# Information Percolation in Segmented Markets*

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**Abstract**

We calculate equilibria of dynamic double-auction markets in which agents are distinguished by their preferences and information. Over time, agents are privately informed by bids and offers. Investors are segmented into groups that differ with respect to characteristics determining information quality, including initial information precision as well as market "connectivity," the expected frequency of their trading opportunities. Investors with superior information sources attain higher expected profits, provided their counterparties are unable to observe the quality of those sources. If, however, the quality of bidders' information sources are commonly observable, then, under conditions, investors with superior information sources have lower expected profits.

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## 1 Introduction

We calculate equilibria of dynamic double-auction markets in which agents are distinguished by their preferences and information. As in an opaque over-the-counter market, agents gather information over time from the bids and offers of their counterparties. We characterize the effect of segmentation of investors into groups that differ by their initial information endowment or by their "connectivity," which depends on the expected frequency with which they trade with other investors, and the quality of the information they obtain through their counterparties' bids. More informed and better connected agents attain higher expected future profits, provided they are able to disguise the characteristics determining the quality of their information. If, however, the characteristics determining the quality of information of the bidders are commonly observable, then investors that are better connected or have better initial information quality can attain lower expected future trading profits, under stated conditions.

We model $N$ classes of agents that are distinguished by their preferences for the asset to be auctioned, by the expected rates at which they have trading opportunities with each of other classes of agents, and by the quality of their initial information about a random variable $Y$, which determines the ultimate utilities of the agents for the asset. Over time, a particular agent of class $i$ meets other agents at a sequence of Poisson arrival times with mean arrival rate $\lambda_i$. At each meeting, a counterparty of class-$j$ is selected with probability $\kappa_{ij}$. The two agents are given the opportunity to trade one unit of the asset in a double auction.

Based on their initial information and on the information gathered from bids in prior auctions with other agents, the two agents typically assign different conditional expectations to $Y$. Because the preference parameters are commonly observed by the two agents participating in the auction, it is common knowledge which of the two agents is the prospective buyer and which is the prospective seller. Trade occurs on the event that the price $\beta$ bid by the buyer is above the seller's offer price $\sigma$, in which case the buyer pays $\sigma$ to the seller. This double-auction format is known as the "seller's price auction."

We provide technical conditions under which the double auctions have a unique equilibrium in undominated strategies. We show how to compute the offer price $\sigma$ and the bid price $\beta$, state by state, by solving an ordinary differential equation. These prices are strictly monotonically decreasing with respect to the seller's and buyer's conditional expectations of $Y$, respectively. The bids therefore reveal these conditional expectations, which are then used to update priors for purposes of subsequent auctions. The technical conditions that we impose in order to guarantee the existence of such an equilibrium also imply that this particular equilibrium uniquely maximizes expected gains from trade in each auction and, consequently, total welfare.

Because our strictly monotone double-auction equilibrium fully reveals the bidders' conditional beliefs for $Y$, we are able to explicitly calculate the evolution over time of the cross-sectional distribution of posterior beliefs of the population of agents, by extending the results of Duffie and Manso (2007) and Duffie, Giroux, and Manso (2008) to $N$ classes of investors. We can calculate the Fourier transforms of the cross-sectional distributions of posterior beliefs of investors in each of the $N$ different classes at each time $t$ as the solution of a $N$-dimensional Riccati ordinary differential equation in $t$. We can solve this equation and then invert the transforms. In order to characterize the solutions, we also extend the Wild summation method of Duffie, Giroux, and Manso (2008) to directly solve the evolution equation for the cross-sectional distribution of beliefs.

The double-auction equilibrium characterization, together with the characterization of the dynamics of the cross-sectional distribution of posterior beliefs of each class of agents, permits a calculation of the expected lifetime utility of each class of agent, including the manner in which utility depends on the class characteristics determining information quality, namely the precision of the initial information endowment and the connectivity of that agent. Whether an agent profits from better information quality is shown to depend on whether auction counterparties are able to pin down the quality of that agent's sources of information. Under specified conditions, well informed agents may prefer that the quality of their information be less precisely determined. An implication is that investors in over-the-counter markets that trade more actively (thus gathering more information from counterparty bids and offers) or have better fundamental research, may prefer to obscure the quality of their information in order to avoid the impact of adverse selection. By doing so, they may increase the probability that they can execute a trade, or better the price execution of their trades. For example, a highly informed investor might prefer to trade anonymously through a proxy, such as a broker, even at a fee. (We do not, however, model proxy trading.)

Although an agent commonly known to have superior information is in some cases punished by discriminatory quotes due to adverse selection, to the point of lowering the agent's expected profits, we also show (under technical conditions) that if gains from trade are sufficiently large, then superior information leads to higher expected profits. That is, under some circumstances, investors with superior information quality prefer to have the characteristics determining the quality of their information sources publicly observable.

Finally, we investigate whether investors with similar preference parameters are influenced to engage in trading with each other in order to gather information that benefits their expected profits from future trading opportunities with other investors. For example, in functioning over-the-counter markets such as those for government bonds, the informational advantage of participating in more trades is sometimes said to be sufficient to cause dealers to narrow quoted bid-ask spreads in order to increase counterparty contacts. We analyze a stylized example of trading that is motivated by informational gains, and is undertaken even by pairs of counterparties with similar preferences for the asset. Although this does not contradict the No-Trade Theorem of Milgrom and Stokey (1982), the intuition runs in the opposite direction: In an over-the-counter market, trade between asymmetrically informed investors that is not based not on gains from exchange of the asset can occur in order to gather information that increases expected profits from future private trading opportunities.

## 2 Related Literature

A large literature in economics and finance addresses learning from market prices of transactions that take place in centralized exchanges.[1] Less attention, however, is given to information transmission in over-the-counter markets. Private information sharing is typical in functioning over-the-counter markets for many types of financial assets, including bonds and derivatives. In these markets, trades occur at private meetings in which counterparties offer prices that reveal information to each other, but not to other market participants.

Wolinsky (1990), Blouin and Serrano (2001), Duffie and Manso (2007), Golosov, Lorenzoni, and Tsyvinski (2008), Duffie, Giroux, and Manso (2008), and Duffie, Malamud, and Manso (2009a,b) are among the few studies that have investigated the issue of learning in over-the-counter markets. The models of search and random matching used in these studies are unsuitable for the analysis of the effects of segmentation of investors into groups that differ by connectivity and initial information quality. In the current paper, we are able to study these effects, as our model allows for $N$ classes of investors with different preferences, initial information quality, and connectivity.

[1] See, for example, Grossman (1976), Grossman and Stiglitz (1980), Wilson (1977), Milgrom (1981), Pesendorfer and Swinkels (1997), and Reny and Perry (2006).

In our model, whenever two agents meet, they have the opportunity to participate in a double auction. Chatterjee and Samuelson (1983) are among the first to study double auctions. The case of independent private values has been extensively analyzed by Williams (1987), Satterthwaite and Williams (1989), and Leininger, Linhart, and Radner (1989). Kadan (2007) studies the case of correlated private values. We extend these previous studies by providing conditions for the existence of a unique strictly monotone equilibrium in undominated strategies of a double auction with common values. Bid monotonicity is natural in our setting given the strict monotone dependence on $Y$ of each agent's ex-post utility for a unit of the asset. Strictly monotone equilibria are not typically available, however, in more general double auctions with a common value component, as indicated by, for example, Reny and Perry (2006).

Our paper solves for the dynamics of information transmission in partially segmented over-the-counter markets. Our model of information transmission is also suitable for other settings in which learning is through successive local interactions, such as bank runs, knowledge spillovers, social learning, and technology diffusion. For example, Banerjee and Fudenberg (2004) and Duffie, Malamud, and Manso (2009) study social learning through word-of-mouth communication, but do not consider situations in which agents differ with respect to connectivity. In social networks, agents naturally differ with respect to connectivity. DeMarzo, Vayanos, and Zwiebel (2003), Gale and Kariv (2003), Acemoglu, Dahleh, Lobel, and Ozdaglar (2008), and Golub and Jackson (2009) study learning in social networks. Our model provides an alternative tractable framework to study the dynamics of social learning when different groups of agents in the population differ in connectivity with other groups of agents.

The conditions provided here for fully-revealing double auctions carry over to a setting in which the transactions prices of a finite sample of trades are publicly revealed, as is often the case in functioning over-the-counter markets. With this mixture of private and public information sharing, the information dynamics can be analyzed by the methods[2] of Duffie, Malamud, and Manso (2009b).

## 3 The Model

This section specifies the economy and characterizes equilibrium behavior. The following section lays out special cases in which we are able to provide more insights.

[2]One obtains an evolution equation for the cross-sectional distribution of beliefs that is studied by Duffie, Malamud, and Manso (2009b) for the case $N = 1$, and easily extended to the case of general $N$.

### 3.1 The Double Auctions

A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is fixed. An economy is populated by a continuum (a non-atomic measure space) of risk-neutral agents who are randomly paired over time for trade, in a manner that will be described. There are $N$ different classes of agents that differ according to the quality of their initial information, their preferences for the asset to be traded, and the expected rate at which they meet each of other classes of agents for trade. At some future time $T$, the economy ends and the utility realized by an agent of class $i$ for each additional unit of the asset is

$$U_i = v_i Y + v^H (1 - Y),$$

measured in units of consumption, for strictly positive constants $v^H$ and $v_i < v^H$, where $Y$ is a non-degenerate 0-or-1 random variable whose outcome will be revealed immediately after time $T$.

Whenever two agents meet at some particular time before $T$, they are given the opportunity to trade one unit of the asset in a double auction. The auction format allows (but does not require) the agents to submit a bid or an offer price for a unit of the asset. (That agents trade at most one unit of the asset at each encounter is an artificial restriction designed to simplify the model. One could suppose, alternatively, that the agents bid for the opportunity to produce a particular service for their counterparty.) Bids are observed by both agents participating in the auction. If an agent submits a bid price that is higher than the offer price submitted by the other agent, then one unit of the asset is assigned to that agent submitting the bid price, in exchange for an amount of consumption equal to the ask price. Certain other auction formats would be satisfactory for our purposes; we chose this format, known as the "seller's price auction," for simplicity. Bids and offers in an auction are only observed by agents participating in the auction.

When a class-$i$ and a class-$j$ agent meet, their preference parameters $v_i$ and $v_j$ are assumed to be commonly observable. Based on their initial information and on the information that they have received from prior auctions held with other agents, the two agents typically assign different conditional expectations to $Y$. From the no-speculative-trade theorem of Milgrom and Stokey (1982), as extended by Serrano-Padial (2007) to our setting of risk-neutral investors,[3] the two counterparties decline the opportunity to bid if they have identical preferences, that is, if $v_i = v_j$. If $v_i \neq v_j$, then it is common

[3]Milgrom and Stokey (1982) assume strictly risk-averse investors. Serrano-Padial (2007) shows that for investors with identical preferences, even if risk-neutral, if the distributions of counterparties' pos-

knowledge which of the two agents is the prospective buyer ("the buyer") and which is the prospective seller ("the seller"). The buyer is of class $j$ whenever $v_j > v_i$.

The seller has an information set $\mathcal{F}_S$ that consists of his initially endowed signals relevant to the conditional distribution of $Y$, as well any bids and offers that he has observed at his previous auctions. The seller's offer price $\sigma$ must be based only on (must be measurable with respect to) the information set $\mathcal{F}_S$. The buyer, likewise, bids on the basis of her information set $\mathcal{F}_B$. The prices $(\sigma, \beta)$ constitute an equilibrium for a seller of class $i$ and a buyer of class $j$ provided that, fixing $\beta$, the offer $\sigma$ maximizes[4] the seller's conditional expected gain,

$$E\left[(\sigma - E(U_i \,|\, \mathcal{F}_S \cup \{\beta\}))1_{\{\sigma<\beta\}} \,|\, \mathcal{F}_S\right], \tag{1}$$

and fixing $\sigma$, the bid $\beta$ maximizes the buyer's conditional expected gain

$$E\left[(E(U_j \,|\, \mathcal{F}_B \cup \{\sigma\}) - \sigma)1_{\{\sigma<\beta\}} \,|\, \mathcal{F}_B\right]. \tag{2}$$

The seller's conditional expected utility for the asset, $E(U_i \,|\, \mathcal{F}_S \cup \{\beta\}))$, once having conducted a trade, incorporates the information $\mathcal{F}_S$ that the seller held before the auction as well as the bid $\beta$ of the buyer. Similarly, the buyer's utility is affected by the information contained in the seller's offer. The informational advantage conferred by more frequent participation in auctions with well informed bidders is a key focus here.

In Section 3.4, we demonstrate technical conditions under which there are equilibria in which the offer price $\sigma$ and bid price $\beta$ can be computed, state by state, by solving an ordinary differential equation, and are strictly monotonically decreasing with respect to $E(Y \,|\, \mathcal{F}_S)$ and $E(Y \,|\, \mathcal{F}_B)$, respectively. This bid monotonicity is natural given the strict monotone decreasing dependence on $Y$ of $U_i$ and $U_j$. Strictly monotone equilibria are not typically available, however, in more general settings explored in the double-auctions literature, as indicated by, for example, Reny and Perry (2006). Because our strictly monotone equilibria fully reveal the bidders' conditional beliefs for $Y$, we will be able to explicitly calculate the evolution over time of the cross-sectional distribution of posterior beliefs of the population of agents, by extending results in Duffie and Manso (2007) and Duffie, Giroux, and Manso (2008). This, in turn, permits a characterization of the expected lifetime utility of each type of agent, including the manner in which

teriors have a density, as here, then there is no mechanism leading to trade with positive probability in which both agents weakly prefer the final allocation over the initial allocation.

[4]Here, to "maximize" means, as usual, to achieve, almost surely, the essential supremum of the conditional expectation.

utility depends on the quality of the initial information endowment and the "market connectivity" of that agent.

### 3.2 Information Setting

Agents are initially informed by signals drawn from a common infinite pool of 0-or-1 random variables that are $Y$-conditionally independent.[5] Each signal is received by at most one agent. Each agent is initially allocated a randomly selected finite subset of these signals. For almost every pair of agents, the numbers of signals received by each of them is assumed to be independent of each other, and of the signals. (The number of signals received by an agent is allowed to be deterministic.) The signals need not have the same probability distributions. Without loss of generality, for any signal $Z$, we suppose that

$$\mathbb{P}(Z = 1 \,|\, Y = 0) \geq \mathbb{P}(Z = 1 \,|\, Y = 1).$$

Whenever finite, we define the "information type" of an arbitrary finite set $K$ of random variables to be

$$\log \frac{\mathbb{P}(Y = 0 \,|\, K)}{\mathbb{P}(Y = 1 \,|\, K)} - \log \frac{\mathbb{P}(Y = 0)}{\mathbb{P}(Y = 1)}, \tag{3}$$

the difference between the conditional and unconditional log-likelihood ratios. The conditional probability that $Y = 0$ associated with the information type $\theta$ is thus

$$P(\theta) = \frac{Re^{\theta}}{1 + Re^{\theta}}, \tag{4}$$

where $R = \mathbb{P}(Y = 0)/\mathbb{P}(Y = 1)$, and the information type of a collection of signals is one-to-one with the conditional probability that $Y = 0$ given the signals. Proposition 3 of Duffie and Manso (2007) implies that whenever a collection of signals of type $\theta$ is combined with a disjoint collection of signals of type $\phi$, the type of the combined set of signals is $\theta + \phi$. More generally, we will use the following result from Duffie and Manso (2007).

**Lemma 3.1** *Let $S_1, \ldots, S_n$ be disjoint sets of signals with respective types $\theta_1, \ldots, \theta_n$. Then the union $S_1 \cup \cdots \cup S_n$ of the signals has type $\theta_1 + \cdots + \theta_n$. Moreover, the type of the information set $\{\theta_1, \theta_2, \ldots, \theta_n\}$ is also $\theta_1 + \theta_2 + \cdots + \theta_n$.*

[5] To be more precise, there is a continuum of signals, indexed by a non-atomic measure space, say $[0, 1]$. Almost every pair of signals is $Y$-conditionally independent.

The Lemma has two key implications for our analysis. First, if two agents meet and reveal all of their endowed signals, they both achieve posterior types equal to the sum of their respective prior types. Second, for the purpose of determining posterior types, revealing one's prior type (or any random variable such as a bid that is strictly monotone with respect to type) is payoff-equivalent to revealing all of one's signals.

An agent of class $i$ is matched with other agents at each of a sequence of Poisson arrival times with a mean arrival rate (intensity) $\lambda_i > 0$. At each meeting time, the agent's counterparty is randomly selected from the population of agents. The probability that a class-$j$ counterparty is selected is denoted $\kappa_{ij}$. Without loss of generality for the purposes of analyzing the evolution of information, we take $\kappa_{ij} = 0$ whenever $v_i = v_j$, because of the no-trade result for agents with the same preferences. A primitive $\kappa$ that does not satisfy this property can without loss of generality be adjusted so as to satisfy this property by conditioning, case by case, on the event that the agents matched have $v_i \neq v_j$.

As is standard in search models of markets, we assume that, for almost every pair of agents, the matching times and the counterparties of one agent are independent of those of the other. We do not show the existence of such a random-matching process, although Duffie and Sun (2007) show the existence of a model with this random matching property for a continuum-of-agents in a discrete-time setting, as well as the associated law of large numbers for random matching on which we rely. Further, the limit behavior of the discrete-agent matching models as the number of agents gets large is shown by Reminik (2009) to coincide with the matching behavior on which we rely in our continuous-time model.[6]

In this random-matching setting, a given pair of agents that have been matched will almost surely never be matched again nor will their respective lifetime sets of trading counterparties overlap. Thus, equilibrium bidding behavior in the multi-period setting is characterized by equilibrium bidding behavior in each individual auction, as described

[6]See also Ferland and Giroux (2008). Taking $G$ to be the set of agents, we assume throughout the joint measurability of agents' type processes $\{\theta_{it} : i \in G\}$ with respect to a $\sigma$-algebra $\mathcal{B}$ on $\Omega \times G$ that allows the Fubini property that, for any measurable subset $A$ of types,

$$\int_G \mathbb{P}(\theta_{\alpha t} \in A)\, d\gamma(\alpha) = E\left(\int_G 1_{\theta_{\alpha t} \in A}\, d\gamma(\alpha)\right),$$

where $\gamma$ is the measure on the agent space. Sun (2006) provides a condition on $\mathcal{B}$, which we assume, that is consistent with the exact law of large numbers. In our setting, if almost every pair of types from $\{\theta_{\alpha t} : \alpha \in G\}$ is independent, this law implies that $E\left(\int_G 1_{\theta_{\alpha t} \in A}\, d\gamma(\alpha)\right) = \int_G 1_{\theta_{\alpha t} \in A}\, d\gamma(\alpha)$ almost surely. Sun (2006) further proves the existence of a model with this property.

above. Later, we will provide primitive technical conditions on the preference parameters $v^H$ and $v_i$, as well as the cross-sectional distribution of initially endowed information types, that imply the existence of an equilibrium with strictly monotone bidding strategies. In this setting, bids therefore reveal types. Lemma 3.1 and induction thus imply that agents' types add up from auction to auction. Specifically, an agent leaves any auction with a type that is the sum of his type immediately before the auction and the type of the other agent bidding at the auction. This fact now allows us to characterize the dynamics of the cross-sectional evolution of posterior types.

### 3.3 Evolution of Type Distributions

For each class $i$, we suppose that the initial cross-sectional distribution of types of the class-$i$ agents has some density $\psi_{i0}$. We do not require that the individual class-$i$ agents have types with the same probability distribution. Nevertheless, our independence and measurability assumptions imply the exact law of large numbers, by which the density function $\psi_{i0}$ has two deterministic outcomes, almost surely, one on the event that $Y = 0$, denoted $\psi_{i0}^H$, the other on the event that $Y = 1$, denoted $\psi_{i0}^L$. That is, for any real interval $(a, b)$, the fraction of class-$i$ agents whose type is initially between $a$ and $b$ is almost surely $\int_a^b \psi_{i0}^H(\theta)\, d\theta$ on the event that $Y = 0$, and is almost surely $\int_a^b \psi_{i0}^L(\theta)\, d\theta$ on the event that $Y = 1$. We make the further assumption that $\psi_{i0}^H$ and $\psi_{i0}^L$ have moment-generating functions that are finite on a neighborhood of zero. Special cases satisfying this condition are the basis for illustrative examples in Section 4.

Our objective now is to calculate, for any time $t > 0$, the cross-sectional density $\psi_{it}$ of the types of class-$i$ agents. This cross-sectional density has (almost surely) only two outcomes, one on the event $Y = 0$ and one on the event $Y = 1$, denoted $\psi_{it}^H$ and $\psi_{it}^L$, respectively.

Assuming that the asset auctions are fully revealing, which will be confirmed under technical conditions, the evolution equation for the cross-sectional densities is

$$\frac{d\psi_{it}}{dt} \;=\; -\lambda_i\, \psi_{it} \;+\; \lambda_i\, \psi_{it} * \sum_{j=1}^{N} \kappa_{ij}\, \psi_{jt}, \qquad i \in \{1, \ldots, N\}, \tag{5}$$

where $*$ denotes convolution. We offer a brief explanation of this evolution equation. The first term on the righthand side captures the outward migration of agents of any given information type $\theta$ at rate $\lambda_i \psi_{it}(\theta)$, that is caused by a change to some other information type due to information gathered at auctions, which occur at the total proportional rate $\lambda_i$. Here, we use the law of large numbers, which almost surely equates the mean rate of change for each agent with the total population rate. The second term captures the inward migration of agents of a given information type due to learning from bids at auctions. The second term is easily understood by noting that auctions with class-$j$ counterparties occur at rate $\lambda_i \kappa_{ij}$. At such an encounter, in a fully revealing equilibrium, bids reveal the types of both agents, which are then added to get the posterior types of each. A class-$i$ agent of type $\theta$ is thus created if a class-$i$ agent of some type $\phi$ meets a class-$j$ agent of type $\theta - \phi$. Because this is true for any possible $\phi$, we integrate over $\phi$ with respect to the population densities. Thus, the total rate of increase of the density of class-$i$ agents of type-$\theta$ agents due to the information released at auctions with class-$j$ agents is

$$\lambda_i \kappa_{ij} \int_{-\infty}^{+\infty} \psi_{it}(\phi) \psi_{jt}(\theta - \phi)\, d\phi = \lambda_i \kappa_{ij} (\psi_i * \psi_j)(\theta).$$

Adding over $j$ gives the second term on the righthand side of the evolution equation (5). For the case $N = 1$, this evolution model is motivated in more detail, and solved, by Duffie and Manso (2007) and Duffie, Giroux, and Manso (2008).

Equation (5) can be solved in terms of the moment generating function of $\psi_{it}$ or, by the same calculation, the Fourier transform $\hat{\psi}_{it}$ of $\psi_{it}$. We have

$$\frac{d\hat{\psi}_{it}}{dt} \;=\; -\lambda_i \, \hat{\psi}_{it} \;+\; \lambda_i \, \hat{\psi}_{it} \sum_{j=1}^{N} \kappa_{ij} \, \hat{\psi}_{jt}, \qquad i \in \{1, \ldots, N\}, \tag{6}$$

using the fact that the Fourier transform of a convolution of two measures is the product of their Fourier transforms. Now, (6) is a Riccati ordinary differential equation in $t$ for the $N$-dimensional vector $\hat{\psi}_t(z) = (\hat{\psi}_{1t}(z), \ldots, \hat{\psi}_{Nt}(z))$. We can solve this equation, numerically if necessary, and then invert the transform to compute the type densities. In special cases, we have an explicit solution, for example as follows.

**Proposition 3.2** *Suppose that $N = n + m$, with $n$ classes of buyers, all with $v_i = \bar{v}$, and with $m$ classes of sellers, all with $v_j = \underline{v} < \bar{v}$. Suppose that all classes have the same mean contact rate $\lambda$. We assume that the class selection probability $\kappa_{ij} = k_j$ for buyer-to-seller contacts does not depend on the buyer class $i$, and likewise that $\kappa_{ji} = k_i$ for seller-to-buyer contacts. The initial type densities can vary across the $n + m$ classes without restriction. We let*

$$\phi_{1t} \;=\; \sum_{i=1}^{n} k_i \, \psi_{it}$$

*and*

$$\phi_{2t} \;=\; \sum_{j=n+1}^{n+m} k_j \, \psi_{jt}.$$

*We calculate that*

$$\hat{\phi}_{1t} = \frac{e^{-\lambda t}\,(\hat{\phi}_{20} - \hat{\phi}_{10})}{\hat{\phi}_{20} e^{-\hat{\phi}_{20}(1-e^{-\lambda t})} - \hat{\phi}_{10} e^{-\hat{\phi}_{10}(1-e^{-\lambda t})}}\,\hat{\phi}_{10}\, e^{-\hat{\phi}_{10}(1-e^{-\lambda t})}$$

$$\hat{\phi}_{2t} = \frac{e^{-\lambda t}\,(\hat{\phi}_{20} - \hat{\phi}_{10})}{\hat{\phi}_{20} e^{-\hat{\phi}_{20}(1-e^{-\lambda t})} - \hat{\phi}_{10} e^{-\hat{\phi}_{10}(1-e^{-\lambda t})}}\,\hat{\phi}_{20}\, e^{-\hat{\phi}_{20}(1-e^{-\lambda t})}.$$

*We then have the solution*

$$\hat{\psi}_{it} = \frac{\hat{\psi}_{i0}}{\hat{\phi}_{10}}\,\hat{\phi}_{1t}, \qquad 1 \le i \le n,$$

$$\hat{\psi}_{jt} = \frac{\hat{\psi}_{j0}}{\hat{\phi}_{20}}\,\hat{\phi}_{2t}, \qquad n+1 \le j \le n+m.$$

For general $N$, $\lambda_i$, $\kappa_{ij}$, and $\psi_{i0}$, an alternative to inverting the transform $\hat{\psi}$ is to directly solve the evolution equation for the type distributions by extending the Wild summation method of Duffie, Giroux, and Manso (2008). The Wild-sum representation also allows us, in Section 4, to characterize expected auction profits in special cases. In order to calculate the Wild-sum representation of type densities, we proceed as follows. For an $N$-tuple $k = (k_1, \ldots, k_N)$ of nonnegative integers, let $a_{it}(k)$ denote the fraction of class-$i$ agents who by time $t$ have collected (directly, or indirectly through auctions) the originally endowed signal information of $k_1$ class-1 agents, of $k_2$ class-2 agents, and so on, including themselves. This means that $|k| = k_1 + \cdots + k_N$ is the number of agents whose originally endowed information has been collected by such an agent. To illustrate, consider an example agent of class 1 who, by a particular time $t$ has met one agent of class 2, and nobody else, with that agent of class 2 having beforehand met 3 agents of class 4 and nobody else, and with those class-4 agents not having met anyone before they met the class-2 agent. The class-1 agents with this precise scenario of meeting circumstances would contribute to $a_{1t}(k)$ for $k = (1, 1, 0, 3, 0, 0, \ldots, 0)$. We can view $a_{it}$ as a measure on $\mathbb{Z}_+^N$, the set of $N$-tuples of nonnegative integers. By essentially the same reasoning used to explain the evolution equation (5), we have

$$a'_{it} = -\lambda_i\, a_{it} + \lambda_i\, a_{it} * \sum_{j=1}^{N} \kappa_{ij}\, a_{jt}, \qquad a_{i0} = \delta_{e_i}, \tag{7}$$

where

$$(a_{it} * a_{jt})(k_1, \ldots, k_N) = \sum_{\{l=(l_1,\ldots,l_N)\,\in\,\mathbb{Z}_+^N\,,\,|l|\le|k|\}} a_{it}(l)\, a_{jt}(k-l).$$

Here, $\delta_{e_i}$ is the dirac measure placing all mass on $e_i$, the unit vector whose $i$-th coordinate is 1.

**Theorem 3.3** *There is a unique solution of (5), given by*

$$\psi_{it} \;=\; \sum_{k\in\mathbb{Z}_+^N} a_{it}(k)\, \psi_{10}^{*k_1} * \cdots * \psi_{N0}^{*k_N}, \tag{8}$$

*where $\psi_{i0}^{*n}$ denotes n-fold convolution.*

That (8) solves (5) follows from substitution and the use of (7). A complete proof is given in the Appendix. The system (7) of equations for the discrete measures admits a closed-form solution via the following recursive procedure. First, $a_i(0) = 0$ for all $i$, and, because the probability that a class-$i$ agent has met nobody by time $t$ is $e^{-\lambda_i t}$, we have

$$a_{it}(e_i) \;=\; e^{-\lambda_i t}\, a_{i0}(e_i).$$

Thus, we have $a_i(k)$ for all $k$ with $|k| \le 1$. Then, we can solve (7) inductively: Having found $a_i(k)$ whenever $|k| \le \overline{k}$, for some $\overline{k}$, we calculate it for any $k$ with $|k| = \overline{k} + 1$ by solving (7), using the fact that the right-hand side is an ODE for $a_i(k)$ that is linear in $a_i(k)$ and otherwise involves $a_i(l)$ only for $|l| \le \overline{k}$.

