Procurement Mechanisms for Assortments of Differentiated Products

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We consider the problem faced by a procurement agency that runs a mechanism for constructing an assortment of differentiated products with posted prices, from which heterogeneous consumers buy their most preferred alternative. Procurement mechanisms used by large organizations, including framework agreements (FAs), which are widely used in the public sector, often take this form. When choosing the assortment, the procurement agency must optimize the tradeoff between offering a richer menu of products for consumers and offering less variety, hoping to engage the suppliers in more aggressive price competition. We formulate the problem faced by the procurement agency as a mechanism design problem, and we progressively incorporate more complex and often more realistic implementation constraints, including that the allocations should be *decentralized* (that is, consumers *choose* what to buy) and that payments must be implemented through *linear pricing* (in particular, no up-front payments are allowed). We characterize the optimal buying mechanisms that highlight the importance of restricting the entry of close-substitute products to the assortment as a way to increase price competition without much damage to variety. Motivated by the implementation of the Chilean FAs, which are being used to acquire around US$3 billion in goods and services per year, we leverage our characterization of the optimal mechanism to study the design of first-price-auction-type mechanisms that are commonly used in public settings. Our results shed light on simple ways to improve their performance.

1 Introduction

Procurement mechanisms in which individual consumers affiliated to an organization (such as a government or a private company) must make their purchases through assortments previously chosen by their organization are widely used in the public and private sectors. For example, private firms and universities typically use assortments of selected suppliers and products from which their workers or units can buy computers and other supplies.[1] Health plans maintain drug formularies, lists of prescription drugs available to enrollees for free or at a minimum co-pay, to help

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[1]See, e.g., Stanford University, University of Minnesota.

manage drug costs.[2] Governments worldwide use *framework agreements* (FAs), in which the central government procurement agency first selects an assortment of products through competitive bidding in an auction mechanism, and then public organizations buy from this assortment as needed. FAs are now recognized as a fundamental tool in public procurement: the European Union awarded €80 billion using FAs in 2010, which accounts for 17% of the total value of all public procurement (European Commision 2012); similarly, the Chilean government procurement agency (Dirección ChileCompra), purchased goods for US$3 billion though FAs in 2018, 22% of the value of all public procurement in Chile (Área de Estudios e Inteligencia de Negocios, Dirección ChileCompra 2019).

Deciding assortments centrally is useful even in settings where, in principle, each consumer could be in charge of his own purchases; by aggregating individual purchases through the assortment, the organization may be able to exploit the purchasing power of a large buyer. However, at the same time, these assortments must contain adequate variety to satisfy the possibly heterogeneous needs of consumers, which is an important concern in the settings previously described. For example, while some public organizations or individuals may need to buy laptops with attractive graphics features, others may need laptops with high processing power. Similarly, consumers buying from a food assortment may have specific dietary constraints, such as those arising in hospitals or in environments with kids. Therefore, the procurement agency faces the following tradeoff. On the one hand, consumers have heterogeneous preferences; hence, increasing product variety may increase consumer welfare. On the other hand, reducing the number of products in the assortment may increase suppliers' incentives to aggressively compete in prices, so that their products have a better chance of being part of the small selection of items. The main objective of this paper is to provide insights into how to achieve (some) variety to satisfy consumers' heterogeneous preferences in a cost-efficient way.

This paper is among the first to provide a formal analysis of procurement mechanisms for constructing assortments of products. Our main contributions are to introduce a model of the problem faced by the procurement agency, to characterize the optimal mechanisms under progressively more complex implementation constraints, and to study the design of simpler first-price auction mechanisms that are commonly used in public procurement. We describe these contributions in more detail next.

We propose a model in which suppliers offer differentiated products within a certain category (e.g., computers) and have a private cost for producing a unit of product. Consumers have heterogeneous preferences for specific products within the category (e.g., for different computer models),

[2] For example, Aetna, Blue Shield of California, United Healthcare.

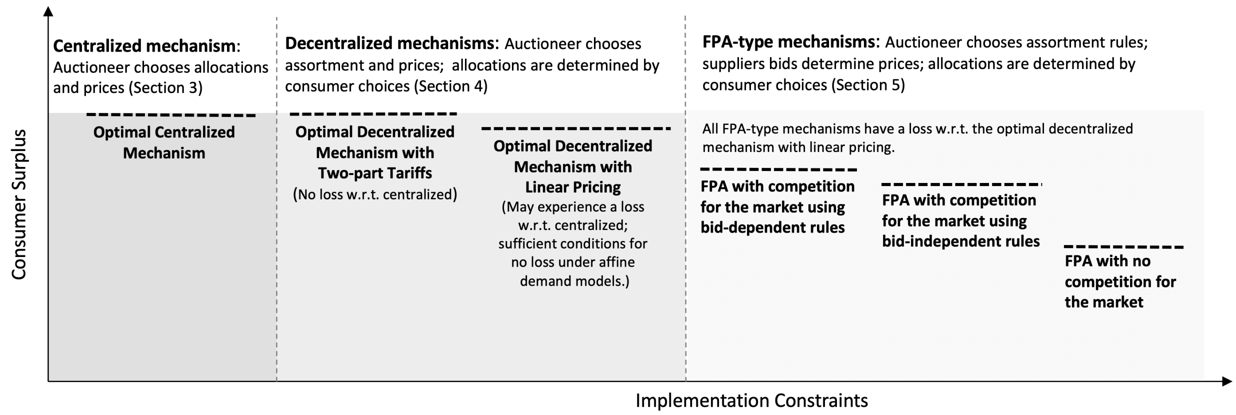

Figure 1: Achievable expected consumer surplus (represented by the dashed lines) for the different classes of mechanisms considered in the paper. The centralized mechanism achieves highest consumer surplus and losses occur as we impose more complex implementation constraints. This figure also illustrates the roadmap for the paper.

and their aggregate preferences are summarized through a demand model. The procurement agency must design a mechanism for selecting an assortment of products and their unit prices. Then, consumers buy their most preferred alternative in the assortment as prescribed by the demand model. The objective of the procurement agency is to maximize the expected consumer surplus, which depends on the aggregate value derived from the consumption of products in the assortment and the total procurement cost, crucially incorporating both variety and cost considerations.

In the above setting, suppliers face two sources of competition: (1) the competition to be part of the assortment, which we refer to as *competition for the market*; and, once in the assortment, (2) the *competition in the market* against the other suppliers in the assortment for consumer demand. Both sources of competition directly impact the consumer surplus as they determine the assortment of products (and thus the offered variety) and also their prices. Therefore, they must be carefully balanced by the mechanism designer when optimizing the tradeoff between variety and prices.

To better understand this tradeoff, we study the design of optimal mechanisms under increasingly more complex (and often more realistic) implementation constraints (see Figure 1). We start with the *centralized mechanism*, in which the auctioneer acts as a central planner in the sense that she determines the fraction of the aggregate demand to be satisfied by each of the suppliers and their appropriate monetary compensations. Centralized mechanisms are used in some contexts. For instance, Cenabast is a procurement organization within the Chilean goverment in charge of making large health-related purchases to be distributed among different organizations; Cenabast runs centralized auctions which take into account variety considerations.

The characterization of the optimal centralized mechanism allows us to understand the optimal tradeoff between variety and price competition, which crucially depends on the level of substitution across products. If products are close substitutes, demand is exclusively allocated to suppliers with the lowest virtual costs. By restricting entry to the assortment in this way, the auctioneer can provide incentives to low (virtual) cost suppliers by awarding them higher quantities, thus reducing the expected payments. Moreover, as products are close substitutes, restricting entry in this way is not very damaging to consumers in terms of variety. By contrast, when substitution across products is low, the demand is typically split between suppliers; in this case, the upside of providing more variety prevails over the decrease in expected payments that can be achieved by restricting entry. When products are also vertically differentiated, the quality advantage is factored in by allowing higher-quality suppliers to be in the assortment even if their virtual costs are higher.

Motivated by the important applications discussed earlier, we then study a *decentralized* setting, in which the auctioneer chooses the assortment of suppliers with their unit prices but, in contrast to the centralized setting, she *cannot* directly choose allocations: these are determined by the choices of the consumers. As illustrated in Figure 1, we show that when the auctioneer can compensate suppliers using two-part tariffs, the same expected consumer surplus as in the centralized setting can be achieved by using the same market structure (thus producing the same insights).

By contrast, when payments to suppliers must be implemented through *linear pricing* (i.e., payments are equal to the unit prices paid by consumers times the quantity demanded), which is the payment structure broadly used in practice, the consumer surplus in the centralized setting is not necessarily achieved. However, when the demand is represented by an affine model and the total demand is inelastic, we provide mild sufficient conditions on the cost distributions for which there is no performance loss associated with decentralizing the allocations and imposing linear pricing (see Figure 1). Moreover, we show that this can be done by preserving the assortment and induced demand in the centralized setting, thus preserving the market structure. We complement these theory results with numerical analyses under more general demand assumptions and under additional constraints, which suggest that the insights from the centralized mechanism also hold in many relevant decentralized settings.

Finally, we study how the consumer surplus is affected when the menu must be decided through *first-price-auction*-type (or *pay-as-bid*) mechanisms, which are prevalent in public procurement and constitute the standard implementation of FAs, one of our leading applications. In these mechanisms, the auctioneer designs the rules to determine the assortment based on bids submitted by suppliers and, possibly, on the characteristics of the products and the demand. Importantly, if a supplier is added to the menu, his bid is taken as the posted price. As illustrated in Figure 1, we show that, in general, imposing the first-price-auction constraint leads to a performance loss, even with respect to the decentralized mechanisms with linear pricing. This is because, in contrast to the optimal mechanism design setting, now the auctioneer can directly control only the competition for the market; competition in the market is determined by the suppliers' bids and the demand system.

We show that, when substitution across products is either high or low, a mechanism that does not impose competition for the market and always includes all suppliers in the menu (which closely resembles the way FAs are awarded in Chile) performs reasonably well. If products are close substitutes, consumers are highly price sensitive and the competition in the market provides sufficient incentives for suppliers to bid aggressively. In turn, when substitution across products is low, restricting entry is not profitable anyway because consumers derive a high value from variety. By contrast, when substitution across products is intermediate, we find that emulating the optimal mechanism in order to introduce competition for the market can lead to substantial improvements in consumer surplus. We show that by using simple rules to restrict the entry to the assortment, the auctioneer can achieve a decrease in suppliers' bids that outweighs the loss of reducing variety.

Overall, our results allow us to formalize and understand the tradeoff between increasing variety and inducing price competition when constructing assortments for heterogeneous consumers under progressively more complex practical constraints. Motivated by the implementation of such mechanisms in public settings, and in particular in the Chilean setting, we further show how introducing competition in first-price auctions through simple rules can result in a significant increase in expected consumer surplus. The analytical insights obtained in this paper played a crucial role in the redesign of the Chilean Framework Agreement for food in 2017.

Related literature. Our work is related to several streams of literature in economics and operations. First, our work extends classic work in mechanism design in the tradition of Myerson (1981) by considering an endogenous demand system; this difference poses significant challenges when solving for the optimal mechanism under linear pricing in the decentralized setting. Furthermore, in our problem the designer maximizes consumer surplus (as opposed to just minimizing payments to suppliers), which also depends on the underlying preferences of consumers.

Our work is closely related to some previous papers in procurement and regulation economics. Dana and Spier (1994) study how to allocate production rights to firms that have private cost information. An important insight of theirs is that the optimal market structure may depend on the firms' bids, which is similar to our result that the optimal allocation depends on suppliers' cost declarations. However, their auction determines the market structure and lump-sum fees only, while an exogenous competition model determines the unit prices paid by consumers. By contrast, our decentralized model captures two realistic features of FAs: linear pricing and the fact that these unit prices are endogenously determined by the mechanism. As it will become clear in Section 4, incorporating these features poses significant challenges when characterizing the optimal mechanism, as we now have one instrument (unit prices) influencing both the demand (allocation) and the payments to suppliers.

Similarly, Anton and Gertler (2004) and McGuire and Riordan (1995) study the optimal mechanism with an endogenous market structure in a Hotelling model. However, unit prices are not part of the mechanism, and allocations are determined by the designer and not endogenously as in our decentralized setting. Closer to our work, Wolinsky (1997) studies a spatial duopoly model where firms compete in both prices and quality. While it considers endogenous demands, it restricts analysis to solutions in which both firms have positive demands. By contrast, we are particularly interested in solutions in which some firms may be left out of the assortment to induce more competition. In fact, in our model, the optimal assortment typically does not contain all suppliers.

Another stream of related work on endogenous market structures is that of split-award auctions or dual sourcing (Chaturvedi et al. 2014, Li and Debo 2009, Elmaghraby 2000, Riordan and Sappington 1989, Anton and Yao 1989). However, purchases are decided by the auctioneer (closer to our centralized setting) and do not consider an underlying set of heterogeneous consumers.

Our work is also related to the operations literature on assortment planning decisions (see, e.g., Kök et al. (2009)). In these settings, decisions are made by one retailer that carries all products, and has full information on their unit costs. By contrast, we construct an assortment using a mechanism that elicits private cost information from many different suppliers.

Our analysis in Section 5 is closely related to Demsetz auctions (Demsetz 1968), which introduce competition for the market; Engel et al. (2002) also study a similar problem in a stylized model. This also relates to papers in group buying showing that committing to a single seller can be convenient for the group even if the members have heterogeneous preferences, as this can reduce buying prices (Dana 2012, Chen and Li 2013). However, these papers study complete information models; we extend their analysis to an auction setting with asymmetric cost information.

Finally, to the best of our knowledge, FAs are directly studied by only two prior mathematical modeling papers. Albano and Sparro (2008) consider a complete-information Hotelling model with equidistant firms, where only the subset of suppliers with lowest bids is added to the assortment. By contrast, we consider an incomplete information setting with a richer set of rules in which the assortment can depend on product characteristics. Gur et al. (2017) consider a model of FAs that studies the cost uncertainty faced by a supplier over the FA time horizon when selling a single item, but does not consider multiple differentiated products nor heterogeneous consumers.

Overall, to the best of our knowledge, our work is the first to study optimal buying mechanisms in an asymmetric information setting, with an endogenous market structure, an endogenous demand for differentiated products, and in which unit prices are determined by the mechanism.

2 Model

We introduce a model of procurement mechanisms for differentiated products. In our setting, the auctioneer runs a mechanism for satisfying the demand of consumers with diverse preferences for the suppliers' heterogeneous products. Therefore, the actors in our model are (*i*) an auctioneer (or designer), (*ii*) suppliers (or agents), and (*iii*) consumers. We describe each of these next.

Suppliers. There is an exogenous set N of n potential *suppliers* indexed by i. Suppliers offer differentiated products that are imperfect substitutes for each other. The number of suppliers and the characteristics of their products are fixed and common knowledge. We assume that suppliers are risk-neutral and seek to maximize expected profits. To simplify the exposition, we assume that each supplier offers exactly one product, so that firms and products share the same indices. In a separate electronic companion we discuss the extension to the multi-product setting, and show that our main results on optimal mechanism design hold under this extension.

Following the tradition in the auction literature (see, e.g., Krishna (2009)), we assume that suppliers have production costs drawn independently from common-knowledge distributions, whose realizations are the private information of each supplier. Formally, supplier i has a private cost $\theta_i \in \Theta_i$ to produce one unit mass of his product, where Θ_i is a finite set of strictly positive real numbers. We index the elements of Θ_i such that $\theta_i^j < \theta_i^k$ whenever $j < k$, for all $\theta_i^j, \theta_i^k \in \Theta_i$. We say that supplier i is of type θ_i if his cost is θ_i. Let f_i be a probability mass function over Θ_i, where $f_i(\theta_i)$ represents the probability that supplier i is of type θ_i. Let $F_i(\theta_i^j) = \sum_{k \leq j} f_i(\theta_i^k)$ be the cumulative probability distribution. Let $\Theta = \Pi_i \Theta_i$ denote the type space. We use discrete distributions for technical convenience, as explained in Section 4. Because suppliers' types are independent, the joint probability of $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ is equal to $f(\boldsymbol{\theta}) = \Pi_{i=1}^n f_i(\theta_i)$. We denote the probability that all suppliers other than i have type $\boldsymbol{\theta}_{-i}$ by $f_{-i}(\boldsymbol{\theta}_{-i})$. We use boldface to denote vectors and matrices throughout the paper.

We assume that suppliers have constant marginal costs of production and do not face capacity constraints.[3] Therefore, the products included in the assortment are always available and their production costs do not depend on the quantity demanded. These assumptions are reasonable for many of the settings we have in mind; for example, usually the quantities that suppliers sell through FAs represent only a small fraction of their total overall sales (Gur et al. 2017).

Consumers. Recall that the auctioneer wants to provide adequate variety to satisfy the (possibly heterogeneous) needs of the consumers while managing costs. As argued in the Introduction, we assume that the auctioneer maximizes consumer surplus when solving for the optimal mechanism. In order to define consumer surplus, we start by specifying how consumers' purchasing decisions translate into aggregate demands for the goods. These demands reveal consumers' preferences; hence, they will be directly related to consumer surplus. Consumer demand will also play an important role in Sections 4 and 5, when we discuss the implementation of decentralized mechanisms.

In the tradition of the assortment planning literature (see Kök et al. (2009) for a comprehensive survey) and in oligopoly pricing literature (e.g., Tirole (1988)), we assume that aggregate demand functions are common knowledge and are an input to our model. This assumption also seems reasonable in the contexts discussed in the Introduction, as a demand system can typically be estimated using available historical data or consumer surveys (Ackerberg et al. (2006)).

Let $\boldsymbol{p} = (p_i)_{i \in N}$ be the vector of unit prices associated with the set of potential suppliers. Suppose that, from the set of potential suppliers N, a subset $Q \subseteq N$ of suppliers is in the assortment. Then, for every such subset Q and vector of prices $\boldsymbol{p}$, the vector of demand functions is given by $\boldsymbol{d}(Q, \boldsymbol{p}) = (d_i(Q, \boldsymbol{p}))_{i \in N}$, where $d_i(Q, \boldsymbol{p})$ denotes the expected demand for product i under assortment Q and prices $\boldsymbol{p}$. Note that the demand functions can naturally change with the set of available products in the assortment.

Given a vector of prices $\boldsymbol{p} = (p_i)_{i \in N}$ and a suppliers' set Q, let $\boldsymbol{p}_Q = (p_i)_{i \in Q}$ be the subvector of prices of the suppliers in Q. We introduce the following assumption, which we keep throughout:

Assumption 2.1 (Demand system). *We assume that (i) $d_i(Q, \boldsymbol{p}) = d_i(Q, \boldsymbol{p}')$ for every $\boldsymbol{p}'$ such that $p'_Q = p_Q$, that is, demand is determined by the prices of products in the assortment; and (ii) $d_i(Q, \boldsymbol{p}_Q) = 0$ for $i \notin Q$, that is, products that are not in the assortment cannot be purchased.*

The assumption is natural and as we illustrate later in this section, is satisfied by many commonly used demand models, including those studied in this paper.

[3] We explain how suppliers' capacity constraints can be incorporated in the 'Consumers' subsection.

Given a demand system, the study of the "integrability problem" provides conditions under which the demand functions can be derived from the maximization of a single utility function (see, e.g., Mas-Colell et al. (1995) and Anderson et al. (1992)); for the demand systems that we consider in this paper, this utility function in fact corresponds to the consumer surplus function. We formalize this in the following assumption, which we keep throughout the paper.

Let $CS(\boldsymbol{x}, \boldsymbol{p})$ be the consumer surplus given consumption quantities $\boldsymbol{x} = (x_i)_{i \in N}$ (where x_i represents the total consumption from supplier i) and prices $\boldsymbol{p}$. Let $\mathcal{X}$ denote the set of feasible consumption quantities. To simplify the exposition, we assume that $\mathcal{X}$ is a compact and convex subset of the Euclidean space. For example, a natural choice would be $\boldsymbol{x} \geq 0$ and $\sum_{i \in N} x_i \leq M$, where M is the market size (i.e., the total mass of potential consumers). We could also incorporate firm-specific capacity constraints, such as $x_i \leq K_i$ for some K_i. An important setting that we focus on is one with no outside option. In this case, normalizing the total population of consumers to be 1, we have that[4] $\mathcal{X} = \{\boldsymbol{x} : \ \boldsymbol{x} \geq 0, \ \mathbf{1}'\boldsymbol{x} = 1\}$. We assume that the set $\mathcal{X}$ is common knowledge.

Assumption 2.2 (Consumer surplus)**.** *There exists a function $GCS(\boldsymbol{x})$ of the consumption quantities $\boldsymbol{x}$ such that:*

$$CS(\boldsymbol{x}, \boldsymbol{p}) \quad = \quad GCS(\boldsymbol{x}) - \sum_{i=1}^{n} p_i x_i \ , \tag{1}$$

that is, consumer surplus is quasi-linear in prices.[5] *We refer to the function $GCS(\cdot)$ as the* gross consumer surplus. *Moreover, the expression for consumer surplus must satisfy, for all $\boldsymbol{p}$ and Q,*

$$\begin{aligned} \boldsymbol{d}(Q, \boldsymbol{p}) \quad \in \quad & \underset{\boldsymbol{x} \in \mathcal{X}}{\text{argmax}} \ \ CS(\boldsymbol{x}, \boldsymbol{p}) \ , \\ & s.t. \qquad x_i = 0 \quad \forall i \notin Q \ , \end{aligned} \tag{2}$$

where $\mathcal{X}$ is a compact and convex subset of the Euclidean space.

The first part of the assumption requires consumer surplus to be quasi-linear in prices. Note that, in this case, the function $GCS(\boldsymbol{x})$ provides a measure of the value derived from the aggregate consumption vector $(\boldsymbol{x})$ that is independent of the prices, and thus the consumer surplus expression transparently captures the tradeoff between variety and prices. The second part of the assumption states that $\boldsymbol{d}(Q, \boldsymbol{p})$, the quantities demanded given assortment Q and prices $\boldsymbol{p}$, maximize consumer surplus given those prices and assuming that products not in the assortment get zero

[4]This case is a reasonable approximation for a variety of settings in public procurement where buying organizations cannot easily adjust the total quantity purchased based on prices, e.g., when buying medicines and school meals.

[5]The latter assumption is useful for solving the optimal mechanism design problems.

demand. (Note that the solution of this maximization problem may also set some of the demanded quantities for products in the assortment equal to zero.) This implies that the demand function is consistent with the consumer surplus function, in a way that one would naturally expect. A common approach to guaranteeing this consistency is to micro-found the aggregate demand system and the associated consumer surplus function through a discrete choice model describing individual consumption decisions; see Anderson et al. (1992) for a detailed discussion.

We illustrate this approach using a simple example of a Hotelling demand model of horizontal differentiation with two suppliers, linear transportation costs, no outside option, and no capacity constraints.[6]

Example 2.1 (Hotelling model with two suppliers)**.** *Consider the unit interval as the product space, with two potential suppliers located at the extremes of the interval. There is a continuum of consumers uniformly distributed on the product space. Each consumer demands one unit of good and incurs transportation costs that are linear in the distance between the consumer and the supplier. Consumer j located at ℓ_j derives the following utilities from consuming from the set of suppliers $N = \{1, 2\}$:*

$$u_{j1}(p_1) = -\left(\delta\ell_j + p_1\right) \qquad \text{and} \qquad u_{j2}(p_2) = -\left(\delta(1-\ell_j) + p_2\right),$$

where supplier 1 (resp. 2) is assumed to be located at 0 (resp. 1) and δ is the transportation cost. As consumers are uniformly distributed on the $[0, 1]$ segment, the aggregate demands can be derived from individual utilities as follows:

$$d_1(N, \boldsymbol{p}) = \max\left\{0, \min\left\{1, \frac{p_2 - p_1 + \delta}{2\delta}\right\}\right\} \qquad \text{and} \qquad d_2(N, \boldsymbol{p}) = \max\left\{0, \min\left\{1, \frac{p_1 - p_2 + \delta}{2\delta}\right\}\right\}.$$

As there is no outside option, when assortments consist of a unique supplier his demand equals one. In addition, aggregating the individual utilities we can derive the expression for consumer surplus:

$$CS(\boldsymbol{x}, \boldsymbol{p}) = -\left(\frac{\delta}{2}\left(x_1^2 + x_2^2\right) + p_1 x_1 + p_2 x_2\right),$$

where the first terms represent the transportation costs and the latter terms the monetary costs. In this example, $GCS(\boldsymbol{x}) = -\frac{\delta}{2}(x_1^2 + x_2^2)$, which is equal to the total transportation cost. It is simple to verify that the Hotelling model satisfies Assumption 2.2.

[6]Similarly, the expected demands resulting from a multinomial logit model will also satisfy the above assumptions. However, we decided not to use MNL because of its limited ability to capture substitution across products.

Our second example is an affine demand system, which generalizes the Hotelling model by allowing for vertical differentiation and more general substitution patterns across products. Traditionally, an affine demand function is one where the relation $\boldsymbol{d}(\boldsymbol{p}) = \boldsymbol{\alpha} - \boldsymbol{\Gamma}\boldsymbol{p}$ holds for all $\boldsymbol{p} \in \{\boldsymbol{p} \in \mathbb{R} : \boldsymbol{\alpha} - \boldsymbol{\Gamma}\boldsymbol{p} \geq 0\}$, where $\boldsymbol{\alpha} \geq 0$ represents a quality component and $\boldsymbol{\Gamma}$ is a matrix that captures substitution patterns across products. We assume that the products are substitutes, hence $\boldsymbol{\Gamma}_{ij} \leq 0$ for $i \neq j$, and that $\boldsymbol{\Gamma}$ is symmetric and positive definite. Since we are particularly interested in solutions in which not necessarily all suppliers have positive demand, it is important to consider the extension of the affine demand specification to price vectors under which some products get zero demand; see Shubik and Levitan (1980) and Soon et al. (2009). We formalize this extension by assuming that a single representative consumer maximizes consumer surplus (see Farahat and Perakis (2010)) and that the demand function is defined as the solution to the representative consumer's maximization problem. (We could also micro-found this aggregate demand system starting from a discrete choice model describing individual consumption decisions; see Armstrong and Vickers (2014).)

Example 2.2 (Affine demand model)**.** *Let $\boldsymbol{\alpha} \geq 0$ represent a quality component and let $\boldsymbol{\Gamma}$ be a positive definite symmetric matrix with $\boldsymbol{\Gamma}_{ij} \leq 0$ for $i \neq j$ that captures substitution patterns across products. Given a vector of prices $\boldsymbol{p}$, suppose that a single representative consumer maximizes consumer surplus, which is given by*

$$GCS(\boldsymbol{x}) = \boldsymbol{c}'\boldsymbol{x} - \frac{1}{2}\boldsymbol{x}'\boldsymbol{D}\boldsymbol{x}, \qquad \text{and} \qquad CS(\boldsymbol{x}, \boldsymbol{p}) = GCS(\boldsymbol{x}) - \boldsymbol{p}'\boldsymbol{x} \ , \tag{3}$$

where $\boldsymbol{D} = \boldsymbol{\Gamma}^{-1}$ and $\boldsymbol{c} = \boldsymbol{\Gamma}^{-1}\boldsymbol{\alpha}$ have been renamed to ease notation.

Consistent with Assumption 2.2, the demand function is defined as the solution of the representative consumer's maximization problem. Hence, for any $\boldsymbol{p} \in \mathbb{R}^n$, let $\boldsymbol{d}(Q, \boldsymbol{p})$ be defined as the solution to $\max_{\boldsymbol{x} \in \mathcal{X}} \ CS(\boldsymbol{x}, \boldsymbol{p})$, s.t. $x_i = 0, \ \forall i \notin Q$. Clearly, this problem has a unique solution for every $\boldsymbol{p} \in \mathbb{R}^n$ (provided that $\mathcal{X}$ is a compact and convex set). Hence, the demand function $\boldsymbol{d}(Q, \boldsymbol{p})$ is well defined for all $Q \subseteq N$ and all $\boldsymbol{p}$.[7]

Auctioneer. The role of the auctioneer is to design a mechanism to satisfy the purchasing needs of the heterogeneous consumers. The auctioneer is risk-neutral and her objective is to maximize

[7] In the separate electronic companion, we show that whenever $\mathcal{X} = \{\boldsymbol{x} : \ \boldsymbol{x} \geq 0, \sum_{i \in N} x_i = 1\}$, the positive part of $\boldsymbol{d}(Q, \boldsymbol{p})$ is an affine function of only the prices of the set of suppliers in Q and, therefore, can be written as $\boldsymbol{d}(Q, \boldsymbol{p}) = \boldsymbol{a} - \boldsymbol{G}\boldsymbol{p}_Q$ for some $\boldsymbol{a}$ and $\boldsymbol{G}$ that depend only on the set Q. We also show that demand for a product is weakly decreasing in its own price and increasing in others' prices. Importantly, we note that cross-price elasticities change as a function of the assortment. Analogous results were established by Farahat and Perakis (2010) for $\mathcal{X} = \{\boldsymbol{x} : \ \boldsymbol{x} \geq 0\}$.

expected consumer surplus. This objective is appropriate for the applications described in the Introduction as it incorporates both variety and cost considerations: the aggregate value derived by consumers from variety is captured by the gross consumer surplus term, while the total procurement cost is captured by the transfers to suppliers.

As we discussed in the Introduction, we consider two classes of procurement mechanisms that we call centralized and decentralized, respectively. In both settings, the rules of the mechanism are common knowledge. In a *centralized* setting, the auctioneer runs a mechanism for deciding the fraction of the aggregate demand that will be satisfied by each of the suppliers and their appropriate compensations. In this case, the auctioneer decides how to distribute the goods among the different consumers, that is, the auctioneer acts as a central planner who can decide how to allocate goods. While the auctioneer has the ability to allocate demand, she does not have access to suppliers' private information and thus the mechanism is used for price discovery. The optimal centralized mechanism is formulated and analyzed in Section 3, and allows us to derive useful insights into the optimal market structure and hence into the tradeoff between variety and payments to suppliers.

By contrast, in a *decentralized* setting, the auction is run to construct the menu of products based on suppliers' bids. The menu consists of a subset of suppliers and prices for their products. Once the menu is fixed, consumers choose which product in the menu to purchase. The main difference from the centralized setting is that the auctioneer *cannot* directly determine the resulting allocations; allocations are determined by the choices of consumers. A decentralized setting corresponds to the way big organizations or governments typically build assortments of products. We will study three decentralized implementations. In Section 4, we study optimal decentralized mechanisms when payments to suppliers can be implemented using two-part tariffs and when they must be implemented using linear prices. In Section 5, we study decentralized first-price-auction mechanisms, which brings us one step closer to the implementation of practical FAs.

3 Centralized Procurement

3.1 Mechanism Design Problem Formulation

We provide a mechanism design formulation for the centralized auctioneer's problem, considering Bayes-Nash implementation. By invoking the revelation principle, we restrict attention to direct-revelation mechanisms without loss of optimality. Hence, for given cost declarations, the designer selects an allocation of consumptions from each supplier, as well as their appropriate compensations. Formally, a direct-revelation centralized mechanism can be specified by (a) the *allocation functions*

$x_i : \Theta \to [0, M]$ (recall that M is the market size), where $x_i(\boldsymbol{\theta})$ is the quantity allocated to supplier i when cost declarations are $\boldsymbol{\theta}$; and (b) the *price functions* $p_i : \Theta \to \mathbb{R}$, where $p_i(\boldsymbol{\theta})$ denotes the unit price of supplier i when cost declarations are $\boldsymbol{\theta}$. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{p} = (p_1, \ldots, p_n)$.

In the optimal mechanism design problem, the designer maximizes her objective (in our case, expected consumer surplus) subject to the usual constraints in mechanism design theory: incentive compatibility (IC), individual rationality (IR), and feasibility of allocations (Feas). To express these constraints, we define the *interim expected utility* for supplier i of type θ_i and report θ_i' as follows:

$$U_i(\theta_i'|\theta_i) = \sum_{\boldsymbol{\theta}_{-i} \in \Theta_{-i}} f_{-i}(\boldsymbol{\theta}_{-i}) \big(p_i(\theta_i', \boldsymbol{\theta}_{-i}) - \theta_i\big) x_i(\theta_i', \boldsymbol{\theta}_{-i}) \ . \tag{4}$$

The IC constraints can be expressed in terms of the interim expected utilities as $U_i(\theta_i|\theta_i) \geq U_i(\theta_i'|\theta_i)$, for all $i \in N$ and all $\theta_i, \theta_i' \in \Theta_i$, whereas the IR constraints are given by $U_i(\theta_i|\theta_i) \geq 0$, for all $i \in N$ and all $\theta_i \in \Theta_i$. Note that, in all the IC and IR constraints, prices always appear multiplied by the corresponding allocations, and these quantities represent the net transfers to the suppliers. In addition, by Assumption 2.2, the same is true for the objective. Therefore, we can formulate the centralized problem in terms of the allocations and the transfers to suppliers.

Formally, define the *transfer functions* $t_i : \Theta \to \mathbb{R}$, as $t_i(\boldsymbol{\theta}) := x_i(\boldsymbol{\theta}) p_i(\boldsymbol{\theta})$; that is, $t_i(\boldsymbol{\theta})$ denotes the payment to supplier i when cost declarations are $\boldsymbol{\theta}$. Let $\boldsymbol{t} = (t_1, \ldots, t_n)$. Then, the auctioneer's optimal mechanism design problem can be formulated in terms of $\boldsymbol{x}$ and $\boldsymbol{t}$ as follows:

$$[Cent] \qquad \max_{\boldsymbol{x}, \boldsymbol{t}} \quad \mathbb{E}_{\boldsymbol{\theta}} \left[GCS(\boldsymbol{x}(\boldsymbol{\theta})) - \sum_{i=1}^{n} t_i(\boldsymbol{\theta}) \right]$$

$$\text{s.t.} \quad U_i(\theta_i|\theta_i) \geq U_i(\theta_i'|\theta_i) \qquad \forall i \in N, \ \forall \theta_i, \theta_i' \in \Theta_i \tag{IC}$$

$$U_i(\theta_i|\theta_i) \geq 0 \qquad \forall i \in N, \ \forall \theta_i \in \Theta_i \tag{IR}$$

$$x(\boldsymbol{\theta}) \in \mathcal{X} \qquad \forall \boldsymbol{\theta} \in \Theta \tag{Feas}$$

The above formulation differs from the classic mechanism design formulation in the objective function only: while in the latter expected transfers are minimized, *Cent* maximizes expected gross consumer surplus minus transfers. Therefore, the optimal solution to *Cent* can be obtained by extending standard arguments based on the envelope theorem (Myerson 1981) adapted to the setting of discrete distributions (Vohra 2011) to determine which IC constraints are binding.

Analogously to the setting of continuous cost distributions, we introduce the following definition of the virtual cost function for cost distributions with discrete support (see Vohra (2011)).

Definition 3.1. *For $\theta_i \in \Theta_i$, let $\rho_i(\theta_i) = \max\{\theta' \in \Theta_i : \theta' < \theta_i\}$, that is, $\rho_i(\theta_i)$ is the predecessor of θ_i in Θ_i. (If θ_i is the lowest cost in the support, we define $\rho_i(\theta_i) := \theta_i$.) Let $v_i(\theta_i) := \theta_i + \frac{F_i(\rho_i(\theta_i))}{f_i(\theta_i)}(\theta_i - \rho_i(\theta_i))$ be the* virtual cost *function of supplier i. Let $\boldsymbol{v}(\boldsymbol{\theta})$ be defined as the vector of virtual costs, i.e., $\boldsymbol{v}(\boldsymbol{\theta}) = (v_1(\theta_1), \ldots, v_n(\theta_n))$.*

We make the standard regularity assumption in mechanism design, which we keep throughout.

Assumption 3.1. *The function $v_i(\theta_i)$ is strictly increasing for all $i \in N$.*

Finally, we also define the *interim expected allocations* and *interim expected transfers* as follows:

$$X_i(\theta_i) := \sum_{\boldsymbol{\theta}_{-i} \in \Theta_{-i}} f_{-i}(\boldsymbol{\theta}_{-i}) x_i(\theta_i, \boldsymbol{\theta}_{-i}), \qquad T_i(\theta_i) := \sum_{\boldsymbol{\theta}_{-i} \in \Theta_{-i}} f_{-i}(\boldsymbol{\theta}_{-i}) t_i(\theta_i, \boldsymbol{\theta}_{-i}). \tag{5}$$

Then, the optimal solution to Problem *Cent* can be characterized as follows.

Proposition 3.1. *Suppose that $(\boldsymbol{x}, \boldsymbol{t})$ satisfy the following conditions:*

1. *The allocation function satisfies, for all $\boldsymbol{\theta} \in \Theta$,*

$$\boldsymbol{x}(\boldsymbol{\theta}) \in \underset{\boldsymbol{x}' \in \mathcal{X}}{\operatorname{argmax}} \; CS(\boldsymbol{x}', \boldsymbol{v}(\boldsymbol{\theta})), \tag{6}$$

2. *Interim expected allocations are monotonically decreasing for all $i \in N$, that is, $X_i(\theta) \geq X_i(\theta')$ for all $\theta, \theta' \in \Theta_i$ such that $\theta \leq \theta'$.*

3. *Interim expected transfers satisfy, for all $i \in N$ and $\theta_i^j \in \Theta_i$,*

$$T_i(\theta_i^j) = \theta_i^j X_i(\theta_i^j) + \sum_{k=j+1}^{|\Theta_i|} (\theta_i^k - \theta_i^{k-1}) X_i(\theta_i^k) \tag{7}$$

Then, $(\boldsymbol{x}, \boldsymbol{t})$ is an optimal mechanism for the centralized procurement problem Cent.

The proof of this and other main results can be found in the Appendix. Omitted proofs are provided in the separate electronic companion. Condition 1 in Proposition 3.1 states that, for each $\boldsymbol{\theta} \in \Theta$, the optimal vector of allocations $\boldsymbol{x}(\boldsymbol{\theta})$ coincides with the demand functions defined by (2) when unit prices are equal to virtual costs and all products are included in the assortment, that is, when[8] $Q = N$. This follows from classic mechanism design arguments, i.e., the equilibrium ex-ante expected payment that the auctioneer makes to a bidder is equal to the ex-ante expectation of the virtual cost times the allocation. However, even though the method of analysis of the centralized

[8]Even if $Q = N$, the maximization problem may set the demand for some products equal to zero.

mechanism is quite standard, we have not seen such a transparent characterization of the tradeoff between variety and payment to suppliers in the literature before.

While the result holds for general demand models that satisfy Assumptions 2.1 and 2.2, to clarify the intuition we next discuss the structure of the optimal centralized mechanism for the demand models introduced in Section 2. Before proceeding, we show the following technical result.

Proposition 3.2. *Consider the centralized problem when the consumer surplus is either that of the Hotelling model or of the affine demand model introduced in Section 2. Then, there exists a feasible solution* $(\boldsymbol{x}, \boldsymbol{t})$ *that:*

1. *Satisfies the optimality conditions stated in Proposition 3.1.*

2. *For all* $i \in N$, $\theta_i \in \Theta_i$, *and* $\boldsymbol{\theta}_{-i} \in \Theta_{-i}$, *we have that* $t_i(\theta_i, \boldsymbol{\theta}_{-i}) \geq \theta_i x_i(\theta_i, \boldsymbol{\theta}_{-i})$ *and* $t_i(\theta_i, \boldsymbol{\theta}_{-i}) = 0$ *if* $x_i(\theta_i, \boldsymbol{\theta}_{-i}) = 0$.

Furthermore, let $\boldsymbol{T}_i := (T_i(\theta_i^j))_{j=1,\ldots,|\Theta_i|}$ *be the vector of expected transfers to supplier* i *and let* $\boldsymbol{T} := (\boldsymbol{T}_i)_{i \in N}$. *Then, for every feasible solution* $(\boldsymbol{x}, \boldsymbol{t})$ *satisfying the conditions stated in Proposition 3.1, we have that* $\boldsymbol{x}$ *and* $\boldsymbol{T}$ *are unique.*

The result shows that, for the main models used in the paper, an optimal solution as characterized in Proposition 3.1 indeed exists and, furthermore, $(\boldsymbol{x}, \boldsymbol{T})$ is unique.

3.2 Examples of Optimal Centralized Mechanisms

3.2.1 Optimal Centralized Mechanism under the Hotelling Demand Model

Example 3.1. *Consider the Hotelling model with the two suppliers introduced in Example 2.1 and suppose that the suppliers have the same cost distribution. Let* θ_1 *and* θ_2 *be the cost realizations of suppliers* 1 *and* 2, *respectively. By Proposition 3.1, for any cost realization* $\boldsymbol{\theta}$, *the optimal allocations are given by the Hotelling demands when prices are equal to the vector of virtual costs* $\boldsymbol{v}(\boldsymbol{\theta})$. *In this case, the centralized problem yields an optimal allocation characterized as: (i) if* $\delta > |v(\theta_1) - v(\theta_2)|$, *the demand is split between the two suppliers with* $x_1(\boldsymbol{\theta}) = (v(\theta_2) - v(\theta_1) + \delta)/(2\delta)$ *and* $x_2(\boldsymbol{\theta}) = (v(\theta_1) - v(\theta_2) + \delta)/(2\delta)$; *and (ii) if* $\delta < |v(\theta_2) - v(\theta_1)|$, *all the demand is awarded to the supplier with the lowest (virtual) cost.*

As illustrated, an important feature of the optimal centralized solution is that the decision of whether to split the demand depends on the cost realizations. If the transportation cost is small relative to the differences in the virtual costs, then it is optimal to have a unique supplier, the one with the lowest virtual cost. In this case, it is worth paying the cost of having less variety in the assortment with the upside of decreasing the expected payments to suppliers. By contrast, if the transportation cost is high relative to the differences in the virtual costs, the demand is split between both suppliers to increase variety. In this sense, the optimal solution to the centralized problem optimizes the tradeoff between variety and payments to suppliers: by restricting the entry to the assortment in some scenarios, the auctioneer can reduce expected payments while still providing incentives for truthful cost revelation.

