Diversity and Freedom:
A Fundamental Tradeoff in Wireless Systems

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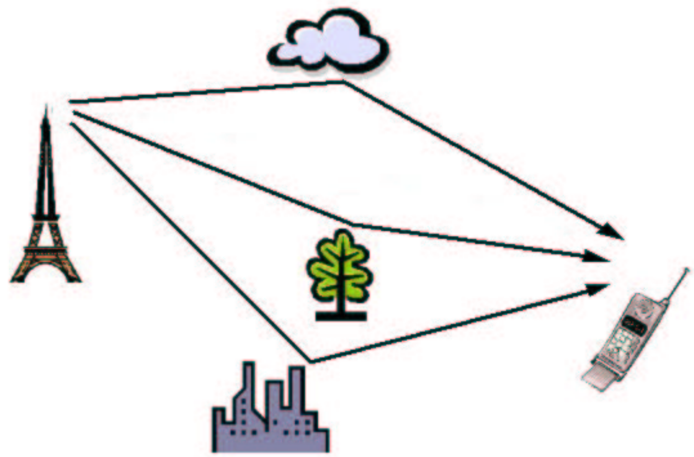
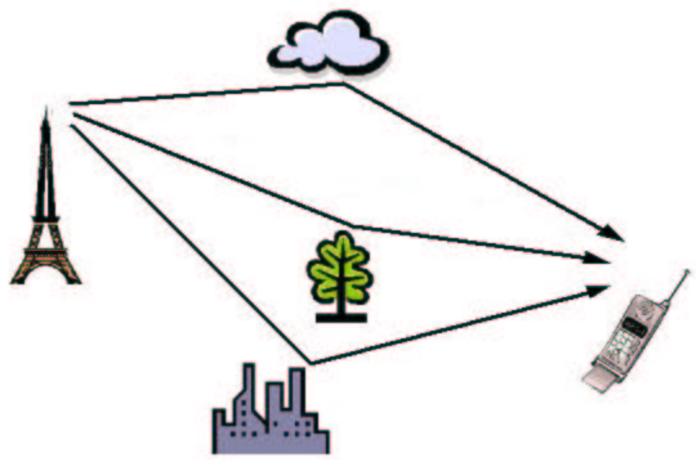
Wireless Fading Channels



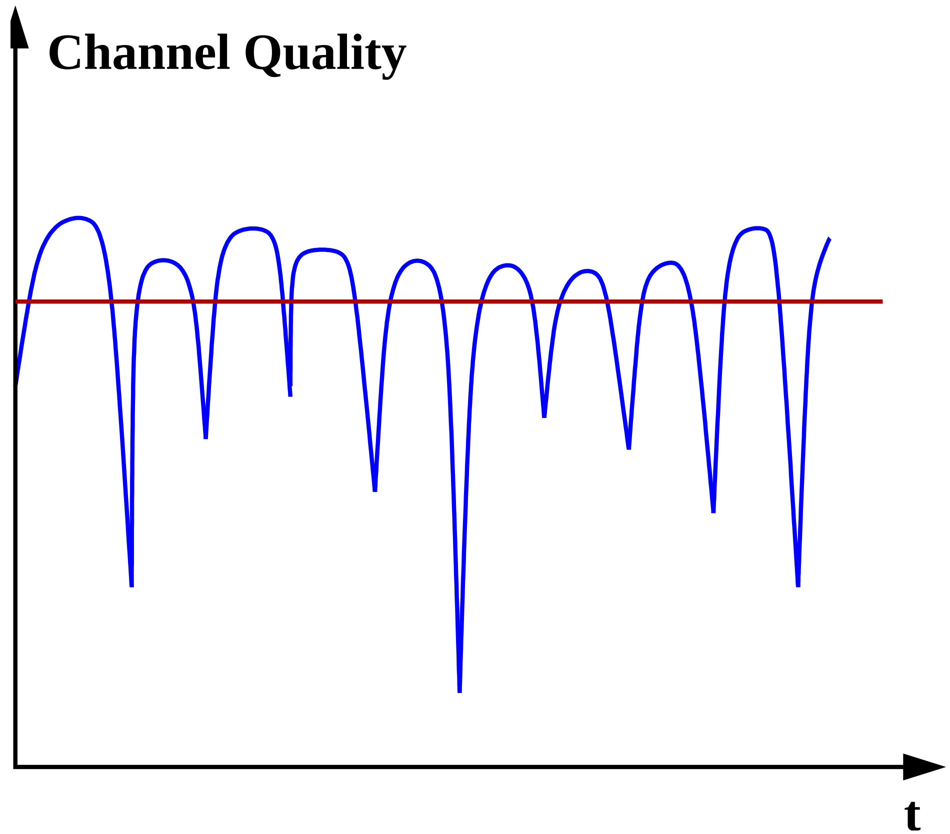
- Fundamental characteristic of wireless channels: multi-path fading.

Wireless Fading Channels



- Fundamental characteristic of wireless channels: multi-path fading.
- Two important resources of a fading channel: diversity and degrees of freedom.

Diversity



A channel with more diversity has smaller probability in deep fades.

Example: Spatial Diversity

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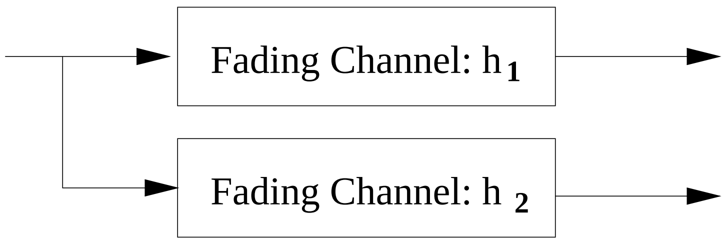
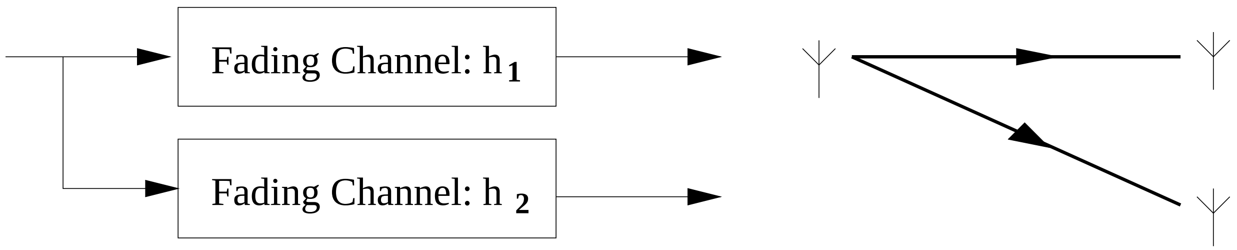


- Additional independent fading channels increase diversity.

Example: Spatial Diversity



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- Spatial diversity

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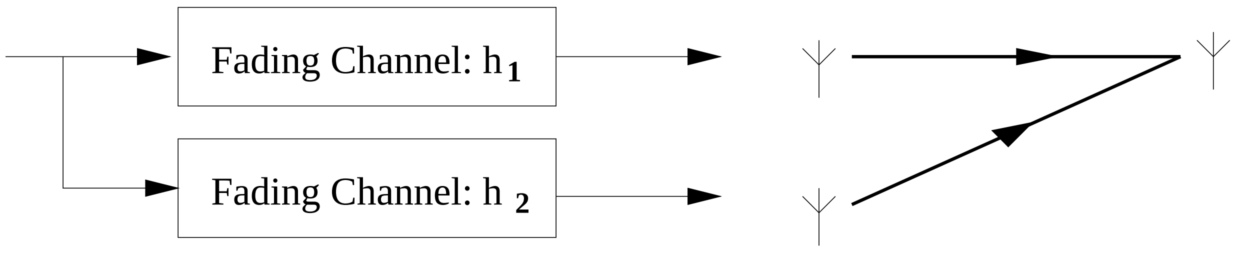

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- Spatial diversity: receive, transmit

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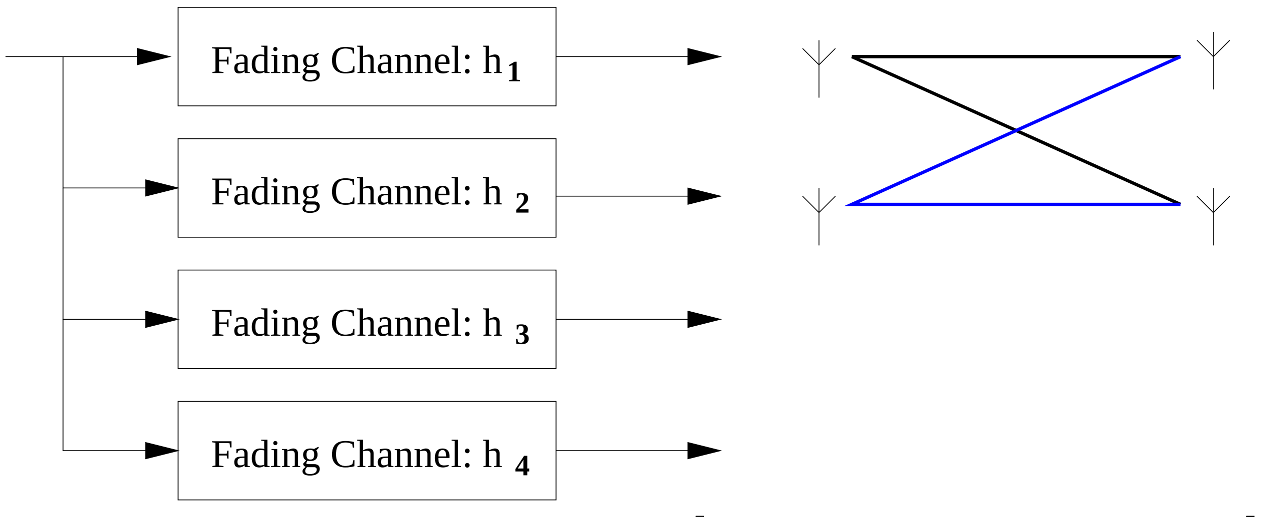

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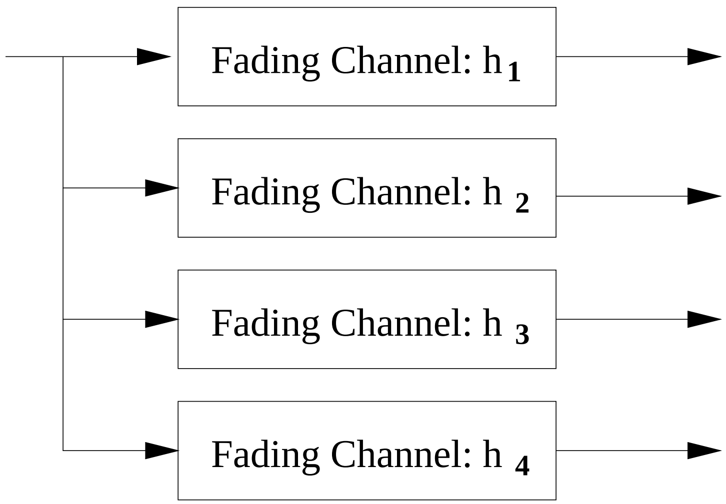
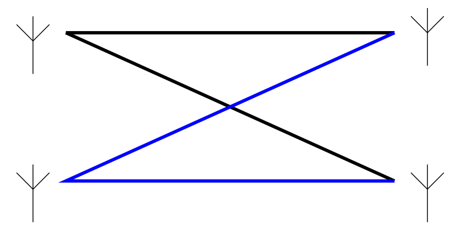
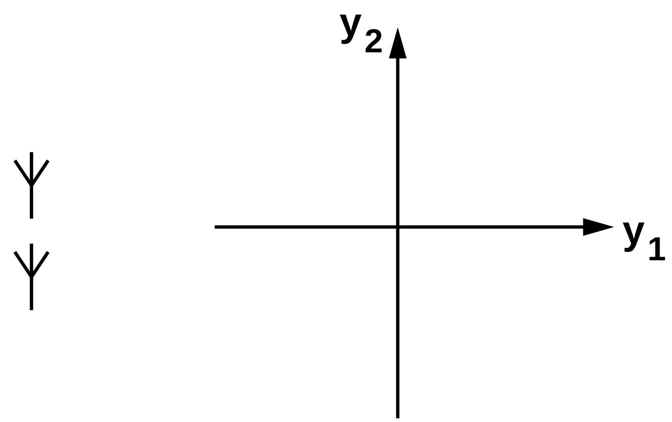




- Additional independent fading channels increase diversity.
- Spatial diversity: receive, transmit or both.
- Repeat and Average: compensate against channel unreliability.

Degrees of Freedom



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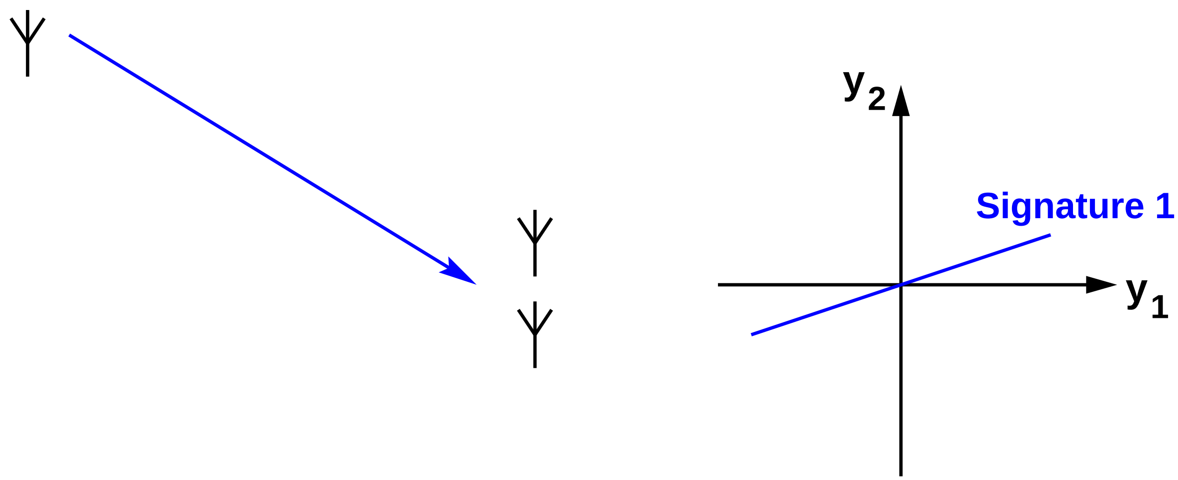

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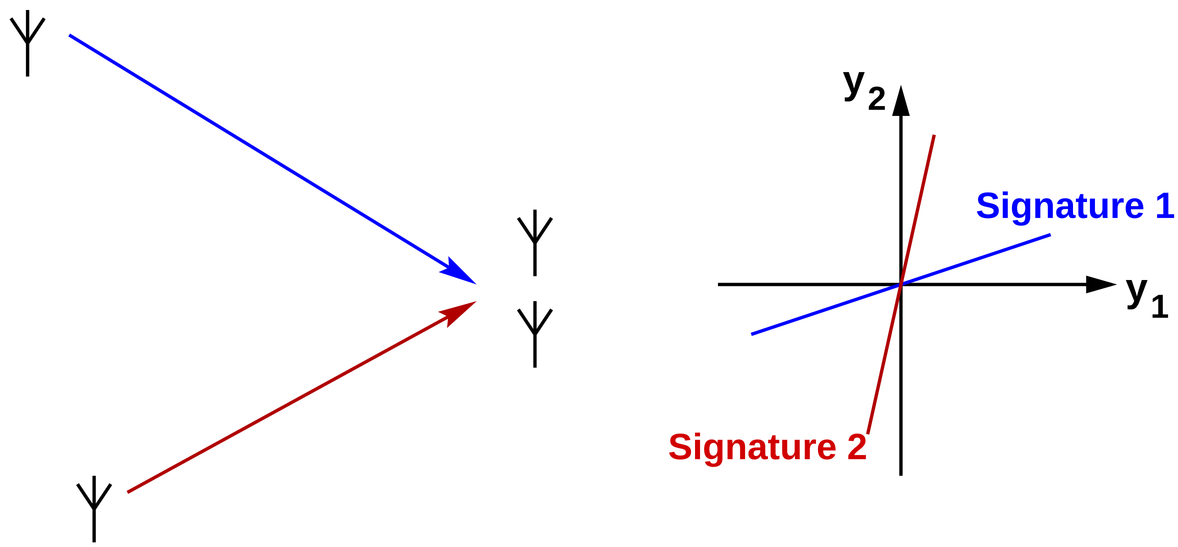
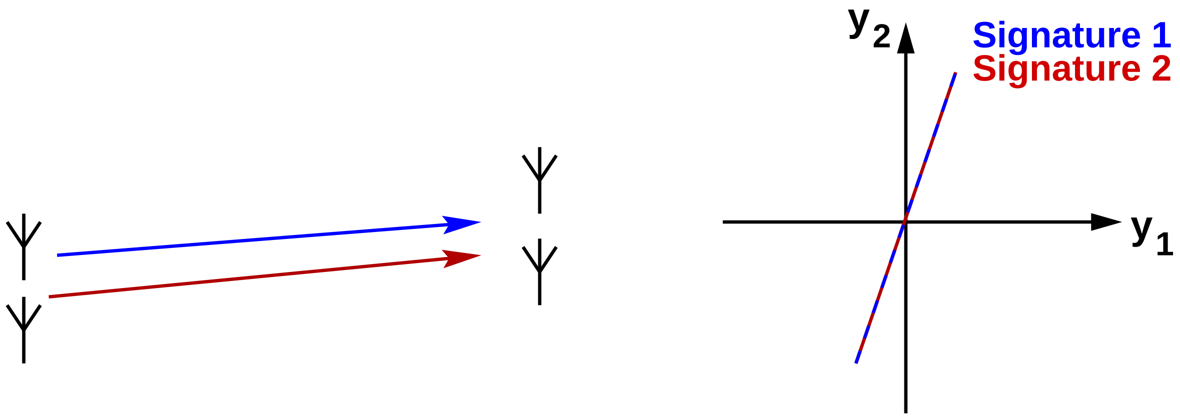


Signals arrive in multiple directions provide multiple degrees of freedom for communication.

