Final Exam

You have 3 hour to complete this open book exam of 160 points. Be concise and, above all, clear. Be sure that your name is printed on your book, and that you are neat. You must show all work to get full credit. Best of luck.

Question 1 (20 points)
Are the following statements true or false? Explain.

1. Because the t distributions have thicker tails than the normal distribution, the null hypothesis is less likely to be rejected if we use the t distributions instead of the normal distribution. Therefore it is always better to use the t distributions than the normal distribution for testing hypotheses.

2. Let $X_i, i = 1, \ldots, n$ be i.i.d $N(\mu, \sigma^2)$ with known $\sigma^2$. Since a test with rejection area of the form \{ $\bar{X} > d$ \} for some $d$ is uniformly most powerful for $H_0 : \mu = 0$ against $H_1 : \mu > 0$, it can also be used to test $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$.

3. For an i.i.d sample $X_i \sim N(\mu, \sigma)$, the sample mean $\bar{X}$ is a better estimator of $\mu$ than $\frac{1}{2} \bar{X}$.

Question 2 (30 points)
Let $X_i$ be i.i.d uniform $(0, \theta)$. The parameter to estimate is $\mu \equiv \frac{\theta}{2}$. Define $\hat{\mu}_1 = \bar{X}$ and $\hat{\mu}_2 = \frac{1}{2} \hat{Z} \equiv \frac{1}{2} \max(Z_1, \ldots, Z_n)$.

1. Using the mean square error as the criterion of comparison, which estimator do you prefer and why?

2. Is it possible to find the minimum variance unbiased estimator for $\mu^2$? Explain your reasoning.
3. Is it possible to find the uniformly most powerful test for testing between $H_0 : \theta = 2$ against $H_1 : \theta > 2$?

Question 3 (45 points)

Assume $Y_i, i = 1, \ldots, n$ be i.i.d. distributed according to the exponential following distribution with parameter $\lambda$:

$$f_{Y_i}(y) = \frac{1}{\lambda} \exp\left(-\frac{y}{\lambda}\right) \text{ for } y \geq 0. \tag{1}$$

1. Find the MLE for $\lambda$, denoted by $\hat{\lambda}_{MLE}$ and derive its asymptotic distribution.

2. Suppose you are given the value of $\hat{\lambda}_{MLE}$ but no other information from the sample of $Y_i, i = 1, \ldots, n$. Is it possible to construct a confidence interval for the parameter $\lambda$, of a given size $\alpha$ that is asymptotically valid using the asymptotic distribution you derived above?

3. Suppose now your assumption of model (1) turns out to be wrong. Instead the true data generating process is such that $Y_i$ is from a Poisson distribution (which is discrete) with parameter $\theta$: for each $i$ and for $y = 0, 1, \ldots,$

$$P(Y_i = y) = \exp\left(-\theta\right) \frac{\theta^y}{y!}. \tag{2}$$

$Y_i$ are still i.i.d. distributed across $i = 1, \ldots, n$. However, you are not aware that the true distribution is Poisson but instead continue to use model (1). Does the confidence interval constructed above provide an asymptotically valid confidence interval (of the given size $\alpha$) for the parameter $\theta$? If not, is it possible to modify it so that it is asymptotically valid?

4. If model (1) is correctly specified, is it possible to construct an exact finite sample confidence interval for the parameter $\lambda$? (In the sense that the converge probability is exactly $\alpha$ for each sample size $n$, rather than only converges to $\alpha$ as $n \to \infty$.)
5. Is the exact finite sample confidence interval still a valid exact finite sample confidence interval for $\theta$ if the true data generating process is the Poisson distribution?

6. Is the exact finite sample confidence interval still an asymptotically valid confidence interval for $\theta$ if the true data generating process is the Poisson distribution?

**Question 4 (35 points)**

Let $X_i, i = 1, \ldots, n$ be i.i.d, where each $X_i$ has the following probability distribution:

$$X_i = \begin{cases} 
3 \text{ with prob. } \lambda \\
2 \text{ with prob. } \mu \\
1 \text{ with prob. } \rho \\
0 \text{ with prob. } 1 - \lambda - \mu - \rho.
\end{cases}$$

You want to test the null hypothesis $H_0 : \exp(\lambda) = 4\mu^2$ against the alternative hypothesis of $H_1 : \exp(\lambda) \neq 4\mu^2$.

1. Derive a Wald test statistic and its asymptotic distribution. Derive the rejection area of your test.

2. Describe a likelihood ratio test statistic and its asymptotic distribution. Describe the rejection area of your test.

3. What is the power of your tests when $n \to \infty$ under the alternative hypothesis that $\exp \lambda \neq \mu^2$.

4. (*) Is it possible to derive the power functions of your tests under the local alternative hypothesis of $\exp(\lambda) = 4\mu^2 + \frac{c}{\sqrt{n}}$, as a function of $c$?

**Question 5 (30 points)**

You are interested in modeling the number of eggs laid by an insect, using a two stage hierarchical model. Let $Y =$ number of eggs laid, $\Lambda =$ a positive random variable. You assume that
Λ is distributed as exponential with hazard parameter $1/\beta$: $f_\Lambda(\lambda) = \frac{1}{\beta} e^{-\frac{1}{\beta} \lambda} 1 (\lambda > 0)$. Conditional on Λ, Y is a Poisson random variable with parameter Λ: $P(Y = y|\Lambda) = e^{-\Lambda} \Lambda^y / y!$ for $y = 0, 1, \ldots$. You are given an i.i.d sample of observations $Y_i, i = 1, \ldots, n$, but you do not observe Λ.

1. Derive the maximum likelihood estimator for $\beta$ and its asymptotic distribution.

2. Constructive a confidence interval for $\beta$ that is asymptotically valid.

3. Derive a test statistic to test the null hypothesis of $H_0: \beta = 1$ against the alternative hypothesis $H_1: \beta > 1$. Is it possible to come up with a uniformly most powerful test?