This insight generalizes to the case with more than two suppliers. Consider a general Hotelling demand model with n suppliers located at $0 \leq \ell_1 < \ldots < \ell_n \leq 1$; the location represents the horizontal characteristic of the product offered by the supplier relative to the product space. The closer two suppliers are in the product space, the closer substitutes their products are. A continuum of consumers, all of whom buy one unit of product, are uniformly distributed on the product space. The utility consumer j obtains from buying from i is given by $u_{ji}(p_i) = -\left(\delta|\ell_i - \ell_j| + p_i\right)$, where δ is the transportation cost and ℓ_j is the location of consumer j in the unit line. Using these utility functions we can characterize the demand function and the optimal centralized solution.

Suppose that suppliers have fixed unit prices given by $\boldsymbol{p}$. It is easy to see that supplier i will have positive demand if and only if i is the preferred choice for the consumer located at ℓ_i. Therefore, the set of suppliers with positive demand as a function of prices $\boldsymbol{p}$ is given by $Q(\boldsymbol{p}) = \{i \in N : \ p_i \leq \min_{k \neq i} \{p_k + \delta|\ell_k - \ell_i|\}\}$. In addition, two consecutive suppliers $i, j \in Q(\boldsymbol{p})$ split the segment between ℓ_i and ℓ_j proportionally to their prices: i obtains $\frac{p_j - p_i + \delta|\ell_j - \ell_i|}{2\delta}$ and j the rest. (The demand equations can be easily derived by determining the location of an indifferent consumer between two active neighboring suppliers.)

Then, by Proposition 3.1, for any cost realization $\boldsymbol{\theta}$, the optimal allocations are given by the Hotelling demands when prices are equal to the vector of virtual costs $\boldsymbol{v}(\boldsymbol{\theta})$. Therefore, for a given $\boldsymbol{\theta} \in \Theta$, the optimal assortment is given by $Q(\boldsymbol{v}(\boldsymbol{\theta})) = \{i \in N : \ v_i(\theta_i) - v_j(\theta_j) \leq \delta|\ell_j - \ell_i| \quad \forall j \in N\}$, which corresponds to the above definition of $Q(\cdot)$ when prices are replaced by virtual costs. That is, if two products are close substitutes, i.e., $\delta|\ell_j - \ell_i|$ is relatively small, the auctioneer will not purchase the product with the highest virtual cost. On the other hand, when two products are not close substitutes, i.e., $\delta|\ell_j - \ell_i|$ is relatively big, then the (virtual) cost of one product has less of an effect in determining whether the other product is included or not in the assortment.

3.2.2 Optimal Centralized Mechanism under General Affine Demand Models

We now turn our attention to the more general affine demand models introduced in Section 2, which allow us to combine both vertical and horizontal sources of differentiation. Recall that for a general affine demand model, the demand functions are obtained by solving Problem (2) with GCS given by (3) when all products are considered in the assortment. To gain intuition, we discuss a simple example with two suppliers.

Example 3.2. *We consider a duopoly where* $\boldsymbol{\alpha} = (a_1, a_2)$ *and* $\boldsymbol{\Gamma} = \begin{pmatrix} r_1 & -\gamma \\ -\gamma & r_2 \end{pmatrix}$*, with all the parameters positive and with* $r_1 + r_2 \geq 2\gamma$*. Under these parameters, we have that* $\boldsymbol{D} = \frac{1}{r_1 r_2 - \gamma^2}\begin{pmatrix} r_2 & \gamma \\ \gamma & r_1 \end{pmatrix}$ *and* $\boldsymbol{c} = \frac{1}{r_1 r_2 - \gamma^2}\begin{pmatrix} r_2 a_1 + \gamma a_2 \\ r_1 a_2 + \gamma a_1 \end{pmatrix}$*. Suppose that* $\mathcal{X} = \{\boldsymbol{x} : \ \boldsymbol{x} \geq 0, \mathbf{1}'\boldsymbol{x} = 1\}$*, that is, there is no outside option. For any given* $\boldsymbol{p}$*, the demand functions* $\boldsymbol{d}(N, \boldsymbol{p})$ *are given by*

$$d_i(N, \boldsymbol{p}) = \max\left\{0,\ \min\left\{\frac{(r_j - \gamma)a_i - (r_i - \gamma)a_j + r_i - \gamma - (r_i r_j - \gamma^2)(p_i - p_j)}{r_i + r_j - 2\gamma},\ 1\right\}\right\}, i, j \in \{1, 2\}.$$

Recall that, for a given cost realization $\boldsymbol{\theta}$, the optimal allocations in the centralized problem equal the demand characterized above with prices equal to the vector of virtual costs $\boldsymbol{v}(\boldsymbol{\theta})$. To illustrate the tradoff, we start by discussing the structure of the optimal solution in Example 3.2, focusing on supplier 1 and assuming that suppliers have the same own-substitution patterns (i.e., $r_1 = r_2$) but different qualities. In this case, for a given $\boldsymbol{\theta}$, we have that supplier 1 will be in the assortment ($x_1(\boldsymbol{\theta}) > 0$) only if[9] $v_1(\theta_1) - v_2(\theta_2) \leq \frac{(a_1 - a_2) + 1}{r + \gamma}$. From this expression one can see that there is a natural bias towards the highest-quality supplier. For instance, if $(a_1 - a_2) + 1 \leq 0$ or, equivalently, $a_1 \leq a_2 - 1$, then supplier 1 can be in the assortment only if he is the one with the lowest virtual cost. Once $(a_1 - a_2) + 1 > 0$, supplier 1 can be part of the assortment even if his virtual cost is greater than that of supplier 2; note that this is possible even if he is still lower quality than supplier 2 (i.e., if $a_1 < a_2$). As the difference in qualitiy between suppliers 1 and 2 ($a_1 - a_2$) increases, the auctioneer becomes more tolerant to larger differences in (virtual) cost between suppliers 1 and 2. In addition, as substitution across products (γ) increases, supplier 1's virtual cost is required to be closer to that of supplier 2 in order for supplier 1 to be part of the assortment, which agrees with the intuition derived from the Hotelling model.

In the more general case, for a given $\boldsymbol{\theta}$, supplier 1 will be in the assortment only if $(r_1 r_2 - \gamma^2)(v_1(\theta_1) - v_2(\theta_2)) \leq (r_2 - \gamma)a_1 - (r_1 - \gamma)a_2 + r_1 - \gamma$. Therefore, for him to be in the assortment, the difference in virtual cost must be bounded by a quantity that is increasing in the normalized difference in quality $(r_2 - \gamma)a_1 - (r_1 - \gamma)a_2$. Hence, the larger this difference in quality (e.g., if

[9] If we let $a_1 = a_2 = 0$, $r_1 = r_2 = \frac{1}{\delta}$, and $\gamma = 0$, we obtain the same expression as in the Hotelling model.

a_1 grows), the larger is the difference in virtual cost that the auctioneer allows in order to keep supplier 1 in the assortment.

Note that, again, the optimal centralized solution restricts the entry of a supplier to the assortment to decrease expected payments. The structure of the optimal centralized allocation also generalizes to the case of more products, but the discussions are omitted for the sake of brevity.

4 Decentralized Procurement

We now study the optimal mechanism problem in a decentralized setting. In this setting, for given cost declarations, the auctioneer selects a menu that consists of an assortment of products (or suppliers) and their unit prices. The auctioneer does not directly decide allocations; instead, purchasing decisions are decentralized: based on the products and prices in the menu, consumers decide which products to buy through the demand system.

Formally, we consider Bayes-Nash implementation and restrict attention to direct-revelation mechanisms without loss of optimality. A decentralized direct-revelation mechanism can be specified by (a) the *assortment functions* $q_i : \Theta \to \{0, 1\}$ that are equal to 1 if and only if supplier i is included in the assortment when cost declarations are $\boldsymbol{\theta}$; and (b) the *price functions* $p_i : \Theta \to \mathbb{R}$, where $p_i(\boldsymbol{\theta})$ is the unit price for the item offered by supplier i when cost declarations are $\boldsymbol{\theta}$. Note that this formulation allows for multiple suppliers to be in the menu. We define $\boldsymbol{q} := (q_1, ..., q_n)$ and $\boldsymbol{p} := (p_1, ..., p_n)$. For given cost declarations $\boldsymbol{\theta}$, the menu is given by $(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{p}(\boldsymbol{\theta}))$. Analogously to the centralized setting, let $x_i : \Theta \to [0, M]$ denote the allocation functions, i.e., $x_i(\boldsymbol{\theta})$ is the quantity allocated to supplier i when cost declarations are $\boldsymbol{\theta}$. Let $\boldsymbol{x} := (x_1, \ldots, x_n)$.

In the decentralized optimal mechanism design problem, the auctioneer chooses the assortment and price functions to maximize expected consumer surplus subject to incentive compatibility (IC), individual rationality (IR), feasibility of allocations (Feas), plus an extra set of constraints that links the allocations to the demand system. In particular, for every vector of cost realizations $\boldsymbol{\theta}$, the allocation to suppliers must be given by the consumer demand associated with the menu $(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{p}(\boldsymbol{\theta}))$, as determined by the underlying demand system. We capture this by imposing the following set of demand constraints on our decentralized problem:

$$\boldsymbol{x}(\boldsymbol{\theta}) = \boldsymbol{d}(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{p}(\boldsymbol{\theta})) \qquad \forall \boldsymbol{\theta} \in \Theta,$$

where we slightly abuse notation and denote by $\boldsymbol{q}(\boldsymbol{\theta})$ the set of suppliers that are in the assortment given costs $\boldsymbol{\theta}$. In other words, the value of $\boldsymbol{x}$ is completely determined[10] by $\boldsymbol{q}$ and $\boldsymbol{p}$.

By imposing these additional constraints, the decentralized problem deviates from the centralized and the classic mechanism design settings, where the designer selects both the payment and the allocation function. In our problem, the designer does not directly select the allocation function; instead, he chooses an assortment and unit prices and, given these, allocations are *endogenously* determined through the demand system. Thus, one can easily observe that the centralized problem is a relaxation of the decentralized problem where the demand constraints are ignored.[11]

While the auctioneer determines the unit prices that will induce demand, the payments to the suppliers need not be linear in prices (i.e., equal to unit price times quantity sold) as the she may choose to implement a more general payment structure. In Section 4.1 we show that if the auctioneer can compensate the suppliers using two-part tariffs, then the optimal mechanism can be easily characterized and the market structure of the centralized mechanism continues to hold.

In Section 4.2 we study a setting where payments to the suppliers need to be linear in prices and upfront fees are not allowed. This compensation structure closely resembles that used in real-world FAs where the procurement agency essentially acts as a platform and does not provide direct payments; see Chapter 6 in Albano and Nicholas (2016) for a summary of FA implementations in different countries. Imposing that payments to suppliers must be linear in prices can result in a loss with respect to the centralized setting, and it also introduces significant technical challenges in characterizing the optimal mechanism. Despite this, we provide mild sufficient conditions under which, for a broad class of models, there is no performance loss associated with linear pricing. We complement these results with simulations illustrating that the performance loss associated with linear prices, if any, appears to be small. This suggests that the centralized mechanism may be used as a somewhat reliable upper bound when thinking about designing mechanisms in practice.

4.1 Two-part Tariffs

As mentioned above, one could consider general payment structures to compensate suppliers. Perhaps a sensible structure is a two-part tariff, in which the auctioneer receives (or pays) a fixed transfer from every firm participating in the assortment and, in addition, every firm receives a linear transfer from consumers (equal to the unit prices set by the mechanism times the demands).

[10] As $\boldsymbol{x}$ is fully determined by $\boldsymbol{q}$ and $\boldsymbol{p}$, one could formulate the decentralized problem without including $\boldsymbol{x}$ as a decision variable. As will become clear later in this section, we decided to keep it as part of our formulation to be able to obtain a cleaner comparison with the centralized mechanism.

[11] As demands are obtained by maximizing consumer surplus and the auctioneer seeks to maximize expected consumer surplus, it may appear that the demand constraints are redundant. Later in this section, it will become clear that this is not the case because of the presence of the IC constraints.

This payment structure has been frequently used in the regulation literature (e.g., Dana and Spier (1994)), where the regulators can choose lump-sum fees that firms must pay to participate in the market.

Formally, define the *upfront payment functions* $y_i : \Theta \to \mathbb{R}$, where $y_i(\boldsymbol{\theta})$ is the upfront payment received (or given to the platform, if negative) by supplier i when cost declarations are $\boldsymbol{\theta}$. Then, suppliers' interim utilities can be written as

$$U_i(\theta_i'|\theta_i) = \sum_{\boldsymbol{\theta}_{-i} \in \Theta_{-i}} f_{-i}(\boldsymbol{\theta}_{-i}) \big(\big(p_i(\theta_i', \boldsymbol{\theta}_{-i}) - \theta_i\big)\, x_i(\theta_i', \boldsymbol{\theta}_{-i}) + y_i(\theta_i', \boldsymbol{\theta}_{-i})\big) \tag{8}$$

and the objective of the auctioneer can be rewritten as

$$\mathbb{E}_{\boldsymbol{\theta}} \left[CS(\boldsymbol{x}(\boldsymbol{\theta}), \boldsymbol{p}(\boldsymbol{\theta})) - \sum_{i \in N} y_i(\boldsymbol{\theta}) \right] = \mathbb{E}_{\boldsymbol{\theta}} \left[GCS(\boldsymbol{x}(\boldsymbol{\theta})) - \sum_{i \in N} (x_i(\boldsymbol{\theta}) p_i(\boldsymbol{\theta}) + y_i(\boldsymbol{\theta})) \right],$$

where the equality follows from Assumption 2.2. Then, the auctioneer's problem is given by:

$$\begin{aligned}
[DecTwoPart] \qquad \max_{\boldsymbol{q},\boldsymbol{y},\boldsymbol{p},\boldsymbol{x}} \quad & \mathbb{E}_{\boldsymbol{\theta}} \left[GCS(\boldsymbol{x}(\boldsymbol{\theta})) - \sum_{i \in N} (x_i(\boldsymbol{\theta}) p_i(\boldsymbol{\theta}) + y_i(\boldsymbol{\theta})) \right] & \\
\text{s.t.} \quad & U_i(\theta_i|\theta_i) \geq U_i(\theta_i'|\theta_i) \qquad \forall i \in N,\ \forall \theta_i, \theta_i' \in \Theta_i & \text{(IC)} \\
& U_i(\theta_i|\theta_i) \geq 0 \qquad \forall i \in N,\ \forall \theta_i \in \Theta_i & \text{(IR)} \\
& \boldsymbol{x}(\boldsymbol{\theta}) \in \mathcal{X} \qquad \forall \boldsymbol{\theta} \in \Theta & \text{(Feas)} \\
& \boldsymbol{x}(\boldsymbol{\theta}) = \boldsymbol{d}(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{p}(\boldsymbol{\theta})) \qquad \forall \boldsymbol{\theta} \in \Theta, & \text{(Demand)}
\end{aligned}$$

where the suppliers' interim utilities are given by (8). The differences from the centralized problem are that (i) we impose an additional set of constraints, the demand constraints, that essentially require the allocations to be consistent with consumer choices, and (ii) we impose more structure on the transfers to suppliers.

However, we show that this problem can still be easily solved by using the solution to the centralized mechanism as follows.

Proposition 4.1. *Let $(\boldsymbol{x}^\star, \boldsymbol{t}^\star)$ be an optimal solution to the centralized problem defined in Section 3. Then, there exists an optimal solution $(\boldsymbol{q}, \boldsymbol{y}, \boldsymbol{p}, \boldsymbol{x})$ to the decentralized problem with two-part tariffs, DecTwoPart, where, for all $\boldsymbol{\theta} \in \Theta$, we have that:*

1. *$q_i(\boldsymbol{\theta}) = 1$ if $x_i^\star(\boldsymbol{\theta}) > 0$ and $q_i(\boldsymbol{\theta}) = 0$ otherwise, for all $i \in N$,*
2. *$\boldsymbol{x}(\boldsymbol{\theta}) = \boldsymbol{x}^\star(\boldsymbol{\theta})$,*

3. $\boldsymbol{p}(\boldsymbol{\theta}) = \boldsymbol{v}(\boldsymbol{\theta})$*, and,*

4. $y_i(\boldsymbol{\theta}) = t_i^\star(\boldsymbol{\theta}) - x_i^\star(\boldsymbol{\theta}) p_i(\boldsymbol{\theta})$ *for all* $i \in N$*.*

Moreover, the objective value of $(\boldsymbol{q}, \boldsymbol{y}, \boldsymbol{p}, \boldsymbol{x})$ *in DecTwoPart is equal to the objective value of* $(\boldsymbol{x}^\star, \boldsymbol{t}^\star)$*; thus, OPT(DecTwoPart) = OPT(Cent), where* $OPT(P)$ *denotes the optimal value in problem* P*.*

The result in Proposition 4.1 establishes that the centralized mechanism can also be implemented using a decentralized mechanism with a two-part tariff payment structure. (Note that the result holds for any demand model satisfying Assumptions 2.1 and 2.2.) This result is perhaps not very surprising as, even though more constraints are imposed on the payment structure, the auctioneer still has two instruments (upfront fees and unit prices) to satisfy two sets of constraints (the suppliers' IC constraints and the demand constraints). In fact, one can see that the auctioneer can use the unit prices to satisfy the demand constraints by setting $\boldsymbol{p}(\boldsymbol{\theta}) = \boldsymbol{v}(\boldsymbol{\theta})$ (Condition 3) and then use the upfront fees to guarantee that the incentive constraints are satisfied (Condition 4).

4.2 Linear Pricing

We now focus on a setting where the auctioneer can only use linear pricing to compensate suppliers and these prices must agree with those quoted to consumers. Linear pricing is a prevalent practice in many of the environments we are trying to capture, and thus an important operational constraint that deserves to be studied.

Using the linear-pricing assumption, the interim expected utility for supplier i of type θ_i and report θ_i' defined in Eq. (4) is given by $U_i(\theta_i'|\theta_i) = \sum_{\boldsymbol{\theta}_{-i} \in \Theta_{-i}} f_{-i}(\boldsymbol{\theta}_{-i}) \big((p_i(\theta_i', \boldsymbol{\theta}_{-i}) - \theta_i)\, x_i(\theta_i', \boldsymbol{\theta}_{-i}) \big)$. In addition, we must include constraints to ensure that the allocations are consistent with the underlying demand system (Demand).

The auctioneer's optimal mechanism design problem can now be formulated as follows:

$$
\begin{aligned}
[DecLin] \qquad & \max_{\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{x}} \quad \mathbb{E}_{\boldsymbol{\theta}}[CS(\boldsymbol{x}(\boldsymbol{\theta}), \boldsymbol{p}(\boldsymbol{\theta}))] \\
& \text{s.t.} \quad U_i(\theta_i|\theta_i) \geq U_i(\theta_i'|\theta_i) \qquad \forall i \in N,\ \forall \theta_i, \theta_i' \in \Theta_i \qquad \text{(IC)} \\
& \qquad\quad U_i(\theta_i|\theta_i) \geq 0 \qquad \forall i \in N,\ \forall \theta_i \in \Theta_i \qquad \text{(IR)} \\
& \qquad\quad \boldsymbol{x}(\boldsymbol{\theta}) \in \mathcal{X} \qquad \forall \boldsymbol{\theta} \in \Theta \qquad \text{(Feas)} \\
& \qquad\quad \boldsymbol{x}(\boldsymbol{\theta}) = \boldsymbol{d}(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{p}(\boldsymbol{\theta})) \qquad \forall \boldsymbol{\theta} \in \Theta. \qquad \text{(Demand)}
\end{aligned}
$$

Problem *DecLin* differs from *DecTwoPart* only in the way payments to suppliers are implemented. In the latter, the auctioneer has two instruments (unit prices and upfront payments) to decentralize allocations and to satisfy the suppliers' incentive constraints. However, in *DecLin*, there is only one instrument available (unit prices) to satisfy both sets of constraints. This results in a significant difference: while in *DecTwoPart* the auctioneer is always able to achieve the consumer surplus generated in a centralized setting, this is not necessarily true when the auctioneer can only rely on linear prices. (We briefly discuss an example later in this section.)

As *DecLin* requires choosing an assortment function $\boldsymbol{q}$, it is a mixed integer program that takes a demand model as an input.[12] Moreover, the presence of the demand constraints prevents us from directly applying the standard mechanism design arguments used in the centralized case; under these additional constraints it is not possible to establish a priori which IC constraints are binding in the optimal solution. Therefore, the auctioneer's problem appears to be challenging to solve.

Our approach is to provide sufficient conditions under which we can characterize an optimal solution to *DecLin*. To that end, we exploit the following result, which provides necessary and sufficient conditions under which *DecLin* attains the optimal objective of *Cent*.

Corollary 4.1. *Let* $(\boldsymbol{x}, \boldsymbol{T})$ *be the unique optimal solution to the centralized problem Cent, where* $\boldsymbol{T}$ *denotes the vector of interim expected transfers.*[13] *Define*

$$q_i(\boldsymbol{\theta}) = 1 \text{ if and only if } x_i(\boldsymbol{\theta}) > 0, \ \forall i \in N, \ \boldsymbol{\theta} \in \Theta. \tag{9}$$

Suppose that for all $\boldsymbol{\theta} \in \Theta$*, there exist prices* $\boldsymbol{p}(\boldsymbol{\theta})$ *such that*

$$\begin{aligned} \boldsymbol{x}(\boldsymbol{\theta}) &= \boldsymbol{d}(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{p}(\boldsymbol{\theta})) \qquad \forall \boldsymbol{\theta} \in \Theta, \text{ and} & (10) \\ T_i(\theta_i) &= \sum_{\boldsymbol{\theta}_{-i} \in \Theta_{-i}} p_i(\theta_i, \boldsymbol{\theta}_{-i}) x_i(\theta_i, \boldsymbol{\theta}_{-i}) f_{-i}(\boldsymbol{\theta}_{-i}), \qquad \forall i \in N, \ \forall \theta_i \in \Theta_i \ . & (11) \end{aligned}$$

Then, the optimal objective of DecLin is equal to the optimal objective of Cent. Moreover, an optimal solution to DecLin is given by $(\boldsymbol{q}, \boldsymbol{p})$ *(characterized by Eqs.* (9), (10), *and* (11)*), and the corresponding optimal allocation* $\boldsymbol{x}$ *of Cent. Furthermore, the optimal objective of DecLin is equal to the optimal objective of Cent if and only if such a* $(\boldsymbol{q}, \boldsymbol{p})$ *solution exists.*

The corollary suggests the following approach to solving for the optimal decentralized mechanism. First, solve the centralized problem, which can be viewed as a relaxation of the decentralized

[12] Even if relaxing the integrality of the variables $\boldsymbol{q}$ is possible (by adjusting the definition of demand accordingly), the program is typically nonconvex because the demand constraints are often nonlinear, even in simple cases such as the Hotelling model with two suppliers in Example 2.1.

[13] Problem *Cent* admits a unique optimal solution $(\boldsymbol{x}, \boldsymbol{T})$ for all demand systems considered in the paper; see Proposition 3.2. If *Cent* admits more than one solution, our arguments can be easily extended accordingly.

one where the demand constraints are ignored. Then, find unit prices that allow us to decentralize the optimal solution by: (i) making the aggregate demands under such prices agree with the optimal centralized allocations, as specified by Eqs. (9) and (10); and (ii) satisfying the individual rationality and incentive compatibility constraints for suppliers' truthful revelation of information through the interim expected transfers (Eqs. (11)). This is at the heart of the technical challenge in solving for the optimal mechanism in this setting: under a linear pricing structure we only have one instrument (unit prices) to accomplish these two tasks, and such prices may not exist. The above discussion also highlights why, even though the demands functions maximize consumer surplus and the auctioneer's objective is to maximize consumer surplus, the demand constraints are not redundant in the presence of the IC constraints under linear pricing.

4.2.1 Decentralized Mechanisms with Linear Prices under Affine Demand Systems

For the results in this section we will focus on affine demand systems, including the Hotelling model and the affine demand models introduced earlier. In this case, Corollary 4.1 introduces a system of linear equations (given by Eqs. (10) and (11)) that unit prices must satisfy for a solution to the decentralized problem to achieve the optimal centralized objective. To see why, note that Eqs. (10) require that prices induce the optimal allocations $\boldsymbol{x}$ of *Cent*, i.e., we must find unit prices such that $\boldsymbol{d}(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{v}(\boldsymbol{\theta})) = \boldsymbol{d}(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{p}(\boldsymbol{\theta}))$, for all $\boldsymbol{\theta} \in \Theta$. As the demand function is assumed to be affine in prices, these equations yield linear constraints in prices as they require us to find prices to generate a *given* vector of demands for firms with strictly positive demand. Moreover, given an optimal solution to *Cent*, $(\boldsymbol{x}, \boldsymbol{T})$, Eqs. (11) are also linear in prices. By the above observations, verifying whether $OPT(DecLin) = OPT(Cent)$ for affine demand models is equivalent to establishing whether the linear system of equations defined by Eqs. (10) and Eqs. (11) admits a solution.[14]

Unfortunately, such system of equations does not always admit a solution. Hence, the restriction of implementing payments through linear pricing can result in a loss, as in the following example.[15]

Example 4.1. *Consider the Hotelling model introduced in Example 2.1. Let $\delta = 1$ be the transportation cost. Define $\Theta_1 = \{1, 2.5\}$ and $\Theta_2 = \{1, 2, 2.3\}$, probability functions $f_1 = \{1/2, 1/2\}$ and*

[14] Assuming discrete types allows us to work with a finite-dimensional system of equations and to use finite-dimensional linear algebra. In the continuous-type setting, we would have to deal with an infinite-dimensional space for price variables, and the results would be technically more involved.

[15] Alternatively, one could think of a scheme where the unit prices posted to consumers differ from those used to compensate suppliers, and "budget balance" is required (i.e., for all cost realizations, the sum of consumers' prices times quantities sold must be equal to the sum of suppliers' prices times quantities sold). It is easy to see that such a scheme is less restrictive than linear pricing. In fact, we can show that when demand is given by a Hotelling model, it always achieves the centralized optimum. While this scheme somewhat resembles the current practice in the ride-sharing industry, we are unaware of practical procurement mechanisms implemented in this way. Hence, we focus on the commonly used linear-pricing setting where suppliers garner the unit prices paid by consumers.

$f_2 = \{1/2, 1/3, 1/6\}$*, and let the virtual costs be* $v_1 = \{1, 4\}, v_2 = \{1, 3.5, 3.8\}$*. In this instance, we can show that* $OPT(Cent) > OPT(DecLin)$*, as there is not enough freedom to choose unit prices that simultaneously implement both constraints. We defer a detailed discussion to Appendix E.*

In the remainder of this section, we provide additional (mild) conditions under which one can guarantee that the system of equations in Corollary 4.1 admits a solution and, therefore, that the optimal mechanism can be characterized. Recall that this solution will have the same intuitive interpretation as the centralized solution, as the assortment, allocations, and expected payments agree. We start by providing these conditions for the Hotelling model.[16]

Theorem 4.1. *Consider the general Hotelling model in which suppliers have arbitrary locations and cost distributions. Let* $c^* = \min_{1 \leq i \leq n-1}(\ell_{i+1} - \ell_i)$*. Suppose that the following conditions are simultaneously satisfied:*

1. *There is at least one profile* $\boldsymbol{\theta} \in \Theta$ *such that* $|v_{i+1}(\theta_{i+1}) - v_i(\theta_i)| \leq \delta(\ell_{i+1} - \ell_i)/4$ *for all* $i \in N$*.*
2. *For every* $i \in N$*, we have that* $|\Theta_i| \geq 3$ *and that* $v_i(\theta_i^{j+1}) - v_i(\theta_i^j) \leq \frac{\delta c^*}{8}$ *for every* $1 \leq j < |\Theta_i|$*.*

Then, $OPT(DecLin) = OPT(Cent)$*.*

The proof of Theorem 4.1 can be found in Appendix F. To better understand the conditions in the theorem, we briefly discuss the intuition behind them. The first condition implies the existence of an "interior solution" in which all n agents are in the assortment of the optimal centralized solution. This is automatically satisfied if there is a profile of costs for which the virtual costs of all firms coincide, e.g., if all suppliers have the same cost distribution. The second condition essentially requires the difference in the virtual cost between adjacent points in the support to be bounded by a function of δ such that, the smaller δ is, the closer the virtual costs should be. In general, if we think of the discrete distribution as an approximation of an underlying continuous distribution, this condition is equivalent to requiring that the grid of points in the support be thin enough with respect to[17] δ. These conditions together imply the existence of enough interrelated price vectors, to provide sufficient degrees of freedom to satisfy the demand constraints and the interim expected transfer constraints simultaneously. As the conditions in Theorem 4.1 will be satisfied provided

[16]In the electronic companion we provide a different set of conditions. In particular, we consider a Hotelling model with n suppliers such that supplier i is located at $\ell_i = \frac{(i-1)}{(n-1)}$ (that is, suppliers are equidistant). Further, we assume that the cost distributions are identical. Then, we have $OPT(DecLin) = OPT(Cent)$.

[17]For example, if costs are uniformly distributed in [0,1], we can construct a grid consisting of k equidistant costs such that the distance between adjacent costs is $1/(k-1)$. Using the definition of virtual costs (Definition 3.1), it is easy to see that the difference between adjacent virtual costs $v_i(\theta_i^{j+1}) - v_i(\theta_i^j)$ is bounded by $2/(k-1)$. Therefore, for every δ, we can define k large enough so that Condition 2 is satisfied (e.g., $k \geq 16/\delta c^*$).

that for at least one cost realization firms have similar virtual costs, and that the cost distribution grids are granular enough, they do not apper to be too restrictive.

We now turn our attention to the more general affine demand models, and again ask when does the solution to the decentralized problem agree with the centralized solution.

Theorem 4.2. *Consider the general setting with $N \geq 2$ agents, general cost distributions, and $\mathbf{\Gamma}$ is strictly diagonally dominant. Moreover, suppose that there are no outside option and no capacity constraints, i.e., $\mathcal{X} = \{\boldsymbol{x} : \boldsymbol{x} \geq 0, \sum_{i \in N} x_i = 1\}$. Suppose that the following conditions are simultaneously satisfied:*

1. *There exists a profile $\boldsymbol{\theta} \in \Theta$ such that the set of active firms in the optimal centralized solution is $Q(\boldsymbol{\theta}) = N$. In addition, there exists a $d^* \in \mathbb{R}$ such that, for all $\boldsymbol{\theta}' \in \Theta$ with $|v_i(\theta_i') - v_i(\theta_i)| \leq d^*$ for all $i \in N$, we have that $Q(\boldsymbol{\theta}') = N$.*
2. *For every $i \in N$ we have that $|\Theta_i| \geq 3$ and that $v_i(\theta_i^{j+1}) - v_i(\theta_i^j) \leq d^*/2$ for every $1 \leq j < |\Theta_i|$.*

Then, $OPT(DecLin) = OPT(Cent)$.

Although here d^* depends on the primitives of the problem, the intuition behind the conditions is similar to that in the Hotelling model: the first condition implies the existence of a set of "interior solutions" for which all firms are active, and the second controls the distance between the virtual costs corresponding to adjacent costs. As discussed after Theorem 4.1, the latter condition can always be satisfied by specifying a thin enough cost discretization, such that the larger the set of interior solutions in (1) is, the coarser the discretization can be. The proof and the explicit characterization of d^* for some classes of instances are deferred to the electronic companion.

Ex-post IR constraints. One potential practical drawback of the decentralized mechanism with linear pricing is that unit-optimal prices are not necessarily transparent and intuitive; in particular, unit prices could even be below unit costs as IR constraints must be satisfied only at the interim level. To address this concern, we study a model in which we require that a supplier's price weakly exceed his cost for every vector of cost realizations, which may be a desirable property in practice. That is, we ask what happens if we require the IR constraints to be satisfied ex post as opposed to ad interim. That is, in our original *DecLin* formulation, we require that $U_i(\theta_i|\theta_i) \geq 0,\ \forall i \in N,\ \forall \theta_i \in \Theta_i$. Now we impose that $(p_i(\boldsymbol{\theta}) - \theta_i)\, x_i(\boldsymbol{\theta}) \geq 0,\ \forall i \in N,\ \forall \boldsymbol{\theta} \in \Theta$.

We first observe that an optimal solution to *DecLin* can violate ex-post IR; this is in contrast to the centralized mechanism and two-part tariff decentralized mechanism, both of which admit optimal solutions that are ex-post individually rational (see Proposition 3.2 for the centralized mechanism, and Propositions 3.2 and 4.1 for the two-part tariff case).

Proposition 4.2. *The optimal decentralized mechanism with linear prices may violate ex-post IR. In particular, in a Hotelling model with two suppliers as in Example 2.1, when both suppliers have the same cost distribution and* $\Theta = \{\theta_L, \theta_H\}$*, any optimal decentralized mechanism with linear prices violates ex-post IR whenever* $\delta > v(\theta_H) - v(\theta_L)$.

To better understand the loss incurred by imposing ex-post IR constraints, we numerically solve for the optimal decentralized mechanism with linear prices and ex-post IR constraints and compare it to the solution to the optimal solution to *DecLin*. Recall that, in general, finding the optimal decentralized mechanism requires solving a nonlinear mixed-integer optimization problem, where the number of variables is $2 \times N \times |\Theta|$. Due to the computational complexity of finding an optimal solution to such a problem, we limit our analysis to cases with 2 or 3 suppliers, with at most 4 types. For simplicity, we assume that all suppliers have the same cost distributions but we allow them to be asymmetric in terms of the demand parameters.[18]

We find that, in general, the decrease in consumer surplus resulting from imposing the ex-post IR constraints is typically negligible. In particular, we find that, for most of the instances, the GAP is virtually nonexistent (the GAP at the 80^{th} percentile was 0.1%) and in all cases the GAP was at most 5%. This suggests that our decentralized mechanism with linear pricing provides a reasonable benchmark for the one where ex-post IR constraints are also imposed.

Elastic Demand. In addition, we considered an extension to elastic demand by relaxing the constraint that demand should add up to one.[19] In this setting, in general, it is not possible to find prices that simultaneously implement the optimal centralized allocations and satisfy the constraints on interim expected transfers, a limitation resulting in a loss with respect to the optimal centralized solution. To gain a better understanding of the loss, we numerically solve for the optimal decentralized mechanism over a big set of instances with the general affine demand model. For each instance, we compute and compare the solutions to the centralized and decentralized problem. We

[18]For a given number of suppliers and a given number of types, we create different instances by varying the parameters of the cost distributions and the demand model primitives. Overall, we run thousands of instances covering the parameter space with different cost structures, different levels of vertical differentiation that could vary across firms, and different own and cross-price sensitivities. We use the nonlinear solver KNITRO to solve for the optimal decentralized mechanism (Byrd et al. 2006) in each instance.

[19]When all consumers still buy one unit but we allow each product offered by a supplier to also be obtained in the outside market at a fixed known price, all theorems extend almost straightforwardly. The main difference is that the virtual costs of the products offered by suppliers are now compared to both the virtual costs of other suppliers and the prices of the outside options. We omit the proofs due to lack of space.

find that the average GAP between these problems is below 4.5% and, for the vast majority of the instances, it does not exceed 10%. In almost all instances, the same assortments arise in both problems for most realizations of $\boldsymbol{\theta}$; however, suppliers with lower costs charge higher prices (thus serve less demand) in the decentralized problem than in the centralized one. These results suggest that the centralized relaxation may provide an approximately optimal market structure that could serve as an input to simplify the solution to the decentralized problem. Further understanding optimal mechanisms under elastic demand is an interesting direction for future research.

5 First-Price Auction Implementation

We now study what happens when the menus must be decided through a *first-price-auction*-type mechanism. In such a mechanism, suppliers submit bids representing the unit prices of their products and, if a product is added to the menu, the bid is the posted price. The role of the auctioneer is to design the rules to decide which products to include in the assortment based on suppliers' bids and, possibly, on characteristics of the products offered and of the demand system. First-price-auction-type (FPA) mechanisms are prevalent in public procurement, and is how FAs are implemented in practice (see Albano and Nicholas (2016)).

Recall that in the optimal decentralized mechanisms studied in the previous section, the auctioneer chooses both the rules to decide who is in the assortment and the prices. By contrast, in a FPA mechanism the auctioneer can choose *only* the rules to decide which products to include in the assortment; prices are determined by the suppliers through their bids. In general, this constraint results in a loss of consumer surplus: the optimal decentralized mechanism with linear-pricing *cannot* generally be implemented using a FPA mechanism, even for the simple demand systems considered in this section. One can see this because the optimal mechanism is generally not ex-post IR (see Proposition 4.2) and, in a FPA mechanism, a supplier never bids below his cost.

In the rest of this section, we rely on a combination of a simple theoretical model and numerical simulations to understand how the rules of a FPA mechanism impact consumer surplus. The characterization of the optimal decentralized mechanism with linear pricing is crucial for our purpose: it serves as a benchmark of what is achievable and its structure provides insights into how to modify the traditional first-price auctions to enhance performance. As we briefly discuss in Section 6, the insights from the present section played a crucial role in the redesign of the Chilean FAs.

Competition for the Market and Competition in the Market. Following up on the discussion presented in the Introduction, in general there are two different (but possibly complementary) types of incentives for the suppliers to aggressively compete in prices. First, suppliers compete at the auction stage to become part of the assortment. Whether a supplier is included or not in the assortment depends on the rules of the auction and the bids; by placing a lower bid, a supplier typically increases his chances of being part of the assortment. We refer to the competition at the auction stage as *competition for the market.* However, even if a supplier is added to the assortment, he is not guaranteed any fixed amount of demand: once in the menu, there is competition between imperfect substitute products. Naturally, one would expect that by placing a lower bid, a supplier can (weakly) increase his market share. We refer to the competition for demand once in the menu as *competition in the market.*

In an optimal decentralized mechanism, the auctioneer controls the level of competition for the market by restricting the entry of some suppliers to the menu, and facilitates competition in the market by choosing the prices. However, in a FPA mechanism, the auctioneer can directly control only the competition for the market.

In the rest of the section, we study the effect these two types of competition have on bids and consumer surplus under different simple FPA designs.

5.1 Analytical Evaluation of Different FPA Designs in a Simple Model

We assume that firms have private costs and that, given a mechanism, they play a *pure strategy Bayesian Nash equilibrium* (BNE). Unfortunately, deriving the equilibrium bidding strategies analytically under general model primitives is challenging as profits are a function of all bids through the demand system. Moreover, analytically characterizing bidding strategies in simple single-unit FPAs when bidders are asymmetric[20] is, except for a couple of special cases, analytically intractable.

Therefore, to derive analytical results we restrict our attention to the simple Hotelling model with two suppliers introduced in Example 2.1. Suppliers have the same cost distribution with two possible cost realizations θ_L, θ_H. As before, δ is the transportation cost. The analysis of this simple model provides essential insights. Then, we test the robustness of these insights with numerical experiments in more general models. All proofs and more details regarding the numerical experiments corresponding to statements in this section can be found in the electronic companion.

No competition for the market (NC mechanism). Perhaps the simplest auction design is one with no competition for the market: every supplier whose price does not exceed a reserve price is added to the menu and bids are taken as posted prices. However, suppliers still compete in the

[20] Other than in simple settings like the one studied in this section, suppliers are typically asymmetric due to their product characteristics.

