomes the same as above. □

Proposition 3 (Delivery Phase) *Each destination node can receive packets from the highway at rate*

$$\Omega\left((\log\sqrt{n})^{-\alpha-2}(\log n)^{-1}n^{-c\kappa\gamma/2}\right),$$

for any constant $\kappa > 0$ and c so large that (2) holds.

Proof: The delivery phase consists in bringing the packets from the highway system to the actual destination. We proceed exactly in the same way as in Proposition 1, but in the other direction (horizontal delivery from the vertical highways).

We divide the network area into $\beta\sqrt{n}/c\sqrt{2}$ vertical slabs, and define a one-to-one mapping between slabs and vertical paths. We assume that packets have been transported by the highway system to their *exit point*, which is defined as the node of the vertical path closest to the horizontal line drawn from the destination. Again, the distance between exit points and destination is at most $\kappa \log \frac{\sqrt{n}}{\sqrt{2}c}$ squares. We can thus let $d = \kappa \log \frac{\sqrt{n}}{\sqrt{2}c}$ in Corollary 1, and conclude that each square can be served with rate $R(d) = \Omega\left((\log\sqrt{n})^{-\alpha-2}n^{-c\kappa\gamma/2}\right)$. As there are at most $c^2 \log n$ in each square (Lemma 3), the throughput per node is at least equal to $R(d)/c^2 \log n$. □

We are now ready to complete the proof of Theorem 1.

Proof of Theorem 1: We observe in Propositions 1 and 3 that if $c\kappa\gamma/2 < 1/2$, the asymptotic throughput per node decreases slowlier than $1/\sqrt{n}$. In this case, the overall throughput of the protocol is limited by the highway phase only, and the first part of Theorem 1 immediately follows from Proposition 2. We thus have to make sure that we can choose values of c and κ such that Inequality (2) is verified, and such that

$$\frac{c\kappa\gamma}{2} < \frac{1}{2}.$$

A possible choice is $c > 2\gamma + \sqrt{4\gamma^2 + \log 6}$ and $\kappa = \frac{1}{2c\gamma}$. We remark that these choices depend on the physical attenuation factor γ.

As for the second part of the theorem, we note that since we route packets along wandering paths of the highways, it may seem possible to have a delay higher than the optimal $nT(n) = \sqrt{n}$, which is achieved by straight line routes [5]. However, a simple counting argument can show the opposite.

First, we can bound the average number of hops of a packet in the network. This is given by the number of hops on the highway, which is at most twice the number of hops of a crossing path in the rectangle $\overline{R}_n$. As there are $\beta \log m$ disjoint crossing paths in $\overline{R}_n$, we have

$$E(\text{path length}) \leq \frac{2}{\beta\kappa \log m} \sum_{i=1}^{\beta\kappa \log m} \text{path length}_i \leq \frac{2m \log m}{\beta \log m} = \frac{2}{\beta} m,$$

where the last inequality holds because the total number of relay nodes in rectangle $\overline{R}_n$ is $m \log m$ and the crossing paths are disjoint. Since the packet size scales with the throughput, each packet is forwarded in a constant time at each relay, and the average delay is at most proportional to $\sqrt{n}$, as $m = \sqrt{2n}/c$. □

6 Dense Networks

In this section, we consider the model where nodes are distributed according to a Poisson point process of intensity n over a square of unit area. Furthermore, we take an attenuation function l of the form

$$l(d) = d^{-\alpha} e^{-\gamma d}.$$

In this case, we divide that network into squares of size $c/\sqrt{n}$. We obtain thus the same number of little squares as in the previous model. The average number of nodes in each little square is also the same, namely c^2. Therefore, all the percolation results above still hold for this model, and we can find as many highways as above.

To derive the lower bound on the capacity, we have to compute the throughput along the highways, as well as the rate at which nodes can send data towards the highways. In fact, both of these throughputs were computed using Theorem 4, so it is enough here to give an adapted version of such theorem.