Degrees of Freedom



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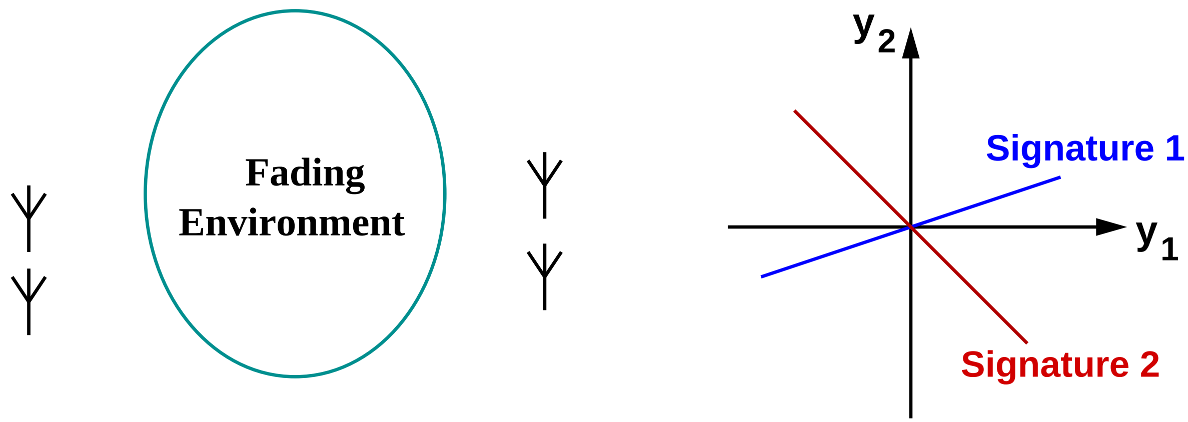
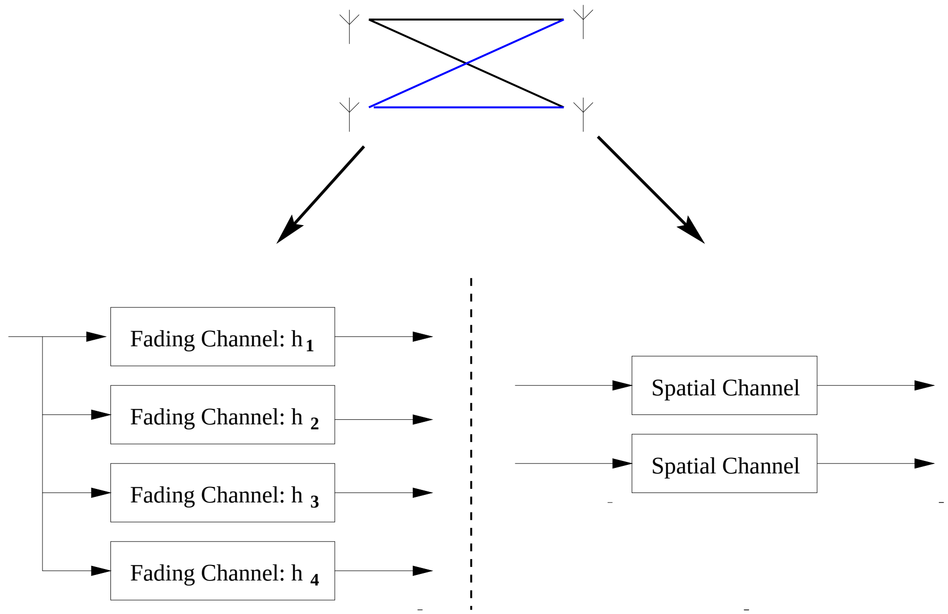


Signals arrive in multiple directions provide multiple degrees of freedom for communication.

Same effect can be obtained via scattering even when antennas are close together.

Diversity vs. Freedom



The two resources have been considered mainly in isolation: existing schemes focus on maximizing either the diversity gain or the multiplexing gain.

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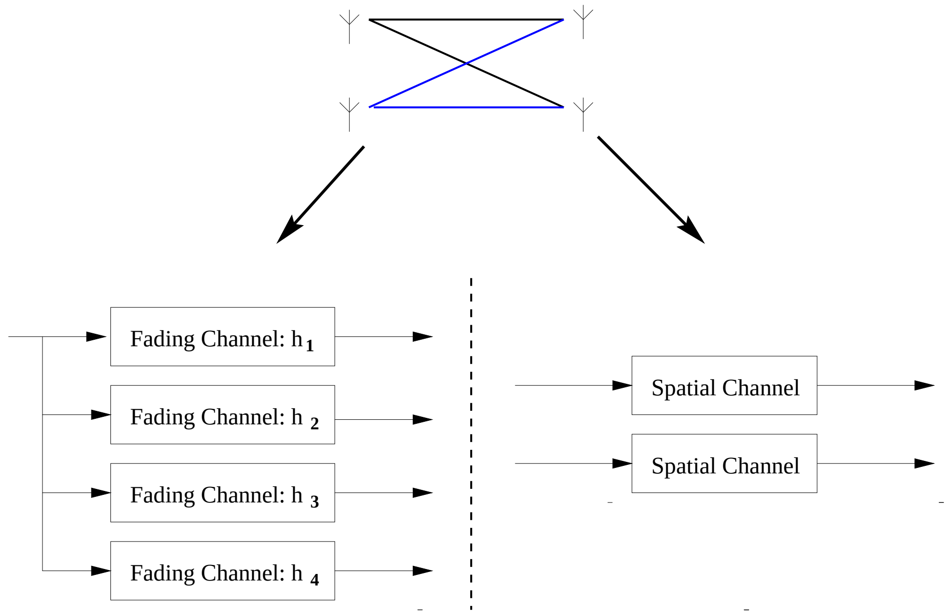

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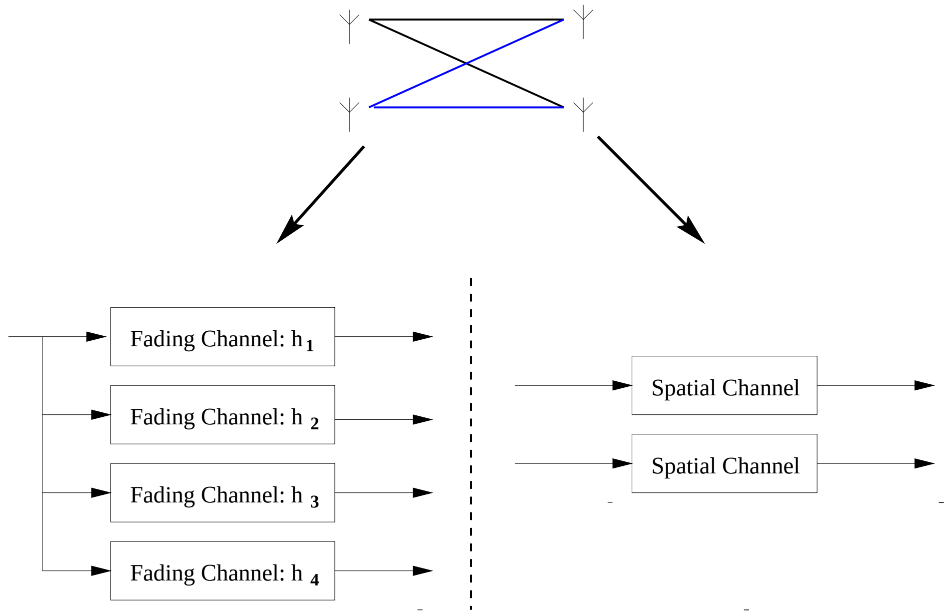

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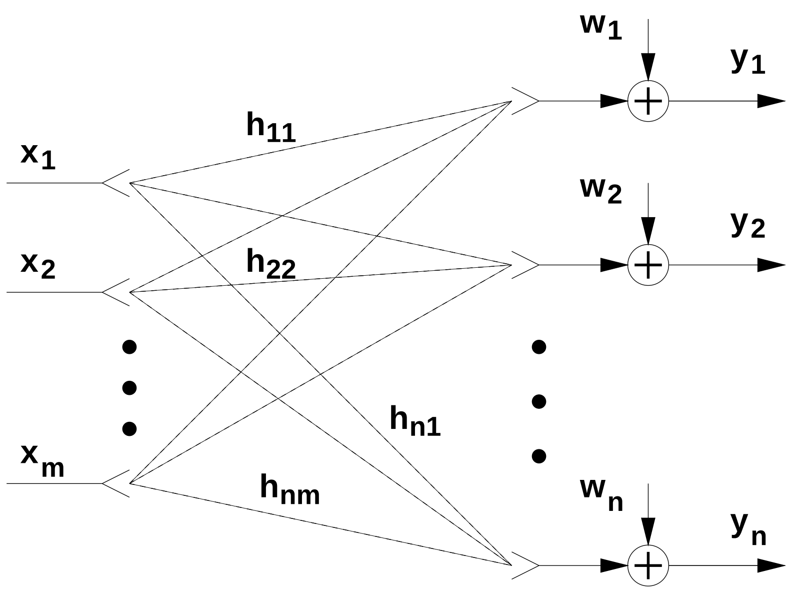
The right way of looking at the problem is a tradeoff between the two types of gain.

The optimal tradeoff achievable by a coding scheme gives a fundamental performance limit on communication over fading channels.

Talk Outline

- point-to-point MIMO channels (Zheng and Tse 02)
- multiple access MIMO channels (Tse, Viswanath, Zheng 03)
- cooperative relaying systems (Laneman,Tse, Wornell 02)

Point-to-point MIMO Channel



$$\mathbf{y}_t = \mathbf{H}_t \mathbf{x}_t + \mathbf{w}_t, \qquad \mathbf{w}_t \sim \mathcal{CN}(0, 1)$$

- Rayleigh flat fading i.i.d. across antenna pairs ($h_{ij} \sim \mathcal{CN}(0, 1)$).
- SNR is the average signal-to-noise ratio at each receive antenna.

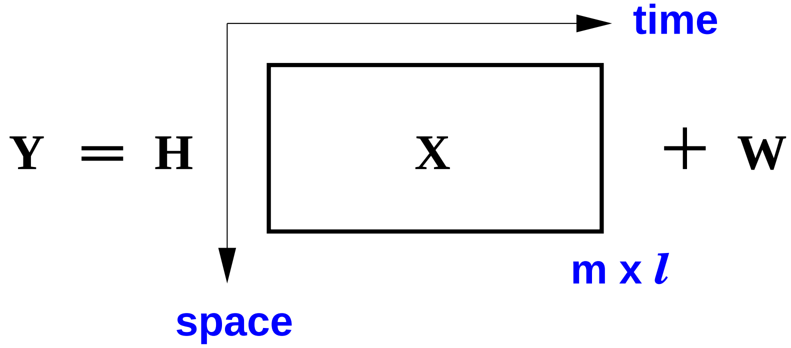
Coherent Block Fading Model

- Focus on codes over l symbols, where $\mathbf{H}$ remains constant.
- $\mathbf{H}$ is known to the receiver but not the transmitter.
- Assumption valid as long as

$$l \ll \text{coherence time } \times \text{ coherence bandwidth.}$$

Space-Time Block Code

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W}$$



Focus on coding over a single block of length l.

Diversity Gain

Motivation: Binary Detection

$$\mathbf{y} = \mathbf{h}\mathbf{x} + \mathbf{w} \qquad\qquad P_e \approx P(\|\mathbf{h}\| \text{ is small }) \propto \mathsf{SNR}^{-1}$$

$$\left.\begin{aligned} \mathbf{y}_1 &= \mathbf{h}_1\mathbf{x} + \mathbf{w}_1 \\ \mathbf{y}_2 &= \mathbf{h}_2\mathbf{x} + \mathbf{w}_2 \end{aligned}\right\} \qquad \begin{aligned} P_e \quad &\approx P(\|\mathbf{h}_1\|, \|\mathbf{h}_2\| \text{ are both small}) \\ &\propto \mathsf{SNR}^{-2} \end{aligned}$$

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General Definition

A space-time coding scheme achieves diversity gain d, if

$$P_e(\text{SNR}) \sim \text{SNR}^{-d}$$

Spatial Multiplexing Gain

Motivation: Channel capacity (Telatar '95, Foschini'96)

$$C(\mathsf{SNR}) \approx \min\{m, n\} \log \mathsf{SNR} \qquad \text{(bps/Hz)}$$

$\min\{m, n\}$ degrees of freedom to communicate.

Spatial Multiplexing Gain

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$$C(\mathsf{SNR}) \approx \min\{m, n\} \log \mathsf{SNR} \qquad \text{(bps/Hz)}$$

$\min\{m, n\}$ degrees of freedom to communicate.

Definition A space-time coding scheme achieves spatial multiplexing gain r, if

$$R(\mathsf{SNR}) = r \log \mathsf{SNR}$$

Fundamental Tradeoff

A space-time coding scheme achieves

Spatial Multiplexing Gain r	:	$R = r \log \mathsf{SNR}$
and		
Diversity Gain d	:	$P_e \approx \mathsf{SNR}^{-d}$

Fundamental Tradeoff

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$$r \longrightarrow d^*_{m,n}(r)$$

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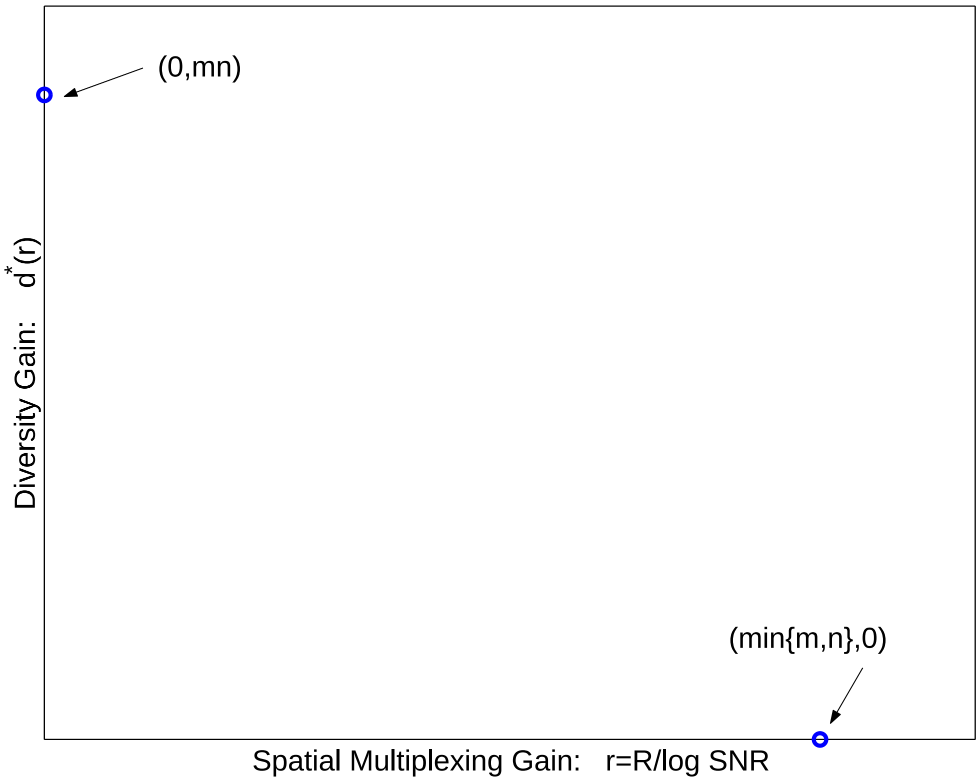
$$r \longrightarrow d^*_{m,n}(r)$$

It is a tradeoff between data rate and error probability.

