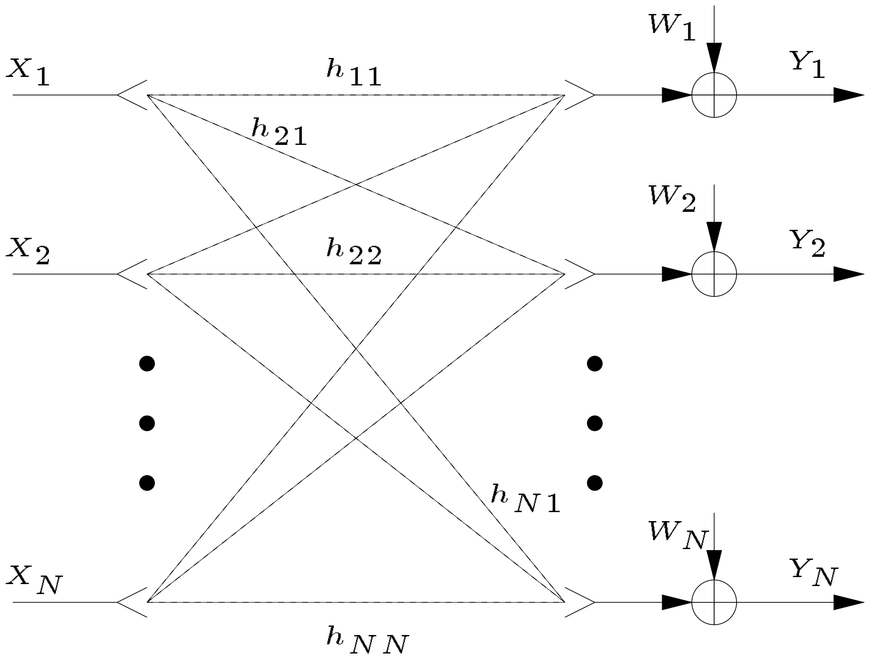
Packing Spheres in the Grassmann Manifold
A Geometric Approach to the Noncoherent Multi-Antenna Channel

Lizhong Zheng David Tse
Dept. of EECS, U.C. Berkeley

EPFL

Multi-antenna Rayleigh Fading Channel



$$y = Hx + w,$$

where H is the $N \times N$ Rayleigh fading gain matrix. (i.i.d. complex Gaussian $\mathcal{CN}(0, 1)$ entries)

Coherent Capacity

Assume H is perfectly tracked at the receiver:

$$
\begin{aligned}
C &= \mathbf{E} \log\det\left(I + \frac{\rho}{N} H H^\dagger\right) \\
&= \sum_{i=1}^{N} E \log(1 + \rho\lambda_i),
\end{aligned}
$$

where ρ is the total SNR at each receive antenna. $\lambda_i, i = 1, \ldots N$ are the eigenvalues of $\frac{1}{N} H H^\dagger$.

Equivalent to N sub-channels. Spatial diversity provides N degrees of freedom to communicate.

(Foschini 96)

High SNR Coherent Capacity

At high SNR,

$$C = \mathbf{E} \log \det \frac{1}{N} H H^{\dagger} + N \log \rho + o(1),$$

where $\mathbf{E} \log \det H H^{\dagger} = \sum_{i=1}^{N} \mathbf{E} \log \chi_{2i}^{2}$.

Every 3db increase in SNR yields N additional bps/Hz.

Assumption of Perfect Knowledge of H

This result assumes perfect tracking of the channel gains at the receiver.

In practice:

- channel may vary rapidly
- many parameters to estimate in large antenna arrays

Marzetta and Hochwald's Noncoherent Model

- Channel remains constant for intervals of T symbol times, and is independent from interval to interval.
- No side information of the channel state is assumed at the receiver or the transmitter.

Interpretations:

- 1) rough approximation of a continuous fading channel with coherence time T;
- 2) model for frequency-hopped, TDMA or packet-based systems;
- 3) model for noncoherent communication. (Think Differential PSK)

Model

Channel over a coherent interval:

$$Y = HX + W$$

$$X = \begin{bmatrix} x_1^t \\ x_2^t \\ . \\ x_N^t \end{bmatrix}, \qquad Y = \begin{bmatrix} y_1^t \\ y_2^t \\ . \\ y_N^t \end{bmatrix}$$

The i th row of X is the transmitted signal x_i from the ith antenna over the coherent interval.