The following result will be useful in Section 4.

**Proposition 3.4** *The measures $a_{it}$ are monotone increasing in time $t$ and in the meeting intensities $\lambda_i$, in the sense of first order stochastic dominance.*

### 3.4 Double Auction Solution

Fixing a particular time $t$, suppose that a class-$i$ and a class-$j$ agent meet, and that the prospective buyer is of class $i$ (that is, $v_i > v_j$). We now calculate their equilibrium bidding strategies. Naturally, we look for equilibria in which the outcome of the offer $\sigma$ for a seller of type $\theta$ is $S(\theta)$ and the outcome of the bid $\beta$ of a buyer of type $\phi$ is $B(\phi)$, where $S(\,\cdot\,)$ and $B(\,\cdot\,)$ are some strictly monotone increasing functions on the real line. In this case, if $(\sigma, \beta)$ is an equilibrium, we also say that $(S, B)$ is an equilibrium.

We assume for the results in this section that whenever two agents are in contact, each can observe all of the primitive characteristics, $\psi_{i0}, \lambda_i, \kappa_i$, and $v_i$, of the class of the counterparty. In the following section, we consider variants of the model in which the initial type density $\psi_{i0}$, the mean trading rate $\lambda_i$ of one's counterparty, and the probabilities $\kappa_i = (\kappa_{i1}, \ldots, \kappa_{iN})$ that govern the distribution of the classes of matched counterparties need not be observable.

Given a candidate pair $(S, B)$ of such bidding policies, a seller of type $\theta$ who offers the price $s$ has an expected increase in utility, defined by (1), of

$$\int_{B^{-1}(s)}^{+\infty} (s - v_j - \Delta_j P(\theta + \phi)) \; \Psi_i(P(\theta), \phi)\, d\phi, \tag{9}$$

where $\Delta_j = v^H - v_j$ and where $\Psi_i(P(\theta), \cdot\,)$ is the seller's conditional probability density for the unknown type of the buyer, defined by

$$\Psi_i(p, \phi) \; = \; p\,\psi_{it}^H(\phi) \, + \, (1-p)\,\psi_{it}^L(\phi). \tag{10}$$

Likewise, from (2), a buyer of type $\phi$ who bids $b$ has an expected increase in utility for the auction of

$$\int_{-\infty}^{S^{-1}(b)} \big(v_i \, + \, \Delta_i P(\theta + \, \phi) \; - \; S(\theta)\big)\, \Psi_j(P(\phi), \theta)\, d\theta. \tag{11}$$

The pair $(S, B)$ therefore constitutes an equilibrium if, for almost every $\phi$ and $\theta$, these gains from trade are maximized with respect to $b$ and $s$ by $B(\phi)$ and $S(\theta)$, respectively.

The hazard rate $h_{it}^L(\theta)$ associated with $\psi_{it}^L$ is defined as usual by

$$h_{it}^L(\theta) \; = \; \frac{\psi_{it}^L(\theta)}{G_{it}^L(\theta)},$$

where $G_{it}^L(\theta) = \int_\theta^\infty \psi_{it}(x)\, dx$. That is, given $Y = 1$, $h_{it}^L(\theta)$ is the probability density for the type $\theta$ of a randomly selected buyer, conditional on this type being at least $\theta$. We likewise define the hazard rate $h_{it}^H(\theta)$ associated with $\psi_{it}^H$. We say that $\psi_{it}$ satisfies the hazard-rate ordering if, for all $\theta$, we have $h_{it}^H(\theta) \leq h_{it}^L(\theta)$. The appendix provides a proof of the following.

**Lemma 3.5** *Suppose that each signal $Z$ satisfies*

$$\mathbb{P}(Z = 1 \,|\, Y = 0) + \mathbb{P}(Z = 1 \,|\, Y = 1) = 1. \tag{12}$$

*Then, for each agent class $i$ and time $t$, the type density $\psi_{it}$ satisfies the hazard-rate ordering as well as the property*

$$\psi_{it}^H(x) = e^x \psi_{it}^H(-x), \qquad \psi_{it}^L(x) \; = \; \psi_{it}^H(-x), \qquad x \in \mathbb{R}. \tag{13}$$

The restriction (12) on signal distributions is somewhat typical of learning models, for example those of Bikhchandani, Hirshleifer and Welch (1992) and Chamley (2004, p. 24). We now adopt this assumption, as well as a technical regularity condition on initial type densities.

**Standing Assumption:** Any signal $Z$ satisfies (12). Moreover, the initial type densities are strictly positive and twice differentiable, with

$$\int_{\mathbb{R}} e^{kx} \left( \left| \frac{d}{dx} \psi_{i0}^H(x) \right| + \left| \frac{d^2}{dx^2} \psi_{i0}^H(x) \right| \right) dx \; < \; \infty \tag{14}$$

for any $k < \alpha_{i0}$, where $\alpha_{i0} \; = \; \sup\{k \, : \, \hat{\psi}_{i0}^H(k) \; < \; \infty\}$.

The calculation of an equilibrium is based on the ODE, stated in the following result, for the type $V_2(b)$ of a buyer who optimally bids $b$. That is, $V_2$ is the inverse $B^{-1}$ of the candidate equilibrium bid policy function $B$.

**Lemma 3.6** *For any $V_0 \in \mathbb{R}$, there exists a unique solution $V_2(\,\cdot\,)$ on $[v_i, v^H)$ to the ODE*

$$V_2'(z) \; = \; \frac{1}{v_i - v_j} \left( \frac{z - v_i}{v^H - z} \frac{1}{h_{it}^H(V_2(z))} + \frac{1}{h_{it}^L(V_2(z))} \right), \qquad V_2(v_i) \; = \; V_0. \tag{15}$$

*This solution, also denoted $V_2(V_0, z)$, is monotone increasing in both $z$ and $V_0$. Further,* $\lim_{z \to v^H} V_2(V_0, z) \; = \; +\infty$ *. The limit $V_2(-\infty, z) \; = \; \lim_{V_0 \to -\infty} V_2(V_0, z)$ exists. Moroever, $V_2(-\infty, z)$ is continuously differentiable with respect to $z$.*

As shown in the proof of the next proposition, found in the appendix, the ODE (15) arises from the first-order optimality conditions for the buyer and seller. The solution of the ODE can be used to characterize equilibria in the double auction, as follows.

**Proposition 3.7** *Suppose that $(S, B)$ is a continuous equilibrium such that $S(\theta) \; \leq \; v^H$ for all $\theta \in \mathbb{R}$. Let $V_0 = B^{-1}(v_i) \geq -\infty$. Then,*

$$B(\phi) \; = \; V_2^{-1}(\phi), \quad \phi \; > \; V_0.$$

*Further, $S(-\infty) = \lim_{\theta \to -\infty} S(\theta) \; = \; v_i$ and $S(+\infty) = \lim_{\theta \to -\infty} S(\theta) \; = \; v^H$. For any $\theta$, we have $S(\theta) \; = \; V_1^{-1}(\theta)$, where*

$$V_1(z) \; = \; \log \frac{z - v_i}{v^H - z} \; - \; V_2(z) \, - \, \log R \, , \quad z \, \in \, (v_i, v^H) \, .$$

*Any buyer of type $\phi < V_0$ will not trade, and has a bidding policy $B$ that is not uniquely determined at types below $V_0$.*

In our double-auction setting, welfare is increasing in the probability of trade conditional on $Y = 1$. We are therefore able to rank the equilibria of our model in terms of welfare, because, from the following corollary of Proposition 3.7, we can rank the equilibria in terms of the probability of trade conditional on $Y = 1$.

**Corollary 3.8** *Let $(S, B)$ be a continuous equilibrium with $V_0 = B^{-1}(v_i)$. Then $S(\phi)$ is strictly increasing in $V_0$ for all $\phi$, while $B(\phi)$ is strictly decreasing in $V_0$ for all $\phi > V_0$. Consequently, the probability of trade conditional on $Y = 1$ is strictly decreasing in $V_0$.*

Buyers and sellers bid more aggressively in equilibria with lower $V_0$. Thus, the probability of trade conditional on $Y = 1$ and total welfare are strictly decreasing in $V_0$.

We turn to the study of particular equilibria, providing conditions for the existence of equilibria in strictly monotone undominated strategies. We also give sufficient conditions for the *failure* of such equilibria to exist. We focus on the welfare-maximizing equilibria.

From Proposition 3.7, the bidding policy $B$ is not uniquely determined at types below $B^{-1}(v_i)$, because agents with these types do not trade in equilibrium. Nevertheless, the equilibrium bidding policy $B$ satisfying $B(\phi) = v_i$ whenever $\phi < V_0$ weakly dominates any other equilibrium bidding policy. That is, an agent whose type is below $V_0$ and who bids less than $v_i$ can increase his bid to $v_i$, thereby increasing the probability of buying the asset, without affecting the price, which will be at most the lowest valuation $v_i$ of the bidder. An equilibrium in strictly monotone undominated strategies is therefore only possible if $V_0 = -\infty$. We now provide technical conditions supporting the existence of such welfare-maximizing equilibria.

We say that a function $g(\,\cdot\,)$ on the real line or the integers is of exponential type $\alpha$ at $+\infty$ if, for some constants $c > 0$ and $\gamma > -1$,

$$\lim_{x \to +\infty} \frac{g(x)}{x^{\gamma}\, e^{\alpha x}} \;=\; c. \tag{16}$$

In this case, we write $g(x) \sim \mathrm{Exp}_{+\infty}(c, \gamma, \alpha)$. We say that a family $\{g_t : t \in [0, T]\}$ of functions satisfies the condition $g_t(x) \sim \mathrm{Exp}_{+\infty}(c_t, \gamma_t, \alpha_t)$ uniformly in $t$ if the convergence in (16) is uniform in $t$.

The tail condition (16), which we will use as a technical regularity assumption on type densities, arises naturally in information percolation models, as we show in the following simple case, in which we also characterize the tail parameters $\alpha$, $c$, and $\gamma$.

**Lemma 3.9** *Suppose $N = 1$, and let $\lambda = \lambda_1$ and $\psi_t = \psi_{1t}$. The Laplace transform $\hat{\psi}_t$ of $\psi_t$ is given by*

$$\hat{\psi}_t(z) \;=\; \frac{e^{-\lambda t}\,\hat{\psi}_0(z)}{1-(1-e^{-\lambda t})\hat{\psi}_0(z)}$$

*and $\psi_t(x) \;\sim\; \mathrm{Exp}_{+\infty}(c_t, 0, -\alpha_t)$, where $\alpha_t$ is the unique positive number $z$ solving*

$$\hat{\psi}_0(z) \;=\; \frac{1}{1-e^{-\lambda t}},$$

*and where*

$$c_t \;=\; \frac{e^{-\lambda t}}{(1-e^{-\lambda t})^2\;\frac{d}{dz}\hat{\psi}_0(\alpha_t)}.$$

*Further, $\alpha_t$ is monotone decreasing in $t$, with $\lim_{t\to\infty}\alpha_t = 0$. Moreover, if $\psi_0 \sim \mathrm{Exp}(c_0, 0, \alpha_0))$, then $\psi_t(x) \;\sim\; \mathrm{Exp}_{+\infty}(c_t, 0, -\alpha_t)$ uniformly in $t$.*

The tail condition (16) also applies to the type density $\psi_{it}$ in more general cases, such as the multi-class example considered in Proposition 3.2, as shown in the Appendix. We conjecture that the tail condition (16) holds for any of the information percolation models considered in this paper, but we have not been able to prove this conjecture.

The following proposition provides conditions for the existence of a unique welfare-maximizing equilibrium in strictly monotone strategies, which are therefore fully revealing. For this purpose, we define $\alpha^*$ to be the unique positive solution to $\alpha^* = 1 + 1/(\alpha^* 2^{\alpha^*})$, which is approximately 1.31. Our result depends in part on a sufficiently high level of

$$G(v) = \frac{v_i - v_j}{v^H - v_i},$$

a measure of the relative gain from trade between buyers and sellers.

**Proposition 3.10** *Suppose that, for all $i$ and $t$, there are $\alpha_{it}$, $c_{it}$, and $\gamma_{it}$ such that, uniformly in $t$,*

$$\psi^H_{it}(x) \sim \mathrm{Exp}_{+\infty}(c_{it}, \gamma_{it}, -\alpha_{it}). \tag{17}$$

*If $\alpha_{iT} < 1$, then there is no equilibrium associated with $V_0 = -\infty$. Suppose, however, that $\alpha_{iT} > \alpha^*$ and that, for all $t$,*

$$-\gamma_{it} \;<\; \frac{(\alpha_{it}+1)\log\alpha_{it}}{\log(\alpha_{it}+1)-\log\alpha_{it}}, \qquad \text{if } \alpha_{it} \;\geq\; 2$$

$$-\gamma_{it} \;<\; \frac{\log(\alpha_{it}^2-\alpha_{it})\,2^{\alpha_{it}}}{\log(\alpha_{it}+1)-\log\alpha_{it}}, \qquad \text{if } \alpha_{it} \;<\; 2.$$

*Then, if the gain from trade $G(v)$ is sufficiently large, there exists a unique strictly monotone equilibrium associated with $V_0 = -\infty$. This equilibrium is in undominated strategies, and maximizes total welfare among all continuous equilibrium bidding policies.*

The technical regularity conditions of the proposition, combined with a sufficiently large trading motive as measured by $G(v)$, together guarantee that $V_2(z)$ does not grow too fast in $z$, leading $V_1(z)$ to be monotone increasing, and thus allowing a welfare-maximizing fully-revealing equilibrium in strictly monotone strategies. Under the conditions of Proposition 3.10, there may be other equilibria in undominated strategies that are associated with a finite $V_0$. The equilibrium with $V_0 = -\infty$, however, maximizes the probability of trade conditional on $Y = 1$, uniquely so for $t > 0$, and consequently also maximizes welfare.

## 4 Connectedness, Information Quality, and Profitability

We now study whether an agent with more precise initial information or with a higher expected frequency of opportunities to gather information from trading attains higher total expected future trading profits. We will also show, in an extension of our model that allows an agent to hide his initial information quality or his expected frequencies of auction observations with each of the other agent classes, whether this can increase the agent's expected trading profits, through the increased uncertainty of the agent's counterparties regarding the quality of the agent's information. This is relevant in functioning markets through the decision of an investor of whether to trade openly with a given reputation for market connectedness, or whether to trade through proxy investors whose quality of information is more uncertain, or through other indirect forms of trade execution.

For these purposes, we first need to characterize investors' expected utilities. We assume throughout this section the existence of a unique welfare-maximizing equilibrium in strictly monotone bidding strategies for each time $t < T$, sufficient conditions for which are given by Proposition 3.10. Our utility calculations are based throughout on these equilibrium bidding strategies.

The stochastic type process $\Theta$ of any particular class-$i$ agent is a Markov process. The transition distribution function of $\Theta$ is determined by the probability density of $\Theta_t - \Theta_s$ given $\Theta_s$, for any times $s$ and $t > s$. We let $\rho_{s,t}(\,\cdot\,|\,\Theta_s)$ denote this conditional density function, and calculate it as follows.

**Lemma 4.1** *We have*

$$\rho_{s,t}(y\,|\,\Theta_s) = P(\Theta_s)h^H_{s,t}(y) + (1 - P(\Theta_s))h^L_{s,t}(y),$$

*where, for $K = H$ or $K = L$, the density $h^K_{s,t}(\,\cdot\,)$ satisfies, for each fixed $s$, the evolution equation*

$$\frac{d}{dt}\, h^K_{s,t} \;=\; -\lambda_i\, h^K_{s,t} \;+\; \lambda_i\, h^K_{s,t} * \sum_j \kappa_{ij}\, \psi^K_{jt}\,, \tag{18}$$

*with an initial condition at $t = s$ given by the Dirac measure at 0. The Fourier transform of $h^K_{s,t}$ is*

$$\hat{h}^K_{s,t} = \; e^{-\lambda_i\,(t-s)}\, \exp\left(\lambda_i \int_s^t \sum_j \kappa_{ij}\, \hat{\psi}^K_{j\tau}\, d\tau\right). \tag{19}$$

*We therefore have the solution*

$$h^K_{s,t} = \; e^{-\lambda_i\,(t-s)} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\lambda_i \int_s^t \sum_j \kappa_{ij}\, \psi^K_{j\tau}\, d\tau\right)^{*k}. \tag{20}$$

The ODE (18) follows from an argument like that given for (5). The corresponding ODE for $\hat{h}^K_{s,t}$ is linear, and thus has the solution (19). The solution (20) arises from the series definition of the exponential function and the fact that multiplication on the Fourier side corresponds to convolution for the inverse Fourier transform.

The expected future profit at time $t$ of this agent is

$$\mathcal{U}_i(t, \Theta_t) \;=\; E\left[\sum_{\tau_k > t} \sum_j \kappa_{ij}\, \pi_{ij}(\tau_k, \Theta_{\tau_k}) \;\Big|\; \Theta_t\right],$$

where $\tau_k$ is this agent's $k$-th auction time and $\pi_{ij}(t, \theta)$ is the expected profit of a class-$i$ agent of type $\theta$ entering an auction at time $t$ with a class-$j$ agent. Given our equilibrium bidding functions $(B, S)$ for such an auction, we can calculate $\pi_{ij}(t, \theta)$ in the obvious way.[7] Because our class-$i$ agent enters auctions at Poisson times with an intensity of $\lambda_i$, we have

$$\mathcal{U}_i(t, \Theta_t) \;=\; \lambda_i \int_t^T \int_{\mathbb{R}} \rho_{t,\tau}(\theta - \Theta_t \,|\, \Theta_t)\, \pi_i(\tau, \theta)\, d\theta\, d\tau\,, \tag{21}$$

where $\pi_i(t, \theta) \;=\; \sum_j \kappa_{ij}\, \pi_{ij}(t, \theta)$.

To this point, we have always assumed that whenever two agents are in contact, each can observe all of the primitive characteristics, $\psi_{i0}, \lambda_i, \kappa_i$, and $v_i$, of the class $i$ of the counterparty. We now consider a variant of the model in which the initial type density $\psi_{i0}$, the mean trading rate $\lambda_i$, and the vector $\kappa_i$ of counterparty selection probabilities are not observable. These characteristics affect the quality of the counterparty's information,

[7] That is, for $v_i < v_j$ we have $\pi_{ij}(t, \theta) = \int_{B^{-1}(S(\theta))}^{+\infty} (s - v_i - \Delta_i P(\theta + \phi))\, \Psi_i(P(\theta), \phi)\, d\phi$, and for $v_i > v_j$ we have $\pi_{ij}(t, \theta) = \int_{-\infty}^{S^{-1}(B(\theta))} \big(v_i + \Delta_i P(\theta + \phi) - S(\phi)\big)\, \Psi_i(P(\theta), \phi)\, d\phi$.

and therefore affect bidding strategies. For this purpose, we assume for the remainder of this section that two classes, say classes 1 and 2, have $v_1 = v_2$, but may differ with respect to $\lambda_i$ and $\psi_{i0}$, and moreover, that their counterparties may or may not be able to distinguish between classes 1 and 2. In a setting in which this distinction cannot be made, a class-$j$ agent therefore assigns the probabilities $\kappa_{j1}/(\kappa_{j1}+\kappa_{j2})$ and $\kappa_{j2}/(\kappa_{j1}+\kappa_{j2})$ of facing a class-1 and class-2 counterparty, respectively. We let $\hat{\pi}_{ij}(t,\theta)$ be the expected profit of a class-$i$ agent of type $\theta$ entering an auction at time $t$ with a class-$j$ agent, when primitive characteristics $\psi_{i0}$, $\lambda_i$, and $\kappa_i$ are not observable.

We isolate for utility comparison a particular class-1 agent with type process $\Theta_1$ and a particular class-2 agent with type process $\Theta_2$. For simplicity, we assume that for each class, the initial types of almost every pair of agents in the class are identically and independently distributed given $Y$. It follows from the law of large numbers that the probability distribution of the initial type $\Theta_{10}$ has a density equal to the cross-sectional type density $\psi_{10}$ of class 1, and likewise that $\Theta_{20}$ has the probability density $\psi_{20}$. Because class-1 and class-2 agents are mutually indistinguishable from the viewpoint of their counterparties, at any given auction they bid or offer according to a pooled bid policy $B$ and a pooled offer policy $S$.

For the remaining results, we suppose that the initial $Y$-conditional type density $\psi_{10}$ of class-1 agents is that associated with receiving a random number of signals that is identically distributed across class-1 agents, with a density $p_1$ on the positive integers. That is, $p_1(k)$, also denoted $p_{ik}$, is the probability of receiving $k$ signals at time zero. Class-2 agents are initially informed in the same manner, except that the probability density of the number of signals that they receive is $p_2$. The signals given to each agent are drawn at random from a common pool of signals whose joint distributions with $Y$ vary cross-sectionally so that the type of a randomly selected signal has some fixed $Y$-conditional probability density $f(\,\cdot\,)$, with outcome $f^H(\,\cdot\,)$ on the event $Y = 0$ and $f^L(\,\cdot\,)$ on the event $Y = 1$. Thus, for any positive integers $m$ and $n > m$, receiving $n$ signals implies strictly better information precision than receiving $m$ signals.

We continue to let $\mathcal{U}_i(t,\theta)$ denote the expected future profit of a class-$i$ agent of type $\theta$ at time $t$, in our usual setting of completely observable agent characteristics, and we let $\hat{\mathcal{U}}_i(t,\theta)$ denote the utility of a class-$i$ agent in the alternative market setting, in which the characteristics $\psi_{i0}$, $\lambda_i$, and $\kappa_{ij}$ of class-1 and class-2 agents are not distinguishable. We now show that when one's quality of information cannot be distinguished by one's counterparty, better quality information, whether due to a higher expected frequency of trading encounters or to better initial information, increases total expected auction profits.

**Theorem 4.2** *If $\lambda_2 \geq \lambda_1$ and if the initial type densities $\psi_{10}$ and $\psi_{20}$ are distinguished by the fact that the density $p_2$ of the number of signals received by class-2 agents has first-order stochastic dominance over the density $p_1$ of the number of signals by class-1 agents, then*

$$\frac{E[\hat{\mathcal{U}}_2(t, \Theta_{2t})]}{\lambda_2} \geq \frac{E[\hat{\mathcal{U}}_1(t, \Theta_{1t})]}{\lambda_1}, \quad t \in [0, T]. \tag{22}$$

*The inequality (22) holds strictly if, in addition, $\lambda_2 > \lambda_1$ or if $p_2$ has strict dominance over $p_1$.*

The comparison (22) implies that the utility advantage of class-2 agents holds even after adjusting for their higher expected frequency of auction opportunities. The intuition is that class-2 investors are expected to be more informed than class-1 investors at any point in time, either because, in expectation, they will learn more in auctions than class-1 investors or because they are initially better informed than class-1 investors. Because class-2 investors cannot be distinguished from class-1 investors by their counterparties, the class-2 investors attain higher total expected profits than class-1 investors.

The previous result shows that better informed and better connected investors have higher expected trading profits if they are able to hide the characteristics determining the quality of their information. We now show that if investors must trade openly with respect to their connectivity and initial information quality, then having better initial information and more opportunities to collect information from trades can in some cases lead to *lower* expected trading profits.

For the remainder of this section, we further restrict our economy so as to allow a total of $N = 3$ classes of agents. We assume that $v_1 = v_2 \equiv \bar{v} > v_3$, so that the only trades are those in which class-3 agents sell to class-1 or class-2 agents.

The next example describes a situation in which better informed buyers have a lower utility than worse informed buyers, provided that the characteristics determining their information quality are commonly observable. An analogous example can be obtained based on a comparison of the matching intensities $\lambda_i$, as in Theorem 4.2.

**Example 4.3** *Suppose that $\kappa_1 = \kappa_2$ and $\lambda_1 = \lambda_2$, so that classes 1 and 2 differ only with respect to their initial cross-sectional type densities $\psi_{10}$ and $\psi_{20}$. In particular, we suppose that the number of initial signals received by class-2 investors has first-order dominance over the number received by class-1 investors such that*

$$\psi_{10}^H(x) \;=\; 12\,\frac{e^{3x}}{(1+e^x)^5}, \qquad \psi_{10}^L(x) \;=\; \psi_{10}^H(-x),$$

*and*

$$\psi_{20}^H \;=\; \psi_{10}^H * \psi_{10}^H.$$

*Moreover, we assume that the seller's type distribution* $\psi_{30}^H$ *corresponds to a distribution sufficiently close in total-variation norm to the convex combination of Dirac measures given by*

$$(1+e^{-A})^{-1}\left(e^{-A}\,\delta_{-A} + \delta_A\right), \tag{23}$$

*for a constant* $A$. *Taking* $v_3 = 0$, $v_1 = v_2 = 1.6$, *and* $A = 1$, *we have* $E[\,\pi_{13}(0,\theta)] = 0.38331$, $E[\,\pi_{23}(0,\theta)] = 0.37232$, $E[\,\hat{\pi}_{13}(0,\theta)] = 0.38150$, *and* $E[\,\hat{\pi}_{23}(0,\theta)] = 0.40038$. *Therefore, by continuity, there exists a sufficiently small time horizon* $T$ *such that, for any time* $t$,

$$E[\mathcal{U}_2(t,\Theta_{2t})] \;<\; E[\hat{\mathcal{U}}_1(t,\Theta_{1t})] \;<\; E[\mathcal{U}_1(t,\Theta_{1t})] \;<\; E[\hat{\mathcal{U}}_2(t,\Theta_{2t})], \quad t \in [0,T]. \tag{24}$$

Class-3 investors face greater adverse selection from class-2 counterparties than from class-1 counterparties, given the relative information precision of the class-2 investors. In order to mitigate this increased adverse selection, class-3 investors tend to bid more conservatively when facing class-2 investors, if they can distinguish them, thus lowering the expected profit to a class-2 investor. On the other hand, in order to benefit from completing a sale on the event $Y = 1$, class-3 investors must bid more aggressively against class-2 investors than against class-1 investors whenever they believe that the event $Y = 1$ is relatively likely. This aggressive bidding brings extra expected benefits to class-2 investors conditional on the event $Y = 1$. In Example 4.3, the first effect dominates the second, and class-2 investors attain lower expected profits than those of class-1 investors, as stated by (24), when their information quality can be distinguished.

In Example 4.3, if class-1 investors have the choice, they would prefer to operate in a market in which the quality of counterparty information is revealed. In this situation, class-1 investors avoid the adverse selection problem of being pooled with class-2 investors.

Although Example 4.3 provides conditions under which better informed buyers attain lower profits than worse informed buyers when their information quality can be distinguished, the opposite can happen if the gain from trade is so large as to cause the opportunity value of an exchange to dominate the adverse selection effect.

In order to state an associated result, we introduce the following notation. For two densities $g_1$ and $g_2$ on the real line, we say that $g_2$ has a fatter right tail than $g_1$, and write $\text{Tail}(g_1) \prec \text{Tail}(g_2)$, if $g_i \sim \text{Exp}_{+\infty}(c_i, \gamma_i, -\alpha_i)$ and if

$$\lim_{x \to +\infty} \frac{g_2(x)}{g_1(x)} = +\infty.$$

This fatter-tail condition applies if either $\alpha_2 < \alpha_1$ or both $\alpha_1 = \alpha_2$ and $\gamma_2 > \gamma_1$. The weak version of this ordering is defined by writing $\text{Tail}(g_1) \preceq \text{Tail}(g_2)$ if $\alpha_2 \leq \alpha_1$ or if both $\alpha_1 = \alpha_2$ and $\gamma_2 \geq \gamma_1$.

From this point, we assume that for each of classes 1 and 2, $\psi_{i0}^H$ satisfies an exponential tail condition $\psi_{i0}^H \sim \text{Exp}_{+\infty}(c_i, \gamma_i, -\alpha_i)$. For this, if the random number of signals received by an agent is bounded, it suffices that the probability density $f^H$ of the type of a single randomly selected signal, given $Y = 0$, satisfies an exponential tail condition. This result is stated and proved as Appendix Lemma E.1, which also gives an alternative sufficient condition for cases in which the random number of signals is not bounded, but has a density with a tail "close to" that of the geometric distribution, in a sense made precise in Lemma E.1.

**Lemma 4.4** *If the density $p_2$ of the number of signals endowed to class-2 agents has first-order stochastic dominance over the density $p_1$ of the number of signals endowed to class-1 agents, then $\text{Tail}(\psi_{10}^H) \preceq \text{Tail}(\psi_{20}^H)$. Furthermore, if either*

$$\sup\{k : p_{1k} > 0\} < \sup\{k : p_{2k} > 0\}$$

*or if $p_1(k)$ and $p_2(k)$ are strictly positive for sufficiently large $k$, with*

$$\lim_{k \to \infty} \frac{p_1(k+1)}{p_1(k)} < \lim_{k \to \infty} \frac{p_2(k+1)}{p_2(k)},$$

*then $\text{Tail}(\psi_{10}^H) \prec \text{Tail}(\psi_{20}^H)$.*

In this sense, being more informed means having fatter-tailed information types.