Main Result: Optimal Tradeoff

(Zheng and Tse 02)

As long as block length $l \geq m + n - 1$:



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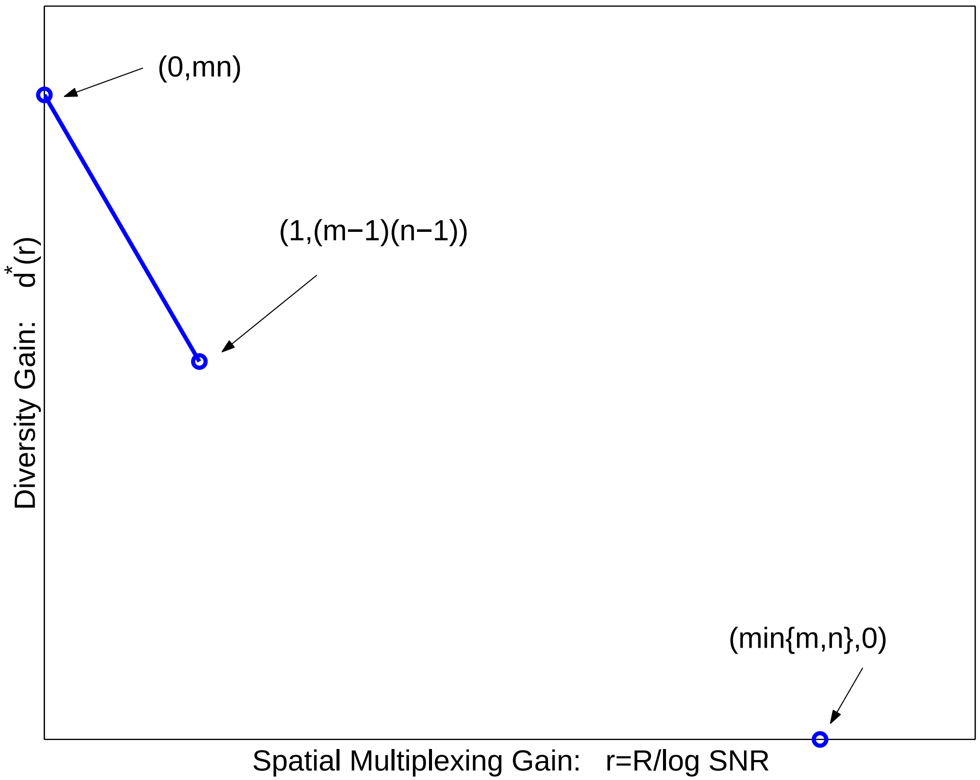

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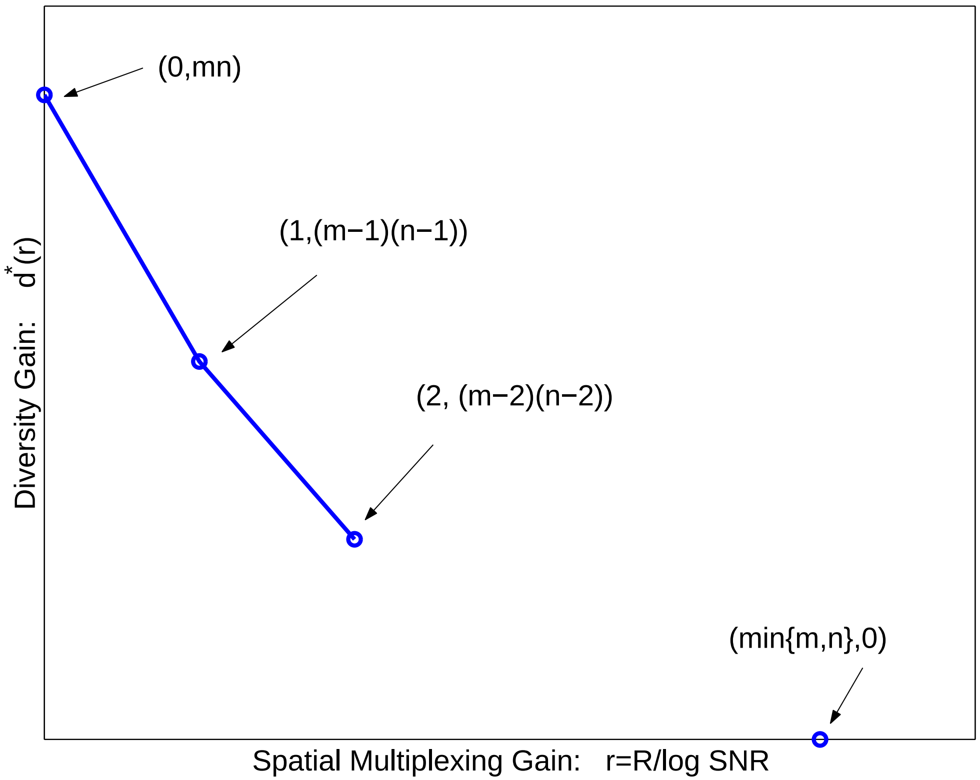

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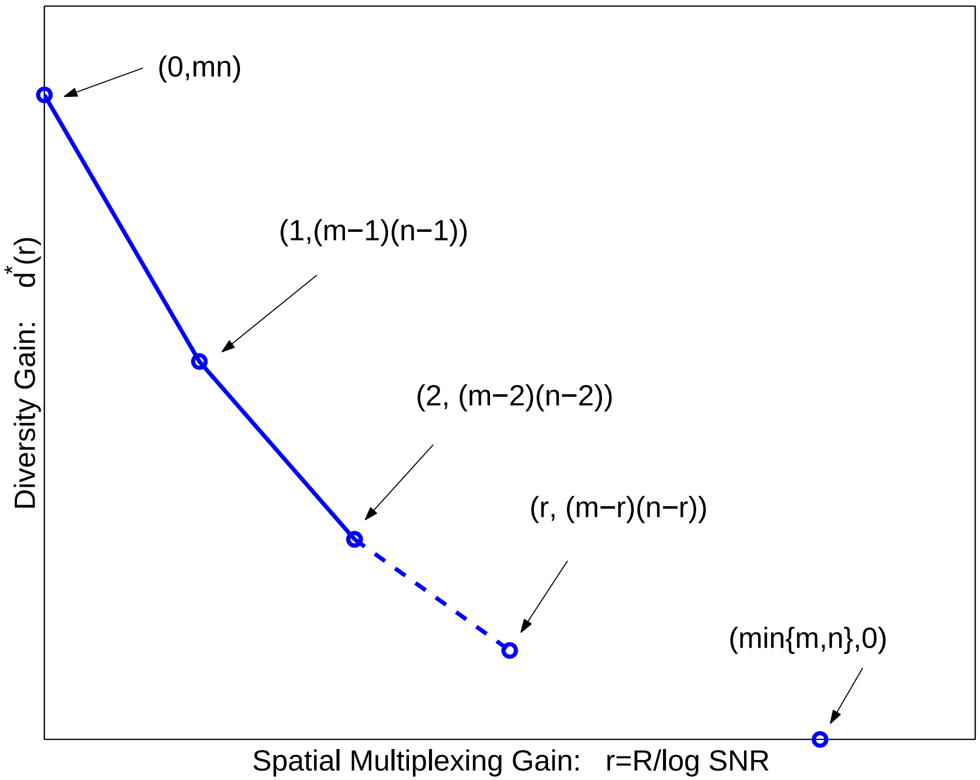

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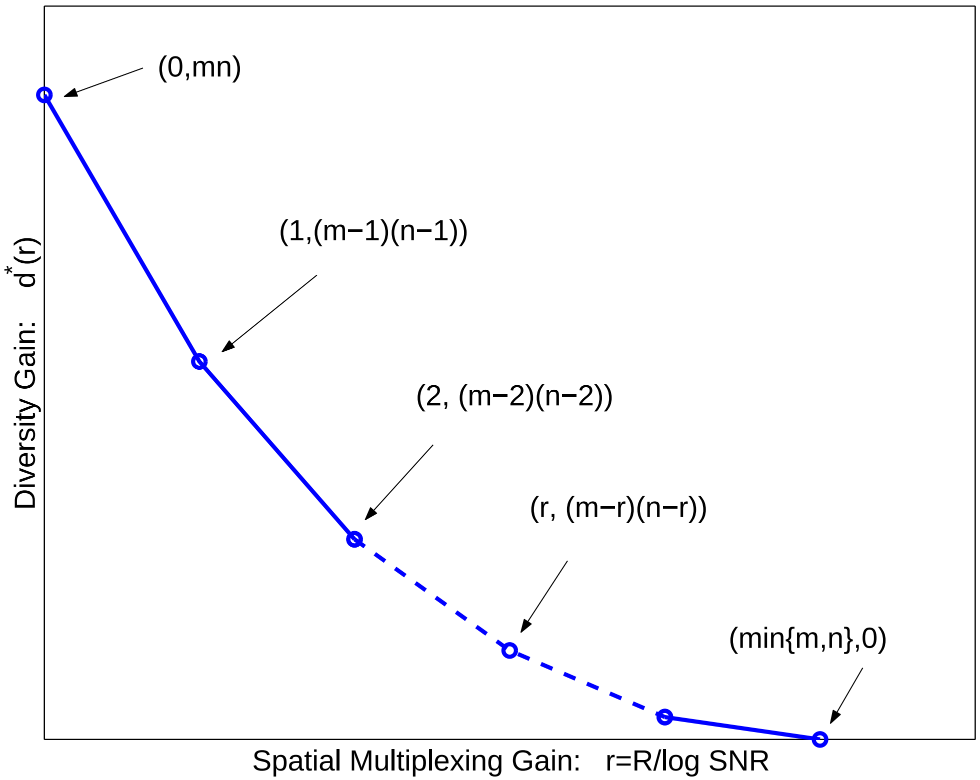

For integer r, it is *as though* r transmit and r receive antennas were dedicated for multiplexing and the rest provide diversity.

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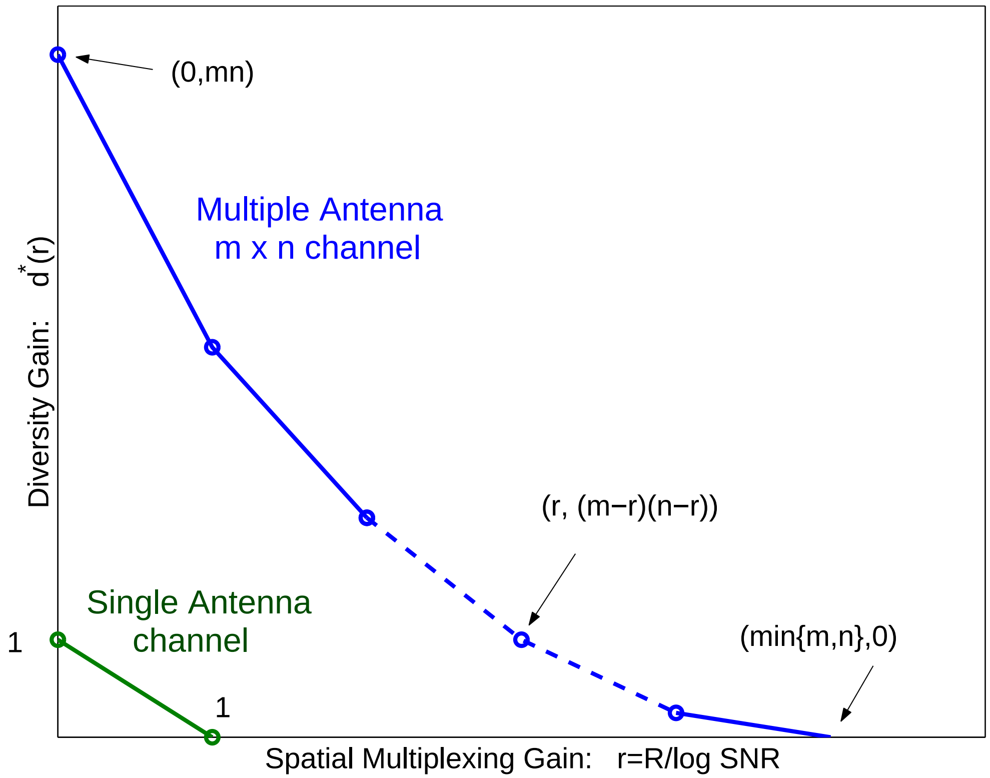

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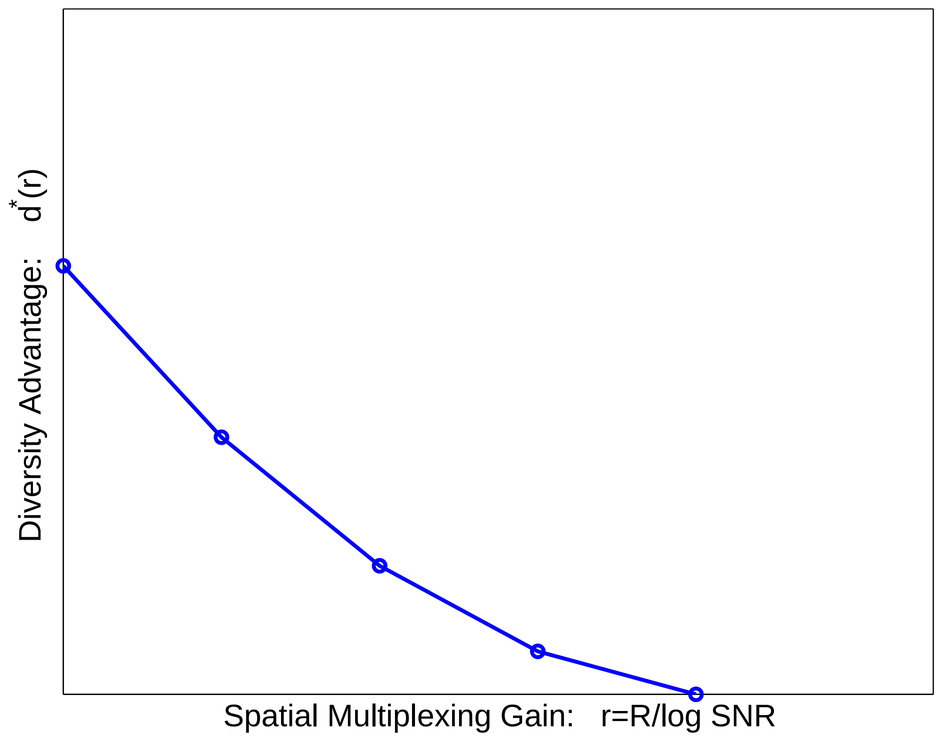
What do I get by adding one more antenna at the transmitter and the receiver?

Adding More Antennas



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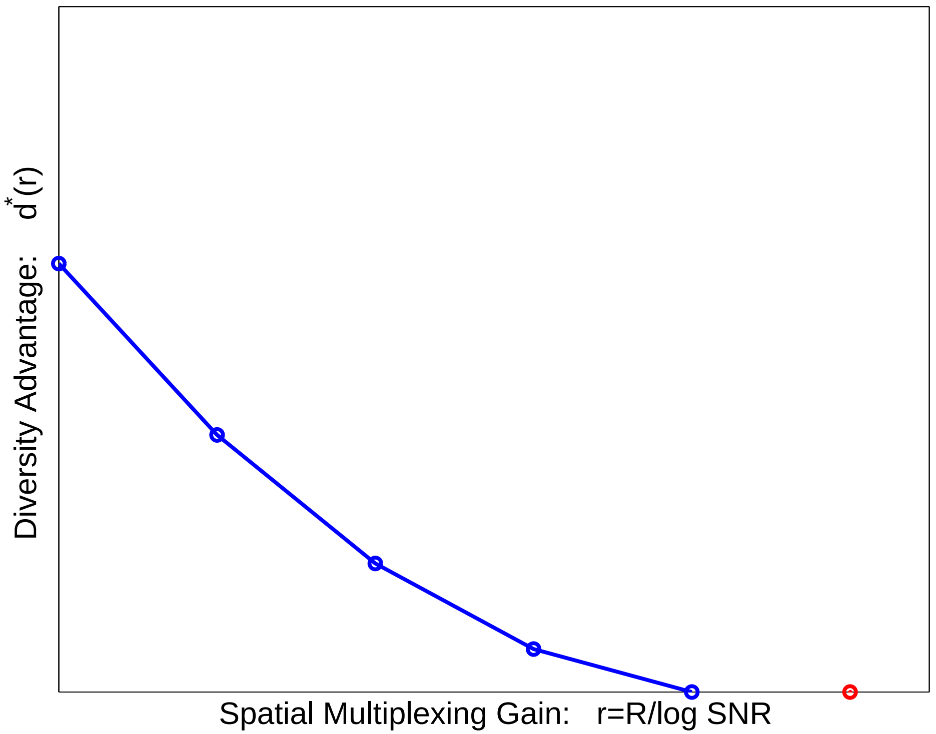

- Capacity result : increasing $\min\{m, n\}$ by 1 adds 1 more degree of freedom.

