

# Modeling Term Structures of Defaultable Bonds<sup>1</sup>

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<sup>1</sup>The authors are at The Graduate School of Business, Stanford University, Stanford CA 94305-5015. This paper is a revised and extended version of the theoretical results from our earlier paper “Econometric Modeling of Term Structures of Defaultable Bonds,” June, 1994. The empirical results from that paper, also revised and extended, are now found in “An Econometric Model of the Term Structure of Interest Rate Swap Yields,” *Journal of Finance*, October, 1997. We are grateful for comments from many, including the anonymous referee, the Editor Ravi Jagannathan, Peter Carr, Ian Cooper, Qiang Dai, Ming Huang, Farshid Jamshidian, Joe Langsam, Francis Longstaff, Amir Sadr, Craig Gustaffson, Michael Boulware, Arthur Mezhlumian, and especially Dilip Madan; and from seminar participants at The University of Arizona, Boston University, Carnegie Mellon University, The Norwegian School of Economics and Business Administration, Queen’s University, The University of Wisconsin, The University of California at Berkeley, The University of Chicago,

**Abstract:** This paper presents convenient reduced-form models of the valuation of contingent claims subject to default risk, focusing on applications to the term structure of interest rates for corporate or sovereign bonds. Examples include the valuation of a credit-spread option.

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# 1 Introduction

This paper presents a new approach to modeling term structures of bonds and other contingent claims that are subject to default risk. As in previous “reduced-form” models, we treat default as an unpredictable event governed by a hazard-rate process.<sup>1</sup> Our approach is distinguished by the parameterization of losses at default in terms of the fractional reduction in market value that occurs at default.

Specifically, we fix some contingent claim that, in the event of no default, pays  $X$  at time  $T$ . We take as given an arbitrage-free setting in which all securities are priced in terms of some short-rate process  $r$  and equivalent martingale measure<sup>2</sup>  $Q$ . Under this “risk-neutral” probability measure, we let  $h_t$  denote the hazard-rate for default at time  $t$ , and let  $L_t$  denote the expected fractional loss in market value if default were to occur at time  $t$ , conditional on the information available up to time  $t$ . We show that this claim may be priced as if it were default-free by replacing the usual short-term interest rate process  $r$  with the default-adjusted short-rate process  $R = r + hL$ . That is, under technical conditions, the initial market value of the defaultable claim to  $X$  is

$$V_0 = E_0^Q \left[ \exp \left( - \int_0^T R_t dt \right) X \right], \quad (1)$$

where  $E_0^Q$  denotes risk-neutral, conditional expectation at date 0. This is natural, in that  $h_t L_t$  is the “risk-neutral mean-loss rate” of the instrument due to default. Discounting at the adjusted short rate  $R$  therefore accounts for both the probability and timing of default,

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<sup>1</sup>Examples of reduced-form models include those of Artzner and Delbaen (1995), Das and Tufano (1995), Fons (1994), Jarrow and Turnbull (1995), Jarrow, Lando, and Turnbull (1997), Lando (1994, 1997, 1998), Martin (1997), Litterman and Iben (1988), Madan and Unal (1993), Nielsen and Ronn (1995), Pye (1974), and Schönbucher (1997). Ramaswamy and Sundaresan (1986) and Cooper and Mello (1996) directly assumed that defaultable bonds can be valued by discounting at an adjusted short rate. Among other results, this paper provides a particular kind of reduced-form model that justifies this assumption. Litterman and Iben (1991) arrived at a similar model in a simple discrete time setting by assuming zero recovery at default.

<sup>2</sup>See Harrison and Kreps (1979) and Harrison and Pliska (1981).

as well as for the effect of losses on default. Pye (1974) developed a pre-cursor to this modeling approach, in a discrete-time setting in which interest rates, default probabilities, and credit spreads all change only deterministically.

A key feature of the valuation equation (1) is that, provided we take the mean-loss rate process  $hL$  to be given exogenously,<sup>3</sup> standard term-structure models for default-free debt are directly applicable to defaultable debt by parameterizing  $R$  instead of  $r$ . After developing the general pricing relation (1) with exogenous  $R$  in Section 2.3, special cases with Markov diffusion or jump-diffusion state dynamics are presented in Section 2.4.

The assumption that default hazard rates and fractional recovery do not depend on the value  $V_t$  of the contingent claim is typical of reduced-form models of defaultable bond yields. There are, however, important cases for which this exogeneity assumption is counterfactual. For instance, as discussed by Duffie and Huang (1996) and Duffie and Singleton (1997),  $h_t$  will depend on  $V_t$  in the case of swap contracts with asymmetric counterparty credit quality. In Section 2.5, we extend our framework to the case of price-dependent  $(h_t, L_t)$ . We show that the absence of arbitrage implies that  $V_t$  is the solution to a non-linear partial differential equation. For example, with this nonlinear dependence of the price on the contractual payoffs, the value of a coupon bond in this setting is not simply the sum of the modeled prices of individual claims to the principal and coupons.

Section 3 presents several applications of our framework to the valuation of corporate bonds. First, in Section 3.1, we discuss the practical implications of our “loss-of-market” value assumption, compared to a “loss-of-face” value assumption, for the pricing of non-callable corporate bonds. Calculations with illustrative pricing models suggest that these alternative recovery assumptions generate rather similar par yield spreads, even for the same fractional loss coefficients. This robustness suggests that, for some pricing problems, one can exploit the analytical tractability of our “loss-of-market” pricing framework for estimating default hazard rates, even when “loss-of-

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<sup>3</sup>By “exogenous,” we mean that  $h_t L_t$  does not depend on the value of the defaultable claim itself.

face” value is the more appropriate recovery assumption. For deep-discount or high-premium bonds, differences in these formulations can be mitigated by compensating changes in recovery parameters.

Second we discuss several econometric formulations of models for pricing of non-callable corporate bonds. In pricing corporate debt using (1), one can either parameterize  $R$  directly, or parameterize the component processes  $r$ ,  $h$ , and  $L$  (which implies a model for  $R$ ). The former approach was pursued in Duffie and Singleton (1997) and Dai and Singleton (1998) in modeling the term structure of interest-rate swap yields. By focusing directly on  $R$ , these pricing models combine the effects of changes in the default-free short rate rate ( $r$ ) and risk-neutral mean loss rate ( $hL$ ) on bond prices. In contrast, in applying our framework to the pricing of corporate bonds, Duffee (1997) and Collin-Dufresne and Solnik (1998) parameterize  $r$  and  $hL$  separately. In this way they are able to “extract” information about mean loss rates from historical information on defaultable bond yields. All of these applications are special cases of the affine family of term-structure models.<sup>4</sup>

In Section 3.2 we explore, along several dimensions, the flexibility of affine models to describe basic features of yields and yield spreads on corporate bonds. First, using the canonical representations of affine term-structure models in Dai and Singleton (1998), we argue that the *CIR*-style models used by Duffee (1997) and Collin-Dufresne and Solnik (1998) are theoretically incapable of capturing the negative correlation between credit spreads and U.S. Treasury yields documented in Duffee (1998), while maintaining non-negative default hazard rates. Several alternative, affine formulations of credit spreads are introduced with the properties that  $hL$  is strictly positive and that the conditional correlation between changes in  $r$  and  $hL$  is unrestricted *a priori* as to sign.

Second, we develop a defaultable version of the Heath, Jarrow, and Morton (1992) (*HJM*) model based on the forward-rate process associated with  $R$ . In developing this model, we derive the counterpart to the usual HJM risk-neutralized drift restriction for

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<sup>4</sup>See, for example, Duffie and Kan (1996) for a characterization of the affine class of term-structure models, and Dai and Singleton (1998) for a complete classification of the admissible affine term-structure models and a specification analysis of three-factor models for the swap yield curve.

defaultable bonds.

Third, we apply our framework to the pricing of callable corporate bonds. We show that, as with non-callable bonds, the hazard rate  $h_t$  and fractional default loss  $L_t$  cannot be separately identified from data on the term structure of defaultable bond prices alone, because  $h_t$  and  $L_t$  enter the pricing relation (1) only through the mean-loss rate  $h_t L_t$ .

The pricing of derivatives on defaultable claims in our framework is explored in Section 4. The underlying could be, for example, a corporate or sovereign bond on which a derivative such as an option is written (by a defaultable or non-defaultable) counterparty. In order to illustrate these ideas, we price a credit-spread put option on a defaultable bond, allowing for correlation between the hazard rate  $h_t$  and short rate  $r_t$ . The nonlinear dependence of the option payoffs on  $h_t$  and  $L_t$  implies that, in contrast to bonds, the default hazard rate and fractional loss rate are separately identified from option price data. Numerical calculations for the spread put option are used to illustrate this point, as well as several other features of credit derivative pricing.

## 2 Valuation of Defaultable Claims

In order to motivate our valuation results, we first provide a heuristic derivation of our basic valuation equation (1) in a discrete-time setting. Then we formalize this intuition in continuous time. For the case of exogenous default processes, the implied pricing relations are derived for special cases in which  $(h, L, r)$  is a Markov diffusion or, more generally, a jump-diffusion.

### 2.1 A Discrete-Time Motivation

Consider a defaultable claim that promises to pay a (possibly contingent amount)  $X_{t+\mathcal{T}}$  at maturity date  $t + \mathcal{T}$ , and nothing before date  $t + \mathcal{T}$ . For any time  $s \geq t$ , let

- $h_s$  be the conditional probability at time  $s$ , under a risk-neutral probability measure  $Q$ , of default between  $s$  and  $s + 1$ , given

the information available at time  $s$ , in the event of no default by  $s$ .

- $\varphi_s$  denote the recovery in units of account, say dollars, in the event of default at  $s$ .
- $r_s$  be the default-free short rate.