<table>
<tr><th rowspan="2">Value of δ</th><th colspan="2">Optimal</th><th colspan="2">NC (ChileCompra)</th></tr>
<tr><th>award</th><th>Avg. price of supplier θ_L</th><th>award</th><th>Equilibrium price of θ_L</th></tr>
<tr><td>$\left[\frac{1}{f_H}(\theta_H - \theta_L), \infty\right)$</td><td>split</td><td>$\frac{(f_H/2+f_L(1-x))\theta_H+(x-1/2)\theta_L}{f_L/2+f_H x}$
where $x = \frac{1/f_H(\theta_H-\theta_L)+\delta}{2\delta}$</td><td rowspan="3">split</td><td rowspan="2">θ_H</td></tr>
<tr><td>$\left[(\theta_H - \theta_L), \frac{1}{f_H}(\theta_H - \theta_L)\right]$</td><td rowspan="5">single</td><td rowspan="5">$\frac{\theta_L+f_H\theta_H}{1+f_H}$</td></tr>
<tr><td>$\left[\frac{(\theta_H-\theta_L)}{2+f_H}, (\theta_H - \theta_L)\right]$</td><td>$\frac{\theta_L+f_H\theta_H+\delta}{1+f_H}$</td></tr>
<tr><td>$\left[\frac{f_L}{2}(\theta_H - \theta_L), \frac{(\theta_H-\theta_L)}{2+f_H}\right)$</td><td rowspan="2">single</td><td>$\theta_H - \delta$</td></tr>
<tr><td>$\left[\frac{f_H f_L(\theta_H-\theta_L)}{\frac{1}{2}(1+f_H)^2+f_H f_L}, \frac{f_L}{2}(\theta_H - \theta_L)\right]$</td><td>$\theta_L + \delta\frac{1+f_H}{f_L}$</td></tr>
<tr><td>$\left[0, \frac{f_H f_L(\theta_H-\theta_L)}{\frac{1}{2}(1+f_H)^2+f_H f_L}\right]$</td><td>no PSBNE</td><td>-</td></tr>
</table>

Table 1: Comparison of the expected prices for the low type in the optimal mechanism and in the no-competition (NC) mechanism with reserve price θ_H. In all cases, the expected price of an item of cost θ_H is θ_H.

market as the demand is split among the firms in the assortment according to their bids and the demand model. This mechanism is equivalent to a pricing game with private costs and a reserve, and is an interesting benchmark as it closely resembles the way FAs are awarded in Chile.[21]

We analytically calculated the BNE pricing strategies of this game when the reserve price is equal to[22] θ_H. Using the equilibrium prices, we computed the expected consumer surplus (i.e., the negative of the expected purchasing cost plus the overall transportation cost) and compared it to that of the optimal mechanism. Due to lack of space, we defer the characterization to the electronic companion, but the results are summarized in Table 1. Generally, the gaps between the optimal decentralized mechanism with linear pricing and the NC mechanism ranged between 2.5% and 18% for different parameters of this model.[23]

In this simplified setting we say that the outcome of the mechanism is a *single award* if, whenever agents have different types, the low-cost agent obtains all the demand; otherwise, we say that the outcome of the mechanism is a *split award*. A key difference between NC and the optimal mechanism is that split-award outcomes occur more frequently in the former. This difference is key

[21] To award the FAs in a given category (e.g., food), ChileCompra first announces the types of products needed within the category (e.g., cereal and pasta). Then, suppliers submit a bid for each *item* they intend to offer; an item stands for a completely specified product (e.g., a 15-oz. box of Kellogg's Corn Flakes and 17-oz. one are two different items). Bids are evaluated using a scoring rule that is dominated by price; all products whose scores are above a threshold are offered in the menu at the price specified by the supplier in his bid. Scores are compared only across *identical* items. As the item definitions are narrow (only identical products are directly compared), in most cases there is a single supplier bidding for an item. In fact, in a recent FA for food products, a total of $8,091$ products were offered by 116 suppliers. Out of those items, $4,549$ were offered by a *unique* supplier, and all such items were added to the menu. Even for items with at least two bidders, the data suggests that the current rules fail to generate competition for the market. In the food FA, there were over $23,000$ bids and only 5% of these were rejected because bids (prices) were too high.

[22] Alternatively, one could calculate these prices using an arbitrary reserve price; however, one can show that the optimal reserve is indeed θ_H as the equilibrium prices are increasing in the reserve prices.

[23] To obtain the reported results, we considered a wide range of model parameters by varying θ_L in $[10, 19]$, θ_H in $[10.5, 20]$, the probability of having low cost in $[0.05, 0.95]$, δ in $[0.5, 15]$, and the own-price elasticities in $[-8, -0.5]$.

to understanding the optimality gaps: *higher gaps are observed for the values of δ for which NC splits awards and the optimal mechanism does not.*

Intuitively, when δ is close to zero, both mechanisms have a single-award outcome and the optimality gap is small: because consumers are highly price sensitive, competition in the market provides sufficient incentives for suppliers to bid aggressively. By contrast, for large values of δ, both mechanisms have a split-award outcome: restricting entry is not profitable as consumers' value is mostly derived from variety. Finally, for intermediate values of δ, NC splits awards and the optimal mechanism does not (see Table 1 and Figure 2), and the highest optimality gaps are observed. In this regime, NC fails to obtain competitive bids from the low type (relative to the optimum), which increases the purchasing costs and thus deteriorates the performance. This suggests that, in this regime, introducing competition for the market might lead to improvements.

Introducing Competition for the Market. We now show how simple changes to the rules of the NC setting can improve performance. The new auction rules generate competition for the market by restricting the entry of inefficient suppliers in order to obtain lower bids, thus making the single-award outcome more likely. This emulates the findings from the optimal mechanism, which restricts the entry of suppliers in order to reduce expected payments. However, while introducing competition for the market might reduce prices, it might also increase transportation costs (reduce variety) and, therefore, restricting entry does not necessarily translate to higher consumer surplus. This tradeoff is analyzed in detail in the electronic companion; we discuss the main takeaways next.

We consider two possible changes to the rules of the auction: restricting entry *ex ante* (before observing the bids) and restricting it *ex post* (as a function of the observed bids). First, suppose that we decide to *restrict entry ex-ante*. If one can optimize over the assortment size, the ex-ante mechanism will always outperform the NC mechanism, as the latter is an ex-ante mechanism in which all suppliers are added to the assortment. Therefore, in our simple model, we must understand when, in choosing a single winner using, the FPA mechanism outperforms the NC mechanism. We compare these analytically in the electronic companion. We find that restricting entry ex ante is more beneficial when the low-cost outcome is more likely to occur and δ is intermediate: the optimality gaps can be decreased by up to 30%. The main drawback of the ex-ante mechanism, however, is that it always chooses one supplier (or a fixed number of them) even when all have similar (or identical) bids. If two suppliers have similar bids, by adding both to the menu we obtain more variety at a similar purchasing cost, thus improving consumer surplus.

To understand the limitation of this lack of flexibility, we next study a class of mechanisms that *restrict entry ex post*, that is, for which the decision on whom to include in the menu is contingent on the bids submitted. This emulates the optimal mechanism, in which the assortment is decided based on the reported costs. Using the intuition from the optimal mechanism, we propose the following parametric restricted-entry (RE) first-price mechanism. There is a reserve price R (which we assume equal to θ_H) and a split parameter C. If bids satisfy $|b_1 - b_2| < C$, then both suppliers are added to the menu; otherwise, only the supplier with the lowest bid is included in the menu (provided the bid is smaller than R). If both suppliers are in the menu, they still compete in the market as before. Hence, the only difference with the NC mechanism is that we restrict the entry to the menu, and the split parameter C quantifies how restrictive the entry to the market is.[24]

We define the *best restricted-entry mechanism* (BRE) by optimizing over the split parameter C to maximize expected consumer surplus. As $C = \delta$ is always a possibility, the BRE cannot do worse than NC. In fact, whenever BRE outperforms NC it must be by restricting entry. Consistent with our intuition, the regime in which the performance of BRE is superior to that of NC is for intermediate values of δ. This is illustrated in Figure 2, where the BRE mechanism restricts the entry whenever $\delta \leq 4.67$. By doing so, it obtains assortments similar to those obtained by the optimal mechanism, and the expected purchasing cost and the consumer surplus are closer to the optimal ones. When δ exceeds 4.67, the savings obtained from the purchases cannot compensate for the increase in transportation costs and, therefore, BRE and NC coincide beyond that point. The optimality gaps in the instances we analyzed were reduced by at least 20% (and usually more than 50%) for those combinations of parameters in which the optimal mechanism restricted entry and ChileCompra did not. The largest optimality gap was reduced from 20% to 7%.

Overall, our analysis highlights the importance of incorporating competition for the market, in particular in settings where substitution across products is neither too high nor too low.

Robustness Results and More General Settings. To extend our analysis and test the robustness of our insights, we numerically solved for the equilibria for the NC, the ex-ante, and the BRE mechanisms in more involved models, and compared their expected consumer surplus with that of the optimal mechanism. We replicated this simulation exercise for a wide range of environments. The most important common finding is that, as suggested by the theory, restricting entry is highly beneficial in the cases in which the optimal mechanism restricts entry but the NC mechanism does not. Due to lack of space, we provide only a summary of the main findings.

We considered more general cost distributions, and found that the performance of the NC mech-

[24] In the electronic companion, we analytically characterize the equilibrium bids of these auctions.

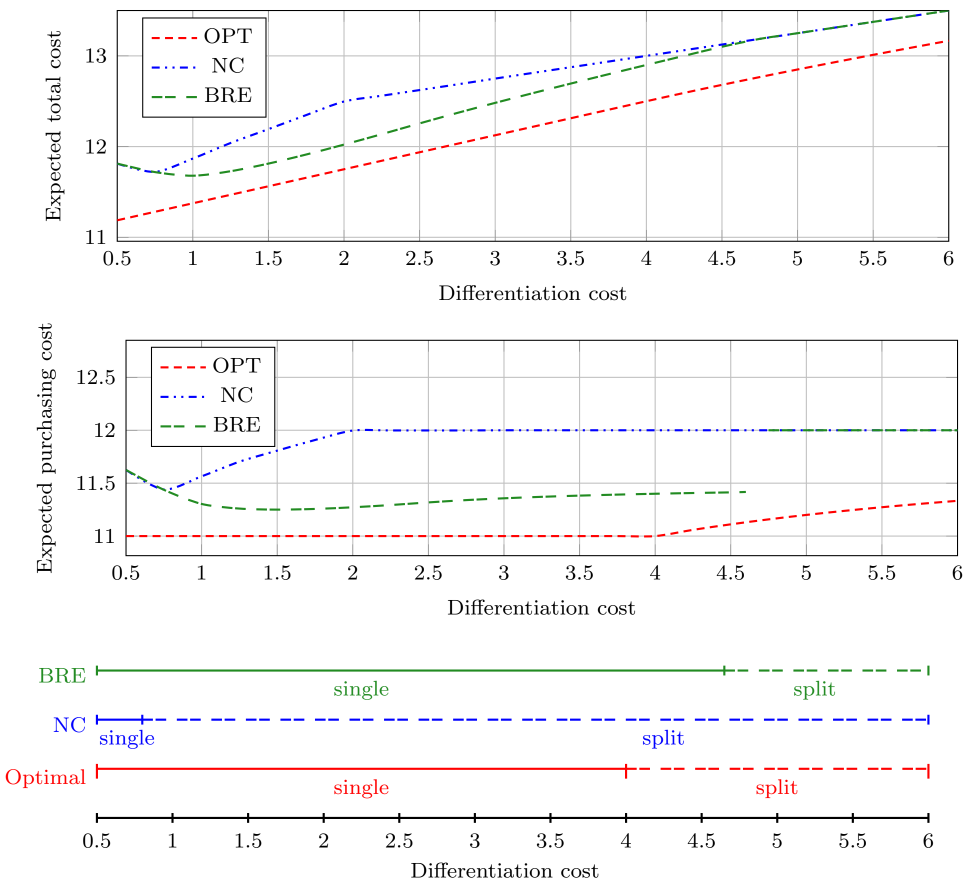

Figure 2: (*Top*) Expected total costs (purchasing plus transportation) or (minus) consumer surplus for the optimal, NC, and BRE mechanisms as a function of the differentiation (transportation) cost δ. The parameters are $\theta_L = 10, \theta_H = 12$ with equal probability. (*Center*) Expected purchasing costs. (*Bottom*) Single-award vs. split-award outcomes in the optimal, NC, and BRE mechanisms.

anism is worse when the distribution is left-skewed or normal-like, with optimality gaps typically above 10%, and as high as 25%; using the BRE, optimality gaps typically decrease by at least 40% in the regime of interest. Also, we find that the gap between the optimal mechanism and NC increases with the number of suppliers and restricting entry performs closer to the optimum than in the two-agent case; this gap was rarely more than 5%, and decreased as the number of suppliers increased.

Finally, we considered a general affine demand model and we varied the vertical qualities of the products and the own and cross-price elasticities. In this setting, introducing competition for the market also improved performance, but the benefits were smaller when the quality difference among products was higher: we obtained an average improvement of 7% in the optimality gap in the cases where the difference between the highest and lowest qualities was more than 20%; when products were of similar quality, the average improvement was 15%. This was to be expected: both the simple ex-ante and BRE mechanisms ignore quality advantages and, therefore, tend to be naturally biased towards the low-cost low-quality suppliers. However, it is still remarkable that such simple mechanisms can achieve significant improvements even under vertical differentiation.

Summary of main insights. To conclude, we summarize the main insights gained from this section. First, there are two main sources of competition in these markets: competition in the market (which naturally arises when substitute products compete in the menu) and competition for the market (which must be enforced through the rules of the auction). When products are very close substitutes, there is no need to introduce competition for the market, as the competition in the market to increase demand ensures low prices. Similarly, when products are very far substitutes, introducing competition for the market is not beneficial as, even though it lowers prices, it damages variety. In the in-between cases, we find that emulating the optimal mechanism with simple FPA mechanisms that introduce competition for the market is highly beneficial. Finally, while introducing competition using simple and anonymous rules in settings with pure horizontal differentiation leads to large improvements, these benefits tend to decrease when there is also vertical differentiation as anonymous rules tend to introduce a bias towards low-quality suppliers. Studying more complex FPA mechanisms for these settings is an interesting avenue for future research.

6 Conclusions and Extensions

We presented a model to study procurement mechanisms for differentiated products. We progressively characterized the optimal mechanisms under an increasing number of practical constraints, and used these results to understand how to better design first-price FA auctions in practice.

When we think more broadly about online two-sided markets, an available design lever is the search algorithm selecting the products shown to consumers. Similar to our assortment decision, Dinerstein et al. (2018) show with a stylized model and through an empirical analysis using eBay data, that lever can be used to find the right balance between variety considerations and price competition. Hence, our optimal mechanism and analysis could serve as a useful benchmark for this setting as well, and it may be worthwhile to further explore this connection.

Moreover, the insights derived in this paper have had a direct practical impact as they have led to concrete changes in the implementation of the Chilean government's FAs. More specifically, we collaborated with the Chilean government to redesign the rules of the new 2017 food FA to introduce more competition at the auction stage. Preliminary analysis suggests that these changes reduced purchasing prices significantly without damaging variety. This implementation together with an empirical analysis will be described in a paper that is currently work in progress.

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Main Appendix

A Proof of Proposition 3.1

Proof of Proposition 3.1. This proof uses the standard arguments from mechanism design theory introduced in Myerson's seminal paper (Myerson 1981). Since the supports of our cost distributions are discrete, we follow the version of these arguments presented by Vohra (2011).

Let m_i denote the number of costs in the support of agent i, that is, $m_i = |\Theta_i|$. We first restate the IC and IR constraints in *Cent* in terms of the expected allocations and transfers:

$$\begin{aligned} \max_{\boldsymbol{x},\boldsymbol{t}} \quad & \mathbb{E}_{\boldsymbol{\theta}}\left[GCS(\boldsymbol{x}(\boldsymbol{\theta})) - \boldsymbol{t}(\boldsymbol{\theta})\right] \\ \text{s.t.} \quad & T_i(\theta_i) - X_i(\theta_i)\theta_i \geq T_i(\theta_i') - X_i(\theta_i')\theta_i \quad \forall i,\ \forall \theta_i, \theta_i' \in \Theta_i \\ & T_i(\theta_i) - X_i(\theta_i)\theta_i \geq 0 \quad \forall i,\ \forall \theta_i \in \Theta_i \\ & \boldsymbol{x}(\boldsymbol{\theta}) \in \mathcal{X} \qquad \forall\ \boldsymbol{\theta} \in \Theta. \end{aligned}$$

Recall that $\Theta_i = \{\theta_i^1, ..., \theta_i^{m_i}\}$. If we add a dummy type per agent $\theta_i^{m_i+1}$ such that $X_i(\theta_i^{m_i+1}) = 0$ and $T_i(\theta_i^{m_i+1}) = 0$, then we can fold the IR constraints into the IC constraints: $T_i(\theta_i^j) - X_i(\theta_i^j)\theta_i^j \geq T_i(\theta_i^k) - X_i(\theta_i^k)\theta_i^j$, for all $j \in \{1, ..., m_i\}$ and all $k \in \{1, ..., m_{i+1}\}$. Applying Theorem 6.2.1 in Vohra (2011) for our procurement setting we obtain that an allocation $\boldsymbol{x}$ is implementable in a Bayes-Nash equilibrium if and only if $X_i(\cdot)$ is monotonically decreasing for all[25] $i = 1, ..., n$. (The results cited in Vohra are for i.i.d. bidders, but the extension to bidders with different distributions is straightforward.) Further, by Theorem 6.2.2 in Vohra (2011), all IC constraints are implied by the following local IC constraints:

$$\begin{cases} T_i(\theta_i^j) - X_i(\theta_i^j)\theta_i^j & \geq & T_i(\theta_i^{j+1}) - X_i(\theta_i^{j+1})\theta_i^j & \quad (BNIC_{i,\theta}^d) \\ T_i(\theta_i^j) - X_i(\theta_i^j)\theta_i^j & \geq & T_i(\theta_i^{j-1}) - X_i(\theta_i^{j-1})\theta_i^j & \quad (BNIC_{i,\theta}^u). \end{cases}$$

Then, we can rewrite the problem as

$$\begin{aligned} \max_{\boldsymbol{x},\boldsymbol{t}} \quad & \mathbb{E}_{\boldsymbol{\theta}}\left[GCS(\boldsymbol{x}(\boldsymbol{\theta}))\right] - \sum_{i=1}^{n}\sum_{j=1}^{m_i} f_i(\theta_i^j) T_i(\theta_i^j) & (obj) \\ \text{s.t.} \quad & T_i(\theta_i^j) - X_i(\theta_i^j)\theta_i^j \geq T_i(\theta_i^{j+1}) - X_i(\theta_i^{j+1})\theta_i^j \quad \forall i \in N,\ \forall j \in \{1, ..., m_i\} & (BNIC_{i,j}^d) \\ & T_i(\theta_i^j) - X_i(\theta_i^j)\theta_i^j \geq T_i(\theta_i^{j-1}) - X_i(\theta_i^{j-1})\theta_i^j \quad \forall i \in N,\ \forall j \in \{2, ..., m_i\} & (BNIC_{i,j}^u) \\ & 0 \leq X_i(\theta^{m_i}) \leq \ldots \leq X_i(\theta^1), \quad \forall i \in N & \text{(M)} \\ & \boldsymbol{x}(\boldsymbol{\theta}) \in \mathcal{X} \qquad \forall\ \boldsymbol{\theta} \in \Theta. \end{aligned}$$

Using standard arguments, we can show that all downward constraints $(BNIC_{i,j}^d)$ are binding in the optimal solution. (A formal proof can be obtained by trivially adapting the Lemma 6.2.4 in

[25] For this result to hold, use that $\mathcal{X}$ is a subset of the Euclidean space. (The result also holds under alternative conditions such as $\mathcal{X}$ having a lattice structure; however, the proof is slightly different and more cumbersome.)

Vohra to the procurement case.) Hence, $T_i(\theta_i^j) - X_i(\theta_i^j)\theta_i^j = T_i(\theta_i^{j+1}) - X_i(\theta_i^{j+1})\theta_i^j \quad \forall i \in N,\ \forall j \in \{1, ..., m_i\}$. Furthermore, it is simple to show that, in this case, the upward constraints ($BNIC_{i,j}^u$) are satisfied. Applying the previous equation recursively we obtain:

$$T_i(\theta_i^j) = \theta_i^j X_i(\theta_i^j) + \sum_{k=j+1}^{m_i} (\theta^k - \theta^{k-1}) X_i(\theta_i^k)\ . \tag{12}$$

Then, we can rexpress the objective as

$$\begin{aligned}
obj &= \mathbb{E}_{\boldsymbol{\theta}}\left[GCS(\boldsymbol{x}(\boldsymbol{\theta}))\right] - \sum_{i=1}^{n}\sum_{j=1}^{m_i} f_i(\theta_i^j) T_i(\theta_i^j) \\
&= \mathbb{E}_{\boldsymbol{\theta}}\left[GCS(\boldsymbol{x}(\boldsymbol{\theta}))\right] - \sum_{i=1}^{n}\sum_{j=1}^{m_i} f_i(\theta_i^j) \left(\theta_i^j X_i(\theta_i^j) + \sum_{k=j+1}^{m_i} (\theta_i^k - \theta_i^{k-1}) X_i(\theta_i^k)\right) \\
&= \mathbb{E}_{\boldsymbol{\theta}}\left[GCS(\boldsymbol{x}(\boldsymbol{\theta}))\right] - \sum_{i=1}^{n}\sum_{j=1}^{m_i} f_i(\theta_i^j) \left(\theta^j X_i(\theta_i^j)\right) - \sum_{i=1}^{n}\sum_{j=1}^{m_i}\sum_{k=0}^{m_i-1} f_i(\theta_i^j) \left(\mathbb{I}\{k \geq j\}(\theta_i^{k+1} - \theta_i^k) X_i(\theta_i^{k+1})\right) \\
&= \mathbb{E}_{\boldsymbol{\theta}}\left[GCS(\boldsymbol{x}(\boldsymbol{\theta}))\right] - \sum_{i=1}^{n}\sum_{j=1}^{m_i} f_i(\theta_i^j) \left(\theta^j X_i(\theta_i^j)\right) - \sum_{i=1}^{n}\sum_{k=1}^{m_i} F_i(\theta_i^{k-1})(\theta_i^k - \theta_i^{k-1}) X_i(\theta_i^k) \\
&= \sum_{\boldsymbol{\theta} \in \Theta} f(\boldsymbol{\theta}) GCS(\boldsymbol{x}(\boldsymbol{\theta})) - \sum_{i=1}^{n}\sum_{j=1}^{m_i} f_i(\theta_i^j) \left(\left(\theta^j + \frac{F_i(\theta_i^{j-1})}{f_i(\theta_i^j)}(\theta_i^j - \theta_i^{j-1})\right) X_i(\theta_i^j)\right) \\
&= \sum_{\boldsymbol{\theta} \in \Theta} f(\boldsymbol{\theta}) GCS(\boldsymbol{x}(\boldsymbol{\theta})) - \sum_{i=1}^{n}\sum_{\theta_i \in \Theta_i} f_i(\theta_i) v_i(\theta_i) X_i(\theta_i) \\
&= \sum_{\boldsymbol{\theta} \in \Theta} f(\boldsymbol{\theta}) \left(GCS(\boldsymbol{x}(\boldsymbol{\theta})) - \sum_{i=1}^{n} v_i(\theta_i) x_i(\boldsymbol{\theta})\right).
\end{aligned}$$

The fourth equality follows from changing the order of summations, and the rest from simple algebra. Hence, if we find an allocation such that for all $\boldsymbol{\theta} \in \Theta$, $\boldsymbol{x}(\boldsymbol{\theta}) \in \text{argmax}\ GCS(\boldsymbol{x}(\boldsymbol{\theta})) - \sum_{i=1}^{n} x_i(\boldsymbol{\theta}) v_i(\theta_i)$ subject to $\boldsymbol{x}(\boldsymbol{\theta}) \in \mathcal{X}$, and such that the interim expected allocations are monotonic for all $i \in N$ (i.e., $X_i(\theta) \geq X_i(\theta')$ for all $\theta \leq \theta' \in \Theta_i$), and such that the interim expected transfers satisfy Eqs. (12), for all $i \in N$ and $\theta \in \Theta_i$, then we have found an optimal solution.

Finally, note that $GCS(\boldsymbol{x}(\boldsymbol{\theta})) - \sum_{i=1}^{n} x_i(\boldsymbol{\theta}) v_i(\theta_i)$ is equal to $CS(\boldsymbol{x}(\boldsymbol{\theta}), \boldsymbol{v}(\boldsymbol{\theta}))$, that is, it is equal to consumer surplus when prices are equal to virtual costs (see Eq. (3)). Therefore, we must have that allocations are a solution to Problem (2) when prices are equal to virtual costs and $Q = N$, which concludes the proof. □

B Proof of Proposition 3.2

Proof of Proposition 3.2. Consider the demand models and the associated consumer surplus functions defined in Examples 2.1 and 2.2. We show that a solution $(\boldsymbol{x}^\star, \boldsymbol{t}^\star)$ satisfying the conditions stated in Proposition 3.1 exists by construction.

First, note that the consumer surplus functions of the models in Examples 2.1 and 2.2 are both quadratic and strictly concave. By our assumptions on $\mathcal{X}$ (i.e., $\mathcal{X}$ it is defined by linear constraints on $\boldsymbol{x}$) and the fact that all other constraints are linear, we have that the auctioneer's problem is a quadratic optimization problem with linear constraints. Therefore, we know that for each $\boldsymbol{\theta} \in \Theta$, there exists a unique solution $\boldsymbol{x}(\boldsymbol{\theta}) \in \operatorname{argmax}_{\boldsymbol{x}' \in \mathcal{X}} CS(\boldsymbol{x}', \boldsymbol{v}(\boldsymbol{\theta}))$. Define $\boldsymbol{x}^\star(\boldsymbol{\theta}) := \operatorname{argmax}_{\boldsymbol{x} \in \mathcal{X}} CS(\boldsymbol{x}, \boldsymbol{v}(\boldsymbol{\theta}))$. Thus, by definition, $\boldsymbol{x}^\star(\boldsymbol{\theta})$ satisfies Condition 1 for all $\boldsymbol{\theta} \in \Theta$.

Second, fix $i \in N$ and also fix $\boldsymbol{\theta}_{-i} \in \Theta_i$. We show that $x_i^\star(\theta_i, \boldsymbol{\theta}_{-i}) \geq x_i^\star(\theta_i', \boldsymbol{\theta}_{-i})$ for every $\theta_i, \theta_i' \in \Theta_i$ with $\theta_i \leq \theta_i'$. First, note that by Assumption 3.1, we have that $v_i(\theta_i') > v_i(\theta_i)$. As the objective in both cases is quadratic and the virtual costs appear as a constant in the linear term, we have that the linear term associated to x_i in the consumer surplus maximization problem associated with $(\theta_i, \boldsymbol{\theta}_{-i})$ is greater than the one accompanying $(\theta_i', \boldsymbol{\theta}_{-i})$. By using standard techniques in the perturbation analysis in quadratic optimization problems (see, e.g., Bonnans and Shapiro (2013)) we thus obtain that $x_i^\star(\theta_i, \boldsymbol{\theta}_{-i}) \geq x_i^\star(\theta_i', \boldsymbol{\theta}_{-i})$, as desired. As this holds for every $\boldsymbol{\theta}_{-i} \in \Theta_i$, we have that $X_i^\star(\theta) \geq X_i^\star(\theta')$, where $X_i^\star(\theta)$ is defined as in Eq. (5), as desired.

Third, for all $i \in N$, $\theta_i^j \in \Theta_i$ and $\boldsymbol{\theta}_{-i} \in \Theta_{-i}$ define

$$t_i^\star(\theta_i^j, \boldsymbol{\theta}_{-i}) := \theta_i^j x_i^\star(\theta_i^j, \boldsymbol{\theta}_{-i}) + \sum_{k=j+1}^{|\Theta_i|} (\theta_i^k - \theta_i^{k-1}) x_i^\star(\theta_i^k, \boldsymbol{\theta}_{-i}).$$

By the definition of the interim allocations and transfers in Eq. (5), we then have that $T_i^\star(\theta_i^j)$ satisfies the third condition, i.e., $T_i^\star(\theta_i^j)$ satisfies Eq. (7) for every $i \in N$ and $\theta_i^j \in \Theta_i$, as desired.

To establish the second part of the claim, note that the monotonicity of $x^\star$ implies that if $x_i^\star(\theta_i^j, \boldsymbol{\theta}_{-i}) = 0$ then $x_i^\star(\theta_i^k, \boldsymbol{\theta}_{-i}) = 0$ for all $k > j$ and thus, by definition, $t_i^\star(\theta_i^j, \boldsymbol{\theta}_{-i}) = 0$. Moreover, we have that

$$t_i^\star(\theta_i^j, \boldsymbol{\theta}_{-i}) \geq \theta_i^j x_i^\star(\theta_i^j, \boldsymbol{\theta}_{-i}).$$

To conclude, note that every optimal solution must have the same $\boldsymbol{x}^\star(\boldsymbol{\theta})$ for the first condition to be satisfied. Therefore, all optimal solutions must have the same $X_i^\star(\theta_i^j)$ for every $i \in N$ and $\theta_i^j \in \Theta_i$. This implies that, by Eq. (7), the expected transfers $T_i^\star(\theta_i^j)$ in every optimal solution must also agree, which completes our proof. □

C Proof of Proposition 4.1

Proof of Proposition 4.1. First, note that problem *Cent* is a relaxation of *DecTwoPart*, where we (i) relax the demand constraints and (ii) define $t_i(\boldsymbol{\theta}) := x_i(\boldsymbol{\theta}) p_i(\boldsymbol{\theta}) + y_i(\boldsymbol{\theta})$ (hence, the definition of the objective and that of the interim utilities agree with those of the centralized problem defined in Section 3). Therefore, the value of the objective associated with $(\boldsymbol{x}^\star, \boldsymbol{t}^\star)$ is an upper bound on that of *DecTwoPart*.

Let $(\boldsymbol{q}, \boldsymbol{y}, \boldsymbol{p}, \boldsymbol{x})$ be as defined in statement of the proposition. We now argue that this is a feasible solution to *DecTwoPart* and, because its objective value in *DecTwoPart* agrees with that of *Cent* under $(\boldsymbol{x}^\star, \boldsymbol{t}^\star)$, that would imply that it is also optimal.

Fix $\boldsymbol{\theta} \in \Theta$. As $\boldsymbol{x}(\boldsymbol{\theta}) = \boldsymbol{x}^\star(\boldsymbol{\theta})$, by Proposition 3.1 we know that $\boldsymbol{x}(\boldsymbol{\theta}) \in \operatorname{argmax}_{\boldsymbol{x}\in\mathcal{X}} CS(\boldsymbol{x}, \boldsymbol{v}(\boldsymbol{\theta}))$. By Assumptions 2.1 and 2.2 and the definition of $\boldsymbol{q}$, we have that $\boldsymbol{x}(\boldsymbol{\theta}) = \boldsymbol{d}(\boldsymbol{1}, v(\boldsymbol{\theta})) = \boldsymbol{d}(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{v}(\boldsymbol{\theta}))$, where $\boldsymbol{1}$ is the vector of 1's. By the definition of $\boldsymbol{p}(\boldsymbol{\theta})$ we therefore have that $\boldsymbol{x}(\boldsymbol{\theta}) = \boldsymbol{d}(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{p}(\boldsymbol{\theta}))$. As the above holds for any $\boldsymbol{\theta} \in \Theta$, the demand constraints are satisfied. All other constraints are satisfied by the feasibility of $(\boldsymbol{x}^\star, \boldsymbol{t}^\star)$ in *Cent*, which completes the proof. □

D Proof of Proposition 4.2

Proof of Proposition 4.2. Consider a Hotelling model with two suppliers as in Example 2.1. Suppose that both suppliers have the same cost distribution, and let $\Theta_1 = \Theta_2 = \{\theta_L, \theta_H\}$. Fix $f(\theta_L)$ and $f(\theta_H)$, and suppose that $\delta > v(\theta_H) - v(\theta_L)$.

Let $(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{x})$ be an optimal solution to *DecLin*. By Theorem H.1 (it is stated and proved in the electronic companion), we know that the optima of the centralized and the decentralized mechanism with linear pricing agree. Therefore, $(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{x})$ must satisfy the conditions in Corollary 4.1. In particular, we must have that $\boldsymbol{x}(\boldsymbol{\theta}) = \boldsymbol{d}(N, v(\boldsymbol{\theta}))$. Thus, by our assumption on δ, we have that $\boldsymbol{q}(\boldsymbol{\theta}) = \{1, 2\}$ for all $\boldsymbol{\theta} \in \Theta$, that is, both suppliers are in the optimal assortment, regardless of their cost realizations. Suppose, towards a contradiction, that $(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{x})$ satisfies ex-post IR, that is, for $i = 1, 2$ we have that $p_i(\theta_i, \theta_j) \geq \theta_i$ for all $\theta_i \in \Theta$ and all $\theta_j \in \Theta$. By Corollary 4.1, we have that

$$T_1(\theta_H) = p_1(\theta_H, \theta_L)x_1(\theta_H, \theta_L)f(\theta_L) + p_1(\theta_H, \theta_H)x_1(\theta_H, \theta_H)f(\theta_H)$$

and, by Proposition 3.1, we have that $T_i(\theta_H) = \theta_H X_i(\theta_H)$. Thus, the only way in which both these contraints are simultaneously satisfied while satisfying ex-post IR is if $p_1(\theta_H, \theta_L) = \theta_H$ and $p_1(\theta_H, \theta_H) = \theta_H$.

However, by Corollary 4.1, we know that $x_1(\theta_H, \theta_L) = \frac{v(\theta_L) - v(\theta_H) + \delta}{2\delta}$ and, by the Hoteling demand model, we need $x_1(\theta_H, \theta_L) = \frac{p_2(\theta_H, \theta_L) - p_1(\theta_H, \theta_L) + \delta}{2\delta}$. Thus we must have that $p_2(\theta_H, \theta_L) - p_1(\theta_H, \theta_L) = v(\theta_L) - v(\theta_H)$ or, equivalently,

$$p_2(\theta_H, \theta_L) = \theta_H - v(\theta_H) + \theta_L = \theta_L - \frac{f(\theta_L)}{f(\theta_H)}(\theta_H - \theta_L) < \theta_L,$$

where the first equality uses the fact that $p_1(\theta_H, \theta_L) = \theta_H$ and $v(\theta_L) = \theta_L$ and the second equality uses the definition of virtual cost, i.e., $v(\theta_H) = \theta_H + \frac{f(\theta_L)}{f(\theta_H)}(\theta_H - \theta_L)$. Therefore, we have reached a contradiction to the fact that $(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{x})$ satisfies ex-post IR. As we have shown this for any arbitrary optimal solution $(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{x})$, the claim follows. □

E Example of Suboptimality of Linear Pricing

Example E.1. *Consider the Hotelling model introduced in Example 2.1. Let $\delta = 1$ be the transportation cost. Let $\Theta_1 = \{1, 2.5\}$ and $\Theta_2 = \{1, 2, 2.3\}$. The probability functions f_1, f_2, and the virtual costs v_1, v_2 are summarized in the following tables:*

Θ_1	1	2.5
f_1	1/2	1/2
v_1	1	4

(a)

Θ_2	1	2	2.3
f_2	1/2	1/3	1/6
v_2	1	3.5	3.8

(b)

To show that a gap exists between Cent and DecLin, we show that it is not possible to find prices satisfying the conditions in Corollary 4.1. To that end, note that the set of possible cost realizations is $\Theta = \{(1,1),(1,2),(1,2.3),(2.5,1),(2.5,2),(2.5,2.3)\}$*. Whenever* $\theta_1 = 1$ *or* $\theta_2 = 1$ *(but not both), only the agent with cost equal to* 1 *is in the optimal assortment as the difference between virtual costs exceeds* δ*. Therefore, whenever agent* 2 *has cost* $\theta_2 = 2$ *he is inthe assortment only in profile* $(2.5, 2)$*. By Eq.* (11)*, the price* $p_2(2.5, 2)$ *is completely determined, and then Eq.* (10) *fixes* $p_1(2.5, 2)$*. Similarly, when agent* 2 *has cost* $\theta_2 = 2.3$ *he is in the assortment only in profile* $(2.5, 2.3)$*. Using the same arguments as before, Eq.* (11) *pins down* $p_2(2.5, 2.3)$ *and hence Eq.* (10) *fixes price* $p_1(2.5, 2.3)$*. However, once the values of* $p_1(2.5, 2)$ *and* $p_1(2.5, 2.3)$ *are fixed as explained above, the expected transfer constraint for* $T_1(2.5)$ *fails to hold and a gap between both problems must exist. In this case, the optimal objective value of Cent and DecLin are* -2.0638 *and* -2.0645*, respectively.*[26]

F Proof of Theorem 4.1

In this section we prove Theorem 4.1. Recall that the idea of the proof is to show that the system of linear equations defined by Eqs. (10) and (11) is consistent (see Corollary 4.1). We start by describing these equations for the Hotelling model. Naturally, we use several basic definitions and concepts from linear algebra throughout this section. We refer the reader to Strang (1988).

F.1 System of Linear Equations for the Hotelling Demand Model

Recall that, by Proposition 3.1, the optimal allocations in the centralized problem are equal to the Hotelling demands when prices are equal to the vector of virtual costs. For a given vector of costs $\boldsymbol{\theta}$, the optimal centralized assortment is given by $Q(\boldsymbol{\theta}) = \{i \in N : \ v_i(\theta_i) - v_j(\theta_j) \leq \delta|\ell_j - \ell_i|, \ \forall j \in N\}$. Given $\boldsymbol{\theta}$, we say that a supplier is *active* if he is in $Q(\boldsymbol{\theta})$. Let $k = |Q(\boldsymbol{\theta})|$ be the number of suppliers in the centralized-optimal assortment, and let $1(\boldsymbol{\theta}), 2(\boldsymbol{\theta}), \ldots, k(\boldsymbol{\theta})$ be the set of active suppliers, where $1(\boldsymbol{\theta})$ and $k(\boldsymbol{\theta})$ denote the leftmost and rightmost suppliers, respectively. When clear from the context, we drop the $\boldsymbol{\theta}$ from the notation and refer to the suppliers as $(1), (2), \ldots, (k)$.

Recall that, in the Hotelling model, active suppliers split the market with their immediate active neighbors through a linear demand system. In particular, recall that if i is the leftmost active supplier, he obtains all the demand in the $[0, \ell_i]$ segment; similarly, if he is the rightmost active supplier, he obtains all the demand in $[\ell_i, 1]$. Therefore, the demand for the leftmost active

[26] It is easy to verify that Condition 2 in Theorem 4.1 is violated in Example E.1. In particular, $|\Theta_1| = 2$ and, furthermore, the difference between consecutive virtual costs in general exceeds $\frac{\delta c^*}{8} = \frac{1}{8}$.

supplier (1) is $\ell_{(1)}+\frac{p_{(2)}-p_{(1)}+\delta|\ell_{(2)}-\ell_{(1)}|}{2\delta}$. Similarly, the demand for supplier (2) is $\frac{p_{(1)}-p_{(2)}+\delta|\ell_{(2)}-\ell_{(1)}|}{2\delta}+\frac{p_{(3)}-p_{(2)}+\delta|\ell_{(3)}-\ell_{(2)}|}{2\delta}$, and so on.

By Corollary 4.1, Eqs. (10) require that the unit prices in the optimal solution decentralize the centralized-optimal demands (where unit prices correspond to virtual costs). Because of the structure of the Hotelling demands as described, the constraints corresponding to Eqs. (10) are given by

$$\begin{aligned} p_{(2)}(\boldsymbol{\theta}) - p_{(1)}(\boldsymbol{\theta}) &= v_{(2)}(\theta_{(2)}) - v_{(1)}(\theta_{(1)}) \\ p_{(i-1)}(\boldsymbol{\theta}) - 2p_{(i)}(\boldsymbol{\theta}) + p_{(i+1)}(\boldsymbol{\theta}) &= v_{(i-1)}(\theta_{(i-1)}) - 2v_i(\theta_i) + v_{(i+1)}(\theta_{(i+1)}) \qquad \text{for } 2 \leq i < k = |Q(\boldsymbol{\theta})| \\ p_{(k-1)}(\boldsymbol{\theta}) - p_{(k)}(\boldsymbol{\theta}) &= v_{(k-1)}(\theta_{(k-1)}) - v_{(k)}(\theta_{(k)}). \end{aligned} \tag{$M_i(\boldsymbol{\theta})$}$$

We refer to the constraint associated with the cost vector $\boldsymbol{\theta}$ and supplier $i \in Q(\boldsymbol{\theta})$ as $M_i(\boldsymbol{\theta})$. Note that, for a vector of costs $\boldsymbol{\theta}$, the above equations can be represented in matrix form as $\boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{p}(\boldsymbol{\theta}) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{v}(\boldsymbol{\theta})$, where the matrix $\boldsymbol{A}(\boldsymbol{\theta})$ is a $\mathbb{R}^{k\times k}$ matrix defined as

$$\boldsymbol{A}(\boldsymbol{\theta}) = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix} \tag{13}$$

For each $\boldsymbol{\theta}$, these equations impose $|Q(\boldsymbol{\theta})|$ constraints on the prices $\boldsymbol{p}(\boldsymbol{\theta})$ corresponding to firms with strictly positive demands. (Recall that only prices associated with active suppliers appear in the demand equations.) However, as the allocations must add up to one, one of these constraints is redundant: the demands for $|Q(\boldsymbol{\theta})| - 1$ suppliers determine the demand for the remaining active supplier. Therefore, Eqs. (10) impose $|Q(\boldsymbol{\theta})| - 1$ constraints on prices $\boldsymbol{p}(\boldsymbol{\theta})$. The redundancy of one constraint plays an important role because it induces degrees of freedom that can be used to satisfy the constraints on interim expected transfers.

In addition, by Corollary 4.1, we also need to guarantee that the interim expected transfers coincide with the optimal ones from *Cent* (Eqs. (11)). We abuse notation and refer to the equality constraint on the expected transfers corresponding to supplier i and cost $\theta_i^j \in \Theta_i$ by $T_i(\theta_i^j)$. Recall that this constraint can be expressed as

$$\sum_{\boldsymbol{\theta}_{-i}\in\Theta_{-i}} f_{-i}(\boldsymbol{\theta}_{-i}) x_i(\theta_i^j, \boldsymbol{\theta}_{-i}) p_i(\theta_i^j, \boldsymbol{\theta}_{-i}) \;=\; T_i(\theta_i^j) \qquad \forall i \in N,\ \forall \theta_i^j \in \Theta_i. \tag{$T_i(\theta_i^j)$}$$

Note that, if in the optimal solution we have that $x_i(\theta_i^j, \boldsymbol{\theta}_{-i}) = 0$ for all $\boldsymbol{\theta}_{-i} \in \Theta_{-i}$, then, by Conditions 2 and 3 in Proposition 3.1, it must be that $T_i(\theta_i^j) = 0$. Hence, the previous equations impose $\sum_{i\in N}\sum_{\theta_i\in\Theta_i} \mathbb{I}[\exists\, \boldsymbol{\theta}_{-i} :\ i \in Q(\theta_i, \boldsymbol{\theta}_{-i})] \equiv K$ constraints ($\mathbb{I}[\cdot]$ denotes the indicator function); prices of inactive suppliers can be discarded, as all the coefficients of such columns are zero.