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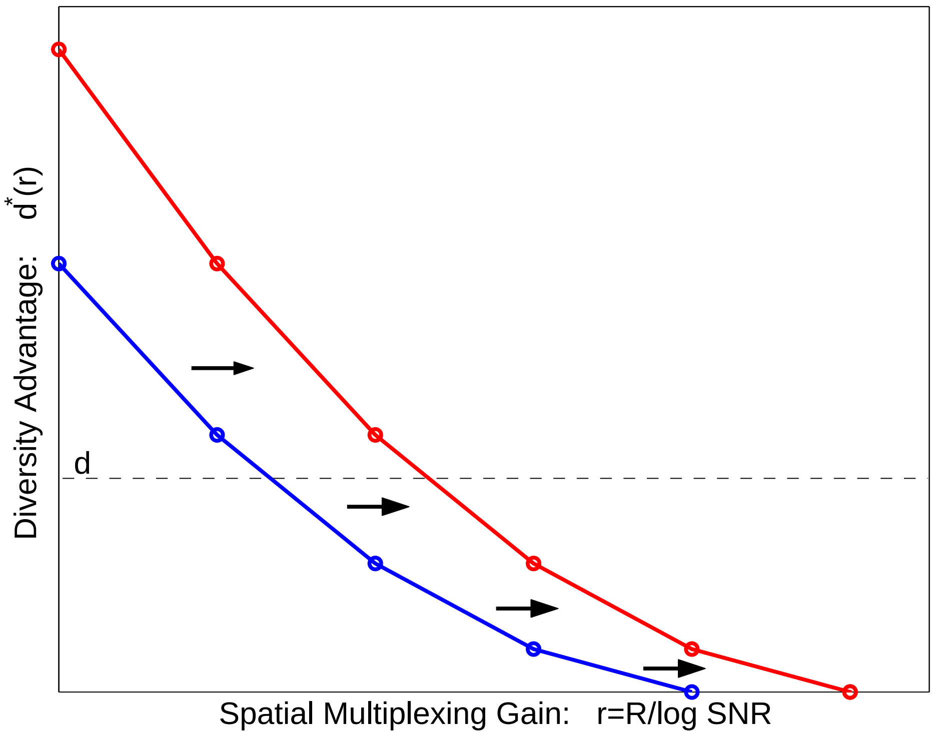
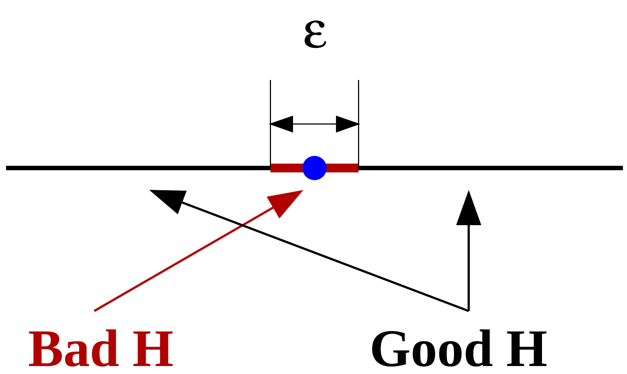


- Capacity result: increasing $\min\{m, n\}$ by 1 adds 1 more degree of freedom.
- Tradeoff curve: increasing both m and n by 1 yields multiplexing gain $+1$ for any diversity requirement d.

Geometric Picture

Scalar Channel



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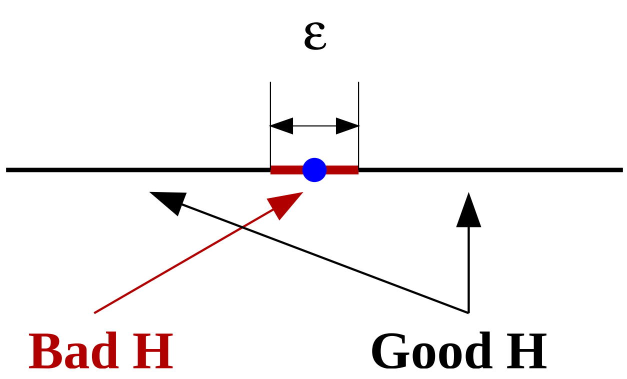

$$\epsilon^2 = \mathsf{SNR}^{-1}$$

$$P_e \sim \mathsf{SNR}^{-1}$$

Geometric Picture

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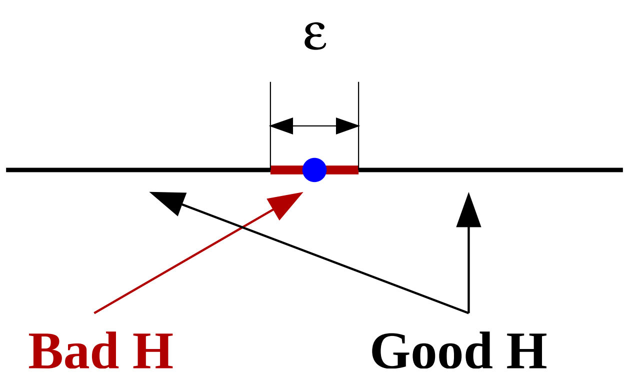

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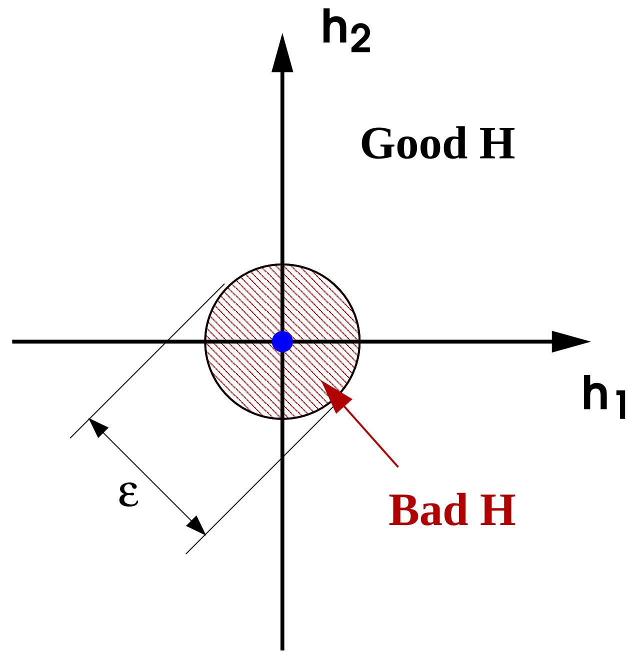
1 x 2 Channel



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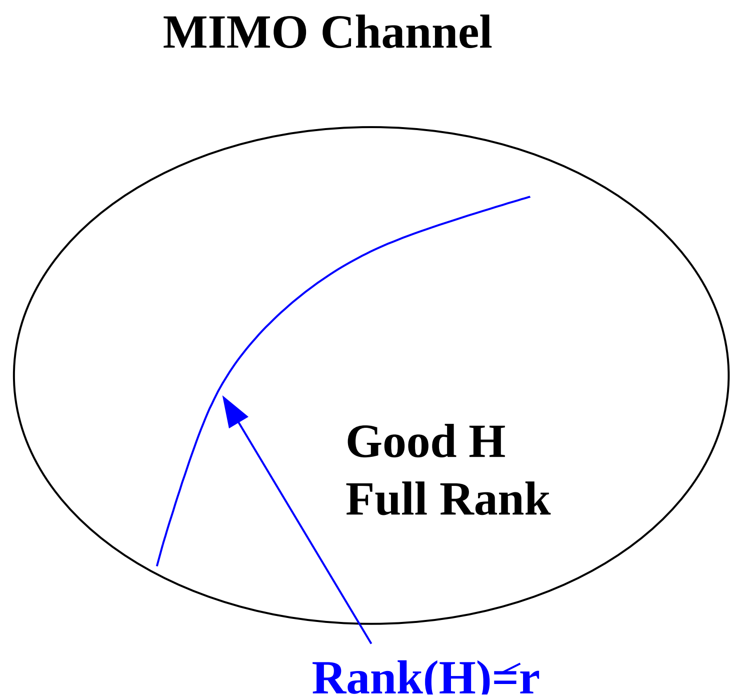
Geometric Picture for General $m \times n$ Channels

Multiplexing gain $= r$ (r integer)



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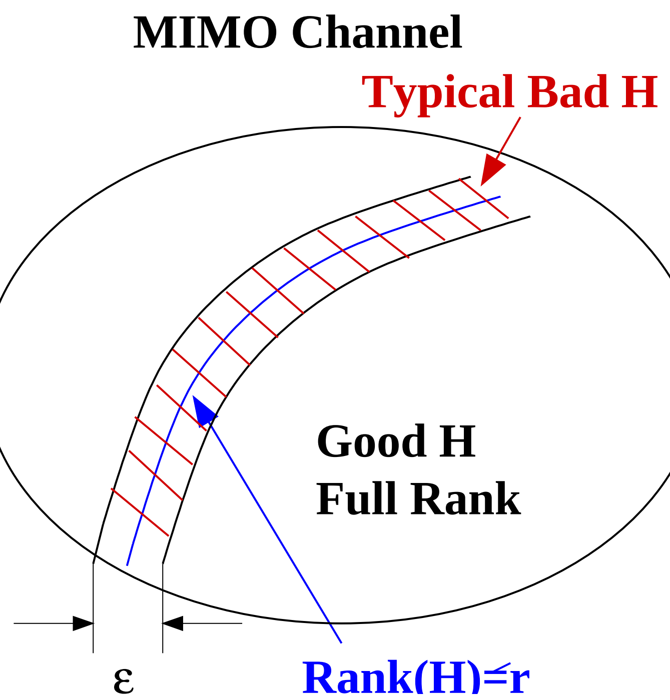

The co-dimension of the sub-manifold of rank r matrices within the set of all $m \times n$ matrices is $(m - r)(n - r)$.

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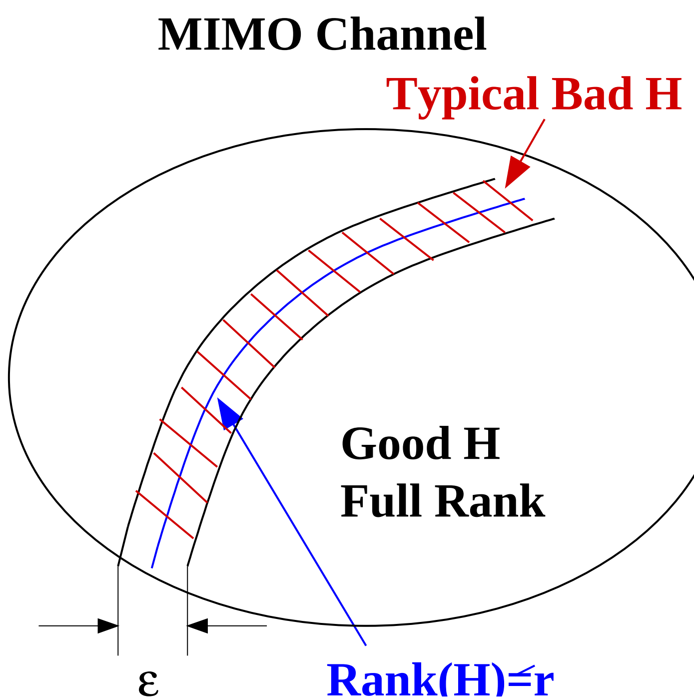

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$$P_e \sim \text{SNR}^{-(m-r)(n-r)}$$

Typical Error Events

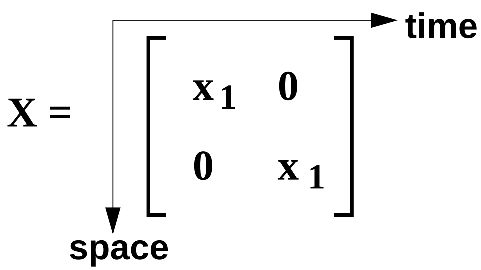
- In 1×1 and $1 \times n$ channels, error occurs when the channel gain $\|\mathbf{h}\|^2$ is small.
- In general $m \times n$ channel, error occurs when some or all of the singular values of $\mathbf{H}$ are small. There are many ways for this to happen.
- High SNR analysis shows that errors occur typically when $\mathbf{H}$ is close to a rank r matrix.

Tradeoff Analysis of Specific Designs

Focus on two transmit antennas.

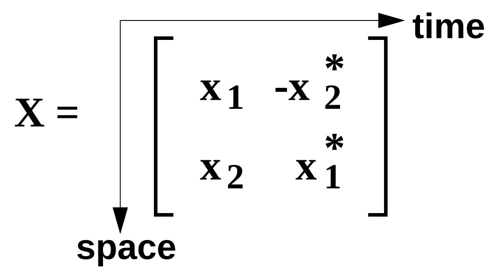
$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W}$$

Repetition Scheme:



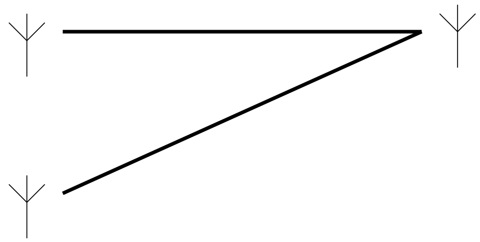
$$\mathbf{y}_1 = \|\mathbf{H}\|_F \mathbf{x}_1 + \mathbf{w}_1$$

Alamouti Scheme:



$$[\mathbf{y}_1 \mathbf{y}_2] = \|\mathbf{H}\|_F [\mathbf{x}_1 \mathbf{x}_2] + [\mathbf{w}_1 \mathbf{w}_2]$$

Comparison: 2×1 System



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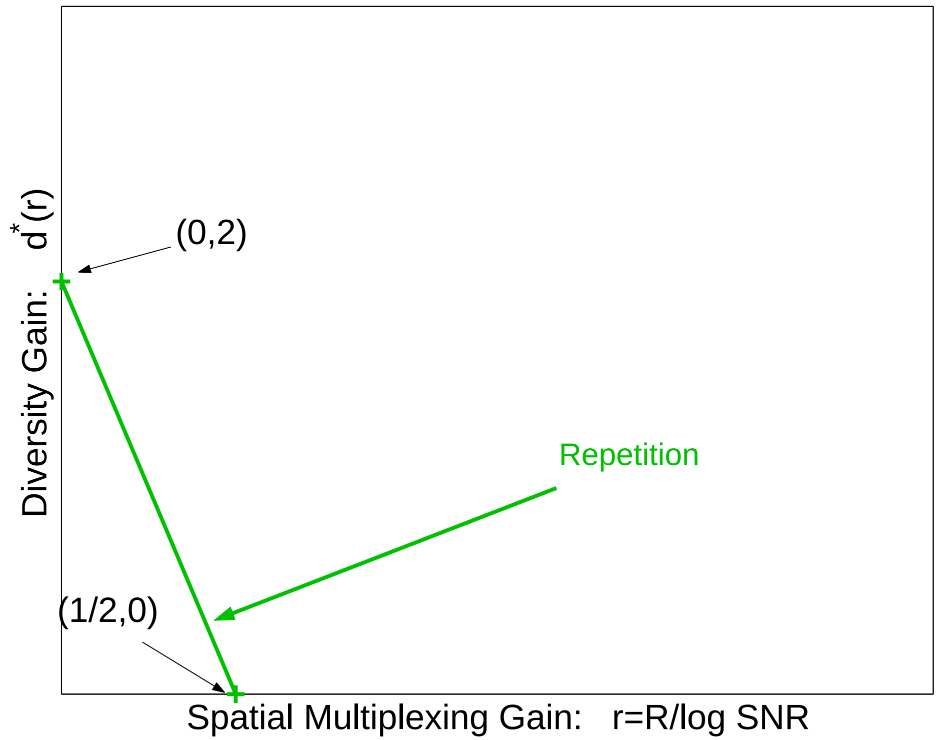