**Proposition 4.5** *Suppose that $\kappa_1 = \kappa_2$ and $\lambda_1 = \lambda_2$, so that classes 1 and 2 differ only with respect to their initial cross-sectional type densities $\psi_{10}$ and $\psi_{20}$. We also suppose that the number of initial signals received by class-2 investors has first-order dominance over the number received by class-1 investors, that*

$$\frac{\alpha_{1t} + 1}{\alpha_{1t} - 1} > \alpha_{3t}, \quad t \in [0, T],$$

*and that* $\text{Tail}(\psi_{10}^H) \prec \text{Tail}(\psi_{20}^H)$ *(more informative tails for class-2 agents).*[8] *Then, if the gain-from-trade meaure* $G(v)$ *is sufficiently large,*

$$E[\mathcal{U}_1(t,\Theta_{1t})] \;<\; E[\hat{\mathcal{U}}_1(t,\Theta_{1t})] \;<\; E[\hat{\mathcal{U}}_2(t,\Theta_{2t})] \;<\; E[\mathcal{U}_2(t,\Theta_{2t})], \quad t \in [0,T].$$

The same two partially offsetting effects highlighted in the discussion after Example 4.3 continue to play a role here. The gain-from-trade measure $G(v)$ can be made so large, however, that the expected loss associated with a failure to exchange the asset dominates the adverse-selection effect, allowing class-2 investors to attain higher profits than class-1 investors even when the determinants of information quality are commonly observed.

Under the conditions of Proposition 4.5, class-1 investors prefer to be in a market in which the quality of information is not revealed. Again, the adverse selection effect is dominated by the loss-from-no-trade effect, reversing the result of Example 4.3.

Analogous results can be obtained when agents differ only in terms of the mean arrival rates of their opportunities to gather information from trading, as we show with the next proposition.

**Proposition 4.6** *Suppose that* $\kappa_1 = \kappa_2$ *and* $\lambda_1 < \lambda_2$*, and that class-1 and class-2 investors have the same initial information quality, that is,* $\psi_{10} = \psi_{20}$*. We further assume the exponential tail condition* $\psi_{it}^H \sim \text{Exp}_{+\infty}(c_{it}, \gamma_{it}, -\alpha_{it})$ *for all* $i$ *and* $t$*, with* $\alpha_{10} < 3$,

$$\alpha_{30} > \frac{\alpha_{10}-1}{3-\alpha_{10}},$$

*and*

$$\frac{\alpha_{1t}+1}{\alpha_{1t}-1} > \alpha_{3t}, \quad t \in [0,T].$$

*If the gain-from-trade measure* $G(v)$ *is sufficiently large, then for any time* $t$ *we have*

$$\frac{E[\mathcal{U}_2(t,\Theta_{2t})]}{\lambda_2} \;>\; \frac{E[\hat{\mathcal{U}}_2(t,\Theta_{2t})]}{\lambda_2} \;>\; \frac{E[\hat{\mathcal{U}}_1(t,\Theta_{1t})]}{\lambda_1} \;>\; \frac{E[\mathcal{U}_1(t,\Theta_{1t})]}{\lambda_1}.$$

Many of the results of this section can also be stated in the form of comparisons of the conditional expected utilities, $\mathcal{U}_i(t,\Theta_{it})$ and $\hat{\mathcal{U}}_i(t,\Theta_{it})$. We avoid this for brevity.

[8] In fact, this condition is "almost" unnecessary, in that we have already assumed that $p_2$ has first-order dominance over $p_1$. With this dominance, it is enough for $\text{Tail}(\psi_{10}^H) \prec \text{Tail}(\psi_{20}^H)$ that $\lim_{k\to\infty} p_1(k+1)/p_1(k) < \lim_{k\to\infty} p_2(k+1)/p_2(k)$. As a substitute for the condition $\text{Tail}(\psi_{10}^H) \prec \text{Tail}(\psi_{20}^H)$, it suffices that $\alpha_1 = \alpha_2$, $\gamma_1 = \gamma_2$, and $c_2 > c_1$.

# 5 Subsidizing Order-Flow Information

So far, in meetings between agents $i$ and $j$ with $v_i = v_j$, no trade takes place. In this section we investigate the possibility that agents with similar preference parameters engage in trading with the sole purpose of obtaining more information about $Y$ from their counterparties. In functioning over-the-counter markets, such as those for government bonds, the informational advantage of handling more trades is sometimes said to be a sufficient advantage to cause dealers to narrow quoted bid-ask spreads in order to increase counterparty contacts.

Because of our continuum-of-agents assumption, an agent is indifferent to the amount of information revealed to a counterparty, because this information has at most an infinitesimal impact on that agent's expected future terms of trade. We now describe a simple mechanism that induces agents to strictly prefer to truthfully reveal information to their counterparties. This mechanism can be interpreted as the trading of a contingent claim.

Suppose that upon meeting, two agents $i$ and $j$ with similar parameter preferences can enter a "swap" agreement by which the amount

$$k\left[(p_j(t) - Y)^2 - (p_i(t) - Y)^2\right],$$

will be paid by investor $i$ to investor $j$ at time $T$, where $p_i(t)$ and $p_j(t)$ are real variables reported by investors $i$ and $j$ at time $t$, and where $k > 0$ is a coefficient. The protocol is that the players first negotiate the multiplier $k$, and then both agents simultaneously submit their respective "reports" $p_i(t)$ and $p_j(t)$. Provided that $k$ is strictly greater than zero and that both agents have agreed to enter, in equilibrium player $i$ optimally submits a report $p_i(t)$ that is his or her conditional expectation of $Y$ (or equivalently, the conditional probability of the event $Y = 1$).

For the above mechanism to induce truthful revelation of posteriors in each auction, we must show that, at any particular meeting there exists some $k > 0$ such that both agents are willing to enter the swap agreement voluntarily. Lemma G.1 in the appendix shows that, keeping fixed the bidding policy of other investors in the economy, an investor attains strictly higher profits if he learns information from another investor in a meeting. Because this information gathering activity is not observable by other investors in the economy, it is a dominating strategy for investors to subsidize order flow with the purpose of learning information from investors with similar preferences, as long as the cost of the subsidy, although strictly positive, is sufficiently small. The net expected cost of the subsidy can indeed be made arbitrarily small in each auction, so that the benefits in terms of information gathering are greater than the costs in terms of the potential loss to the counterparty. If, for example, we let $k$ be the minimum of two coefficients $k_i > 0$ announced by the two agents when they meet and before they enter the swap agreement, then there is an equilibrium in which both agents select a small enough $k_i$ such that they are willing to participate in the swap agreement.

Therefore, there exists an equilibrium in which investors always subsidize order flow with counterparties with similar preference parameters, and counterparties treat investors as if they have been engaging in this activity.

The ability to subsidize order flow may have a negative impact on investors expected profits. For example, under the conditions of Example 4.3, an investor attains higher profits if he is less informed. However, as shown in this section, if investors have the ability to subsidize order flow to get more information, they will engage in this behavior, and may thus end up with a lower profit than if they did not have the ability to subsidize order flow.

# Appendices

## A Information Percolation

**Proof of Proposition 3.2.** For simplicity, by abuse of notation, we omit everywhere in this proof the superscript "$H$" on densities, writing $\psi_t$ in place of $\psi_t^H$, and so on.

Passing to Laplace transforms and adding up the equations for $\hat{\psi}_{it}$ over $i$ and the equation for $\hat{\psi}_{jt}$ over $j$ we get the system

$$\begin{aligned} \frac{d}{dt}\hat{\phi}_{1t} &= -\lambda\,\hat{\phi}_{1t} + \lambda\,\hat{\phi}_{1t}\,\hat{\phi}_{2t} \\ \frac{d}{dt}\hat{\phi}_{2t} &= -\lambda\,\hat{\phi}_{2t} + \lambda\,\hat{\phi}_{1t}\,\hat{\phi}_{2t}. \end{aligned} \tag{25}$$

Subtracting,

$$\hat{\phi}_{1t} - \hat{\phi}_{2t} = e^{-\lambda\,t}\,\hat{\nu},$$

where $\hat{\nu} = \hat{\phi}_{10} - \hat{\phi}_{20}$ satisfies $\hat{\nu}(0) = 0$. That is, in this case $\hat{\phi}_{1t}$ converges exponentially to $\hat{\phi}_{2t}$. Thus,

$$\frac{d}{dt}\hat{\phi}_{1t} = \lambda\,\hat{\phi}_{1t}(-1 + \hat{\phi}_{1t} - e^{-\lambda t}\hat{\nu})\,.$$

Denote $\xi_t = \hat{\phi}_{1t}e^{\lambda t}$. Then,

$$\frac{d}{dt}\xi_t = \lambda e^{-\lambda t}\xi_t(\xi_t - \hat{\nu}).$$

Integrating, we get

$$\frac{\xi}{\xi - \hat{\nu}} = \frac{\hat{\phi}_{10}}{\hat{\phi}_{20}}\,e^{-\hat{\nu}(1-e^{-\lambda t})}.$$

That is,

$$\hat{\phi}_{1t} = e^{-\lambda t}\hat{\xi}_t = \frac{e^{-\lambda t}\,(\hat{\phi}_{20} - \hat{\phi}_{10})}{\hat{\phi}_{20}e^{-\hat{\phi}_{20}(1-e^{-\lambda t})} - \hat{\phi}_{10}e^{-\hat{\phi}_{10}(1-e^{-\lambda t})}}\,\hat{\phi}_{10}\,e^{-\hat{\phi}_{10}(1-e^{-\lambda t})}.$$

On the other hand, integrating (25), we get

$$\hat{\phi}_{1t} = \hat{\phi}_{10}\,e^{-\lambda t}\,e^{\lambda\int_0^t \hat{\phi}_{2s}\,ds}$$

and therefore

$$e^{-\lambda t}\,e^{\lambda\int_0^t \hat{\phi}_{2s}\,ds} = \frac{\hat{\phi}_{1t}}{\hat{\phi}_{10}}.$$

Similarly,

$$e^{-\lambda t}\,e^{\lambda\int_0^t \hat{\phi}_{1s}\,ds} = \frac{\hat{\phi}_{2t}}{\hat{\phi}_{20}}\,.$$

Thus, integrating the equation for the Laplace transform of $\psi_{it}$, we get

$$\hat{\psi}_{it} \;=\; \hat{\psi}_{i0}\, e^{-\lambda t}\, e^{\lambda \int_0^t \hat{\phi}_{2s}\, ds} \;=\; \frac{\hat{\psi}_{i0}}{\hat{\phi}_{10}}\, \hat{\phi}_{1t},$$

and similarly for $\psi_{jt}$. ■

**Proof of Theorem 3.3.** Let the probability measures $\{a_{it}(k) \,:\, k \in \mathbb{Z}_+^N,\, i \in \{1,\ldots,N\}\}$ on $\mathbb{Z}_+^N$ satisfy the system of ODEs:

$$a'_{it} \;=\; -\lambda_i\, a_{it} \;+\; \lambda_i\, a_{it} * \sum_{j=1}^{N} \kappa_{ij}\, a_{jt}$$

or, coordinate-wise,

$$\frac{d}{dt} a_{it}(k) \;=\; -\lambda_i\, a_{it}(k) \;+\; \lambda_i \sum_{j=1}^{N} \kappa_{ij} \sum_{\{l_1,\, l_2 \in \mathbb{Z}_+^N \,:\, l_1 + l_2 = k\}} a_{it}(l_1)\, a_{jt}(l_2).$$

Let

$$\psi_{it} \;=\; \sum_{k \in \mathbb{Z}_+^N} a_{it}(k) \psi_0^{*k},$$

where

$$\psi_0^{*k} \;\stackrel{def}{=}\; \psi_{10}^{*k_1} * \cdots * \psi_{N0}^{*k_N}\,.$$

The series is well defined and convergent because $a_{it}$ is a probability measure. Then,

$$\begin{aligned}
\frac{d}{dt}\, \psi_{it} \quad &= \quad \sum_{k \in \mathbb{Z}_+^N} \frac{d}{dt} a_{it}(k)\, \psi_0^{*k} \\
&= \quad \sum_{k \in \mathbb{Z}_+^N} \left( -\lambda_i a_{it}(k) \;+\; \lambda_i \sum_{j=1}^{N} \kappa_{ij} \sum_{l_1 + l_2 = k} a_{it}(l_1)\, a_{jt}(l_2) \right) \psi_0^{*k} \\
&= \quad -\lambda_i\, \psi_{it} \;+\; \lambda_i \sum_{j=1}^{N} \kappa_{ij} \left( \sum_{l_1 \in \mathbb{Z}_+^N} a_{it}(l_1)\, \psi_0^{*l_1} \right) * \left( \sum_{l_2 \in \mathbb{Z}_+^N} a_{jt}(l_2) \psi_0^{*l_2} \right) \\
&= \quad -\lambda_i\, \psi_{it} \;+\; \lambda_i \sum_{j=1}^{N} \kappa_{ij}\, \psi_{it} * \psi_{jt}\,.
\end{aligned}$$

Uniqueness follows by standard arguments. ■

**Proof of Proposition 3.4.** Let $f : \mathbb{Z}_+ \to \mathbb{R}$ be monotone increasing and bounded. Let also $Y_{it}$ be a random variable (taking values in $\mathbb{Z}_+$) distributed with the measure $a_{it}$.

By (7),

$$\begin{aligned}\frac{d}{dt}\sum_k a_{it}(k)\,f(k) &= -\lambda_i \sum_k a_{it}(k)\,f(k) \;+\; \lambda_i \sum_{j=1}^N \kappa_{ij}(a_{it} * a_{jt})(k)\,f(k) \\ &= -\lambda_i E[f(Y_{it})] \;+\; \lambda_i \sum_{j=1}^N \kappa_{ij}\,E[f(Y_{it}+Y_{jt})] \\ &\geq -\lambda_i E[f(Y_{it})] \;+\; \lambda_i \sum_{j=1}^N \kappa_{ij}\,E[f(Y_{it})] \;=\; 0,\end{aligned}$$

and the stipulated monotonicity in time follows.

Now, define (for the moment, formally), for $p \in \{1, \ldots, N\}$,

$$b_{it}^{(p)} \;=\; \frac{\partial}{\partial \lambda_p}\,a_{it}.$$

Differentiating (formally) (7) with respect to $\lambda_p$, for $i \neq p$ we get

$$\frac{d}{dt}\,b_{it}^{(p)} \;=\; -\,\lambda_i\,b_{it}^{(p)} \;+\; \lambda_i\,b_{it}^{(p)} \;*\; \sum_{j=1}^N \kappa_{ij}\,a_{jt} \;+\; \lambda_i\,a_{it} \;*\; \sum_{j=1}^N \kappa_{ij}\,b_{jt}^{(p)}, \quad b_{i0}^{(p)} \;=\; 0, \tag{26}$$

and otherwise we get

$$\frac{d}{dt}\,b_{pt}^{(p)} \;=\; a_{pt} * \sum_{j=1}^N \kappa_{pj}\,a_{jt} - a_{pt} - \lambda_p\,b_{pt}^{(p)} + \lambda_p\,b_{pt}^{(p)} * \sum_{j=1}^N \kappa_{pj}\,a_{jt} + \lambda_p\,a_{pt} * \sum_{j=1}^N \kappa_{pj}\,b_{jt}^{(p)}, \tag{27}$$

with the same initial condition $b_{p0}^{(p)} \;=\; 0$. This is a system of linear equations for the vector $b_t^{(p)} \;=\; (b_{it}^{(p)})$. Following standard arguments, for example those of Duffie, Manso and Malamud (2009b), this equation indeed has a unique solution, which is a finite measure, and this solution measure is indeed the derivative of $b_{it}$ with respect to $\lambda_p$.

Denoting

$$c_{it}^{(p)} \;=\; e^{\lambda_i t}\,b_{it}^{(p)},$$

we get that

$$\frac{d}{dt}\,c_{it}^{(p)} \;=\; \lambda_i\,e^{\lambda_i t}\,c_{it}^{(p)} \;*\sum_{j=1}^N \kappa_{ij}\,a_{jt} \;+\; \lambda_i\;a_{it} \;*\; \sum_{j=1}^N \kappa_{ij}\,e^{(\lambda_i-\lambda_j)t}c_{jt}^{(p)}\;, \quad c_{i0}^{(p)} \;=\; 0,$$

and similarly for $i = p$.

Now, let us pass to the moment-generating functions $\hat{c}_{it}^{(p)}$ and $\hat{a}_{it}$ of these measures. Define the matrix

$$\hat{K}(t) \;=\; (\hat{R}_{ij}(t)),$$

where

$$\hat{R}_{ij}(t) \;=\; \kappa_{ij} e^{(\lambda_i - \lambda_j)t}\, \lambda_i\, \hat{a}_{it} \;+\; \delta_{ij}\, \lambda_i \sum_{k=1}^{N} \kappa_{ik} \hat{a}_{kt}$$

and let

$$\hat{\alpha}(t) \;=\; (\delta_{ip})\, e^{\lambda_p t} \left( \hat{a}_{pt} \,*\, \sum_{j=1}^{N} \kappa_{pj}\, \hat{a}_{jt} - \,\hat{a}_{pt} \right).$$

Then, the system (26)-(27) is equivalent to the following system for the moment-generating functions:

$$\frac{d}{dt}\, \hat{c}_t^{(p)} \;=\; \hat{K}(t)\, \hat{c}_t^{(p)} \;+\; \hat{\alpha}(t). \tag{28}$$

Consider the fundamental solution $\Phi(t, \tau)$ to the equation

$$\frac{d}{dt}\, \hat{\Phi}(t, \tau) \;=\; \hat{K}(t)\, \hat{\Phi}(t, \tau)\;, \quad \hat{\Phi}(t, t) \;=\; I_{N \times N}.$$

Then, the unique solution to (28) is given by

$$\hat{c}_t^{(p)} \;=\; \int_0^t \hat{\Phi}(t, \tau)\, \hat{\alpha}(\tau)\, d\tau\,.$$

Once again, a standard argument implies that the matrix $\hat{\Phi}(t, \tau)$ consists of moment generating functions of measures $\Phi_{ij}(t, \tau)$ that solve the system of equations

$$\frac{d}{dt}\, \Phi(t, \tau) \;=\; K(t) \,*\, \Phi(t)\;, \quad \Phi(t, t) \;=\; \mathrm{Id}_{N \times N},$$

where $\mathrm{Id}_{N \times N}$ has the Dirac measure $\delta_0$ for each diagonal element, and zero off-diagonal elements. Since $K(t)$ consists of positive measures, it follows (for example, from the Euler scheme for constructing the solution) that $\Phi(t, \tau)$ is a matrix of positive measures. Hence,

$$b_t^{(p)} \;=\; \mathrm{diag}(e^{-\lambda_i t}) \int_0^t \Phi(t, \tau) \,*\, \alpha(\tau)\, d\tau.$$

Thus, for any monotone increasing bounded $f : \mathbb{Z}_+^N \;\to\; \mathbb{R}$,

$$\frac{\partial}{\partial \lambda_p} \sum_k a_{it}(k)\, f(k) \;=\; \sum_k b_{it}^{(p)}(k)\, f(k) \;=\; e^{-\lambda_i t} \int_0^t \sum_j \sum_k (\Phi_{ij}(t, \tau) * \alpha_j(\tau))(k)\, f(k)\, d\tau.$$

Let $Z$ be a random variable with distribution $\Phi_{ij}(t, \tau)$ (normalized, if necessary, to have mass one) and let $X$ be an independent variable whose distribution is

$$\sum_{j=1}^{N} \kappa_{pj}\, a_{jt}\,.$$

Then,

$$
\begin{aligned}
&\sum_j \sum_k (\Phi_{ij}(t,\tau) \,*\, \alpha_j(\tau))(k)\, f(k) \\
=\;& e^{\lambda_p t} \sum_k \left( \Phi_{ip}(t,\tau) \,*\, \left( \hat{a}_{pt} \,*\, \sum_{j=1}^{N} \kappa_{pj}\, \hat{a}_{jt} - \hat{a}_{pt} \right) \right)(k)\, f(k) \\
=\;& E[f(Z + X + Y_{pt})] \;-\; E[f(Z + Y_{pt})] \;\geq\; 0.
\end{aligned}
$$

The claim follows. ■

**Proof of Lemma 3.5.** First, we say that a pair $(F^H, F^L)$ of cumulative distribution functions (CDFs) on the real line is *amenable* if

$$dF^L(y) = dF^H(-y) = e^{-y}\, dF^H(y), \tag{29}$$

that is, if for any bounded measurable function $g$,

$$\int_{-\infty}^{+\infty} g(y)\, dF^L(y) = \int_{-\infty}^{+\infty} g(-y)\, dF^H(y) = \int_{-\infty}^{+\infty} e^{-y} g(y)\, dF^H(y).$$

It is immediate that the set of amenable pairs of CDFs is closed under mixtures, in the following sense.

**Fact 1**. *Suppose $(A, \mathcal{A}, \eta)$ is a probability space and $F^H : \mathbb{R}\times A \to [0,1]$ and $F^L : \mathbb{R}\times A \to [0,1]$ are jointly measurable functions such that, for each $\alpha$ in $A$, $(F^H(\,\cdot\,,\alpha), F^L(\,\cdot\,,\alpha))$ is an amenable pair of CDFs. Then an amenable pair of CDFs is defined by $(\overline{F}^H, \overline{F}^L)$, where*

$$\overline{F}^H(y) = \int_A F^H(y,\alpha)\, d\eta(\alpha), \qquad \overline{F}^L(y) = \int_A F^L(y,\alpha)\, d\eta(\alpha).$$

The set of amenable pairs of CDFs is also closed under finite convolutions.

**Fact 2**. *Suppose that $X_1, \ldots, X_n$ are independent random variables and $Y_1, \ldots, Y_n$ are independent random variables such that, for each $i$, the CDFs of $X_i$ and $Y_i$ are amenable. Then the CDFs of $X_1 + \cdots + X_n$ and $Y_1 + \cdots + Y_n$ are amenable.*

For a particular signal $Z$ with type $\theta_Z$, let $F^H_Z$ be the CDF of $\theta_Z$ conditional on $Y = 0$, and let $F^L_Z$ be the CDF of $\theta_Z$ conditional on $Y = 1$.

**Fact 3**. *If $Z$ satisfies (12), then $(F^H_Z, F^L_Z)$ is an amenable pair of CDFs.*

In order to verify Fact 3, we let $\theta$ be the outcome of the type $\theta_Z$ on the event $\{Z = 1\}$, so that

$$\theta = \log \frac{\mathbb{P}(Y = 0 \,|\, Z = 1)}{\mathbb{P}(Y = 1 \,|\, Z = 1)} - \log \frac{\mathbb{P}(Y = 0)}{\mathbb{P}(Y = 1)} = \log \frac{\mathbb{P}(Z = 1 \,|\, Y = 0)}{\mathbb{P}(Z = 1 \,|\, Y = 1)}.$$

Because $Z$ satisfies (12), $-\theta$ is the outcome of $\theta_Z$ associated with observing $Z = 0$, so we have the following:

$$\mathbb{P}(Z = 1 \,|\, Y = 0) = \frac{e^\theta}{1 + e^\theta}, \qquad \mathbb{P}(Z = 0 \,|\, Y = 0) = \frac{1}{1 + e^\theta},$$

$$\mathbb{P}(Z = 1 \,|\, Y = 1) = \frac{1}{1 + e^\theta}, \quad \text{and} \quad \mathbb{P}(Z = 0 \,|\, Y = 1) = \frac{e^\theta}{1 + e^\theta}.$$

We can then write the CDFs $F_Z^H$ and $F_Z^L$ as

$$F_Z^H(y) = \frac{e^\theta}{1 + e^\theta}\, 1_{\{\theta \le y\}} \; + \; \frac{1}{1 + e^\theta}\, 1_{\{-\theta \le y\}}$$

and

$$F_Z^L(y) = \frac{1}{1 + e^\theta}\, 1_{\{\theta \le y\}} \; + \; \frac{e^\theta}{1 + e^\theta}\, 1_{\{-\theta \le y\}}.$$

These CDFs are each piece-wise constant, and jump only twice, at $y = -\theta$ and $y = \theta$. We let $\Delta F(y) = F(y) - \lim_{z \uparrow y} F(z)$. At $y = -\theta$ and $y = \theta$, we have $\Delta F_Z^H(-y) = e^{-y} \Delta F_Z^H(y)$ and $\Delta F_Z^L(y) = \Delta F_Z^H(-y)$, completing the proof of Fact 3.

Now, we recall that a particular agent receives at time 0 a random number, say $N$, of signals, where $N$ is independent of all else, and can have a distribution that depends on the agent. By assumption, although the signals need not have the same joint distributions with $Y$, all signals satisfy (12). The type of the set of signals received by the agent is, by Lemma 3.1, the sum of the types of the individual signals. Thus, conditional on $N$, the type $\theta$ of this agent's signal set has a CDF conditional on $Y = 0$, denoted $F_N^H$, and a CDF conditional on $Y = 1$, denoted $F_N^L$, that are the convolutions of the conditional distributions of the underlying $N$ signals given $Y = 0$ and given $Y = 1$, respectively. Thus, by Facts 2 and 3, conditional on $N$, $(F_N^H, F_N^L)$ is an amenable pair of CDFs. Now, we can average these CDFs over the distribution of $N$ to see by Fact 1 that this agent's type has CDFs given $Y = 0$ and $Y = 1$, respectively, that are amenable.

Now, let us consider the cross-sectional distribution of agent types of a given class $i$ at time 0, across the population. Recall that the agent space is the measure space $(G, \mathcal{G}, \gamma)$. Let $\gamma_i$ denote the restriction of $\gamma$ to the subset of class-$i$ agents, normalized by the total mass of this subset. Because of the exact law of large numbers of Sun (2006), we have, almost surely, that on the event $Y = 0$, the fraction $\gamma_i(\{\alpha : \theta_{\alpha 0} \leq y\})$ of class-$i$ agents whose types are less than a given number $y$ is

$$F^H(y) \equiv \int_G F^H_\alpha(y)\, d\gamma_i(\alpha),$$

where $F^H_\alpha$ is the conditional CDF of the type $\theta_{\alpha 0}$ of agent $\alpha$ given $Y = 0$. We similarly define $F^L$ as the cross-sectional distribution of types on the event $Y = 1$. Now, by Fact 1, $(F^H, F^L)$ is an amenable pair of CDFs. By assumption, these CDFs have densities denoted $\psi^H_{i0}$ and $\psi^L_{i0}$, respectively, for class $i$. The definition (29) of amenability implies that

$$\psi^L_{i0}(y) \;=\; \psi^H_{i0}(-y) \;=\; \psi^H_{i0}(y)\, e^{-y}\,,$$

as was to be demonstrated. That $\psi^H_{it}$ satisfies $\psi^H_{it}(-x) = e^{-x}\psi^H_{it}(x) = \psi^L_{it}(x)$ for any $t > 0$ now follows from the Wild sum solution (8) and from the fact that amenability is preserved under convolutions (Fact 2) and mixtures (Fact 1). That the hazard-rate ordering property is satisfied for any density satisfying (13) follows from the calculation (suppressing subscripts for notational simplicity):

$$\frac{G^L(x)}{\psi^L(x)} \;=\; \frac{\int_x^{+\infty} \psi^L(y)\, dy}{\psi^L(x)} \;=\; \frac{\int_x^{+\infty} \psi^H(y)\, e^{(x-y)}\, dy}{\psi^H(x)} \;\leq\; \frac{\int_x^{+\infty} \psi^H(y)\, dy}{\psi^H(x)} \;=\; \frac{G^H(x)}{\psi^H(x)}\,.$$

■

# B ODE and Equilibrium

**Proof of Lemma 3.6.** By the assumptions made, the right-hand side of equation (15) is Lipschitz-continuous, so local existence and uniqueness follow from standard results. To prove the claim for finite $V_0$, it remains to show that the solution does not blow up for $z < v^H$. By Lemma 3.5,

$$\frac{1}{h^H_{it}(V_2(z))} \;\geq\; \frac{1}{h^L_{it}(V_2(z))},$$

and therefore

$$\begin{aligned} V_2'(z) \;&=\; \frac{1}{v_i - v_j} \left( \frac{z - v_i}{v^H - z} \frac{1}{h^H_{it}(V_2(z))} + \frac{1}{h^L_{it}(V_2(z))} \right) \\ &\leq\; \frac{1}{h^H_{it}(V_2(z))} \frac{v^H - v_i}{(v_i - v_j)\,(v^H - z)}\,. \end{aligned} \tag{30}$$

That is,

$$\frac{d}{dz}\left(-\log G_H(V_2(z))\right) \;\leq\; \frac{v^H - v_i}{(v_i - v_j)\,(v^H - z)}.$$

Integrating this inequality, we get

$$\log\left(\frac{G_H(V_0)}{G_H(V_2(z))}\right) \;\le\; \frac{v^H - v_i}{v_i - v_j}\log\frac{v^H - v_i}{v^H - z}.$$

That is,

$$G_H(V_2(z)) \;\ge\; G_H(V_0)\left(\frac{v^H - z}{v^H - v_i}\right)^{\frac{v^H - v_i}{v_i - v_j}},$$

or equivalently,

$$V_2(V_0, z) \;\le\; G_H^{-1}\left(G_H(V_0)\left(\frac{v^H - z}{v^H - v_i}\right)^{\frac{v^H - v_i}{v_i - v_j}}\right).$$

Similarly, we get a lower bound

$$V_2(V_0, z) \;\ge\; G_L^{-1}\left(G_L(V_0)\left(\frac{v^H - z}{v^H - v_i}\right)^{\frac{v^H - v_i}{v_i - v_j}}\right). \tag{31}$$

The fact that $V_2$ is monotone increasing in $V_0$ follows from a standard comparison theorem for ODEs (for example, (Hartman (1982), Theorem 4.1, p. 26). Furthermore, as $V_0 \to -\infty$, the lower bound (31) for $V_2$ converges to

$$G_L^{-1}\left(\left(\frac{v^H - z}{v^H - v_i}\right)^{\frac{v^H - v_i}{v_i - v_j}}\right).$$

Hence, $V_2$ stays bounded from below and, consequently, converges to some function $V_2(-\infty, z)$. Since $V_2(V_0, z)$ solves the ODE (15) for each $V_0$ and the right-hand side of (15) is continuous, $V_2(-\infty, z)$ is also continuously differentiable and solves the same ODE (15). ■

**Proof of Proposition 3.7.** Suppose that $(S, B)$ is a strictly increasing continuous equilibrium and let $V_1(z)$, $V_2(z)$ be the corresponding (strictly increasing and continuous) inverse functions defined on the intervals $(a_1, A_1)$ and $(a_2, A_2)$ respectively, where one or both ends of the intervals may be infinite.