Capacity per coherence time:

$$C(\rho) = \sup_{p_X} I(X; Y)$$

This is not a Gaussian channel!

Main Questions

- What is the high SNR capacity of this non-coherent channel?
- How many degrees of freedom are available for noncoherent communications?
- How to achieve capacity?

Known Results

- Capacity for $N > T$ transmit antennas can be achieved using only T of the N transmit antennas.
- Partial characterization of optimal input distribution.
- High SNR capacity obtained for single antenna case $(N = 1, T \geq 2)$.

High SNR Capacity for $N \leq T/2$

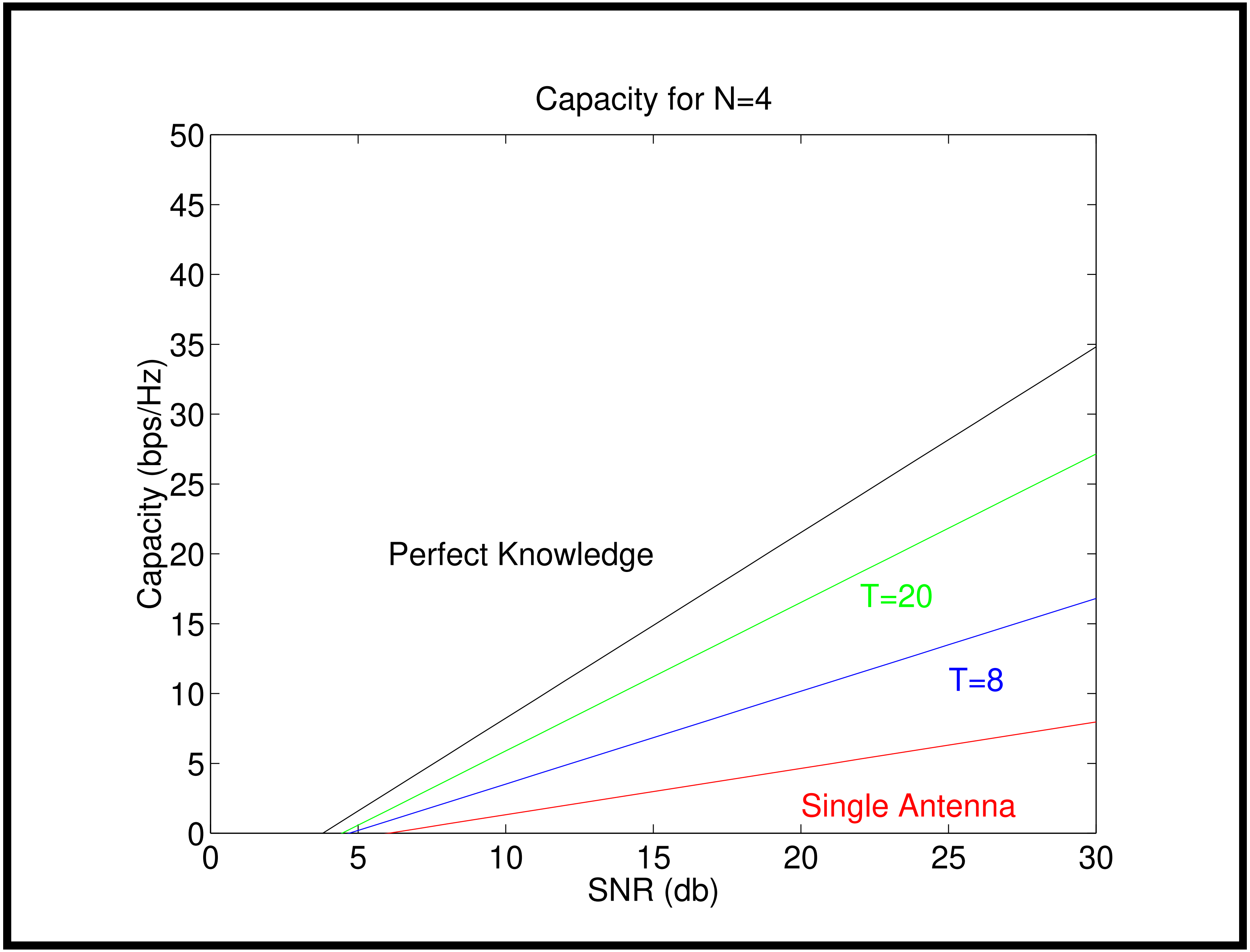
Theorem:

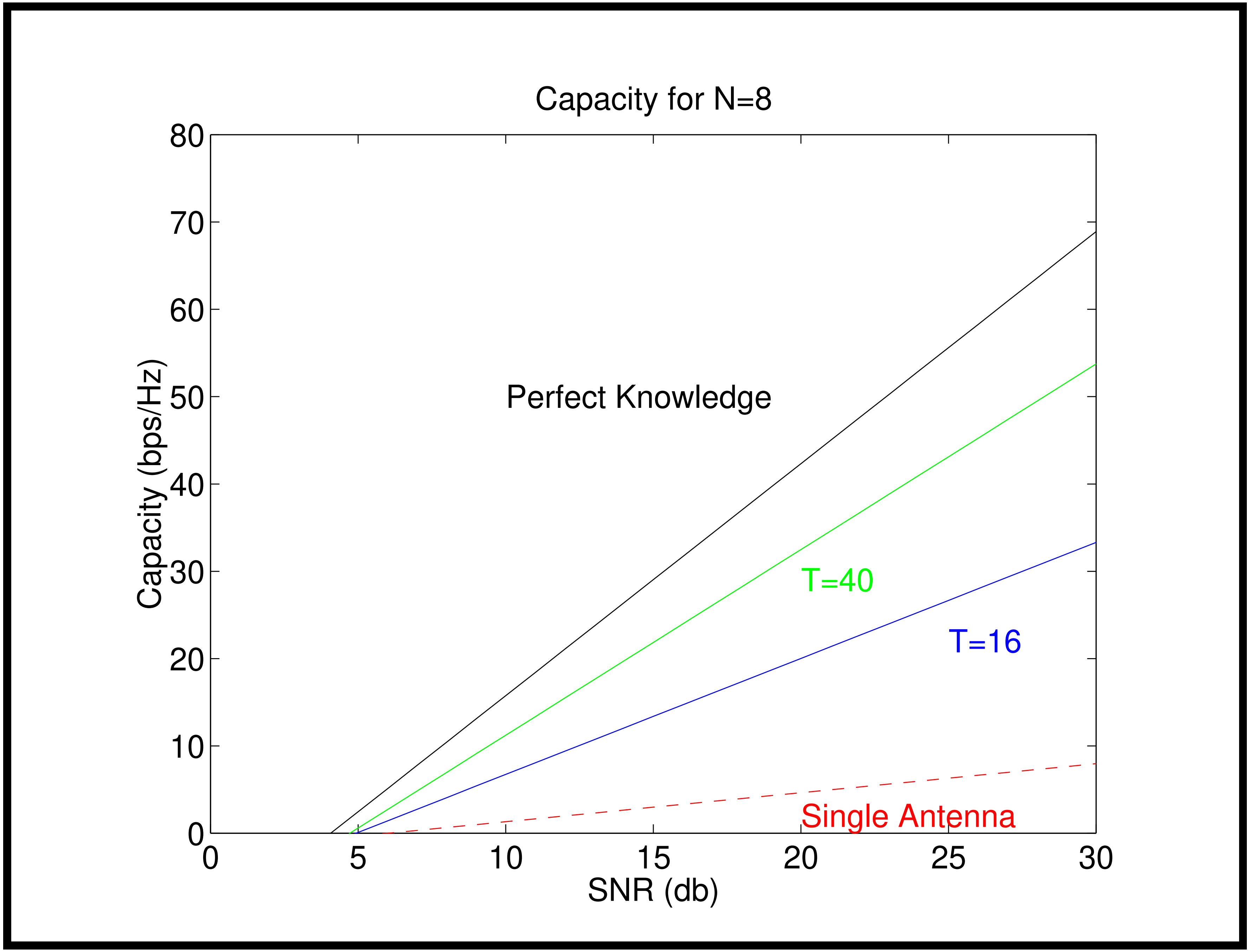
For $N \leq T/2$, the capacity $C(\rho)$ at high SNR ρ is :

$$C(\rho) = C_0 + N(T - N)\log\rho + o(1)$$

where C_0 is an explicitly computable constant.

Every 3dB increase in SNR yields $N(1 - N/T)$ additional bps/Hz.