If the asset has not defaulted by time  $t$ , its market value  $V_t$  would be the present value of receiving  $\varphi_{t+1}$  in the event of default between  $t$  and  $t + 1$  plus the present value of receiving  $V_{t+1}$  in the event of no default, meaning that

$$V_t = h_t e^{-r_t} E_t^Q(\varphi_{t+1}) + (1 - h_t) e^{-r_t} E_t^Q(V_{t+1}), \quad (2)$$

where  $E_t^Q(\cdot)$  denotes expectation under  $Q$ , conditional on information available to investors at date  $t$ . By recursively solving (2) forward over the life of the bond,  $V_t$  can be expressed equivalently as:

$$\begin{aligned} V_t = & E_t^Q \left[ \sum_{j=0}^{\mathcal{T}-1} h_{t+j} e^{-\sum_{k=0}^j r_{t+k}} \varphi_{t+j+1} \prod_{\ell=0}^j (1 - h_{t+\ell-1}) \right] \\ & + E_t^Q \left[ e^{-\sum_{k=0}^{\mathcal{T}-1} r_{t+k}} X_{t+\mathcal{T}} \prod_{j=1}^{\mathcal{T}} (1 - h_{t+j-1}) \right]. \end{aligned} \quad (3)$$

Evaluation of the pricing formula (3) is complicated in general by the need to deal with the joint probability distribution of  $\varphi$ ,  $r$ , and  $h$ , over various horizons. The key observation underlying our pricing model is that (3) can be simplified by taking the risk-neutral expected recovery at time  $s$ , in the event of default at time  $s + 1$ , to be a fraction of the risk-neutral expected survival-contingent market value at time  $s + 1$  (“recovery of market value” or *RMV*). Under this assumption, there is some adapted process  $L$ , bounded by 1, such that

$$\mathbf{RMV:} \quad E_s^Q(\varphi_{s+1}) = (1 - L_s) E_s^Q(V_{s+1}).$$

Substituting *RMV* into (3) leaves

$$\begin{aligned} V_t &= (1 - h_t) e^{-r_t} E_t^Q(V_{t+1}) + h_t e^{-r_t} (1 - L_t) E_t^Q(V_{t+1}) \\ &= E_t^Q \left( e^{-\sum_{j=0}^{\mathcal{T}-1} R_{t+j}} X_{t+\mathcal{T}} \right), \end{aligned} \quad (4)$$

where

$$e^{-R_t} = (1 - h_t)e^{-r_t} + h_t e^{-r_t}(1 - L_t). \quad (5)$$

For annualized rates but time periods of small length, it can be seen that  $R_t \simeq r_t + h_t L_t$ , using the approximation of  $e^c$ , for small  $c$ , given by  $1 + c$ ,

Equation (4) says that the price of a defaultable claim can be expressed as the present value of the promised payoff  $X_{t+\mathcal{T}}$ , treated as if it were default free, discounted by the default-adjusted short rate  $R_t$ . We will show technical conditions under which the approximation  $R_t \simeq r_t + h_t L_t$  of the default-adjusted short rate is in fact precise and justified in a continuous-time setting. This implies, under the assumption that  $h_t$  and  $L_t$  are exogenous processes, that one can proceed as in standard valuation models for default-free securities, using a discount rate that is the default-adjusted rate  $R_t = r_t + h_t L_t$  instead of the usual short rate  $r_t$ . For instance,  $R$  can be parameterized as in a typical single- or multi-factor model of the short rate, including the Cox-Ingersoll-Ross (1985) (*CIR*) model and its extensions, or as in the HJM model. The body of results regarding default-free term-structure models is immediately applicable to pricing defaultable claims.

The *RMV* formulation accommodates general state-dependence of the hazard-rate process  $h$  and recovery rates without adding computational complexity beyond the usual burden of computing the prices of riskless bonds. Moreover,  $(h_t, L_t)$  may depend on, or be correlated with, the riskless term structure. Some evidence consistent with the state-dependence of recovery rates is presented in Figure 1, based on recovery rates compiled by Moody's for the period 1974 through 1997.<sup>5</sup> The square boxes represent the range between the twenty-fifth and seventy-fifth percentiles of the recovery distributions. Comparing senior secured and unsecured bonds, for example, one sees that the recovery distribution for the latter is more spread out and has a longer lower tail. However, even for senior secured

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<sup>5</sup>These figures are constructed from revised and updated recovery rates as reported in "Corporate Bond Defaults and Default Rates 1938-1995," Moody's Investor's Services, January, 1996. Moody's measures the recovery rate as the value of a defaulted bond, as a fraction of \$100 face, recorded in its secondary market subsequent to default.

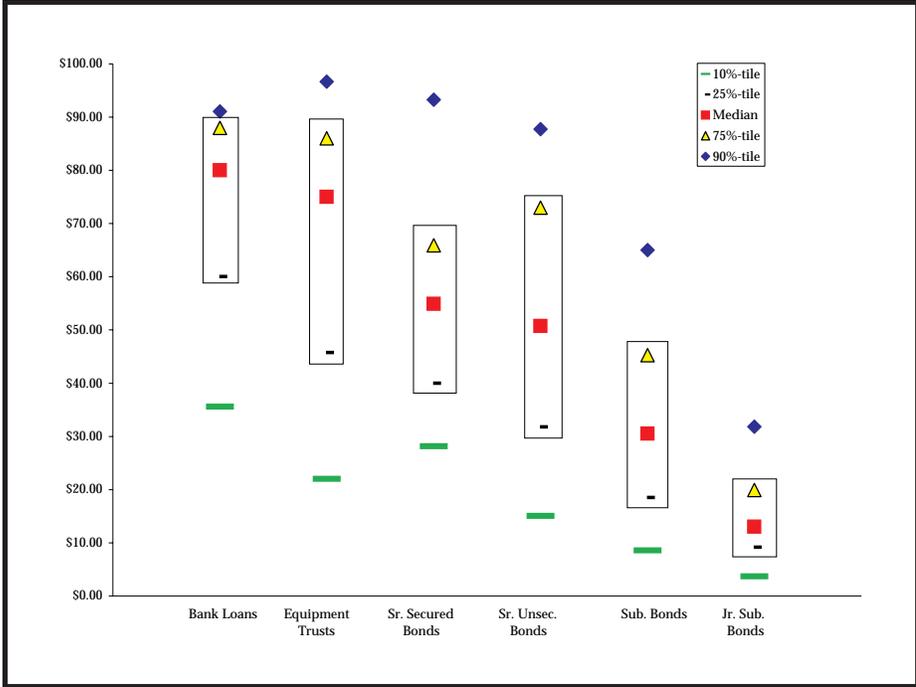


Figure 1: Distributions of Recovery by Seniority

bonds, there was substantial variation in the actual recovery rates. Although these data are also consistent with cross-sectional variation in recovery that is not associated with stochastic variation in time of expected recovery, Moody’s recovery data (not shown in Figure 1) also exhibit a pronounced cyclical component.

There is equally strong evidence that hazard rates for default of corporate bonds vary with the business cycle (as is seen, for example, in Moody’s data). Speculative-grade default rates tend to be higher during recessions, when interest rates and recovery rates are typically below their long-run means. Thus, allowing for correlation between default hazard-rate processes and riskless interest rates also seems desirable. Partly in recognition of these observations, Das and Tufano (1996) allowed recovery to vary over time so as to induce a non-zero correlation between credit spreads and the riskless term structure. However, for computational tractability, they maintained

the assumption of independence of  $h_t$  and  $r_t$ .

In allowing for state-dependence of  $h$  and  $L$ , we do not model the default time directly in terms of the issuer's incentives or ability to meet its obligations (in contrast to the corporate-debt pricing literature beginning with Black and Scholes (1973) and Merton (1974)). Our modeling approach and results are nevertheless consistent with a direct analysis of the issuer's balance sheet and incentives to default, as shown by Duffie and Lando (1997), using a version of the models of Fisher, Heinkel, and Zechner (1989) and Leland (1994) that allows for imperfect observation of the assets of the issuer. A general formula can be given for the hazard rate  $h_t$  in terms of the default boundary for assets, the volatility of the underlying asset process  $V$  at the default boundary, and the risk-neutral conditional distribution of the level of assets given the history of information available to investors. This makes precise one sense in which we are proposing a reduced-form model. While, following our approach, the behavior of the hazard-rate process  $h$  and fractional loss process  $L$  may be fitted to market data and allowed to depend on firm-specific or macroeconomic variables (as in Bijnen and Wijn (1994), Lundstedt and Hillgeist (1998), McDonald and Van de Gucht (1996), and Shumway (1996)), we do not constrain this dependence to match that implied by a formal structural model of default by the issuer.

Our discussion so far presumes the exogeneity of the hazard rate and fractional recovery. There are important circumstances in which these assumptions are counterfactual, and failure to accommodate endogeneity may lead to mispricing. For instance, if the market value of recovery at default is fixed, and does not depend on the pre-default price of the defaultable claim itself, then the fractional recovery of market value cannot be exogenous. Alternatively, in the case of some *OTC* derivatives, the hazard and recovery rates of the counterparties are different and the operative  $h$  and  $L$  for discounting depends on which counterparty is in the money.<sup>6</sup> While

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<sup>6</sup>This would be the case with a swap or forward contract between counterparties  $A$  and  $B$  of different credit quality. As the market value of the contract to counterparty  $A$  changes from positive to negative, the expected loss rate applied to the swap switches from that of counterparty  $B$  to that of counterparty  $A$ . Using the framework developed in this paper, Duffie and Huang (1996) present several numerical examples of the consequences of this switching for swap prices.

(1) (and (4)) apply with price-dependent hazard and recovery rates, this dependence makes the pricing equation a nonlinear difference equation that must typically be solved by recursive methods. In Section 2.5 we characterize the pricing problem with endogenous hazard and recovery rates and describe methods for pricing in this case.