Abusing notation, let $\boldsymbol{M}$ and $\boldsymbol{m}$ be the coefficient matrix and the corresponding RHS respectively defined by the linear equations in ($M_i(\boldsymbol{\theta})$) and ($T_i(\theta_i^j)$), where each column is associated with a price $p_i(\boldsymbol{\theta})$ with $i \in Q(\boldsymbol{\theta})$. The goal of the proof is to show that the system of linear equations given by $(\boldsymbol{M}, \boldsymbol{m})$ has a solution. By the previous discussion, the resulting matrix $\boldsymbol{M}$ has $\sum_{\theta\in\Theta} |Q(\boldsymbol{\theta})|$ columns and $\sum_{\boldsymbol{\theta}} |Q(\boldsymbol{\theta})| - |\Theta| + K$ rows. It is easy to verify that $K \leq |\Theta|$. By the Rouché–Frobenius theorem, a system of linear equations $\boldsymbol{M}\boldsymbol{p} = \boldsymbol{m}$ is consistent (has a solution) if and only if the rank of its coefficient matrix $\boldsymbol{M}$ is equal to the rank of its augmented matrix $[\boldsymbol{M}|\boldsymbol{m}]$. To show that the system of equations has a solution, we use an equivalent definition of consistency.

Lemma F.1 (Consistency of a system of linear equations)**.** *Consider the system of linear equations $\boldsymbol{M}\boldsymbol{p} = \boldsymbol{m}$. Let $\boldsymbol{M}_{i,*}$ denote the i^{th} row of $\boldsymbol{M}$. Then, the system is consistent (has a solution) if and only if for every vector $\boldsymbol{y}$ such that $\sum_i y_i \boldsymbol{M}_{i,*} = \boldsymbol{0}$, we have that $\sum_i y_i m_i = 0$.*

To apply the above lemma, we define the following coefficients. For each row $M_i(\boldsymbol{\theta})$, let $a^i_{\boldsymbol{\theta}}$ denote its associated coefficient. Similarly, we denote by $b^i_{\theta^j_i}$ the coefficient associated with row $T_i(\theta^j_i)$. Let $(\boldsymbol{a}, \boldsymbol{b})$ be the vector of coefficients we just described. Now, rephrasing Lemma F.1 for our setting, for a system to be consistent we must have that for every vector $(\boldsymbol{a}, \boldsymbol{b})$ such that

$$\sum_{\boldsymbol{\theta}\in\Theta} \sum_{i\in Q(\boldsymbol{\theta})} a^i_{\boldsymbol{\theta}} M_i(\boldsymbol{\theta}) + \sum_{i\in N} \sum_{\theta^j_i \in\Theta_i} b^i_{\theta^j_i} T_i(\theta^j_i) \;=\; 0 \;, \tag{14}$$

the linear combination of the right-hand side also equals zero, that is,

$$\sum_{\boldsymbol{\theta}\in\Theta} \sum_{i\in Q(\boldsymbol{\theta})} a^{(i)}_{\boldsymbol{\theta}} \boldsymbol{A}_{i,*}(\boldsymbol{\theta}) \boldsymbol{v}(\boldsymbol{\theta}) + \sum_{i\in N} \sum_{\theta^j_i\in\Theta_i} b^i_{\theta^j_i} \left(\theta^j_i X_i(\theta^j_i) + \sum_{k=j+1}^{|\Theta_i|} (\theta^k_i - \theta^{k-1}_i) X_i(\theta^k_i) \right) = 0, \tag{15}$$

where $\boldsymbol{A}_{i,*}(\boldsymbol{\theta})$ represents the ith row of the matrix $\boldsymbol{A}(\boldsymbol{\theta})$ as defined in Eq. (13).

F.2 Preliminary Lemmas

In this section we prove important lemmas that will be useful for proving the main result. We start with some definitions. Let $\underline{\theta_i}$ and $\overline{\theta_i}$ denote the lowest and highest values in Θ_i, respectively. For each $j \in N$, let θ^u_j be the maximum $\theta_j \in \Theta_j$ under which there exists a profile $\boldsymbol{\theta} = (\theta_j, \boldsymbol{\theta}_{-j})$ such that $j \in Q(\boldsymbol{\theta})$. We may assume that $\underline{\theta_j} \leq \theta^u_j$ for all agents $j \in N$, as otherwise we can consider (w.l.o.g.) the reduced problem in which all agents for which the condition is violated are removed. In addition, note that, for agent j, all constraints and coefficients associated with $\theta_j > \theta^u_j$ do not play a role in our analysis, because agent j is inactive in all profiles with $\theta_j > \theta^u_j$.

The conditions of Theorem 4.1 imply two properties that will be useful for proving the result, as stated by the following lemma.

Lemma F.2. *Under the conditions of Theorem 4.1, the following two properties must be satisfied:*

1. *There exists a subset of profiles $\tilde{\Theta} = \Pi_{i\in N}\tilde{\Theta}_i \subseteq \Theta$ such that $Q(\boldsymbol{\theta}) = N$ for every $\boldsymbol{\theta} \in \tilde{\Theta}$, $|\tilde{\Theta}_i| \geq 3$ for every $i \in N$, and, for every $\theta_i \in \Theta_i$, such that $\min \tilde{\Theta}_i \leq \theta_i \leq \max \tilde{\Theta}_i$, we must have that $\theta_i \in \tilde{\Theta}_i$. That is, each $\tilde{\Theta}_i$ must be a (discrete) interval.*

2. *Let* $\theta_i \in \Theta_i$ *with* $\max \tilde{\Theta}_i \leq \theta_i \leq \theta_i^u$, *and let* $\boldsymbol{\theta} = (\theta_i, \boldsymbol{\theta}_{-i})$ *be a profile such that* $i \in Q(\boldsymbol{\theta})$. *Then, there exists a sequence of profiles* $\{\boldsymbol{\theta}_0 = \boldsymbol{\theta}', \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_K = \boldsymbol{\theta}\}$ *such that* $\boldsymbol{\theta}_0 \in \tilde{\Theta}$, $Q(\boldsymbol{\theta}_k) \subseteq Q(\boldsymbol{\theta}_{k-1})$ *for all* $1 \leq k \leq K$, *and such that the subprofiles* $(\boldsymbol{\theta}_{k-1})_{Q(\boldsymbol{\theta}_k)}$ *and* $(\boldsymbol{\theta}_k)_{Q(\boldsymbol{\theta}_k)}$ *differ in at most one component; that is, at most one agent among those active in* $\boldsymbol{\theta}_k$ *has a different cost in both profiles.*

Intuitively, Property 1 states that there exists a set of profiles in which all agents are active. Property 2 states that for every θ_i and $\boldsymbol{\theta} = (\theta_i, \boldsymbol{\theta}_{-i})$ such that $i \in Q(\boldsymbol{\theta})$, there exists a sequence of profiles that can take us from $\boldsymbol{\theta}'$ to $\boldsymbol{\theta}$, where we move from one profile to the next by changing the cost of at most one active supplier at a time. Both these properties will be useful for showing that the corresponding system of linear equations admits a solution and therefore the solutions of the centralized and the linear-pricing decentralized problems coincide. We defer the proof of the lemma to the electronic companion due to lack of space.

We now state and prove the following lemma, which plays a key role in the proof of the main theorem.

Lemma F.3. *Suppose that the coefficients* $(\boldsymbol{a}, \boldsymbol{b})$ *are such that Eq. (14) holds. For each* $i \in N$ *and each* $\theta_i \in \Theta_i$, *let* $g_i(\theta_i)$ *be defined as* $g_i(\theta_i) = \frac{b^i_{\theta_i}}{f_i(\theta_i)}$. *Then for each* $\boldsymbol{\theta} \in \Theta$, *we must have that*

$$\sum_{i \in Q(\boldsymbol{\theta})} g_i(\theta_i) x_i(\boldsymbol{\theta}) = 0. \tag{16}$$

Proof. Fix $\boldsymbol{\theta} \in \Theta$. As Eq. (14) holds, for each $j \in Q(\boldsymbol{\theta})$ we must have that

$$\begin{array}{rclll} b^j_{\theta_j} f_{-j}(\boldsymbol{\theta}_{-j}) x_j(\boldsymbol{\theta}) - a^{(1)}_{\boldsymbol{\theta}} + a^{(2)}_{\boldsymbol{\theta}} & = & 0 & & \text{if } j = (1) \\ b^j_{\theta_j} f_{-j}(\boldsymbol{\theta}_{-j}) x_j(\boldsymbol{\theta}) - 2a^{(i)}_{\boldsymbol{\theta}} + a^{(i+1)}_{\boldsymbol{\theta}} + a^{(i-1)}_{\boldsymbol{\theta}} & = & 0 & & \text{if } j = (i),\ 2 \leq i < k = |Q(\boldsymbol{\theta})| \\ b^j_{\theta_j} f_{-j}(\boldsymbol{\theta}_{-j}) x_j(\boldsymbol{\theta}) - a^{(k)}_{\boldsymbol{\theta}} + a^{(k-1)}_{\boldsymbol{\theta}} & = & 0 & & \text{if } j = (k). \end{array}$$

Adding up all the above equations for $j = (1), \ldots, (k)$ and noting that the coefficients of each row $M(\boldsymbol{\theta})_{(i)}$ add up to zero, we then have $\sum_{j \in Q(\boldsymbol{\theta})} b^j_{\theta_j} f_{-j}(\boldsymbol{\theta}_{-j}) x_j(\boldsymbol{\theta}) = 0$. To complete the proof, note that $\sum_{j \in Q(\boldsymbol{\theta})} b^j_{\theta_j} f_{-j}(\boldsymbol{\theta}_{-j}) x_j(\boldsymbol{\theta}) = f(\boldsymbol{\theta}) \left(\sum_{j \in Q(\boldsymbol{\theta})} g_j(\theta_j) x_j(\boldsymbol{\theta}) \right) = 0$. Hence, $\sum_{j \in Q(\boldsymbol{\theta})} g_j(\theta_j) x_j(\boldsymbol{\theta}) = 0$, as desired. □

F.3 Proof of Theorem 4.1

We can now prove Theorem 4.1.

Proof of Theorem 4.1. To show that $OPT(Cent) = OPT(DecLin)$, we show that the system of equations is consistent. Let $(\boldsymbol{a}, \boldsymbol{b})$ be a vector of coefficients satisfying Eq. (14). Let $g_i(\theta_i)$ be as defined in the statement of Lemma F.3. The idea of the proof is as follows. First, we show that under the assumptions of Theorem 4.1, all $g_i(\theta_i)$ must be zero. Then, we show that if $g_i(\theta_i) = 0$, for all $\theta_i \in \Theta_i$ and all $i \in N$, the system is consistent as desired. Consequently, the proof is divided into the following steps:

Step 1: Show that if $(\boldsymbol{a}, \boldsymbol{b})$ satisfies Eq. (14), all $g_i(\theta_i)$ must be zero. Let $\tilde{\Theta} \subseteq \Theta$ be as in the statement of Lemma F.2. Step 1 is subdivided into the following two steps:

(a) **Step 1.a:** Show that $g_i(\theta_i) = 0$ for all $\theta_i \in \tilde{\Theta}_i$ and all $i \in N$.

(b) **Step 1.b:** Show that $g_i(\theta_i) = 0$ for all $\theta_i \notin \tilde{\Theta}_i$ and all $i \in N$.

Step 2: Show that $g_i(\theta_i) = 0$, for all $\theta_i \in \Theta_i$ and all $i \in N$, implies consistency of the system of linear equations.

Step 1.a : Show that $g_i(\theta_i) = 0$, for all $\theta_i \in \tilde{\Theta}_i$ and all $i \in N$. By the definition of $\tilde{\Theta}$, for every $\boldsymbol{\theta} \in \tilde{\Theta}$ we must have that $Q(\boldsymbol{\theta}) = N$. Consider two profiles $\boldsymbol{\theta} = (\theta_i, \boldsymbol{\theta}_{-i})$ and $\boldsymbol{\theta}' = (\theta_i', \boldsymbol{\theta}_{-i})$ that differ only in agent i's cost and such that $\boldsymbol{\theta}, \boldsymbol{\theta}' \in \tilde{\Theta}$. By the definition of $\tilde{\Theta}$, such a pair of profiles exists. By Eq. (16), we must have that $g_i(\theta_i)x_i(\boldsymbol{\theta}) + \sum_{j \neq i} g_j(\theta_j)x_j(\boldsymbol{\theta}) = 0$ and $g_i(\theta_i')x_i(\boldsymbol{\theta}') + \sum_{j \neq i} g_j(\theta_j)x_j(\boldsymbol{\theta}') = 0$. Hence, by subtracting the second equality from the first one we obtain

$$g_i(\theta_i)x_i(\boldsymbol{\theta}) - g_i(\theta_i')x_i(\boldsymbol{\theta}') = \sum_{j \neq i} g_j(\theta_j) \left[x_j(\boldsymbol{\theta}') - x_j(\boldsymbol{\theta})\right].$$

Recall that, in the Hotelling model, when all agents are active, a change in agent i's cost affects only the demand of agents $i-1, i, i+1$. Therefore, for agent $i = 1$, we have that

$$g_1(\theta_1)x_1(\boldsymbol{\theta}) - g_1(\theta_1')x_1(\boldsymbol{\theta}') = \frac{v_1(\theta_1') - v_1(\theta_1)}{2\delta} g_2(\theta_2). \tag{17}$$

Let θ_2 be the cost of agent 2 in both $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$, where the cost profiles are as defined above. Let $\theta_2' \in \Theta_2$ be such that $\theta_2' \neq \theta_2$ and $\theta_2' \in \tilde{\Theta}_2$ (by Lemma F.2, such a θ_2' exists). Define $\tilde{\boldsymbol{\theta}} = (\theta_1, \theta_2', \boldsymbol{\theta}_{-1,2})$ and $\tilde{\boldsymbol{\theta}}' = (\theta_1', \theta_2', \boldsymbol{\theta}_{-1,2})$. The only assumption on θ_2 was $\theta_2 \in \tilde{\Theta}_2$. Therefore, the above equality must also hold for any ${\theta_2}' \in \tilde{\Theta}_2$. That is, $g_1(\theta_1)x_1(\tilde{\boldsymbol{\theta}}) - g_1(\theta_1')x_1(\tilde{\boldsymbol{\theta}}') = \frac{v_1(\theta_1') - v_1(\theta_1)}{2\delta} g_2(\theta_2')$. By subtracting the inequality when agent 2 has cost θ_2 from the one when his cost is θ_2', we get $\frac{g_1(\theta_1)\left(x_1(\tilde{\boldsymbol{\theta}}) - x_1(\boldsymbol{\theta})\right) - g_1(\theta_1')\left(x_1(\tilde{\boldsymbol{\theta}}') - x_1(\boldsymbol{\theta}')\right)}{v_1(\theta_1') - v_1(\theta_1)} = \frac{1}{2\delta}\left(g_2(\theta_2') - g_2(\theta_2)\right)$. Note that $x_1(\tilde{\boldsymbol{\theta}}) - x_1(\boldsymbol{\theta}) = \frac{v_2(\theta_2') - v_2(\theta_2)}{2\delta}$. Therefore,

$$\frac{g_1(\theta_1) - g_1(\theta_1')}{v_1(\theta_1') - v_1(\theta_1)} = \frac{g_2(\theta_2') - g_2(\theta_2)}{v_2(\theta_2') - v_2(\theta_2)} \qquad \forall \theta_1, \theta_1' \in \tilde{\Theta}_1, \forall \theta_2, \theta_2' \in \tilde{\Theta}_1. \tag{18}$$

We now show that an inductive version of Eq. (17) holds. Consider two profiles profiles $\boldsymbol{\theta} = (\theta_i, \boldsymbol{\theta}_{-i})$ and $\boldsymbol{\theta}' = (\theta_i', \boldsymbol{\theta}_{-i})$, which only differ in terms of agent i's cost. Note that only the demands of agents $i-1, i, i+1$ change in $\boldsymbol{\theta}'$ when compared to $\boldsymbol{\theta}$. Repeating a similar argument to the one for Eq. (17), we have that

$$g_i(\theta_i)x_i(\boldsymbol{\theta}) - g_i(\theta_i')x_i(\boldsymbol{\theta}') = \frac{v_i(\theta_i') - v_i(\theta_i)}{2\delta} \left(g_{i-1}(\theta_{i-1}) + g_{i+1}(\theta_{i+1})\right) \quad \forall 2 \leq i \leq n-1. \tag{19}$$

To complete the proof that $g_i(\theta_i) = 0$ for all $\theta_i \in \tilde{\Theta}_i$, we are going to consider two options: either $g_1(\theta_1) - g_1(\theta_1') = 0$ for at least one pair of $g_1(\theta_1), g_1(\theta_1')$, or $g_1(\theta_1) - g_1(\theta_1') \neq 0$ for all $\theta_1, \theta_1' \in \tilde{\Theta}_1$.

We explore both options next.

Case $g_1(\theta_1) - g_1(\theta_1') = 0$ for some $\theta_1, \theta_1' \in \tilde{\Theta}_1$. Let $k = g_1(\theta_1)$ for $\theta_1 \in \tilde{\Theta}_1$. By Eq. (18), we must have that $k = g_1(\theta_1')$ for every $\theta_1' \in \tilde{\Theta}_1$. Next, note that when $g_i(\theta_i) = g_i(\theta_i')$, we have that $g_i(\theta_i)x_i(\boldsymbol{\theta}) - g_i(\theta_i')x_i(\boldsymbol{\theta}') = g_i(\theta_i)\left(v_i(\theta_i') - v_i(\theta_i)\right)/(2\delta)$. Then, by Eq. (17), we must have that $g_2(\theta_2) = k$ for every $\theta_2 \in \tilde{\Theta}_2$. Inductively, using $g_j(\theta_j) = k$ for every $j < i$, by Eq. (19) we must have that $k = g_i(\theta_i)$ for all $i \in N$. Using Lemma F.3, for $\boldsymbol{\theta} \in \tilde{\Theta}$ we have that $0 = \sum_{i \in N} g_i(\boldsymbol{\theta})x_i(\boldsymbol{\theta}) = k\left(\sum_{i \in N} x_i(\boldsymbol{\theta})\right) = k$, and thus $g_i(\theta_i) = 0$ for all $i \in N$ with $\theta_i \in \tilde{\Theta}_i$, as desired. $\lhd$

Case $g_1(\theta_1) - g_1(\theta_1') \neq 0$ for all $\theta_1, \theta_1' \in \tilde{\Theta}_1$. Let the pair $g_1(\theta_1), g_1(\theta_1')$ be such that $\frac{g_1(\theta_1) - g_1(\theta_1')}{v_1(\theta_1') - v_1(\theta_1)} = k \neq 0$, and rewrite $g_1(\theta_1) = g_1(\theta_1') + k[v_1(\theta_1') - v_1(\theta_1)]$. Let $\theta_1, \theta_1', \theta_1'' \in \tilde{\Theta}_1$ (these three distinct θ_1's exist, as stated in Lemma F.2), and let $\theta_{-1} \in \tilde{\Theta}_{-1}$. Then, using Eq. (17), we have that

$$(v_1(\theta_1') - v_1(\theta_1))g_2(\theta_2)/(2\delta) = g_1(\theta_1')(v_1(\theta_1') - v_1(\theta_1))/(2\delta) + k(v_1(\theta_1') - v_1(\theta_1))x_1(\boldsymbol{\theta}).$$

By dividing on both sides by $(v_1(\theta_1') - v_1(\theta))/(2\delta)$ we obtain $g_2(\theta_2) = g_1(\theta_1') + 2\delta k x_1(\boldsymbol{\theta})$. Since $\theta_1'' \in \tilde{\Theta}_1$, by Eq. (18) we have that $\frac{g_1(\theta_1'') - g_1(\theta_1')}{v_1(\theta_1') - v_1(\theta_1'')} = k$. Thus, by repeating the above steps, we have that $g_2(\theta_2) = g_1(\theta_1') + 2\delta k x_1(\boldsymbol{\theta}'')$, which is a contradiction: the virtual costs are strictly increasing and hence $x_1(\boldsymbol{\theta}) \neq x_1(\boldsymbol{\theta}'')$. $\lhd$

Therefore, we have shown that (i) in the first case we must have that $g_i(\theta_i) = 0$ for all $\theta_i \in \tilde{\Theta}_i$ and all $i \in N$, and that (ii) the second case cannot arise as it will result in a contradiction. This completes the proof of Step 1.a: we have established that $g_i(\theta_i) = 0$, for all $\theta_i \in \tilde{\Theta}_i$ and all $i \in N$. $\lhd$

Step 1.b: $g_i(\theta_i) = 0$ for all $\theta_i \notin \tilde{\Theta}_i$ and all $i \in N$. Next, we show that $g_j(\theta_j) = 0$ whenever $\theta_j < \min \tilde{\Theta}_j$ or $\theta_j > \max \tilde{\Theta}_j$. For $\theta_j < \min \tilde{\Theta}_j$ consider a profile $\boldsymbol{\theta} = (\theta_j, \boldsymbol{\theta}_{-j})$ such that $\theta_i \in \tilde{\Theta}_i$ for all $i \neq j$. (Note that this implies that $g_i(\theta_i) = 0\ \forall i \neq j$.) By the definition of $\tilde{\Theta}_j$ and the monotonicity of the Hotelling demand—it is easy to see that, if we decrease the cost of an agent while keeping all other costs constant, his demand can only (weakly) increase—we must have that $x_j(\boldsymbol{\theta}) > 0$. By Lemma F.3 and Step 1.a we have that $0 = \sum_{i \in Q(\boldsymbol{\theta})} g_i(\theta_i)x_i(\boldsymbol{\theta}) = g_j(\theta_j)x_j(\boldsymbol{\theta})$ and, therefore, $g_j(\theta_j) = 0$ for all $\theta_j < \min \tilde{\Theta}_j$ and all $j \in N$.

Let $\theta_j > \max \tilde{\Theta}_j$ and $\theta_j \leq \theta_j^u$, as defined at the beginning of this section. By Property 2 in Lemma F.2, there exists a profile $\boldsymbol{\theta} = (\theta_j, \theta_{-j})$ and a profile $\boldsymbol{\theta}' \in \tilde{\Theta}$ such that there exists a sequence of profiles $\{\boldsymbol{\theta}_0 = \boldsymbol{\theta}', \ldots, \boldsymbol{\theta}_K = \boldsymbol{\theta}\}$ satisfying that two consecutive profiles differ in at most one cost. Given that $\boldsymbol{\theta}' \in \tilde{\Theta}$, we must have that $g_i(\theta_i') = 0$ for all $i \in N$. We will inductively show that $g_i((\boldsymbol{\theta}_k)_i) = 0$ for every $k = 1, \ldots, K$ and every $i \in Q(\boldsymbol{\theta}_k)$. As $j \in Q(\boldsymbol{\theta}_K)$, this will establish the result. Let k be the component in which $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_1$ differ, and $k \in Q(\boldsymbol{\theta}_1)$. (If no such k exists, then all active agents share the same cost and thus the claim follows from the base case.) By Lemma F.3

we have that $\sum_{i \in Q(\boldsymbol{\theta}_1)} g_i((\theta_1)_i) x_i(\boldsymbol{\theta}_1) = 0$. As $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_1$ differ only in the kth component, $\boldsymbol{\theta}_0 \in \tilde{\Theta}$, and $k \in Q(\boldsymbol{\theta}_1)$, we must have that $g_k((\boldsymbol{\theta}_1)_k) = 0$. We can inductively repeat this argument to show that all the g's corresponding to a profile in the path between $\boldsymbol{\theta}'$ and $\boldsymbol{\theta}$ must be zero, which implies that $g_j(\theta_j) = 0$. $\lhd$

We have shown that both the statements described in Steps 1.a and 1.b hold. Hence, if $(\boldsymbol{a}, \boldsymbol{b})$ satisfies Eq. (14), then $g_i(\theta_j) = 0$ for all $i \in N$ and all $\theta_i \in \Theta_i$, completing the proof of Step 1. $\diamondsuit$

Step 2: $g_i(\theta_i) = 0$ for all $i \in N$ and all $\theta_i \in \Theta_i$ implies that the system is consistent. So far we have shown that $g_i(\theta_i) = 0$ for all $i \in N$ and all $\theta_i \in \Theta_i$. By the definition of g_i, this implies $b^i_{\theta_i} = 0$ for all $i \in N$ and all $\theta_i \in \Theta_i$. To conclude the proof, we show that $b^i_{\theta_i} = 0$ for all $i \in N$ and all $\theta_i \in \Theta_i$ implies that the system is consistent. To that end, consider a vector $(\boldsymbol{a}, \boldsymbol{0})$ satisfying Eq. (14). Let $\boldsymbol{A}(\boldsymbol{\theta})$ be the coefficient matrix associated with the vector of prices $\boldsymbol{p}(\boldsymbol{\theta})$ as defined in Eq. (13). Then,

$$\sum_{i=1}^{|Q(\boldsymbol{\theta})|} a^i_{\boldsymbol{\theta}} \left(\sum_{j \in Q(\boldsymbol{\theta})} \boldsymbol{A}_{ij}(\boldsymbol{\theta}) v_j(\theta_j) \right) = \sum_{j \in Q(\boldsymbol{\theta})} v_j(\theta_j) \left(\sum_{i=1}^{|Q(\boldsymbol{\theta})|} a^i_{\boldsymbol{\theta}} \boldsymbol{A}_{ij}(\boldsymbol{\theta}) \right) = 0,$$

as $(\boldsymbol{a}, \boldsymbol{0})$ satisfying Eq. (14) implies that $\sum_{i=1}^{|Q(\boldsymbol{\theta})|} a^i_{\boldsymbol{\theta}} \boldsymbol{A}_{ij}(\boldsymbol{\theta}) = 0$. Hence, we have shown that $(\boldsymbol{a}, \boldsymbol{0})$ also satisfies Eq. (15). Therefore, the system is consistent and $OPT(Cent) = OPT(DecLin)$, as desired. $\square$

Electronic Companion: Procurement Mechanisms for Differentiated Products

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G Proof of Main Theorems

In this section we prove our main theorems. In particular, we prove a more general theorem (Theorem G.1), which generalizes the statements of Theorem 4.1 (for the Hotelling model) and Theorem 4.2 (for general affine demands).

The rest of the section is organized as follows. Recall that the idea of the proof is to show that the system of linear equations defined by Eqs. (10) and (11) is consistent (see Corollary 4.1). Therefore, in Section G.1 we start by describing the coefficient matrix of the associated system of equations, and deriving some properties of the matrix that will be useful to prove the theorem. In Section G.2, we state some definitions needed for our proof. In Section G.3, we state and prove a preliminary lemma that plays an important role in our proof. Finally, in Section G.4, we state and prove the main theorem. Naturally, we use several basic definitions and concepts from linear algebra throughout this section. We refer the reader to Strang (1988).

G.1 The Coefficient Matrix and the System of Equations

Given $\boldsymbol{\theta} \in \Theta$ and $i \in N$, let $\boldsymbol{A}_{ij}(\boldsymbol{\theta})$ denote the coefficient of $v_j(\theta_j)$ corresponding to the left hand side of Eqs. (10); that is, the coefficient of $v_j(\theta_j)$ in $d_i(N, v(\boldsymbol{\theta}))$. Recall that $Q(\boldsymbol{\theta})$ is the set of active firms (i.e., those with positive demand) in the centralized-optimal solution under profile $\boldsymbol{\theta}$. Also, recall that in all demand models considered in the paper, $\boldsymbol{A}_{ij}(\boldsymbol{\theta}) = 0$ for every $i \in Q(\boldsymbol{\theta})$ and $j \notin Q(\boldsymbol{\theta})$ (i.e. if a supplier has zero demand, then its price does not play a role in the demand equations of competitors).

For a given $\boldsymbol{\theta}$ and a given $i \in Q(\boldsymbol{\theta})$, the constraints imposed by Eqs. (10) can be expressed as:

$$\sum_{j \in Q(\boldsymbol{\theta})} \boldsymbol{A}_{ij}(\boldsymbol{\theta}) p_j(\boldsymbol{\theta}) = \sum_{j \in Q(\boldsymbol{\theta})} \boldsymbol{A}_{ij}(\boldsymbol{\theta}) v_j(\boldsymbol{\theta}) \qquad (M_i(\boldsymbol{\theta}))$$

We refer to the constraint associated with the cost vector $\boldsymbol{\theta}$ and supplier $i \in Q(\boldsymbol{\theta})$ as $M_i(\boldsymbol{\theta})$. Any set of prices $\boldsymbol{p}(\boldsymbol{\theta})$ (for all $\boldsymbol{\theta} \in \Theta$) satisfying all these constraints implement the centralized-optimal allocations.

In addition, by Corollary 4.1, we must also guarantee that the expected interim transfers coincide with the optimal ones from *Cent*. We abuse notation and refer to the equality constraint on the expected transfers corresponding to supplier i and cost $\theta_i^j \in \Theta_i$ by $T_i(\theta_i^j)$. Recall that this constraint

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can be expressed as:

$$\sum_{\boldsymbol{\theta}_{-i}\in\Theta_{-i}} f_{-i}(\boldsymbol{\theta}_{-i})x_i(\theta_i^j,\boldsymbol{\theta}_{-i})p_i(\theta_i^j,\boldsymbol{\theta}_{-i}) \;=\; T_i(\theta_i^j) \qquad \forall i\in N,\ \forall\theta_i^j\in\Theta_i, \tag{$T_i(\theta_i^j)$}$$

where $x_i(\theta_i,\boldsymbol{\theta}_{-i})$ is a constant equal to the corresponding centralized-optimal allocation.

Abusing notation, let $\boldsymbol{M}$ and $\boldsymbol{m}$ be the coefficient matrix and the corresponding RHS respectively defined by linear equations in $(M_i(\boldsymbol{\theta}))$, for every $\boldsymbol{\theta}\in\Theta$ and every $i\in Q(\boldsymbol{\theta})$, and $(T_i(\theta_i^j))$, for every $i\in N$ and every $\theta_i^j\in\Theta_i$, where each column is associated with a price $p_i(\boldsymbol{\theta})$ with $i\in Q(\theta)$.[29] The goal of the proof is to show that the system of linear equations given by $(\boldsymbol{M},\boldsymbol{m})$ has a solution. Recall from Section F that the number of columns in $\boldsymbol{M}$ is greater than or equal to the number of rows. By the Rouché-Frobenius theorem, a system of linear equations $\boldsymbol{Mp}=\boldsymbol{m}$ is consistent (has a solution) if and only if the rank of its coefficient matrix $\boldsymbol{M}$ is equal to the rank of its augmented matrix $[\boldsymbol{M}|\boldsymbol{m}]$. To show whether the system of equations has a solution, we use an equivalent definition of consistency.

Lemma G.1 (Consistency of a system of linear equations)**.** *Consider the system of linear equations $\boldsymbol{Mp}=\boldsymbol{m}$. Let $\boldsymbol{M}_{i,*}$ denote the i^{th} row of $\boldsymbol{M}$. Then, the system is consistent (has a solution) if and only if for every vector $\boldsymbol{y}$ such that $\sum_i y_i\boldsymbol{M}_{i,*}=\boldsymbol{0}$, we have $\sum_i y_i m_i = 0$.*

To apply the above lemma, we define the associated coefficients as follows:

Definition G.1 (Associated Coefficients)**.** *For each row $M_i(\boldsymbol{\theta})$, let $a^i_{\boldsymbol{\theta}}$ denote the associated coefficient. Similarly, we denote by $b^i_{\theta_i^j}$ the coefficient associated to row $T_i(\theta_i^j)$. Let $(\boldsymbol{a},\boldsymbol{b})$ be the vector of coefficients we just described.*

Rephrasing Lemma G.1 for our setting, for a system to be consistent we must have that for every vector $(\boldsymbol{a},\boldsymbol{b})$ such that:

$$\sum_{\boldsymbol{\theta}\in\Theta}\sum_{\substack{i\in Q(\boldsymbol{\theta})\\ i\neq\iota(Q(\boldsymbol{\theta}))}} a^i_{\boldsymbol{\theta}}M_i(\boldsymbol{\theta}) + \sum_{i\in N}\sum_{\theta_i^j\in\Theta_i} b^i_{\theta_i^j}T_i(\theta_i^j) \;=\; 0 \tag{20}$$

then the linear combination of the right hand side also equals zero, that is,

$$\sum_{\boldsymbol{\theta}\in\Theta}\sum_{\substack{i\in Q(\boldsymbol{\theta})\\ i\neq\iota(Q(\boldsymbol{\theta}))}} a^i_{\boldsymbol{\theta}}\left(\sum_{j\in Q(\boldsymbol{\theta})}\boldsymbol{A}_{ij}(\boldsymbol{\theta})v_j(\theta_j)\right) + \sum_{i\in N}\sum_{\theta_i^j\in\Theta_i} b^i_{\theta_i^j}\left(\theta_i^jX_i(\theta_i^j) + \sum_{k=j+1}^{|\Theta_i|}(\theta_i^k-\theta_i^{k-1})X_i(\theta_i^k)\right) = 0. \tag{21}$$

Note that whenever the rows of $\boldsymbol{M}$ are linearly independent, the only vector of coefficients satisfying Eq. (20) is $(\boldsymbol{a},\boldsymbol{b})=\boldsymbol{0}$ and, therefore, the system is trivially consistent.

[29]Prices $p_i(\boldsymbol{\theta})$ with $i\notin Q(\boldsymbol{\theta})$ can safely be discarded, as all the coefficients of such columns are zero.

G.1.1 Further Properties of the Coefficient Matrix

Through the rest of the section, we consider the general affine demand model. Given a matrix $\boldsymbol{A}$, we denote the i^{th} row of $\boldsymbol{A}$ by $\boldsymbol{A}_{i,*}$. Similarly, the j^{th} column is denoted by $\boldsymbol{A}_{*,j}$. For a subset of indices $Q \subset N$, $\boldsymbol{A}_Q$ denotes the principal submatrix of $\boldsymbol{A}$ obtained by selecting only the rows and columns in Q. Similarly, $\boldsymbol{c}_Q$ denotes the vector obtained by selecting only the components in Q and $\mathbf{1}_Q$ denotes the vector of ones of dimension $|Q|$. We have the following result that characterizes an affine demand function for the set of active suppliers.

Lemma G.2. *Given a price vector* $\boldsymbol{p}$ *and the associated demand* $\boldsymbol{d}(N, \boldsymbol{p})$*, we denote by* $Q = Q(\boldsymbol{p}) = \{i \in N : d_i(N, \boldsymbol{p}) > 0\}$*. Then, demand* $\boldsymbol{d}_Q(\boldsymbol{p}) = \boldsymbol{d}(Q, \boldsymbol{p})$ *can be expressed as:*

$$\boldsymbol{d}_Q(\boldsymbol{p}_Q) = \boldsymbol{d}_Q(\boldsymbol{p}_Q) = (\boldsymbol{D}_Q)^{-1} \left(\boldsymbol{c}_Q - \boldsymbol{p}_Q + \left(\frac{1 - \mathbf{1}_Q'(\boldsymbol{D}_Q)^{-1}\left(\boldsymbol{c}_Q - \boldsymbol{p}_Q\right)}{\mathbf{1}_Q'(\boldsymbol{D}_Q)^{-1}\mathbf{1}_Q} \right) \mathbf{1}_Q \right). \tag{22}$$

Proof. We start by stating the KKT conditions for problem in Example 2.2:

$$\begin{aligned} \boldsymbol{c} - \boldsymbol{D}\boldsymbol{x} - \boldsymbol{p} + \lambda \mathbf{1} + \boldsymbol{q} &= \mathbf{0} \\ \mathbf{1}'\boldsymbol{x} &= 1 \\ \boldsymbol{x} &\geq \mathbf{0} \\ \boldsymbol{x}'\boldsymbol{q} &= \mathbf{0} \\ \boldsymbol{q} &\geq 0, \end{aligned} \tag{23}$$

where λ is the multiplier associated to the equality constraint and $\boldsymbol{q}$ is the vector of multipliers associated to the non-negativity constraints. Define $\boldsymbol{v} = \boldsymbol{c} - \boldsymbol{D}\boldsymbol{x} - \boldsymbol{p} + \lambda\mathbf{1}$. By the KKT conditions we must have that $v_i = c_i - \boldsymbol{D}_{i,*}\boldsymbol{x} - p_i + \lambda = 0$, for all $i \in Q$. Therefore,

$$\mathbf{0} = \boldsymbol{v}_Q = \boldsymbol{c}_Q - \boldsymbol{D}_Q\boldsymbol{x}_Q - \boldsymbol{p}_Q + \lambda\mathbf{1}_Q.$$

As $\boldsymbol{D}$ is positive definite and $\boldsymbol{D}_Q$ is a principal submatrix of $\boldsymbol{D}$ we have that $(\boldsymbol{D}_Q)^{-1}$ exists and, furthermore,

$$\boldsymbol{x}_Q = (\boldsymbol{D}_Q)^{-1}\left(\boldsymbol{c}_Q - \boldsymbol{p}_Q + \lambda\mathbf{1}_Q\right)$$

In addition, by the feasibility constraint, we must have $\mathbf{1}_Q'\boldsymbol{x}_Q = 1$ and hence,

$$1 = \mathbf{1}_Q'\boldsymbol{x}_Q = \mathbf{1}_Q'(\boldsymbol{D}_Q)^{-1}\left(\boldsymbol{c}_Q - \boldsymbol{p}_Q + \lambda\mathbf{1}_Q\right)$$

which implies

$$\lambda = \frac{1 - \mathbf{1}_Q'(\boldsymbol{D}_Q)^{-1}\left(\boldsymbol{c}_Q - \boldsymbol{p}_Q\right)}{\mathbf{1}_Q'(\boldsymbol{D}_Q)^{-1}\mathbf{1}_Q}.$$

Therefore,

$$\boldsymbol{x}_Q = (\boldsymbol{D}_Q)^{-1}\left(\boldsymbol{c}_Q - \boldsymbol{p}_Q + \left(\frac{1 - \mathbf{1}_Q'(\boldsymbol{D}_Q)^{-1}(\boldsymbol{c}_Q - \boldsymbol{p}_Q)}{\mathbf{1}_Q'(\boldsymbol{D}_Q)^{-1}\mathbf{1}_Q}\right)\mathbf{1}_Q\right),$$

as desired. □

The above demand specification exhibits a natural regularity property: *if there is no demand for a particular product, the price of that product does not affect the demand for other products.* In addition, it is simple to observe that *any increase in price of a product with zero demand will not have an impact on the demand function either.*

From Eq. (22), it should be clear that whenever two vector of prices $\boldsymbol{p}_Q$ and $\hat{\boldsymbol{p}}_Q$ satisfy

$$(\boldsymbol{D}_Q)^{-1}\left(\boldsymbol{p}_Q - \frac{\mathbf{1}_Q'(D_Q)^{-1}\boldsymbol{p}_Q}{\mathbf{1}_Q'(\boldsymbol{D}_Q)^{-1}\mathbf{1}_Q}\mathbf{1}_Q\right) = (\boldsymbol{D}_Q)^{-1}\left(\hat{\boldsymbol{p}}_Q - \frac{\mathbf{1}_Q'(D_Q)^{-1}\hat{\boldsymbol{p}}_Q}{\mathbf{1}_Q'(\boldsymbol{D}_Q)^{-1}\mathbf{1}_Q}\mathbf{1}_Q\right), \tag{24}$$

we must have that $\boldsymbol{d}_Q(\boldsymbol{p}_Q) = \boldsymbol{d}_Q(\hat{\boldsymbol{p}}_Q)$. This observation is useful: it states that *demands only depend on price differences.* This freedom in setting unit prices is essential to our proof technique, as we will find unit prices that satisfy the same differences induced by the virtual costs and that simultaneously satisfy the expected interim transfer constraints.

By the previous observation, for $\boldsymbol{\theta} \in \Theta$ and each $i \in Q(\boldsymbol{\theta})$, the coefficient matrix $\boldsymbol{M}$ will consist of at most $Q(\boldsymbol{\theta})$ non-zero rows: $Q(\boldsymbol{\theta}) - 1$ correspond to the demand equations[30] and the remaining one corresponding to the expected transfer constraint. Note that for given $\boldsymbol{\theta} \in \Theta$, the demand equations are given by Eq. (24) where we replace Q by $Q(\boldsymbol{\theta})$ and $\boldsymbol{p}_Q$ by $\boldsymbol{p}_{Q(\boldsymbol{\theta})}(\boldsymbol{\theta})$ in the left hand side. In the right hand side we replace prices $\hat{\boldsymbol{p}}_Q$ by virtual costs $\boldsymbol{v}_{Q(\boldsymbol{\theta})}(\boldsymbol{\theta})$.

We now define the the demand submatrix associated to cost $\boldsymbol{\theta} \in \Theta$ as follows.

Definition G.2 (Demand submatrix of cost vector $\boldsymbol{\theta}$)**.** *For a given $\boldsymbol{\theta} \in \Theta$, we denote by $\boldsymbol{A}(\boldsymbol{\theta})$ the submatrix of $\boldsymbol{M}$ that contains the demand constraints for $\boldsymbol{\theta}$, that is, $\boldsymbol{A}(\boldsymbol{\theta})$ equals the left hand side of $(M_i(\boldsymbol{\theta}))_{i \in Q(\boldsymbol{\theta})}$.*

The following corollary of Lemma G.2 characterizes the matrix $\boldsymbol{A}(\boldsymbol{\theta})$ for the general affine demand models. We include all demand equations in this matrix, even though as previously discussed, one of them is redundant.