Capacity for N=4
Capacity (bps/Hz)
SNR (db)
Perfect Knowledge
T=20
T=8
Single Antenna
0
5
10
15
20
25
30
35
40
45
50
0
5
10
15
20
25
30


Capacity for N=8
Capacity (bps/Hz)
SNR (db)
0
10
20
30
40
50
60
70
80
0
5
10
15
20
25
30
Perfect Knowledge
T=40
T=16
Single Antenna

New Coordinate System

For matrix $R \in \mathcal{C}^{N \times T}$, consider the coordinate change

$$R \to (\Omega_R, C_R),$$

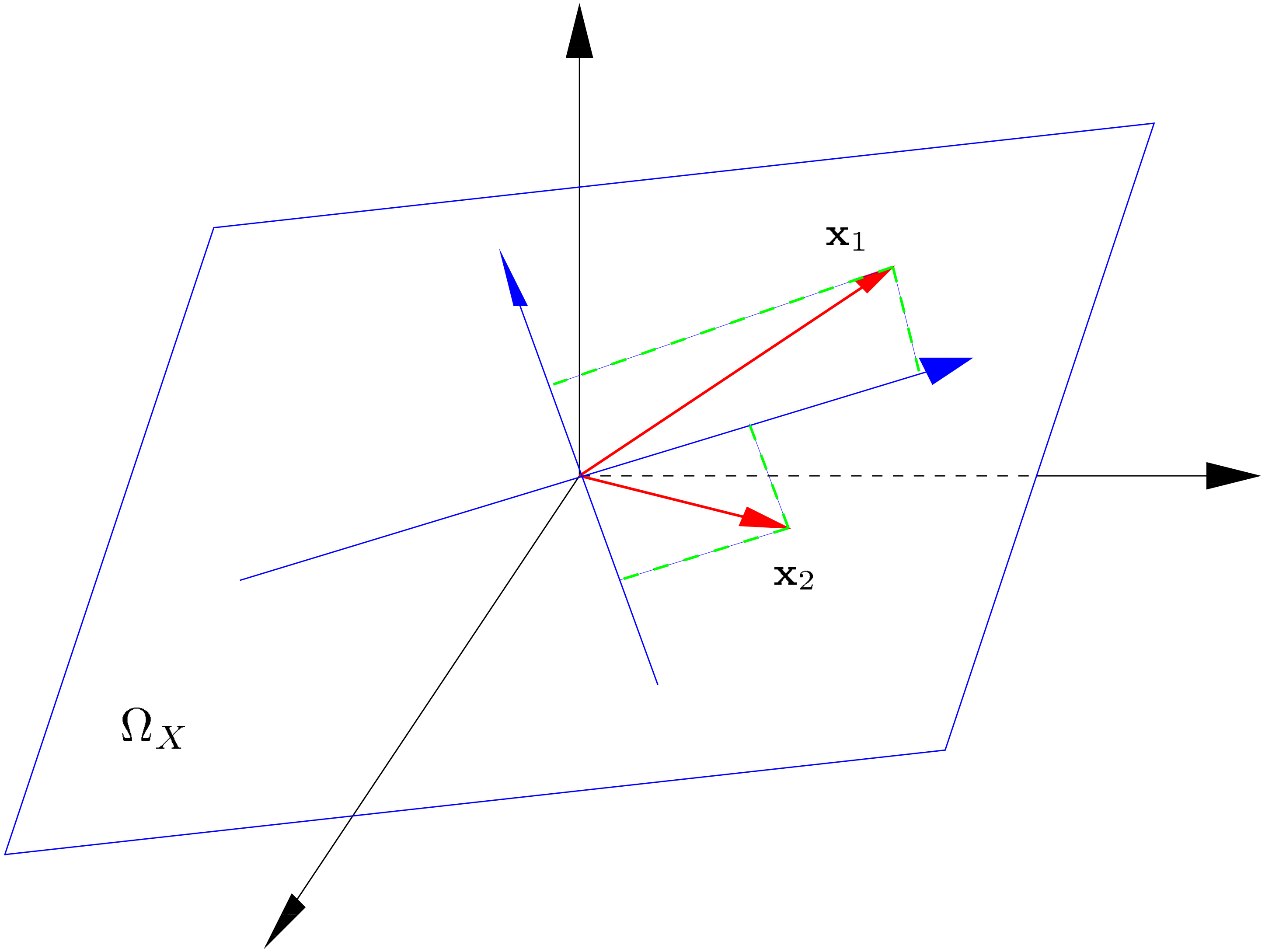
- Ω_R is the N dimension subspace of $\mathcal{C}^T$ spanned by the row vectors of R.
- C_R is a $N \times N$ matrix, the expansion of row vectors w.r.t a canonical orthonormal basis of Ω_R

Key Observation:

$$\Omega_{HX} = \Omega_X \qquad \text{with probability 1.}$$

The row space of X is invariant to the channel fading H, and is perturbed only by the additive noise W.