The optimization problems for auction participants are

$$\max_s f_S(s) \equiv \max_s \int_{V_2(s)}^{+\infty} (s - v_j - \Delta_j P(\theta + \phi))\; \Psi_i(P(\theta), \phi)\, d\phi \tag{32}$$

and

$$\max_b f_B(b) \;\equiv\; \max_b \int_{-\infty}^{V_1(b)} \big(v_i + \Delta_i P(\theta + \phi) - S(\theta)\big)\, \Psi_j(P(\phi), \theta)\, d\theta. \tag{33}$$

First, we note that the assumption that $A_1 \le v^H$ implies a positive trading volume. Indeed, by strict monotonicity of $S$, there is a positive probability that the selling price is below $v^H$. Therefore, for buyers of sufficiently high type, it is optimal to participate in trade.

In equilibrium, it can never happen that the seller trades with buyers of all types. Indeed, if that were the case, the seller's utility would be

$$\int_{\mathbb{R}} (s - v_j - \Delta_j P(\theta + \phi)) \; \Psi_i(P(\theta), \phi) \, d\phi,$$

which is impossible because the seller can then attain a larger utility by increasing $s$ slightly. Thus, $a_1 \;\ge\; a_2$. Furthermore, given the assumption $S \le v^H$, buyers of sufficiently high types find it optimal to trade with sellers of arbitrarily high types. That is, $A_2 \;=\; \sup_\theta B(\theta) \;\ge\; \sup_\theta S(\theta) \;=\; A_1$. Thus,

$$A_2 \;\ge\; A_1 \;>\; a_1 \;\ge\; a_2.$$

Let $\theta_l \;=\; V_2(a_1),\; \theta_h \;=\; V_2(A_1)$. (Each of these numbers might be infinite if either $A_2 = A_1$ or $a_2 = a_1$.) By definition, $V_1(a_1) = -\infty,\; V_1(A_1) = +\infty$. Furthermore, $f_B(b)$ is locally monotone increasing in $b$ for all $b$ such that

$$v_i \;+\; \Delta_i P(V_1(b) + \phi) \;-\; S(V_1(b)) \;>\; 0.$$

Further, $f_B(b)$ is locally monotone decreasing in $b$ if

$$v_i \;+\; \Delta_i P(V_1(b) + \phi) \;-\; S(V_1(b)) \;<\; 0.$$

Hence, for any type $\phi \in (\theta_l, \theta_h)$, $B(\phi)$ solves the equation

$$v_i \;+\; \Delta_i \, P(V_1(B(\phi)) \;+\; \phi)) \;=\; B(\phi).$$

Letting $B(\phi) = z \in (a_1, A_1)$, we get that

$$v_i \;+\; \Delta_i \, P(V_1(z) \;+\; V_2(z)) \;=\; z\,. \tag{34}$$

Now, as $\phi \uparrow \theta_h$, we have $B(\phi) \uparrow A_1$ and therefore $V_1(B(\phi)) \uparrow +\infty$. Thus,

$$A_1 \;=\; \lim_{\phi \uparrow \theta_h} B(\phi) \;=\; \lim_{\phi \uparrow \theta_h} (v_i \;+\; \Delta_i \, P(V_1(B(\phi)) \;+\; \phi))) \;=\; v^H,$$

and similarly, $a_1 \;=\; v_i$

We now turn to the first-order condition of the seller. Because $V_2$ is strictly increasing and continuous, it is differentiable Lebesque-almost everywhere by the Lebesque

Theorem (see, for example, Theorem 7.2 of Knapp (2005), p. 359). Let $X \subset (a_2, A_2)$ be the set on which $V_2'$ exists and is finite. Then, for all $\theta \in V_1(X)$ the first-order condition holds for the seller. For a seller of type $\theta$, because the offer price $s$ affects the limit of the integral defining the seller's utility (9) as well as the integrand, there are two sources of marginal utility associated with increasing the offer $s$: (i) losing the gains from trade with the marginal buyers, who are of type $B^{-1}(s)$), and (ii) increasing the gain from every infra-marginal buyer type $\phi$. At an optimal offer $S(\theta)$, these marginal effects are equal in magnitude. This leaves the seller's first-order condition

$$G_i(P(\theta), V_2(S(\theta))) \;=\; V_2'(S(\theta))\left(S(\theta) - v_j - \Delta_j\, P(\theta + V_2(S(\theta)))\right)\Psi_i(P(\theta)\,,\, S(\theta)), \quad (35)$$

where

$$G_i(p, x) \;=\; \int_x^{+\infty} \Psi_i(p, y)\, dy.$$

Letting $z = S(\theta)$, we have $\theta = V_1(z)$ and hence

$$\frac{G_i(P(V_1(z)), V_2(z))}{\Psi_i(P(V_1(z)), V_2(z))} \;=\; V_2'(z)\left(z \,-\, v_j \,-\, \Delta_j\, P(V_1(z) \,+\, V_2(z))\right). \quad (36)$$

Now, if $V_2(z)$ were not absolutely continuous, it would have a singular component and therefore, by the de la Valée Poussin Theorem (Saks (1937), p.127) there would be a point $z_0$ where $V_2'(z_0) \;=\; +\infty$. Let $\theta \;=\; V_1(z_0)$. Then, $S(\theta)$ cannot be optimal because there will an inequality $<$ in (35) and therefore there will always be an incentive to deviate. Thus, $V_2(z)$ is absolutely continuous and, since the right-hand side of (36) is continuous and (36) holds almost everywhere in $(a_2, A_2)$, identity (36) actually holds for all $z \in (a_2, A_2)$.

Now, using the first order condition (34) for the buyer, we have

$$z \,-\, v_j \,-\, \Delta_j\, P(V_1(z) \,+\, V_2(z)) \;=\; z \,-\, v_j \,-\, \frac{\Delta_j}{\Delta_i}\,(z \,-\, v_i) \;=\; \frac{v_i \,-\, v_j}{v^H - v_i}\,(v^H \,-\, z). \quad (37)$$

Furthermore, (34) implies that

$$P(V_1(z){+}V_2(z)) \;=\; \frac{R\, e^{V_1(z)+V_2(z)}}{1 \,+\, R\, e^{V_1(z)+V_2(z)}} \;=\; \frac{z - v_i}{v^H - v_i} \;\Leftrightarrow\; V_1(z){+}V_2(z) \;=\; \log\frac{z - v_i}{v^H - z} - \log R\,.$$

That is,

$$V_1(z) \;=\; \log\frac{z - v_i}{v^H - z} \;-\; V_2(z) \;-\; \log R\,.$$

Therefore,

$$P(V_1(z)) \;=\; \frac{e^{-V_2(z)}\,\frac{z - v_i}{v^H - z}}{1 \,+\, e^{-V_2(z)}\,\frac{z - v_i}{v^H - z}} \;=\; \frac{(z - v_i)e^{-V_2(z)}}{v^H - z \,+\, e^{-V_2(z)}\,(z - v_i)}.$$

Using the fact that $\Psi_i^L(V_2(z)) \;=\; e^{-V_2(z)}\,\Psi_i^H(V_2(z))$, we get

$$\begin{aligned}\Psi_i(P(V_1(z)),V_2(z)) \;&=\; P(V_1(z))\,\Psi_i^H(V_2(z)) \,+\, (1-P(V_1(z)))\,\Psi_i^L(V_2(z))\\ &=\; \frac{(z-v_i)\,e^{-V_2(z)}}{v^H - z \,+\, e^{-V_2(z)}\,(z-v_i)}\,\Psi_i^H(V_2(z))\\ &\qquad +\frac{(v^H-z)\,e^{-V_2(z)}}{v^H - z \,+\, e^{-V_2(z)}\,(z-v_i)}\,\Psi_i^H(V_2(z))\\ &=\; \frac{v^H - v_i}{v^H - z \,+\, e^{-V_2(z)}\,(z-v_i)}\,\Psi_i^L(V_2(z))\,.\end{aligned}$$

Similarly,

$$\begin{aligned}G_i(P(V_1(z)),V_2(z)) \;&=\; P(V_1(z))\,G_i^H(V_2(z)) \,+\, (1-P(V_1(z)))\,G_i^L(V_2(z))\\ &=\; \frac{(z-v_i)e^{-V_2(z)}\,G_i^H(V_2(z)) \,+\, (v^H-z)\,G_i^L(V_2(z))}{v^H - z \,+\, e^{-V_2(z)}\,(z-v_i)}\,.\end{aligned} \tag{38}$$

Consequently,

$$\begin{aligned}\frac{G_i(P(V_1(z)),V_2(z))}{\Psi_i(P(V_1(z)),V_2(z))} \;&=\; \frac{P(V_1(z))\,G_i^H(V_2(z)) \,+\, (1-P(V_1(z)))\,G_i^L(V_2(z))}{P(V_1(z))\,\Psi_i^H(V_2(z)) \,+\, (1-P(V_1(z)))\,\Psi_i^L(V_2(z))}\\ &=\; \frac{(z-v_i)e^{-V_2(z)}\,G_i^H(V_2(z)) \,+\, (v^H-z)\,G_i^L(V_2(z))}{(v^H-v_i)\,\Psi_i^L(V_2(z))}\\ &=\; (v^H-v_i)^{-1}\left((z-v_i)\,\frac{1}{h_i^H(V_2(z))} \,+\, (v^H-z)\,\frac{1}{h_i^L(V_2(z))}\right).\end{aligned}$$

Thus, by (37), the ODE (36) takes the form

$$\begin{aligned}V_2'(z) \;&=\; \frac{G_i(P(V_1(z)),V_2(z))}{\Psi_i(P(V_1(z)),V_2(z))\,\big(z \,-\, v_j \,-\, \Delta_j\,P(V_1(z) \,+\, V_2(z))\big)}\\ &=\; (v^H-v_i)^{-1}\left((z-v_i)\,\frac{1}{h_i^H(V_2(z))} \,+\, (v^H-z)\,\frac{1}{h_i^L(V_2(z))}\right)\frac{1}{\frac{v_i-v_j}{v^H-v_i}\,(v^H \,-\, z)}\\ &=\; \frac{1}{v_i-v_j}\left(\frac{z-v_i}{v^H-z}\,\frac{1}{h_i^H(V_2(z))} \,+\, \frac{1}{h_i^L(V_2(z))}\right),\quad z\in(a_1,A_1) \;=\; (v_i,v^H).\end{aligned}$$

Consequently, $V_2(z)$ solves (15). By Lemma 3.6, $V_2(v^H)=+\infty$. Thus $A_2=v^H$ and the proof is complete. ■

**Proof of Corollary 3.8.** By Proposition 3.7, $V_2(V_0,z)$ is monotone increasing in $V_0$. Consequently, $B \;=\; V_2^{-1}$ is monotone decreasing in $V_0$. Similarly,

$$V_1(V_0,z) \;=\; \log\frac{z-v_i}{v^H-z} \;-\; V_2(V_0,z) \,-\, \log R$$

is monotone decreasing in $V_0$ and therefore $S \;=\; V_1^{-1}$ is monotone increasing in $V_0$. ■

In order to prove Proposition 3.10, we will need the following auxiliary result

**Lemma B.1** *Suppose that* $B, S : \mathbb{R} \to (v_i, v^H)$ *are strictly increasing and that their inverses* $V_1$ *and* $V_2$ *satisfy*

$$v_i \ + \ \Delta_i \, P(V_1(z) \ + \ V_2(z)) \ = \ z.$$

*Suppose further that* $V_2'(z)$ *solves* (15) *for all* $z \in (v_i, v^H)$*. Then* $(B\ ,\ S)$ *is an equilibrium.*

**Proof.** Recall that the seller maximizes

$$f_S(s) \ = \ \int_{V_2(s)}^{+\infty} (s - v_j \ - \Delta_j P(\theta + \phi)) \ \Psi_i(P(\theta), \phi) \, d\phi. \tag{39}$$

To show that $S(\theta)$ is indeed optimal, it suffices to show that $f_S'(s) \ \geq \ 0$ for $s \ \leq \ S(\theta)$ and that $f_S'(s) \ \leq \ 0$ for $s \ \geq \ S(\theta)$. We prove only the first inequality. A proof of the second is analogous. So, let $s \leq S(\theta) \ \Leftrightarrow \ V_1(s) \ \leq \ \theta$. Then,

$$\begin{aligned} f_S'(s) \ &= \ V_2'(s) \ (-s \ + \ v_j \ + \Delta_j P(\theta + V_2(s))) \ \Psi_i(P(\theta), V_2(s)) \ + \ G_i(P(\theta), V_2(s)) \\ &= \ V_2'(s) \Psi_i(P(\theta), V_2(s)) \left( -s + v_j + \Delta_j P(\theta + V_2(s)) + \frac{1}{V_2'(s) h_i(P(\theta), V_2(s))} \right). \end{aligned}$$

By Lemma 3.5,

$$\frac{1}{h_i(p, V_2(s))}$$

is monotone increasing in $p$. Therefore, by (36),

$$\frac{1}{V_2'(s) \, h_i(P(\theta), V_2(s))} \ \geq \ \frac{1}{V_2'(s) \, h_i(P(V_1(S)), V_2(s))} \ = \ s \ - \ v_j \ - \ \Delta_j \, P(V_1(s) + V_2(s)).$$

Hence,

$$\begin{aligned} f_S'(s) \quad &\geq \quad V_2'(s) \, \Psi_i(P(\theta), V_2(s)) \\ &\qquad \times (-s + v_j \ + \Delta_j P(\theta + V_2(s)) \ + \ \ s \ - \ v_j \ - \ \Delta_j \, P(V_1(s) + V_2(s))) \ \geq \ 0, \end{aligned}$$

because $\theta \ \geq \ V_1(s)$.

For the buyer, it suffices to show that

$$f_B(b) \ = \ \max_b \int_{-\infty}^{V_1(b)} \big( v_i \ + \ \Delta_i P(\theta + \ \phi) \ \ - \ S(\theta) \big) \, \Psi_j(P(\phi), \theta) \, d\theta \tag{40}$$

satisfies $f_B'(b) \ \geq \ 0$ for $b \ \leq \ B(\phi)$ and $f_B'(b) \ \leq \ 0$ for $b \ \geq \ B(\phi)$. That is,

$$v_i \ + \ \Delta_i \, P(\phi \ + \ V_1(b)) \ - \ S(V_1(b)) \ = \ v_i \ + \ \Delta_i \, P(\phi \ + \ V_1(b)) \ - \ b \ \geq \ 0$$

for $b \ \leq B(\phi)$, and the reverse inequality for $b \geq B(\phi)$. For $b \ \leq B(\phi)$, we have $\phi \ \geq \ V_2(b)$ and therefore

$$v_i \ + \ \Delta_i \, P(\phi \ + \ V_1(b)) \ - \ b \ \geq \ v_i \ + \ \Delta_i \, P(V_2(b) \ + \ V_1(b)) \ - \ b \ = \ 0,$$

as claimed. The case of $b \geq B(\phi)$ is analogous. ■

## C Exponential Tails

**Lemma C.1** *The Laplace transform* $\hat{\psi}^H_{it}(k)$ *is monotone increasing in* $k$ *for each* $i$ *and all* $t \geq 0$.

**Proof.** Using the identity $\psi^H_{it}(-x) = e^{-x}\,\psi^H_{it}(x)$, we get

$$\begin{aligned} \frac{d}{dk}\,\hat{\psi}^H_{it}(k) \quad &= \quad \int_{\mathbb{R}} x\, e^{kx}\, \psi^H_{it}(x)\, dx \;=\; \int_{-\infty}^{0} x\, e^{kx}\, \psi^H_{it}(x)\, dx \;+\; \int_{0}^{+\infty} x\, e^{kx}\, \psi^H_{it}(x)\, dx \\ &= \quad \int_{0}^{+\infty} x\,(1-e^{-x})\, e^{kx}\, \psi^H_{it}(x)\, dx \;>\; 0. \end{aligned}$$

■

Lemma 3.9 is a direct consequences of the following result, which also gives the exponential tail property for $\psi^H_{it}$.

**Proposition C.2 (Exponential tails)** *Let* $k = \alpha(t)$ *be the unique solution to*

$$\hat{\phi}^H_{10}(k)\, e^{-\hat{\phi}^H_{10}(k)\,(1-e^{-\lambda t})} \;=\; \hat{\phi}^H_{20}(k)\, e^{-\hat{\phi}^H_{20}(k)\,(1-e^{-\lambda t})}$$

*satisfying* $\min_{i\in\{1,2\}}\, \phi^H_{i0}(\alpha(t)) \;\leq\; (1-e^{-\lambda t})^{-1}$. *Then, for all* $t$ *in* $(0,T]$,

$$\begin{aligned} \psi^H_{it} \quad &\sim \quad \mathrm{Exp}_{+\infty}(c_i(t)\,,\, 0\,,\, -\alpha(t)) \\ \frac{d}{dx}\psi^H_{it} \quad &\sim \quad \mathrm{Exp}_{+\infty}(-\alpha(t)\, c_i(t), 0, -\alpha(t)), \end{aligned} \tag{41}$$

*with*[9]

$$c_i(t) \;=\; \frac{e^{-\lambda t}\, \hat{\psi}^H_{i0}(\alpha(t))\, (\hat{\phi}^H_{10} - \hat{\phi}^H_{20})(\alpha(t))}{\hat{\phi}^H_{10}(\alpha(t))\, \frac{d}{dk}\left((1-e^{-\lambda t})(\hat{\phi}^H_{10}(k) - \hat{\phi}^H_{20}(k)) \;-\; \log\left(\frac{\hat{\phi}^H_{10}}{\hat{\phi}^H_{20}}\right)(k)\right)|_{\,k=\alpha(t)}}\,.$$

*Furthermore, these exponential tails are uniform in* $t$ *if* (41) *holds for* $t = 0$.

**Proof.** Since the functions $\hat{\phi}_{10}(k)$ and $\hat{\phi}_{20}(k)$ are analytic in $k$ in the stripe

$$\mathcal{H} \;:=\; \Re k \,\in\, \left(-\min_{i\,\in\,\{1,\dots,n+m\}} \alpha_{i0} \,-\, 1,\;\; \min_{i\,\in\,\{1,\dots,n+m\}} \alpha_{i0}\right)\,,$$

[9]For the case in which $\hat{\phi}^H_{i0}(\alpha(t)) \;=\; (1-e^{-\lambda t})^{-1}$ for $i = 1, 2$, both the numerator and the denominator of $c_i(t)$ are zero, so the stated formula must be understood as a limit as $k \uparrow \alpha(t)$. In this case,

$$c_i(t) \;=\; \frac{e^{-\lambda t}\, \hat{\psi}^H_{i0}(\alpha(t))}{(1-e^{-\lambda t})\, \frac{1}{2}\frac{d}{dk}\left(\hat{\phi}^H_{20}(k) + \hat{\phi}^H_{10}(k)\right)|_{k=\alpha(t)}}.$$

it follows directly from their definitions that the functions $\hat{\phi}_{1t}(k)$ and $\hat{\phi}_{2t}(k)$ are meromorphic in $k$ for $k \in \mathcal{H}$.

Let

$$\mathcal{T} \;=\; \{t > 0 : \exists k > 0 \;:\; \hat{\phi}_{10}(k) \;=\; \hat{\phi}_{20}(k) \;=\; (1 - e^{-\lambda t})^{-1}\}\,.$$

By analyticity, $\mathcal{T}$ is at most countable. For any $t \in \mathcal{T}$, define

$$\alpha(t) \;=\; \min\{k \,:\, \hat{\phi}_{10}(k) \;=\; \hat{\phi}_{20}(k) \;=\; (1 - e^{-\lambda t})^{-1}\}$$

and

$$n(t) \;=\; \max\left\{l \,\geq\, 0 \,:\, \frac{d^m}{dk^m}\left(\hat{\phi}_{10}(k) \,-\, \hat{\phi}_{20}(k)\right)|_{k=\alpha(t)} \;=\; 0, \quad m \leq l\right\}.$$

For any $t \notin \mathcal{T}$, let $\alpha(t)$ be the unique solution to

$$\hat{\phi}_{10}(k)\, e^{-\hat{\phi}_{10}(k)\,(1-e^{-\lambda t})} \;=\; \hat{\phi}_{20}(k)\, e^{-\hat{\phi}_{20}(k)\,(1-e^{-\lambda t})}$$

and let

$$\begin{aligned} n(t) \;=\; & \max\Big\{l \geq 0 : \\ & \frac{d^m}{dk^m}\left(\hat{\phi}_{10}(k)\, e^{-\hat{\phi}_{10}(k)(1-e^{-\lambda t})} - \hat{\phi}_{20}(k) e^{-\hat{\phi}_{20}(k)(1-e^{-\lambda t})}\right)|_{k=\alpha(t)} \;= 0,\; m \leq l\Big\}. \end{aligned}$$

By Lemma C.1, $\hat{\phi}_{i0}(k)$ is monotone increasing in $k$. Thus, either $\hat{\phi}_{10}(k)$ and $\hat{\phi}_{20}(k)$ hit the level $(1-e^{-\lambda t})^{-1}$ together, in which case we are in the set $\mathcal{T}$, or one of them crosses this level earlier than the other. The function $x \mapsto xe^{-x(1-e^{-\lambda t})}$ is monotone increasing for $x < (1-e^{-\lambda t})^{-1}$, and monotone decreasing for $x > (1-e^{-\lambda t})^{-1}$. Hence, when $k$ reaches the level $\alpha(t)$,

$$\frac{d}{dk}\left(\hat{\phi}_{10}(k)\, e^{-\hat{\phi}_{10}(k)\,(1-e^{-\lambda t})}\right)|_{k=\alpha(t)} \;\text{ and }\; \frac{d}{dk}\left(\hat{\phi}_{20}(k)\, e^{-\hat{\phi}_{20}(k)\,(1-e^{-\lambda t})}\right)|_{k=\alpha(t)}$$

have opposite signs, implying that $n(t) = 0$.

First, consider some $t \in \mathcal{T}$. By assumption,

$$\hat{\phi}_{20}(\alpha(t)) - \hat{\phi}_{10}(\alpha(t)) \;\approx\; \frac{1}{(n(t)+1)!}\,\frac{d^{n(t)+1}}{dk^{n(t)+1}}\left(\hat{\phi}_{20}(k) - \hat{\phi}_{10}(k)\right)|_{k=\alpha(t)}.$$

Similarly, a direct calculation shows that, for any smooth function $f$ such that

$$f'((1-e^{-\lambda t})^{-1}) = 0, \quad f''((1-e^{-\lambda t})^{-1}) \neq 0,$$

we have

$$f(\hat{\phi}_{20}(k)) \; - \; f(\hat{\phi}_{10}(k)) \; \approx \; \frac{1}{(n(t)+1)!}\,(k-\alpha(t))^{n(t)+2}\, f''((1-e^{-\lambda t})^{-1})\xi_k,$$

where

$$\xi_k = \frac{d^{n(t)+1}}{dk^{n(t)+1}}\left(\hat{\phi}_{20}(k) - \hat{\phi}_{10}(k)\right)|_{k=\alpha(t)}\,\frac{1}{2}\frac{d}{dk}\left(\hat{\phi}_{20}(k) + \hat{\phi}_{10}(k)\right)|_{k=\alpha(t)}.$$

In our case,

$$f(x) \; = \; x\,e^{-x(1-e^{-\lambda t})} \;\Rightarrow\; f''((1-e^{-\lambda t})^{-1}) \; = \; -(1-e^{-\lambda t})^{-1}\,e^{-1}\,.$$

Thus, the leading term of the asymptotic behavior at $k = \alpha(t)$ is given by

$$\hat{\psi}_{it} \underset{k\uparrow\alpha(t)}{\sim} \frac{e^{-\lambda t}\,\hat{\psi}_{i0}(\alpha(t))}{(1-e^{-\lambda t})\,\frac{1}{2}\frac{d}{dk}\left(\hat{\phi}_{20}(k)+\hat{\phi}_{10}(k)\right)|_{k=\alpha(t)}}\,\frac{1}{\alpha(t)-k}.$$

Similarly, for $t \notin \mathcal{T}$,

$$\hat{\psi}_{it} \underset{k\uparrow\alpha(t)}{\sim} \frac{e^{-\lambda t}\,\hat{\psi}_{i0}(\alpha(t))\,(\hat{\psi}_{20}-\hat{\psi}_{10})(\alpha(t))\,e^{-\hat{\phi}_{10}(\alpha(t))\,(1-e^{-\lambda t})}}{\frac{d}{dk}\left(\hat{\phi}^{H}_{10}(k)\,e^{-\hat{\phi}^{H}_{10}(k)\,(1-e^{-\lambda t})} \; - \; \hat{\phi}_{20}(k)\,e^{-\hat{\phi}_{20}(k)\,(1-e^{-\lambda t})}\right)|_{k=\alpha(t)}\,(\alpha(t)-k)}\,.$$

By Theorem 3.3, we can write

$$\psi_{it} \; = \; \sum_{k} a_{it}(k)\,\psi_{10}^{*k_1}\,\cdots\,\psi_{N0}^{*k_N}$$

and $a_{it}(k) = 0$ if $k_1 = 0$. Thus,

$$\psi_{it} \; = \; \psi_{i0} \, * \, \zeta,$$

with

$$\zeta \; \stackrel{def}{=} \; \sum_{k} a_{it}(k)\,\psi_{10}^{*(k_1-1)}\,\cdots\,\psi_{N0}^{*k_N}\,.$$

Then,

$$\hat{\zeta} \; = \; \frac{\hat{\psi}_{it}}{\hat{\psi}_{i0}}.$$

By Theorem 3.3, $\zeta$ is the density of a probability measure. A Tauberian Theorem (Proposition 1 in Aramaki (1983))[10] implies that, for any $\varepsilon > 0$,

$$X(y) \; \stackrel{def}{=} \; \int_{-\infty}^{y} e^{\varepsilon+\alpha(t)\,x}\,\zeta(x)\,dx$$

[10] In fact, we could have directly used Ikehara's Tauberian Theorem (see, for example, Theorem 4.2 of Korevaar (2004), p.124). However, we appeal to the higher order version of Ikehara's Theorem to show that our result does not depend on the fact that $n(t) = 0$.

satisfies the asymptotic

$$X(y) \sim \tilde{c}_i(t)\,\varepsilon^{-1}\,e^{\varepsilon y},$$

where

$$\tilde{c}_i(t) = \frac{c_i(t)}{\hat{\psi}_{i0}(\alpha(t))}.$$

Thus,[11]

$$\begin{aligned}
\psi_{it}(x) &= \int_{\mathbb{R}} \psi_{i0}(x-y)\,\zeta(y)\,dy = \int_{\mathbb{R}} \psi_{i0}(x-y)\,e^{-(\alpha(t)+\varepsilon)\,y}\,dX(y) \\
&= \int_{\mathbb{R}} e^{-(\alpha(t)+\varepsilon)y}\,X(y)\left(\frac{d}{dx}\psi_{i0}(x-y) + (\alpha(t)+\varepsilon)\psi_{i0}(x-y)\right)dy \qquad (42)\\
&= e^{-(\alpha(t)+\varepsilon)x}\int_{\mathbb{R}} e^{(\alpha(t)+\varepsilon)y}\,X(x-y)\left(\frac{d}{dy}\psi_{i0}(y) + (\alpha(t)+\varepsilon)\psi_{i0}(y)\right)dy.
\end{aligned}$$

Therefore, by the Lebesque dominated convergence theorem,

$$\lim_{x\to+\infty}\frac{\psi_{it}(x)}{e^{-\alpha(t)x}} = \int_{\mathbb{R}} e^{\alpha(t)y}\,\varepsilon^{-1}\,\tilde{c}_i(t)\left(\frac{d}{dy}\psi_{i0}(y) + (\alpha(t)+\varepsilon)\psi_{i0}(y)\right)dy. \qquad (43)$$

But

$$\int_{\mathbb{R}} e^{\alpha(t)y}\left(\frac{d}{dy}\psi_{i0}(y) + \alpha(t)\psi_{i0}(y)\right)dy = \int_{\mathbb{R}}\left(\frac{d}{dy}(e^{\alpha(t)y}\,\psi_{i0}(y))\right)dy = 0,$$

and therefore

$$\lim_{x\to+\infty}\frac{\psi_{it}(x)}{e^{-\alpha(t)x}} = \tilde{c}_i(t)\int_{\mathbb{R}} e^{\alpha(t)y}\psi_{i0}(y)\,dy = c_i(t).$$

The asymptotic behavior of

$$\frac{d}{dx}\psi_{it}(x) = \int_{\mathbb{R}}\frac{d}{dx}\,\psi_{i0}(x-y)\,\zeta(y)\,dy$$

is proved analogously. The fact that the tails are uniform follows from a standard proof of the Tauberian Theorem (see the proof of Proposition 1 of Aramaki (1983)).[12]

■

**Corollary C.3** *Let $\alpha_*$ be as in Proposition 3.10. Suppose that*

$$T < \frac{1}{\lambda}\log\left(\frac{\max\{\hat{\phi}^H_{10}(\alpha_*),\,\hat{\phi}^H_{20}(\alpha_*)\}}{1-\max\{\hat{\phi}^H_{10}(\alpha_*),\,\hat{\phi}^H_{20}(\alpha_*)\}}\right).$$

[11] We note that $\alpha_i(0) > \alpha_i(t)$, and therefore the boundary terms arising from integration by parts vanish for sufficiently small $\varepsilon$.