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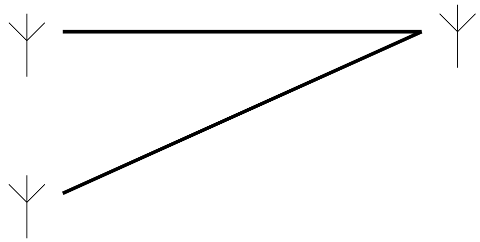

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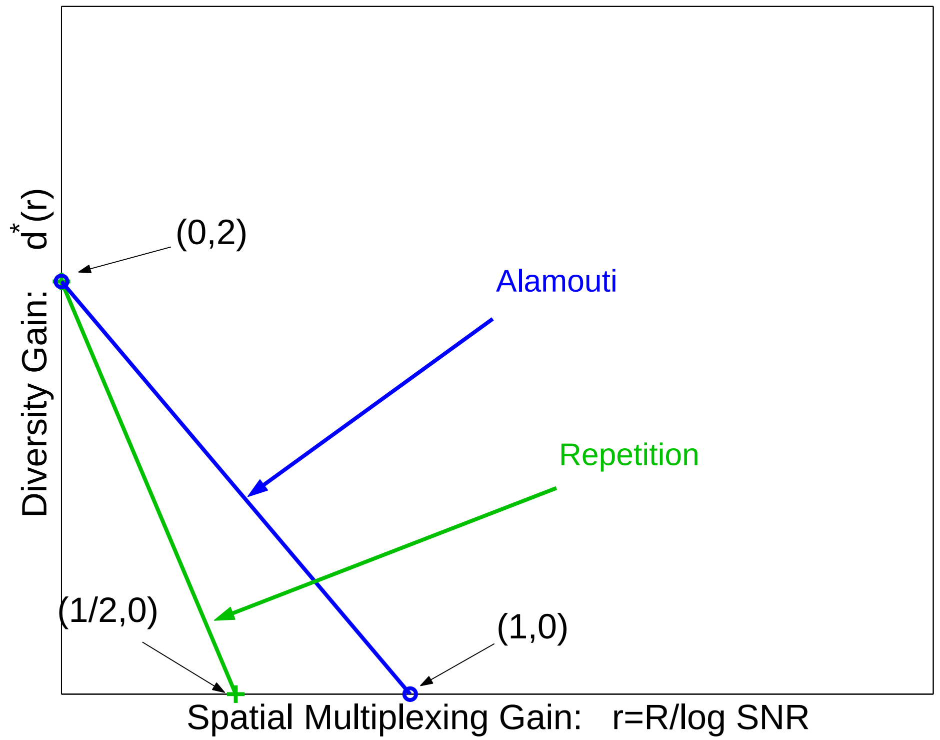

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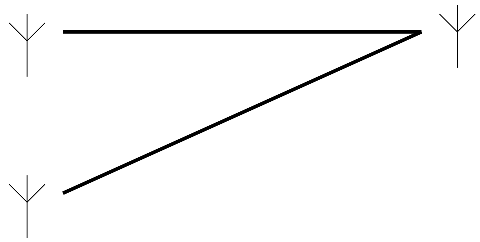

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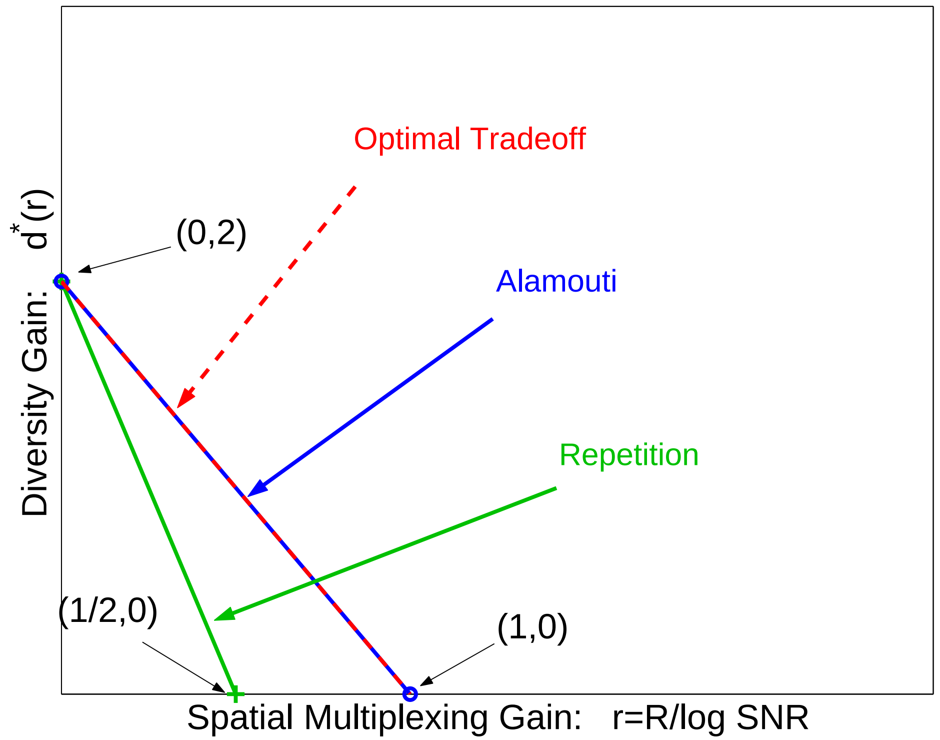
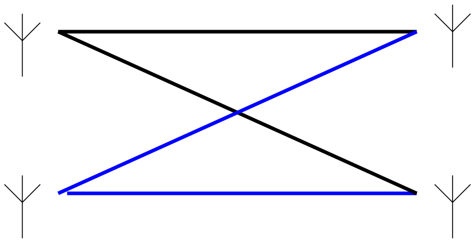


Comparison: 2×2 System



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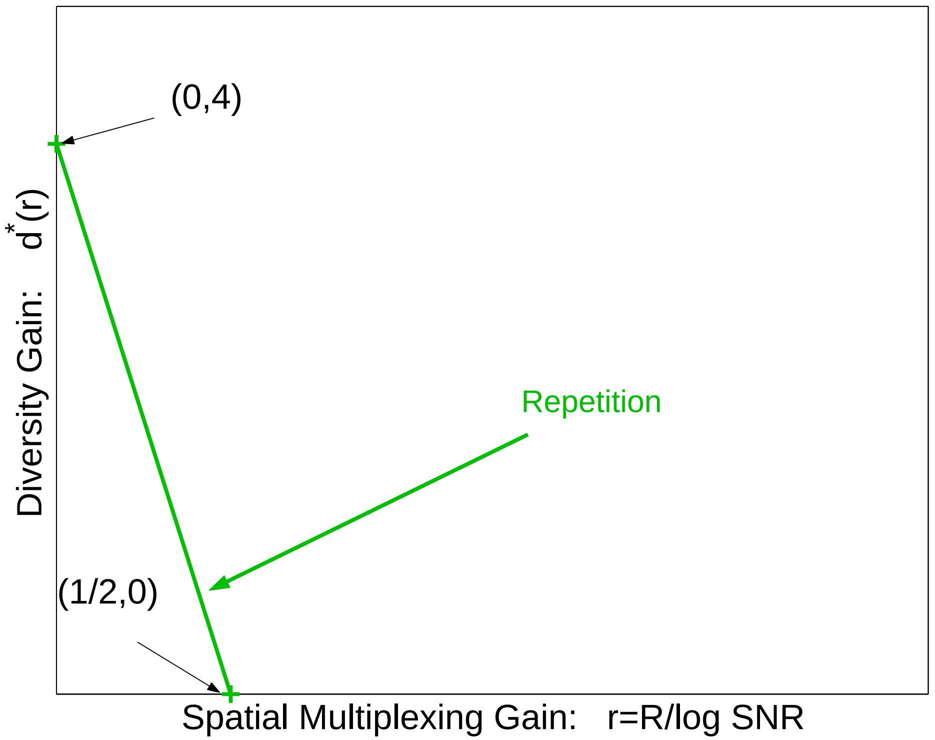

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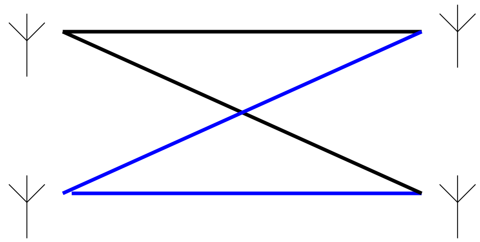

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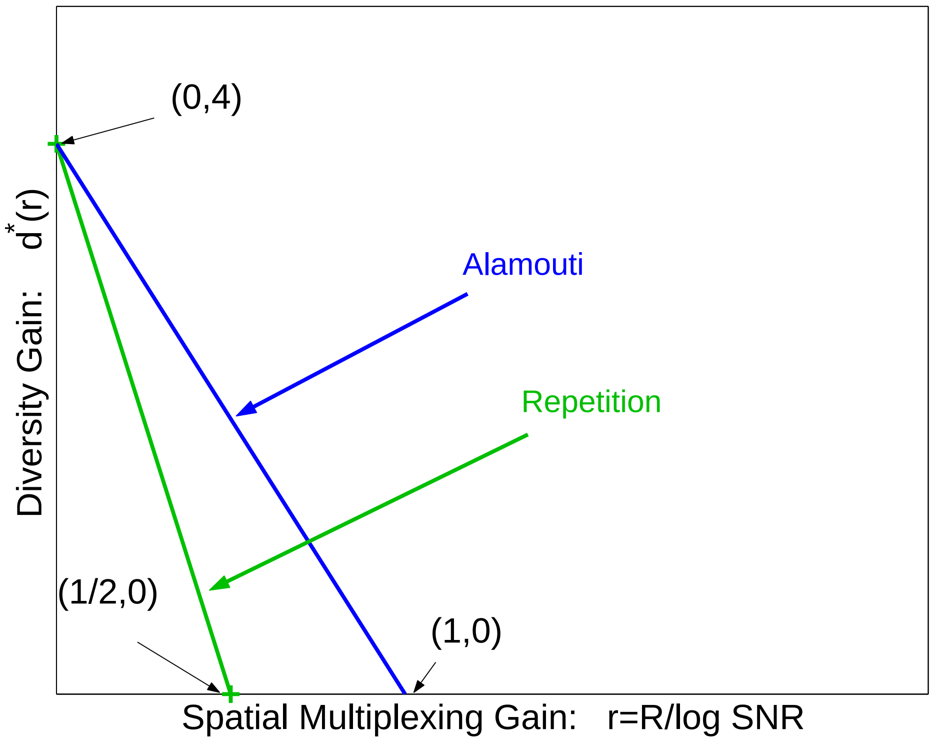

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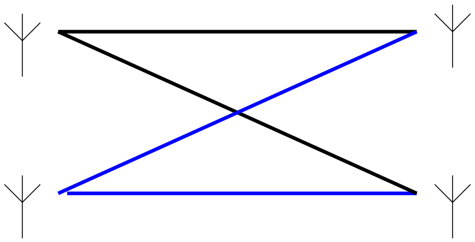
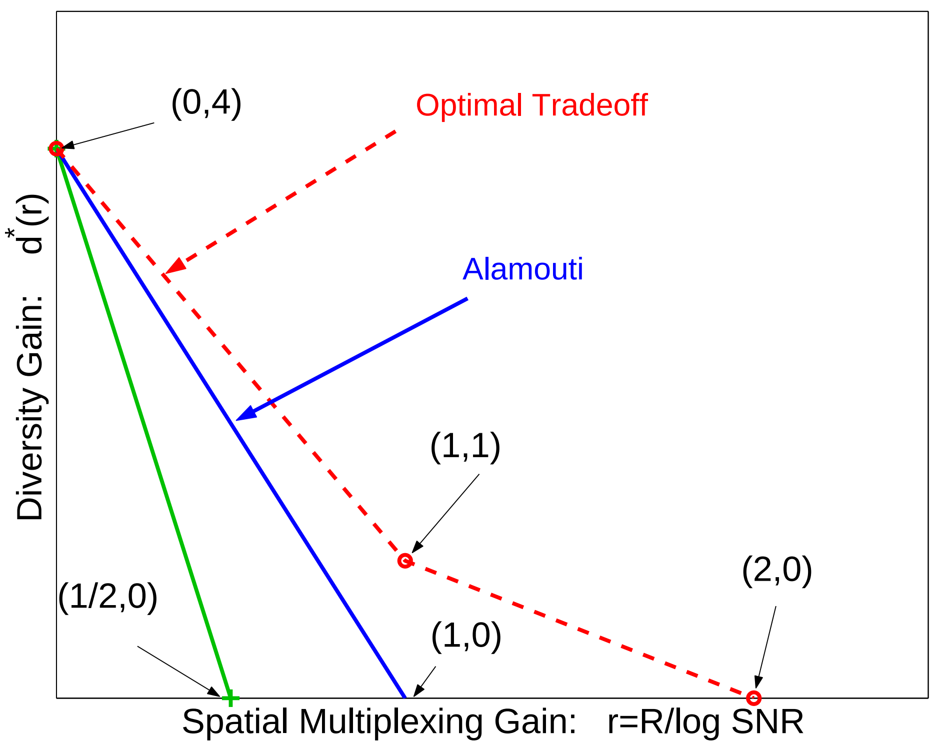


Repetition: $\mathbf{y}_1 = \|\mathbf{H}\|_F \mathbf{x}_1 + \mathbf{w}$

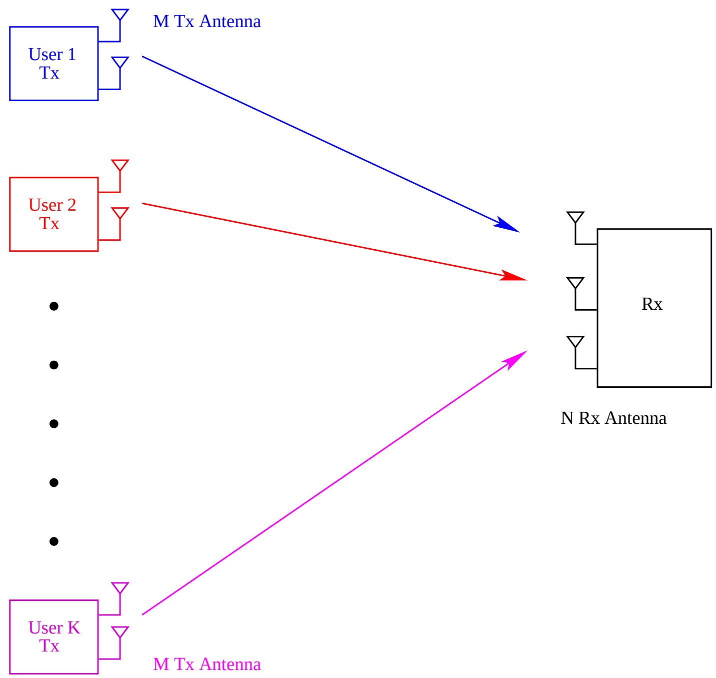
Alamouti: $[\mathbf{y}_1 \mathbf{y}_2] = \|\mathbf{H}\|_F [\mathbf{x}_1 \mathbf{x}_2] + [\mathbf{w}_1 \mathbf{w}_2]$



Talk Outline

- point-to-point MIMO channels
- **multiple access MIMO channels**
- cooperative relaying systems

Multiple Access



In a point-to-point link, multiple antennas provide diversity and multiplexing gain.

In a system with K users, multiple antennas can be used to discriminate signals from different users too.

Continue assuming i.i.d. Rayleigh fading, n receive antennas, m transmit antennas per user.