One can also allow for “liquidity” effects by introducing a stochastic process  $\ell$  as the fractional carrying cost of the defaultable instrument.<sup>7</sup> Then, under mild technical conditions, the valuation model (1) applies with the “default-and-liquidity-adjusted” short-rate process

$$R = r + hL + \ell.$$

In practice, it is common to treat spreads relative to treasury rates rather than to “pure” default-free rates. In that case, one may treat the “treasury short rate”  $r^*$  as itself defined in terms of a spread (perhaps negative) to a pure default-free short rate  $r$ , reflecting (among other effects) repo specials. Then we can also write  $R = r^* + hL + \ell^*$ , where  $\ell^*$  absorbs the relative effects of repo specials and other determinants of relative carrying-costs.

## 2.2 Continuous-time Valuation

This section formalizes the heuristic arguments presented in the preceding section. We fix a probability space  $(\Omega, \mathcal{F}, P)$ , and a family  $\{\mathcal{F}_t : t \geq 0\}$  of  $\sigma$ -algebras satisfying the *usual* conditions. (See, for example, Protter (1990) for technical details.) A predictable short-rate process  $r$  is also fixed, so that it is possible at any time  $t$  to invest one unit of account in default-free deposits and “roll over” the proceeds until a later time  $s$  for a market value at that time of  $\exp\left(\int_t^s r_u du\right)$ .<sup>8</sup> At this point, we do not specify whether  $r_t$  is determined in terms of a Markov state-vector, an *HJM* forward-rate model, or by some other approach.

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<sup>7</sup>Formally, in order to invest in a given bond with price process  $U$ , this assumption literally means that one must continually make payments at the rate  $\ell U$ .

<sup>8</sup>We assume that this integral exists.

A contingent claim is a pair  $(Z, \tau)$  consisting of a random variable  $Z$  and a stopping time  $\tau$  at which  $Z$  is paid. We assume that  $Z$  is  $\mathcal{F}_\tau$ -measurable (so that the payment can be made based on currently available information). We take as given an equivalent martingale measure  $Q$  relative to the short-rate process  $r$ . This means, by definition, that the ex-dividend price process  $U$  of any given contingent claim  $(Z, \tau)$  is defined by  $U_t = 0$  for  $t \geq \tau$  and

$$U_t = E_t^Q \left[ \exp \left( - \int_t^\tau r_u du \right) Z \right], \quad t < \tau, \quad (6)$$

where  $E_t^Q$  denotes expectation under the risk-neutral measure  $Q$ , given  $\mathcal{F}_t$ . Included in the assumption that  $Q$  exists is the existence of the expectation in (6) for any traded contingent claim. (Later, we extend the definition of a contingent claim to include payments at different times.)

We define a defaultable claim to be a pair  $((X, T), (X', T'))$  of contingent claims. The underlying claim  $(X, T)$  is the obligation of the issuer to pay  $X$  at date  $T$ . The secondary claim  $(X', T')$  defines the stopping time  $T'$  at which the issuer defaults and claimholders receive the payment  $X'$ . This means that the actual claim  $(Z, \tau)$  generated by a defaultable claim  $((X, T), (X', T'))$  is defined by

$$\tau = \min(T, T'); \quad Z = X 1_{\{T < T'\}} + X' 1_{\{T \geq T'\}}. \quad (7)$$

We can imagine the underlying obligation to be a zero-coupon bond ( $X = 1$ ) maturing at  $T$ , or some derivative security based on other market prices, such as an option on an equity index or a government bond, in which case  $X$  is random and based on market information at time  $T$ . One can apply the notion of a defaultable claim  $((X, T), (X', T'))$  to cases in which the underlying obligation  $(X, T)$  is itself the actual claim generated by a more primitive defaultable claim, as with an OTC option or credit derivative on an underlying corporate bond. The issuer of the derivative may or may not be the same as that of the underlying bond.

Our objective is to define and characterize the price process  $U$  of the defaultable claim  $((X, T), (X', T'))$ . We suppose that the default time  $T'$  has a risk-neutral default hazard-rate process  $h$ , which means

that the process  $\Lambda$  which is 0 before default and 1 afterward (that is,  $\Lambda_t = 1_{\{t \geq T'\}}$ ) can be written in the form

$$d\Lambda_t = (1 - \Lambda_t)h_t dt + dM_t, \quad (8)$$

where  $M$  is a martingale under  $Q$ . One may safely think of  $h_t$  as the jump arrival intensity at time  $t$  (under  $Q$ ) of a Poisson process whose first jump occurs at default.<sup>9</sup> Likewise, the risk-neutral conditional probability, given the information  $\mathcal{F}_t$  available at time  $t$ , of default before  $t + 1$ , in the event of no default by  $t$ , is approximately  $h_t$  for small time intervals.<sup>10</sup>

We will first characterize, and then (under technical conditions) prove the existence of the unique arbitrage-free price process  $U$  for the defaultable claim. For this, one additional piece of information is needed: the payoff  $X'$  at default. If default occurs at time  $t$ , we will suppose that the claim pays

$$X' = (1 - L_t)U_{t-}, \quad (9)$$

where  $U_{t-} = \lim_{s \uparrow t} U_s$  is the price of the claim “just before” default,<sup>11</sup> and  $L_t$  is the random variable describing the fractional loss of market value of the claim at default. We assume that the fractional loss process  $L$  is bounded by 1 and predictable, which means roughly that the information determining  $L_t$  is available before time  $t$ . Section 2.6 provides an extension to handle fractional losses in market value that are uncertain even given all information available up to the time of default.

As a preliminary step, it is useful to define a process  $V$  with the property that, if there has been no default by time  $t$ , then  $V_t$  is the market value of the defaultable claim.<sup>12</sup> In particular  $V_T = X$  and  $U_t = V_t$  for  $t < T'$ .

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<sup>9</sup>The process  $\{(1 - \Lambda_{t-})h_t : t \geq 0\}$  is the intensity process associated with  $\Lambda$ , and is by definition non-negative and predictable with  $\int_0^t h_s ds < \infty$  almost surely for all  $t$ . See Brémaud (1980). Artzner and Delbaen (1995) showed that, if there exists an intensity process under  $P$ , then there exists an intensity process under any equivalent probability measure, such as  $Q$ .

<sup>10</sup>This is true in a limiting sense, for example, if  $h$  is right continuous.

<sup>11</sup>We will also show that the left limit  $U_{t-}$  exists.

<sup>12</sup>Because  $V(\omega, t)$  is arbitrary for those  $\omega$  for which default has occurred before  $t$ , the process  $V$  need not be uniquely defined. We will show, however, that  $V$  is uniquely defined up to the default time, under weak regularity conditions.

### 2.3 Exogenous Expected Loss Rate

From the heuristic reasoning used in Section 2.1, we conjecture the continuous-time valuation formula

$$V_t = E_t^Q \left[ \exp \left( - \int_t^T R_s ds \right) X \right], \quad (10)$$

where

$$R_t = r_t + h_t L_t. \quad (11)$$

In order to confirm this conjecture, we use the fact that the gain process (price plus cumulative dividend), after discounting at the short-rate process  $r$ , must be a martingale under  $Q$ . This discounted gain process  $G$  is defined by

$$\begin{aligned} G_t &= \exp \left( - \int_0^t r_s ds \right) V_t (1 - \Lambda_t) \\ &+ \int_0^t \exp \left( - \int_0^s r_u du \right) (1 - L_s) V_{s-} d\Lambda_s. \end{aligned} \quad (12)$$

The first term is the discounted price of the claim; the second term is the discounted payout of the claim upon default. The property that  $G$  is a  $Q$ -martingale and the fact that  $V_T = X$  together provide a complete characterization of arbitrage-free pricing of the defaultable claim.

Let us suppose that  $V$  does not itself jump at the default time  $T'$ . From (10), this is a primitive condition on  $(r, h, X)$  and the information filtration  $\{\mathcal{F}_t : t \geq 0\}$ . This means essentially that, although there may be “surprise” jumps in the conditional distribution of the market value of the default-free claim  $(X, T)$ ,  $h$ , or  $L$ , these surprises occur precisely at the default time with probability zero. This is automatically satisfied in the diffusion settings described in Section 2.4.1, since in that case  $V_t = J(Y_t, t)$ , where  $J$  is continuous and  $Y$  is a diffusion process. This condition is also satisfied in the jump-diffusion model of Section 2.4.2 provided jumps in the conditional distribution of  $(h, L, X)$  do not occur at default.<sup>13</sup>

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<sup>13</sup>Kusuoka (1998) gives an example in which a jump in  $V$  at default is induced

Applying Ito's Formula<sup>14</sup> to (12), using (9) and our assumption that  $V$  does not jump at  $T'$ , we can see that for  $G$  to be a  $Q$ -martingale, it is necessary and sufficient that

$$V_t = \int_0^t R_s V_s ds + m_t, \quad (13)$$

for some  $Q$ -martingale  $m$ . (Since  $V$  jumps at most a countable number of times, we can replace  $V_{s-}$  in (12) with  $V_s$  for purposes of this calculation.) Given the terminal boundary condition  $V_T = X$ , this implies (10)-(12). The uniqueness of solutions of (13) with  $V_T = X$  can be found, for example, in Antonelli (1993). Thus, we have shown the following basic result.