[12] In fact, Subhankulov (1976) (Theorem 5.1.2, p. 196) establishes strong bounds on the tails that can be used to determine the exact speed of convergence to exponential tails.

*Then, there exists an $A > 0$ such that, for any*

$$\frac{v_i - v_j}{v^H - v_i} \; > \; A,$$

*there exists a unique continuous equilibrium. By contrast, if*

$$T \; > \; \frac{1}{\lambda} \log \left( \frac{\min\{\hat{\phi}_{10}^H(1)\,,\,\hat{\phi}_{20}^H(1)\}}{1 \, - \, \min\{\hat{\phi}_{10}^H(1)\,,\,\hat{\phi}_{20}^H(1)\}} \right),$$

*then there exist no continuous equilibria.*

# D Proof of Proposition 3.10

**Proof of Proposition 3.10.** It follows from Proposition 3.7 and Lemma B.1 that a strictly monotone equilibrium in undominated strategies exists if and only if there exists a solution $V_2(z)$ to (15) such that $V_2(v_i) \; = \; -\infty$ and

$$V_1(z) \; = \; \log \frac{z - v_i}{v^H - z} \; - \; V_2(z) \, - \, \log R$$

is monotone increasing in $z$ and satisfies $V_1(v_i) = -\infty$ , $V_1(v^H) = +\infty$. Furthermore, such an equilibrium is unique if the solution to the ODE (15) with $V_2(v_i) \; = \; -\infty$ is unique.

Fix a $t \leq T$ and denote for brevity $\alpha \; = \; \alpha_{it}$ , $\gamma \; = \; \gamma_{it}$ , $c \; = \; c_{it}$. Let also

$$g(z) \; = \; e^{(\alpha+1)\, V_2(z)} \,.$$

Then, a direct calculation shows that $V_2(z)$ solves (15) with $V_2(v_i) \; = \; -\infty$ if and only if $g(z)$ solves

$$\begin{aligned} &g'(z) \\ &= g(z) \, \frac{\alpha+1}{v_i - v_j} \left( \frac{z - v_i}{v^H - z} \, \frac{1}{h_i^H((\alpha+1)^{-1} \log g(z))} \, + \, \frac{1}{h_i^L((\alpha+1)^{-1} \log g(z))} \right), \end{aligned} \tag{44}$$

with $g(v_i) \; = \; 0$. By assumption and Lemma 3.5,

$$h_i^H(V) \; \sim \; c_i \, |V|^{\gamma} \, e^{(\alpha+1)V} \quad \text{and} \quad h_i^L(V) \; \sim \; c_i \, |V|^{\gamma} \, e^{\alpha V} \tag{45}$$

as $V \to -\infty$ because $G_i^{H,L}(V) \, \to \, 1$. Hence, the right-hand side of (44) is continuous and the existence of a solution follows from the Euler theorem. Therefore, when studying the asymptotic behavior of $g(z)$ as $z \downarrow v_i$, we can replace $h_i^H$ and $h_i^L$ by their respective asymptotics (45).

Indeed, let us consider

$$\tilde g'(z) \;=\; (\alpha+1)\,\tilde g(z)\,\frac{1}{v_i-v_j}\left(\frac{z-v_i}{v^H-z}\,\frac{1}{c\,((\alpha+1)^{-1}\log 1/\tilde g)^{\gamma}\,\tilde g} + \frac{1}{c((\alpha+1)^{-1}\log 1/\tilde g)^{\gamma}\,\tilde g^{\alpha/(\alpha+1)}}\right), \tag{46}$$

with the initial condition $\tilde g(v_i) \;=\; 0$. We consider only values of $z$ sufficiently close to $v_i$, so that $\log \tilde g(z) \;<\; 0$.

It follows from standard ODE comparison arguments and the results below that for any $\varepsilon > 0$ there exists a $\bar z > v_i$ such that

$$\left|\frac{g(z)}{\tilde g(z)} - 1\right| \;+\; \left|\frac{g'(z)}{\tilde g'(z)} - 1\right| \;\le\; \varepsilon \tag{47}$$

for all $z \in (v_i, \bar z)$. The assumptions of the Proposition guarantee that the same asymptotics hold for the derivatives of the hazard rates, which implies that the estimates obtained in this manner are uniform.

First, we will consider the case of general (not necessarily large) $v_i - v_j$ and show that, when $\alpha < 1$, $g(z)$ decays so fast as $z \downarrow v_i$ that $V_1(z)$ cannot remain monotone increasing.

At points in the proof, we will define suitable positive constants denoted $C_1$, $C_2$, $C_3, \ldots$ without further mention.

Denote

$$\zeta \;=\; \frac{(\alpha+1)^{\gamma+1}}{c\,(v_i-v_j)}\,. \tag{48}$$

Then, we can rewrite (46) in the form

$$\tilde g'(z) \;=\; \frac{\zeta}{(\log 1/\tilde g)^{\gamma}}\left(\frac{z-v_i}{v^H-z} \;+\; \tilde g^{1/(\alpha+1)}\right). \tag{49}$$

From this point, throughout the proof, without loss of generality, we assume that $v_i = 0$. Furthermore, after rescaling if necessary, we may assume that $v^H - v_i = 1$. Then, the same asymptotic considerations as above imply that, when studying the behavior of $\tilde g$ as $z \downarrow v_i$, we may replace $v^H - z \approx v^H - v_i$ in (46) by 1.

Let $A(z)$ be the solution to

$$z \;=\; \int_0^{A(z)} \zeta^{-1}\,(-\log x)^{\gamma}\,x^{-1/(\alpha+1)}\,dx\,.$$

A direct calculation shows that

$$B(z) \stackrel{def}{=} \int_0^z \zeta^{-1} (-\log x)^\gamma x^{-1/(\alpha+1)} \, dx \sim \zeta^{-1} \frac{\alpha+1}{\alpha} (-\log z)^\gamma z^{\alpha/(\alpha+1)} .$$

Conjecturing the asymptotics

$$A(z) \sim K (-\log z)^{\gamma(\alpha+1)/\alpha} z^{(\alpha+1)/\alpha} \tag{50}$$

and substituting these into $B(A(z)) = z$, we get

$$K = \zeta^{\frac{\alpha+1}{\alpha}} \left( \frac{\alpha}{\alpha+1} \right)^{\frac{(\gamma+1)(\alpha+1)}{\alpha}} .$$

Standard considerations imply that this is indeed the asymptotic behavior of $A(z)$. It is then easy to see that

$$A'(z) \sim K \frac{\alpha+1}{\alpha} (-\log z)^{\gamma(\alpha+1)/\alpha} z^{1/\alpha}. \tag{51}$$

By (49),

$$\tilde{g}'(z) \geq \frac{\zeta}{(\log 1/\tilde{g})^\gamma} \tilde{g}^{1/(\alpha+1)}.$$

Integrating this inequality, we get $\tilde{g}(z) \geq A(z)$. Now, the factor $(\log 1/\tilde{g})^\gamma$ is asymptotically negligible as $z \downarrow v_i$. Namely, for any $\varepsilon > 0$ there exists a $C_1 > 0$ such that

$$C_1 \tilde{g}^{1/(\alpha+\varepsilon+1)} \geq \frac{\zeta}{(\log 1/\tilde{g})^\gamma} \tilde{g}^{1/(\alpha+1)} \geq C_1^{-1} \tilde{g}^{1/(\alpha-\varepsilon+1)}.$$

Thus,

$$\left( (\tilde{g})^{\frac{\alpha-\varepsilon}{1+\alpha-\varepsilon}} \right)' \geq C_2 .$$

Integrating this inequality, we get that

$$\tilde{g}(z) \geq C_3 (z - v_i)^{\frac{\alpha-\varepsilon+1}{\alpha-\varepsilon}} . \tag{52}$$

Let

$$l(z) = B(\tilde{g}(z)) - z .$$

Then, for small $z$, by (50),

$$\begin{aligned} l'(z) &= \tilde{g}'(z) \zeta^{-1} (-\log \tilde{g})^\gamma \tilde{g}^{-1/(\alpha+1)} - 1 \\ &= \frac{\zeta}{(\log 1/\tilde{g})^\gamma} \left( \frac{z}{v^H - z} + \tilde{g}^{1/(\alpha+1)} \right) \zeta^{-1} (-\log \tilde{g})^\gamma \tilde{g}^{-1/(\alpha+1)} - 1 \\ &= \frac{z}{1-z} \frac{1}{\tilde{g}^{1/(\alpha+1)}} \\ &= \frac{z}{1-z} \frac{1}{(A(l(z)+z))^{1/(\alpha+1)}} \\ &\leq \frac{z}{1-z} \frac{1}{(A(l(z)))^{1/(\alpha+1)}} , \end{aligned} \tag{53}$$

where we have used the fact that $l(z) \geq 0$ because $h(0) = 0$ and $l'(z) \geq 0$. Integrating this inequality, we get that, for small $z$,

$$l(z) \ \leq \ C_4 \, z^{2(\alpha-\varepsilon)/(\alpha-\varepsilon+1)}.$$

Hence, for small $z$,

$$\tilde{g}(z) \ = \ A(l(z) \, + \, z) \ \leq \ A((C_4+1)z^{2(\alpha-\varepsilon)/(\alpha-\varepsilon+1)}) \ \leq \ C_5 \, z^{2-\varepsilon}. \tag{54}$$

Let $C(z)$ solve

$$\int_0^{C(z)} (-\log x)^\gamma \, dx \ = \ \zeta \int_0^z \frac{x}{1-x} \, dx \, .$$

A calculation similar to that for the function $A(z)$ implies that

$$C(z) \ \sim \ C_6 \, (-\log z)^\gamma \, z^2 \tag{55}$$

as $z \to 0$. Integrating the inequality

$$\tilde{g}'(z) \ \geq \ \frac{\zeta}{(-\log g)^\gamma} \frac{z}{1-z},$$

we get that

$$\tilde{g}(z) \ \geq \ C(z).$$

Let now $\alpha < 1$. Then, (54) immediately yields that the second term in the brackets in (46) is asymptotically negligible and, consequently,

$$\frac{\zeta}{(\log 1/\tilde{g})^\gamma} \frac{z}{1-z} \ \leq \ \tilde{g}'(z) \ \leq \ \frac{(1+\varepsilon)\,\zeta}{(\log 1/\tilde{g})^\gamma} \frac{z}{1-z} \tag{56}$$

holds for sufficiently small $z$. Integrating this inequality implies that

$$C(z) \ \leq \ \tilde{g}(z) \ \leq \ (1+\varepsilon)\, C(z).$$

Now, (56) implies that

$$(1-\varepsilon)\, 2\, C(z) z^{-1} \ \leq \ \tilde{g}'(z) \ \leq \ 2\,(1+\varepsilon)\, C(z)\, z^{-1}$$

for sufficiently small $z$.[13]

Using the asymptotics (45) and repeating the same argument implies that $g(z)$ also satisfies these bounds. (The calculations for $g$ are lengthier and omitted here.)

[13] We are using the same $\varepsilon$ in all of these formulae. This can be achieved by shrinking if necessary the range of $z$ under consideration.

Now,

$$V_2'(z) \;=\; \frac{g'(z)}{(\alpha+1)\,g(z)} \;\geq\; (1-\varepsilon)\,\frac{2}{\alpha+1}\,z^{-1}.$$

Therefore,

$$V_1'(z) \;=\; \frac{1}{z\,(1-z)} \;-\; V_2'(z) \;<\; 0$$

for sufficiently small $z$. Thus, $V_1(z)$ cannot be monotone increasing and the equilibrium does not exist.

Let now $\alpha > 1$. We will now show that there exists a unique solution to (44) with $g(0) = 0$. Since the right-hand side loses Lipschitz continuity only at $z = 0$, it suffices to prove local uniqueness at $z = 0$. Hence, we need only consider the equation in a small neighborhood of $z = 0$. (It is recalled that we assume $v_i = 0$.)

As above, we prove the result directly for the ODE (46), and then explain how the argument extends directly to (44).

Suppose, to the contrary, that there exist two solutions $\tilde{g}_1$ and $\tilde{g}_2$ to (46). Define the corresponding functions $l_1$ and $l_2$ via $l_i \;=\; B(\tilde{g}_i) - z$. Both functions solve (53). Integrating over a small interval $[0, l]$, we get

$$|l_1(x) \;-\; l_2(x)| \;\leq\; \int_0^x \frac{z}{1-z}\left|\frac{1}{(A(l_1(z)+z))^{1/(\alpha+1)}} \;-\; \frac{1}{(A(l_2(z)+z))^{1/(\alpha+1)}}\right| dz\,. \quad (57)$$

Now, we will use the following elementary inequality: There exists a constant $C_6 > 0$ such that

$$a^{1/\alpha} - b^{1/\alpha} \;\leq\; \frac{C_6\,(a-b)}{a^{(\alpha-1)/\alpha} \;+\; b^{(\alpha-1)/\alpha}} \quad (58)$$

for $a > b > 0$. Indeed, let $x = b/a$ and $\beta = 1/\alpha$. Then, we need to show that

$$(1 + x^{1-\beta})\,(1 \;-\; x^{\beta}) \;\leq\; C_6\,(1-x)$$

for $x \in (0, 1)$. That is, we must show that

$$x^{1-\beta} \;-\; x^{\beta} \;\leq\; (C_6 - 1)\,(1-x).$$

By continuity and compactness, it suffices to show that the limit

$$\lim_{x \to 1} \frac{x^{1-\beta} \;-\; x^{\beta}}{1-x}$$

is finite. This follows from L'Hôpital's rule.

By (50) and (51), we can replace the function $A(z)$ in (57) by its asymptotics (50) at the cost of getting a finite constant in front of the integral. Thus, for small $z$,

$$\begin{aligned} &|l_1(x) \; - \; l_2(x)| \\ &\leq \; C_7 \int_0^x z \left| \frac{((-\log(l_1+z))^\gamma \, (l_1+z))^{1/\alpha} - ((-\log(l_2+z))^\gamma \, (l_2+z))^{1/\alpha}}{((-\log(l_1+z))^\gamma \, (l_1+z))^{1/\alpha}((-\log(l_2+z))^\gamma \, (l_2+z))^{1/\alpha}} \right| dz \, . \end{aligned} \tag{59}$$

By (58),

$$\begin{aligned} &|((-\log(l_1+z))^\gamma \, (l_1+z))^{1/\alpha} - ((-\log(l_2+z))^\gamma \, (l_2+z))^{1/\alpha}| \\ &\leq \; C_6 \, \frac{|(-\log(l_1+z))^\gamma \, (l_1+z) \; - \; (-\log(l_2+z))^\gamma \, (l_2+z)|}{((-\log(l_1+z))^\gamma \, (l_1+z))^{(\alpha-1)/\alpha} + ((-\log(l_2+z))^\gamma \, (l_2+z))^{(\alpha-1)/\alpha}} \, . \end{aligned} \tag{60}$$

Now, consider some $\gamma > 0$. Then, for any sufficiently small $a > b > 0$, a direct calculation shows that

$$0 \; < \; (\log(1/a))^\gamma \, a \; - \; (\log(1/b))^\gamma \, b \; \leq \; ((\log(1/a))^\gamma \; + \; (\log(1/b))^\gamma) \, (a-b).$$

If, instead, $\gamma \leq 0$, then the function $a \mapsto (\log(1/a))^\gamma \, a$ is continuously differentiable at $a = 0$, and hence

$$0 \; < \; (\log(1/a))^\gamma \, a \; - \; (\log(1/b))^\gamma \, b \; \leq \; C_8 \, (a-b) \, .$$

Since $\alpha > 1$, the same calculation as that preceding (56) implies that

$$A(z) \; \leq \; \tilde{g}_i(z) \; = \; A(z + l_i(z)) \; \leq \; (1+\varepsilon) \, A(z) \; , \; i \; = \; 1 \, , 2$$

for sufficiently small $z$.

Thus,

$$\begin{aligned} &\left| \frac{((-\log(l_1+z))^\gamma \, (l_1+z))^{1/\alpha} - ((-\log(l_2+z))^\gamma \, (l_2+z))^{1/\alpha}}{((-\log(l_1+z))^\gamma \, (l_1+z))^{1/\alpha}((-\log(l_2+z))^\gamma \, (l_2+z))^{1/\alpha}} \right| \\ &\qquad \leq \; C_9 \, |l_1(z) \; - \; l_2(z)| \, \frac{1}{z^{((\alpha+1)/\alpha)-\varepsilon}} \\ &\qquad \leq \; C_9 \left( \sup_{z \in [0, \bar{\varepsilon}]} |l_1(z) \; - \; l_2(z)| \right) \frac{1}{z^{((\alpha+1)/\alpha)-\varepsilon}} \end{aligned} \tag{61}$$

for $z \in [0, \bar{\varepsilon}]$. Thus, (59) implies that

$$\begin{aligned} |l_1(x) \; - \; l_2(x)| \leq \; & C_{10} \left( \sup_{z \in [0, \bar{\varepsilon}]} |l_1(z) \; - \; l_2(z)| \right) \int_0^x z \, \frac{1}{z^{((\alpha+1)/\alpha)+\varepsilon}} \, dz \\ = \; & C_{11} \, (\bar{\varepsilon})^{\frac{\alpha-1}{\alpha}-\varepsilon} \sup_{z \in [0, \bar{\varepsilon}]} |l_1(z) \; - \; l_2(z)| \end{aligned} \tag{62}$$

for all $l \le \bar{\varepsilon}$. Taking the supremum over $l \in [0, \bar{\varepsilon}]$, we get

$$\sup_{z \in [0,\bar{\varepsilon}]} |l_1(z) - l_2(z)| \le C_{11} (\bar{\varepsilon})^{\frac{\alpha-1}{\alpha}-\varepsilon} \sup_{z \in [0,\bar{\varepsilon}]} |l_1(z) - l_2(z)| .$$

Picking $\bar{\varepsilon}$ so small that $C_{11} (\bar{\varepsilon})^{\frac{\alpha-1}{\alpha}-\varepsilon} < 1$ immediately yields that $l_1 = l_2$ on $[0, \bar{\varepsilon}]$ and hence, since the right-hand side of (46) is Lipschitz continuous for $z\, l \neq 0$, we have $l_1 = l_2$ for all $z$ by a standard uniqueness result for ODEs.

The fact that the same result holds for the original equation (44) follows by the same arguments as above.

It remains to prove the last claim, namely the existence of equilibrium for sufficiently large $v_i - v_j$. By Proposition 3.7, it suffices to show that

$$V_1'(z) = \frac{1}{z\,(1-z)} - V_2'(z) > 0 \tag{63}$$

for all $z \in (0,1)$ provided that $v_i - v_j$ is sufficiently large.

It follows from the proof of Lemma 3.5 that

$$G_L^{-1}\left((1-z)^{\frac{1}{(v_i-v_j)}}\right) \le V_2(z) \le G_H^{-1}\left((1-z)^{\frac{1}{(v_i-v_j)}}\right) .$$

Thus, as $v_i - v_j \uparrow +\infty$, $V_2(z)$ converges to $-\infty$ uniformly on compact subsets of $[0,1)$. By assumption,

$$\lim_{V \to +\infty} \frac{1}{h_i^H(V)} = \frac{1}{\alpha}, \qquad \lim_{V \to +\infty} \frac{1}{h_i^L(V)} = \frac{1}{\alpha+1}.$$

Thus, as $z \uparrow 1$,

$$V_2'(z) \sim \frac{1}{\alpha\,(v_i - v_j)} \frac{1}{1-z} < \frac{1}{z(1-z)}.$$

Fixing a sufficiently small $\varepsilon > 0$, we will show below that there exists a threshold $W$ such that (63) holds for all $v_i - v_j > W$ and all $z$ such that $V_2(z) \le -\varepsilon^{-1}$. Since, by the assumptions made, $1/h_i^H(V)$ and $1/h_i^L(V)$ are uniformly bounded from above for $V \ge -\varepsilon^{-1}$, it will immediately follow from (15) that (63) holds for all $z$ with $V_2(z) \ge -\varepsilon^{-1}$ as soon as $v_i - v_j$ is sufficiently large.

Thus, it remains to prove (63) when $V_2(z) \le -\varepsilon^{-1}$. We pick an $\varepsilon$ so small that we can replace the ODE (44) by (46) when proving (63). That is, once we prove the claim for the "approximate" solution $\tilde{g}(z)$, the actual claim will follow from (47).

Let

$$\tilde{g}(z) = \frac{\zeta}{(-\log \zeta)^{\gamma}} f(z) \stackrel{def}{=} \delta\, f(z).$$

Then, (44) is equivalent to the ODE

$$f'(z) \;=\; \left(\frac{\log(1/\zeta)}{\log(1/\zeta)\,+\,\log(1/f(z))}\right)^{\gamma}\left(\frac{z}{1-z}\,+\,\delta^{1/(\alpha+1)}\,f(z)^{1/(1+\alpha)}\right). \qquad (64)$$

As $v_i - v_j \to +\infty$, we get that $\zeta, \delta \to 0$. Let

$$f_0(z) \;\stackrel{def}{=}\; \int_0^z \frac{x}{1-x}\,dx \;=\; -\log(1-z)\,-\,z\,.$$

Using bounds analogous to that preceding (56), it is easy to see that

$$\lim_{v_i - v_j \to +\infty} f(z) \;=\; f_0(z)\,, \qquad \lim_{v_i - v_j \to +\infty} f'(z) \;=\; f_0'(z),$$

and that the convergence is uniform on compact subsets of $(0,1)$. Fixing a small $\varepsilon_1 > 0$, we have, for $z > \varepsilon_1$,

$$\begin{aligned}
\lim_{v_i - v_j \to \infty} V_2'(z) \;&=\; \lim_{v_i - v_j \to \infty} \frac{\tilde{g}'(z)}{(\alpha+1)\tilde{g}(z)} \\
&=\; \lim_{v_i - v_j \to \infty} \frac{f'(z)}{(\alpha+1)f(z)} \\
&=\; \frac{f_0'(z)}{(\alpha+1)f_0(z)} \\
&=\; \frac{z}{(\alpha+1)\,(1-z)\,(-\log(1-z)-z)}.
\end{aligned}$$

We then have

$$\frac{d^2}{dz^2}(-\log(1-z)) \;=\; \frac{1}{(1-z)^2} \;\geq\; 1.$$

Therefore, by Taylor's formula,

$$-\log(1-z) - z \;\geq\; \frac{1}{2}\,z^2\,.$$

Hence,

$$\frac{z}{(\alpha+1)\,(1-z)\,(-\log(1-z)-z)} \;\leq\; \frac{2}{\alpha+1}\,\frac{1}{z(1-z)}.$$

Therefore (63) holds for large $v_i - v_j$ because $\alpha > 1$. This argument does not work as $z \to 0$ because $f(0) = f_0(0) = 0$. So, we need to find a way to get uniform upper bounds for $f'(z)/f(z)$ when $z$ is small. By the comparison argument used above, and picking $\varepsilon_1$ sufficiently small, since our goal is to prove inequality (63), we can replace $1-z$ by 1 in (64).

In this part of the proof, it will be more convenient to deal with $\tilde{g}$ instead of $f$. By the above, we may replace $\tilde{g}$ by the function $g_1$ solving

$$g_1'(z) \;=\; \frac{\zeta}{(-\log(g_1))^\gamma}\,\left(z \,+\, g_1^{1/(1+\alpha)}\right)\,.$$

Let

$$d(z) \;=\; \int_0^z \left(\log\left(\frac{1}{x}\right)\right)^\gamma\,dx\;,$$

$D(z) \;=\; d^{-1}(z)$, and $k(z) \;=\; D(g_1(z))$. Then, we can rewrite the ODE for $g_1$ as

$$k'(z) \;=\; \zeta\,\left(z \,+\, (D(k(z)))^{1/(\alpha+1)}\right)\;, \qquad k(0) = 0.$$

Define $L(z)$ via

$$\int_0^{L(z)} (D(x))^{-1/(\alpha+1)}\,dx \;=\; z,$$

and let

$$\phi(z) \;=\; L(\zeta\,z) \,+\, \frac{1}{2}\zeta\,z^2 \;\geq\; L(\zeta z).$$

Then, by the monotonicity of $D(z)$,

$$\phi'(z) \;=\; \zeta\,L'(\zeta z) \,+\, \zeta\,z \;=\; \zeta\,\left(z \,+\, (D(L(\zeta z)))^{1/(\alpha+1)}\right) \;\leq\; \zeta\,(z \,+\, (D(\phi(\zeta z)))^{1/(\alpha+1)}),$$

By a comparison theorem for ODEs (for example, Hartman (1982), Theorem 4.1, p. 26),[14] we have

$$k(z) \;\geq\; \phi(z) \;\Leftrightarrow\; g_1(z) \;=\; D(k(z)) \;\geq\; D(\phi(z))\,. \tag{65}$$

Therefore, since the functions $x(-\log x)^\gamma$ and $x^{\alpha/(\alpha+1)}\,(-\log x)^\gamma$ are monotone increasing for small $x$, we have

$$\begin{aligned}(1+\alpha)\,V_2'(z) \;&=\; \frac{g'(z)}{g(z)}\\ &\leq\; (1+\varepsilon)\,\frac{g_1'(z)}{(\alpha+1)\,g_1(z)}\\ &=\; \frac{(1+\varepsilon)\zeta\,z}{g_1\,(-\log g_1)^\gamma} \,+\, \frac{(1+\varepsilon)\zeta}{g_1^{\alpha/(\alpha+1)}\,(-\log g_1)^\gamma}\\ &\leq\; \frac{(1+\varepsilon)\zeta\,z}{D(\phi(z))\,(-\log D(\phi(z)))^\gamma} \,+\, \frac{(1+\varepsilon)\zeta}{D(\phi(z))^{\alpha/(\alpha+1)}\,(-\log D(\phi(z)))^\gamma}.\end{aligned} \tag{66}$$

[14]Even though the right-hand side of the ODE in question is not Lipschitz continuous, the proof of this comparison theorem easily extends to our case because of the uniqueness of the solution, due to (62).