Corollary G.1. *Let $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{\theta}) = (\boldsymbol{D}_{Q(\boldsymbol{\theta})})^{-1}$. Then, for every $j \in Q(\boldsymbol{\theta})$ and every i such that $1 \le i \le Q(\boldsymbol{\theta})$, the coefficient for $p_j(\boldsymbol{\theta})$ in equation i is given by:*

$$\boldsymbol{A}(\boldsymbol{\theta})_{ij} = -\boldsymbol{F}_{ij} + \frac{(\mathbf{1}_{Q(\boldsymbol{\theta})}' \cdot \boldsymbol{F}_{*,j})(\boldsymbol{F}_{i,*} \cdot \mathbf{1}_{Q(\boldsymbol{\theta})})}{\mathbf{1}_{Q(\boldsymbol{\theta})}'\boldsymbol{F}\mathbf{1}_{Q(\boldsymbol{\theta})}}. \tag{25}$$

[30]Note that if we can find prices p_Q satisfying the constraints imposed by $x_1, \ldots, x_{|Q|-1}$, then the last constraint will also be satisfied as $x_Q = 1 - \sum_{j=1}^{|Q|-1} x_j$.

We now show that the associated demand vectors satisfy the original properties we wanted: the demand for a product is (weakly) decreasing in its own-price and (weakly) increasing in others' prices.

Lemma G.3 (Monotonicity)**.** *For every $\boldsymbol{\theta} \in \Theta$ and every $i \in Q(\boldsymbol{\theta})$ we have $A(\boldsymbol{\theta})_{ii} < 0$ and $A(\boldsymbol{\theta})_{ij} \geq 0$ for every $j \in Q(\boldsymbol{\theta})$ with $j \neq i$.*

Proof. We start by noting that, given the conditions imposed to matrix $\boldsymbol{\Gamma}$ in Section 2, we have that such matrix is a symmetric, non-singular, strictly-diagonally dominant M-matrix. In particular, an M-matrix with such properties has strictly positive diagonal elements, and non-positive off-diagonal elements. This proof uses several properties of M-matrices; the reader is referred to Horn and Johnson (1991) for the details.

Fix an arbitrary $\boldsymbol{\theta} \in \Theta$. By Corollary G.1, we have that

$$\boldsymbol{A}(\boldsymbol{\theta})_{ij} = -\boldsymbol{F}_{ij} + \frac{(\mathbf{1}'_{Q(\boldsymbol{\theta})} \cdot \boldsymbol{F}_{*,j})(\boldsymbol{F}_{i,*} \cdot \mathbf{1}_{Q(\boldsymbol{\theta})})}{\mathbf{1}'_{Q(\boldsymbol{\theta})}\boldsymbol{F}\mathbf{1}_{Q(\boldsymbol{\theta})}},$$

where $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{\theta}) = (\boldsymbol{D}_{Q(\boldsymbol{\theta})})^{-1}$. The proof will consist of two steps. First, we argue that, if $\boldsymbol{F}$ is a symmetric, strictly diagonally dominant M-matrix, we have $A(\boldsymbol{\theta})_{ii} < 0$ and $A(\boldsymbol{\theta})_{ij} \geq 0$ for every $i, j \in Q(\boldsymbol{\theta})$ with $j \neq i$ as desired. Second, we show that $\boldsymbol{F}$ is indeed a symmetric, strictly diagonally dominant M-matrix.

To that end, suppose $\boldsymbol{F}$ is a symmetric, strictly diagonally dominant M-matrix. Then, $\boldsymbol{F}_{ii} > 0$ and $\boldsymbol{F}_{ij} \leq 0$ and we must have that, for every row, the sum of the elements in a row must be strictly positive. By symmetry, this is true also for the sum of the elements in a column. In turn, this implies that the sum of all elements in the matrix is strictly positive and hence,

$$\boldsymbol{A}(\boldsymbol{\theta})_{ij} = -\boldsymbol{F}_{ij} + \frac{(\mathbf{1}'_{Q(\boldsymbol{\theta})} \cdot \boldsymbol{F}_{*,j})(\boldsymbol{F}_{i,*} \cdot \mathbf{1}_{Q(\boldsymbol{\theta})})}{\mathbf{1}'_{Q(\boldsymbol{\theta})}\boldsymbol{F}\mathbf{1}_{Q(\boldsymbol{\theta})}} < -\boldsymbol{F}_{ij} + \boldsymbol{F}_{i,*} \cdot \mathbf{1}_{Q(\boldsymbol{\theta})} \tag{26}$$

Then, we have $\boldsymbol{A}(\boldsymbol{\theta})_{ii} < -\boldsymbol{F}_{ii} + \boldsymbol{F}_{i,*} \cdot \mathbf{1}_{Q(\boldsymbol{\theta})} \leq 0$, where the last inequality follows because $F_{ij} \leq 0$, $\forall i \neq j$. Hence $\boldsymbol{A}(\boldsymbol{\theta})_{ii} < 0$ as desired. Similarly, $\boldsymbol{A}(\boldsymbol{\theta})_{ij} \geq 0$ follows from the fact that both terms in the summation are non-negative. Therefore, we have shown that if $\boldsymbol{F}$ is a symmetric, strictly diagonally dominant M-matrix, the result follows.

To complete the proof, we show that $\boldsymbol{F}$ satisfies the stated properties. As usual, for the given $\boldsymbol{\theta} \in \Theta$, let $Q = Q(\boldsymbol{\theta}) = \{k \in N : x_k(\boldsymbol{\theta}) > 0\}$ and $\overline{Q} = \overline{Q}(\boldsymbol{\theta}) = \{k \in N : x_k(\boldsymbol{\theta}) = 0\}$. From now on, we omit the dependence on $\boldsymbol{\theta}$ to simplify notation. By the KKT conditions (Eq. (23)), we have that $\boldsymbol{x} = \boldsymbol{D}^{-1}(\boldsymbol{c} - \boldsymbol{p} + \lambda\mathbf{1} + \boldsymbol{q})$. (Recall that $\boldsymbol{D}^{-1} = \boldsymbol{\Gamma}$.) By definition, we have that

$$\mathbf{0} = \boldsymbol{x}_{\overline{Q}} = \boldsymbol{\Gamma}_{\overline{Q},*}(\boldsymbol{c} - \boldsymbol{p} + \lambda\mathbf{1} + \boldsymbol{q}) = \boldsymbol{\Gamma}_{\overline{Q},\overline{Q}}(\boldsymbol{c}_{\overline{Q}} - \boldsymbol{p}_{\overline{Q}} + \lambda\mathbf{1}_{\overline{Q}} + \boldsymbol{q}_{\overline{Q}}) + \boldsymbol{\Gamma}_{\overline{Q},Q}(\boldsymbol{c}_Q - \boldsymbol{p}_Q + \lambda\mathbf{1}_Q),$$

where we used $\boldsymbol{q}_Q = 0$ by the KKT conditions. Note that $\boldsymbol{\Gamma}_{\overline{Q},\overline{Q}}$ is a principal submatrix of a non-singular M-matrix. Thus, $(\boldsymbol{\Gamma}_{\overline{Q},\overline{Q}})^{-1}$ exists and:

$$(\boldsymbol{c}_{\overline{Q}} - \boldsymbol{p}_{\overline{Q}} + \lambda \boldsymbol{1}_{\overline{Q}} + \boldsymbol{q}_{\overline{Q}}) = -(\boldsymbol{\Gamma}_{\overline{Q},\overline{Q}})^{-1}\boldsymbol{\Gamma}_{\overline{Q},Q}(\boldsymbol{c}_Q - \boldsymbol{p}_Q + \lambda \boldsymbol{1}_Q) \quad (27)$$

In addition, we have

$$\begin{aligned}
\boldsymbol{x}_Q &= \boldsymbol{\Gamma}_{Q,*}(\boldsymbol{c} - \boldsymbol{p} + \lambda \boldsymbol{1} + \boldsymbol{q}) \\
&= \boldsymbol{\Gamma}_{Q,Q}(\boldsymbol{c}_Q - \boldsymbol{p}_Q + \lambda \boldsymbol{1}_Q) + \boldsymbol{\Gamma}_{Q,\overline{Q}}(\boldsymbol{c}_{\overline{Q}} - \boldsymbol{p}_{\overline{Q}} + \lambda \boldsymbol{1}_{\overline{Q}} + \boldsymbol{q}_{\overline{Q}}) \\
&= \boldsymbol{\Gamma}_{Q,Q}(\boldsymbol{c}_Q - \boldsymbol{p}_Q + \lambda \boldsymbol{1}_Q) - \boldsymbol{\Gamma}_{Q,\overline{Q}}(\boldsymbol{\Gamma}_{\overline{Q},\overline{Q}})^{-1}\boldsymbol{\Gamma}_{\overline{Q},Q}(\boldsymbol{c}_Q - \boldsymbol{p}_Q + \lambda \boldsymbol{1}_Q) \\
&= \left(\boldsymbol{\Gamma}_{Q,Q} - \boldsymbol{\Gamma}_{Q,\overline{Q}}(\boldsymbol{\Gamma}_{\overline{Q},\overline{Q}})^{-1}\boldsymbol{\Gamma}_{\overline{Q},Q}\right)(\boldsymbol{c}_Q - \boldsymbol{p}_Q + \lambda \boldsymbol{1}_Q),
\end{aligned}$$

where the second to last equality follows from Eq. (27). By the definition of $\boldsymbol{D}_Q$, we have $(\boldsymbol{D}_Q)^{-1} = \left(\boldsymbol{\Gamma}_{Q,Q} - \boldsymbol{\Gamma}_{Q,\overline{Q}}(\boldsymbol{\Gamma}_{\overline{Q},\overline{Q}})^{-1}\boldsymbol{\Gamma}_{\overline{Q},Q}\right)$. In turn, this implies that $(\boldsymbol{D}_Q)^{-1}$ is the Schur complement of $\boldsymbol{\Gamma}_{\overline{Q},\overline{Q}}$ in $\boldsymbol{\Gamma}$. In particular, we have that the Schur complement of a M-matrix is also a M-matrix, and non-singularity, symmetry and strict diagonal dominance are preserved in Schur complementation (Carlson and Markham 1979, Horn and Johnson 1991). Therefore, the matrix $\boldsymbol{F}$ satisfies the desired properties, which completes the proof. □

Next, we establish two useful properties on the demand submatrices associated to cost profiles $\boldsymbol{\theta}$ such that all agents are active in $\boldsymbol{\theta}$.

Lemma G.4. *Let $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{\theta})$ for any $\boldsymbol{\theta} \in \Theta$ such that $Q(\boldsymbol{\theta}) = N$ be as defined by Corollary G.1. Then, $\boldsymbol{A}$ is symmetric and has rank $n-1$.*

Proof. Note that, whenever $Q(\boldsymbol{\theta}) = N$, we have $\boldsymbol{F} = \boldsymbol{D}^{-1}$ where $\boldsymbol{F}$ is as defined in Claim G.1. By assumption, $\boldsymbol{D}^{-1}$ is symmetric and positive definite. Therefore, $\boldsymbol{A}$ is also symmetric by definition.

To show that $\boldsymbol{A}$ has rank $n-1$, let $\boldsymbol{I}$ denote the identity matrix of size n. Note that $\boldsymbol{A} = \boldsymbol{D}^{-1}\left(-\boldsymbol{I} + \boldsymbol{1}\frac{\boldsymbol{1}'\boldsymbol{D}^{-1}}{\boldsymbol{1}'\boldsymbol{D}^{-1}\boldsymbol{1}}\right)$. Therefore,

$$rank(\boldsymbol{A}) \geq rank(\boldsymbol{D}^{-1}) + rank\left(-\boldsymbol{I} + \boldsymbol{1}\frac{\boldsymbol{1}'\boldsymbol{D}^{-1}}{\boldsymbol{1}'\boldsymbol{D}^{-1}\boldsymbol{1}}\right) - n = rank\left(-\boldsymbol{I} + \boldsymbol{1}\frac{\boldsymbol{1}'\boldsymbol{D}^{-1}}{\boldsymbol{1}'\boldsymbol{D}^{-1}\boldsymbol{1}}\right),$$

as $\boldsymbol{D}^{-1}$ has full rank. In addition, we have[31]

$$rank\left(-\boldsymbol{I} + \boldsymbol{1}\frac{\boldsymbol{1}'\boldsymbol{D}^{-1}}{\boldsymbol{1}'\boldsymbol{D}^{-1}\boldsymbol{1}}\right) \geq \left| n - rank\left(\boldsymbol{1}\frac{\boldsymbol{1}'\boldsymbol{D}^{-1}}{\boldsymbol{1}'\boldsymbol{D}^{-1}\boldsymbol{1}}\right)\right| \geq n-1,$$

as the matrix $\boldsymbol{1}\frac{\boldsymbol{1}'\boldsymbol{D}^{-1}}{\boldsymbol{1}'\boldsymbol{D}^{-1}\boldsymbol{1}}$ has rank exactly one. The converse follows just from the definition of $\boldsymbol{A}$, as we know that one row must be redundant as all demands must some up to one. □

In order to show that the system of equations is consistent, we want to find prices $\boldsymbol{p}$ such that $\boldsymbol{x}(\boldsymbol{p}) = \boldsymbol{x}(v(\boldsymbol{\theta}))$, where $v(\boldsymbol{\theta}) = (v_1(\boldsymbol{\theta}), \ldots, v_n(\boldsymbol{\theta}))$ is defined as the vector of virtual costs. That is,

[31] Matrix property: $rank(A - B) \geq |rank(A) - rank(B)|$

we must have $\boldsymbol{Ap} = \boldsymbol{A}v(\boldsymbol{\theta})$. Therefore, Lemma G.4 states that for $\boldsymbol{\theta} \in \Theta$ such that $Q(\boldsymbol{\theta}) = N$, the dimension of prices satisfying those demand constraints is exactly one, as $\boldsymbol{A}$ has rank $n-1$.

G.1.2 Coefficient Matrix for Hotelling Model

We provide a brief note on the Hotelling model. While all the material in this section is presented with the general affine demand model in mind to avoid cumbersome notation, we now show that all the properties of matrix $\boldsymbol{A}(\boldsymbol{\theta})$ shown above (that we use to prove our main result) also hold for the Hotelling Model.

Remark G.1. *Given $\boldsymbol{\theta}$, let $Q(\boldsymbol{\theta})$ be the set of active agents ordered from leftmost to rightmost. Then,*

$$\boldsymbol{A}(\boldsymbol{\theta}) = \frac{1}{2\delta}\begin{bmatrix} -1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -1 \end{bmatrix}$$

This follows from the fact that, in the Hotelling model, suppliers split the market with their immediate active neighbors; in particular, i obtains $\frac{p_j - p_i + \delta|\ell_j - \ell_i|}{2\delta}$ units from the segment $[\ell_i, \ell_j]$ and j the rest. If i is the leftmost active supplier, he obtains all the demand in the $[0, \ell_i]$ segment; similarly, if he is the rightmost active supplier, he obtains all the demand in $[\ell_i, 1]$. Renaming the suppliers in $Q(\boldsymbol{\theta})$ as $1, 2, \ldots$, with numbers increasing from left to right suppliers, we have that the demand for the leftmost active supplier 1 is $\ell_1 + \frac{p_2 - p_1 + \delta|\ell_2 - \ell_1|}{2\delta}$. Note that the coefficient of the prices in this equation are represented by the first row of the matrix. Similarly, the demand for supplier 2 is $\frac{p_1 - p_2 + \delta|\ell_2 - \ell_1|}{2\delta} + \frac{p_3 - p_2 + \delta|\ell_3 - \ell_2|}{2\delta}$; this is summarized by the second row, and so on.

It is immediate to see that, under the Hotelling model, Lemmas G.3 and G.4 hold. Similarly, for a fixed set Q, the demands only depend on price differences and not on actual prices.

G.2 Definitions and Notation

We now state some definitions that we will use to prove the main theorem. Let $\underline{\theta_i}$ and $\overline{\theta_i}$ denote the lowest and highest values in Θ_i. For each $j \in N$, let θ_j^u be the maximum $\theta_j \in \Theta_j$ under which there exists a profile $\boldsymbol{\theta} = (\theta_j, \boldsymbol{\theta}_{-j})$ such that $j \in Q(\boldsymbol{\theta})$. We may assume that $\underline{\theta_j} \leq \theta_j^u$ for all agents $j \in N$, as otherwise we can consider (w.l.o.g.) the reduced problem in which all agents for which the condition is violated are removed. In addition, note that for agent j all constraints and coefficients associated to $\theta_j > \theta_j^u$ will not play a role in our analysis, because agent j is inactive for all profiles with $\theta_j > \theta_j^u$.

Two profiles $\boldsymbol{\theta}, \boldsymbol{\theta}' \in \Theta$ are defined to be *adjacent* if and only if $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ only differ in one component and $Q(\boldsymbol{\theta}) = Q(\boldsymbol{\theta}')$, where $Q(\boldsymbol{\theta})$ is the set of active firms in the relaxed optimal solution under profile $\boldsymbol{\theta}$. Given two profiles $\boldsymbol{\theta}, \boldsymbol{\theta}'$, we define $\boldsymbol{\theta}$ to be *reachable* from $\boldsymbol{\theta}'$ if there exists a sequence of profile $\{\boldsymbol{\theta}_0 = \boldsymbol{\theta}', \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_K = \boldsymbol{\theta}\}$ such that $Q(\boldsymbol{\theta}_k) \subseteq Q(\boldsymbol{\theta}_{k-1})$ for all $1 \leq k \leq K$, and the sub-profiles $(\boldsymbol{\theta}_{k-1})_{Q(\boldsymbol{\theta}_k)}$ and $(\boldsymbol{\theta}_k)_{Q(\boldsymbol{\theta}_k)}$ differ in at most one component; that is, at most one agent among those active in $\boldsymbol{\theta}_k$ has a different cost in both profiles.

Definition G.3 (Acceptable set)**.** *A subset of profiles $\tilde{\Theta} \subseteq \Theta$ is an* acceptable set *if the following conditions are simultaneously satisfied:*

1. *$Q(\boldsymbol{\theta}) = N$ for every $\boldsymbol{\theta} \in \tilde{\Theta}$.*
2. *For each agent i, let $\tilde{\Theta}_i = \{\theta_i \in \Theta_i : \exists \boldsymbol{\theta}_{-i}$ such that $(\theta_i, \boldsymbol{\theta}_{-i}) \in \tilde{\Theta}\}$. Then, for every $\theta_i \in \Theta_i$ such that $\min \tilde{\Theta}_i \leq \theta_i \leq \max \tilde{\Theta}_i$ we must have $\theta_i \in \tilde{\Theta}_i$. That is, each $\tilde{\Theta}_i$ must be a (discrete) interval.*
3. *For every profile $\boldsymbol{\theta}$ such that $\theta_i \in \tilde{\Theta}_i$ for all $i \in N$, we must have $\boldsymbol{\theta} \in \tilde{\Theta}$.*

We abuse notation to denote $\min \tilde{\Theta}_i = \min\{\theta_i : \theta_i \in \tilde{\Theta}_i\}$ and $\max \tilde{\Theta}_i = \max\{\theta_i : \theta_i \in \tilde{\Theta}_i\}$. The above definition of acceptable set will help us characterize sufficient conditions under which the optima of the *Cent* and *DecLin* problems agree. In particular, let a market be defined by the set of suppliers, their product characteristics and cost distributions, as well as the demand model. We define a *relaxation-is-optimal market* (RIOM) as follows.

Definition G.4 (RIOM)**.** *A market is* relaxation-is-optimal market *(RIOM) if there exists an acceptable set $\tilde{\Theta}$ under which the following (additional) conditions are satisfied:*

4. *For every $i \in N$ we have $|\tilde{\Theta}_i| \geq 3$.*
5. *For all $i \in N$ and θ_i such that $\max \tilde{\Theta}_i \leq \theta_i \leq \theta_i^u$, there exists a profile $\boldsymbol{\theta} = (\theta_i, \theta_{-i})$ with $i \in Q(\boldsymbol{\theta})$ and a profile $\boldsymbol{\theta}' \in \tilde{\Theta}$ such that profile $\boldsymbol{\theta}$ is reachable from $\boldsymbol{\theta}'$.*

Intuitively, a market will be RIOM if (1) there exists a solution in which all agents are active, and (2) the difference in virtual costs between adjacent points in the support is "small enough". If the difference between adjacent virtual costs is small, then by changing a cost by the following (or preceding) one, we do not expect the allocation (and hence the set of active suppliers) to change much. Therefore, Conditions (2) and (3) will be satisfied. Similarly, if there exists a cost profile $\boldsymbol{\theta}$ for which all agents are active, one would expect that this will also be true for the cost profiles close to $\boldsymbol{\theta}$ provided adjacent virtual costs are close enough. Therefore, Condition (4) will be satisfied. Finally, a small difference between adjacent virtual costs also implies Condition (5); we can change one cost at a time by an adjacent one while having some control over the set of active suppliers, and therefore we can construct a path of profiles that can take us from $\boldsymbol{\theta}'$ to $\boldsymbol{\theta}$.

Our main theorem will state that, if the market is RIOM, then we have that the optima of the *DecLin* and *Cent* agree and thus we can characterize the optimal decentralized mechanism under linear pricing. Therefore, we now show that the conditions of Theorem 4.1 for the Hotelling model and Theorem 4.2 for the general affine model imply that the markets are RIOM.

Lemma G.5. *Any market satisfying the conditions of Theorem 4.1 is RIOM.*

Proof. Recall that all firms are active in the centralized-optimal solution under profile θ, if $|v_j(\theta_j) - v_i(\theta_i)| \leq \delta|\ell_j - \ell_i|$, for all $i, j \in N$. Hence, by Condition (1) in the statement of the theorem, a profile $\boldsymbol{\theta}$ in which $Q(\boldsymbol{\theta}) = N$ must exist. Furthermore, $|v_{i+1}(\theta_i) - v_i(\theta_i)| \leq \delta(\ell_{i+1} - \ell_i)/4$ for all $i \in N$. Condition (2) in the statement of the theorem states that $v_i(\theta_i^{j+1}) - v_i(\theta_i^j) \leq \frac{\delta c^*}{8}$ for all $i \in N$, and $\theta_i^j \in \Theta_i$. Using the two conditions it is simple to show that, by letting θ_i^k denote θ_i, we must have $Q(\theta_i^{k+2}, \boldsymbol{\theta}_{-i}) = Q(\theta_i^{k-2}, \boldsymbol{\theta}_{-i}) = N$, provided these exist. Further, let $\boldsymbol{\theta} = (\theta_i, \boldsymbol{\theta}_{-i})$ and define $\tilde{\Theta}_i = \{\theta'_i \in \Theta_i : |v_i(\theta'_i) - v_i(\theta_i)| \leq \frac{\delta c^*}{4}\}$. As $|\Theta_i| \geq 3$, we must have that $|\tilde{\Theta}_i| \geq 3$. In addition, notice that for every pair $\theta'_i, \theta'_j \in \tilde{\Theta}_i \times \tilde{\Theta}_j$ we have that $|v_i(\theta'_i) - v_j(\theta'_j)| \leq |v_i(\theta_i) - v_j(\theta_j)| + \frac{\delta c^*}{2} \leq \frac{3|\ell_i - \ell_j|}{4}\delta$. Therefore, defining $\tilde{\Theta} = \prod_{i \in N} \tilde{\Theta}_i$, we get that $Q(\theta) = N$, $\forall \theta \in \tilde{\theta}$, and $\tilde{\Theta}$ is an acceptable set satisfying Condition (4). Finally, we show that the reachability requirement (Condition (5)) is satisfied.

To that end, we explicitly construct a sequence of profiles that are reachable from a $\boldsymbol{\theta}' \in \tilde{\Theta}$, and such that for all $i \in N$ and all θ_i with $\max \tilde{\Theta}_i \leq \theta_i \leq \theta_i^u$, there exists a profile in the sequence of profiles for which i has cost θ_i and is active. Let $\boldsymbol{\theta}_0$ be a profile such that $(\boldsymbol{\theta}_0)_i = \max \tilde{\Theta}_i$ for all $i \in N$. Note that $\boldsymbol{\theta}_0 \in \tilde{\Theta}$ by construction. From $\boldsymbol{\theta}_0$, we construct a profile $\boldsymbol{\theta}_1$ by selecting the agent j with the minimum virtual cost, and increasing his cost to the adjacent one, call it $(\boldsymbol{\theta}_1)_j$. The costs of all other agents do not change in the new profile. Note that, for all $i \neq j$:

$$v_j((\boldsymbol{\theta}_1)_j) - v_i((\boldsymbol{\theta}_1)_i) = v_j((\boldsymbol{\theta}_1)_j) - v_j((\boldsymbol{\theta}_0)_j) + v_j((\boldsymbol{\theta}_0)_j) - v_i((\boldsymbol{\theta}_1)_i) \leq \delta c^*/8 \; , \tag{28}$$

because the difference between adjacent virtual costs is bounded by $\delta c^*/8$ by assumption, and $v_j((\boldsymbol{\theta}_0)_j) \leq v_i((\boldsymbol{\theta}_1)_i) = v_i((\boldsymbol{\theta}_0)_i)$, by construction. Hence, agent j remains active and all other agents remain active by monotonicity of Hotelling demand. We can inductively apply this procedure —select the agent with lowest virtual cost and increase his cost to the adjacent one— to obtain profiles that are adjacent and in which all agents are active.

Eventually, we will reach a profile $\boldsymbol{\theta}_K$ for which we cannot increase the cost of the agent j with lowest virtual cost; this means that $(\boldsymbol{\theta}_K)_j = \overline{\theta_j}$. Further, it must be that $\theta_j^u = \overline{\theta_j}$. Thus, we have shown that for all θ_j with $\max \tilde{\Theta}_j \leq \theta_j \leq \theta_j^u$, there exists a profile in the sequence of profiles for which j has cost θ_j, j is active, and such profile is reachable from $\boldsymbol{\theta}_0$.

Let $U = \{j\}$; from now on, the set U will contain all agents who have reached θ^u. Construct a profile $\boldsymbol{\theta}_{K+1}$ by selecting the agent j' with the lowest virtual cost among those in $N \backslash U$, and increasing his virtual cost to the adjacent one. Now three possibilities arise:

1. If the cost of agent j' can be increased and j' remains active, then we just increase his cost and repeat.

2. If the cost of such agent cannot be increased further, this implies that we have shown our claim for j', because we have reached $\overline{\theta_{j'}}$; hence, we can add him to U and repeat.

3. Finally, we consider the case in which the cost of j' can be increased but in doing so we have $j' \notin Q(\boldsymbol{\theta}_{K+1})$; then, we must have $\theta_{j'}^u = (\boldsymbol{\theta}_K)_{j'}$. To see why this holds, note that as j' is the agent with lowest virtual cost among the ones in $N \backslash U$, he can only be inactive in the new

profile if an agent in U (agent j) grabs the demand j' had in the old profile ($\boldsymbol{\theta}_K$) (by a similar argument to equation (28)). As a consequence, it is simple to observe that agent j will keep j' inactive even if the virtual costs of other agents increase. This together with $(\boldsymbol{\theta}_K)_j = \overline{\theta}_j$ shows our claim for j'.

We proceed by adding j' to U and defining $\boldsymbol{\theta}_{K+1} = (\overline{\theta_{j'}}, (\boldsymbol{\theta}_K)_{-j'})$; by construction, $\boldsymbol{\theta}_{K+1}$ is reachable from $\boldsymbol{\theta}_0$. We conclude the proof by noting that we can inductively apply this procedure. Each time we add an agent to U, we have shown the claim for such agent. Specifically, every time the cost of an agent cannot be increased because he will become inactive, it must be caused by the fact that one of the costs of at least one agent in U is preventing for doing so. However, in the current profile all agents in U are at their maximum costs by construction; thus, such agent has reached θ^u and we have shown that the statement is true for him as well. □

We now show an analogous result for the case of affine demand models. Let $\overline{\boldsymbol{\theta}} = (\overline{\theta_j})_{j\in N}$. To prove the following lemma we will assume that $Q(\overline{\boldsymbol{\theta}}) = N$, that is all suppliers are active in the centralized-optimal solution at the largest cost profile. This assumption significantly simplifies the proof and the notation required. However, one can also show that any market satisfying the conditions of Theorem 4.2 is RIOM even if $Q(\overline{\boldsymbol{\theta}}) \subset N$.

Lemma G.6. *Any market satisfying the conditions of Theorem 4.2 and for which $Q(\overline{\boldsymbol{\theta}}) = N$ is RIOM.*

Proof. First, note that the existence of d^* and Condition (1) in the statement of the theorem defines an acceptable set $\tilde{\Theta}$. Furthermore, note that Condition (2) in the statement of the theorem implies that Condition (4) in the definition of RIOM will be satisfied. Finally, Condition (5) in the definition of RIOM is trivially satisfied because $\overline{\boldsymbol{\theta}} \in \tilde{\Theta}$. □

Furthermore, in the setting of Lemma G.6 we can also characterize d^* as follows. Let $M = \max_{j\in N} |\tilde{\Theta}_j|$, and let $\boldsymbol{A} = \boldsymbol{A}(\overline{\boldsymbol{\theta}})$ be the coefficient matrix associated with profile $\overline{\boldsymbol{\theta}}$, as defined in Definition G.2. Then, as long as $d^* < \frac{2}{M} \min_{i\in N}\{(-\frac{1}{\boldsymbol{A}_{ii}} x_i(\overline{\boldsymbol{\theta}}))\}$, we have that the market is RIOM. Note that $\boldsymbol{x}(\overline{\boldsymbol{\theta}})$, the optimal allocations for the centralized problem at profile $\overline{\boldsymbol{\theta}}$, depend on the models primitives through the demand system and the virtual costs.

G.3 Auxiliary Lemma

We state and prove the following Lemma, which will play a key role in the proof of the main theorem.

Lemma G.7. *Suppose the coefficients $(\boldsymbol{a}, \boldsymbol{b})$ are such that equality in Eq. (20) holds. For each $i \in N$ and each $\theta_i \in \Theta_i$, let $g_i(\theta_i)$ be defined as $g_i(\theta_i) = \frac{b^i_{\theta_i}}{f_i(\theta_i)}$. Then for each $\boldsymbol{\theta} \in \Theta$, we must have*

$$\sum_{i\in Q(\boldsymbol{\theta})} g_i(\theta_i) x_i(\boldsymbol{\theta}) = 0 \tag{29}$$

Proof. Fix $\boldsymbol{\theta} \in \Theta$. We show the result for the general affine demand model as described in Section 2. Recall that the coefficients of the matrix corresponding to the demand equations (that is, Eqs. ($M_i(\boldsymbol{\theta})$)) are as defined by Eq. (25). As the equality in Eq. (20) holds, for each $j \in Q(\boldsymbol{\theta})$ we must have:

$$b^j_{\theta_j} f(\boldsymbol{\theta}_{-j}) x_j(\boldsymbol{\theta}) + \sum_{i=1}^{Q(\boldsymbol{\theta})-1} a^i_{\boldsymbol{\theta}} \left(-\boldsymbol{F}_{ij} + \frac{(\mathbf{1}'_{Q(\boldsymbol{\theta})} \cdot \boldsymbol{F}_{*,j})(\boldsymbol{F}_{i,*} \cdot \mathbf{1}_{Q(\boldsymbol{\theta})})}{\mathbf{1}'_{Q(\boldsymbol{\theta})} \boldsymbol{F} \mathbf{1}_{Q(\boldsymbol{\theta})}} \right) = 0,$$

where we have used the fact that one constraint is indeed redundant (and thus the summation goes to $Q(\boldsymbol{\theta}) - 1$ instead of $Q(\boldsymbol{\theta})$). Therefore,

$$\begin{aligned}
\sum_{j \in Q(\boldsymbol{\theta})} b^j_{\theta_j} f(\boldsymbol{\theta}_{-j}) x_j(\boldsymbol{\theta}) &= - \sum_{j \in Q(\boldsymbol{\theta})} \sum_{i=1}^{Q(\boldsymbol{\theta})-1} a^i_{\boldsymbol{\theta}} \left(-\boldsymbol{F}_{ij} + \frac{(\mathbf{1}'_{Q(\boldsymbol{\theta})} \cdot \boldsymbol{F}_{*,j})(\boldsymbol{F}_{i,*} \cdot \mathbf{1}_{Q(\boldsymbol{\theta})})}{\mathbf{1}'_{Q(\boldsymbol{\theta})} \boldsymbol{F} \mathbf{1}_{Q(\boldsymbol{\theta})}} \right) \\
&= - \sum_{i=1}^{Q(\boldsymbol{\theta})-1} a^i_{\boldsymbol{\theta}} \left(\sum_{j \in Q(\boldsymbol{\theta})} \left(-\boldsymbol{F}_{ij} + \frac{(\mathbf{1}'_{Q(\boldsymbol{\theta})} \cdot \boldsymbol{F}_{*,j})(\boldsymbol{F}_{i,*} \cdot \mathbf{1}_{Q(\boldsymbol{\theta})})}{\mathbf{1}'_{Q(\boldsymbol{\theta})} \boldsymbol{F} \mathbf{1}_{Q(\boldsymbol{\theta})}} \right) \right) \\
&= - \sum_{i=1}^{Q(\boldsymbol{\theta})-1} a^i_{\boldsymbol{\theta}} \left(-\boldsymbol{F}_{i,*} \cdot \mathbf{1}_{Q(\boldsymbol{\theta})} + \boldsymbol{F}_{i,*} \cdot \mathbf{1}_{Q(\boldsymbol{\theta})} \left(\sum_{j \in Q(\boldsymbol{\theta})} \frac{(\mathbf{1}'_{Q(\boldsymbol{\theta})} \cdot \boldsymbol{F}_{*,j})}{\mathbf{1}'_{Q(\boldsymbol{\theta})} \boldsymbol{F} \mathbf{1}_{Q(\boldsymbol{\theta})}} \right) \right) \\
&= - \sum_{i=1}^{Q(\boldsymbol{\theta})-1} a^i_{\boldsymbol{\theta}} \left(-\boldsymbol{F}_{i,*} \cdot \mathbf{1}_{Q(\boldsymbol{\theta})} + \boldsymbol{F}_{i,*} \cdot \mathbf{1}_{Q(\boldsymbol{\theta})} \right) \\
&= 0
\end{aligned}$$

To complete the proof, note that $\sum_{j \in Q(\boldsymbol{\theta})} b^j_{\theta_j} f(\boldsymbol{\theta}_{-j}) x_j(\boldsymbol{\theta}) = f(\boldsymbol{\theta}) \left(\sum_{j \in Q(\boldsymbol{\theta})} g_j(\theta_j) x_j(\boldsymbol{\theta}) \right) = 0$. Hence, $\sum_{j \in Q(\boldsymbol{\theta})} g_j(\theta_j) x_j(\boldsymbol{\theta}) = 0$ as desired. □

G.4 Main Theorem

We can now state and prove our main theorem.

Theorem G.1. *Consider the general affine demand model in which agents have arbitrary costs distributions. If the market is RIOM, then $OPT(DecLin) = OPT(Cent)$.*

Proof. To show $OPT(DecLin) = OPT(Cent)$, we show that the system of equations is consistent. Let $(\boldsymbol{a}, \boldsymbol{b})$ be a vector of coefficients satisfying Eq. (20). Let $g_i(\theta_i)$ be as defined in the statement of Lemma G.7. The idea of the proof is to first show that, if a market is RIOM, then all $g_i(\theta_i)$ must be zero. Then, we show that if $g_i(\theta_i) = 0$, for all $\theta_i \in \Theta_i$ and all $i \in N$, then the system is consistent and thus $OPT(DecLin) = OPT(Cent)$ as desired. Consequently, the proof is divided into the following steps:

Step 1: Show that if $(\boldsymbol{a}, \boldsymbol{b})$ satisfies Eq. (20) and a market is RIOM all $g_i(\theta_i)$ must be zero. Let $\tilde{\Theta} \subseteq \Theta$ be such that it satisfies Conditions (1)-(5) in Definitions G.3 and G.4 respectively (we know such $\tilde{\Theta}$ exists as the market is RIOM) . Step 1 is further divided into the following two sub-steps:

(a) **Step 1.a:** Show that $g_i(\theta_i) = 0$ for all $\theta_i \in \tilde{\Theta}_i$ and all $i \in N$.

(b) **Step 1.b:** Show that $g_i(\theta_i) = 0$ for all $\theta_i \notin \tilde{\Theta}_i$ and all $i \in N$.

Step 2: Show that $g_i(\theta_i) = 0$, for all $\theta_i \in \Theta_i$ and all $i \in N$, implies consistency of the system of linear equations.

Step 1.a : Show $g_i(\theta_i) = 0$, for all $\theta_i \in \tilde{\Theta}_i$ and all $i \in N$. By assumption, $\tilde{\Theta}$ satisfies conditions (1)-(5). Therefore, for every $\boldsymbol{\theta} \in \tilde{\Theta}$ we must have $Q(\boldsymbol{\theta}) = N$ (by Condition (1)). Consider two profiles $\boldsymbol{\theta} = (\theta_i, \boldsymbol{\theta}_{-i})$ and $\boldsymbol{\theta}' = (\theta_i', \boldsymbol{\theta}_{-i})$ which only differ in agent i's cost and such that $\boldsymbol{\theta}, \boldsymbol{\theta}' \in \tilde{\Theta}$. By the definition of $\tilde{\Theta}$, such pair of profiles exists (Conditions (3) and (4)). By Eq. (29), we must have $g_i(\theta_i)x_i(\boldsymbol{\theta}) + \sum_{j \neq i} g_j(\theta_j)x_j(\boldsymbol{\theta}) = 0$ and $g_i(\theta_i')x_i(\boldsymbol{\theta}') + \sum_{j \neq i} g_j(\theta_j)x_j(\boldsymbol{\theta}') = 0$. Hence, by subtracting the second equality from the first one we obtain

$$g_i(\theta_i)x_i(\boldsymbol{\theta}) - g_i(\theta_i')x_i(\boldsymbol{\theta}') = \sum_{j \neq i} g_j(\theta_j) \left[x_j(\boldsymbol{\theta}') - x_j(\boldsymbol{\theta})\right].$$

For each $j \in N$, we must have $x_j(\boldsymbol{\theta}') - x_j(\boldsymbol{\theta}) = \boldsymbol{A}(\boldsymbol{\theta})_{ji}\left(v_i(\theta_i') - v_i(\theta_i)\right)$, where we used the fact that $\boldsymbol{A}(\boldsymbol{\theta}) = \boldsymbol{A}(\boldsymbol{\theta}')$ by definition, as the same set of agents are active. Let $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{\theta})$, and note that this $\boldsymbol{A}$ agrees with the one in Lemma G.4, because $Q(\boldsymbol{\theta}) = N$. Hence, we can re-write the above equality as:

$$g_i(\theta_i)x_i(\boldsymbol{\theta}) - g_i(\theta_i')x_i(\boldsymbol{\theta}') = \left(v_i(\theta_i') - v_i(\theta_i)\right)\left(\sum_{j \neq i} g_j(\theta_j)\boldsymbol{A}_{ji}\right),$$

and therefore,

$$\frac{g_i(\theta_i)x_i(\boldsymbol{\theta}) - g_i(\theta_i')x_i(\boldsymbol{\theta}')}{v_i(\theta_i') - v_i(\theta_i)} = \left(\sum_{j \neq i} g_j(\theta_j)\boldsymbol{A}_{ji}\right). \tag{30}$$

Fix an arbitrary $j \in N$ with $j \neq i$ and $\boldsymbol{A}_{ij} \neq 0$. By strict diagonal dominance of $\boldsymbol{F}$, such j always exists (see Eq. (26)). Let θ_j be the cost of agent j in both $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$, where the cost profiles are as defined above. Let $\theta_j' \in \Theta_j$ be such that $\theta_j' \neq \theta_j$ and $\theta_j' \in \tilde{\Theta}_j$ (by Conditions (3) and (4), such θ_j' exists). Define $\tilde{\boldsymbol{\theta}} = (\theta_i, \theta_j', \boldsymbol{\theta}_{-i,j})$ and $\tilde{\boldsymbol{\theta}}' = (\theta_i', \theta_j', \boldsymbol{\theta}_{-i,j})$. The only thing we assumed about θ_j was $\theta_j \in \tilde{\Theta}_j$. Therefore, the above equality must also hold for any $\tilde{\theta}_j \in \tilde{\Theta}_j$. That is,

$$\frac{g_i(\theta_i)x_i(\tilde{\boldsymbol{\theta}}) - g_i(\theta_i')x_i(\tilde{\boldsymbol{\theta}}')}{v_i(\theta_i') - v_i(\theta_i)} = g_j(\theta_j')A_{ji} + \sum_{k \neq i,j} g_k(\theta_k)\boldsymbol{A}_{ki}.$$

By subtracting the inequality when j has cost θ_j from the one when his cost is θ_j' we get

$$\frac{g_i(\theta_i)\left(x_i(\tilde{\boldsymbol{\theta}}) - x_i(\boldsymbol{\theta})\right) - g_i(\theta_i')\left(x_i(\tilde{\boldsymbol{\theta}}') - x_i(\boldsymbol{\theta}')\right)}{v_i(\theta_i') - v_i(\theta_i)} = \boldsymbol{A}_{ji}\left(g_j(\theta_j') - g_j(\theta_j)\right).$$

However, note that $x_i(\tilde{\boldsymbol{\theta}}) - x_i(\boldsymbol{\theta}) = \boldsymbol{A}_{ij}\left(v_j(\theta'_j) - v_j(\theta_j)\right)$. Therefore,

$$\boldsymbol{A}_{ij}\frac{g_i(\theta_i) - g_i(\theta'_i)}{v_i(\theta'_i) - v_i(\theta_i)} = \boldsymbol{A}_{ji}\frac{g_j(\theta'_j) - g_j(\theta_j)}{v_j(\theta'_j) - v_j(\theta_j)}.$$

Recall that $\boldsymbol{A}$ is symmetric (Lemma G.4).Therefore, whenever $\boldsymbol{A}_{ij} \neq 0$ we must have:

$$\frac{g_i(\theta_i) - g_i(\theta'_i)}{v_i(\theta'_i) - v_i(\theta_i)} = \frac{g_j(\theta'_j) - g_j(\theta_j)}{v_j(\theta'_j) - v_j(\theta_j)}, \qquad \forall i \neq j,\ \forall \theta_i, \theta'_i \in \tilde{\Theta}_i,\ \forall \theta_j, \theta'_j \in \tilde{\Theta}_j. \tag{31}$$

Furthermore, the above equality should hold for every $i, j \in N$ as we can find a sequence of agents $\{l_0 = i, \ldots, l_K = j\}$ such that $\boldsymbol{A}_{l_k, l_{k+1}} \neq 0$ for all $0 \leq k < K$.[32]

To complete the proof that $g_i(\theta_i) = 0$ for all $\theta_i \in \tilde{\Theta}_i$, we are going to consider two options: either $g_i(\theta_i) - g_i(\theta'_i) = 0$ for at least one pair of $g_i(\theta_i), g_i(\theta'_i)$, or $g_i(\theta_i) - g_i(\theta'_i) \neq 0$ for all $i \in N$ and $\theta_i, \theta'_i \in \tilde{\Theta}_i$. We explore both options next:

Case $g_i(\theta_i) - g_i(\theta'_i) = 0$ for at least one pair of $g_i(\theta_i), g_i(\theta'_i)$. Suppose the numerator is zero for at least one pair of $g_i(\theta_i), g_i(\theta'_i)$. Then, $g_j(\theta_j) - g_j(\theta'_j)$ must be zero for every $j \in N$ and all pairs $\theta_j, \theta'_j \in \tilde{\Theta}_j$.