Example: $N = 2, T = 3$



Decomposition of Mutual information

$$Y = HX + W$$

$$X \to (\Omega_X, C_X).$$

The new coordinate system decomposes the problem into

- communicating *via* transmitted signal subspace Ω_X
- communicating *within* the subspace using C_X

$$I(X;Y) = I(\Omega_X;Y) + I(C_X;Y \mid \Omega_X)$$

Optimal Input

Result:

At high SNR, the optimal input distribution of X is to choose:

- C_X unitary
- Ω_X uniformly distributed in the set of N-dim subspaces in $\mathbf{C}^T$.

Observe that in the new coordinates:

$$HX \to (HC_X, \Omega_X)$$

Conditional on the unitary C_X, HC_X has the same distribution as H; thus the output Y is independent of C_X.

All the information is conveyed by the row space Ω_X

Communicating via Subspaces is not New!

In Differential PSK, encoding data by a phase difference ϕ is equivalent to representing it by the subspace of $\mathbf{C}^2$ spanned by

$$\begin{bmatrix} 1 & e^{j\phi} \end{bmatrix}$$

In multiple symbol DPSK (Divsalar and Simon 90), data is represented by a subspace in $\mathbf{C}^T$ spanned by:

$$\begin{bmatrix} 1 & e^{j\phi_1} & e^{j\phi_2} & \cdot & \cdot & e^{j\phi_{T-1}} \end{bmatrix}$$

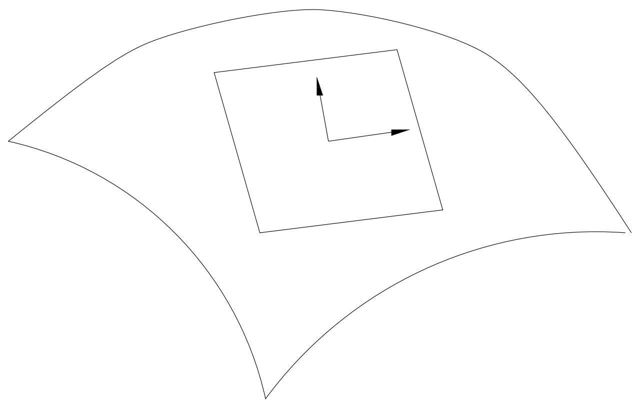
where $\phi_1, \ldots, \phi_{T-1}$ are the sequence of phase differences.

Communicating on Grassmann Manifold

Grassmann manifold $G(T,N)$ = the set of all N-dimensional subspaces of $\mathcal{C}^T$.

$$\dim[G(T,N)] = N(T-N).$$

$$\mathrm{vol}[G(T,N)] = \frac{\prod_{i=T-N+1}^{T} \frac{2\pi^i}{(i-1)!}}{\prod_{i=1}^{N} \frac{2\pi^i}{(i-1)!}}.$$



High SNR Capacity for $N \leq T/2$ (2)

$$C(\rho) \quad = \quad C_0 + N(T-N)\log\rho + o(1)$$

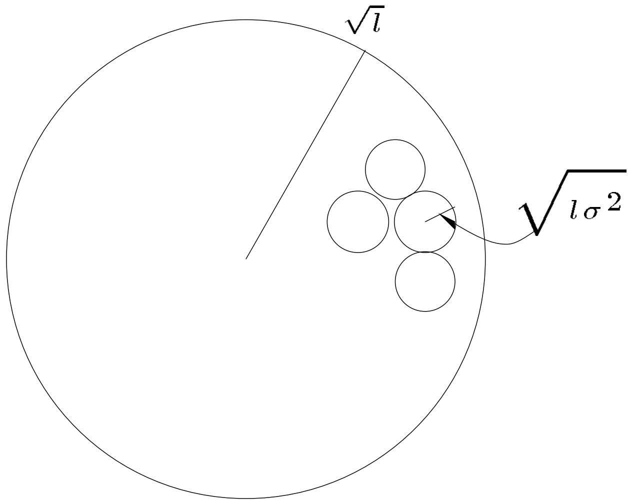
where

$$\begin{aligned} N(T-N) &= \dim[G(T,N)] \\ C_0 &= \log\mathrm{vol}[G(T,N)] + (T-N)\mathbf{E}[\log\det(\frac{T}{N\pi e}HH^\dagger)] \end{aligned}$$

and $\mathbf{E}\log\det HH^\dagger = \sum_{i=1}^{N}\mathbf{E}\log\chi^2_{2i}$.