Thus, it suffices to show that

$$\frac{\zeta\, z^2}{D(\phi(z))\,(-\log D(\phi(z)))^{\gamma}} \;+\; \frac{\zeta\, z}{D(\phi(z))^{\alpha/(\alpha+1)}\,(-\log D(\phi(z)))^{\gamma}} \;<\; (1-\varepsilon)(1+\alpha)$$

for some $\varepsilon > 0$, and for all sufficiently small $z$ and $\zeta$. Now, a direct calculation similar to that for the functions $A(z)$ and $C(z)$ implies that

$$d(z) \;\sim\; z\,(-\log z)^{\gamma}$$

and therefore that

$$D(z) \;\sim\; z\,(-\log z)^{-\gamma}\,.$$

Thus, it suffices to show that

$$\begin{aligned}&\frac{\zeta\, z^2}{\phi(z)\,(-\log\phi)^{-\gamma}\,(-\log(\phi(z)\,(-\log\phi)^{-\gamma}))^{\gamma}}\\ &\qquad + \frac{\zeta\, z}{(\phi(z)\,(-\log\phi)^{-\gamma})^{\alpha/(\alpha+1)}\,(-\log(\phi(z)\,(-\log\phi)^{-\gamma}))^{\gamma}}\\ &\quad <\; (1-\varepsilon)(1+\alpha).\end{aligned} \tag{67}$$

Leaving the leading asymptotic term, we need to show that

$$\frac{\zeta\, z^2}{\phi(z)} \;+\; \frac{\zeta\, z}{(\phi(z))^{\alpha/(\alpha+1)}\,(-\log(\phi(z)))^{\gamma/(\alpha+1)}} \;<\; (1-\varepsilon)(1+\alpha)\,.$$

We have

$$\int_0^z (D(x))^{-1/(\alpha+1)}dx \;\sim\; \frac{\alpha+1}{\alpha}\, z^{\alpha/(\alpha+1)}\,(-\log z)^{\gamma/(\alpha+1)}.$$

Therefore

$$L(z) \;\sim\; \left(\frac{\alpha}{\alpha+1}\, z\right)^{(\alpha+1)/\alpha}\,(-\log z)^{-\gamma/\alpha}\,.$$

Hence, we can replace $\phi(z)$ by

$$\tilde{\phi}(z) \;\stackrel{def}{=}\; \left(\frac{\alpha}{\alpha+1}\,\zeta z\right)^{(\alpha+1)/\alpha}\,(-\log(\zeta z))^{-\gamma/\alpha} \;+\; \frac{1}{2}\,\zeta\, z^2\,.$$

Let

$$x \;=\; \frac{\zeta\, z^2}{(\zeta z)^{(\alpha+1)/\alpha}\,(-\log(\zeta z))^{-\gamma/\alpha}}\,.$$

Then,

$$\begin{aligned}&\frac{\zeta\, z^2}{\tilde{\phi}(z)} \;+\; \frac{\zeta\, z}{(\tilde{\phi}(z))^{\alpha/(\alpha+1)}\,(-\log(\tilde{\phi}(z)))^{\gamma/(\alpha+1)}}\\ &\quad=\; \frac{1}{\left(\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}} + 0.5x\right)^{\alpha/(\alpha+1)}}\left(\frac{-\log(\zeta z)}{-\log\tilde{\phi}}\right)^{\gamma/(\alpha+1)} \;+\; \frac{x}{\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}} + 0.5x}\,.\end{aligned}$$

We have

$$\log(\tilde{\phi}) \;=\; \log(\zeta\, z) \,+\, \log\left(\left(\frac{\alpha}{\alpha+1}\right)^{(\alpha+1)/\alpha} (\zeta z)^{1/\alpha}\,(-\log(\zeta z))^{-\gamma/\alpha} \;+\; 0.5\, z\right)$$
$$\leq\; \log(\zeta z)$$

for small $\zeta, z$. Furthermore, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left(\frac{\alpha}{\alpha+1}\right)^{(\alpha+1)/\alpha} (\zeta z)^{1/\alpha}\,(-\log(\zeta z))^{-\gamma/\alpha} \;\geq\; (\zeta z)^{1/(\alpha-\varepsilon)}$$

for all $\zeta z \leq \delta$. Hence,

$$\frac{\alpha-\varepsilon}{\alpha-\varepsilon+1} \;\leq\; \frac{-\log(\zeta z)}{-\log\tilde{\phi}} \;\leq\; 1$$

for all sufficiently small $\zeta, z$. Consequently, to prove (66) it suffices to show that

$$\sup_{x>0}\; \chi(x) < \; 1+\alpha,$$

where

$$\chi(x) \;=\; \frac{1}{\left(\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}} \,+\, 0.5x\right)^{\alpha/(\alpha+1)}}\, A_\alpha \;+\; \frac{x}{\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}} \,+\, 0.5x},$$

with

$$A_\alpha \;=\; \max\left\{\left(\frac{\alpha}{\alpha+1}\right)^{\gamma/(\alpha+1)},\, 1\right\}.$$

Let

$$K \;=\; \left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}}.$$

Then,

$$\chi'(x) \;=\; -\frac{0.5\, A_\alpha\, \alpha}{\alpha+1}\,\frac{1}{(K+0.5x)^{(2\alpha+1)/(\alpha+1)}} \;+\; \frac{K}{(K+0.5x)^2}\,.$$

Thus, $\chi'(x_*) = 0$ if and only if

$$K + 0.5x_* \;=\; \left(\frac{K}{\frac{0.5\, A_\alpha\, \alpha}{\alpha+1}}\right)^{\alpha+1},$$

which means that

$$x_* \;=\; 2\left(\left(\frac{2}{A_\alpha}\right)^{\alpha+1} - 1\right)\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}}.$$

Then,

$$\begin{aligned}
&\chi(x_*) \\
&= \frac{1}{\left(\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}} + 0.5x_*\right)^{\alpha/(\alpha+1)}} A_\alpha + \frac{x_*}{\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}} + 0.5x_*} \\
&= \frac{1}{\left((2/A_\alpha)^{\alpha+1}(\alpha/(\alpha+1))^{(\alpha+1)/\alpha}\right)^{\alpha/(\alpha+1)}} A_\alpha + \frac{2\left(\left(\frac{2}{A_\alpha}\right)^{\alpha+1} - 1\right)\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}}}{(2/A_\alpha)^{\alpha+1}(\alpha/(\alpha+1))^{(\alpha+1)/\alpha}} \\
&= \left(\frac{A_\alpha}{2}\right)^{\alpha} \frac{\alpha+1}{\alpha} A_\alpha + 2 - 2\left(\frac{A_\alpha}{2}\right)^{\alpha+1} = 2 + \frac{A_\alpha^{\alpha+1}}{2^\alpha \alpha}.
\end{aligned} \tag{68}$$

There are three candidates for $x$ that achieve a maximum of $\chi$, namely $x = 0$, $x = +\infty$, and $x = x_*$, which is positive if and only if $A_\alpha < 2$.

If $\gamma \geq 0$, then $A_\alpha = 1$, so $x = 0$ and $x = +\infty$ satisfy the required inequality as soon as $\alpha > 1$, whereas $\chi(x_*) < \alpha + 1$ if and only if $\alpha > \alpha_*$, where

$$\alpha_* = 1 + \frac{1}{\alpha_* \, 2^{\alpha_*}}.$$

A calculation shows that $\alpha^* \in (1.30, 1.31)$.

If $\gamma < 0$, then

$$\chi(0) = \frac{(\alpha+1)\, A_\alpha}{\alpha}, \qquad \chi(+\infty) = 2,$$

and this gives the condition $A_\alpha < \alpha$. If $A_\alpha > 2$, that is, if

$$-\gamma > (\alpha+1)\frac{\log 2}{\log((\alpha+1)/\alpha)},$$

then we are done. Otherwise, we need the property

$$2 + \frac{A_\alpha^{\alpha+1}}{2^\alpha \alpha} < \alpha + 1 \Leftrightarrow -\gamma < \frac{\log\left((\alpha^2 - \alpha)\, 2^\alpha\right)}{\log((\alpha+1)/\alpha)}.$$

By assumption, $\psi_{it} \sim \mathrm{Exp}_{+\infty}(c_{it}, \gamma_{it}, -\alpha_{it})$ uniformly if $t$. Thus, the arguments above imply that a lower bound for $v_i - v_j$ that is sufficient to guarantee the existence of equilibrium for each fixed $t$ can be chosen, independent of $t$. ■

# E Proofs of Section 4

Everywhere in the sequel, we assume for simplicity that $R = 1$.

**Proof of Theorem 4.2.** The expected utility of a seller of class $i \in \{1,2\}$ of a trade with a class-3 buyer is

$$\begin{aligned} &\frac{1}{2}\lambda_i \int_t^T \int_{\mathbb{R}} (\psi_{i\tau}^H(z) \,+\, \psi_{i\tau}^L(z))\, \Pi(\tau, z, S_\tau(z))\, dz\, d\tau \\ &\quad = \frac{1}{2}\lambda_i \int_t^T \int_{\mathbb{R}} \psi_{i\tau}^H(z)\,(1+e^{-z})\, \Pi(\tau, z, S_\tau(z))\, dz\, d\tau, \end{aligned}$$

where

$$\Pi(\tau, z, S) \;=\; P(z)(S - v^H) G_{3\tau}^H(V_{2\tau}(S)) \,+\, (1 - P(z))\,(S - v_1) G_{3\tau}^L(V_{2\tau}(S)).$$

Let $f^H$ be the probability density of a single signal, so that

$$\psi_{i0}^H \;=\; \sum_{k=1}^{\infty} p_{ik}\,(f^H)^{*k}\;,\; i = 1, 2.$$

Substituting these expansions into (8), we get that

$$\psi_{it} \;=\; \sum_{k \in \mathbb{Z}_+^{N-1}} \tilde{a}_{it}(k)\, f^{*k_1}\, \psi_{30}^{*k_2} \cdots \psi_{N0}^{*k_{N-1}},$$

where the measures $\tilde{a}_{it}$ on $\mathbb{Z}_+^{N-1}$ satisfy the system of equations

$$\tilde{a}_{it}' \;=\; -\lambda_i\, \tilde{a}_{it} \;+\; \lambda_i\, \tilde{a}_{it} * \sum_{j=1}^{N} \kappa_{ij}\, \tilde{a}_{jt}, \tag{69}$$

but with the initial conditions $(\tilde{a}_{i0})(k, 0, \ldots, 0) \;=\; p_{ik}$ and $(\tilde{a}_{i0})(0, k_2, \ldots, k_{N-1}) = 0$. Then, the same argument as in the proof of Proposition 3.4 implies that the measure $\tilde{a}_{2t}$ dominates $\tilde{a}_{1t}$ is the sense of first-order stochastic dominance. Therefore, it suffices to show that

$$\begin{aligned} &\int_{\mathbb{R}} (1+e^{-z})\, \Pi(\tau, z, S_\tau(z))\, ((f^H)^{*k_1} * (\psi_{30}^H)^{*k_2} \cdots * (\psi_{0N}^H)^{*k_{N-1}})(z - x)\, dz \\ &= \int_{\mathbb{R}} (1+e^{-z-x})\, \Pi(\tau, z+x, S_\tau(z+x))\, ((f^H)^{*k_1} * \psi_{30}^{*k_2} \cdots * (\psi_{0N}^H)^{*k_{N-1}})(z)\, dz \end{aligned} \tag{70}$$

is monotone increasing in $k \;=\; (k_1, \ldots, k_N)$.

To show the latter, it suffices to prove that

$$\begin{aligned} &\int_{\mathbb{R}} (1+e^{-x-y-z})\, \Pi(\tau, x+y+z, S_\tau(x+y+z))\, \zeta(z)\, dz \\ &\qquad\qquad > \; (1+e^{-x-y})\, \Pi(\tau, x+y, S_\tau(x+y)), \end{aligned} \tag{71}$$

for any $x, y$ and any probability density $\zeta$ satisfying (13).

Now, by the optimality of $S$, we have that

$$\Pi(\tau, x+y+z, S_\tau(x+y+z)) \;\geq\; \Pi(\tau, x+y+z, S_\tau(x+y)),$$

and the inequality is strict for all $z \neq 0$. Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}} (1+e^{-x-y-z})\, \Pi(\tau, x+y+z, S_\tau(x+y+z))\, \zeta(z)\, dz \\
&> \int_{\mathbb{R}} (1+e^{-x-y-z})\, \Pi(\tau, x+y+z, S_\tau(x+y))\, \zeta(z)\, dz \\
&= \int_{\mathbb{R}} (1+e^{-x-y-z}) \Big( P(x+y+z)(S_\tau(x+y) - v^H) G^H_{3\tau}(V_{2\tau}(S_\tau(x+y))) \\
&\quad + (1 - P(x+y+z))\, (S_\tau(x+y) - v_1) G^L_{3\tau}(V_{2\tau}(S_\tau(x+y))) \Big)\, \zeta(z)\, dz \\
&= (S_\tau(x+y) - v^H) G^H_{3\tau}(V_{2\tau}(S_\tau(x+y))) \int_{\mathbb{R}} (1+e^{-x-y-z})\, P(x+y+z) \zeta(z)\, dz \\
&\quad + (S_\tau(x+y) - v_1) G^L_{3\tau}(V_2(S_\tau(x+y))) \int_{\mathbb{R}} (1+e^{-x-y-z})\, (1 - P(x+y+z)) \zeta(z)\, dz\,.
\end{aligned} \tag{72}$$

The claim follows now from the identities $(1+e^{-x})P(x) \;=\; 1$,

$$(1+e^{-x})(1-P(x)) \;=\; e^{-x},$$

and

$$\int_{\mathbb{R}} \zeta(z)\, dz \;=\; \int_{\mathbb{R}} e^{-z}\, \zeta(z)\, dz \;=\; 1.$$

**Proof of Lemma 4.4.** We have

$$\hat{\psi}^H_{i0}(s) \;=\; \sum_{k=1}^{\infty} p_{ik}\, (\hat{f}^H)^k(s).$$

Standard results (for example, Korevaar (2004), Theorem 15.3, p. 30) imply that

$$\hat{f}^H(s) \;\sim\; \frac{c\,\Gamma(\gamma+1)}{(\alpha - s)^{\gamma+1}}\,.$$

Suppose first that[15]

$$K_1 \;\stackrel{def}{=}\; \sup\{k \,:\, p_{1k} > 0\} \;<\; K_2 \;\stackrel{def}{=}\; \sup\{k \,:\, p_{2k} \,>\, 0\} \;<\; \infty.$$

[15]The case of $\sup\{k \,:\, p_{2k} \,>\, 0\} = \infty$ is analogous.

Then,

$$\hat{\psi}_i^H \;\sim\; \frac{p_{i\,K_i}\,(c\,\Gamma(\gamma+1))^{K_i}}{(\alpha-s)^{K_i(\gamma+1)}}.$$

Therefore (using, for example, Korevaar (2004), Theorem 15.3, p. 30)

$$c_i \;=\frac{p_{i\,K_i}\,(c\,\Gamma(\gamma+1))^{K_i}}{\Gamma(K_i(\gamma+1))}$$

and

$$\gamma_i \;=\; K_i(\gamma+1) \;-\; 1\;,\;\; \alpha_i \;=\; \alpha.$$

The claim follows.

If $K_1 = K_2 = \infty$, we have

$$\inf\{s\,:\,\hat{\psi}_i^H(s) \;=\; \infty\} \;=\; \alpha_i,$$

where $\alpha_i$ is the unique positive number $s$ solving

$$\hat{f}^H(s) \;=\; \lim_{k\to\infty}\frac{p_{ik}}{p_{i\,k+1}},$$

and therefore $\alpha_1 > \alpha_2$. ■

**Lemma E.1** *If $f^H \;\sim\; \mathrm{Exp}_{+\infty}(c,0,-\alpha)$ and if there is a finite maximum number $N(i)$ of signals that an agent of class $i$ receives with strictly positive probability, then $\psi_{i0} \;\sim\; \mathrm{Exp}_{+\infty}(c_{i0}\,,\,N(i)-1\,,\,-\alpha)$, where*

$$c_{i0} \;=\; \frac{p_{i\,N(i)}\,c^{N(i)}}{(N(i)-1)!},\qquad \gamma_{i0} \;=\; N(i)\;-\;1.$$

*Alternatively, suppose that the moment generating function of $f^H$ is finite on $(-\epsilon,\epsilon)$ for some $\epsilon>0$ and that for some positive constants $r<1$ and $R>1$, we have*

$$p_{ik}R^k - 1 = \,\mathrm{O}(r^k). \tag{73}$$

*Then, $\psi_{i0}$ satisfies an exponential tail condition.*

Condition (73) implies that, asymptotically in $k$, the probability of receiving $k$ signals is close to geometric in $k$, in a particularly tight sense.

**Proof.** The second claim follows by the Tauberian arguments used in the proof of Proposition C.2.

For the first claim, it suffices to show that

$$(f^H)^{*k} \;\sim\; \mathrm{Exp}_{+\infty}(c^k/((k-1)!)\,,\,k-1\,,\,-\alpha).$$

We will prove this by induction in $k$. The case of $k = 1$ follows by the assumption on $f^H$. Suppose that we have proved the claim for some positive integer $k$; we will now prove it for $k+1$. Let $\phi \;=\; (f^H)^{*k}$. We use the decomposition

$$(\phi * f^H)(x) \;=\; \left(\int_{-\infty}^{A} + \int_{A}^{+\infty}\right) \phi(x-y)\, f^H(y)\, dy.$$

Now, we fix an $\varepsilon > 0$ and pick some constant $A$ so large that

$$\frac{f^H(y)}{c\, e^{-\alpha y}} \;\in\; (1-\varepsilon, 1+\varepsilon)$$

for all $y > A$. Then,

$$\frac{\int_A^{+\infty} \phi(x-y)\, f^H(y)\, dy}{c \int_A^{+\infty} \phi(x-y)\, e^{-\alpha y}\, dy} \;\in\; (1-\varepsilon, 1+\varepsilon)$$

for all $x$. Changing variables, applying L'Hôpital's rule, and using the induction hypothesis, we get

$$\begin{aligned}
\lim_{x\to+\infty} \frac{\int_A^{+\infty} \phi(x-y)\, e^{-\alpha y}\, dy}{x^k\, e^{-\alpha x}} \;&=\; \lim_{x\to+\infty} \frac{\int_{-\infty}^{x-A} \phi(z)\, e^{-\alpha(x-z)}\, dy}{x^k\, e^{-\alpha x}} \\
&=\; \lim_{x\to+\infty} \frac{\int_{-\infty}^{x-A} \phi(z)\, e^{\alpha z}\, dz}{x^k} \\
&=\; \lim_{x\to+\infty} \frac{\phi(x-A)e^{\alpha(x-A)}}{k\, x^{k-1}} \qquad (74) \\
&=\; \lim_{x\to+\infty} \frac{c^k\, (x-A)^{k-1}\, e^{-\alpha(x-A)}\, e^{\alpha(x-A)}}{k!\, x^{k-1}} \\
&=\; \frac{c^k}{k!}.
\end{aligned}$$

Now, using the Lebesque dominated convergence theorem and the induction hypothesis, we get

$$\begin{aligned}
\lim_{x\to+\infty} \frac{\int_{-\infty}^{A} \phi(x-y)\, f^H(y)\, dy}{x^{k-1}\, e^{-\alpha x}} \;&=\; \lim_{x\to+\infty} \int_{-\infty}^{A} \frac{\phi(x-y)}{x^{k-1}\, e^{-\alpha x}}\, f^H(y)\, dy \\
&=\; \frac{c^k}{(k-1)!} \int_{-\infty}^{A} e^{\alpha y}\, f^H(y)\, dy. \qquad (75)
\end{aligned}$$

Consequently,

$$(1-\varepsilon)\,\frac{c^{k+1}}{k!} \;\le\; \lim\inf_{x\to+\infty} \frac{(f^H)^{*(k+1)}(x)}{x^k\, e^{-\alpha x}} \;\le\; \lim\sup_{x\to+\infty} \frac{(f^H)^{*(k+1)}(x)}{x^k\, e^{-\alpha x}} \;\le\; (1+\varepsilon)\,\frac{c^{k+1}}{k!},$$

and the claim follows because $\varepsilon > 0$ is arbitrary. ■

**Proof of Proposition 4.5.** It will follow from the results below that it suffices to prove the result for a single auction at time zero. For a strictly positive time $t$, for $i = 1$ and $i = 2$,

$$\psi_{it} \;\sim\; \text{Exp}(c_{it}\,,\,\gamma(t)\,,\,-\alpha(t)),$$

with $\gamma(t), \alpha(t)$ and with $c_{2t} \;>\; c_{1t}$. It follows that the monotonicity result holds for any auction at any time $t > 0$. Indeed, it follows directly from (6) that

$$\hat{\psi}_{2t} \;=\; \frac{\hat{\psi}_{20}}{\hat{\psi}_{10}}\,\hat{\psi}_{1t}.$$

Therefore,

$$c_{2t} \;=\; c_{1t}\,\frac{\hat{\psi}_{20}(\alpha(t))}{\hat{\psi}_{10}(\alpha(t))} \;>\; c_{1t},$$

because $p_1 \prec_{fosd} p_2$ and Lemma C.1 together yield, for all $k \;>\; 0$,

$$\hat{\psi}_{10}(k) \;<\; \hat{\psi}_{20}(k).$$

In order to prove Proposition 4.5, we will need a detailed analysis of the asymptotic behavior of $S_i(y)$ as $G(v) \to \infty$. Let

$$\zeta \;=\; \frac{(\alpha+1)^{\gamma+1}}{c\,(\bar{v}-v_3)}\,.$$

Here, we consciously suppress indices for $\alpha$, $\gamma$ and $c$. Namely, if the information type is not hidden, $(c,\gamma,\alpha) \;=\; (c_i,\gamma_i,\alpha_i)$. If the information type is hidden, we will have $(c,\gamma,\alpha) \;=\; (\kappa_2\,c_2,\gamma_2,\alpha_2)$ if $\text{Tail}(\psi_{10}) \;\prec\; \text{Tail}(\psi_{20})$, and we have $(c,\gamma,\alpha) \;=\; (\kappa_1 c_1 + \kappa_2 c_2, \gamma_2, \alpha_2)$ if $(\gamma_1,\alpha_1) = (\gamma_2,\alpha_2)\,.$

As in the proof of Proposition 3.10, we define

$$g(z) \;=\; e^{(\alpha+1)V_2(z)} \;=\; \frac{\zeta}{(-\log\zeta)^{\gamma}}\,f(z) \;\stackrel{def}{=}\; \varepsilon\,f(z)\,.$$

Then, as we have shown in the proof of Proposition 3.10, we may assume that, for large $G(v)$,

$$f'(z) \;=\; \left(\frac{\log(1/\zeta)}{\log(1/\zeta)\;+\;\log(1/f(z))}\right)^{\gamma}\left(\frac{z-\bar{v}}{v^H-z}\;+\;\varepsilon^{1/(\alpha+1)}\,f(z)^{1/(1+\alpha)}\right)\;,\quad f(\bar{v})=0. \tag{76}$$

See (64). Furthermore, as $G(v) \to \infty$, we have $\zeta, \varepsilon \to 0$,

$$\lim_{G(v)\to\infty} f(z) \;=\; f_0(z),$$

where

$$f_0(z) \;=\; (v^H - \bar{v}) \log \frac{v^H - \bar{v}}{v^H - z} \;-\; (z - \bar{v}),$$

and the convergence is uniform on compact subsets of $[\bar{v}, v^H)$.

From this point, for simplicity we take the case $\gamma_i = 0$ for all $i$. The general case follows by similar but lengthier arguments. Hence, we assume that $f$ solves

$$f'(z) \;=\; \frac{z - \bar{v}}{v^H - z} \;+\; \varepsilon^{1/\alpha+1}\, f^{1/(\alpha+1)}. \tag{77}$$

Since the solution $f(z)$ to (77) is uniformly bounded on compact subsets of $[\bar{v}, v^H)$, by integrating (77) we find that

$$0 \;\le\; f(z) \;-\; f_0(z) \;=\; O(\varepsilon^{1/(\alpha+1)}\,(z \;-\; \bar{v})),$$

uniformly on compact subsets of $[\bar{v}, v^H)$. Furthermore, $f_0(z) \;\le\; C_1\,(z - \bar{v})^2$, uniformly on compact subsets of $[\bar{v}, v^H)$. Substituting these bounds into (77), we get

$$\begin{aligned} f(z) \;-\; f_0(z) \quad &\le \quad C_2\,\varepsilon^{1/(\alpha+1)} \int_{\bar{v}}^{z} (\varepsilon^{1/\alpha+1}\,(z - \bar{v}) \;+\; (z - \bar{v})^2)^{1/(\alpha+1)}\, dz \\ &\le \quad C_3\,\varepsilon^{1/(\alpha+1)}\,(z - \bar{v})\,(\varepsilon^{1/(\alpha+1)^2}\,(z - \bar{v})^{1/(\alpha+1)} \;+\; (z - \bar{v})^{2/(\alpha+1)}). \end{aligned}$$

Let now

$$l(z) \;=\; f(z)^{\alpha/(\alpha+1)} \;-\; \frac{\varepsilon^{1/\alpha+1}\alpha}{\alpha+1}\,(z - \bar{v})\,.$$

Then,

$$\begin{aligned} l'(z) \;&=\; \frac{\alpha}{\alpha+1}\, f'(z)\, f^{-1/(\alpha+1)} \;-\; \frac{\varepsilon^{1/\alpha+1}\alpha}{\alpha+1} \\ &=\; \frac{\alpha}{\alpha+1}\, \frac{z - \bar{v}}{\left(\frac{\varepsilon^{1/\alpha+1}\alpha}{\alpha+1}\,(z - \bar{v}) + l(z)\right)^{1/\alpha}} \\ &\le\; \frac{\alpha}{\alpha+1}\, \frac{z - \bar{v}}{(l(z))^{1/\alpha}}\,. \end{aligned} \tag{78}$$

Integrating this inequality, we get

$$l(z) \;\le\; \frac{1}{2}(z - \bar{v})^2,$$

and therefore

$$f(z) \;\le\; C_4\,((z - \bar{v})^2 \;+\; \varepsilon^{1/\alpha}\,(z - \bar{v})^{(\alpha+1)/\alpha})\,. \tag{79}$$

Consequently,

$$e^{V_2(z)} \;=\; \varepsilon^{1/(\alpha+1)}\left(f_0(z)\,+\,o(\varepsilon^{1/(\alpha+1)}\,(z-\bar{v}))\right)^{1/(\alpha+1)}$$

uniformly on compact subsets of $[\bar{v}, v^H)$. Therefore,

$$\lim_{\varepsilon\to 0}\left(V_2(z)\;-\;\frac{1}{\alpha+1}\log\varepsilon\right)\;=\;\frac{1}{\alpha+1}\,\log f_0(z),$$

uniformly on compact subsets of $(\bar{v}, v^H)$.

Now, since $V_2 \to -\infty$ uniformly on compact subsets of $[\bar{v}, v^H)$,

$$V_1(z)\;=\;\log\frac{z-\bar{v}}{v^H-z}\;-\;V_2(z)$$

converges to $+\infty$, uniformly on compact subsets of $(\bar{v}, v^H)$. Pick an $\eta > 0$ and let $\varepsilon$ be so small that $V_2(\bar{v}+\varepsilon) > K$ for some very large $K$. Then, for all $\theta < K$ we have that

$$\bar{v}\;<\;S(\theta)\;<\;S(K)\;<\;S(V_2(\bar{v}+\varepsilon))\;=\;\bar{v}+\varepsilon.$$

Thus, $S(\theta)$ converges to $\bar{v}$ uniformly on compact subsets of $[-\infty, +\infty)$ (with $-\infty$ included). Furthermore,

$$\lim_{\varepsilon\to 0}\left(V_1(z)\;+\;\frac{1}{\alpha+1}\log\varepsilon\right)\;=\;\log\frac{z-\bar{v}}{v^H-z}\;-\;\frac{1}{\alpha+1}\,\log f_0(z)\;\stackrel{def}{=}\;M(z)$$

uniformly on compact subsets of $(\bar{v}, v^H)$. Let $\hat{M}(z) \;=\; M^{-1}(z)$. We claim that

$$\lim_{\varepsilon\to 0} S\left(\theta\;-\;\frac{1}{\alpha+1}\log\varepsilon\right)\;=\;\hat{M}(\theta)\,, \tag{80}$$

uniformly on compact subsets of $\mathbb{R}$. Indeed, $S\left(\theta\;-\;\frac{1}{\alpha+1}\log\varepsilon\right)$ is the unique solution to the equation in $y$ given by

$$\theta\;=\;V_1(y)\;+\;\frac{1}{\alpha+1}\log\varepsilon\,.$$

Since the right-hand side converges uniformly to the strictly monotone function $M(\,\cdot\,)$, this unique solution also converges uniformly to $\hat{M}(\theta)$. Furthermore,

$$\bar{v}\;+\;\Delta_i\,P(V_1(z)\;+\;V_2(z))\;=\;z\;\Leftrightarrow\bar{v}\;+\;\Delta_i\,P(\theta\;+\;V_2(S(\theta)))\;=\;S(\theta)$$

implies that

$$V_2\left(S\left(\theta\;-\;\frac{1}{\alpha+1}\log\varepsilon\right)\right)\;=\;\log\left(\frac{S-\bar{v}}{v^H-S}\right)\;-\;\theta\;+\;\frac{1}{\alpha+1}\log\varepsilon$$

and therefore

$$V_2\left(S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\right) - \frac{1}{\alpha+1}\log\varepsilon \;\to\; \log\left(\frac{\hat{M}(\theta)-\bar{v}}{v^H - \hat{M}(\theta)}\right) - \theta.$$

We have

$$M(z) \;=\; \log\left(\frac{z-\bar{v}}{(v^H - z)\left((v^H-\bar{v})\,\log\left(\frac{v^H-\bar{v}}{v^H-z}\right) - (z-\bar{v})\right)^{1/(\alpha_i+1)}}\right).$$

Now, for $z \approx \bar{v}$,

$$\log\left(\frac{v^H-\bar{v}}{v^H-z}\right) \;=\; -\log\left(1 - \frac{z-\bar{v}}{v^H-\bar{v}}\right) \;\approx\; \frac{z-\bar{v}}{v^H-\bar{v}} + \frac{1}{2}\left(\frac{z-\bar{v}}{v^H-\bar{v}}\right)^2, \tag{81}$$

and therefore

$$M_i(z) \;\approx\; (1+\alpha)^{-1}\log(2(v^H-\bar{v})) \;+\; \frac{\alpha_i-1}{\alpha_i+1}\log\left(\frac{z-\bar{v}}{v^H-\bar{v}}\right) \tag{82}$$

as $z \to \bar{v}$. Consequently, as $\theta \to -\infty$, we have

$$\hat{M}(\theta) \;\sim\; \bar{v} + K\,e^{\frac{\alpha+1}{\alpha-1}\theta}$$

for some constant $K = K(\alpha)$.