Theorem 5 *For any given integer $d > 0$ and when n is sufficiently large, there exists a TDMA scheduling, such that one node per square can transmit to any destination located within a radius of d squares (in Manhattan distance) with a fixed rate independent of n.*

When d goes to infinity, the asymptotic behavior of the rate is given by

$$R(d) = \Omega(d^{-2}).$$

Proof: We set up the same TDMA scheme as in Theorem 4, with k^2 time slots, where $k = 2(d+1)$. Similarly, the 8 closest interferers are located at least $d+2$ squares away from the receiver, the next 16 interferers at distance $3d+3$, and so on. The difference here is that squares have size $c/\sqrt{n}$ here. The sum of the interferences at the receiver can be bounded as follows:

$$\begin{aligned}
I(d,n) &\leq \sum_{i=1}^{\infty} 8i\, Pl\left(\frac{c[ki-(d+1)]}{\sqrt{n}}\right) \\
&= \sum_{i=1}^{\infty} 8i\, Pl\left(\frac{c(2i-1)(d+1)}{\sqrt{n}}\right) \\
&= \sum_{i=1}^{\infty} 8i\, P\left(\frac{c(2i-1)(d+1)}{\sqrt{n}}\right)^{-\alpha} e^{-\gamma c(2i-1)(d+1)/\sqrt{n}} \\
&\leq P\left(\frac{c(d+1)}{\sqrt{n}}\right)^{-\alpha} e^{-\gamma c(d+1)/\sqrt{n}} \sum_{i=1}^{\infty} 8i\,(2i-1)^{-\alpha} e^{-\gamma c(2i-2)(d+1)/\sqrt{n}}.
\end{aligned}$$

This sum clearly converges if $\alpha > 2$ or $\gamma > 0$.

As the receiver is at most d squares away from the emitter, the Euclidean distance between them is less than $c(d+1)/\sqrt{n}$. The strength $S(d,n)$ of the signal at the receiver reads thus

$$S(d,n) = P\left(\frac{c(d+1)}{\sqrt{n}}\right)^{-\alpha} e^{-\gamma c(d+1)/\sqrt{n}}.$$

The SNIR is thus

$$\begin{aligned}
SNIR(d,n) &= \frac{S(d)}{N_0 + I(d)} \\
&\geq \frac{P\left(\frac{c(d+1)}{\sqrt{n}}\right)^{-\alpha} e^{-\gamma c(d+1)/\sqrt{n}}}{N_0 + P\left(\frac{c(d+1)}{\sqrt{n}}\right)^{-\alpha} e^{-\gamma c(d+1)/\sqrt{n}} \sum_{i=1}^{\infty} 8i\,(2i-1)^{-\alpha} e^{-\gamma c(2i-2)(d+1)/\sqrt{n}}} \\
&= \frac{1}{N_0\left(\frac{c(d+1)}{\sqrt{n}}\right)^{\alpha} e^{\gamma c(d+1)/\sqrt{n}}/P + \sum_{i=1}^{\infty} 8i\,(2i-1)^{-\alpha} e^{-\gamma c(2i-2)(d+1)/\sqrt{n}}}
\end{aligned}$$

When n goes to infinity, we observe that the first term of the denominator tends to zero, whereas the second term tends to a constant independent of d. The whole fraction tends thus to a constant $K > 0$, and thus

$$\lim_{n\to\infty} SNIR(d,n) \geq K.$$

It means that for any $\varepsilon > 0$, when n is large enough, the rate

$$R' = \frac{1}{2}\log(1 + K - \varepsilon)$$

is achievable by this scheme during each time slot. This proves the first part of the theorem.

Now if d increases with n, we notice that the above limit still holds whenever $(d+1)/\sqrt{n}$ tends to zero. Therefore, if $d = O(\sqrt{n})$, the rate R' is achievable for each active transmission when n is large enough. However, as there are $k^2 = 4(d+1)^2$ time slots in our TDMA scheme, the actual throughput available for each square is $R(d) = R'/k^2$, and thus

$$R(d) = \Omega(d^{-2}).$$

□

7 Conclusions

We have found that the capacity of wireless networks of randomly located nodes has the same asymptotic behavior of the capacity of arbitrary networks: nodes in a random network can transmit at the same rate than nodes in an arbitrary network and there is no price to pay —at least asymptotically— for the additional randomness present in the system. This result closes a previous gap between upper and lower bounds on the optimal per-node transmission rate that consistently appeared in different proofs proposed in the literature.

We were able to close this gap by exploiting a connection between percolation theory and the way we scale the transmitting power of the nodes. By scaling the power at a sufficiently slow rate, a wireless backbone containing paths that cross the network area exists with high probability, but this covers all the nodes, generating excessive interference. By scaling the power at a higher rate, this backbone does not form at all. Previous results used the first kind of scaling to prove a capacity lower bound. We noticed that percolation theory ensures that a different kind of backbone forms in the transition region between this two extreme scalings. This does not cover all the nodes, nevertheless it is sufficiently rich in crossing paths so that it can transport the total amount of traffic. By operating the network in this transition region between order and disorder, we are able to prove our tight bound.

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