Sphere Packing for AWGN Channel

$$y = x + w$$



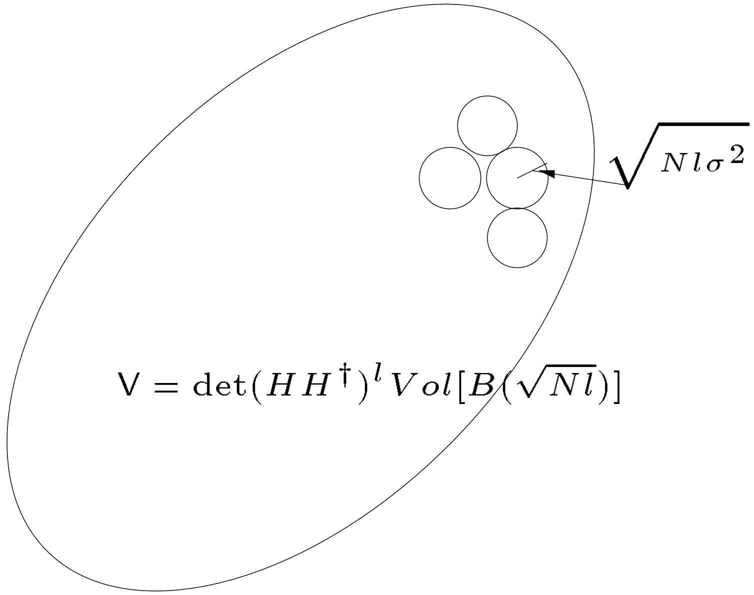
Codeword length $= l$. Packing in l dimensonal space.

$$C \approx \frac{1}{l} \log \frac{\mathrm{vol}[B(\sqrt{l})]}{\mathrm{vol}[B(\sqrt{l\sigma^2})]} = \frac{1}{l} \log \frac{(\sqrt{l})^{2l}}{(\sqrt{l\sigma^2})^{2l}} = \log \rho, \qquad \rho = \frac{1}{\sigma^2}$$

Packing small noise spheres into a large sphere

Sphere Packing for Coherent Multi-Antenna Channel

$$y = Hx + w, \qquad H \text{ is known.}$$



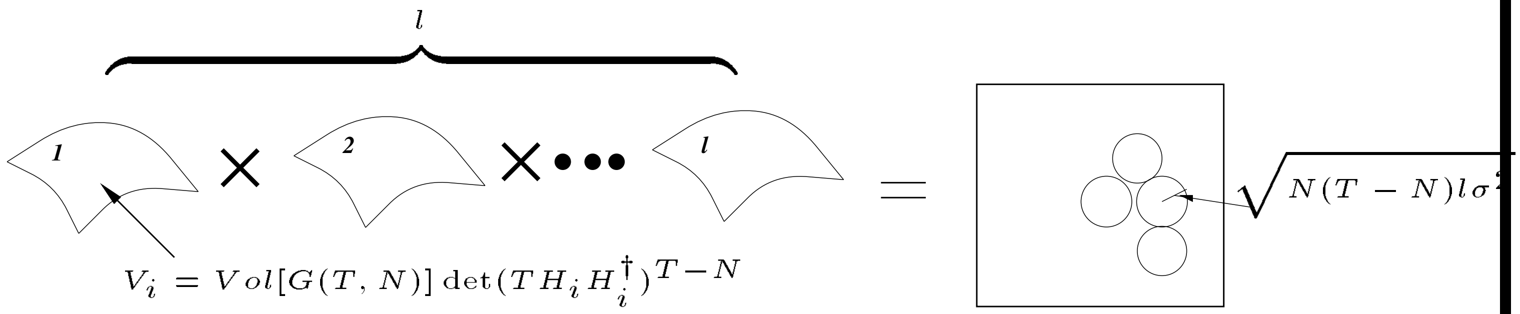
Received signal lies in an ellipsoid of dimension Nl, scaled by H.

$$C \approx \frac{1}{l} \log \frac{V}{\mathrm{vol}[B(\sqrt{Nl\sigma^2})]} = E \log\det HH^\dagger + N \log \frac{\rho}{N}, \quad \rho = \frac{N}{\sigma^2}$$

Packing small noise spheres into a large ellipsoid

Sphere Packing for Non-Coherent Channel

$$Y = HX + W, \quad H \text{ is unknown.}$$



Received signal space is a product of l scaled Grassmann manifolds, of total dimension $lN(T-N)$.

$$C \approx \frac{1}{l}\log\frac{\prod_{i=1}^{N} V_i}{\mathrm{vol}[B(\sqrt{N(T-N)l\sigma^2})]}$$

Packing noise spheres into a product of scaled Grassmann manifolds.