By continuity,[16] it suffices to prove Proposition 4.5 for a single auction at time zero. For brevity, we omit the index 0 in this proof. For example, we write "$\psi_i$" for $\psi_{i0}$.

We use the notation $u_i^{H,L}$ for the pair of expected utilities of a class-$i$ investor in an auction held at time zero, conditional on $Y=0$ and $Y=1$, respectively.[17]

$$\begin{aligned}
u_i^{H,L} \;&=\; \int_{\mathbb{R}} \psi_i^{H,L}(x)\int_{-\infty}^{V_{1,i}(B_i(x))} (\{v^H,\bar{v}\} - S_i(y))\,\psi_3^{H,L}(y)\,dy\,dx \\
&=\; \int_{\mathbb{R}} \psi_3^{H,L}(y)\,(\{v^H,\bar{v}\} - S_i(y))\,G_i^{H,L}(V_{2,i}(S_i(y)))\,dy \\
&=\; \int_{\mathbb{R}} \psi_3^{H,L}(y)\,(\{v^H,\bar{v}\} - S_i(y))\,dy \\
&\qquad - \int_{\mathbb{R}} \psi_3^{H,L}(y)\,(\{v^H,\bar{v}\} - S_i(y))\,F_i^{H,L}(V_{2,i}(S_i(y)))\,dy \\
&=\; (\{v^H,\bar{v}\} - \bar{v}) + \int_{\mathbb{R}} \psi_3^{H,L}(y)\,(\bar{v} - S_i(y))\,dy \\
&\qquad - \int_{\mathbb{R}} \psi_3^{H,L}(y)\,(\{v^H,\bar{v}\} - S_i(y))\,F_i^{H,L}(V_{2,i}(S_i(y)))\,dy\,,
\end{aligned} \tag{83}$$

[16]Because the exponential tails are uniform, it will follow that the convergence proved below is also uniform in time.

[17]Here, $S_1(y)$ and $S_2(y)$ are different if and only if information type is not hidden.

where $F_i \;=\; 1 \;-\; G_i$. Let us first study the asymptotic behavior of the term

$$\int_{\mathbb{R}} \psi_3^{H,L}(y)\,(\bar{v} \;-\; S_i(y))\,dy$$

as $G(v) \to \infty$. We have

$$\begin{aligned}&\int_{\mathbb{R}} \psi_3^{H,L}(y)\,(\bar{v} \;-\; S_i(y))\,dy \\ &= \int_{\mathbb{R}} \psi_3^{H,L}\left(y - \frac{1}{\alpha_i+1}\log\varepsilon_i\right)\,\left(\bar{v} \;-\; S_i\left(y - \frac{1}{\alpha_i+1}\log\varepsilon_i\right)\right)\,dy\,.\end{aligned} \tag{84}$$

Since, by assumption, $\psi_3^{H,L} \sim \mathrm{Exp}_{+\infty}(c_3, \gamma_3, -\{\alpha_3, \alpha_3+1\})$, we get

$$\lim_{\varepsilon\to 0} c_3^{-1}\,\varepsilon^{-\{\alpha_3,\alpha_3+1\}/(\alpha_i+1)}\,\psi_3^{H,L}\left(y - \frac{1}{\alpha_i+1}\log\varepsilon_i\right) \;=\; e^{-\{\alpha_3,\alpha_3+1\}\,y}.$$

By (80),

$$\bar{v} \;-\; S_i\left(y - \frac{1}{\alpha_i+1}\log\varepsilon_i\right) \;\to\; \bar{v} \;-\; \hat{M}_i(y)\,.$$

In order to conclude that

$$\begin{aligned}&\lim_{\varepsilon\to 0} \varepsilon^{-\alpha_3/(\alpha_i+1)} \int_{\mathbb{R}} \psi_3^{H}\left(y - \frac{1}{\alpha_i+1}\log\varepsilon_i\right)\,\left(\bar{v} \;-\; S_i\left(y - \frac{1}{\alpha_i+1}\log\varepsilon_i\right)\right)\,dy \\ &\qquad = \; c_3 \int_{\mathbb{R}} e^{-\alpha_3 y}\,(\bar{v} \;-\; \hat{M}_i(y))\,dy,\end{aligned} \tag{85}$$

and that

$$\int_{\mathbb{R}} \psi_3^{L}\left(y - \frac{1}{\alpha_i+1}\log\varepsilon_i\right)\,\left(\bar{v} \;-\; S_i\left(y - \frac{1}{\alpha_i+1}\log\varepsilon_i\right)\right)\,dy \;=\; o(\varepsilon^{\alpha_3/(\alpha_i+1)}),$$

we will show that the integrands

$$I(y) = \varepsilon^{-\alpha_3/(\alpha_i+1)}\,\psi_3^{H}\left(y - \frac{1}{\alpha_i+1}\log\varepsilon_i\right)\,\left(\bar{v} \;-\; S_i\left(y - \frac{1}{\alpha_i+1}\log\varepsilon_i\right)\right)$$

and

$$\varepsilon^{-(\alpha_3+\varepsilon)/(\alpha_i+1)}\,\psi_3^{L}\left(y - \frac{1}{\alpha_i+1}\log\varepsilon_i\right)\,\left(\bar{v} \;-\; S_i\left(y - \frac{1}{\alpha_i+1}\log\varepsilon_i\right)\right)$$

have an integrable majorant. Then, (85) will follow from the Lebesque dominated convergence theorem.

We decompose the integral in question into three parts, as

$$\int_{-\infty}^{\frac{1}{1+\alpha}\log\varepsilon} I_1(y)\,dy \;+\; \int_{\frac{1}{1+\alpha^L}\log\varepsilon}^{A} I_2(y)\,dy \;+\; \int_{A}^{+\infty} I_3(y)\,dy,$$

and prove the required limit behavior for each integral separately. To this end, we will need to establish sharp bounds for $S(\theta)$ and $V_2(\theta)$.

**Lemma E.2** *Let $\mathcal{L}(\theta,\varepsilon)$ be a function such that*

$$\lim_{\theta\to-\infty,\varepsilon\to 0} \mathcal{L}(\theta,\varepsilon) \;=\; 0.$$

*We have*

$$S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right) \;\le\; \bar{v} \,+\, C_1\,\mathcal{L}(\theta,\varepsilon) \tag{86}$$

*for all sufficiently small $\varepsilon > 0$ and sufficiently small $\theta$ if and only if*

$$\frac{1}{\alpha+1}\log f(\bar{v} \,+\, \mathcal{L}(\theta,\varepsilon)) \;-\; \log(\mathcal{L}(\theta,\varepsilon)) \le\; C_2 \,-\, \theta\,. \tag{87}$$

*If* (86) *holds, we have*

$$V_2\left(S\left(\theta_3 - \frac{1}{\alpha+1}\log\varepsilon\right)\right) \;\le\; \frac{\log\varepsilon}{1+\alpha} \,+\, C_3 \,+\, \log\mathcal{L}(\theta,\varepsilon) \,-\, \theta. \tag{88}$$

**Proof.** Applying $V_1$ to both sides of (86) and using the fact that $V_1$ is strictly increasing, we see that the desired inequality is equivalent to

$$\theta - \frac{1}{\alpha+1}\log\varepsilon \;\le\; V_1(\bar{v} \,+\, \mathcal{L})\,.$$

Now,

$$V_1(z) + \frac{1}{\alpha+1}\log\varepsilon \;=\; \log\frac{z-\bar{v}}{v^H - z} - V_2(z) + \frac{1}{\alpha+1}\log\varepsilon \;=\; \log\frac{z-\bar{v}}{v^H - z} - \frac{1}{\alpha+1}\,\log f(z)\,.$$

The claim follows because we are in the regime when $v^H - z$ is uniformly bounded away from zero.

Furthermore,

$$-\frac{\log\varepsilon}{1+\alpha} \,+\, V_2(S) \;=\; \log\left(\frac{S-\bar{v}}{v^H - S}\right) \,-\, \theta. \tag{89}$$

If $\theta$ is bounded from above, $S$ is uniformly bounded away from $v^H$, and hence

$$\log\left(\frac{S-\bar{v}}{v^H - S}\right) \,-\, \theta \;\le\; C_4 \,+\, \log(S-\bar{v}) \,-\, \theta.$$

The claim follows. ■

**Lemma E.3** *Suppose that $\varepsilon > 0$ is sufficiently small. Then, for*

$$\theta \ge \frac{1}{\alpha+1}\log\varepsilon, \tag{90}$$

*we have*

$$S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right) \;\le\; \bar v \,+\, C_5\, e^{\frac{\alpha+1}{\alpha-1}\theta}\,, \tag{91}$$

*and for*

$$\theta \;\le\; \frac{1}{\alpha+1}\log\varepsilon, \tag{92}$$

*we have that*

$$S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right) \;\le\; \bar v \,+\, C_6\, \varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\, e^{\frac{\alpha}{\alpha-1}\theta}\,. \tag{93}$$

**Proof.** By Lemma E.2, inequality (93) is equivalent to

$$\frac{1}{\alpha+1}\,\log f(\bar v \,+\, C_6\, \varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\, e^{\frac{\alpha}{\alpha-1}\theta}) \;-\; \log(C_6\, \varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\, e^{\frac{\alpha}{\alpha-1}\theta}) \;\le\; -\theta \,+\, C_7\,. \tag{94}$$

Under the condition (92),

$$\max\left\{(z-\bar v)^2\,,\, \varepsilon^{1/\alpha}\,(z-\bar v)^{(\alpha+1)/\alpha}\right\} \;=\; \varepsilon^{1/\alpha}\,(z-\bar v)^{(\alpha+1)/\alpha} \tag{95}$$

for

$$z \;=\; C_8\, \varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\, e^{\frac{\alpha}{\alpha-1}\theta}\,.$$

Hence, by (79),

$$f(z) \;\le\; C_9\, \varepsilon^{1/\alpha}\,(z-\bar v)^{(\alpha+1)/\alpha}\,.$$

Consequently,

$$\begin{aligned}
&\frac{1}{\alpha+1}\,\log f(\bar v \,+\, C_6\, \varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\, e^{\frac{\alpha}{\alpha-1}\theta}) \;-\; \log\left(C_6\, \varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\, e^{\frac{\alpha}{\alpha-1}\theta}\right) \\
&\quad\le\; C_{10} \;+\; \frac{1}{(\alpha+1)\,\alpha}\log\varepsilon \;+\; \frac{1}{\alpha}\left(\frac{\alpha}{\alpha-1}\theta \,+\, \frac{1}{(\alpha+1)(\alpha-1)}\log\varepsilon\right) \\
&\qquad -\; \left(\frac{\alpha}{\alpha-1}\theta \,+\, \frac{1}{(\alpha+1)(\alpha-1)}\log\varepsilon\right) \\
&\quad=\; -\theta \;+\; C_{10}\,,
\end{aligned} \tag{96}$$

and (93) follows.

Similarly, when $\theta$ satisfies (90), a direct calculation shows that

$$\max\left\{(z-\bar v)^2\,,\, \varepsilon^{1/\alpha}\,(z-\bar v)^{(\alpha+1)/\alpha}\right\} \;=\; (z-\bar v)^2 \tag{97}$$

for

$$z \;=\; \bar v \,+\, C_5\, e^{\frac{\alpha+1}{\alpha-1}\theta}\,.$$

Therefore, by (79),

$$\frac{1}{\alpha+1}\,\log f(\bar{v}\;+\;C_5\,e^{\frac{\alpha+1}{\alpha-1}\theta})\;-\;\log(C_5\,e^{\frac{\alpha+1}{\alpha-1}\theta}\,)$$
$$\leq\;C_{11}\;+\;\frac{2}{\alpha-1}\theta\;-\;\frac{\alpha\,+1}{\alpha-1}\theta\;=\;-\theta\;+\;C_{11}, \tag{98}$$

and (91) follows. ■

**Lemma E.4** *If*

$$\frac{\alpha+1}{\alpha-1}\;>\;\alpha_3,$$

*then*

$$\int_{-\infty}^{\frac{1}{\alpha+1}\,\log\varepsilon}\psi_3^{H,L}\left(\theta-\frac{1}{\alpha+1}\log\varepsilon\right)\,\left(\bar{v}\;-\;S\left(\theta-\frac{1}{\alpha+1}\log\varepsilon\right)\right)\,d\theta\;=\;o(\varepsilon^{\alpha_3/(\alpha+1)})\,.$$

**Proof.** By (92), since $\psi_3^{H,L}$ is bounded, we get

$$\begin{aligned}
&\int_{-\infty}^{\frac{1}{\alpha+1}\,\log\varepsilon}\psi_3^{H,L}\left(\theta-\frac{1}{\alpha+1}\log\varepsilon\right)\,\left(\bar{v}\;-\;S\left(\theta-\frac{1}{\alpha+1}\log\varepsilon\right)\right)\,d\theta\\
&\leq\;C_{12}\int_{-\infty}^{\frac{1}{\alpha+1}\,\log\varepsilon}\varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\,e^{\frac{\alpha}{\alpha-1}\theta}\,d\theta\\
&=\;\varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\,\frac{\alpha-1}{\alpha}\,\varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}+\frac{\alpha}{(\alpha+1)(\alpha-1)}}\\
&=\;o(\varepsilon^{\alpha_3/(\alpha+1)})\,.
\end{aligned} \tag{99}$$

■

**Lemma E.5** *If*

$$\frac{\alpha+1}{\alpha-1}\;>\;\alpha_3,$$

*then*

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon^{-\frac{\alpha_3}{\alpha+1}}\int_{\frac{1}{\alpha+1}\,\log\varepsilon}^{A}\psi_3^{H}\left(\theta-\frac{1}{\alpha+1}\log\varepsilon\right)\,\left(\bar{v}\;-\;S\left(\theta-\frac{1}{\alpha+1}\log\varepsilon\right)\right)\,d\theta\\
&\qquad=\;c_3\int_{-\infty}^{A}(\bar{v}-\hat{M}(\theta))\,e^{-\alpha_3\,\theta}\,d\theta
\end{aligned} \tag{100}$$

*and*

$$\int_{\frac{1}{\alpha+1}\,\log\varepsilon}^{A}\psi_3^{L}\left(\theta-\frac{1}{\alpha+1}\log\varepsilon\right)\,\left(\bar{v}\;-\;S\left(\theta-\frac{1}{\alpha+1}\log\varepsilon\right)\right)\,d\theta\;=\;o(\varepsilon^{\alpha_3/(\alpha+1)}).$$

**Proof.** By assumption, as $x \to \infty$,

$$\psi_3^H(x) \ \sim \ c_3\, e^{-\alpha_3 x}.$$

The claim follows from (80) and (90), which provides an integrable majorant. ■

The same argument implies the following result.

**Lemma E.6** *We have*

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \varepsilon^{-\frac{\alpha_3}{\alpha+1}} \int_A^{+\infty} \psi_3^H\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\left(\bar v \, - \, S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\right) d\theta \\
&\qquad = \ c_3 \int_A^{+\infty} (\bar v - \hat M(\theta))\, e^{-\alpha_3\,\theta}\, d\theta
\end{aligned} \tag{101}$$

*and*

$$\int_A^{+\infty} \psi_3^L\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\left(\bar v \, - \, S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\right) d\theta \ = \ o(\varepsilon^{\alpha_3/(\alpha+1)})\,.$$

Finally, to complete the proof, it suffices to show that the term

$$\int_{\mathbb{R}} \psi_3^{H,L}(y)\,(\{\bar v, v^H\} \, - \, S_i(y))\, F_i^{H,L}(V_{2,i}(S_i(y)))\, dy \ = \ o(\varepsilon^{\alpha_3/(\alpha+1)}) \tag{102}$$

in (83) is negligible for large $G(v)$ As $G(v) \to +\infty$, we have $V_{2,i}(S_i(y)) \ \to \ -\infty$. Furthermore, as $x \to -\infty$,

$$F_i^{H,L}(x) \ \sim \ \frac{c_i}{\{\alpha_i + 1, \alpha_i\}}\, e^{x\,\{\alpha_i+1, \alpha_i\}}.$$

The claim then follows by essentially the same arguments used above. Special care is only needed because $(v^H - S)^{-1}$ blows up as $\theta \uparrow +\infty$.

By (89),

$$F_i^H\left(V_2\left(S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\right)\right) \ \le \ C_{13}\,\varepsilon\,\left(\frac{S - \bar v}{v^H - S}\, e^{-\theta}\right)^{\alpha_i+1}\,.$$

Thus, to get an integrable majorant, it would suffice to have a bound

$$v^H - S \ \ge \ C_{14}\, e^{-\beta\theta},$$

for some $\beta > 0$ and for a sufficiently large $\theta$. By the argument used in the proof of Lemma E.2, it suffices to show that for sufficiently large $\theta$,

$$\frac{1}{\alpha+1}\,\log f(v_H \, - \, C_{14}\, e^{-\beta\theta}) \ \le \ C_{15} \, + \, (\beta - 1)\,\theta\,.$$

Now, it follows from (77) that

$$f'(z) \;\le\; f(z)^{1/(\alpha+1)} \;+\; \frac{v^H - \bar{v}}{v^H - z}.$$

Since, for sufficiently small $\varepsilon$, $f(z)$ is uniformly bounded away from zero on compact subsets of $(\bar{v}, v^H]$, we get

$$\frac{d}{dz}\big(f(z)^{\alpha/(\alpha+1)}\big) \;\le\; C_{16}\,(1 \,+\, (v^H - z)^{-1}),$$

for some $K > 0$ when $z$ is close to $v^H$. Integrating this inequality, we get

$$f(z)^{\alpha/(\alpha+1)} \;\le\; C_{17}\,(1 \,-\, \log(v^H - z)).$$

Consequently,

$$\frac{1}{\alpha+1}\log f(v_H \,-\, C_{14}\, e^{-\beta\theta}) \;\le\; C_{18}\, \log\theta$$

if $\theta$ is sufficiently large. Hence, the required inequality holds for any $\beta > 1$ with a sufficiently large $C_{14}$. Pick a $\beta$ so that $(\beta - 1)(\alpha + 1) \;<\; \alpha_3$. Then we get that, for sufficiently large $\theta$,

$$F_i^H\left(V_2\left(S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\right)\right) \;\le\; C_{19}\, e^{(\beta-1)(\alpha+1)\theta},$$

and the claim follows.

Thus, the unconditional expected utility of agent $i$ is approximately

$$0.5\,(v^H - \bar{v}) \;-\; \varepsilon_i^{\alpha_3/(\alpha+1)} \int_{\mathbb{R}} (\hat{M}_i(\theta) - \bar{v})\, e^{-\alpha_3\theta}\, d\theta\,.$$

For the case in which the information characteristics of classes 1 and 2 are not hidden, we need to consider two sub cases. If $\alpha_1 > \alpha_2$, then, since $\varepsilon_1$ and $\varepsilon_2$ differ from each other by a constant proportion, sending $G(v)$ to infinity leads to

$$\varepsilon_1^{\alpha_3/(\alpha_1+1)} \int_{\mathbb{R}} (\hat{M}_1(\theta) - \bar{v})\, e^{-\alpha_3\theta}\, d\theta \;>\; \varepsilon_2^{\alpha_3/(\alpha_2+1)} \int_{\mathbb{R}} (\hat{M}_2(\theta) - \bar{v})\, e^{-\alpha_3\theta}\, d\theta,$$

and the claim follows. If, instead, $\alpha_1 = \alpha_2$ but $c_1 < c_2$, we get that $\varepsilon_1 > \varepsilon_2$ and $\hat{M}_1 = \hat{M}_2$, so the claim also follows in this case.

The case in which information characteristics are hidden is handled analogously.

## F Proof of Proposition 4.6

First, we note that the evolution equations

$$\frac{d}{dt}\hat{\psi}_{it} \;=\; \lambda_i\,\hat{\psi}_{it}\,(-1\,+\,\hat{\psi}_{3t})$$

imply that

$$\hat{\psi}_{2t} \;=\; \hat{\psi}_{20}\,e^{-\lambda_2 t}\,e^{\lambda_2\int_0^t \hat{\psi}_{3\tau}\,d\tau} \;=\; \hat{\psi}_{20}\left(\frac{\hat{\psi}_{1t}}{\hat{\psi}_{10}}\right)^{\lambda_2/\lambda_1}.$$

Since, by assumption, $\psi_{it} \;\sim\; \mathrm{Exp}_{+\infty}(c_{it},\gamma_{it},-\alpha_{it})$, we immediately get (see, for example, Korevaar (2004), Theorem 15.3, p.30) that $\alpha_{1t}=\alpha_{2t}$ and that

$$\hat{\psi}_{it}(k) \underset{k\uparrow\alpha_{1t}}{\sim} \frac{c_{it}\,\Gamma(\gamma_{it}+1)}{(\alpha_{it}-k)^{\gamma_{it}+1}}\,.$$

This immediately yields that

$$\gamma_{2t}+1 \;=\; \frac{\lambda_2}{\lambda_1}(\gamma_{1t}+1) \;\Rightarrow\; \gamma_{2t} \;>\; \gamma_{1t}\,.$$

Consequently, $\mathrm{Tail}(\psi_{1t}) \;\prec\; \mathrm{Tail}(\psi_{2t})$. It follows from the proof of Proposition 4.5 that the required result holds for any positive $t>0$, provided that $\bar{v}-v_3$ is larger than some $t$-dependent threshold.

Thus, it remains to show the required inequality, comparing auction expected utilities, over a sufficiently small time interval $[0,t]$. Thus, from now on, we will assume that $T$ is sufficiently small. Furthermore, we will provide a proof only for the case in which the information characteristics are not hidden. The case of unobservable information characteristics is handled analogously.

We have

$$\lambda_i^{-1}\,E[\mathcal{U}_i(\Theta_{i0})] \;=\; \frac{1}{2}\int_0^T\int_{\mathbb{R}}\big(\psi_{i\tau}^H(\theta)\,\pi_i^H(\tau,\theta)\;+\;\psi_{i\tau}^L(\theta)\,\pi_i^L(\tau,\theta)\big)\;d\theta\,d\tau.$$

Here,

$$\frac{d}{d\tau}\psi_{i\tau}^K \;=\; -\lambda_i\,\psi_{i\tau}^K\;+\;\lambda_i\,\psi_{i\tau}^K*\psi_{3\tau}^K \tag{103}$$

for $K=H$ or $K=L$, and

$$\pi_i^{H,L}(\tau,z) \;=\; \int_{-\infty}^{V_{1,i}(B_i(\tau,z))}(\{v^H,\bar{v}\}\;-\;S_i(\tau,y))\psi_{3\tau}^{H,L}(y)\;dy\,.$$

By assumption,

$$\psi_{10}^{H,L} \;=\; \psi_{2,0}^{H,L}\,.$$

Therefore $V_{2,i}(0,z)$ is also independent of $i$, and we will omit the index $i$ in what follows.

We denote

$$\Pi_i(\tau) \;=\; \int_{\mathbb{R}} \left(\psi^H_{i\tau}(\theta)\,\pi^H_i(\tau,\theta) \;+\; \psi^L_{i\tau}(\theta)\,\pi^L_i(\tau,\theta)\right)\,d\theta.$$

It follows from (103) that, for small $\tau$,

$$\psi^K_{i,\tau} \;=\; (1-\lambda_i\tau)\,\psi_{0i} \;+\; \lambda_i\,\tau\,\psi_{i0} \,*\, \psi^K_{30} \;+\; o(\tau).$$

Consequently,[18]

$$\Pi_i(\tau) \;=\; (1-\lambda_i\tau)\int_{\mathbb{R}} \left(\psi^H_{i0}(\theta)\,\pi^H_i(\tau,\theta) \;+\; \psi^L_{i0}(\theta)\,\pi^L_i(\tau,\theta)\right)\,d\theta$$

$$+\;\lambda_i\tau\int_{\mathbb{R}} \left((\psi_{i0}*\psi_{30})^H(\theta)\,\pi^H_i(\tau,\theta) \;+\; (\psi_{i0}*\psi_{30})^L(\theta)\,\pi^L_i(\tau,\theta)\right)\,d\theta + \; o(\tau)\,. \tag{104}$$

The argument used in the proof of Theorem 4.2 implies that for small $\tau$,

$$\begin{aligned}&\int_{\mathbb{R}} \left((\psi_{i0}*\psi_{30})^H(\theta)\,\pi^H_i(\tau,\theta) \;+\; (\psi_{i0}*\psi_{30})^L(\theta)\,\pi^L_i(\tau,\theta)\right)\,d\theta\\ &\qquad > \int_{\mathbb{R}} \left(\psi^H_{i0}(\theta)\,\pi^H_i(\tau,\theta) \;+\; \psi^L_{i0}(\theta)\,\pi^L_i(\tau,\theta)\right)\,d\theta.\end{aligned} \tag{105}$$

Thus, in order to complete the proof, it remains to show that, for small $\tau$,

$$\begin{aligned}&\int_{\mathbb{R}} \left(\psi^H_{20}(\theta)\,\pi^H_2(\tau,\theta) \;+\; \psi^L_{20}(\theta)\,\pi^L_2(\tau,\theta)\right)\,d\theta\\ &\qquad > \int_{\mathbb{R}} \left(\psi^H_{10}(\theta)\,\pi^H_1(\tau,\theta) \;+\; \psi^L_{10}(\theta)\,\pi^L_1(\tau,\theta)\right)\,d\theta.\end{aligned} \tag{106}$$

As above, for simplicity, we use the normalization $\bar{v}=0,\; v^H=1$. As in the proof of Proposition 3.10, let

$$g_i(\tau,z) \;=\; e^{(\alpha+1)V_{2,i}(\tau,z)},$$

where

$$\alpha \;\stackrel{def}{=}\; \alpha_{10} \;=\; \alpha_{20}\,.$$

Let also

$$w_i(z) \;=\; \frac{d}{d\tau}g_i(\tau,z)\,|_{\tau=0}.$$

[18]The $o(\tau)$ term is a measure and therefore, when integrating against it, the result requires additional justification. This is supplied by using the bounds derived in the proof of Proposition 4.5.