**Theorem 1.** *Given  $(X, T, T', L, r)$ , suppose the default time  $T'$  has a risk-neutral hazard-rate process  $h$ . Let  $R = r + hL$  and suppose that  $V$  is well defined by (10) and satisfies  $\Delta V(T') = 0$  almost surely. Then there is a unique defaultable claim  $((X, T), (X', T'))$  and process  $U$  satisfying (6), (7), and (9). Moreover, for  $t < T'$ ,  $U_t = V_t$ .*

For a defaultable asset, such as a coupon bond, with a series of payments  $X_k$  at  $T_k$ , assuming no default by  $T_k$ , for  $1 \leq k \leq K$ , the claim to all  $K$  payments has a value equal to the sum of the values of each, in this setting in which  $h$  and  $L$  are exogenously given processes. (It may be appropriate to specify recovery assumptions that distinguish the various claims making up the asset.) The proof is an easy extension of the above Theorem, again using the fact that the total gain process, including the jumps associated with interim payments, is a  $Q$ -martingale. This linearity property does not hold,

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by a jump in the risk-premium. This may be appropriate, for example, if the arrival of default changes risk attitudes. In any case, given  $(h, L, X)$ , one can always construct a model in which there is a stopping time  $\tau$  with  $Q$ -hazard-rate process  $h$  and with no jump in  $V$  at  $\tau$ . For this, one can take any exponentially distributed random variable  $z$  with parameter 1, independent under  $Q$  of  $(h, L, X)$ , and let  $\tau$  be defined by  $\int_0^\tau h_s ds = z$ . (This allows for  $\tau = +\infty$  with positive probability.) If necessary, one can re-define the underlying probability space so that there exists such a random variable  $z$ , and minimally enlarge each information set  $\mathcal{F}_t$  so that  $\tau$  is a stopping time.

<sup>14</sup>See Protter (1990) for a version of Ito's Formula that applies in this generality.

however, for the more general case, treated in Section 2.5, in which  $h$  or  $L$  may depend on the value of the claim itself.

## 2.4 Special Cases with Exogenous Expected Loss

Next, we specialize to the case of valuation with dependence of exogenous  $r$ ,  $h$ , and  $L$  on continuous-time Markov state variables.

### 2.4.1 A Continuous-Time Markov Formulation

In order to present our model in a continuous-time state-space setting that is popular in finance applications, we suppose for this section that there is a state-variable process  $Y$  that is Markovian under an equivalent martingale measure  $Q$ . We assume that the promised contingent claim is of the form  $X = g(Y_T)$ , for some function  $g$ , and that  $R_t = \rho(Y_t)$ , for some function<sup>15</sup>  $\rho(\cdot)$ . Under the conditions of Theorem 1, a defaultable claim to payment of  $g(Y_T)$  at time  $T$  has a price at time  $t$ , assuming that the claim has not defaulted by time  $t$ , of

$$J(Y_t, t) = E^Q \left[ \exp \left( - \int_t^T \rho(Y_s) ds \right) g(Y_T) \mid Y_t \right]. \quad (14)$$

Modeling the default-adjusted short rate  $R_t$  directly as a function of the state variable  $Y_t$  allows one to model defaultable yield curves analogously with the large literature on dynamic models of default-free term structures. For example, suppose  $Y_t = (Y_{1t}, \dots, Y_{nt})'$ , for some  $n$ , solves a stochastic differential equation of the form

$$dY_t = \mu(Y_t) dt + \sigma(Y_t) dB_t, \quad (15)$$

where  $B$  is an  $\{\mathcal{F}_t\}$ -standard Brownian motion in  $\mathbb{R}^n$  under  $Q$ , and where  $\mu$  and  $\sigma$  are well behaved functions on  $\mathbb{R}^n$  into  $\mathbb{R}^n$  and  $\mathbb{R}^{n \times n}$ , respectively. Then we know from the ‘‘Feynman-Kac formula’’ that,

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<sup>15</sup>For notational reasons, we have not shown any dependence of  $\rho$  on time  $t$ , which could be captured by including time as one of the state variables. Of course, we assume that  $\rho$  and  $g$  are measurable real-valued functions on the state space of  $Y$ , and that (14) is well defined.

under technical conditions,<sup>16</sup> (14) implies that  $J$  solves the backward Kolmogorov partial differential equation

$$\mathcal{D}^{\mu,\sigma} J(y, t) - \rho(y)J(y, t) = 0, \quad (y, t) \in \mathbb{R}^n \times [0, T], \quad (16)$$

with the boundary condition

$$J(y, T) = g(y), \quad y \in \mathbb{R}^n, \quad (17)$$

where

$$\mathcal{D}^{\mu,\sigma} J(y, t) = J_t(y, t) + J_y(y, t)\mu(y) + \frac{1}{2}\text{trace}[J_{yy}(y, t)\sigma(y, t)\sigma(y, t)']. \quad (18)$$

This is the framework used in models for pricing swaps and corporate bonds discussed in Section 3.

#### 2.4.2 Jump-Diffusion State Process

Because of the possibility of sudden changes in perceptions of credit quality, particularly among low-quality issues such as Brady bonds, one may wish to allow for “surprise” jumps in  $Y$ . For example, one can specify a standard jump-diffusion model for the risk-neutral behavior of  $Y$ , replacing  $\mathcal{D}^{\mu,\sigma}$  in (16), under technical regularity, with the jump-diffusion operator  $\mathcal{D}$  given by

$$\mathcal{D}J(y, t) = \mathcal{D}^{\mu,\sigma} J(y, t) + \lambda(y) \int_{\mathbb{R}^n} [J(y + z, t) - J(y, t)] d\nu_y(z), \quad (19)$$

where  $\lambda : \mathbb{R}^n \rightarrow [0, \infty)$  is a given function determining the arrival intensity  $\lambda(Y_t)$  of jumps in  $Y$  at time  $t$ , under  $Q$ , and where, for each  $y$ ,  $\nu_y$  is a probability distribution for the jump size ( $z$ ) of the state variable. Examples of affine, defaultable term-structure models with jumps are presented in Section 3.

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<sup>16</sup>See, for example, Friedman (1975) or Krylov (1980).

## 2.5 Price-Dependent Expected Loss Rate

If the risk-neutral expected loss rate  $h_t L_t$  is price-dependent, then the valuation model is non-linear in the promised cash flows. We can accommodate this in a model in which default at time  $t$  implies a fractional loss  $L_t = \hat{L}(Y_t, U_t)$  of market value and hazard rate  $h_t = H(Y_t, U_t)$  that may depend on the current price  $U_t$  of the defaultable claim.<sup>17</sup> This would allow, for example, for recovery of an exogenously specified fraction of face value at default.

In our Markov setting, we can now write  $R_t = \rho(Y_t, U_t)$ , where  $\rho(y, u) = H(y, u)L(y, u) + \hat{r}(y)$ , where  $\{\hat{r}(Y_t) : t \geq 0\}$  is the state-dependent default-free short-rate process. By the same reasoning used in Section 2.3, and under technical regularity conditions, the price  $U_t$  of the defaultable claim at any time  $t$  before default is, in our general Markov setting, given by

$$J(Y_t, t) = E^Q \left[ \exp \left( - \int_t^T \rho(Y_s, J(Y_s, s)) ds \right) g(Y_T) \mid Y_t \right]. \quad (20)$$

With the diffusion or jump-diffusion assumption for  $Y$ , and under additional technical regularity conditions (as, for example, in Ivanov (1984)),  $J$  solves the quasi-linear *PDE*

$$\mathcal{D}J(y, t) + \rho(y, J(y, t))J(y, t) = 0, \quad y \in \mathbb{R}^n, \quad (21)$$

where  $\mathcal{D}J(y, t)$  is defined by (19), with boundary condition (17). This *PDE* can be treated numerically, essentially as with the linear case (16). Duffie and Huang (1996) and Huge and Lando (1999)

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<sup>17</sup>We suppose for this section that the price process is left-continuous, so that if default occurs at time  $t$ ,  $U_t$  is the price of the claim just before it defaults, and  $(1 - L_t)U_t$  is the market value just after default. This is simply for notational convenience. We could also take the more usual convention of right-continuous price paths, and let  $h_t$  and  $L_t$  be determined in terms of functions of the left limit  $U_{t-}$ , that is, the limit of  $U_s$  as  $s$  approaches  $t$  from below. The pricing implications would be the same. We could also allow  $L$  to depend on  $U$  and other state information. For example, for a model of a bond collateralized by an asset with price process  $U$ , we could let  $(1 - L_t)U_t = q_t \min(U(t), K)$ , where  $K$  is the maximal effective legal claim at default, say par, and  $q_t$  is the conditional expected fraction recovered at default of the effective legal claim.

have several numerical examples of an application of this framework to defaultable swap rates.

For cases of endogenous dependence of the risk-neutral mean-loss-rate  $hL$  on the price of the claim, not necessarily based on a Markovian state-space, Duffie, Schroder, and Skiadas (1996) provide technical conditions for the existence and uniqueness of pricing, and explore the pricing implications of advancing in time the resolution of information.

## 2.6 Uncertainty about Recovery

We have been assuming that the fractional loss in market value due to default at time  $t$  is determined by the information available up to time  $t$ . An extension of our model to allow for conditionally uncertain jumps in market value at default is due to Schönbucher (1997). A simple version of this extension is provided below for completeness.

Suppose that at default, instead of (9), the claim pays

$$X' = (1 - \ell)S_{T'-}, \quad (22)$$

where  $\ell$  is a bounded random variable<sup>18</sup> describing the fractional loss of market value of the claim at default. It would not be natural to require that  $\ell \geq 0$ , as the onset of default could actually reveal, with non-zero probability, “good” news about the financial condition of the issuer. Given limited liability, we require that  $\ell \leq 1$ .

It can be shown that there exists a process  $L$  such that  $L_t$  is the expectation of the fractional default loss  $\ell$  given all current information up to, *but not including*, time  $t$ . To be precise,  $L$  is a predictable process, and  $L_{T'} = E(\ell | \mathcal{F}_{T'-})$ .

With this change in the definitions of  $X'$  and  $L_t$ , the pricing formula (10) applies as written, with  $R = r + hL$ , under the conditions of Theorem 1. The proof is almost identical to that of Theorem 1.