Multiuser Diversity-Multiplexing Tradeoff

Suppose we want every user to achieve an error probability:

$$P_e \sim \mathsf{SNR}^{-d}$$

and a data rate

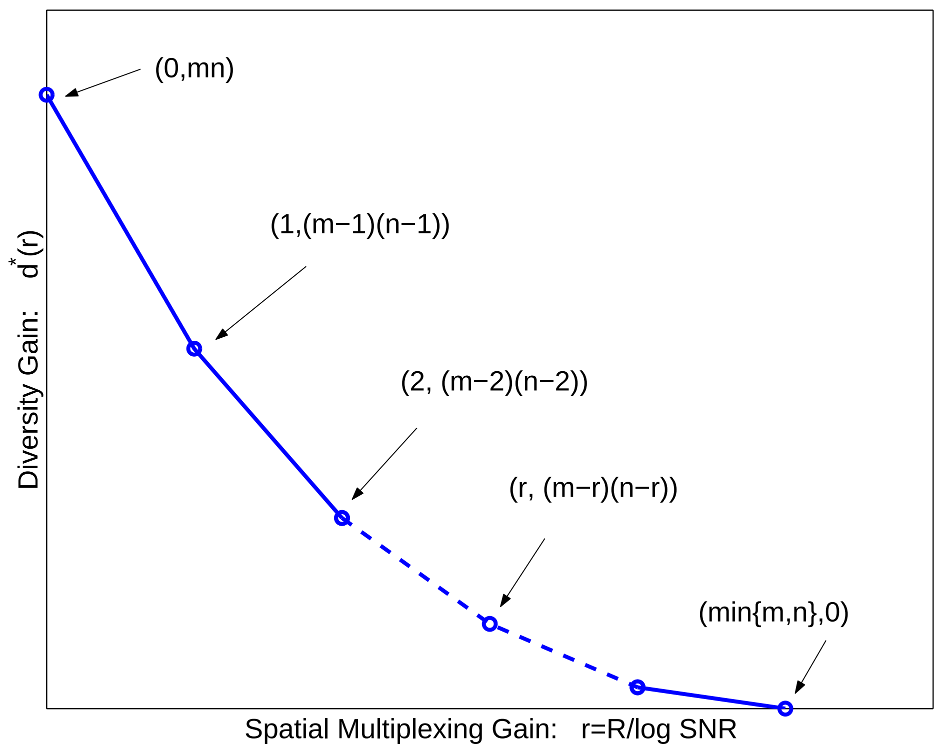
$$R = r \log \mathsf{SNR} \quad \text{bits/s/Hz.}$$

What is the optimal tradeoff between the diversity gain d and the multiplexing gain r?

Assume a coding block length $l \geq Km + n - 1$.

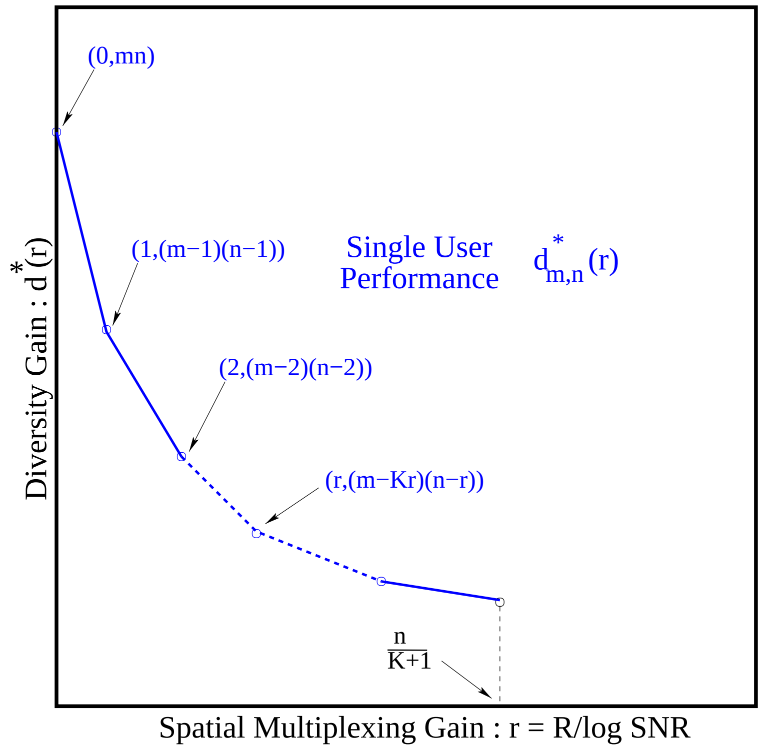
Optimal Multiuser D-M Tradeoff: $m \leq n/(K+1)$

(Tse, Viswanath and Zheng 02)



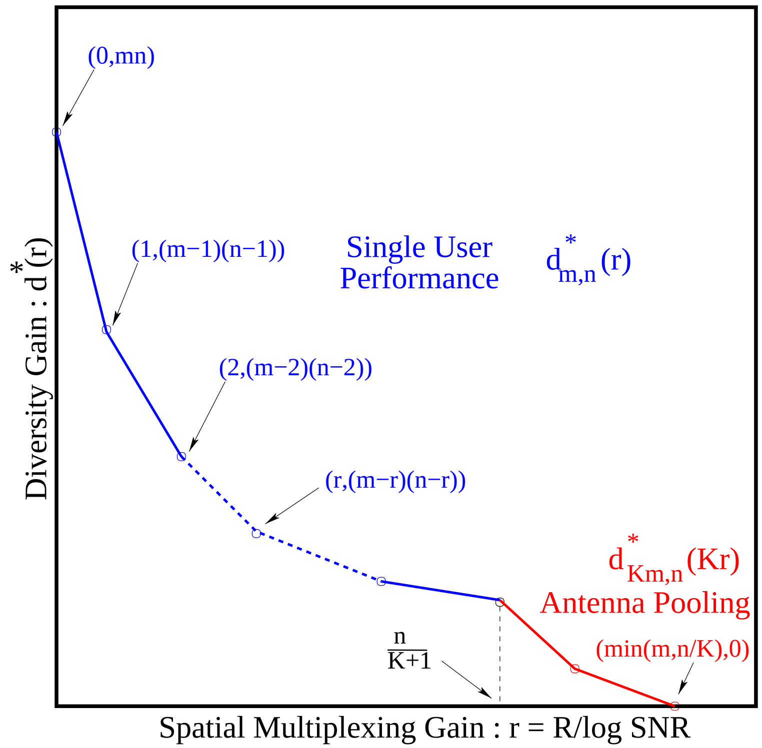
In this regime, the diversity-multiplexing tradeoff of each user is as though it is the only user in the system, i.e. $d^*_{m,n}(r)$

Multiuser Tradeoff: $m > n/(K+1)$



Single-user diversity-multiplexing tradeoff up to $r^* = n/(K+1)$.

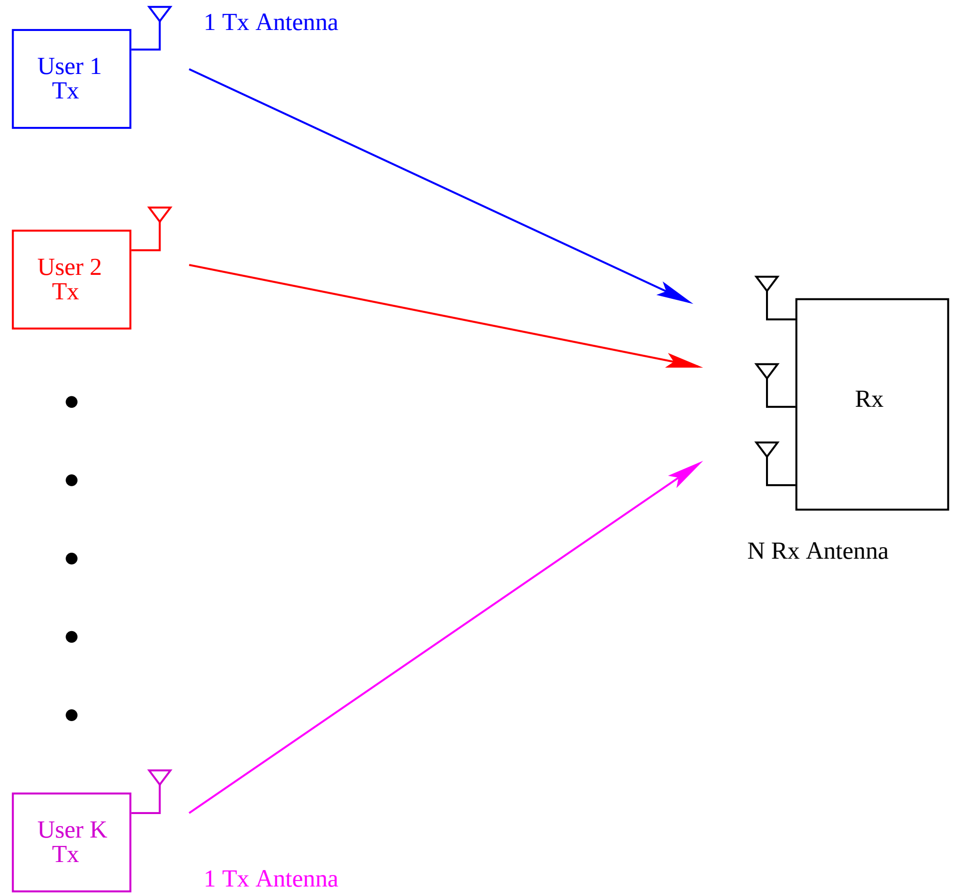
Multiuser Tradeoff: $m > n/(K+1)$



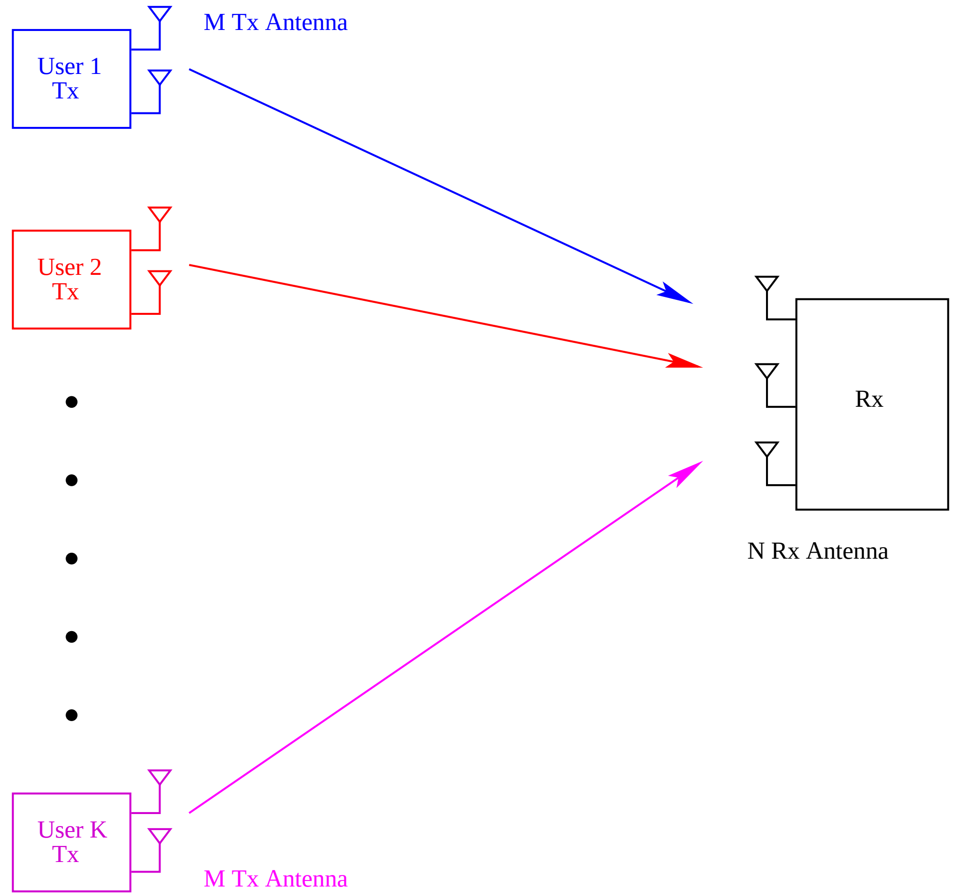
Single-user diversity-multiplexing tradeoff up to $r^* = m/(K+1)$.

For r from $n/(K+1)$ to $\min\{n/K, m\}$, tradeoff is as though the K users are pooled together into a single user with Km antennas and rate Kr, i.e. $d^*_{Km,n}(Kr)$.

Benefit of Dual Transmit Antennas



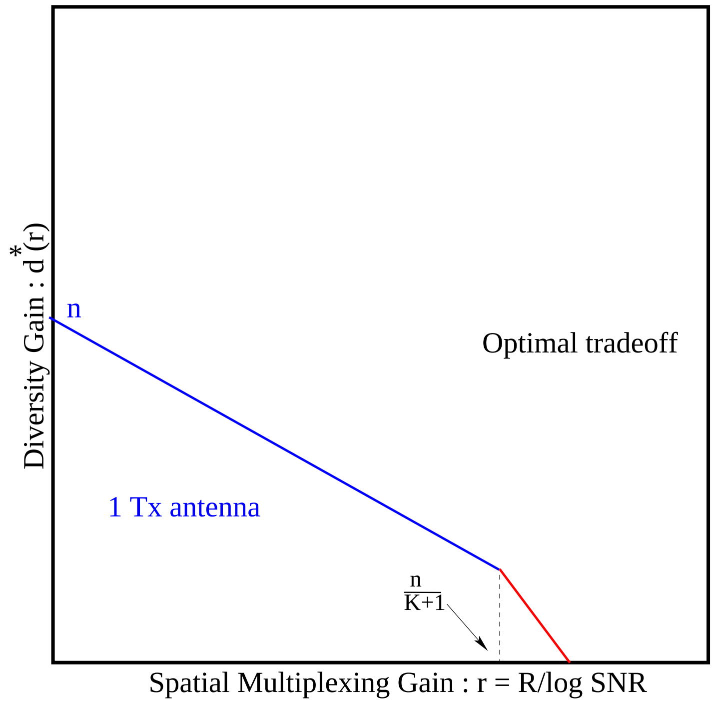
Benefit of Dual Transmit Antennas



Question: what does adding one more antenna at each mobile buy me?

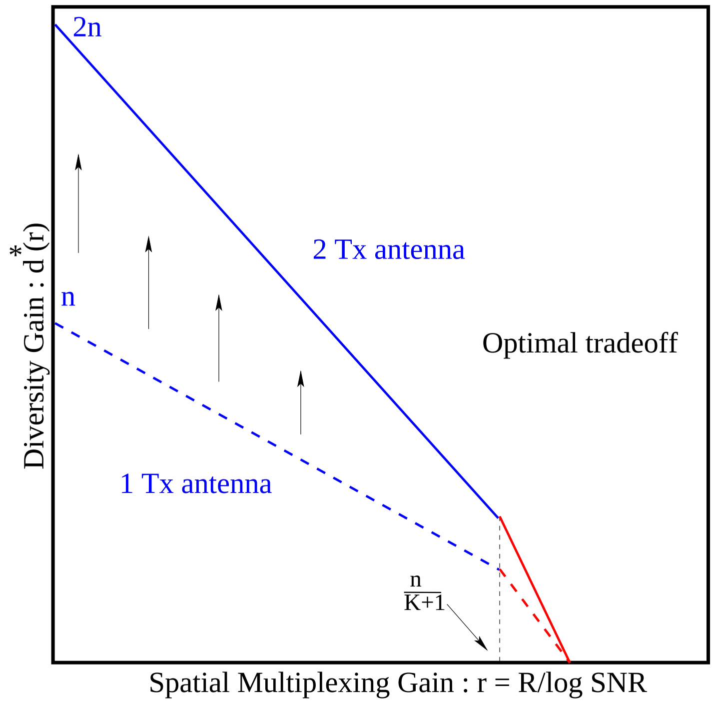
Assume there are more users than receive antennas.