The next step is to show that $g_i(\theta_i) = g_j(\theta_j)$ must hold for every $\theta_i \in \tilde{\Theta}_i$ and $\theta_j \in \tilde{\Theta}_j$ and $i, j \in N$; we will use this fact as an intermediate step to show that indeed we must have $g_i(\theta_i) = 0$ for all $\theta_i \in \tilde{\Theta}_i$ and all $i \in N$.

Showing that $g_i(\theta_i) = g_j(\theta_j)$ is trivial if $i = j$, as $g_i(\theta_i) - g_i(\theta'_i)$ must be zero for every $i \in N$ and all pairs $\theta_i, \theta'_i \in \tilde{\Theta}_i$, by Eq. (31). For the other cases, note that when $g_i(\theta_i) = g_i(\theta'_i)$, we have $g_i(\theta_i)x_i(\boldsymbol{\theta}) - g_i(\theta'_i)x_i(\boldsymbol{\theta}') = g_i(\theta_i)\boldsymbol{A}_{ii}\left(v_i(\theta_i) - v_i(\theta'_i)\right)$. By Eq. (30) the above equality reduces to

$$\sum_{j \in N} g_j(\theta_j)\boldsymbol{A}_{ij} = 0, \tag{32}$$

and this must be true for any $i \in N$. Let $\boldsymbol{A}_R$ denote the submatrix of $\boldsymbol{A}$ consisting of $(n-1)$ linearly independent rows. By Lemma G.4, we know such matrix exists. Furthermore, we can assume that those are the $n-1$ demand equations that appear in the coefficient matrix $\boldsymbol{M}$. Let $\mathbf{g} = (g_1, \ldots, g_n)$ denote the vector of coefficients $g_i = g_i(\theta_i)$ for $\boldsymbol{\theta} \in \Theta$. By Eq. (32), the vector $\mathbf{g}$ must be in the nullspace of $\boldsymbol{A}_R$. However, as $\boldsymbol{A}_R \in \mathbb{R}^{(n-1)\times n}$ has dimension $(n-1)$ the dimension of its nullspace is at most 1. We will show that $\mathbf{1}$ is in Null($\boldsymbol{A}_R$), which implies that *all* g_i with $i \in N$ must be equal.

Consider $(\boldsymbol{A}_R)_{i,*}$, that is, row i of the coefficient matrix $\boldsymbol{A}_R$. We will show that $(\boldsymbol{A}_R)_{i,*} \cdot \mathbf{1} = 0$.

[32] Here we are implicitly assuming that matrix $\boldsymbol{A}$ has only one block. If $\boldsymbol{A}$ has more than one block, then we can use the same argument for each block.

Note that

$$(\boldsymbol{A}_R)_{i,*} \cdot \mathbf{1} = \sum_j \left(-\boldsymbol{F}_{ij} + \frac{(\mathbf{1}'_{Q(\boldsymbol{\theta})} \cdot \boldsymbol{F}_{*,j})(\boldsymbol{F}_{i,*} \cdot \mathbf{1}_{Q(\boldsymbol{\theta})})}{\mathbf{1}'_{Q(\boldsymbol{\theta})} \boldsymbol{F} \mathbf{1}_{Q(\boldsymbol{\theta})}} \right) = -\boldsymbol{F}_{i,*} \cdot \mathbf{1} + \boldsymbol{F}_{i,*} \cdot \mathbf{1} = 0,$$

as desired. Therefore, $\mathbf{1}$ is in $\text{Null}(\boldsymbol{A}_R)$ and thus $g_i(\theta_i) = g_j(\theta_j)$ for all $i, j \in N$, $\theta_i \in \tilde{\Theta}_i$, $\theta_j \in \tilde{\Theta}_j$.

Using that $g_i(\theta_i) = g_j(\theta_j)$ for all $\theta_i \in \tilde{\Theta}_i$ and $\theta_j \in \tilde{\Theta}_j$, we now show that $g_i(\theta_i) = 0$ for all $i \in N$ and all $\theta_i \in \tilde{\Theta}_i$, which implies $b^i_{\theta_i} = 0$ for all $\theta_i \in \tilde{\Theta}_i$. If $g_i(\theta_i) = 0$, for some $i \in N$ and $\theta_i \in \Theta_i$, we are done. Otherwise, suppose that $g_i(\theta_i) = k \neq 0$ for all $i \in N$ and all $\theta_i \in \Theta_i$. By Lemma G.7 we have:

$$0 \;=\; \sum_{j \in Q(\boldsymbol{\theta})} g_j(\theta_j) x_j(\boldsymbol{\theta}) \;=\; k \left(\sum_{j \in Q(\boldsymbol{\theta})} x_j(\boldsymbol{\theta}) \right) \;= k,$$

which is a contradiction ◁

Case $g_i(\theta_i) - g_i(\theta'_i) \neq 0$ for all $i \in N$ and $\theta_i, \theta'_i \in \tilde{\Theta}_i$. Let the pair $g_i(\theta_i), g_i(\theta'_i)$ be such that $\frac{g_i(\theta_i) - g_i(\theta'_i)}{v_i(\theta'_i) - v_i(\theta_i)} = k \neq 0$, and rewrite $g_i(\theta_i) = g_i(\theta'_i) + k[v_i(\theta'_i) - v_i(\theta_i)]$. Let $\theta_i, \theta'_i, \theta''_i \in \tilde{\Theta}_i$ and let $\theta_{-i} \in \tilde{\Theta}_{-i}$. Then, we must have using equation (30):

$$\begin{aligned}
\big(v_i(\theta'_i) - v_i(\theta_i)\big) \sum_{j \neq i} \boldsymbol{A}_{ji} g_j(\theta_j) &= g_i(\theta_i) x_i(\boldsymbol{\theta}) - g_i(\theta'_i) x_i(\boldsymbol{\theta}') \\
&= \big(g_i(\theta'_i) + k[v_i(\theta'_i) - v_i(\theta_i)]\big)\, x_i(\boldsymbol{\theta}) - g_i(\theta'_i) x_i(\boldsymbol{\theta}') \\
&= g_i(\theta'_i)\big(x_i(\boldsymbol{\theta}) - x_i(\boldsymbol{\theta}')\big) + k[v_i(\theta'_i) - v_i(\theta_i)] x_i(\boldsymbol{\theta}) \\
&= g_i(\theta'_i) \boldsymbol{A}_{ii} \big(v_i(\theta_i) - v_i(\theta'_i)\big) + k[v_i(\theta'_i) - v_i(\theta_i)] x_i(\boldsymbol{\theta})
\end{aligned}$$

By dividing on both sides by $v_i(\theta'_i) - v_i(\theta)$ we obtain:

$$\sum_{j \neq i} \boldsymbol{A}_{ji} g_j(\theta_j) \;=\; -g_i(\theta'_i) \boldsymbol{A}_{ii} + k x_i(\boldsymbol{\theta})$$

In addition, since $\theta''_i \in \tilde{\Theta}_i$, we have $\frac{g_i(\theta''_i) - g_i(\theta'_i)}{v_i(\theta'_i) - v_i(\theta''_i)} = k$ by Eq. (31), and thus:

$$\sum_{j \neq i} \boldsymbol{A}_{ji} g_j(\theta_j) \;=\; -g_i(\theta'_i) \boldsymbol{A}_{ii} + k x_i(\boldsymbol{\theta}'')$$

which is a contradiction, because the virtual costs are strictly increasing and therefore $x_i(\boldsymbol{\theta}) \neq x_i(\boldsymbol{\theta}'')$ ◁

Therefore, we have shown that in the first case we must have $g_i(\theta_i) = 0$ for all $\theta_i \in \tilde{\Theta}_i$ and all $i \in N$, and that the second case cannot arise as it will result in a contradiction. Note that this concludes the proof of Step 1.a —we have established that $g_i(\theta_i) = 0$ for all $\theta_i \in \tilde{\Theta}_i$ and all $i \in N$

$\triangleleft$

Step 1.b: $g_i(\theta_i) = 0$ for all $\theta_i \notin \tilde{\Theta}_i$ and all $i \in N$. Next, we show that $g_j(\theta_j) = 0$ whenever $\theta_j < \min \tilde{\Theta}_j$ or $\theta_j > \max \tilde{\Theta}_j$. (Recall that by Condition (2), $\tilde{\Theta}_j$ is an interval.) For $\theta_j < \min \tilde{\Theta}_j$ consider a profile $\boldsymbol{\theta} = (\theta_j, \boldsymbol{\theta}_{-j})$ such that $\theta_i \in \tilde{\Theta}_i$ for all $i \neq j$. By the definition of $\tilde{\Theta}_j$ and the monotonicity of demand (Lemma G.3), we must have $x_j(\boldsymbol{\theta}) > 0$. By Lemma G.7 and Step 1.a we have

$$0 \;=\; \sum_{i \in Q(\boldsymbol{\theta})} g_i(\theta_i) x_i(\boldsymbol{\theta}) \;=\; g_j(\theta_j) x_j(\boldsymbol{\theta}).$$

and therefore $g_j(\theta_j) = 0$ for all $\theta_j < \min \tilde{\Theta}_j$ and all $j \in N$.

Let $\theta_j > \max \tilde{\Theta}_j$ and $\theta_j \leq \theta_j^u$, as defined at the beginning of Section G.2. By Condition (5) of RIOM, there exists a profile $\boldsymbol{\theta} = (\theta_j, \theta_{-j})$ and a profile $\boldsymbol{\theta}' \in \tilde{\Theta}$ such that the profile $\boldsymbol{\theta}$ is reachable from $\boldsymbol{\theta}'$. Hence, there exists a sequence of profiles $\{\boldsymbol{\theta}_0 = \boldsymbol{\theta}', \ldots, \boldsymbol{\theta}_K = \boldsymbol{\theta}\}$ satisfying Condition (5). Given that $\boldsymbol{\theta}' \in \tilde{\Theta}$, we must have that $g_i(\theta_i') = 0$ for all $i \in N$. We will inductively show that $g_i((\boldsymbol{\theta}_k)_i) = 0$ for every $k = 1, \ldots, K$ and every $i \in Q(\boldsymbol{\theta}_k)$. As $j \in Q(\boldsymbol{\theta}_K)$, this will establish the result. Let k be the component in which $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_1$ differ, and $k \in Q(\boldsymbol{\theta}_1)$. (If no such k exists, then all active agents share the same cost and thus the claim follows from the base case.) By Lemma G.7 we have $\sum_{i \in Q(\boldsymbol{\theta}_1)} g_i((\theta_1)_i) x_i(\boldsymbol{\theta}_1) = 0$. As $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_1$ only differ in the k^{th} component, $\boldsymbol{\theta}_0 \in \tilde{\Theta}$, and $k \in Q(\boldsymbol{\theta}_1)$, we must have $g_k((\boldsymbol{\theta}_1)_k) = 0$. We can inductively repeat this argument to show that all the g's corresponding to a profile in the path between $\boldsymbol{\theta}'$ and $\boldsymbol{\theta}$ must be zero, which implies $g_j(\theta_j) = 0$ $\triangleleft$

Therefore, we have shown that both the statements described in Steps 1.a and 1.b hold. Therefore if $(\boldsymbol{a}, \boldsymbol{b})$ satisfies Eq. (20), then $g_i(\theta_j) = 0$ for all $i \in N$ and all $\theta_i \in \Theta_i$. This concludes the proof of Step 1. $\diamondsuit$

Step 2: $g_i(\theta_j) = 0$ for all $i \in N$ and all $\theta_i \in \Theta_i$ implies the system is consistent. So far we have shown that $g_i(\theta_j) = 0$ for all $i \in N$ and all $\theta_i \in \Theta_i$. By the definition of g_i, this implies $b^i_{\theta_i} = 0$ for all $i \in N$ and all $\theta_i \in \Theta_i$. To conclude the proof, we show that $b^i_{\theta_i} = 0$ for all $i \in N$ and all $\theta_i \in \Theta_i$ implies that the system is consistent. To that end, consider a vector $(\boldsymbol{a}, \boldsymbol{0})$ satisfying Eq. (20). For each $\boldsymbol{\theta} \in \tilde{\Theta}$, we have

$$\sum_{i=1}^{|Q(\boldsymbol{\theta})|-1} a^i_{\boldsymbol{\theta}} \left(\sum_{j \in Q(\boldsymbol{\theta})} \boldsymbol{A}_{ij}(\boldsymbol{\theta}) v_j(\theta_j) \right) = \sum_{j \in Q(\boldsymbol{\theta})} v_j(\theta_j) \left(\sum_{i=1}^{|Q(\boldsymbol{\theta})|-1} a^i_{\boldsymbol{\theta}} \boldsymbol{A}_{ij}(\boldsymbol{\theta}) \right) = 0,$$

as $(\boldsymbol{a}, \boldsymbol{0})$ satisfying Eq. (20) implies $\sum_{i=1}^{|Q(\boldsymbol{\theta})|-1} a^i_{\boldsymbol{\theta}} \boldsymbol{A}_{ij}(\boldsymbol{\theta}) = 0$. Hence, we have shown that $(\boldsymbol{a}, \boldsymbol{0})$ also satisfies Eq. (21). Therefore, the system is consistent and $OPT(DecLin) = OPT(Cent)$ as desired. $\square$

H Supplement to Section 4

Theorem H.1. *Consider a Hotelling Model with n suppliers such that supplier i is located at $\ell_i = \frac{(i-1)}{(n-1)}$ (that is, suppliers are equidistant). Further, assume the cost distributions are identical, and let f denote the such cost distribution and $\tilde{\Theta}$ its support. Then, we have $OPT(DecLin) = OPT(Cent)$.*

The proof of Theorem H.1 can be found in Appendix L.3.

I Optimal Mechanisms for Vertical Demand Model

We consider a classic model of pure vertical differentiation (see, e.g. Bresnahan (1987)). There are n potential suppliers, supplier i offering a product of quality α_i. We assume, w.l.o.g., that $\alpha_1 < \ldots < \alpha_n$. The qualities of the products are common-knowledge. There is a continuum of consumers, all wishing to buy one unit of the good (so the market is covered), uniformly distributed on the consumer-type space $Z = [0, 1]$. The type of a consumer indicates her value for quality. In particular, the utility a consumer of type $j \in Z$ obtains from consuming the product offered by supplier i at price p_i is given by:

$$u_{ji}(p_i) = j\alpha_i - p_i, \tag{33}$$

Given a set of potential suppliers with fixed unit prices $\boldsymbol{p} = \{p_i\}_{i \in N}$, the set of active suppliers with strictly positive demand is given by:

$$Q(\boldsymbol{p}) = \left\{ i \in N : \max_{j \in Z} \min_{k \neq i} \{ j (\alpha_i - \alpha_k) - (p_i - p_k) \} > 0 \right\}.$$

Namely, a supplier $i \in N$ will be active only if there exists a $j \in Z$ for which $u_{ji}(p_i) > u_{jk}(p_k)$ for all $k \in N$ with $k \neq i$.

For unit prices $\boldsymbol{p}$ and agent $i \in Q(\boldsymbol{p})$, let $\varrho_{\boldsymbol{p}}(i)$ (resp. $\vartheta_{\boldsymbol{p}}(i)$) denote the agent preceding (resp. following) i in $Q(\boldsymbol{p})$, that is, $\varrho_{\boldsymbol{p}}(i) = \max \{j \in Q(\boldsymbol{p}) : j < i\}$ and $\vartheta_{\boldsymbol{p}}(i) = \min \{j \in Q(\boldsymbol{p}) : j > i\}$. Also, let $\iota(Q(\boldsymbol{p}))$ (resp. $\eta(Q(\boldsymbol{p}))$) denote the highest (resp. lowest) quality agent in $Q(\boldsymbol{p})$. Then, the expected demand for product i is given by:

$$d_i(Q, \boldsymbol{p}) = \begin{cases} 0 & \text{if } i \notin Q(\boldsymbol{p}) \\ 1 & \text{if } Q(\boldsymbol{p}) = \{i\} \\ \frac{p_{\vartheta_{\boldsymbol{p}}(i)} - p_i}{\alpha_{\vartheta_{\boldsymbol{p}}(i)} - \alpha_i} & \text{if } i = \eta(Q(\boldsymbol{p})) \\ \frac{p_{\vartheta_{\boldsymbol{p}}(i)} - p_i}{\alpha_{\vartheta_{\boldsymbol{p}}(i)} - \alpha_i} - \frac{p_i - p_{\varrho_{\boldsymbol{p}}(i)}}{\alpha_i - \alpha_{\varrho_{\boldsymbol{p}}(i)}} & \text{if } i \in Q(\boldsymbol{p}),\ i \neq \eta(Q(\boldsymbol{p})),\ \iota(Q(\boldsymbol{p})) \\ 1 - \frac{p_i - p_{\varrho_{\boldsymbol{p}}(i)}}{\alpha_i - \alpha_{\varrho_{\boldsymbol{p}}(i)}} & \text{if } i = \iota(Q(\boldsymbol{p})) \end{cases} \tag{34}$$

The linear constraints imposed by Eq. (34) that the prices must satisfy so as to have $OPT(DecLin) =$

$OPT(Cent)$ agree with those of Hotelling demand case. That is, the prices must satisfy:

$$p_{\vartheta_{\boldsymbol{\theta}}(i)}(\boldsymbol{\theta}) - p_i(\boldsymbol{\theta}) = v_{\vartheta_{\boldsymbol{\theta}}(i)}(\theta_{\vartheta_{\boldsymbol{\theta}}(i)}) - v_i(\theta_i) \qquad \forall \boldsymbol{\theta} \in \Theta,\ i \in Q(\boldsymbol{\theta}),\ i \neq \iota(\boldsymbol{\theta}), \tag{35}$$

together with the constraints $T_i(\theta_i^j),\ \forall i \in N,\ \forall \theta_i^j \in \Theta_i$. With this in mind, it is simple to derive a result analogous to that of Theorem 4.1.

Theorem I.1. *Consider the general setting in which agents have arbitrary qualities and costs distributions. Let $b^* = \min_{1 \le i \le n-1}(\alpha_{i+1} - \alpha_i)$. Suppose that the following two conditions are simultaneously satisfied:*

1. *There exists $\boldsymbol{\theta} \in \Theta$ and $c^* \in \mathbb{R}$ such that $\frac{v_{i+2}(\theta_{i+2}) - v_{i+1}(\theta_{i+1})}{\alpha_{i+2} - \alpha_{i+1}} > c^* + \frac{v_{i+1}(\theta_{i+1}) - v_i(\theta_i)}{\alpha_{i+1} - \alpha_i}$ for all $1 \le i \le n-2$, $\frac{v_2(\theta_2) - v_1(\theta_1)}{\alpha_2 - \alpha_1} > c^*$, and, $1 - c^* > \frac{v_n(\theta_n) - v_{n-1}(\theta_{n-1})}{\alpha_n - \alpha_{n-1}}$;*
2. *$|\Theta_i| \ge 3$ for all $i \in N$, and for every $i \in N$ and every $\theta^j \in \Theta_i$, we have $v_i(\theta_i^{j+1}) - v_i(\theta_i^j) \le \frac{c^* b^*}{8}$.*

Then, we have $OPT(DecLin) = OPT(Cent)$.

The intuition behind these two requirements is the same as that of Theorem 4.1. As usual, let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$. From the definition of vertical demands (Eq. (34)), it is easy to see that, by condition (1), for $n \ge 2$ we must have $Q(\boldsymbol{\theta}) = N$. Hence, the first condition guarantees the existence of an 'interior solution'. The second condition imposes a 'thin enough' cost discretization.

J Extension to Multi-Product Suppliers

In this section we propose an extension to our model which allows suppliers to offer more than one product. We show that, under this extension, our main result extends accordingly; therefore, we are able to characterize (under additional conditions) the optimal mechanisms for the multiproduct case.

We now discuss how to extend our model to the case where suppliers can offer multiple products. If each agent is assumed to have a different random variable to represent the cost for each product, the problem involves solving a multidimensional mechanism design problem. This problem is recognized to be hard in general. Therefore, our approach is to assume that suppliers' costs can be parametrized by a single type, which can be interpreted as if the auctioneer knows the agents' cost structures but not their underlying cost parameter. This approach is commonly used in the literature to overcome the multidimensional mechanism design problem (Levin 1997).

For $i \in N$, let P_i denote the set of products offered by supplier i. We assume that agent i has cost $c_{ip}(\theta_i)$ for product $p \in P_i$, where θ_i is agent i's type. The utility function of supplier i is given by

$$u_i = t_i - \sum_{p \in P_i} c_{ip}(\theta_i) x_{ip},$$

where x_{ip} is the amount of product p allocated to i, t_i is the payment i receives in the auction, and θ_i is his type. Similarly, the interim utility for supplier i when he reports cost θ_i' and has true cost

θ_i is given by:

$$U_i(\theta_i'|\theta_i) = T_i(\theta_i') - \sum_{p \in P_i} c_{ip}(\theta_i) X_{ip}(\theta_i').$$

For each pair (i, p) with $i \in N$ and $p \in P_i$, we define the modified virtual cost as:

$$v_{ip}(\theta_i) = c_{ip}(\theta_i) + \frac{F_i(\rho(\theta_i))}{f_i(\theta_i)} \left(c_{ip}(\theta_i) - c_{ip}(\rho(\theta_i))\right).$$

As usual, we assume virtual costs to be increasing. Furthermore, we require that the function $h_i : \mathbb{R}^{|P_i|} \times \mathbb{R} \to \mathbb{R}$ defined as $h_i(x_i, \theta_i) = \sum_{p \in P_i} c_{ip}(\theta_i) x_{ip}$ satisfies the increasing differences property. Under these assumptions, the optimal solution to the centralized problem is characterized by the following proposition.

Proposition J.1. *Suppose that $(\boldsymbol{x}, \boldsymbol{t})$ satisfy the following conditions:*

1. *The allocation function satisfies for all $\boldsymbol{\theta} \in \Theta$,*

$$\begin{aligned} x(\boldsymbol{\theta}) \quad &\in \quad \operatorname{argmax} K(x(\boldsymbol{\theta})) - \sum_{i=1}^{n} \sum_{p \in P_i} v_{ip}(\theta_i) x_{ip}(\boldsymbol{\theta}) \\ &s.t. \qquad \sum_{i=1}^{N} \sum_{p \in P_i} x_{ip}(\boldsymbol{\theta}) = 1, \qquad x_{ip}(\boldsymbol{\theta}) \geq 0 \quad \forall i \in N, p \in P_i \ . \end{aligned}$$

2. *Interim expected transfers satisfy for all $i \in N$ and $\theta_i^j \in \Theta_i$:*

$$T_i(\theta_i^j) = \sum_{p \in P_i} c_{ip}(\theta_i^j) X_{ip}(\theta_i^j) + \sum_{k=j+1}^{|\Theta_i|} \sum_{p \in P_i} \left(c_{ip}(\theta_i^k) - c_{ip}(\theta_i^{k-1})\right) X_{ip}(\theta_i^k)$$

Then, $(\boldsymbol{x}, \boldsymbol{t})$ is an optimal mechanism for the centralized problem.

The proof is provided in the last section of this document. Ideally, we would like to use the the characterization of the optimal solution to the centralized problem to study the decentralized problem. The optimal demands for the centralized problem still have an intuitive form, similar to the single-product case. However, the expected transfers constraints differ, because they involve terms for potentially many products. This introduces some additional complexities in the analysis, and the extension of Theorem 4.2 to the multiproduct case is not straightforward.

Surprisingly, under sufficient conditions, we are able to show that our main result still holds. That is, there exists prices under which we have $OPT(DecLin) = OPT(Cent)$. This is formalized by the following theorem.

Theorem J.1. *Consider the general setting in which agents have arbitrary costs distributions and offer any arbitrary number of products. Then, there exists $c^* \in \mathbb{N}$, $d^* \in \mathbb{R}_+$ such that, whenever the following conditions are simultaneously satisfied,*

1. *There exists a profile* $\boldsymbol{\theta} \in \Theta$ *such that* $p_i \in Q(\boldsymbol{\theta})$ *for all* $p_i \in P_i$ *and all* $i \in N$*. Furthermore, there exists a* $d^* \in \mathbb{R}$ *such that, for all* $\boldsymbol{\theta}' \in \Theta$ *with* $|\boldsymbol{\theta} - \boldsymbol{\theta}'|_\infty \leq d^*$ *we have* $Q(\boldsymbol{\theta}') = \cup_{i \in N} P_i$*.*

2. $|\Theta_i| \geq c^*$ *for all* $i \in N$*, and for every* $i \in N$ *and every* $\theta^j \in \Theta_i$*, we have* $\max_{p \in P_i}\{v_{ip}(\theta_i^{j+1}) - v_{ip}(\theta_i^j)\} \leq d^*/3$*.*

we have $OPT(DecLin) = OPT(Cent)$*.*

Although the intuition behind the proof of Theorem J.1 is similar to that of the single-product case, there are some fundamental differences. For example, the set $Q(\boldsymbol{\theta})$ now denotes the active products rather than the active suppliers. Note that a single supplier can simultaneously have many different products in the assortment, which will be reflected in the expected transfer constraints. In addition, as the cost realization of a supplier is simultaneously valid for all his products, we need to guarantee that the grid is thin enough for all products offered by the supplier. A sketch of the proof of Theorem J.1, provided in the last section of this document, shows how to address these differences.

K Supplement to Section 5

K.1 Equilibrium Bidding Strategies Under the NC (ChileCompra) Mechanism

We can analytically calculate the pure strategy Bayes Nash equilibrium (PSBNE) bids for the agents under the NC —hereafter may be also referred to as ChileCompra— mechanism with reserve price θ_H. Using standard arguments, it is straightforward to verify that the equilibrium bid for a high-type agent is θ_H. The characterization for the low-type bid can be found in Table 2. Furthermore, in Table 2, we compare the equilibrium bidding strategy for the low-type agent in NC with reserve price θ_H[33] to the average price per unit payed to a supplier of type θ_L in the optimal mechanism.[34] The pure strategy Bayes Nash equilibrium bids are formally characterized in Lemma L.1, which can be found in the last section of this document.

K.2 Ex-Ante Restricted-Entry Mechanism.

We analyze what happens if competition for the market is induced by restricting entry before bids are placed. Suppose that we decide how many agents will be in the menu before observing the bids and then run a FPA type mechanism to decide the prices. In our two-agent model, this amounts to deciding when does choosing a single winner using a FPA outperforms the NC mechanism.

Recall that, in general, a FPA does not have an equilibrium in pure strategies when types are discrete. However, by allowing equilibria in mixed strategies, expected payments in the FPA are given by $\theta_H - f_L^2(\theta_H - \theta_L)$.[35] By adding the transportation cost, the total expected cost faced by a

[33] Note that for low-values of δ a BNE does not exist for the same reasons a BNE does not typically exist in first price auctions with discrete types Krishna (2009).

[34] We calculate the average price per unit payed to a supplier of type low as $T(\theta_L)/X(\theta_L)$.

[35] This follows from standard arguments.

Value of δ	Optimal		NC	
	award	avg. low price	award	equation strat. low
$\left[\frac{1}{f_H}(\theta_H - \theta_L), \infty\right)$	split	$\frac{(f_H/2+f_L(1-x))\theta_H+(x-1/2)\theta_L}{f_L/2+f_H x}$ where $x = \frac{1/f_H(\theta_H-\theta_L)+\delta}{2\delta}$	split	θ_H
$\left[(\theta_H - \theta_L), \frac{1}{f_H}(\theta_H - \theta_L)\right]$	single	$\frac{\theta_L+f_H\theta_H}{1+f_H}$		
$\left[\frac{(\theta_H-\theta_L)}{2+f_H}, (\theta_H - \theta_L)\right]$				$\frac{\theta_L+f_H\theta_H+\delta}{1+f_H}$
$\left[\frac{f_L}{2}(\theta_H - \theta_L), \frac{(\theta_H-\theta_L)}{2+f_H}\right)$			single	$\theta_H - \delta$
$\left[\frac{f_H f_L(\theta_H-\theta_L)}{\frac{1}{2}(1+f_H)^2+f_H f_L}, \frac{f_L}{2}(\theta_H - \theta_L)\right]$				$\theta_L + \delta\frac{1+f_H}{f_L}$
$\left[0, \frac{f_H f_L(\theta_H-\theta_L)}{\frac{1}{2}(1+f_H)^2+f_H f_L}\right]$			no BNE	-

Table 2: Comparison Optimal mechanism and NC mechanism with reserve price θ_H. In all cases, the expected price for an item of cost θ_H is θ_H.

designer who chooses to run a FPA is $\theta_H - f_L^2(\theta_H - \theta_L) + \frac{\delta}{2}$. Using these analytical expressions, we can characterize the set of parameters for which the FPA outperforms NC. To illustrate, for fixed $\theta_L = 10$ and $\theta_H = 12$, the relative performance of NC and FPA as a function of parameters (f_L, δ) can be seen in Figure 4.[36]

As it can be observed, FPA may or may not improve over NC, depending on the combination of parameters. In particular, NC outperforms the FPA mechanisms when both f_L and the differentiation cost δ are relatively small (the white area). As the differentiation cost increases beyond $\theta_H - \theta_L$ but f_L remains small, the FPA is still worse than NC. In that region (light gray area), the equilibrium strategy for the low-type in NC mechanism is to bid θ_H, which agrees with the bid a low-type agent will place if there was no competition. However, the designer cannot improve by switching to a FPA; in the light gray area, the reduction in purchasing costs that results from the price competition cannot compensate for the large transportation cost, even when bids in the NC mechanism are as high as possible. On the other hand, as f_L increases, it is profitable to restrict the entry using a FPA even if that implies a higher transportation cost (gray area);[37] this is due to the fact that a FPA is able to obtain much lower (expected) bids from the low-type.

K.3 BRE mechanism

We now study when the BRE mechanism (as defined in Section 5.1) outperforms NC as a function of the parameters. We find that when δ is relatively small and NC split-awards, restricting entry improves over NC mechanism regardless of the value of other parameters. In such cases, the decrease in the low-type equilibrium bid results in a considerable decrease in the expected purchasing cost without a major increase in the expected transportation cost. In addition, restricting entry performs better for the middle-values of f_L. If f_L is too low, the savings are less likely to occur and therefore the potential impact is smaller. On the other hand, if f_L is too high, the best-low-type-bid tends to increase and the single-award becomes less profitable. This is illustrated by Figure 5, where we

[36]The black area is omitted from the analysis, as no equilibrium in pure strategies exists in the NC mechanism.

[37]We note that the non-convexity of the areas FPA and NC is due to the fact that, in NC, the equilibrium bidding strategy as a function of δ is decreasing in the interval $\left[\frac{f_L}{2}(\theta_H - \theta_L), \frac{1}{2+f_H}(\theta_H - \theta_L)\right]$.

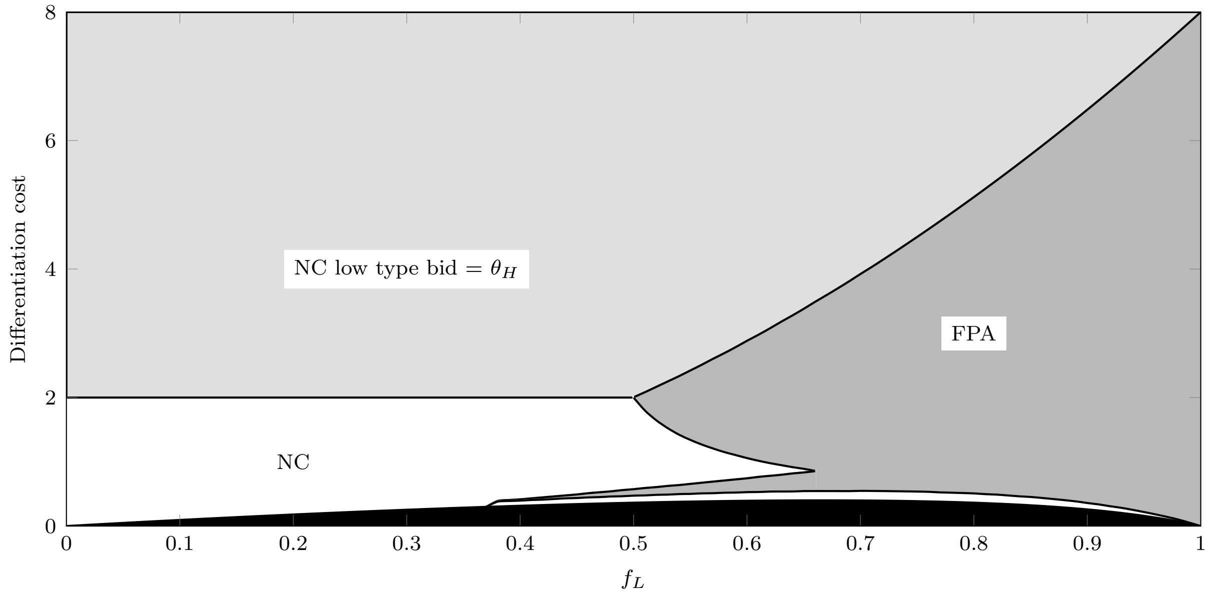

Figure 4: For $\theta_L = 10$, $\theta_H = 12$, we show when it is profitable to restrict the entry using a FPA as a function of f_L and δ. The black area is omitted from the analysis, as no equilibrium in pure strategies exists in the NC mechanism. NC outperfoms the FPA mechanisms only in the white area. The single-winner FPA is better in dark gray area. In the light-gray area, NC has the highest possible low-type bid, but it is still better than a single-winner FPA.

fix $\theta_L = 10$, $\theta_H = 12$, and show when it is profitable to restrict entry as a function of δ and f_L.

Implementation of the BRE The BRE mechanism uses the best split-parameter C that depends on the problem primitives and therefore it may be hard to estimate in practice. However, in the electronic companion we show that even implementing the BRE mechanism with a rough estimate of the best C (but not the exact one) typically improves performance. In particular, if restricting entry is profitable, any smaller C which is relatively close to the best C will induce the equilibrium bid $\theta_H - C$. Therefore, if the parameters are in the interior of the gray area in Figure 5 (where restricting entry improves performance), by choosing a conservative C the auctioneer should be able to increase consumer surplus. Also note that any C larger than the best C yields the same outcome as the current NC mechanism, so it will not damage performance.

L Additional Proofs

L.1 Proofs of Equilibria for Section 5

Lemma L.1. *The unique PSNBE for the NC mechanism with reserve price $R = \theta_H$ are as given by Table 3.*

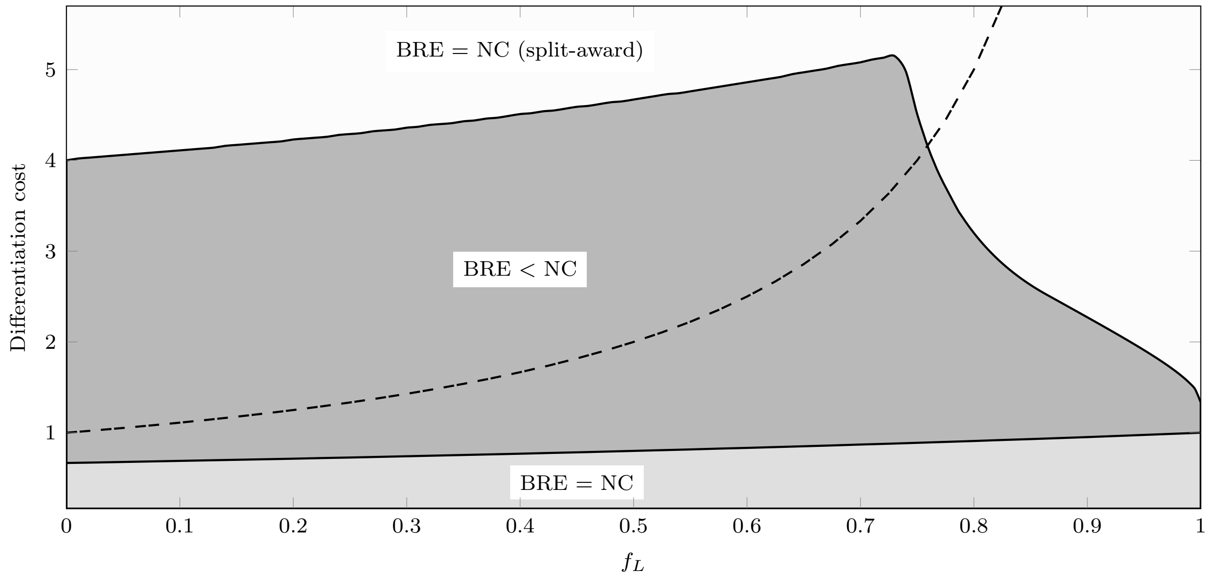

Figure 5: For $\theta_L = 10$, $\theta_H = 12$, we show when it is profitable to restrict the entry as a function of the differentiation cost δ and f_L. The dashed line represents the cutoff between single and split award in the optimal mechanism (i.e., $\delta = \frac{1}{f_H}(\theta_H - \theta_L)$).

Value of δ	Equil. strategy b_L	Award	Expected procurement cost
$[(\theta_H - \theta_L), \infty)$	$b_L = \theta_H$	split	$\theta_H + \frac{\delta}{4}$
$\left[\frac{(\theta_H-\theta_L)}{2+f_H}, (\theta_H - \theta_L)\right]$	$b_L = \frac{f_H\theta_H+\delta+\theta_L}{1+f_H}$	split	$\frac{f_L\theta_L+f_H(4-f_L)\theta_H}{(1+f_H)^2} - \frac{f_Hf_L(\theta_H-\theta_L)^2}{2\delta(1+f_H)^2} + \frac{\delta}{(1+f_H)^2} + \frac{f_L}{4}\delta$
$\left[\frac{f_L}{2}(\theta_H - \theta_L), \frac{(\theta_H-\theta_L)}{2+f_H}\right)$	$b_L = \theta_H - \delta$	single	$\theta_H + \frac{\delta}{4} - \frac{f_L(1+f_L)}{2}\delta$
$\left[\frac{f_Hf_L(\theta_H-\theta_L)}{\frac{1}{2}(1+f_H)^2+f_Hf_L}, \frac{f_L}{2}(\theta_H - \theta_L)\right]$	$b_L = \theta_L + \delta\frac{1+f_H}{f_L}$	single	$f_H^2\theta_H + f_L(1+f_H)\theta_L + \frac{17-10f_L-2f_L^2}{4}\delta$
$\left[0, \frac{f_Hf_L(\theta_H-\theta_L)}{\frac{1}{2}(1+f_H)^2+f_Hf_L}\right]$	No PSBNE	-	-

Table 3: Equilibrium strategies and expected procurement costs in NC mechanism as a function of the transportation cost δ

Proof. Let $\Pi(b, (b_L, b_H))$ denote the profit function when a player's type is θ_L, his adversary plays $(b_L, b_H = \theta_H)$ and his bid is b. We have three different cases depending on the value of b_L. We denote the cases by I, II or III depending on whether $b_L \in [\theta_H - \delta, \theta_H]$, $b_L \in [\theta_H - 2\delta, \theta_H - \delta]$, or, $b_L \in [\theta_L, \theta_H - 2\delta]$ respectively. This cases are described in Table 4.