## 3 Valuation of Defaultable Bonds

An important application of the basic valuation equation (10) with exogenous default risk is the valuation of defaultable corporate bonds.

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<sup>18</sup>Here,  $\ell$  is  $\mathcal{F}_{T'}$ -measurable.

We discuss various aspects of this pricing problem in this section, beginning with the sensitivity of bond prices to the nature of the default recovery assumption. We argue that the tractability of assumption *RMV* may come at a low cost in terms of pricing errors for bonds trading near par even if, in truth, bonds are priced in the markets assuming a given fractional recovery of face value. Then, maintaining our assumption *RMV*, we present several “affine” models for pricing defaultable, non-callable bonds, giving particular attention to parameterizations that allow for flexible correlations among the riskless rate  $r$  and the default hazard rate  $h$ . Additionally, we derive the default-environment counterparts to the *HJM* no-arbitrage conditions for term-structure models based on forward rates. Finally, we discuss the valuation of callable corporate bonds.

### 3.1 Recovery and Valuation of Bonds

The determination of recoveries to creditors during bankruptcy proceedings is a complex process that typically involves substantial negotiation and litigation. No tractable, parsimonious model captures all aspects of this process so, in practice, all models involve tradeoffs regarding how various aspects of default (hazard and recovery rates) are captured. To help motivate our *RMV* convention, consider the following alternative recovery-of-face value (*RFV*) and recovery-of-treasury (*RT*) formulations of  $\varphi_t$ :

**RT:**  $\varphi_t = (1 - L_t)P_t$ , where  $L$  is an exogenously specified fractional recovery process and  $P_t$  is the price at time  $t$  of an otherwise-equivalent, default-free bond (Jarrow and Turnbull (1995)).<sup>19</sup>

**RFV:**  $\varphi_t = (1 - L_t)$ ; the creditor receives a (possibly random) fraction  $(1 - L_t)$  of face (\$1) value immediately upon default (Brennen and Schwartz (1980) and Duffee (1998)).

Under *RT*, the computational burden of directly computing  $V_t$  from (3), for a given fractional recovery process  $(1 - L_t)$ , can be

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<sup>19</sup>In the case of zero-coupon bonds, the recovery-of-treasury assumption can be reinterpreted as one in which the creditor receives the fraction  $(1 - L_t)$  of par, which is paid at the maturity date of the original defaultable bond. In the case of coupon bonds, recovery upon default includes a fraction of the promised post-default coupon payments and not just of the face amount.

substantial. Largely for this reason, various simplifying assumptions have been made in previous studies. Jarrow and Turnbull (1995), for example, assumed that the risk-neutral default hazard-rate process  $h$  is independent (under  $Q$ ) of the short rate  $r$  and, for computational examples, that the fractional loss process  $L$  is constant. Lando (1998) relaxes the Jarrow-Turnbull model within the  $RT$  setting by allowing a random hazard-rate process that need not be independent of the short rate  $r$ , but at the cost of added computational complexity. With  $L_t = \bar{L}$ , a constant, the payoff at maturity in the event of default is  $(1 - \bar{L})$ , *regardless of when the default occurred*. This simplification is lost when  $L_t$  is time-varying, since the payoff at maturity will be indexed by the period in which default occurred. Not only is the time of default relevant, but the joint  $\mathcal{F}_t$ -conditional distributions of  $L_v$ ,  $h_s$  and  $r_u$ , for all  $v, s$ , and  $u$  between  $t$  and  $T$ , play a computationally challenging role in determining  $V_t$ .

Turning to  $RMV$  and  $RFV$ , one basis for choosing between these two assumptions is the legal structure of the instrument to be priced. For instance, Duffie and Singleton (1997) and Dai and Singleton (1998) apply the model in this paper (assumption  $RMV$ ) to the determination of at-market U.S.-dollar, fixed-for-variable swap rates. These authors assume exogenous  $(h_t, L_t, r_t)$  and parameterize the default-adjusted discount rate  $R$  directly as an affine function of a Markov state vector  $Y$ . In this manner they were able to apply frameworks for valuing default-free bonds without modification to the problem of determining at-market swap rates. The  $RMV$  assumption is well matched to the legal structure of swap contracts in that standard agreements typically call for settlement upon default based on an obligation represented by an otherwise equivalent, non-defaulted, swap.

For the case of corporate bonds, on the other hand, we see the choice of recovery assumptions as involving both conceptual and computational trade-offs. The  $RMV$  model is easier to use, because standard default-free term-structure modeling techniques can be applied. If, however, one assumes liquidation at default and that absolute priority applies, then assumption  $RFV$  is more realistic as it implies equal recovery for bonds of equal seniority of the same issuer. Absolute priority, however, is not always maintained by bankruptcy courts and liquidation at default is often avoided.

In the end, is there a significant difference between the pricing implications of models under *RMV* and *RFV*? In order to address this question, we proceed under the assumption of exogenous  $(h_t, L_t)$  (as in Section 2.3) and, for simplicity, take  $L_t = \bar{L}$ , a constant. We adopt for this illustration a “four-factor” model of  $r$  and  $h$  given by

$$r_t = \rho_0 + Y_t^1 + Y_t^2 - Y_t^0, \quad (23)$$

$$h_t = bY_t^0 + Y_t^3, \quad (24)$$

where  $Y_1, Y_2, Y_3$ , and  $Y_4$  are independent “square-root diffusions” under  $Q$ , and  $\rho_0$  and  $b$  are constant coefficients. The degree of *negative* correlation between  $r_t$  and  $h_t$  (consistent with Duffee (1998)) is controlled by choice of  $b$ .<sup>20</sup>

Under assumption *RMV*, the price  $V_t$  at any time  $t$  before default of an  $n$ -year bond with semi-annual coupon payments of  $c$  is, under the regularity conditions of Theorem 1, given by

$$V_{n,t}^{RMV} = cE_t^Q \left( \sum_{j=1}^{2n} e^{-\int_t^{t+.5j} R(s) ds} \right) + E_t^Q \left( e^{-\int_t^{t+n} R(s) ds} \right), \quad (25)$$

where  $R_t = r_t + h_t \bar{L}$ . For the case of assumption *RFV* (zero recovery of coupon payments and recovery of the fraction  $(1 - \bar{L})$  of face value)

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<sup>20</sup>The parameter values were chosen as follows. Using standard notation,  $dY_{it} = \kappa_i(\theta_i - Y_{it}) dt + \sigma \sqrt{Y_{it}} dB_t^{(i)}$ , where  $B^{(i)}$  is a standard Brownian motion under  $P$ , with a market-price-of-risk coefficient  $\lambda_i$ . The associated parameters are listed in the following table.

$Y$	$\kappa_i$	$\theta_i$	$\sigma_i$	$\lambda_i$
$Y_0$	0.25	0.005	0.1	0
$Y_1$	0.544	0.374	0.0199	-0.036
$Y_2$	0.003	0.258	0.0181	-0.004

We set  $\rho = -0.575$ . The initial conditions of  $Y_0, Y_1$ , and  $Y_2$ , and the parameters were chosen to match the level and volatilities of the riskless zero-coupon yield curve, out to 10 years, implied by the Duffie-Singleton (1997) two-factor model, with their state variables evaluated at their long-run means. For our base case, we set  $h_0$  and its long-run mean at 200 basis points, and the initial volatility of  $h$  at 100%. Additionally,  $b$  was adjusted so that the slope coefficient in the “instantaneous” regression at time 0 of the increment of  $h$  on that of  $r$  was -10%. In doing this, we took  $\kappa_3 = 0.25$ , and adjusted  $\theta_3$  and  $\sigma_3$  accordingly.

the results of Lando (1998) imply that the market value of the same bond at any time  $t$  before default is given by

$$V_{nt}^{RFV} = cE_t^Q \left( \sum_{j=1}^{2n} e^{-\int_t^{t+0.5j} (r_s+h_s) ds} \right) + E_t^Q \left( e^{-\int_t^{t+n} (r_s+h_s) ds} \right) + \int_t^{t+n} (1 - \bar{L})\gamma(Y_t, t, s) ds, \quad (26)$$

where

$$\gamma(Y_t, t, s) = E_t^Q \left( h_s e^{-\int_t^s (r_u+h_u) du} \right).$$

For this multi-factor CIR setting,  $\gamma(Y_t, t, s)$  can be computed explicitly, so the computation of  $V_{nt}$  from (26) calls for one numerical integration. With  $\bar{L} = 1$  (zero recovery), all bond prices are clearly identical under the two models.

In calculating par-bond spreads, or the risk-neutral default hazard rates implied by par spreads, for the case of  $\bar{L} < 1$ , we find rather little difference between the *RFV* and *RMV* formulations. This is true even without making compensating adjustments to  $\bar{L}$  across the two models in order to calibrate one to the other. For example, Figure 2 shows the initial (equal to long-run mean) default hazard-rate for both models, implied by a given 10-year spread and a given fractional-recovery coefficient  $(1 - \bar{L})$ . The implied risk-neutral hazard rates are obviously rather close.

The assumption that the initial and long-run mean intensities are equal makes for a rather gentle stress test of the distinction between the *RMV* and *RFV* formulations, as it implies that the term structure of risk-neutral forward<sup>21</sup> default probabilities is rather flat, and therefore that the risk-neutral expected market value given survival to a given time is close to face value. The present value of recoveries for the *RMV* and *RFV* models would therefore be rather similar.

In order to show the impact of upward or downward sloping term structures of forward default probabilities, we provide in Figure 3 the term structures of par-coupon yield spreads (semi-annual, bond equivalent) for cases in which the initial risk-neutral default hazard

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<sup>21</sup>The forward default probability at a future time  $t$ , a definition due to Litterman and Iben (1991), means the probability of default between  $t$  and one “short” unit of time after  $t$ , conditional on survival to  $t$ .