Answer



Diversity Gain : d*(r)
n
Optimal tradeoff
1 Tx antenna
n/(K+1)
Spatial Multiplexing Gain : r = R/log SNR

Answer

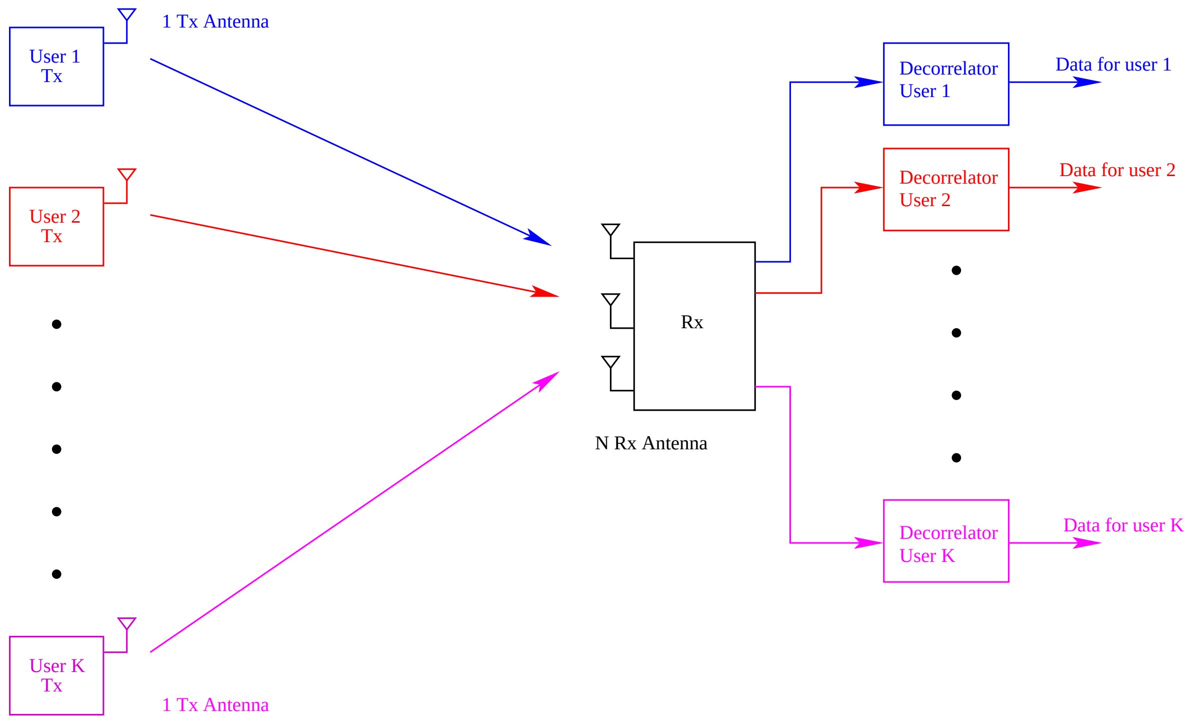


Adding one more transmit antenna does not increase the number of degrees of freedom for each user.

However, it increases the maximum diversity gain from n to $2n$.

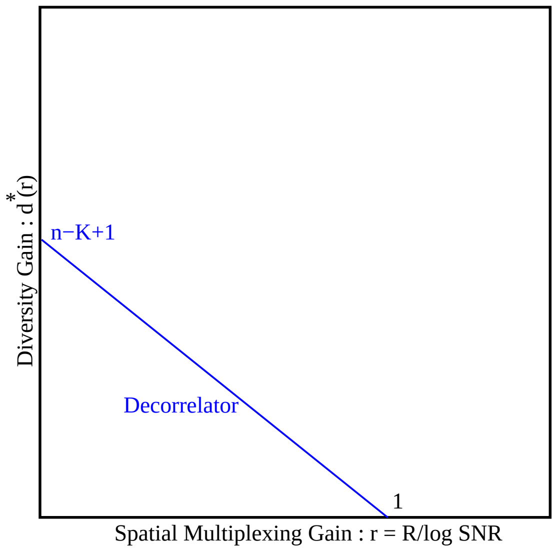
More generally, it improves the diversity gain $d(r)$ for every r.

Suboptimal Receiver: the Decorrelator/Nuller



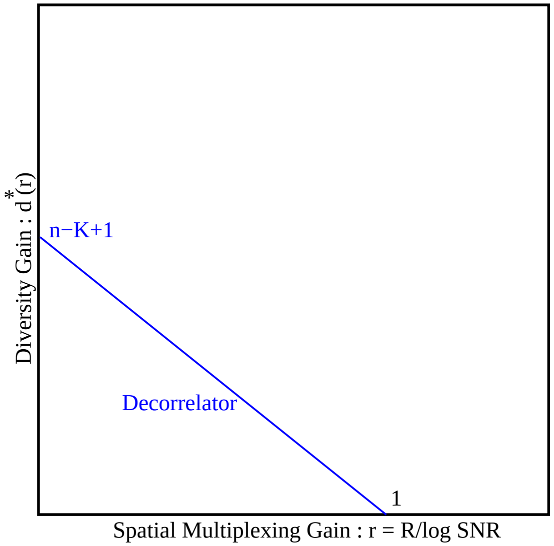
Consider only the case of $m = 1$ transmit antenna for each user and number of users $K < n$.

Tradeoff for the Decorrelator



Maximum diversity gain is $n - K + 1$: "costs $K - 1$ diversity gain to null out $K - 1$ interferers." (Winters, Salz and Gitlin 93)

Tradeoff for the Decorrelator



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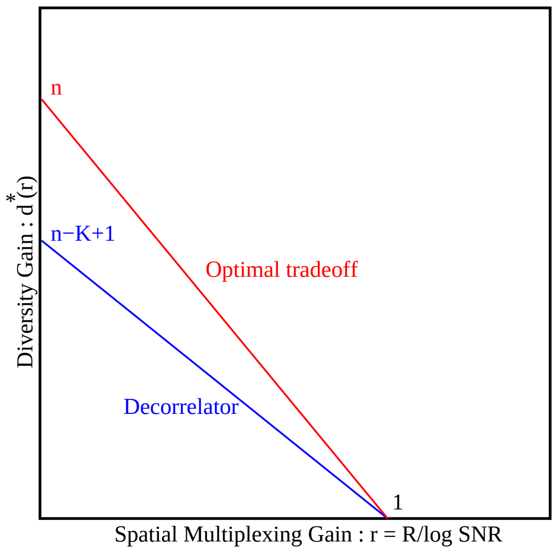
Adding one receive antenna provides either more reliability per user or accommodate 1 more user at the same reliability.

Tradeoff for the Decorrelator



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Adding one receive antenna provides either more reliability per user or accommodate 1 more user at the same reliability.

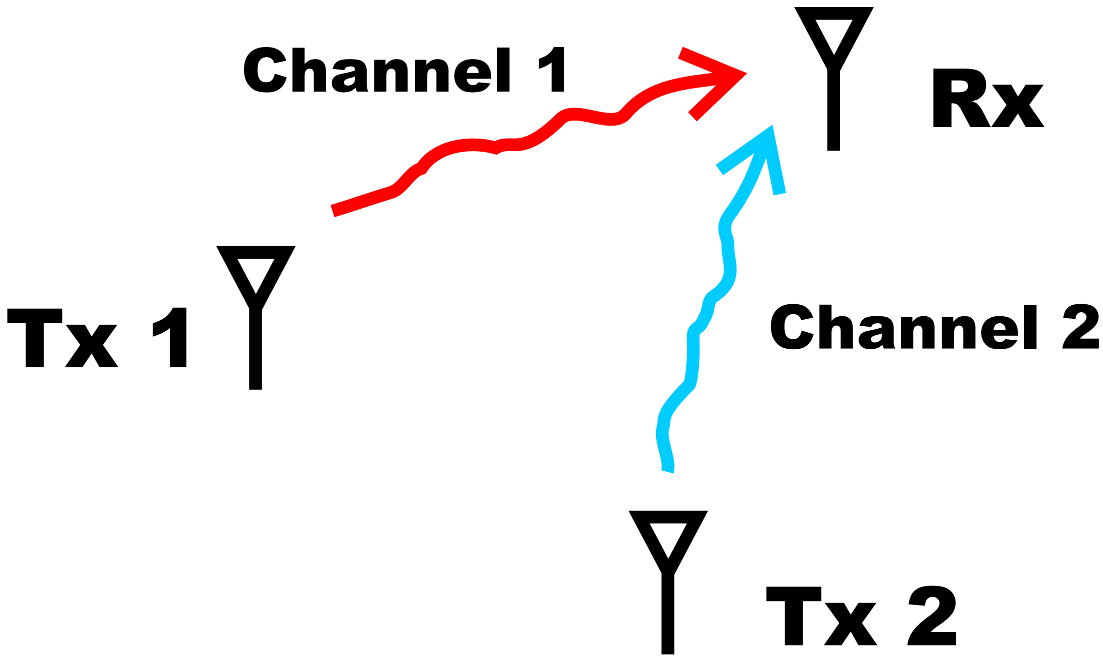
Optimal tradeoff curve is also a straight line but with a maximum diversity gain of n.

Adding one receive antenna provides more reliability per user and accommodate 1 more user.

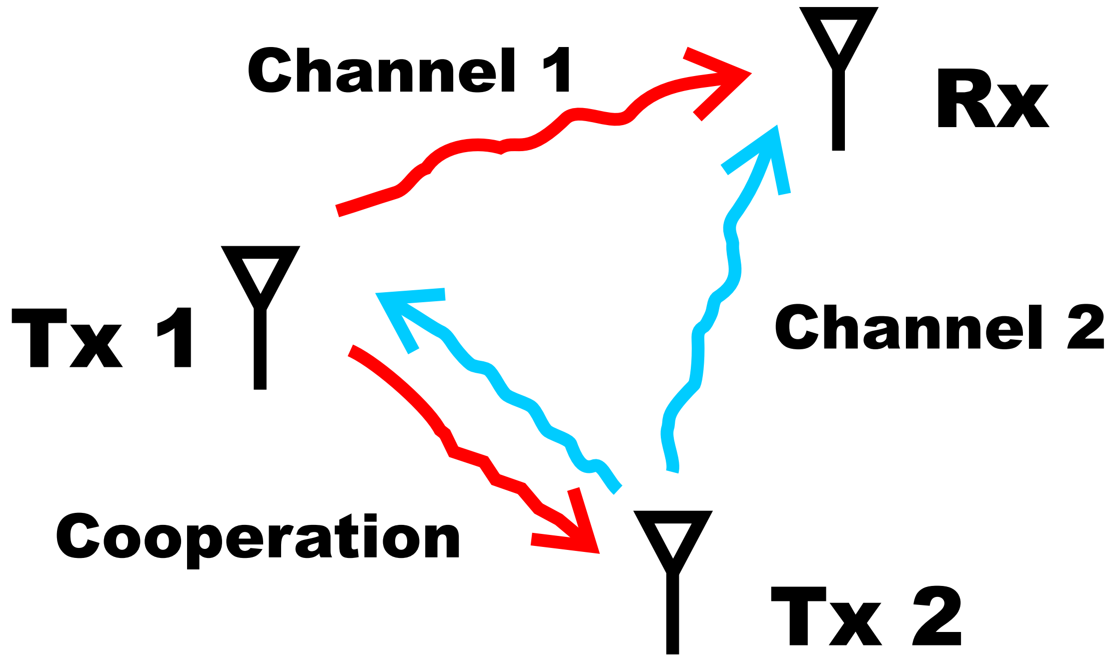
Talk Outline

- point-to-point MIMO channels
- multiple access MIMO channels
- **cooperative relaying systems**

Cooperative Relaying



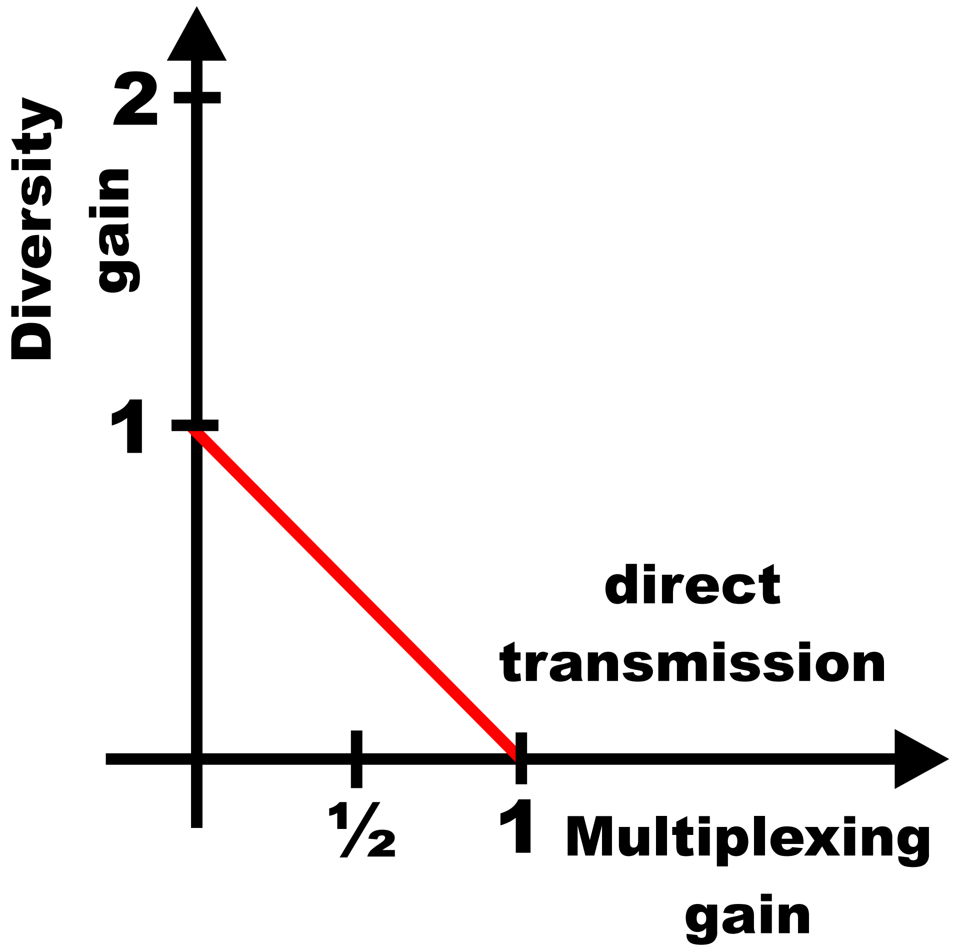
Cooperative Relaying



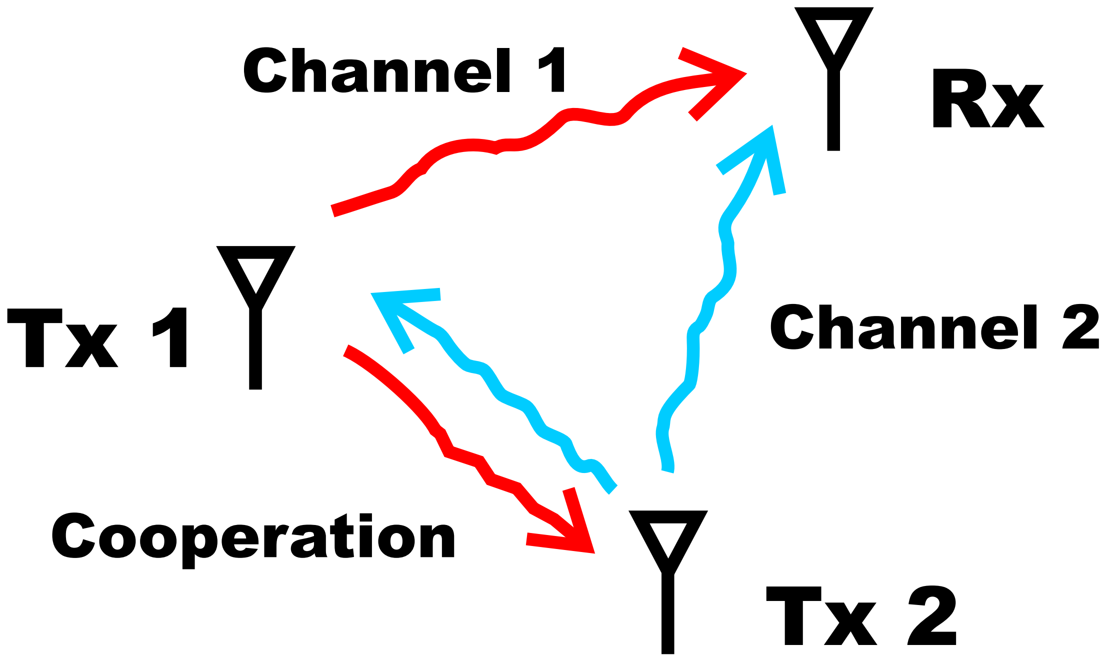
Cooperative relaying protocols can be designed via a diversity-multiplexing tradeoff analysis.