Number of Degrees of Freedom

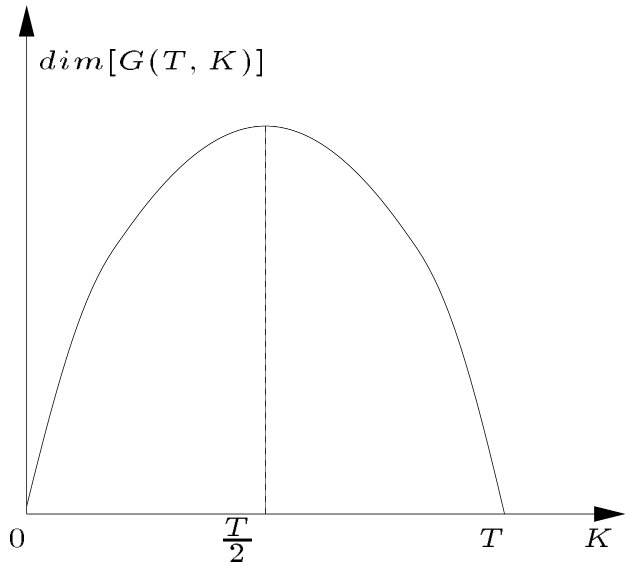
- Communicating on the Grassmann manifold, the number of degrees of freedom in each coherence interval of length T is:

$$\dim[G(T,N)] = N(T-N)$$

- Compare to coherent capacity, we lose N^2 degrees of freedom in each coherence interval.

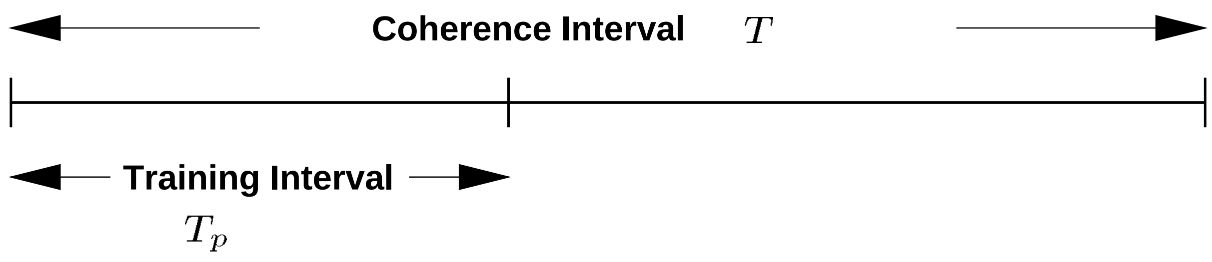
Capacity for $N > T/2$

For fixed T, if we use K transmit antennas, the number of degrees of freedom is $\dim[G(T,K)]$