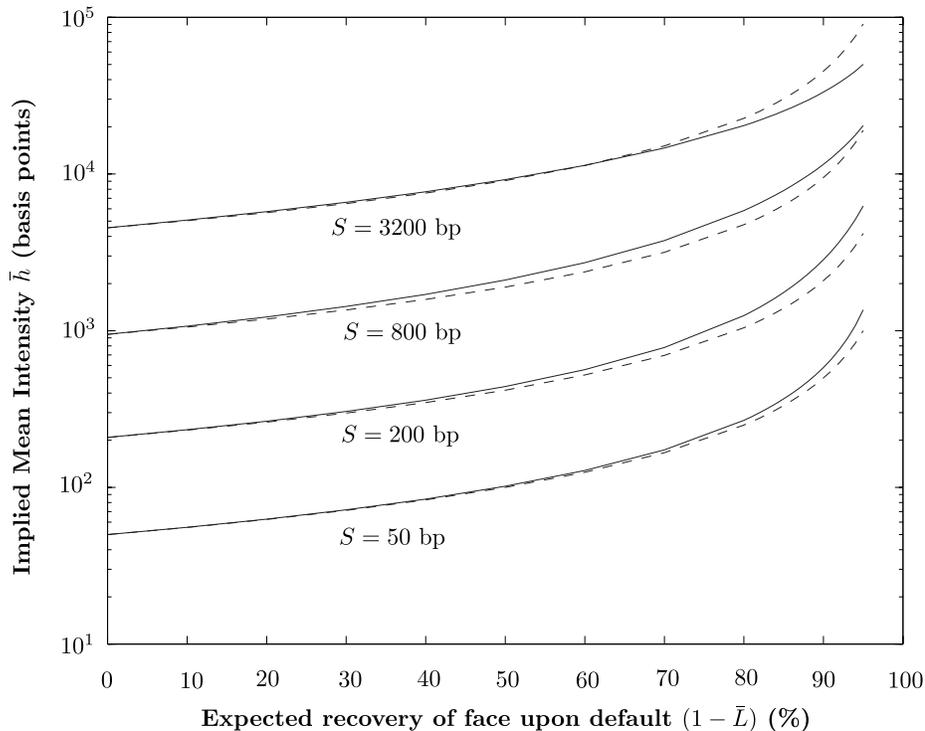


Figure 2: For fixed ten-year par-coupon spreads,  $S$ , this figure shows the dependence of the mean hazard rate  $\bar{h}$  on the assumed fractional recovery  $1 - \bar{L}$ . The solid (dashed) lines correspond to the model  $RFV$  ( $RMV$ ).

rate,  $h_0$ , is much higher or lower than its long run mean  $\theta = 200$  basis points. Under both recovery assumptions,  $\bar{L}$  was set at 50%. With an increasing term structure of default risk ( $h_0 \leq \theta$ ) bond prices under the two recovery assumptions are again rather similar. On the other hand, for a steeply declining term structure of default risk, the implied credit spreads are larger under  $RMV$ , with the maximum difference (at 10 years) of 8.4 basis points, for actual spreads of 168.2 basis points for  $RFV$  and 176.6 basis points for ( $RMV$ ). From (25) and (26) we see that a higher value of  $h_0$  tends to depress  $V^{RFV}$  more than  $V^{RMV}$  through the effect of the present values of the coupons. A higher  $h_0$  also implies a higher contribution from the

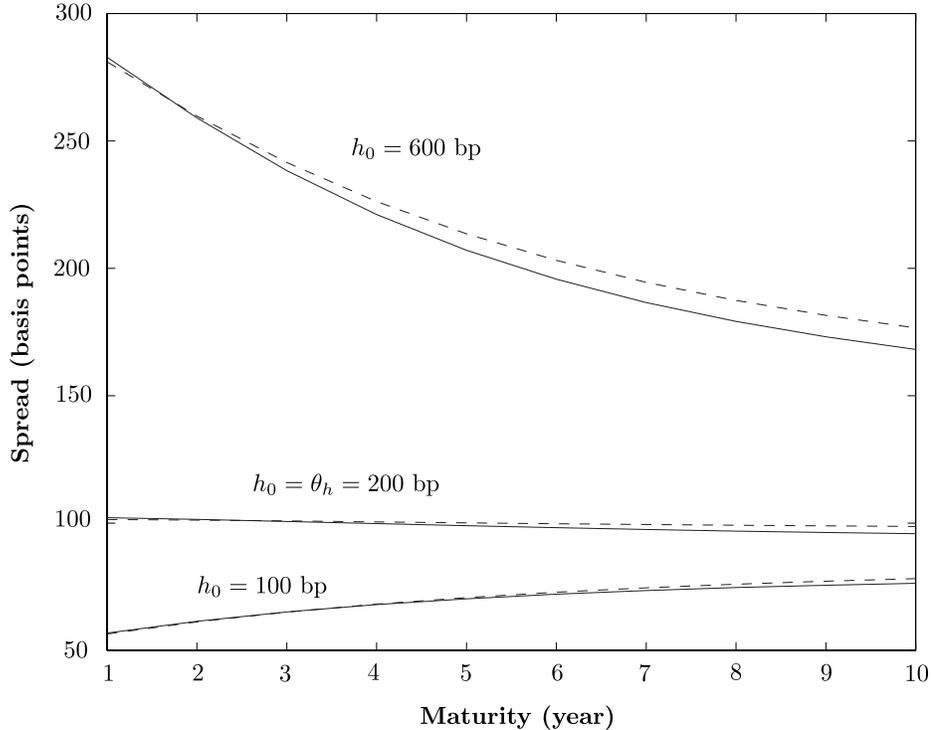


Figure 3: Term structures of par-coupon yield spreads for *RMV* (dashed lines) and *RFV* (solid lines), with 50% recovery upon default, a long-run mean hazard rate of  $\theta_h = 200$  bp, mean reversion rate of  $\kappa = 0.25$ , and an initial hazard-rate volatility of 100%.

accelerated fractional recovery of par (the last term in (26)) under *RFV*. The latter effect dominates, giving a larger spread in the case of assumption *RMV*.

For bonds with a significant premium or discount, or with steeply upward or downward sloping term structures of interest rates, the *RMV* and *RFV* assumptions may have more markedly differing spread implications for a given exogenous loss fraction. For example, a given fractional loss, say 50%, of a premium bond's market value represents a greater loss in market value than does a 50% loss of the same bond's face value. In such cases, some of the distinction between the two model assumptions can be compensated for with different

fractional loss processes.

## 3.2 Valuation of Non-Callable Corporate Bonds

The valuation framework set forth in Section 2.3 with exogenous hazard and recovery rates has been applied by Duffee (1997), Duffee and Liu (1997), and Collin-Dufresne and Solnik (1998) to value non-callable corporate bonds. All of these studies focus on special cases in which  $Y$  is a vector of independent square-root diffusions. In this section, we nest these *RMV* specifications within a more general affine diffusion model and argue that square-root diffusions are limited theoretically in their flexibility to explain term structures of corporate yield spreads. We then introduce an alternative affine formulation, motivated in part by the empirical analysis in Dai and Singleton (1998), that offers greater flexibility in capturing non-zero correlations among the variables  $(h_t, L_t, r_t)$  while preserving positivity of the hazard rate.

At the outset, it is important to note that the hazard-rate process  $h$  and fractional-loss-at-default process  $L$  enter the adjustment for default in the discount rate  $R = r + hL$  in the product form  $hL$ . Furthermore, under the assumption of exogenous  $(h, L, r)$ , the value of a non-callable corporate bond is simply the sum of the present values of the promised coupon payments. It follows that knowledge of defaultable bond prices (before default) alone is not sufficient to separately identify  $h$  and  $L$ . At most, we can extract information about the risk-neutral mean loss rate  $h_t L_t$ . In order to learn more about the hazard and recovery rates implicit in market prices (within our *RMV* pricing framework), it is necessary to examine either a collection of bonds that share some but not all of the same default characteristics, or derivative securities with payoffs that depend in different ways on  $h$  and  $L$  (see Section 4).

As an illustration of the former strategy, suppose that one has prices of junior ( $V_t^J$ ) and senior ( $V_t^S$ ) bonds of the same issuer, along with the prices of one or more default-free (Treasury) bonds. In this case, it seems reasonable to assume that these securities will share a hazard-rate process  $h$ , but will have different conditional expected fractional losses at default,  $L_t^J$  and  $L_t^S$ , respectively, consistent with the evidence in Figure 1. Using models of the type discussed subse-

quently, it will often be possible (for given parameters determining the dynamics of  $r$  and  $hL$ ) to extract observations on  $h_t L_t^J$  and  $h_t L_t^S$  from these prices. In this case, we can infer relative recovery rates, in terms of  $L_t^J/L_t^S$ , but it is still not possible to extract the hazard rate  $h_t$  or the individual levels of recovery rates.<sup>22</sup> Of course, if the hazard rate or either recovery rate were observed, or known functions of observable variables, then the identification problem would be solved. Having prices on both junior and senior debt would then serve to provide more market information for the estimation of  $h$ .

With this identification problem in mind, suppose that one has data on the prices of a collection of defaultable bonds with different maturities and the same associated hazard and fractional loss rates. We also suppose that the objective of the empirical analysis is to model jointly the dynamic properties of  $r_t$  and the “short spread”  $s_t \equiv h_t L_t$ .