(Laneman, Tse and Wornell 02)

Tradeoff Curves of Relaying Strategies



Cooperative Relaying



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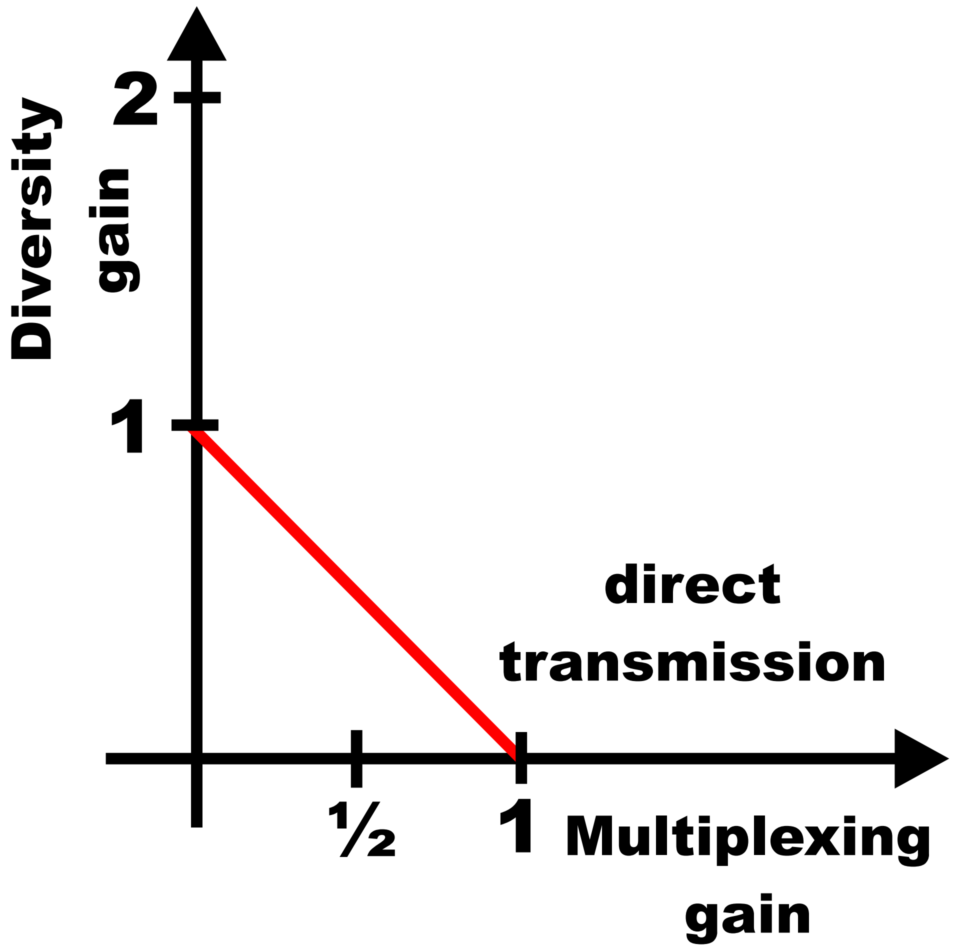

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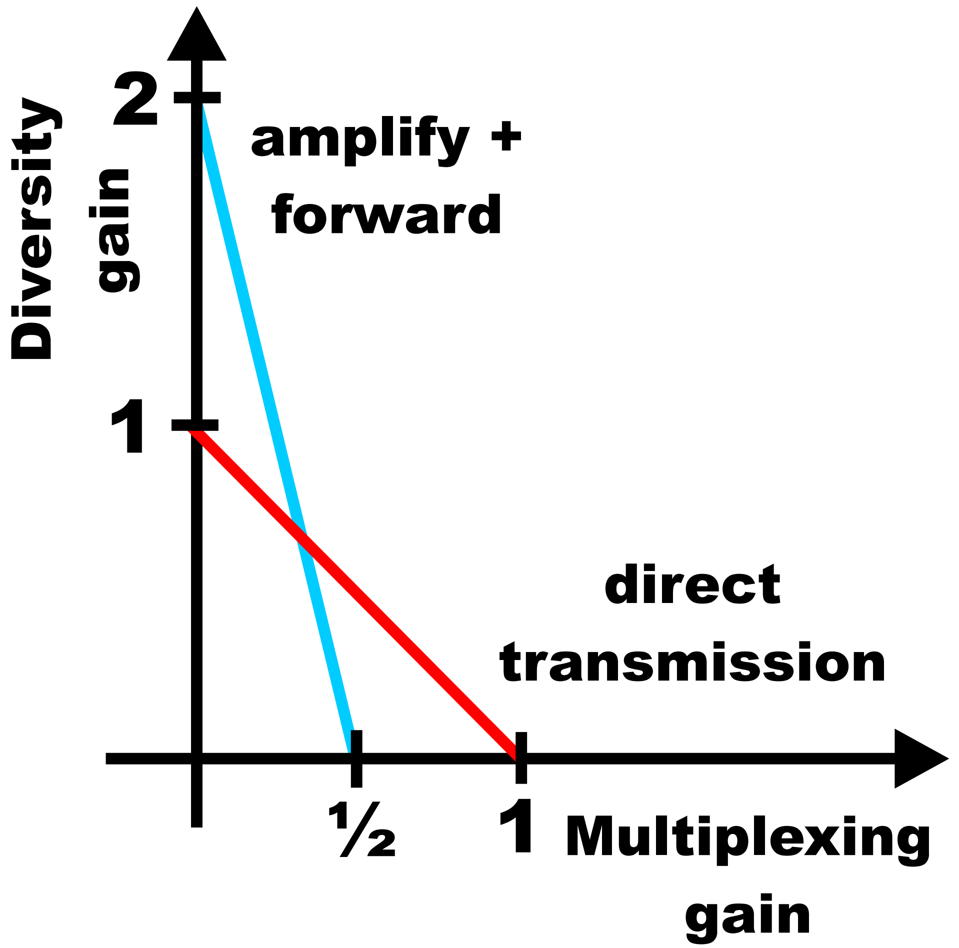
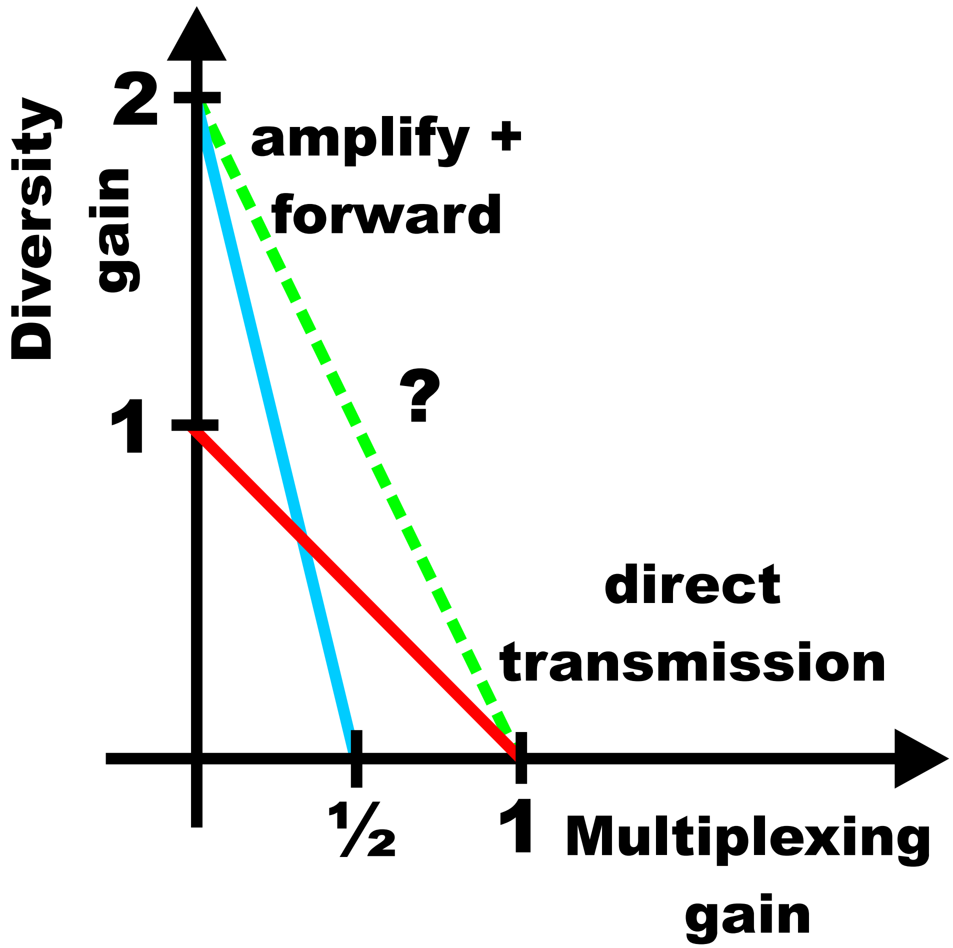


Tradeoff Curves of Relaying Strategies



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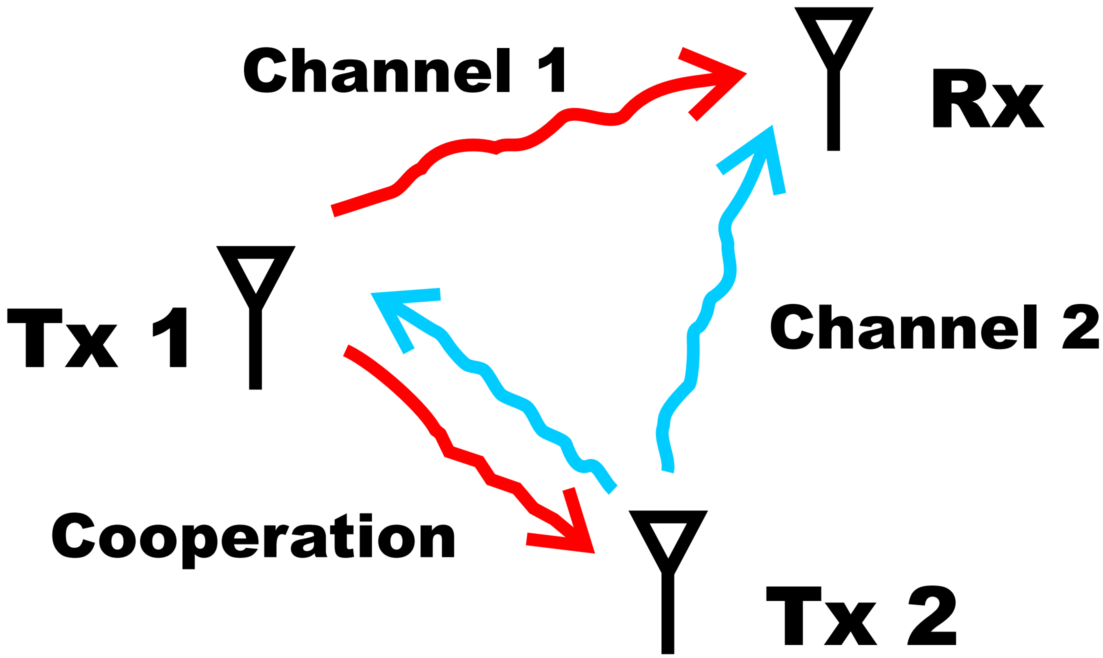

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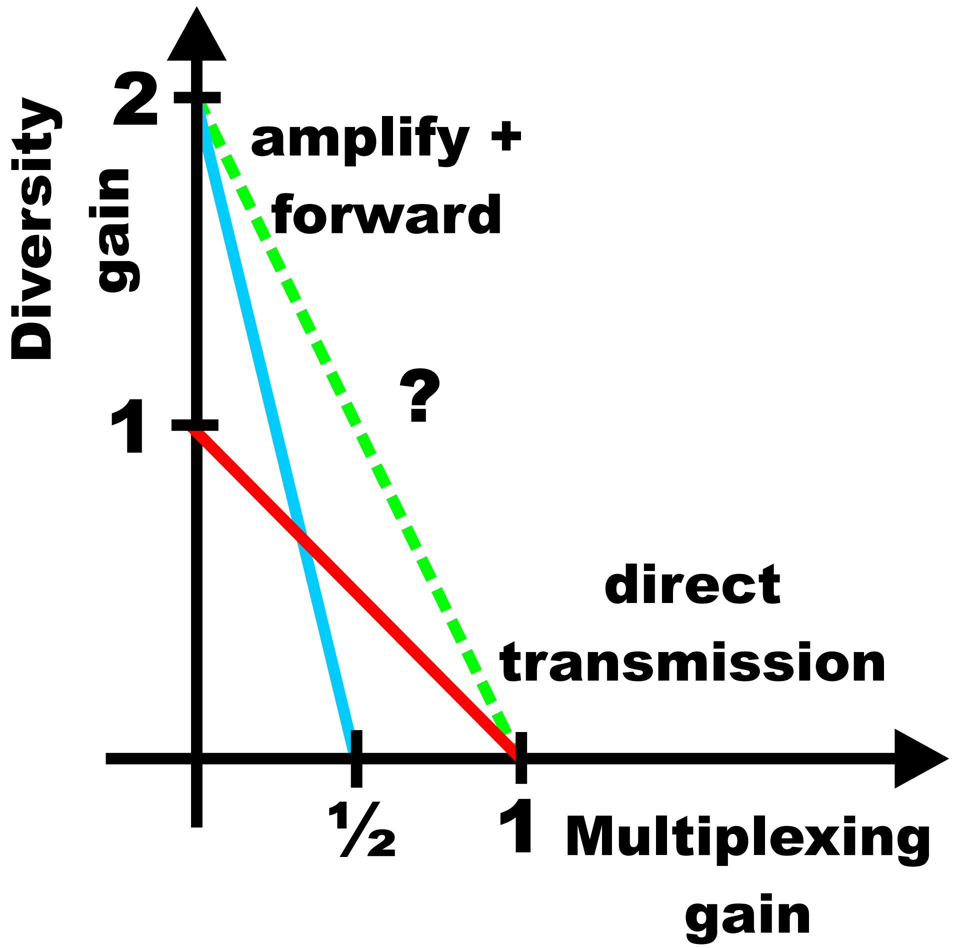

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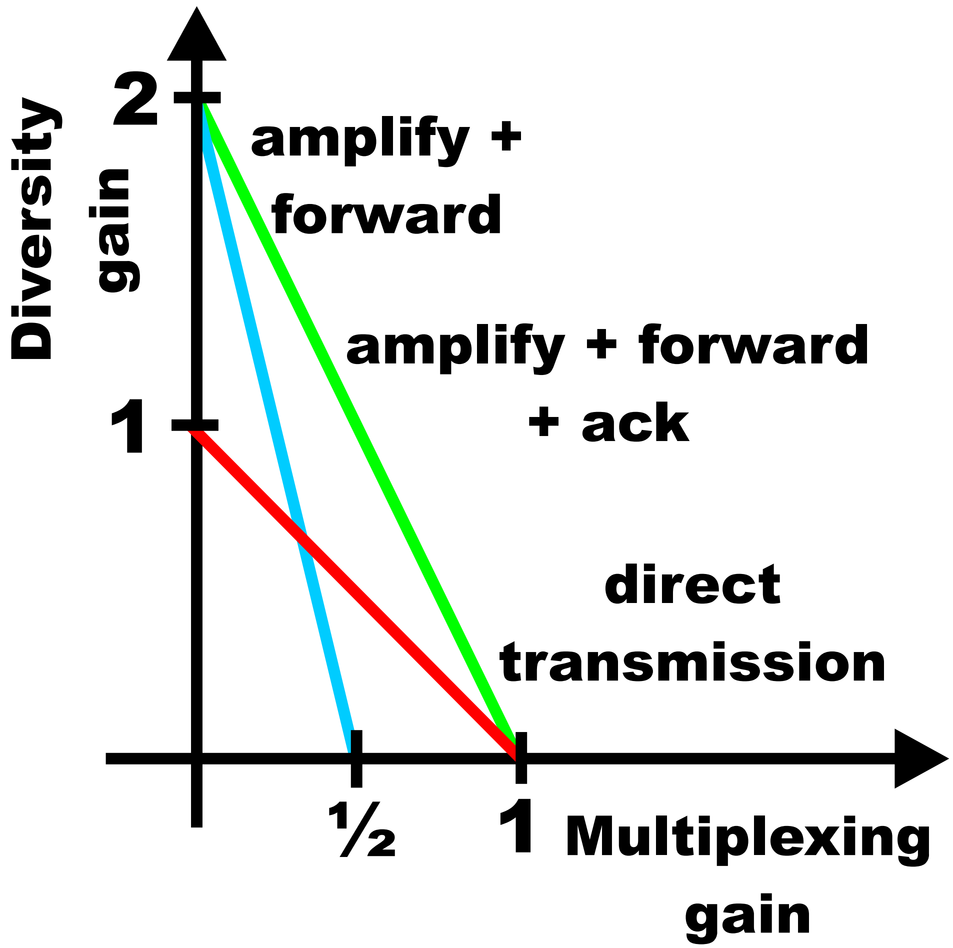

Conclusion

Diversity-multiplexing tradeoff is a unified way to look at performance over wireless channels.

Future work:

- Code and receiver design to achieve good tradeoffs.
- Application to other wireless scenarios.