Case $\delta \in [(\theta_H - \theta_L), \infty)$. We claim that $(b_H, b_L) = (\theta_H, \theta_H)$ is a PSBNE. For a player of type θ_L, the profit function is as defined in case I. However, since $\delta \geq (\theta_H - \theta_L)$ the only meaningful case is the first one, that is: $\Pi(b, (b^*, \theta_H)) = (b - \theta_L)\left(\frac{f_H\theta_H+f_Lb^*+\delta-b}{2\delta}\right)$ for $b \in [\theta_L, \theta_H]$. We now focus on finding a symmetric equilibrium b^*. By the first order conditions we must have $\frac{f_H\theta_H+f_Lb^*+\delta-2b+\theta_L}{2\delta} = 0$, or equivalently, $(1 + f_H)b^* = f_H\theta_H + \delta + \theta_L$. However, as $\delta \geq \theta_H - \theta_L$ we obtain $b^* \geq \theta_H$. Hence, the best response for a player of type θ_L is $b = \theta_H$. Furthermore, the same argument shows that θ_H is the unique symmetric equilibrium. $\Diamond$

Case	b_L	Profit Function
I	$[\theta_H-\delta,\theta_H]$	$\Pi_I(b,(b_L,b_H)) = \begin{cases} (b-\theta_L)\left(\frac{f_H\theta_H+f_Lb_L+\delta-b}{2\delta}\right) & \text{if } b\in[\theta_H-\delta,\theta_H] \\ (b-\theta_L)\left(f_H+f_L\frac{b_L+\delta-b}{2\delta}\right) & \text{if if } b\in[b_L-\delta,\theta_H-\delta] \\ (b-\theta_L) & \text{otherwise} \end{cases}$
II	$[\theta_H-2\delta,\theta_H-\delta]$	$\Pi_{II}(b,(b_L,b_H)) = \begin{cases} (b-\theta_L)\left(f_H\frac{\theta_H+\delta-b}{2\delta}\right) & \text{if } b\in[b_L+\delta,\theta_H] \\ (b-\theta_L)\left(\frac{f_H\theta_H+f_Lb_L+\delta-b}{2\delta}\right) & \text{if } b\in[\theta_H-\delta,b_L+\delta] \\ (b-\theta_L)\left(f_H+f_L\frac{b_L+\delta-b}{2\delta}\right) & \text{if if } b\in[b_L-\delta,\theta_H-\delta] \\ (b-\theta_L) & \text{otherwise} \end{cases}$
III	$[\theta_L,\theta_H-2\delta]$	$\Pi_{III}(b,(b_L,b_H)) = \begin{cases} (b-\theta_L)\left(f_H\frac{\theta_H+\delta-b}{2\delta}\right) & \text{if } b\in[\theta_H-\delta,\theta_H] \\ (b-\theta_L)f_H & \text{if } b\in[b_L+\delta,\theta_H-\delta] \\ (b-\theta_L)\left(f_H+f_L\frac{b_L+\delta-b}{2\delta}\right) & \text{if if } b\in[b_L-\delta,b_L+\delta] \\ (b-\theta_L) & \text{otherwise} \end{cases}$

Table 4: Characterization of the profit functions according to the value of b_L.

Case $\delta \in \left[\frac{(\theta_H-\theta_L)}{2+f_H},(\theta_H-\theta_L)\right]$. We claim that $(b_H,b_L)=\left(\theta_H,\frac{f_H\theta_H+\delta+\theta_L}{1+f_H}\right)$ is the unique PSBNE. Note that $b_L\in[\theta_H-\delta,\theta_H]$ and therefore the profit function is as defined by case I. It can be verified that $\frac{\partial}{\partial b}\left((b-\theta_L)\left(f_H+f_L\frac{b_L+\delta-b}{2\delta}\right)\right)$ is positive at $\theta_H-\delta$ for all δ in the considered interval. Therefore, the best response must be in the interval $[\theta_H-\delta,\theta_H]$, and by deriving the function Π_I in that interval we can see that $\frac{f_H\theta_H+\delta+\theta_L}{1+f_H}$ is indeed a best response. To check uniqueness, we divide it into two cases: $b<\theta_H-\delta$ and $b\geq\theta_H-\delta$. If $b\geq\theta_H-\delta$, the profit function is $\Pi(b,(b^*,\theta_H))=(b-\theta_L)\left(\frac{f_H\theta_H+f_Lb^*+\delta-b}{2\delta}\right)$ and it can be seen that b_L as defined above is the unique b for which the FOCs are satisfied. If $b^*<\theta_H-\delta$, the profit function is $\Pi(b,(b^*,\theta_H))=(b-\theta_L)\left(f_H+f_L\frac{b^*+\delta-b}{2\delta}\right)$. Then, b^* can never be a symmetric equilibrium as $\frac{\partial\Pi}{\partial b}>0$ at $b=b^*$ for any $b^*<\theta_H-\delta$. $\Diamond$

Case $\delta\in\left[\frac{f_L}{2}(\theta_H-\theta_L),\frac{(\theta_H-\theta_L)}{2+f_H}\right]$. We claim that $(b_H,b_L)=(\theta_H,\theta_H-\delta)$ is a PSNE. In this case, the profit function is a particular case of case I. It suffices to show that the left derivative of the profit function is positive in $\theta_H-\delta$ and the right derivative is negative in $\theta_H-\delta$. The right derivative at $\theta_H-\delta$ is $\frac{\partial\Pi}{\partial b}(\theta_H-\delta)=\frac{-\theta_H+\theta_L+(2+f_H)\delta}{2\delta}$, which cannot be positive as long as $\delta\leq\frac{1}{2+f_H}(\theta_H-\theta_L)$. On the other hand, the left derivative is $\frac{\partial\Pi}{\partial b}(\theta_H-\delta)=f_h+\frac{f_L(-\theta_H+\theta_L+2\delta)}{2\delta}$ which is non-negative as long $\delta\geq\frac{f_L}{2}(\theta_H-\theta_L)$. Therefore $\theta_H-\delta$ is a best response.

To show uniqueness, suppose there exists a different symmetric equilibrium with $b_L=b^*$ with $b^*\neq\theta_H-\delta$. First, consider the case in which $b^*>\theta_H-\theta_L$. In that case, the BR function is $\Pi(b,(b^*,\theta_H))=(b-\theta_L)\left(\frac{f_H\theta_H+f_Lb^*+\delta-b}{2\delta}\right)$ for $b\in[\theta_H-\delta,\theta_H]$. By imposing symmetry, the FOCs are $f_H\theta_H+\delta+\theta_L=(1+f_H)b^*$ which implies $f_H\theta_H+\delta+\theta_L>(1+f_H)(\theta_H-\delta)$ as $b^*\in(\theta_H-\delta,\theta_H]$ by assumption. However, this reduces to $(2+f_H)\delta>\theta_H-\theta_L$ which is a contradiction. Next, consider the case $b^*<\theta_H-\delta$. The profit function is $\Pi(b,(b^*,\theta_H))=(b-\theta_L)\left(f_H+f_L\frac{b^*+\delta-b}{2\delta}\right)$ for $b<\theta_H-\delta$. The FOCs are $f_H+f_L\frac{b^*+\delta-2b+\theta_L}{2\delta}$. By imposing symmetry, we must have $(1+f_H)\delta+f_L\theta_L=f_Lb^*<f_L(\theta_H-\delta)$ which is possible only if $2\delta<f_L(\theta_H-\theta_L)$, which is a contradiction. $\Diamond$

Case $\delta\in\left[\frac{f_Hf_L(\theta_H-\theta_L)}{\frac{1}{2}(1+f_H)^2+f_Hf_L},\frac{f_L}{2}(\theta_H-\theta_L)\right]$. We claim that $(b_H,b_L)=\left(\theta_H,\theta_L+\delta\frac{1+f_H}{f_L}\right)$ is a PSNE. Note that $b_L\leq\theta_H-\delta$ for all values of δ considered.

We first show that b_L satisfies the first order conditions. As usual, consider $b^*\leq\theta_H-\delta$.

The profit function is $\Pi(b,(b^*,\theta_H)) = (b-\theta_L)\left(f_H + f_L\frac{b^*+\delta-b}{2\delta}\right)$ for $b \geq \theta_H - \delta$. The FOCs are $f_H + f_L\frac{b^*+\delta-2b+\theta_L}{2\delta}$. By imposing symmetry, we must have $(1+f_H)\delta + f_L\theta_L = f_L b^*$, or equivalently, $b_L = b^* = \theta_L + \delta\frac{1+f_H}{f_L}$ as desired. In addition, we must show that the agent cannot benefit by deviating to $b_L - \delta$. To that end, note that the expected profit at b_L is $(b_L - \theta_L)(f_H + f_L/2) = \delta\frac{(1+f_H)^2}{2f_L}$ and the expected profit at $b_L - \delta$ is $\delta\frac{2f_H}{f_L}$. As $\frac{(1+f_H)^2}{2f_L} > \frac{2f_H}{f_L}$, the deviation is not profitable. Furthermore, if $b_L \leq \theta_H - 2\delta$, we must also guarantee that a deviation in the interval $[b_L+\delta, \theta_H-\delta]$ is not profitable. In that case, the profit function is as described by Case III. Note that this function is strictly increasing in the interval $[b_L+\delta, \theta_H-\delta]$ and therefore we need to compare the max in the interval $[b_L+\delta, \theta_H-\delta]$ with that in the interval $[\theta_L, b_L+\delta]$ to obtain the global maximum and thus the best response. If $\delta \leq \frac{1}{3}(\theta_H - \theta_L)$, the maximum of the interval $[b_L+\delta, \theta_H-\delta]$ will be in $\theta_H - \delta$ as the right derivative at that point is negative. Since $\delta \leq \frac{f_L}{2+f_L}(\theta_H-\theta_L)$ (as otherwise we are in the previous case) and $\frac{f_L}{2+f_L} \leq \frac{1}{3}$, we conclude that the maximum in the interval $[b_L+\delta, \theta_H-\delta]$ is achieved at $\theta_H - \delta$ and the expected revenue is $f_H(\theta_H - \delta - \theta_L)$. Note that for $f_H(\theta_H - \delta - \theta_L) < (b_L - \theta_L)(f_H + f_L/2)$ we must have $\delta \geq \frac{f_H f_L(\theta_H-\theta_L)}{\frac{1}{2}(1+f_H)^2 + f_H f_L}$.

To show uniqueness, we show that there cannot be an equilibrium with $b^* > \theta_H - \delta$. In that case, the FOCs are $f_H\theta_H + \delta + \theta_L = (1+f_H)b^*$ and $f_H\theta_H + \delta + \theta_L > (1+f_H)(\theta_H - \delta)$ only if $(2+f_H)\delta > \theta_H - \theta_L$, which is a contradiction. $\Diamond$

Case $\delta < \frac{f_H f_L(\theta_H-\theta_L)}{\frac{1}{2}(1+f_H)^2+f_H f_L}$. The lack of equilibria follows from the arguments in the previous case.

□

Proposition L.1. *Let r_L and r_U be defined as:*

$$r_L = \frac{f_L(\theta_H-\theta_L) - (1+3f_H)\delta - 2\sqrt{\delta(1+f_H)f_H(2\delta - f_L(\theta_H-\theta_L))}}{f_L^2},$$

$$r_U = \frac{f_L(\theta_H-\theta_L) - (1+3f_H)\delta + 2\sqrt{\delta(1+f_H)f_H(2\delta - f_L(\theta_H-\theta_L))}}{f_L^2}.$$

Then, for every C in the (possibly empty) intervals indicated by Table 5, $\theta_H - C$ is the unique equilibrium bidding (PSBNE) strategy for the low-type in the RE mechanism.

Value of δ	Interval of C
$\left[\frac{\left(2+f_Hf_L+\sqrt{(1+f_H)f_H(2+f_L+f_L^2)}\right)(\theta_H-\theta_L)}{2(1+f_H)}, \frac{f_L^2(\theta_H-\theta_L)}{f_L-2f_H}\right.$ $(f_L > \frac{2}{3})\Big]$	$\left[\frac{f_L}{2+f_L}(\theta_H-\theta_L),\ \min\left(\frac{\delta-(\theta_H-\theta_L)}{f_L},\ \frac{f_H(\theta_H-\theta_L)}{1+f_H-\frac{f_L(\theta_H-\theta_L)}{\delta}}\right)\right]$
$\left[(\theta_H-\theta_L), \frac{2+f_Hf_L+\sqrt{(1+f_H)f_H(2+f_L+f_L^2)}}{2(1+f_H)}(\theta_H-\theta_L)\right]$	$\left[\frac{f_L}{2+f_L}(\theta_H-\theta_L), r_U\right]$
$\left[\frac{2f_L}{(2+f_L)(1+f_H)}(\theta_H-\theta_L), \theta_H-\theta_L\right]$	$\left[\max\left(\frac{f_L}{2+f_L}(\theta_H-\theta_L), r_L\right),\ \min\left(\delta,\ \max\left(r_U,\ \frac{(\theta_H-\theta_L)-\delta}{1+f_H}\right)\right)\right]$
$\left[\frac{f_L}{2}(\theta_H-\theta_L), \frac{2f_L}{(2+f_L)(1+f_H)}(\theta_H-\theta_L)\right]$	$\left[\max\left(\frac{f_L(\theta_H-\theta_L)-(1+f_H)\delta}{f_L},\ r_L\right),\ \min\left(\delta,\ \max\left(r_U,\ \frac{(\theta_H-\theta_L)-\delta}{1+f_H}\right)\right)\right]$

Table 5: For a given C and δ, the intervals for which $\theta_H - C$ is the unique equilibrium bidding strategy for the low-type are given as a function of the parameters.

Proof. We must consider $\delta \geq \frac{f_L}{2}(\theta_H - \theta_L)$, as otherwise we know that the equilibrium bidding strategy for an agent of type low is smaller than $\theta_H - \delta$ and, therefore, smaller than $\theta_H - C$. We first show that, under the stated conditions, $\theta_H - C$ is an equilibrium. The profit function is:

$$\Pi(b,(\theta_H - C,\theta_H)) = \begin{cases} \pi_1(b) = \frac{f_H}{2}(\theta_H-\theta_L) & \text{if } b=\theta_H \\ \pi_2(b) = f_H(b-\theta_L)\frac{\theta_H-b+\delta}{2\delta} + f_L(b-\theta_L)\frac{\theta_H-C-b+\delta}{2\delta} & \text{if } b\in(\theta_H-C,\theta_H) \\ \pi_3(b) = f_H(b-\theta_L) + f_L(b-\theta_L)\frac{\theta_H-C-b+\delta}{2\delta} & \text{if } b\in(\theta_H-2C,\theta_H-C] \\ \pi_4(b) = b-\theta_L & \text{otherwise} \end{cases}$$

For $\theta_H - C$ to be an equilibrium, we need $\theta_H - C$ to be a maximizer of $\Pi(b,(\theta_H - C,\theta_H))$. The following conditions are then necessary (and sufficient):

(a) $\frac{\partial\pi_3(\theta_H - C)}{\partial b} \geq 0$.

(b) $\pi_3(\theta_H - C) \geq \pi_4(\theta_H - 2C)$

(c) $\pi_3(\theta_H - C) \geq \max_{b\in(\theta_H-C,\theta_H]}\pi_2(b)$

We now derive conditions under which $(a)-(c)$ hold:

Condition for (a)**:** $\frac{\partial\pi_3(b)}{\partial b} = f_H + f_L\frac{\theta_H+\theta_L-C-2b+\delta}{2\delta}$. Then, $\frac{\partial\pi_3(\theta_H-C)}{\partial b} = f_H + f_L\frac{\theta_H+\theta_L-C-2(\theta_H-C)+\delta}{2\delta} = f_H + f_L\frac{\theta_L-\theta_H+C+\delta}{2\delta}$ and it is non-negative whenever $C \geq (\theta_H-\theta_L) - \frac{1+f_H}{f_L}\delta$.

Condition for (b)**:** $\pi_3(\theta_H - C) \geq \pi_4(\theta_H - 2C)$ is equivalent to $\left(f_H + \frac{f_L}{2}\right)(\theta_H - C - \theta_L) \geq (\theta_H - \theta_L - 2C)$ which occurs if and only if $\left(1+\frac{f_L}{2}\right)C \geq \frac{f_L}{2}(\theta_H-\theta_L)$ or $C \geq \frac{f_L}{2+f_L}(\theta_H-\theta_L)$.

Condition for (c)**:** We just consider the case in which the max is at θ_H or the case where the max is somewhere in $(\theta_H - C,\theta_H)$. Note that $\frac{\partial\pi_2(b)}{\partial b} = \frac{\partial}{\partial b}\left((b-\theta_L)\frac{\theta_H-b+\delta}{2\delta} - f_L(b-\theta_L)\frac{C}{2\delta}\right) = \frac{\theta_H+\theta_L-2b+\delta}{2\delta} - f_L\frac{C}{2\delta}$.

First, note that if $\frac{\partial\pi_2}{\partial b}(\theta_H - C) \leq 0$, condition (c) is automatically satisfied as $\pi_3(\theta_H - C) \geq \pi_2(\theta_H - C)$. Hence, condition (c) holds whenever $C \leq \frac{(\theta_H-\theta_L)-\delta}{1+f_H}$. Next, consider the case in which $\max_{b\in(\theta_H-C,\theta_H]}\pi_2(b)$ is achieved at $\theta_H - C < b^* < \theta_H$. Then, we must have $\frac{\partial\pi_2}{\partial b}(\theta_H) < 0$, or equivalently, $\delta - (\theta_H - \theta_L) < f_L C$. In that case, we must have $\pi_3(\theta_H - C) \geq \pi_2(b^*)$, or equivalently $\left(f_H + \frac{f_L}{2}\right)(\theta_H - C - \theta_L) \geq \frac{((\theta_H-\theta_L+\delta)-Cf_L)^2}{8\delta}$. Note that this quadratic constraint imposes both a lower and upper bound on C. Finally, if the maximum is achieved at θ_H, we must have $\frac{\partial\pi_2}{\partial b}(\theta_H) \geq 0$, therefore, $\delta - (\theta_H - \theta_L) \geq f_L C$. In addition, we must have $f_H(\theta_H-\theta_L) \geq \left(1 + f_H - \frac{f_L(\theta_H-\theta_L)}{\delta}\right)C$.

We can summarize the conditions $(a)-(c)$ by requiring $C \in \mathcal{C}$, where the set $\mathcal{C}$ is defined as follows:

$$\mathcal{C} = \left\{ \begin{array}{lll} C: & (1) & \max\left((\theta_H - \theta_L) - \frac{1+f_H}{f_L}\delta,\ \frac{f_L}{2+f_L}\left(\theta_H - \theta_L\right)\right) \leq C \leq \delta \text{ and either} \\ & (2A) & \delta - (\theta_H - \theta_L) < f_L C \text{ and } \left(f_H + \frac{f_L}{2}\right)\left(\theta_H - C - \theta_L\right) \geq \frac{((\theta_H - \theta_L + \delta) - C f_L)^2}{8\delta}, \text{ or,} \\ & (2B) & \delta - (\theta_H - \theta_L) \geq f_L C \text{ and } f_H\left(\theta_H - \theta_L\right) \geq \left(1 + f_H - \frac{f_L(\theta_H - \theta_L)}{\delta}\right) C, \text{ or,} \\ & (2C) & C \leq \frac{(\theta_H - \theta_L) - \delta}{1+f_H} \end{array} \right\}$$

Constraint (1) groups the constraints imposed (a) and (b) plus requiring $C \leq \delta$. Constraints $(2A) - (2C)$ represent the (disjoint) constraints imposed in (c). By using algebraic manipulations we can obtain the intervals in Table 5. In particular, r_L and r_U correspond to the roots of the quadratic equation given in Condition $(2A)$ of set $\mathcal{C}$.

We show uniqueness (except in border cases) by contradiction. Suppose there exists a symmetric equilibrium strategy b^* that is an equilibrium. First, it is easy to see that $b^* < \theta_H - C$ is not possible unless $\delta \leq \frac{f_L}{2}(\theta_H - \theta_L)$. Second, we argue that b^* cannot be θ_H. The profit when both players select θ_H is $\frac{\theta_h - \theta_L}{2}$; by deviating to $\theta_H - C$ the profit is $(\theta_H - C - \theta_L)$, which is bigger provided $C < \frac{\theta_H - \theta_L}{2}$. However, note that $C \leq r_U$ for the appropriate δ and r_U as a function of δ is concave, achieves its max at $\frac{1+f_H}{2}(\theta_H - \theta_L)$ and the max value is $\frac{\theta_H - \theta_L}{2}$. Therefore, whenever r_U is binding, $C \leq r_U < \frac{\theta_H - \theta_L}{2}$. as desired. For δs for which $\frac{f_H(\theta_H - \theta_L)}{1+f_H - \frac{f_L(\theta_H - \theta_L)}{\delta}}$ is binding, note that we must $\delta \geq (\theta_H - \theta_L)$ and therefore the condition is satisfied. Finally, for the cases in which $b^* \in (\theta_H - C, \theta_H)$, we have that $b^* = \frac{f_H \theta_H + \delta + \theta_L}{1+f_H}$ (must satisfy the first order conditions) and hence $\delta \leq (\theta_H - \theta_L)$. However, the reader can verify that $\theta_H - C$ is a profitable deviation for the appropriate values of C. In particular, this holds for $C = r_U$. □

As the designer is utilitity-maximizer, we are concerned with the biggest C under which we can have an equilibrium. This yields 3 different cases, as shown by Table 6, 7, and 8 respectively.

Value of δ	Best low-type bid
$\left[\frac{1}{2+f_H}(\theta_H - \theta_L), \frac{(1+f_H)(\sqrt{2}+\sqrt{f_H})^2}{(2+f_L)}(\theta_H - \theta_L)\right]$	$\theta_H - r_U$
$\left[\frac{f_L}{2}(\theta_H - \theta_L), \frac{1}{2+f_H}(\theta_H - \theta_L)\right]$	$\theta_H - \delta$

Table 6: Case 1: $\left(f_L \geq \frac{1}{6}\left(1 - \frac{23}{(181+24\sqrt{78})^{1/3}} + ((181 + 24\sqrt{78})^{1/3}\right) \approx 0.8641\right)$

Value of δ	Best low-type bid
$\left[\frac{2+f_H f_L+\sqrt{(1+f_H)f_H(2+f_L+f_L^2)}}{2(1+f_H)}(\theta_H - \theta_L), \frac{f_L^2}{f_L - 2f_H}(\theta_H - \theta_L)\right]$	$\theta_H - \frac{f_H(\theta_H - \theta_L)}{1+f_H - \frac{f_L(\theta_H - \theta_L)}{\delta}}$
$\left[\frac{1}{2+f_H}(\theta_H - \theta_L), \frac{2+f_H f_L+\sqrt{(1+f_H)f_H(2+f_L+f_L^2)}}{2(1+f_H)}(\theta_H - \theta_L)\right]$	$\theta_H - r_U$
$\left[\frac{f_L}{2}(\theta_H - \theta_L), \frac{1}{2+f_H}(\theta_H - \theta_L)\right]$	$\theta_H - \delta$

Table 7: Case 2: $2/3 < f_L \leq \frac{1}{6}\left(1 - \frac{23}{(181+24\sqrt{78})^{1/3}} + ((181 + 24\sqrt{78})^{1/3}\right) \approx 0.8641$

Value of δ	Best low-type bid
$\left[\frac{2+f_H f_L+\sqrt{(1+f_H)f_H(2+f_L+f_L^2)}}{2(1+f_H)}(\theta_H-\theta_L),\infty\right)$	$\theta_H - \frac{f_H(\theta_H-\theta_L)}{1+f_H-\frac{f_L(\theta_H-\theta_L)}{\delta}}$
$\left[\frac{1}{2+f_H}(\theta_H-\theta_L),\frac{2+f_H f_L+\sqrt{(1+f_H)f_H(2+f_L+f_L^2)}}{2(1+f_H)}(\theta_H-\theta_L)\right]$	$\theta_H - r_U$
$\left[\frac{f_L}{2}(\theta_H-\theta_L),\frac{1}{2+f_H}(\theta_H-\theta_L)\right]$	$\theta_H - \delta$

Table 8: Case 3: $f_L \leq 2/3$

To derive Case 1, we know that $r_U > \frac{f_L}{2+f_L}(\theta_H - \theta_L)$ only if

$$\delta \leq \frac{(2+f_H)(1+f_H)+2\sqrt{2}\sqrt{f_H(4-f_L^2)^2}}{(2+f_L)^2}(\theta_H-\theta_L) = \frac{(1+f_H)(\sqrt{2}+\sqrt{f_H})^2}{(2+f_L)}(\theta_H-\theta_L).$$

Note that, whenever $\frac{(1+f_H)(\sqrt{2}+\sqrt{f_H})^2}{(2+f_L)}(\theta_H-\theta_L) \leq \frac{2+f_H f_L+\sqrt{(1+f_H)f_H(2+f_L+f_L^2)}}{2(1+f_H)}(\theta_H-\theta_L)$ (equivalently, $f_L > l_1 = \frac{1}{6}\left(1-\frac{23}{(181+24\sqrt{78})^{1/3}}+((181+24\sqrt{78})^{1/3}\right)$ or $f_L \approx 0.8641$. In addition, we highlight that $\frac{(1+f_H)(\sqrt{2}+\sqrt{f_H})^2}{(2+f_L)}(\theta_H-\theta_L) > (\theta_H-\theta_L)$ whenever $f_H \geq \sqrt{2}-\sqrt{2\sqrt{2}-1} \approx 0.062$. Therefore, if $f_L > 0.938$, we have that our mechanism will not work better than the original for $\delta \geq (\theta_H - \theta_L)$. Case 2 is derived from the fact that we have an upper bound on the largest interval only if $f_L > 2/3$. Finally, standard calculations allow us to derive Case 3.

Now, we briefly discuss the idea behind the characterization of the best low-type bid. Intuitively, the advantage of bidding at $\theta_H - C$ is to capture the whole demand when the other agent has a high cost. As f_L becomes close to one, this advantage vanishes; for this reason, the best low-type bid is increasing in f_L. In addition, the best low-type bid is also increasing in δ; as the transportation cost increases, demands become less sensitive to prices and, therefore, a supplier can increase his bid without significantly decreasing demand.

L.2 Multiproduct Model - Proofs

L.2.1 Proof of Proposition J.1

Proof. Recall that the utility of supplier i is

$$u_i = t_i - \sum_{p \in P_i} c_{ip}(\theta_i) x_{ip},$$

where x_{ip} is the amount of product p allocated to i, t_i is the payment i receives in the auction, and θ_i is his type. Similarly, the interim utility for supplier i when he reports cost θ_i' and has true cost θ_i is given by:

$$U_i(\theta_i'|\theta_i) = T_i(\theta_i') - \sum_{p \in P_i} c_{ip}(\theta_i) X_{ip}(\theta_i').$$

The centralized problem is then:

$$[P_1] \qquad \max_{\boldsymbol{x},\boldsymbol{t}} \quad \mathbb{E}_{\boldsymbol{\theta}}\left[K(x(\boldsymbol{\theta})) - \sum_{i=1}^{n}\sum_{p\in P_i} t_i(\boldsymbol{\theta})\right]$$

$$\text{s.t.} \quad U_i(\theta_i|\theta_i) \geq U_i(\theta_i'|\theta_i) \qquad \forall i \in N,\ \forall \theta_i, \theta_i' \in \Theta_i \tag{IC}$$

$$U_i(\theta_i|\theta_i) \geq 0 \qquad \forall i \in N,\ \forall \theta_i \in \Theta_i \tag{IR}$$

$$\sum_{i\in N}\sum_{p\in P_i} x_{ip}(\boldsymbol{\theta}) = 1 \qquad \forall \boldsymbol{\theta} \in \Theta, \qquad x_i(\boldsymbol{\theta}) \geq 0 \qquad \forall i \in N,\ \boldsymbol{\theta} \in \Theta, \tag{Feas}$$

where $U_i(\theta_i'|\theta_i)$ is defined accordingly.

Throughout this proof, we define m_i to be the number of costs in the support of agent i, that is, $m_i = |\Theta_i|$. We use the same technique as in the proof of Proposition 3.1, which is based on the works by Myerson (1981) and Vohra (2011). To avoid repetition, the reader will be referred to the proof of Proposition 3.1 when needed. We start by re-stating the IC and IR constraints in terms of the expected allocations and transfers:

$$\begin{aligned}
\max_{\boldsymbol{x},\boldsymbol{t}} \quad & \mathbb{E}_{\boldsymbol{\theta}}\left[K(x(\boldsymbol{\theta})) - \sum_{i=1}^{n}\sum_{p\in P_i} t_i(\boldsymbol{\theta})\right] \\
\text{s.t.} \quad & T_i(\theta_i) - \sum_{p\in P_i} X_{ip}(\theta_i)c_{ip}(\theta_i) \geq T_i(\theta_i') - \sum_{p\in P_i} X_{ip}(\theta_i')c_{ip}(\theta_i) \quad \forall i,\ \forall \theta_i, \theta_i' \in \Theta_i \\
& T_i(\theta_i) - \sum_{p\in P_i} X_{ip}(\theta_i)c_{ip}(\theta_i) \geq 0 \quad \forall i,\ \forall \theta_i \in \Theta_i \\
& \sum_{i\in N}\sum_{p\in P_i} x_{ip}(\boldsymbol{\theta}) = 1 \qquad \forall \boldsymbol{\theta} \in \Theta, \qquad x_i(\boldsymbol{\theta}) \geq 0 \qquad \forall i \in N,\ \boldsymbol{\theta} \in \Theta,
\end{aligned}$$

Recall that $\Theta_i = \{\theta_i^1, ..., \theta_i^{m_i}\}$. If we add a dummy type per agent $\theta_i^{m_i+1}$ such that $X_i(\theta_i^{m_i+1}) = 0$ and $T_i(\theta_i^{m_i+1}) = 0$, then we can fold the IR constraints into the IC constraints.

Using the arguments in Theorem 6.2.1 in Vohra (2011) for our procurement setting we obtain that an allocation $\boldsymbol{x}$ is implementable in Bayes Nash equilibrium if and only if $X_{ip}(\cdot)$ is monotonically decreasing for all $i = 1, ..., n$ and all $p \in P_i$. Further, by Theorem 6.2.2 in Vohra (2011), all IC constraints are implied by the following local adjacent IC constraints $(BNIC^d_{i,\theta})$ and $(BNIC^d_{i,\theta})$[38] for all $i \in N$ and all $\theta \in \Theta_i$.

In addition, using standard arguments, we can show that all downward constraints $(BNIC^d_{i,j})$ bind in the optimal solution.[39] Hence,

$$T_i(\theta_i^j) - \sum_{p\in P_i} X_{ip}(\theta_i^j)c_{ip}(\theta_i^j) = T_i(\theta_i^{j+1}) - \sum_{p\in P_i} X_{ip}(\theta_i^{j+1})c_{ip}(\theta_i^j) \quad \forall i \in N,\ \forall j \in \{1, ..., m_i\}.$$

Further, it is simple to show that in this case, the upward constraints $(BNIC^u_{i,j})$ are satisfied.

[38] These are defined in the proof of Proposition 3.1

[39] A formal proof can be obtained by trivially adapting the Lemma 6.2.4 in Vohra to the procurement case.

Applying the previous equation recursively we obtain:

$$T_i(\theta_i^j) = \sum_{p \in P_i} \left(c_{ip}(\theta_i^j) X_{ip}(\theta_i^j) + \sum_{k=j+1}^{m_i} (c_{ip}(\theta^k) - c_{ip}(\theta^{k-1}) X_{ip}(\theta_i^k) \right) . \tag{36}$$

Replacing in the objective and using simple algebra[40] we obtain

$$obj = \mathbb{E}_{\boldsymbol{\theta}} \left[K(x(\boldsymbol{\theta})) \right] - \sum_{i=1}^{n} \sum_{p \in P_i} \sum_{\theta_i \in \Theta_i} f_i(\theta_i) T_i(\theta_i) = \sum_{\boldsymbol{\theta} \in \Theta} f(\boldsymbol{\theta}) \left(K(x(\boldsymbol{\theta})) - \sum_{i=1}^{n} \sum_{p \in P(i)} v_{ip}(\theta_i) x_{ip}(\boldsymbol{\theta}) \right)$$

where v_{ip} is the modified virtual cost function for each agent-product pair ip, defined as

$$v_{ip}(\theta_i) = c_{ip}(\theta_i) + \frac{F_i(\rho(\theta_i))}{f_i(\theta_i)} \left(c_{ip}(\theta_i) - c_{ip}(\rho(\theta_i)) \right) .$$

Therefore, if we can find an allocation function such that for all $\boldsymbol{\theta} \in \Theta$ and $i \in N$,

$$\begin{aligned} x(\boldsymbol{\theta}) \quad &\in \quad \operatorname{argmax} K(x(\boldsymbol{\theta})) - \sum_{i=1}^{n} \sum_{p \in P_i} v_{ip}(\theta_i) x_{ip}(\boldsymbol{\theta}) \\ &\text{s.t.} \qquad \sum_{i=1}^{N} \sum_{p \in P_i} x_{ip}(\boldsymbol{\theta}) = 1, \qquad x_{ip}(\boldsymbol{\theta}) \geq 0 \quad \forall i \in N, p \in P_i \ . \end{aligned}$$

and such that the interim expected allocations are monotonic for all $i \in N$ and $p \in P_i$, and such that the interim expected transfers are satisfied, then we have found an optimal solution. □

L.2.2 Sketch of Proof of the Extension of the Main Theorem to the Multiproduct Case

We start with some remarks:

- Proposition G.7 extends easily: for every $\boldsymbol{\theta} \in \Theta$, we have $\sum_{(i,p) \in Q(\boldsymbol{\theta})} g_i(\theta_j) x_{ip}(\boldsymbol{\theta}) = 0$.
- All the observations in Section G.1 still hold, as the demand system remains unmodified.

We now provide a sketch of the proof of Theorem J.1.

Sketch of proof of Theorem J.1. To show $OPT(DecLin) = OPT(Cent)$, we show that the system of equations is consistent. Let $(\boldsymbol{a}, \boldsymbol{b})$ be a vector of coefficients satisfying Equation (20). Let $g_i(\theta_i)$ be as defined in the statement of Proposition G.7. As the instance is RIOM, we know that there exists a subset of profiles $\tilde{\Theta} \subseteq \Theta$ that satisfies Conditions (1)-(5). The proof will follow the same structure as the proof of Theorem G.1.

[40] See the proof of Proposition 3.1 for the calculations

We show that $g_i(\theta_i) = 0$ **for all** $\theta_i \in \tilde{\Theta}_i$ **and all** $i \in N$**.** Consider two profiles $\boldsymbol{\theta} = (\theta_i, \boldsymbol{\theta}_{-i})$ and $\boldsymbol{\theta}' = (\theta_i', \boldsymbol{\theta}_{-i})$ which only differ in agent i's cost and such that $\boldsymbol{\theta}, \boldsymbol{\theta}' \in \tilde{\Theta}$. By the definition of $\tilde{\Theta}$, such pair of profiles exists (Condition (4)). By Eq. (29), we must have $g_i(\theta_i) \sum_{p \in P_i} x_{ip}(\boldsymbol{\theta}) + \sum_{j \neq i} g_j(\theta_j) \left(\sum_{p \in P_j} x_{jp}(\boldsymbol{\theta}) \right) = 0$ and $g_i(\theta_i') \sum_{p \in P_i} x_{ip}(\boldsymbol{\theta}') + \sum_{j \neq i} g_j(\theta_j) \left(\sum_{p \in P_j} x_{jp}(\boldsymbol{\theta}) \right) = 0$. Hence, by subtracting the second equality from the first one we obtain

$$g_i(\theta_i) \sum_{p \in P_i} x_{ip}(\boldsymbol{\theta}) - g_i(\theta_i') \sum_{p \in P_i} x_{ip}(\boldsymbol{\theta}') = \sum_{j \neq i} g_j(\theta_j) \left[\sum_{p \in P_j} (x_{jp}(\boldsymbol{\theta}') - x_{jp}(\boldsymbol{\theta})) \right].$$

For each $p_i \in P_i$, each $j \in N$ and each $p_j \in P_j$, we must have $x_{jp}(\boldsymbol{\theta}') - x_{jp}(\boldsymbol{\theta}) = \boldsymbol{A}(\boldsymbol{\theta})_{p_j, p_i} (v_{ip}(\theta_i') - v_{ip}(\theta_i))$, where we used the fact that $A(\boldsymbol{\theta}) = A(\boldsymbol{\theta}')$ by definition (see Claim G.1). Let $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{\theta})$. Re-write the above equality as:

$$g_i(\theta_i) \sum_{p \in P_i} x_{ip}(\boldsymbol{\theta}) - g_i(\theta_1') \sum_{p \in P_i} x_{ip}(\boldsymbol{\theta}') = \sum_{p_i \in P_i} \left(\sum_{j \neq i} g_j(\theta_j) \left(\sum_{p_j \in P_j} \boldsymbol{A}_{p_j, p_i} \right) \right) \left(v_{ip}(\theta_i') - v_{ip}(\theta_i) \right),$$

Fix an arbitrary $j \in N$ with $j \neq i$ and $\boldsymbol{A}_{ij} \neq 0$. Assume that j has cost θ_j in both $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ as defined above. Let $\theta_j' \in \Theta_j$ be such that $\theta_j' \neq \theta_j$ and $\theta_j' \in \tilde{\Theta}_j$. Define $\tilde{\boldsymbol{\theta}} = (\theta_i, \theta_j', \boldsymbol{\theta}_{-i,j})$ and $\tilde{\boldsymbol{\theta}}' = (\theta_i', \theta_j', \boldsymbol{\theta}_{-i,j})$. The only thing we assumed about θ_j was $\theta_j \in \tilde{\Theta}_j$. Therefore, the above equality must also hold for any $\tilde{\Theta}_j$. That is,

$$g_i(\theta_i) \sum_{p \in P_i} x_{ip}(\tilde{\boldsymbol{\theta}}) - g_i(\theta_1') \sum_{p \in P_i} x_{ip}(\tilde{\boldsymbol{\theta}}') =$$
$$= \sum_{p_i \in P_i} \left(g_j(\theta_j') \sum_{p_j \in P_j} \boldsymbol{A}_{p_j, p_i} + \sum_{k \neq i,j} g_k(\theta_k) \left(\sum_{p_k \in P_k} \boldsymbol{A}_{p_k, p_i} \right) \right) (v_{ip}(\theta_i') - v_{ip}(\theta_i)).$$

By subtracting the inequality when j has cost θ_j from the one when his cost is θ_j' we obtain

$$g_i(\theta_i) \sum_{p \in P_i} \left(x_{ip}(\tilde{\boldsymbol{\theta}}) - x_{ip}(\boldsymbol{\theta}) \right) - g_i(\theta_i') \sum_{p \in P_i} \left(x_{ip}(\tilde{\boldsymbol{\theta}}') - x_{ip}(\boldsymbol{\theta}') \right) =$$
$$= \left(g_j(\theta_j') - g_j(\theta_j) \right) \sum_{p_i \in P_i} \sum_{p_j \in P_j} \boldsymbol{A}_{p_j, p_i} (v_{ip}(\theta_i') - v_{ip}(\theta_i)).$$

However, note that $x_{ip_i}(\tilde{\boldsymbol{\theta}}) - x_{ip_i}(\boldsymbol{\theta}) = \boldsymbol{A}_{p_i, p_j} \left(v_{jp_j}(\theta_j') - v_{jp_j}(\theta_j) \right)$. Therefore,

$$(g_i(\theta_i) - g_i(\theta_i'))) \sum_{p_i \in P_i} \sum_{p_j \in P_j} \boldsymbol{A}_{p_j, p_i} \left(v_{jp_j}(\theta_j') - v_{jp_j}(\theta_j) \right) =$$
$$= \left(g_j(\theta_j') - g_j(\theta_j) \right) \sum_{p_i \in P_i} \sum_{p_j \in P_j} \boldsymbol{A}_{p_j, p_i} (v_{ip}(\theta_i') - v_{ip}(\theta_i)),$$

or, equivalently, as long as $\boldsymbol{A}_{p_i, p_j} \neq 0$ for some $p_i \in P_i$ and $p_j \in P_j$ we must have

$$\frac{g_i(\theta_i) - g_i(\theta_i')}{\sum_{p_i \in P_i} \sum_{p_j \in P_j} \boldsymbol{A}_{p_j, p_i} (v_{ip}(\theta_i') - v_{ip}(\theta_i))} = \frac{g_j(\theta_j') - g_j(\theta_j)}{\sum_{p_i \in P_i} \sum_{p_j \in P_j} \boldsymbol{A}_{p_j, p_i} \left(v_{jp_j}(\theta_j') - v_{jp_j}(\theta_j) \right)}. \tag{37}$$

Furthermore, the above equality should hold for every $i, j \in N$ as we can find a sequence of agents $\{l_0 = i, \ldots, l_K = j\}$ such that $\boldsymbol{A}_{l_k, l_{k+1}} \neq 0$ for all $0 \leq k < K$.[41]

We now use Eq. (37) to show that $g_i(\theta_i) = 0$ for all $\theta_i \in \tilde{\Theta}_i$ as desired. Similarly to the single-product case, we consider two cases: that the numerator is zero for at least one pair $g_i(\theta_i), g_i(\theta_i')$, or that is non-zero for all pairs.