*Case 1: Square-root Diffusion Model of  $Y$ .*

Consider the case of a three-factor model in which the instantaneous, default-free short rate process  $r$  is given by

$$r_t = \delta_0 + \delta_1 Y_{1t} + \delta_2 Y_{2t} + \delta_3 Y_{3t}, \quad (27)$$

for state variables  $Y_1, Y_2$ , and  $Y_3$  that are “square-root diffusions,” in the sense that the conditional volatility of the  $i^{\text{th}}$  state variable is proportional to  $\sqrt{Y_{it}}$ . Also, suppose that

$$s_t = \gamma_0 + \gamma_1 Y_{1t} + \gamma_2 Y_{2t} + \gamma_3 Y_{3t}. \quad (28)$$

Dai and Singleton (1998) provide a sense in which the “most flexible” affine term-structure model with this volatility structure and well-defined bond prices has<sup>23</sup>

$$dY_t = \mathcal{K}(\Theta - Y_t) dt + \sqrt{S_t} dB_t, \quad (29)$$

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<sup>22</sup>Using an approach that has elements of both reduced-form and structural models, Madan and Unal (1993) used data on junior and senior CD issues from thrift institutions to estimate default hazard rates, assuming strict priority and recovery by both junior and senior debt from a given pool of assets.

<sup>23</sup>In the notation of Dai and Singleton (1998), this is the  $\mathbb{A}_3(3)$  family of models.

where  $\mathcal{K}$  is a  $3 \times 3$  matrix with positive diagonal and non-positive off-diagonal elements;  $\Theta$  is a vector in  $\mathbb{R}_+^3$ ;  $S_t$  is the  $3 \times 3$  diagonal matrix with diagonal elements  $(Y_{1t}, Y_{2t}, \text{ and } Y_{3t})$ ; and  $B$  is a standard Brownian motion in  $\mathbb{R}^3$  under  $Q$ .

Duffee (1997) considered the special case of (27)-(28) in which  $\delta_0 = -1$  and  $\delta_3 = 0$ , so that  $r_t$  could take on negative values and depends only on the first two state variables. He assumed, moreover, that  $\mathcal{K}$  is diagonal (so that  $Y_1, Y_2,$  and  $Y_3$  are  $Q$ -independent square-root diffusions, as commonly assumed in *CIR*-style models).<sup>24</sup> Given the independence of the state variables and Duffee's normalizations of the  $\delta_i$  coefficients to unity, the only means of introducing negative correlation among  $r_t$  and  $s_t$  in this model is to allow for negative  $\gamma_i$ 's. His estimates of some of the  $\gamma_i$ 's were in fact negative, implying that the default hazard rates may take on negative values, a technical impossibility. With his formulation, the average error in fitting non-callable corporate bond yields was less than ten basis points.

The possibility of negative hazard rates in Duffee's model is not a consequence of the particular restricted version of (27)-(28) that he chose to study. More generally, within this correlated square-root model of  $(r_t, s_t)$ , one cannot simultaneously have a non-negative hazard-rate process and negatively correlated increments of  $r$  and  $h$ . This is an immediate implication of the observation in Dai and Singleton (1998) that well-defined correlated square-root models do not allow for negative correlation among any of the state variables, because the off-diagonal elements of  $\mathcal{K}$  must be non-positive for the model to be well defined. Therefore,  $r$  and  $s$  cannot have negatively correlated increments (in the usual "instantaneous" sense) in this model unless one or more of the  $\delta_i$  or  $\gamma_i$  is negative.

*Case 2: Models with More Flexible Correlation Structures for  $(r_t, s_t)$*

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<sup>24</sup>A potential drawback of imposing these over-identifying restrictions is that they unnecessarily constrain the joint conditional distribution of  $r_t$  and  $s_t$ . A key benefit is that they allowed Duffee to estimate the parameters of  $(Y_{1t}, Y_{2t})$  governing the default-free Treasury using data on Treasury prices alone, while still allowing for a non-trivial idiosyncratic factor driving  $s_t$  and correlation between  $r_t$  and  $s_t$  through non-zero  $\gamma_i$  in (28). This two-step estimation strategy would nevertheless have been feasible with non-zero  $(\kappa_{12}, \kappa_{21})$  and, thus, a more flexible correlation structure could have been introduced.

In an important respect, the limitations of the correlated square-root model are due to the assumed structure of the stochastic volatility in  $Y$ . Dai and Singleton (1998) show that, within the affine family of term-structure models, more flexibility in specifying the correlations among the state variables is gained by restricting the dependence of the conditional variances of the state variables on  $Y$ .<sup>25</sup>

Suppose, for instance that, instead of (29), we assume that

$$dY_t = \mathcal{K}(\Theta - Y_t) dt + \Sigma \sqrt{S_t} dB_t, \quad (30)$$

where  $\mathcal{K}$  and  $\Theta$  are as in (29),  $\Sigma$  is a  $3 \times 3$  matrix and

$$S_{11}(t) = Y_1(t), \quad (31)$$

$$S_{22}(t) = [\beta_2]_2 Y_2(t), \quad (32)$$

$$S_{33}(t) = \alpha_3 + [\beta_3]_1 Y_1(t) + [\beta_3]_2 Y_2(t), \quad (33)$$

with strictly positive coefficients,  $[\beta_i]_j$ . Also, suppose that

$$r_t = \delta_0 + \delta_1 Y_{1t} + Y_{2t} + Y_{3t}, \quad (34)$$

$$s_t = \gamma_0 + \gamma_1 Y_{1t} + \gamma_2 Y_{2t}, \quad (35)$$

with all of  $\delta_0, \delta_1, \gamma_0, \gamma_1$ , and  $\gamma_2$  strictly positive. Then Dai and Singleton (1998) show a sense in which the most flexible, admissible affine term-structure model based on (30) with (31)-(33) has

$$\mathcal{K} = \begin{bmatrix} \kappa_{11} & \kappa_{12} & 0 \\ \kappa_{21} & \kappa_{22} & 0 \\ 0 & 0 & \kappa_{33} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sigma_{31} & \sigma_{32} & 1 \end{bmatrix}, \quad (36)$$

with the off-diagonal elements of  $\mathcal{K}$  being non-positive.

The short spread rate  $s_t$  is strictly positive in this model, because it is a positive affine function of a correlated square-root diffusion. At the same time, the signs of  $\sigma_{31}$  and  $\sigma_{32}$  are unconstrained, so the third

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<sup>25</sup>Motivated in part by the limitations of the square-root diffusion model, Duffie and Liu (1997) use the framework in this paper to study the pricing of floating-rate corporate debt with  $r_t$  assumed to be an affine function of squared Gaussian variables – sometimes called the “quadratic-Gaussian” model. This state process offers an alternative way of introducing negative correlation among the state variables, though does not allow for stochastic variation in the variances of the state process.

state variable may have increments that are negatively correlated with those of the first two. This may induce negative correlation between the increments of  $r$  and  $s$ . Given these interdependencies among the state variables, the parameters must be estimated using corporate and Treasury price data simultaneously.

With the imposition of over-identifying restrictions, we can specialize this model to one in which the riskless term-structure can be estimated independently of the corporate-spread component  $s_t$ . Specifically, suppose that  $\delta_1 = 0$  so that  $Y_3$  and  $Y_1$  are idiosyncratic risk factors for  $r$  and  $s$ , respectively. Also, set  $\kappa_{21} = 0$ ,  $[\beta_3]_1 = 0$ , and  $\sigma_{31} = 0$ . Then  $r_t = \delta_0 + Y_{2t} + Y_{3t}$ ,  $S_{33}(t) = \alpha_3 + [\beta_3]_2 Y_{2t}$ , and

$$\mathcal{K} = \begin{bmatrix} \kappa_{11} & \kappa_{12} & 0 \\ 0 & \kappa_{22} & 0 \\ 0 & 0 & \kappa_{33} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \sigma_{32} & 1 \end{bmatrix}. \quad (37)$$

Under this parameterization, the model of the riskless term-structure is a two-factor affine model with  $r$  determined by  $(Y_2, Y_3)$ . All of the parameters of this two-factor Treasury model can be estimated without using corporate bond data.

Corporate bond price data is necessary to estimate the parameters of the diffusion representation of  $Y_1$ , as well as the parameters of (35). A non-zero  $\kappa_{12}$  induces positive correlation between the increments of  $r$  and  $s$  (recall that  $\kappa_{12}$  cannot be positive). On the other hand, negative correlation in the increments of  $r$  and  $s$  is induced if  $\sigma_{32} < 0$ . By construction, this model also has the property that hazard rates are strictly positive. We do stress, however, that the restrictions leading to this special case are testable and may restrict the joint distribution of  $(r_t, s_t)$  in ways that are not supported by the data. In particular, compared to the preceding model, the degree of negative correlation between the increments of  $r$  and  $s$  is limited by the restriction that  $\sigma_{31} = 0$ .

Neither of these cases considers the possibility of jumps. Duffie and Kan (1996) showed that introducing jumps into an affine term-structure model preserves the affine dependence of yields on state variables provided the jump-arrival intensity is an affine function of the state vector and the distribution of the jump sizes depends only on time. Thus, we could easily extend these parameterizations to

incorporate jumps, provided the jump-size distributions respect the positivity of the hazard rate  $h_t$ . Duffie, Pan, and Singleton (1998) extend this model and further discuss parameterizations of affine jump-diffusions with jumps that preserve the positivity of a subset of the state variables. Their examples could be adapted to our defaultable bond pricing problem.

### 3.3 A Defaultable *HJM* Model

Given an exogenous risk-neutral mean-loss-rate process  $hL$ , one can treat the dynamics of the term structure of interest rates on defaultable debt using the same model developed for default-free forward rates by *HJM*. This section formalizes this idea by deriving counterparts to the *HJM* no-arbitrage risk-neutral drift restriction on forward rates.