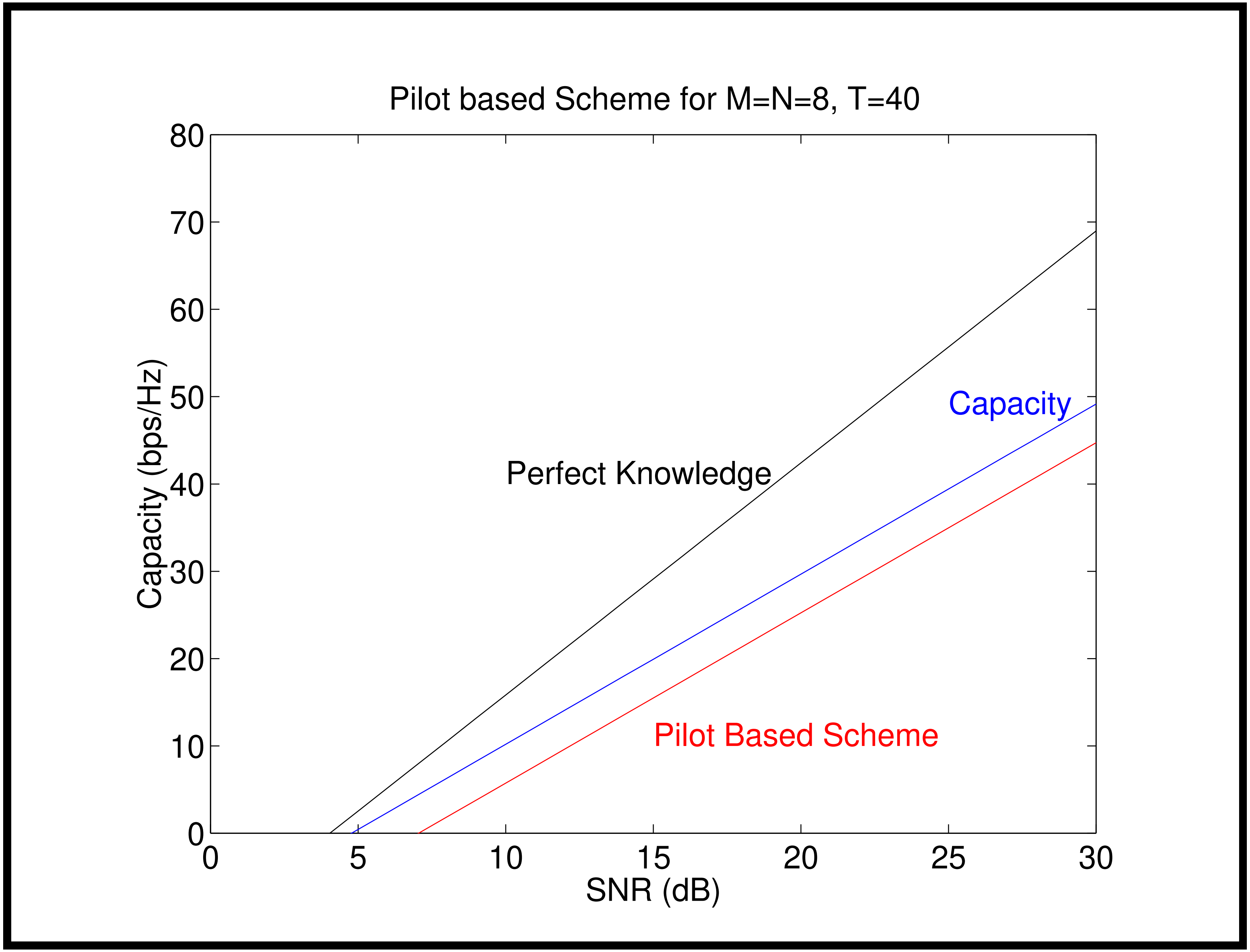


When $N \geq T/2$, use only $K = \lfloor \frac{T}{2} \rfloor$ of the antennas to maximize the number of degrees of freedom $K(T-K)$.

Training-Based Schemes



- Suppose we use M transmit antennas.
- N measurements are obtained at each symbol time during training.
- Training duration T_p has to be at least M to estimate all the MN parameters.
- degrees of freedom left for communication $= M(T - M)$
- To maximize the number of d.o.f., use $M^* = \min\{N, \lfloor \frac{T}{2} \rfloor\}$ transmit antennas, and set $T_p = M^*$.
- **Achieve the same number of d.o.f. as the channel capacity**


Pilot based Scheme for M=N=8, T=40
Capacity (bps/Hz)
SNR (dB)
Perfect Knowledge
Capacity
Pilot Based Scheme
0
10
20
30
40
50
60
70
80
0
5
10
15
20
25
30

Rectangular Systems

- Can compute high SNR capacity for the general case when number of transmit M is not equal to number of receive N.
- For $M > N$, optimal strategy is to use only N of the M transmit antennas.
- In general, use $M^* = \min\{M, N, \lfloor \frac{T}{2} \rfloor\}$ transmit antennas. The information carrying object is in $G(T, M^*)$. Achieve $M^*(T - M^*)$ d.o.f. in each coherence interval.

Conclusions

- For the noncoherent $N \times M$ multi-antenna channel with coherence time T, every 3dB increase in SNR yields

$$M^*(1 - M^*/T) \quad \text{bps/Hz}, \qquad M^* \equiv \min(M, N, \lfloor T/2 \rfloor)$$

- Given a fixed coherence time, capacity at high SNR is maximized by using $T/2$ transmit antennas. System is **coherence-time limited** rather than antenna limited.
- Geometric interpretation of capacity expression in terms of **sphere packing in the Grassmann manifold.**