Suppose the numerator is zero for at least one pair of $g_i(\theta_i), g_i(\theta_i')$. Then, $g_j(\theta_j) - g_j(\theta_j')$ must be zero for every $j \in N$ and all pairs $\theta_j, \theta_j' \in \tilde{\Theta}_j$. We now show that $g_i(\theta_i) = g_j(\theta_j)$ must hold for every $\theta_i \in \tilde{\Theta}_i$ and $\theta_j \in \tilde{\Theta}_j$ and $i, j \in N$. This is trivial if $i = j$, as $g_i(\theta_i) - g_i(\theta_i')$ must be zero for every $i \in N$ and all pairs $\theta_i, \theta_i' \in \tilde{\Theta}_i$. Otherwise, note that when $g_i(\theta_i) = g_i(\theta_i')$, we have

$$g_i(\theta_i) \sum_{p \in P_i} x_{ip}(\boldsymbol{\theta}) - g_i(\theta_i') \sum_{p \in P_i} x_{ip}(\boldsymbol{\theta}') = g_i(\theta_i) \sum_{p \in P_i} \left(\sum_{p' \in P_i} \boldsymbol{A}_{p,p'} \right) \left(v_{ip}(\theta_i) - v_{ip}(\theta_i') \right).$$

Therefore,

$$\sum_{p_i \in P_i} \left(\sum_{j \in N} g_j(\theta_j) \left(\sum_{p_j \in P_j} \boldsymbol{A}_{p_j, p_i} \right) \right) \left(v_{ip_i}(\theta_i') - v_{ip_i}(\theta_i) \right) = 0,$$

or, equivalently,

$$\sum_{j \in N} g_j(\theta_j) \left(\sum_{p_j \in P_j} \sum_{p_i \in P_i} \boldsymbol{A}_{p_j, p_i} \left(v_{ip_i}(\theta_i') - v_{ip_i}(\theta_i) \right) \right) = 0,$$

and this must be true for any $i \in N$. By Lemma G.4, we know that $\boldsymbol{A}$ has rank $n - 1$. Consider the matrix $\boldsymbol{R}$ obtained from $\boldsymbol{A}$ as follows: for each $i \in N$ and each $p_i \in P_i$, multiply the column corresponding to p_i by $(v_{ip_i}(\theta_i') - v_{ip_i}(\theta_i))$ where $\theta_i, \theta_i' \in \tilde{\Theta}_i$. Note that, since $(v_{ip_i}(\theta_i') - v_{ip_i}(\theta_i)) \neq 0$, the matrix $\boldsymbol{R}$ has rank $n - 1$. Furthermore, we can re-write the above equation as:

$$\sum_{j \in N} g_j(\theta_j) \left(\sum_{p_j \in P_j} \sum_{p_i \in P_i} \boldsymbol{R}_{p_j, p_i} \right) = 0, \tag{38}$$

and this must be true for any $i \in N$. Let $\mathbf{g} = (g_1, \ldots, g_n)$ denote the vector of coefficients $g_i = g_i(\theta_i)$ for $\boldsymbol{\theta} \in \Theta$. By equation (38), the vector $\mathbf{g}$ must be in the nullspace of $\boldsymbol{R}'$. However, as $\boldsymbol{R}$ has rank $(n-1)$, the dimension of its nullspace is 1. Using the same arguments as in the proof of the main Theorem, it is easy to see that $\mathbf{1} \in \text{Null}(\boldsymbol{R}')$, which implies that *all* g_i with $i \in N$ must be equal. Furthermore, let $g = g_i(\theta_i)$. As $g_i(\theta_i) = g_j(\theta_j)$ for all $\theta_i \in \tilde{\Theta}_i$ and $\theta_j \in \tilde{\Theta}_j$,

$$0 = \sum_{i \in N} g_i(\theta_i) \sum_{p \in P_i} x_{ip}(\boldsymbol{\theta}) = g \left(\sum_{i \in N} \sum_{p \in P_i} x_{ip}(\boldsymbol{\theta}) \right) = g$$

[41]Here we are implicitly assuming that matrix $\boldsymbol{A}$ has only one block. If $\boldsymbol{A}$ has more than one block, then we can use the same argument for each block.

which implies $g_i(\theta_i) = 0$ for all $i \in N$ and $\theta_i \in \tilde{\Theta}_i$.

The rest of the proof generalizes straightforward from the single-product case. $\square$

L.3 Proof of Theorem H.1

Before presenting the proof of Theorem H.1, we state and and prove some propositions.

Proposition L.2. *If the coefficients* (a, b) *are such that equality* (20) *holds, then for each* $\boldsymbol{\theta} \in \Theta$, *we must have:*

$$\sum_{j \in Q(\boldsymbol{\theta})} b^j_{\theta_j} f(\boldsymbol{\theta}_{-j}) x_j(\boldsymbol{\theta}) = 0 \tag{39}$$

and

$$a^i_{\boldsymbol{\theta}} = \sum_{\{j \in Q(\boldsymbol{\theta}):\ j \le i\}} b^j_{\theta_j} f(\boldsymbol{\theta}_{-j}) x_j(\boldsymbol{\theta}) \qquad \forall\ i \in Q(\boldsymbol{\theta}), i \neq \iota(\boldsymbol{\theta}). \tag{40}$$

The proof is omitted, as it follows the same step as the proof of Lemma G.7. We highlight that the conditions stated in Proposition L.2 are necessary but need not to be sufficient.

Proposition L.3. *Consider the setting of Theorem H.1 and let* $(\boldsymbol{a}, \boldsymbol{b})$ *be a vector of coefficients satisfying Eq.* (20). *Then, for every* $\theta_k \in \tilde{\Theta}$, *we must have* $b^1_{\theta_k} + b^n_{\theta_k} + 2\sum_{i=2}^{n-1} b^i_{\theta_k} = 0$.

Proof. Consider the profile $\boldsymbol{\theta}$ such that agent i has cost $\theta_i = \theta_k$ for all $1 \le i \le n$. By Proposition 3.1, the optimal allocations are $x_1(\boldsymbol{\theta}) = x_n(\boldsymbol{\theta}) = \frac{1}{2(n-1)}$ and $x_i(\boldsymbol{\theta}) = \frac{1}{(n-1)}$ for $2 \le i \le n-1$. By equation (39) in Proposition L.2 we must have that

$$0 = \sum_{j \in Q(\boldsymbol{\theta})} b^j_{\theta_k} f(\boldsymbol{\theta}_{-j}) x_j(\boldsymbol{\theta}) = \sum_{j=1}^{n} b^j_{\theta_k} f(\theta_k)^{n-1} x_j(\boldsymbol{\theta}) = \frac{f(\theta_k)^{n-1}}{2(n-1)} \left(b^1_{\theta_k} + b^n_{\theta_k} + 2\sum_{i=2}^{n-1} b^i_{\theta_k} \right) \quad \square$$

Proposition L.4. *Consider the setting of Theorem H.1 and let* $(\boldsymbol{a}, \boldsymbol{b})$ *be a vector of coefficients satisfying equation* (20). *Let* $\overline{\theta}$ *be the highest value in* $\tilde{\Theta}$. *Then, for every* $\theta_k \in \tilde{\Theta}$ *such that* $v(\overline{\theta}) - v(\theta_k) \ge \frac{\delta}{n-1}$, *we have* $b^i_{\theta_k} = 0$ *for all* $1 \le i \le n$.

Proof. We show by induction in the agent number that $b^i_{\theta_k} = b^1_{\theta_k}$ for all $2 \le i \le n$. This fact, together with Proposition L.3 imply the result. The equality $b^i_{\theta_k} = b^1_{\theta_k}$ trivially holds for $i = 1$. Consider now the profile $\boldsymbol{\theta}$ in which $\theta_1 = \overline{\theta}$ and $\theta_j = \theta_k$ for all $j \neq 1$. As $v(\overline{\theta}) - v(\theta_k) \ge \frac{\delta}{n-1}$. By Proposition 3.1, the optimal allocations are defined as $x_1(\boldsymbol{\theta}) = 0$, $x_2(\boldsymbol{\theta}) = \frac{3}{2(n-1)} =$, $x_j(\boldsymbol{\theta}) = \frac{1}{n-1}$ for $2 \le j \le n-1$, and $x_n(\boldsymbol{\theta}) = \frac{1}{2(n-1)}$. We can then write $\sum_{j \in Q(\boldsymbol{\theta})} b^j_{\theta_k} f(\boldsymbol{\theta}_{-j}) x_j(\boldsymbol{\theta}) = \frac{f(\overline{\theta}) f(\theta_k)^{n-2}}{2(n-1)} \left(3b^2_{\theta_k} + 2\sum_{i=3}^{n-1} b^i_{\theta_k} + b^n_{\theta_k} \right)$. By Eq. (39), we must have $3b^2_{\theta_k} + 2\sum_{i=3}^{n-1} b^i_{\theta_k} + b^n_{\theta_k} = 0$. By Proposition L.3, we must have $b^1_{\theta_k} + b^n_{\theta_k} + 2\sum_{i=2}^{n-1} b^i_{\theta_k} = 0$. By considering the difference between both equalities, we obtain $b^2_{\theta_k} = b^1_{\theta_k}$.

Suppose that for $j \le i$ we have $b^j_{\theta_k} = b^1_{\theta_k}$. We now want to show it holds $b^{i+1}_{\theta_k} = b^1_{\theta_k}$ for $i+1 < n$. To that end, consider the profile $\boldsymbol{\theta}$ in which $\theta_i = \overline{\theta}$ and $\theta_j = \theta_k$ for all $j \neq i$, and define $(\theta_k)^n$ to be the profile in which every agent has cost θ_k. Recall that $x_j((\theta_k)^n) = \frac{1}{2(n-1)}$ if $j = 1$ or $j = n$ and $x_j((\theta_k)^n) = \frac{1}{(n-1)}$ otherwise. Then, compared to the profile $(\theta_k)^n$, all optimal allocations will remain the same except for those of agents $i-1$, i and $i+1$. While the optimal allocation of agent i will now be zero, the allocations of both $i-1$ and $i+1$ will increase by $\frac{1}{2(n-1)}$. By Eq. (39), we must have $0 = \sum_{j \in Q(\boldsymbol{\theta})} b^j_{\theta_k} f(\boldsymbol{\theta}_{-j}) x_j(\boldsymbol{\theta}) = f(\overline{\theta}) f(\theta_k)^{n-2} \left(\sum_{j \neq i} b^j_{\theta_k} x_j(\boldsymbol{\theta})\right)$. Therefore, $0 = \sum_{j \neq i} b^j_{\theta_k} x_j(\boldsymbol{\theta})$ which implies

$$0 = \frac{1}{2(n-1)} \left(b^1_{\theta_k} + b^n_{\theta_k} + \sum_{j=2}^{i-2} 2b^j_{\theta_k} + 3b^{i-1}_{\theta_k} + 3b^{i+1}_{\theta_k} \sum_{j=i+2}^{n-1} 2 + b^j_{\theta_k} \right).$$

By multiplying the above equation by $2(n-1)$ and considering difference between that equation and $b^1_{\theta_k} + b^n_{\theta_k} + 2\sum_{i=2}^{n-1} b^i_{\theta_k} = 0$ (given by Proposition L.3), we obtain $b^{i-1}_{\theta_k} - 2b^i_{\theta_k} + b^{i+1}_{\theta_k} = 0$. Therefore, by inductive hypothesis, we conclude $b^{i+1}_{\theta_k} = b^1_{\theta_k}$. Finally, the case $i+1 = n$ follows by the same arguments as before, with the only difference that the coefficient of $b^{i+1}_{\theta_k}$ in the main equation of the proof will be 2 instead of three. □

We can now proceed to prove Theorem H.1 by dividing it into the following two (disjoint) Lemmas.

Lemma L.2. *Consider the setting of Theorem H.1. and let $\underline{\theta}$ and $\overline{\theta}$ be respectively the lowest and highest values in $\tilde{\Theta}$. If $v(\overline{\theta}) - v(\underline{\theta}) \leq \frac{\delta}{n-1}$, then $OPT(DecLin) = OPT(Cent)$.*

Proof. Let us start by highlighting that, although costs are IID, the agents are not (ex-ante) symmetric as they have different locations. In this particular setting $Q(\boldsymbol{\theta}) = \{1, \ldots, n\}$ for every cost profile $\boldsymbol{\theta} \in \Theta$, so players 1 and n are ex-ante symmetric, and all agents i, j such that $2 \leq i, j \leq n-1$ are ex-ante symmetric as well. This implies that, for every $\theta_k \in \tilde{\Theta}$, we have $X_1(\theta_k) = X_n(\theta_k)$ and $X_i(\theta_k) = X_j(\theta_k)$ for all $2 \leq i, j \leq n-1$. We now show that $X_1(\theta_k) = 2X_2(\theta_k)$. By definition:

$$X_1(\theta_k) = \sum_{\theta_{-1} \in \Theta_{-1}} f(\boldsymbol{\theta}_{-1}) x_1(\theta_k, \boldsymbol{\theta}_{-1}) = \sum_{\theta_q \in \tilde{\Theta}} f(\theta_q) x_1(\theta_k, \theta_q),$$

where the last equality follows from the fact that all agents are always active, hence the allocation of agent 1 depends only on his own cost and the cost of agent 2. Similarly, the demand for agent 2 only depends on his cost and the costs of the adjacent agents. Let $x^l_i(\boldsymbol{\theta})$ (resp. $x^r_i(\boldsymbol{\theta})$) denote the demand that i experiences to his left (resp. his right) in the $[0,1]$ segment. Then, we have:

$$X_2(\theta_k) = \sum_{\theta_q \tilde{\Theta}} \sum_{\theta_p \in \tilde{\Theta}} f(\theta_q) f(\theta_p) \left(x^l_2(\theta_q, \theta_k) + x^r_2(\theta_k, \theta_p) \right) = 2 \sum_{\theta_q} f(\theta_q) x_1(\theta_k, \theta_q),$$

where the last equality follows from the fact that $x^l_2(\theta_q, \theta_k) = x^r_1(\theta_k, \theta_q)$ and $x^r_2(\theta_k, \theta_p) = x^l_2(\theta_p, \theta_k)$.

Let $(\boldsymbol{a}, \boldsymbol{b})$ be a vector of coefficients satisfying the equality in Eq. (20). We want to show that the equality in Eq. (21) is also satisfied. To do so, we first show that

$$\sum_{i \in N} \sum_{\theta^j_i \in \Theta_i} b^i_{\theta^j_i} \left(\theta^j_i X_i(\theta^j_i) + \sum_{k=j+1}^{|\Theta_i|} (\theta^k_i - \theta^{k-1}_i) X_i(\theta^k_i) \right) = 0.$$

In particular, we show that for every $\theta_k \in \tilde{\Theta}$, the sum over all $i \in N$ of the RHS of the equations corresponding to those in $T_i(\theta_k)$ is zero. By our previous result, we have that $X_i(\theta^k) = 2X_1(\theta^k)$ for all $2 \leq i \leq n$ and $X_n(\theta^k) = X_1(\theta^k)$. Hence,

$$\begin{aligned}
\sum_{i=1}^{n} b^i_{\theta_k}\left(\theta^k X_i(\theta^k) + \sum_{j=k+1}^{m}(\theta^j - \theta^{j-1})X_i(\theta^j)\right) &= b^1_{\theta_k}\left(\theta^k X_1(\theta^k) + \sum_{j=k+1}^{m}(\theta^j - \theta^{j-1})X_1(\theta^j)\right) + \\
&\quad b^n_{\theta_k}\left(\theta^k X_1(\theta^k) + \sum_{j=k+1}^{m}(\theta^j - \theta^{j-1})X_1(\theta^j)\right) + \\
&\quad \sum_{i=2}^{n-1} b^i_{\theta_k}\left(\theta^k 2X_1(\theta^k) + \sum_{j=k+1}^{m}(\theta^j - \theta^{j-1})2X_1(\theta^j)\right) \\
&= \left(\theta^k X_1(\theta^k) + \sum_{j=k+1}^{m}(\theta^j - \theta^{j-1})X_1(\theta^j)\right)\left(b^1_{\theta_k} + \sum_{i=2}^{n-1} 2b^i_{\theta_k} + b^n_{\theta_k}\right) \\
&= 0
\end{aligned}$$

where the last equality follows from Proposition L.3. As we have shown that, for every $\theta_k \in \tilde{\Theta}$, the sum over all $i \in N$ of the RHS of the equations corresponding to those in $T_i(\theta_k)$ is zero we have:

$$\sum_{i \in N} \sum_{\theta_i^j \in \Theta_i} b^i_{\theta_i^j}\left(\theta_i^j X_i(\theta_i^j) + \sum_{k=j+1}^{|\Theta_i|}(\theta_i^k - \theta_i^{k-1})X_i(\theta_i^k)\right) = 0.$$

To complete the proof, we must show that $\sum_{\boldsymbol{\theta} \in \Theta} \sum_{\substack{i \in Q(\boldsymbol{\theta}) \\ i \neq \iota(Q(\boldsymbol{\theta}))}} a^i_{\boldsymbol{\theta}}\left(v_{\vartheta_{\boldsymbol{\theta}}(i)}(\theta_{\vartheta_{\boldsymbol{\theta}}(i)}) - v_i(\theta_i)\right)$. As in our case we have $Q(\boldsymbol{\theta}) = \{1, \ldots, n\}$ for all $\boldsymbol{\theta} \in \Theta$, this reduces to show that $\sum_{\boldsymbol{\theta} \in \Theta} \sum_{i<n} a^i_{\boldsymbol{\theta}}\left(v(\theta_{i+1}) - v(\theta_i)\right) = 0$. We do that next.

$$
\begin{aligned}
\sum_{\boldsymbol{\theta}\in\Theta}\left(\sum_{i=1}^{n-1} a^i_{\boldsymbol{\theta}}\left(v(\theta_{i+1})-v(\theta_i)\right)\right) &= \sum_{\boldsymbol{\theta}\in\Theta}\left(\sum_{i=1}^{n-1}\left(\sum_{j=1}^{i} b^j_{\theta_j} f(\boldsymbol{\theta}_{-j})x_j(\boldsymbol{\theta})\right)\left(v(\theta_{i+1})-v(\theta_i)\right)\right)\\
&= \sum_{\boldsymbol{\theta}\in\Theta}\left(v(\theta_n)\left(\sum_{j=1}^{n-1} b^j_{\theta_j} f(\boldsymbol{\theta}_{-j})x_j(\boldsymbol{\theta})\right) - \sum_{i=1}^{n-1} v(\theta_i) b^i_{\theta_i} f(\boldsymbol{\theta}_{-i})x_i(\boldsymbol{\theta})\right)\\
&= -\sum_{\boldsymbol{\theta}\in\Theta}\sum_{i=1}^{n} v(\theta_i) b^i_{\theta_i} f(\boldsymbol{\theta}_{-i})x_i(\boldsymbol{\theta})\\
&= -\sum_{\theta_k\in\Theta}\sum_{i=1}^{n} v(\theta_k) b^i_{\theta_k}\left(\sum_{\theta_{-i}\in\Theta} f(\boldsymbol{\theta}_{-i})x_i(\theta_k,\theta_{-i})\right)\\
&= -\sum_{\theta_k\in\Theta} v(\theta_k)\left(\sum_{i=1}^{n} b^i_{\theta_k} X_i(\theta_k)\right)\\
&= -\sum_{\theta_k\in\Theta} v(\theta_k) X_1(\theta_k)\left(b^1_{\theta_k} + b^n_{\theta_k} + \sum_{i=2}^{n-1} 2 b^i_{\theta_k}\right)\\
&= 0.
\end{aligned}
$$

where the first and third equalities follow from the definitions of $a^i_{\boldsymbol{\theta}}$ and $b^n_{\boldsymbol{\theta}}$ given in Proposition L.2 respectively; and the second to last equality follows from the fact that $2X_1(\theta_k) = X_2(\theta_k)$. Finally, the last equality follows from Proposition L.3. Therefore, we have show that every vector of coefficients $(\boldsymbol{a}, \boldsymbol{b})$ that satisfies the equality in Eq. (20), must also satisfy the equality in Eq. (21). Hence, the system of linear equations is consistent and $OPT(P_0) = OPT(P_1)$ as desired. □

Lemma L.3. *Consider the setting of Theorem H.1. and let $\underline{\theta}$ and $\overline{\theta}$ be defined as the lowest and highest values in $\tilde{\Theta}$ respectively. If $v(\overline{\theta}) - v(\underline{\theta}) \geq \frac{\delta}{n-1}$, then $OPT(DecLin) = OPT(Cent)$.*

Proof. Let $(\boldsymbol{a}, \boldsymbol{b})$ be a vector of coefficients satisfying the equality in Eq. (20). As usual, we want to show that $(\boldsymbol{a}, \boldsymbol{b})$ also satisfies the equality in Eq. (21). Let $\theta_k = \text{argmax}\{\theta_j \in \tilde{\Theta} : \ v(\overline{\theta}) - v(\theta_j) \geq \frac{\delta}{n-1}\}$. As $v(\overline{\theta}) - v(\underline{\theta}) \geq \frac{\delta}{n-1}$, such θ_k must exist. By Proposition L.4, we have that $b^i_{\theta_j} = 0$ for every agent $i \in N$ and for all $\theta_j \in \tilde{\Theta}$ such that $\theta_j \leq \theta_k$. We shall prove the result by considering three separate cases: $v(\theta_{k+1}) - v(\theta_k) \geq \delta$, $v(\theta_{k+1}) - v(\theta_k) < \frac{\delta}{2}$ and $\frac{\delta}{2} \leq v(\theta_{k+1}) - v(\theta_k) < \delta$.

Case $\mathbf{v}(\theta_{\mathbf{k+1}}) - \mathbf{v}(\theta_{\mathbf{k}}) \geq \delta$: Then, an agent with cost θ_{k+1} can be active only if all other agents have costs at least θ_{k+1}. Further, by the definition of θ_k, we know that $v(\overline{\theta}) - v(\theta_{k+1}) < \frac{\delta}{n-1}$ which implies that an agent with cost θ_{k+1} will be active in all profiles consisting of costs greater or equal than θ_{k+1}. Therefore, an agent with cost θ_{k+1} can be active only if all other agents have costs at least θ_{k+1}. By mimicking the proof of Lemma L.2, we can show that $X_i(\theta_j) = 2X_1(\theta_j) = 2X_n(\theta_j)$ for all $2 \leq i \leq n-1$ and all $\theta_j \geq \theta_{k+1}$, and that $b^i_{\theta_j} = 0$ for all $i \in N$ and for all $\theta_j \in \tilde{\Theta}$ such that $\theta_j > \theta_k$. Hence, we have show that $b^i_{\theta_j} = 0$ for all $i \in N$ and for all $\theta_j \in \tilde{\Theta}$. By Proposition L.2, this implies that $a^i_{\boldsymbol{\theta}} = 0$ for every $\boldsymbol{\theta} \in \Theta$ and every $i \in Q(\boldsymbol{\theta})$. Therefore, $(\boldsymbol{a}, \boldsymbol{b}) = 0$ and the result holds.

Case $\mathbf{v}(\theta_{\mathbf{k+1}}) - \mathbf{v}(\theta_{\mathbf{k}}) < \frac{\delta}{\mathbf{2}}$: . Again, we show that the system of equations is consistent by showing that $b^i_{\theta_j} = 0$ for all agent i and all $\theta_j \in \tilde{\Theta}$. By Proposition L.4, this holds for all $\theta_j \leq \theta_k$. By induction in i, we next show that $b^i_{\theta_{k+1}}$ for every $i \in N$. By symmetry, it suffices to show it for every $i \leq \frac{n}{2} + 1$.

Let $q \in \mathbb{N}_0$ be such that $q\frac{\delta}{(n-1)} \leq v(\theta_{k+1}) - v(\theta_k) < (q+1)\frac{\delta}{(n-1)}$. Consider the profile $\boldsymbol{\theta}$ in which $\theta_1 = \theta_{k+1}$, $\theta_i = \theta_{k+1}$ for the next q agents (that is, for $2 \leq i \leq q+1$) and $\theta_i = \theta_k$ for the remaining agents. As $q\frac{\delta}{(n-1)} \leq v(\theta_{k+1}) - v(\theta_k)$, any agent i with cost θ_{k+1} will be active only if every agent j such that $|i-j| \leq q$ has cost θ_{k+1} as well. Further, as $v(\theta_{k+1}) - v(\theta_k) < (q+1)\frac{\delta}{(n-1)}$, if every agent j such that $|i-j| \leq q$ has cost θ_{k+1}, agent i will be active. Therefore, the definition of q implies $Q(\boldsymbol{\theta}) = \{1\} \cup \{i : \ q+1 < i \leq n\}$. By Proposition L.2, we have

$$0 \ = \sum_{j \in Q(\boldsymbol{\theta})} b^j_{\theta_j} f(\boldsymbol{\theta}_{-j}) x_j(\boldsymbol{\theta}) \ = \ b^1_{\theta_{k+1}} f(\theta_{-1}) x_1(\boldsymbol{\theta}) + \sum_{q+1<j\leq n} b^j_{\theta_k} f(\boldsymbol{\theta}_{-j}) x_j(\boldsymbol{\theta}) \ = \ b^1_{\theta_{k+1}} f(\theta_{-1}) x_1(\boldsymbol{\theta}).$$

Therefore, we must have $b^1_{\theta_{k+1}} = 0$, which establishes the base case.

Now suppose that $b^j_{\theta_{k+1}} = 0$ for all $j \leq i$. We show that $b^{i+1}_{\theta_{k+1}} = 0$ it holds for $i+1 \leq \frac{n}{2}$ as well. By hypothesis, $v(\theta_{k+1}) - v(\theta_k) < \frac{\delta}{2}$. By the definition of q, we must have $2q < (n-1)$ and therefore $q+1 \leq n-(i+1)$. Hence, the profile $\boldsymbol{\theta}$ in which $\theta_j = \theta_{k+1}$ for all $j \leq i+1+q$ and $\theta_j = \theta_k$ for all $j > i+1+q$ is well defined. By the same arguments as before, $Q(\boldsymbol{\theta}) = \{1, \ldots i+1\} \cup \{i+1+q+1, \ldots, n\}$. Then, by Proposition L.2 we have

$$0 \ = \sum_{j \in Q(\boldsymbol{\theta})} b^j_{\theta_j} f(\boldsymbol{\theta}_{-j}) x_j(\boldsymbol{\theta}) \ = \ \sum_{j=1}^{i+1} b^j_{\theta_{k+1}} f(\boldsymbol{\theta}_{-j}) x_j(\boldsymbol{\theta}) + \sum_{q+i+1<j\leq n} b^j_{\theta_k} f(\boldsymbol{\theta}_{-j}) x_j(\boldsymbol{\theta}) \ = \ b^{i+1}_{\theta_{k+1}} f(\boldsymbol{\theta}_{-1}) x_{i+1}(\boldsymbol{\theta}),$$

where the last equality follows from the inductive hypothesis. Then, $b^{i+1}_{\theta_{k+1}} = 0$ and our proof yields $b^j_{\theta_{k+1}} = 0$ for every $j \in N$.

To conclude the proof for this case, we show that $b^i_{\theta_q} = 0$ for each i and each θ_q with $q > k+1$. To do so, construct a profile $\boldsymbol{\theta}$ in which $\theta_i = \theta_j$ and $\theta_l = \theta_{k+1}$ for every agent $l \in N$ with $l \neq i$. Note that $v(\theta_j) - v(\theta_{k+1}) \leq v(\overline{\theta}) - v(\theta_{k+1}) < \frac{\delta}{n-1}$, where the last inequality follows from definition of θ_k. Thus, all agents are active in $\boldsymbol{\theta}$. By Proposition L.2:

$$0 \ = \sum_{l \in Q(\boldsymbol{\theta})} b^l_{\theta_l} f(\boldsymbol{\theta}_{-l}) x_l(\boldsymbol{\theta}) \ = \ b^i_{\theta_j} f(\boldsymbol{\theta}_{-i}) x_i(\boldsymbol{\theta}) + \sum_{l \neq i} b^l_{\theta_{k+1}} f(\boldsymbol{\theta}_{-l}) x_l(\boldsymbol{\theta}) \ = \ b^i_{\theta_j} f(\boldsymbol{\theta}_{-i}) x_i(\boldsymbol{\theta}),$$

and $b^i_{\theta_j} = 0$ as desired. So far, we have shown that $b^i_{\theta_j} = 0$ for all $i \in N$ and for all $\theta_j \in \tilde{\Theta}$. By Proposition L.2, we must have $a^i_{\boldsymbol{\theta}} = 0$ for every $\boldsymbol{\theta} \in \Theta$ and every $i \in Q(\boldsymbol{\theta})$. Hence, $(\boldsymbol{a}, \boldsymbol{b}) = 0$ and the result follows.

Case $\frac{\delta}{2} \leq \mathbf{v}(\theta_{\mathbf{k+1}}) - \mathbf{v}(\theta_{\mathbf{k}}) < \delta$: We know that $b^i_{\theta_j} = 0$ for all agents $i \in N$ and all $\theta_j \in \tilde{\Theta}$ such that $j \leq k$. We now see what happens for θ_j with $j \geq k+1$. For each $j \geq k+1$, let $q(j) \in \mathbb{N}_0$ be such that $q(j)\frac{\delta}{(n-1)} \leq v(\theta_j) - v(\theta_k) < (q(j)+1)\frac{\delta}{(n-1)}$. Consider the partition of N into $\{N_1(j), N_2(j)\}$ defined as follows: $N_1(j) = \{1, \ldots, n-q(j)-1\} \cup \{q(j)+2, \ldots, n\}$ and $N_2(j) = \{n-q(j), \ldots, q(j)+1\}$. We first show that $b^i_{\theta_j} = 0$ for all agent $i \in N_1(j)$ and all $j \geq k+1$.

If $N_1(j) = \emptyset$, the claim vacuously holds. Therefore, we may assume that $N_1 \neq \emptyset$ and hence $\{1, n\} \subseteq N_1(j)$. Consider the profile $\boldsymbol{\theta}$ in which $\theta_1 = \theta_j$, $\theta_i = \theta_j$ for the next $q(j)$ agents (that is, for $2 \leq i \leq q(j)+1$) and $\theta_i = \theta_k$ for the remaining agents. By using the same arguments as in the previous case, we can show that $b^1_{\theta_j} = 0$. Once this has been established, we can use the same inductive argument to show that $b^i_{\theta_j} = 0$ holds for every $i \in N_1(j)$ such that $i \leq n - (q(j)+1)$. Furthermore, by symmetry, we must also have $b^i_{\theta_j} = 0$ for every $i \in N_1(j)$ such that $i > q(j)+1$. Therefore, $b^i_{\theta_j} = 0$ for all agent $i \in N_1(j)$.

Next, we show that for every $i, i' \in N_2(j)$ we must have $X_i(\theta_j) = X_{i'}(\theta_j)$ if $N_1(j) \neq \emptyset$ or $2X_1(\theta_j) = 2X_n(\theta_j) = X_i(\theta_j)$ for every $i \in N$ such that $2 \leq i \leq n-1$ if $N_1(j) = \emptyset$. To that end, note that whenever the cost of $i \in N_2$ is θ_j, he can only be active if all other agents have cost at least θ_{k+1}. Moreover, by definition of θ_k we have $v(\overline{\theta}) - v(\theta_{k+1}) < \frac{\delta}{n-1}$. Hence, in any profile $\boldsymbol{\theta}$ in which i is active, we must have $Q(\boldsymbol{\theta}) = N$. This implies that for every $i \in N_2$ and any $\boldsymbol{\theta} \in \Theta$, we have either $x_i(\boldsymbol{\theta}) = 0$, or $x_i(\boldsymbol{\theta})$ only depends on the values $\theta_{i-1}, \theta_i, \theta_{i+1}$, (in the cases of $i = 1$ or $i = n$, this reduces to only the costs of i and that of his immediate neighbor). By mimicking the proof of Lemma L.2, we can easily show that $X_i(\theta_j) = X_{i'}(\theta_j)$ every $i, i' \in N_2(j)$ when $N_1(j) \neq \emptyset$ or $2X_1(\theta_j) = 2X_n(\theta_j) = X_i(\theta_j)$ for every $i \in N$ such that $2 \leq i \leq n-1$ when $N_1(j) = \emptyset$.

Let $J_1 = \{j \geq k+1 : \; N_1(j) \neq \emptyset\}$ and $J_2 = \{j \geq k+1 : \; N_1(j) \neq \emptyset\}$. As the virtual costs are increasing, we have that $N_1(j') \subseteq N_1(j)$ whenever $j' \geq j$. Hence, J_1, J_2 can be seen as a partition of the indices j with $k+1 \geq j \leq |\tilde{\Theta}|$ with the property that $j_1 \leq j_2$ for every $j_1 \in J_1$, $j_2 \in J_2$. We highlight that, by hypothesis, $J_1 \neq \emptyset$, but J_2 might be empty. For $j \in J_1$, let $(\theta_j)^n$ be the profile in the cost of every agent is θ_j. By Proposition L.3, we must have $b^1_{\theta_j} + b^n_{\theta_j} + 2\sum_{i=2}^{n-1} b^i_{\theta_j} = 0$, and therefore $\sum_{i \in N_2(j)} b^i_{\theta_j} = 0$. Similarly, for $j \in J_2$ we have $b^1_{\theta_j} + b^n_{\theta_j} + 2\sum_{i=2}^{n-1} b^i_{\theta_j} = 0$.

We now show that for every $\theta_j \in \tilde{\Theta}$, the sum over all $i \in N$ of the RHS of the equations corresponding to those in $T_i(\theta_j)$ is zero, that is,

$$\sum_{i=1}^{n} b^i_{\theta_j} \left(\theta^j X_i(\theta^j) + \sum_{j'=j}^{|\tilde{\Theta}|} (\theta^{j'} - \theta^{j'-1}) X_i(\theta^{j'}) \right) = 0.$$

If $j \leq k$, we have $b^i_{\theta_j} = 0$ for all $i \in N$ and the claim follows. Otherwise, we have only those $i \in N_2(j)$ can have positive coefficients. Therefore, the above equation reduces to

$$\sum_{i \in N_2(j)} b^i_{\theta_j} \left(\theta^j X_i(\theta^j) + \sum_{j'=j}^{|\tilde{\Theta}|} (\theta^{j'} - \theta^{j'-1}) X_i(\theta^{j'}) \right) = 0.$$

Suppose $j \in J_1$. We have shown that $X_i(\theta^j) = X_{i'}(\theta^j)$ for every $i, i' \in N_2(j)$ and $j \in J_1$.

As $N_2(j) \subseteq N_2(j')$ for every $j' \geq j$, we must have $X_i(\theta^{j'}) = X_{i'}(\theta^{j'})$. In addition, we know that $\sum_{i \in N_2(j)} b^i_{\theta_j} = 0$. Hence, if $j \in J_1$,

$$\sum_{i \in N_2(j)} b^i_{\theta_j} \left(\theta^j X_i(\theta^j) + \sum_{j'=j}^{|\tilde{\Theta}|} (\theta^{j'} - \theta^{j'-1}) X_i(\theta^{j'})\right) =$$

$$\left(\sum_{i \in N_2(j)} b^i_{\theta_j}\right) \left(\theta^j X_i(\theta^j) + \sum_{j'=j}^{|\tilde{\Theta}|} (\theta^{j'} - \theta^{j'-1}) X_i(\theta^{j'})\right) = 0,$$

as desired. Similarly, if $j \in J_2$, $2X_1(\theta_j) = 2X_n(\theta_j) = X_i(\theta_j)$ for every $i \in N$ such that $2 \leq i \leq n-1$ and this must hold for every $j' \geq j$. In addition, $b^1_{\theta_j} + b^n_{\theta_j} + 2\sum_{i=2}^{n-1} b^i_{\theta_j} = 0$. Therefore,

$$\sum_{i \in N} b^i_{\theta_j} \left(\theta^j X_i(\theta^j) + \sum_{j'=j}^{|\tilde{\Theta}|} (\theta^{j'} - \theta^{j'-1}) X_i(\theta^{j'})\right) =$$

$$= \left(b^1_{\theta_j} + b^n_{\theta_j} + 2\sum_{i=2}^{n-1} b^i_{\theta_j}\right) \left(\theta^j X_1(\theta^j) + \sum_{j'=j}^{|\tilde{\Theta}|} (\theta^{j'} - \theta^{j'-1}) X_1(\theta^{j'})\right) = 0$$

So far we have shown that, for every $\theta_j \in \tilde{\Theta}$, the sum over all $i \in N$ of the RHS of the equations corresponding to those in $T_i(\theta_j)$ is zero and therefore:

$$\sum_{i \in N} \sum_{\theta_i^j \in \Theta_i} b^i_{\theta_i^j} \left(\theta_i^j X_i(\theta_i^j) + \sum_{k=j+1}^{|\Theta_i|} (\theta_i^k - \theta_i^{k-1}) X_i(\theta_i^k)\right) = 0.$$

To complete the proof, we must show that $\sum_{\boldsymbol{\theta} \in \Theta} \sum_{\substack{i \in Q(\boldsymbol{\theta}) \\ i \neq \iota(Q(\boldsymbol{\theta}))}} a^i_{\boldsymbol{\theta}} \left(v_{\vartheta_{\boldsymbol{\theta}}(i)}(\theta_{\vartheta_{\boldsymbol{\theta}}(i)}) - v_i(\theta_i)\right)$.

$$\begin{aligned}
\sum_{\boldsymbol{\theta} \in \Theta} \left(\sum_{\substack{i \in Q(\boldsymbol{\theta}) \\ i \neq \iota(\boldsymbol{\theta})}} a^i_{\boldsymbol{\theta}} \left(v_{\vartheta_{\boldsymbol{\theta}}(i)}(\theta_{\vartheta_{\boldsymbol{\theta}}(i)}) - v_i(\theta_i)\right)\right) &= \sum_{\boldsymbol{\theta} \in \Theta} \left(\sum_{\substack{i \in Q(\boldsymbol{\theta}) \\ i \neq \iota(\boldsymbol{\theta})}} \left(\sum_{\substack{j \in Q(\boldsymbol{\theta}) \\ j \leq i}} b^j_{\theta_j} f(\boldsymbol{\theta}_{-j}) x_j(\boldsymbol{\theta})\right) \left(v_{\vartheta_{\boldsymbol{\theta}}(i)}(\theta_{\vartheta_{\boldsymbol{\theta}}(i)}) - v_i(\theta_i)\right)\right) \\
&= \sum_{\boldsymbol{\theta} \in \Theta} \left(v(\theta_{\iota(\boldsymbol{\theta})}) \left(\sum_{\substack{i \in Q(\boldsymbol{\theta}) \\ i \neq \iota(\boldsymbol{\theta})}} b^j_{\theta_j} f(\boldsymbol{\theta}_{-j}) x_j(\boldsymbol{\theta})\right) - \sum_{\substack{i \in Q(\boldsymbol{\theta}) \\ i \neq \iota(\boldsymbol{\theta})}} v(\theta_i) b^i_{\theta_i} f(\boldsymbol{\theta}_{-i}) x_i(\boldsymbol{\theta})\right) \\
&= -\sum_{\boldsymbol{\theta} \in \Theta} \sum_{i \in Q(\boldsymbol{\theta})} v(\theta_i) b^i_{\theta_i} f(\boldsymbol{\theta}_{-i}) x_i(\boldsymbol{\theta}) \\
&= -\sum_{\boldsymbol{\theta} \in \Theta} \sum_{i=1}^{n} v(\theta_i) b^i_{\theta_i} f(\boldsymbol{\theta}_{-i}) x_i(\boldsymbol{\theta})
\end{aligned}$$

where the first and third equalities follow from the definitions of $a^i_{\boldsymbol{\theta}}$ and b^i_{θ} given in Proposition L.2 respectively, and the last equality follows from the fact that $x_i(\boldsymbol{\theta}) = 0$ if $i \notin Q(\boldsymbol{\theta})$. Therefore, we just need to show that $\sum_{\boldsymbol{\theta} \in \Theta} \sum_{i=1}^{n} v(\theta_i) b^i_{\theta_i} f(\boldsymbol{\theta}_{-i}) x_i(\boldsymbol{\theta}) = 0$. To that end, note that $b^i_{\theta_j} \neq 0$ only if $i \in N_2(j)$. Then,

$$
\begin{aligned}
\sum_{\boldsymbol{\theta}\in\Theta}\sum_{i=1}^{n} v(\theta_i) b^i_{\theta_i} f(\boldsymbol{\theta}_{-i}) x_i(\boldsymbol{\theta}) &= \sum_{\theta_k\in\tilde{\Theta}}\sum_{i=1}^{n} v(\theta_k) b^i_{\theta_k}\left(\sum_{\theta_{-i}\in\Theta} f(\boldsymbol{\theta}_{-i}) x_i(\theta_k,\theta_{-i})\right) \\
&= \sum_{j\geq k+1}\sum_{i\in N_2(j)} v(\theta_j) b^i_{\theta_j}\left(\sum_{\theta_{-i}\in\Theta} f(\boldsymbol{\theta}_{-i}) x_i(\theta_j,\theta_{-i})\right) \\
&= \sum_{j\in J(1)} v(\theta_j)\left(\sum_{i\in N_2(j)} b^i_{\theta_j} X_i(\theta_j)\right) + \sum_{j\in J(2)} v(\theta_j)\left(\sum_{i=1}^{n} b^i_{\theta_j} X_i(\theta_j)\right) \\
&= \sum_{j\in J(1)} v(\theta_j) X_{\frac{n}{2}}(\theta_j)\left(\sum_{i\in N_2(j)} b^i_{\theta_j}\right) + \sum_{j\in J(2)} v(\theta_j) X_1(\theta_j)\left(b^1_{\theta_j} + b^n_{\theta_j} + \sum_{i=2}^{n-1} 2b^i_{\theta_j}\right) \\
&= 0.
\end{aligned}
$$

Therefore, we have show that every vector of coefficients $(\boldsymbol{a},\boldsymbol{b})$ that satisfies Eq. (20), must also satisfy (21). Thus, the system of linear equations is consistent and $OPT(DecLin) = OPT(Cent)$ as desired. □