It follows from the proof of Proposition 3.10[19] that this derivative is well-defined and we can differentiate (46) to obtain

$$\frac{d}{dz}w_i(z) \;=\; (\alpha+1)w(z)\,\frac{1}{(\bar v - v_3)}\,\left(\frac{z}{1-z}\,\frac{G_0^H(V_2(0,z))}{\psi_0^H(V_2(0,z))} \;+\; \frac{G_0^L(V_{2,i}(\tau,z))}{\psi_0^L(V_{2,i}(\tau,z))}\right)$$
$$+\;\frac{(\alpha+1)\,g(0,z)}{(\bar v - v_3)}\left(\frac{z}{(1-z)(\psi_0^H(V_2(0,z)))^2}\times\right.$$
$$\left(\left(\left(\frac{d}{d\tau}G_0^H\right)(V_2(0,z)) \;-\; \psi_0^H(V_2(0,z))\,(\alpha+1)^{-1}w_i(z)g(0,z)^{-1}\right)\psi_0^H(V_2(0,z))\right.$$
$$\left.-\;G_0^H(V_2(0,z))\,\left(\left(\frac{d}{d\tau}\psi_0^H\right)(V_2(0,z)) \;+\; \frac{d}{dV}\psi_0^H(V_2(0,z))\,(\alpha+1)^{-1}w(z)g(0,z)^{-1}\right)\right)$$
$$+\;\left(\left(\left(\frac{d}{d\tau}G_0^L\right)(V_2(0,z)) \;-\; \psi_{i0}^L(V_2(0,z))\,(\alpha+1)^{-1}w_i(z)g(0,z)^{-1}\right)\right.$$
$$\left.\left.-\;\frac{G_0^L(V_2(0,z))}{\psi_0^L(V_2(0,z))}\left(\left(\frac{d}{d\tau}\psi_0^L\right)(V_2(0,z)) \;+\; \frac{d}{dV}\psi_0^L(V_2(0,z))\,(\alpha+1)^{-1}w_i(z)g(0,z)^{-1}\right)\right)\right), \tag{107}$$

with $w(0) \;=\; 0$. This is a linear ODE. Solving it, we obtain

$$w(z) \;=\; \int_0^z e^{\int_y^z \chi(x)dx}\,\mu_i(y)\,dy,$$

where

$$\chi(z) \;=\; (\alpha+1)\,\frac{1}{(\bar v - v_3)}\left(\frac{z}{1-z}\,\frac{G_\tau^H(V_2(0,z))}{\psi_0^H(V_2(0,z))} \;+\; \frac{G_0^L(V_2(0,z))}{\psi_0^L(V_2(0,z))}\right)$$
$$-\;\frac{1}{\bar v - v_3}\left(\frac{z}{(1-z)(\psi_0^H(V_2(0,z)))^2}\times\right.$$
$$\left((\psi_0^H(V_2(0,z)))^2 \;+\; G_0^H(V_2(0,z))\,\frac{d}{dV}\psi_0^H(V_2(0,z))\right)$$
$$\left.+\,(\psi_0^L(V_2(0,z)))^{-2}\left((\psi_0^L(V_2(0,z)))^2 \;+\; G_0^L(V_2(0,z))\,\frac{d}{dV}\psi_0^L(V_2(0,z))\right)\right) \tag{108}$$

[19]This claim follows from implicit function theorem if we rewrite the required ODE as an integral equation and use the arguments from the proof of Proposition 3.10.

is independent of $i$ and where

$$\mu_i(z) \;=\; \frac{(\alpha+1)\,g(0,z)}{(\bar v - v_3)} \left( \frac{z}{(1-z)(\psi_0^H(V_2(0,z)))^2} \times \right.$$

$$\left( \left(-\lambda_i\, G_0^H + \lambda_i G_0^H * \psi_{30}^H\right)(V_2(0,z))\,\psi_0^H(V_2(0,z)) \right.$$

$$\left. -\; G_0^H(V_2(0,z))\,\left(-\lambda_i\,\psi_0^H + \lambda_i \psi_0^H * \psi_{30}^H\right)(V_2(0,z)) \right) \tag{109}$$

$$+\;(\psi_0^L(V_2(0,z)))^{-2} \left( \left(-\lambda_i\, G_0^L + \lambda_i G_0^L * \psi_{30}^L\right)(V_2(0,z))\psi_0^L(V_2(0,z)) \right.$$

$$\left.\left. -\; G_0^L(V_2(0,z))\,\left(-\lambda_i\,\psi_0^L + \lambda_i \psi_0^L * \psi_{30}^L\right)(V_2(0,z)) \right)\right).$$

By definition,

$$\frac{d}{d\tau}\,V_{2,i}(\tau,z)|_{\tau=0} \;=\; \frac{w_i(z)}{(\alpha+1)\,g(0,z)} \;\stackrel{def}{=}\; W_i(z)\,. \tag{110}$$

For brevity, we use the notation $g(z) = g(0,z)$, $S(z) = S(0,z)$, and $B(z) = B(0,z)$.

We have

$$\frac{d}{d\tau} V_{1,i}(\tau,z)|_{\tau=0} \;=\; -\,W_i(z).$$

Therefore, differentiating the identity

$$V_{1,i}(\tau, S_i(\tau,z)) \;=\; z,$$

we get

$$\begin{aligned} \frac{d}{d\tau}\,S_i(\tau,z)|_{\tau=0} \;&=\; \frac{W_i(S(z))}{\frac{d}{dz}(V_1)(S(z))} \\ &=\; \frac{W_i(S(z))}{\frac{1}{S(z)(1-S(z))} \;-\; (\bar v - v_3)^{-1}\left(\frac{S(z)}{1-S(z)}\,\frac{1}{h^H(V_2(S(z)))} \;+\; \frac{1}{h^L(V_2(S(z)))}\right)}\,. \end{aligned} \tag{111}$$

Differentiating the identity

$$V_{2,i}(S_i(\tau,z)) \;=\; \log \frac{S_i(\tau,z)}{1-S_i(\tau,z)} \;-\; z$$

with respect to $\tau$, we get

$$\begin{aligned} &\frac{d}{d\tau}\,(V_{2,i}(S_i(\tau,z)))\,|_{\tau=0} \\ &=\; \frac{W_i(S(z))}{\frac{1}{S(z)(1-S(z))} \;-\; (\bar v - v_3)^{-1}\left(\frac{S(z)}{1-S(z)}\,\frac{1}{h^H(V_2(S(z)))} \;+\; \frac{1}{h^L(V_2(S(z)))}\right)}\,\frac{1}{S(z)\,(1-S(z))}. \end{aligned} \tag{112}$$

Therefore,

$$
\begin{aligned}
&\frac{d}{d\tau}\left(\int_{\mathbb{R}} \psi_0^{H,L}(z)\, \pi_i^{H,L}(\tau,z) dz\right)|_{\tau=0} \\
&= \frac{d}{d\tau}\left(\int_{\mathbb{R}} G_i^H(V_{2,i}(\tau, S_i(\tau,y))(\{v^H, \bar v\} \,-\, S_i(\tau,y))\, \psi_{3\tau}^{H,L}(y) dy\right)|_{\tau=0} \\
&= -\int_{\mathbb{R}} \psi_0^{H,L}(V_2(S(y)))\, \frac{W_i(S(y))}{\frac{1}{S(y)(1-S(y))} \,-\, (\bar v - v_3)^{-1}\left(\frac{S(y)}{1-S(y)}\,\frac{1}{h^H(V_2(S(y)))} \,+\, \frac{1}{h^L(V_2(S(y)))}\right)} \\
&\quad \times \frac{1}{S(y)\,(1-S(y))}\,(\{1,0\} - S(y))\,\psi_{30}^{H,L}(y)\, dy \\
&\quad - \int_{\mathbb{R}} G_0^{H,L}(V_2(S(y)))\, \frac{W_i(S(y))}{\frac{1}{S(y)(1-S(y))} \,-\, (\bar v - v_3)^{-1}\left(\frac{S(y)}{1-S(y)}\,\frac{1}{h^H(V_2(S(y)))} \,+\, \frac{1}{h^L(V_2(S(y)))}\right)} \\
&\quad \times \psi_{30}^{H,L}(y)\, dy \\
&\quad + \int_{\mathbb{R}} G_0^{H,L}(V_2(S(y)))\,(\{1,0\} - S(y)) \\
&\quad \times \lambda_3\left(-\psi_{30}^{H,L}(y) \,+\, \psi_{30}^{H,L} * \sum_k \kappa_{3k}\, \psi_{k0}^{H,L}(y)\right) dy \\
&\overset{def}{=} \tilde\pi_i^{H,L}.
\end{aligned}
\tag{113}
$$

We now define

$$
\tilde\pi_i \;=\; \frac{1}{2}\,(\tilde\pi_i^H \;+\; \tilde\pi_i^L)\,. \tag{114}
$$

In order to prove (106), it remains to show that

$$
\tilde\pi_2 \;>\; \tilde\pi_1\,.
$$

As in the proof of Proposition 4.5, we assume for simplicity that $\gamma_i = 0$ for all $i$ (that is, no power tails). Recall also that, by assumption, $\psi_{10} = \psi_{20}$. Hence, $(c_1, \alpha_1) = (c_2, \alpha_2) = (c, \alpha)$.

We will use the same bounds and asymptotic results that were derived in the proof of Proposition 4.5.

Let us first understand the behavior of $W_i(z)$ as $G(v) \to \infty$. We have

$$
V_2(z) \;\approx\; \frac{1}{\alpha+1}\log(\varepsilon\, f_0(z)).
$$

Therefore,

$$
\begin{aligned}
g(0,z) \quad &\approx \quad e^{(\alpha+1)V_2(z)} \;\sim\; \varepsilon\, f_0(z) \\
\psi_0^{H,L}(V_2(z)) \quad &\sim \quad c\, e^{\{\alpha+1,\alpha\}\left(\frac{1}{\alpha+1}\log(\varepsilon\, f_0(z))\right)} \;=\; c\,\varepsilon^{\{\alpha+1,\alpha\}/(\alpha+1)}\, f_0(z)^{\{\alpha+1,\alpha\}/(\alpha+1)},
\end{aligned}
$$

and

$$\frac{d}{dV}\,\psi_0^{H,L}(V_2(z)) \;\sim\; c\,\{\alpha+1,\alpha\}\,e^{\{\alpha+1,\alpha\}\left(\frac{1}{\alpha+1}\log(\varepsilon\,f_0(z))\right)} \tag{115}$$
$$\;=\; c\,\{\alpha+1,\alpha\}\,\varepsilon^{\{\alpha+1,\alpha\}/(\alpha+1)}\,f_0(z)^{\{\alpha+1,\alpha\}/(\alpha+1)},$$

and

$$\frac{1}{\bar{v}-v_3} \;=\; \frac{c}{\alpha+1}\,\varepsilon\,.$$

Therefore,

$$\begin{aligned}
&\chi(z)\\
&= (\alpha+1)\,\frac{1}{(\bar{v}-v_3)}\,\left(\frac{z}{1-z}\,\frac{G_0^H(V_2(0,z))}{\psi_0^H(V_2(0,z))} \;+\; \frac{G_0^L(V_2(0,z))}{\psi_0^L(V_2(0,z))}\right)\\
&-\;\frac{1}{\bar{v}-v_3}\left(\frac{z}{(1-z)(\psi_0^H(V_2(0,z)))^2}\right.\\
&\quad\times\left((\psi_0^H(V_2(0,z)))^2 \;+\; G_0^H(V_2(0,z))\,\frac{d}{dV}\psi_0^H(V_2(0,z))\right)\\
&\quad\left.+\,(\psi_0^L(V_2(0,z)))^{-2}\left((\psi_0^L(V_2(0,z)))^2 \;+\; G_0^L(V_2(0,z))\,\frac{d}{dV}\psi_0^L(V_2(0,z))\right)\right)
\end{aligned} \tag{116}$$

$$\begin{aligned}
&= c\,\varepsilon\,\left(\frac{z}{1-z}\,\frac{1}{c\,\varepsilon\,f_0(z)} \;+\; \frac{1}{c\,\varepsilon^{\alpha/(\alpha+1)}\,f_0(z)^{\alpha/(\alpha+1)}}\right)\\
&\quad-\;\frac{c}{\alpha+1}\,\varepsilon\left(\frac{z}{(1-z)(c\,\varepsilon\,f_0(z))^2}\left((c\,\varepsilon\,f_0(z))^2 \;+\; (\alpha+1)\,c\,\varepsilon\,f_0(z)\right)\right.\\
&\quad\left.+\,(c\varepsilon^{\alpha/(\alpha+1)}f_0^{\alpha/(\alpha+1)})^{-2}\left((c\varepsilon^{\alpha/(\alpha+1)}f_0^{\alpha/(\alpha+1)})^2 \;+\; c\,\alpha\,\varepsilon^{\alpha/(\alpha+1)}f_0^{\alpha/(\alpha+1)}\right)\right)\\
&= \frac{1}{\alpha+1}\,\varepsilon^{1/(\alpha+1)}\,f_0(z)^{-\alpha/(\alpha+1)} \;+\; O(\varepsilon)\,.
\end{aligned} \tag{117}$$

For simplicity, we assume that $\alpha_{30} \neq \alpha$. (If $\alpha_{30} = \alpha$, then power tails will appear. The analysis is in that case analogous, but technically more involved.) We then have, as $x \to -\infty$,

$$\begin{aligned}
(\psi_0^H * \psi_{30}^H)(x) \;&=\; \int_{\mathbb{R}} \psi_0^H(x-y)\,\psi_{30}^H(y)\,dy\\
&\sim\; \begin{cases} c\,e^{(\alpha+1)x}\,\hat{\psi}_{30}^H(\alpha) & , \quad \alpha < \alpha_{30} \\ c_{30}\,e^{(\alpha_{30}+1)x}\,\hat{\psi}_0^H(\alpha_{30}) & , \quad \alpha > \alpha_{30} \end{cases} \;\equiv\; d\,e^{(\beta+1)\,x},
\end{aligned}$$

where

$$(d,\beta) \;\stackrel{def}{=}\; \begin{cases} (c\hat{\psi}_{30}^H(\alpha),\alpha)\;, & \alpha < \alpha_{30} \\ (c_{30}\hat{\psi}_0^H(\alpha_{30}),\alpha_{30})\;, & \alpha > \alpha_{30} \end{cases}$$

and where we have used the fact that

$$\hat{\psi}^H(k) \;=\; \hat{\psi}^H(-k-1).$$

In this case,

$$(G_0^H * \psi_{30}^H)(x) - G_0^H(x) \;=\; F_0^H(x) - (F_0^H * \psi_{30})(x) \underset{x\downarrow -\infty}{\sim} \frac{1}{\alpha+1}\, c\, e^{(\alpha+1)x} - \frac{1}{\beta+1}\, d\, e^{(\beta+1)\,x}.$$

Thus, in complete analogy with (116),

$$\begin{aligned}
\mu_i(z) \;=\; & \frac{(\alpha+1)\, g(0,z)}{(\bar{v}-v_3)} \Bigg( \frac{z}{(1-z)(\psi_0^H(V_2(0,z)))^2} \\
& \times \Bigg( \left(-\lambda_i\, G_0^H + \lambda_i G_0^H * \psi_{30}^H\right) (V_2(0,z))\, \psi_0^H(V_2(0,z)) \\
& \quad -\; G_0^H(V_2(0,z)) \left(-\lambda_i\, \psi_0^H + \lambda_i \psi_0^H * \psi_{30}^H\right) (V_2(0,z)) \Bigg) \\
& + \; (\psi_0^L(V_2(0,z)))^{-2} \Bigg( \left(-\lambda_i\, G_0^L + \lambda_i G_0^L * \psi_{30}^L\;\right) (V_2(0,z)) \psi_0^L(V_2(0,z)) \\
& \quad -\; G_0^L(V_2(0,z)) \left(-\lambda_i\, \psi_0^L + \lambda_i \psi_0^L * \psi_{30}^L\right) (V_2(0,z)) \Bigg) \Bigg)
\end{aligned} \tag{118}$$

$$\begin{aligned}
\sim\; & c\,\varepsilon^2\, f_0(z) \Bigg( \frac{z}{(1-z)\,(c\,\varepsilon\, f_0(z))^2} \\
& \times \Bigg( \lambda_i \left( \frac{1}{\alpha+1}\, c\,\varepsilon\, f_0(z) \;-\; \frac{1}{\beta+1}\, d\,\varepsilon^{(\beta+1)/(\alpha+1)}\, f_0(z)^{(\beta+1)/(\alpha+1)} \right) c\,\varepsilon\, f_0(z) \\
& \quad -\; \lambda_i\; \left( d\,\varepsilon^{(\beta+1)/(\alpha+1)}\, f_0(z)^{(\beta+1)/(\alpha+1)} \;-\; c\,\varepsilon\, f_0(z) \right) \Bigg) \\
& + \; (c\,\varepsilon^{\alpha/(\alpha+1)}\, f_0(z)^{\alpha/(\alpha+1)})^{-2} \\
& \times \Bigg( \lambda_i \left( \frac{1}{\alpha}\, c\,\varepsilon^{\alpha/(\alpha+1)}\, f_0(z)^{\alpha/(\alpha+1)} \;-\; \frac{1}{\beta}\, d\,\varepsilon^{\beta/(\alpha+1)}\, f_0^{\beta/(\alpha+1)} \right) c\,(\varepsilon\, f_0(z))^{\alpha/(\alpha+1)} \\
& \quad -\; \lambda_i\; \left( d\,(\varepsilon\, f_0(z))^{\beta/(\alpha+1)} \;-\; c\,(\varepsilon f_0(z))^{\alpha/(\alpha+1)} \right) \Bigg) \Bigg) \\
=\; & -\lambda_i\; b\, \frac{z}{(1-z)\, c}\, \varepsilon^{(\beta+1)/(\alpha+1)}\, f_0(z)^{(\beta-\alpha)/(\alpha+1)} \;+\; o(\varepsilon^{(\beta+1)/(\alpha+1)}) \,,
\end{aligned} \tag{119}$$

where

$$b \overset{def}{=} \begin{cases} d\,, & \alpha_{30} < \alpha \\ c\,(\hat{\psi}^H_{30}(\alpha) - 1)\,, & \alpha_{30} > \alpha \end{cases}$$

is a positive constant.

Therefore,

$$\begin{aligned} w_i(z) \;&=\; \int_0^z e^{\int_y^z \chi(x)dx}\, \mu_i(y)\, dy \\ &\approx\; -\,\varepsilon^{(\beta+1)/(\alpha+1)}\lambda_i\; b \int_0^z \frac{y}{(1-y)\,c}\, f_0(y)^{(\beta-\alpha)/(\alpha+1)}\, dy \;+\; o\left(\varepsilon^{(\beta+1)/(\alpha+1)}\right). \end{aligned} \tag{120}$$

Therefore,

$$W_i(z) \;=\, -\,\varepsilon^{(\beta-\alpha)/(\alpha+1)}\lambda_i\; X(z) \;+\; o\left(\varepsilon^{(\beta-\alpha)/(\alpha+1)}\right),$$

with

$$X(z) \overset{def}{=} b\, \frac{\int_0^z \frac{y}{(1-y)\,c}\, f_0(y)^{(\beta-\alpha)/(\alpha+1)}\, dy}{(\alpha+1) f_0(z)}\,.$$

Since $f_0(z) \;=\; \log(1-z)^{-1} - z$, a direct calculation shows that

$$X(z) \;\sim\; C_1\, z^{\frac{2(\beta-\alpha)}{\alpha+1}}$$

as $z \downarrow 0$, and that $X(z)$ is bounded when $z \uparrow 1$.

Now,

$$
\begin{aligned}
&\int_{\mathbb{R}} \psi_0^H(V_2(S(y)))\, \frac{W_i(S(y))}{\frac{1}{S(y)(1-S(y))} \;-\; (\bar v - v_3)^{-1}\left(\frac{S(y)}{1-S(y)}\,\frac{1}{h^{H,L}(V_2(S(y)))} \;+\; \frac{1}{h^L(V_2(S(y)))}\right)} \\
&\qquad \times \frac{1}{S(y)\,(1-S(y))}\,(\{1,0\} - S(y))\,\psi_{30}^{H,L}(y)\,dy \\
&= \int_{\mathbb{R}} \psi_0^{H,L}(V_2(S(y-(\alpha+1)^{-1}\log\varepsilon)))\, W_i(S(y-(\alpha+1)^{-1}\log\varepsilon)) \\
&\qquad \times \left(\frac{1}{S(y-(\alpha+1)^{-1}\log\varepsilon)(1-S(y-(\alpha+1)^{-1}\log\varepsilon))}\right. \\
&\qquad\qquad -\; (\bar v - v_3)^{-1}\left(\frac{S(y-(\alpha+1)^{-1}\log\varepsilon)}{1-S(y-(\alpha+1)^{-1}\log\varepsilon)}\,\frac{1}{h^{H,L}(V_2(S(y-(\alpha+1)^{-1}\log\varepsilon)))}\right. \\
&\qquad\qquad \left.\left. +\; \frac{1}{h^L(V_2(S(y-(\alpha+1)^{-1}\log\varepsilon)))}\right)\right)^{-1} \psi_{30}^{H,L}(y-(\alpha+1)^{-1}\log\varepsilon)\,dy \\
&\sim\; c\varepsilon^{\{1,\alpha/(\alpha+1)\}} \int_{\mathbb{R}} \left(\frac{\hat M(y)}{1-\hat M(y)}\,e^{-y}\right)^{\{\alpha+1,\alpha\}} W_i(\hat M(y))\left(\frac{1}{\hat M(y)(1-\hat M(y))}\right. \\
&\qquad \left. -\; (\alpha+1)^{-1}c\varepsilon\left(\frac{\hat M(y)}{1-\hat M(y)}\,\frac{1}{c\,\varepsilon\,\left(\frac{\hat M(y)}{1-\hat M(y)}e^{-y}\right)^{\alpha+1}} \;+\; \frac{1}{c\,\varepsilon^{\alpha/(\alpha+1)}\,\left(\frac{\hat M(y)}{1-\hat M(y)}e^{-y}\right)^{\alpha}}\right)\right)^{-1} \\
&\qquad \times\, c_{30}\,\varepsilon^{\{\alpha_{30},\alpha_{30}+1\}/(\alpha+1)}\,e^{-\{\alpha_{30},\alpha_{30}+1\}y}\,dy \\
&\sim\; -c\varepsilon^{(\alpha_{30}+\beta+1)/(\alpha+1)} \int_{\mathbb{R}} \left(\frac{\hat M(y)}{1-\hat M(y)}\,e^{-y}\right)^{\{\alpha+1,\alpha\}} \lambda_i\, X(\hat M(y)) \\
&\qquad \times \left(\frac{1}{\hat M(y)\,(1-\hat M(y))} \;-\; \frac{1}{\alpha+1}\,\frac{e^{(\alpha+1)y}(1-\hat M(y))^\alpha}{(\hat M(y))^\alpha}\right)^{-1} c_{30}\,e^{-\{\alpha_{30},\alpha_{30}+1\}y}\,dy,
\end{aligned}
\tag{121}
$$

where, by arguments used in the proof of Proposition 4.5, this asymptotic relationship holds provided that $\alpha < 3$ and $2(\alpha+1)/(\alpha-1) > \alpha_{30}$.

Similarly,

$$\int_{\mathbb{R}} G_0^H(V_2(S(y)))\,\frac{W_i(S(y))}{\frac{1}{S(y)(1-S(y))}\;-\;(\bar v - v_3)^{-1}\left(\frac{S(y)}{1-S(y)}\,\frac{1}{h^H(V_2(S(y)))}\;+\;\frac{1}{h^L(V_2(S(y)))}\right)}$$
$$\times\,\psi_{30}^H(y)\,dy$$
$$= \int_{\mathbb{R}} G_0^H(V_2(S(y-(\alpha+1)^{-1}\log\varepsilon)))$$
$$\times\,W_i(S(y-(\alpha+1)^{-1}\log\varepsilon))\Bigg(\frac{1}{S(y-(\alpha+1)^{-1}\log\varepsilon)(1-S(y-(\alpha+1)^{-1}\log\varepsilon))}$$
$$-\;(\bar v - v_3)^{-1}\Bigg(\frac{S(y-(\alpha+1)^{-1}\log\varepsilon))}{1-S(y-(\alpha+1)^{-1}\log\varepsilon))}\,\frac{1}{h^H(V_2(S(y-(\alpha+1)^{-1}\log\varepsilon))))}$$
$$+\;\frac{1}{h^L(V_2(S(y-(\alpha+1)^{-1}\log\varepsilon))))}\Bigg)\Bigg)^{-1}\,\psi_{30}^H(y-(\alpha+1)^{-1}\log\varepsilon))\,dy \tag{122}$$

$$\sim\;\int_{\mathbb{R}} W_i(\hat M(y))\Bigg(\frac{1}{\hat M(y)(1-\hat M(y))}$$
$$-\;(\alpha+1)^{-1}c\varepsilon\left(\frac{\hat M(y)}{1-\hat M(y)}\,\frac{1}{c\,\varepsilon\,\left(\frac{\hat M(y)}{1-\hat M(y)}e^{-y}\right)^{\alpha+1}}\;+\;\frac{1}{c\,\varepsilon^{\alpha/(\alpha+1)}\,\left(\frac{\hat M(y)}{1-\hat M(y)}e^{-y}\right)^{\alpha}}\right)\Bigg)^{-1}$$
$$\times\,\varepsilon^{\alpha_{30}/(\alpha+1)}\,c_{30}\,e^{-\alpha_{30}y}\,dy$$
$$\sim\;\varepsilon^{\alpha_{30}/(\alpha+1)}\int_{\mathbb{R}} W_i(\hat M(y))\,\left(\frac{1}{\hat M(y)\,(1-\hat M(y))}\;-\;\frac{1}{\alpha+1}\,\frac{e^{(\alpha+1)y}(1-\hat M(y))^\alpha}{(\hat M(y))^\alpha}\right)^{-1}$$
$$\times\;c_{30}\,e^{-\alpha_{30}y}\,dy$$
$$\sim\;-\varepsilon^{(\alpha_{30}+\beta-\alpha)/(\alpha+1)}\int_{\mathbb{R}}\lambda_i X(\hat M(y))$$
$$\times\left(\frac{1}{\hat M(y)\,(1-\hat M(y))}\;-\;\frac{1}{\alpha+1}\,\frac{e^{(\alpha+1)y}(1-\hat M(y))^\alpha}{(\hat M(y))^\alpha}\right)^{-1}\,c_{30}\,e^{-\alpha_{30}y}\,dy, \tag{123}$$

and the corresponding term for $L$ state is of order $o(\varepsilon^{(\alpha_{30}+\beta-\alpha)/(\alpha+1)})$ provided that $\alpha<3$ and

$$\frac{\alpha+1}{\alpha-1}\;>\;\alpha_{30}\;>\;\frac{\alpha-1}{3-\alpha}\,.$$

Gathering all the terms from (113), we get that the terms (121) are asymptotically negligible. Furthermore, for the terms (122), the senior term comes from the state $H$-

contribution, and is given by

$$\lambda_i\,\varepsilon^{(\alpha_{30}+\beta-\alpha)/(\alpha+1)}\int_{\mathbb{R}} X(\hat{M}(y)) \times \left(\frac{1}{\hat{M}(y)\,(1-\hat{M}(y))} - \frac{1}{\alpha+1}\,\frac{e^{(\alpha+1)y}(1-\hat{M}(y))^{\alpha}}{(\hat{M}(y))^{\alpha}}\right)^{-1} c\,e^{-\alpha_{30}y}\,dy. \tag{124}$$

It follows from the proof of Proposition 3.10 that the comparison $V_1'(z) > 0$ for large $\bar{v} - v_3$ is equivalent to

$$\frac{1}{\hat{M}(y)\,(1-\hat{M}(y))} - \frac{1}{\alpha+1}\,\frac{e^{(\alpha+1)y}(1-\hat{M}(y))^{\alpha}}{(\hat{M}(y))^{\alpha}} > 0.$$

The claim follows.

# G Proofs of Section 5

**Lemma G.1** *Suppose that an agent of type $\theta$ decides to exchange information with another agent. Then, his future expected profit will be strictly larger than if he does not exchange information.*

**Proof.** Let the other agent's type density, conditional on state $Y = 0$, be $\psi^H(z)$. Then, if the agent decides to exchange information, his unconditional type distribution is

$$P(\theta)\psi^H(z-\theta) + (1-P(\theta))\psi^L(z-\theta) = P(\theta)\,(1+e^{-z})\,\psi^H(z-\theta).$$

Let also $\Pi(\tau, z)$ be the agent's profit at time $\tau$, given that his type at time $\tau$ is equal to $z$. Then, if he does not exchange information, his expected utility is

$$P(\theta)\int_t^T \int_{\mathbb{R}} h^H(t,\tau,z-\theta)\,(1+e^{-z})\,\Pi(\tau,z)\,dz\,d\tau,$$

where $h^{H,L}(t,\tau,z-\theta)$ is his type density at time $\tau$ given his type $\theta$ at time $t$ and where we have used the identity

$$P(\theta)\,h^H(t,\tau,z-\theta) + (1-P(\theta))\,h^L(t,\tau,z-\theta) = P(\theta)\,(1+e^{-z})\,h^H(t,\tau,z-\theta)\,.$$

If he does decide to exchange information, the same argument implies that his expected utility is

$$P(\theta)\int_t^T \int_{\mathbb{R}} (h^H * \psi^H)(z-\theta)\,(1+e^{-z})\,\Pi(\tau,z)\,dz\,d\tau$$

Thus, it suffices to show that

$$\int_{\mathbb{R}} (h^H * \psi^H)(z-\theta)\,(1+e^{-z})\,\Pi(\tau,z)\,dz \;\geq\; \int_{\mathbb{R}} h^H(t,\tau,z-\theta)\,(1+e^{-z})\,\Pi(\tau,z)\,dz\,.$$

But,

$$\begin{aligned}
&\int_{\mathbb{R}} (h^H * \psi^H)(z-\theta)\,(1+e^{-z})\,\Pi(\tau,z)\,dz \\
&\quad = \int_{\mathbb{R}}\int_{\mathbb{R}} h^H(t,\tau,z-\theta-x)\,\psi^H(x)\,(1+e^{-z})\,\Pi(\tau,z)\,dx\,dz \\
&\quad = \int_{\mathbb{R}} h^H(t,\tau,y-\theta) \int_{\mathbb{R}} \psi^H(x)\,(1+e^{-x-y})\,\Pi(\tau,x+y)\,dx\,dy,
\end{aligned}$$

where we have used the change of variables $z-x=y$. Thus, it suffices to show that

$$\int_{\mathbb{R}} \psi(x)\,(1+e^{-x-y})\,\Pi(\tau,x+y)\,dx \;>\; (1+e^{-y})\,\Pi(\tau,y).$$

This inequality follows from (71). ■

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