Suppose the defaultable discount function is modeled by taking the price at a given time  $t$  before default of a zero-coupon defaultable bond (of a given homogeneous class of defaultable debt) maturing at  $T$  to be

$$p_{t,T} = \exp \left( - \int_t^T F(t, u) du \right), \quad (38)$$

where

$$F(t, T) = F(0, T) + \int_0^t \mu(s, T) ds + \int_0^t \sigma(s, T) dB_s, \quad (39)$$

where  $B$  is a standard Brownian motion in  $\mathbb{R}^n$  under  $Q$  and, for each fixed maturity  $T$ , the real-valued process  $\mu(\cdot, T)$  and the  $\mathbb{R}^n$ -valued process  $\sigma(\cdot, T)$  satisfy the technical regularity conditions imposed by Heath, Jarrow, and Morton (1992) and by Carverhill (1995). We note that  $F(t, T)$  does not literally correspond to the interest rate on forward bond contracts unless one allows for special language in the forward-rate agreement regarding the obligations of the counterparties in the event of default of the underlying bond before  $T$ . We can nevertheless take  $F$  as a process that describes, through (38) and (39), the behavior prior to default of the discount function of a given class of defaultable debt.

We first show that, with given processes  $h$  and  $L$  for the risk-neutral hazard rate and fractional loss of market value, respectively, and under technical regularity conditions, we have the usual HJM risk-neutralized drift restriction

$$\mu(t, T) = \sigma(t, T) \cdot \int_t^T \sigma(t, u) du. \quad (40)$$

This restriction is derived, under the conditions of Theorem 1, as follows.

As in Section 2.3, the discounted gain process  $G$  for the zero-coupon corporate bond maturing at  $T$  is

$$G_t = (1 - \Lambda_t) D_t p_{t,T} + \int_0^t (1 - L_s) D_s p_{s-,T} d\Lambda_s, \quad (41)$$

where

$$D_t = \exp \left( - \int_0^t r_s ds \right). \quad (42)$$

Since  $G$  is a  $Q$ -martingale, its drift is zero, or, using Ito's Formula, and taking  $t < T'$ , we have (almost surely)

$$0 = \int_0^t D_s p_{s,T} \alpha_{s,T} ds, \quad (43)$$

where, after an application of Fubini's Theorem for stochastic integrals, as in Protter (1990), we have

$$\begin{aligned} \alpha_{t,T} &= F(t, t) - r_t - \int_t^T \mu(t, u) du \\ &\quad + \frac{1}{2} \left( \int_t^T \sigma(t, u) du \right) \cdot \left( \int_t^T \sigma(t, u) du \right) - h_t L_t. \end{aligned} \quad (44)$$

(Once again, we can ignore the distinction between  $p_{s-,T}$  and  $p_{s,T}$  for purposes of this calculation.) From (43), we have  $\alpha_{t,T} = 0$  (almost everywhere). Taking partial derivatives of  $\alpha_{t,T}$  with respect to  $T$  leaves (40).

From (44), we can also see that the risk-neutral hazard-rate process  $h$  implied by the models for  $F$  and  $r$  is given, under regularity, by

$$h(t) = \frac{F(t, t) - r_t}{L_t}. \quad (45)$$

Suppose, alternatively, that one specifies a model of the Jarrow-Turnbull (1995) variety, in which default of a corporate zero-coupon bond at time  $t$  implies recovery of an exogenously specified fraction  $\delta_t$  of a default-free zero-coupon bond of the same maturity, where  $\delta$  is a non-negative stochastic process satisfying regularity conditions.<sup>26</sup> This is a version of the recovery formulation RT discussed in Section 3.1. In this case, lack of homogeneity implies a correction term to the usual HJM drift restriction, which is given instead by

$$\mu^*(t, T) = \sigma(t, T) \cdot \int_t^T \sigma(t, u) du + h_t \delta_t \frac{q_{t,T}}{p_{t,T}} [F(t, T) - f(t, T)], \quad (46)$$

where, for any  $t$  and  $s$ ,  $q_{t,s} = \exp(-\int_t^s f(t, u) du)$  is the price at time  $t$  of a default-free zero-coupon bond maturing at  $T$ , and  $f(t, T)$  is the associated default-free forward rate.<sup>27</sup>

### 3.4 Valuation of Defaultable Callable Bonds

The majority of dollar-denominated corporate bonds are callable.<sup>28</sup> In this section we extend our pricing results to the case of defaultable bonds with embedded call options. This extension requires an

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<sup>26</sup>Provided  $\delta_t$  is independent of maturity, the same recovery model would apply for straight coupon bonds as well. Callable debt may need to be treated separately.

<sup>27</sup>The restriction (46) can be derived by using the fact that the associated gain process  $\Gamma$ , defined by

$$\Gamma_t = (1 - \Lambda_t) D_t p_{t,T} + \int_0^t \delta_s D_s q_{s-,T} d\Lambda_s,$$

is a  $Q$ -martingale.

<sup>28</sup>For the incidence of callability, see G. Duffee (1997).

assumption about the call policy of the issuer. In order to minimize the total market value of a portfolio of corporate liabilities, it may not be optimal for the issuer of the liabilities to call in a particular bond so as to minimize the market value of that particular bond. For simplicity, however, we will assume a callable bond is called so as to minimize its market value. The resulting pricing model is easily extended to the case in which any issuer options embedded in a portfolio of liabilities issued by a given corporation are exercised so as to minimize the total market value of the portfolio.

At each time  $t$ , the issuer minimizes the market value of the liability represented by a corporate bond by exercising the option to call in the bond if and only if its market price, if not called, is higher than the strike price on the call, as implied by Bellman's principle of optimality. We will take the simple discrete-time setting of Section 2.1. During the time window of "callability," we thus have the recursive pricing formula

$$V_t = \min \left[ \bar{V}_t, e^{-R_t} E_t^Q (V_{t+1} + d_{t+1}) \right], \quad (47)$$

where  $d_t$  is the coupon on the bond at time  $t$ ;  $V_t$  is the bond price at time  $t$ , after the coupon is paid, assuming that the bond has not defaulted by  $t$ ;  $\bar{V}_t$  is the exercise price at time  $t$  (often par); and  $R_t$  is the discrete default-adjusted short rate at time  $t$ , defined by (5). Outside the callability window,

$$V_t = e^{-R_t} E_t^Q (V_{t+1} + d_{t+1}). \quad (48)$$

The "boundary condition" at maturity  $T$  is that  $V_T$  is the face value of the bond.

In a more general continuous-time context, suppose that a callable bond maturing at time  $T$  with first call at  $\underline{T}$  has a coupon of size  $c_i$  (as a fraction of face value) at time  $T(i)$ , for  $T(1) \leq T(2) \leq \dots \leq T$ . At time  $t$ , we let  $\mathcal{T}(t, T)$  denote the set of feasible call policies. (These are the stopping times that are bounded above by  $T$  and below by  $\max(t, \underline{T})$ .) By standard arguments for non-defaultable securities, provided default has not occurred by time  $t$ , the market price at  $t$ ,

as a fraction of face value, is

$$V_t = \min_{\tau \in \mathcal{T}(t,T)} E_t^Q \left[ \sum_{t < T(i) \leq \tau} \gamma_{t,T(i)} c_i + \gamma_{t,\tau} \right], \quad (49)$$

where

$$\gamma_{t,s} = \exp \left( - \int_t^s R_u du \right), \quad (50)$$

and where  $R$  is the default-adjusted (and, perhaps, liquidity-adjusted) short-rate process associated with the bond. The pricing relation (49) applies under the technical regularity conditions given in Section 2, assuming that the minimum in (49) is attained<sup>29</sup> by some stopping time. This approach, minimization of value over stopping times, was proposed by Merton (1974) in a structural model for the pricing of callable defaultable corporate bonds. In practice, (49) could be solved by a discrete algorithm such as (47) for sufficiently small time periods.

Convertible bonds, whose option to convert is exercised by the investor when the relative value of equity is high, are more common among issuers of low credit quality. For simplicity, one could model the risk-neutral default hazard rate  $h_t$  and fractional loss rate  $L_t$  so that they depend in part on the equity price process underlying conversion. This captures the idea that the bond is of lower credit quality when the equity price is lower. See Brennan and Schwartz (1980) for a full convertible pricing model in a defaultable setting.

Notable in our setting is the absence of a role for the default variables  $h$  and  $L$  in callable bond pricing, other than through the definition of the default-adjusted short rate  $R$ . This is evident from equations (10) and (49). It follows that  $h$  and  $L$  cannot be identified directly from the market prices for undefaulted corporate bonds, callable or not. Changes in credit spreads of undefaulted bonds reflect changes in the joint distribution of the risk-neutral mean loss rate process  $hL$  and short-rate process  $r$ .

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<sup>29</sup>For technical regularity conditions, see Karatzas (1988).

## 4 Pricing Bond and Credit Derivatives

The inability to separately identify  $h_t$  and  $L_t$  using defaultable bond yields is not an issue for some derivative securities. For example, the payoff on a corporate bond option depends directly on the value of the underlying corporate debt. Due to the nonlinearity of the option payoff, the default characteristics  $(h, L)$  of the issuer of debt (as contrasted to those for the writer of the option) may in some cases be identified.

For illustration, suppose that there is a constant treasury yield of 6 percent and a constant risk-neutral mean loss rate of  $s = hL = 2$  percent, with  $h = 0.08$  and  $L = 0.25$ . The underlying defaultable bond is therefore priced at a constant yield of 8 percent until maturity or default. Consider a default-free put option on the bond struck at a yield of 11 percent. Because the option is initially out of the money, and all yields are constant except at the bond's default, the option will pay off only if the bond defaults. If the bond default occurs at  $\tau$ , before the expiration of the option, the option pays  $\pi(L, \tau) \equiv p(0.11, \tau) - (1 - L)p(0.08, \tau)$ , where  $p(y, t)$  is the price of the corporate at time  $t$  at a bond-equivalent yield of  $y$ . Because  $\pi(L, \tau)$  is not linear in  $L$ , whereas the risk-neutral probability of default is approximately linear in the hazard rate  $h$  (for small  $h$ ), the option price depends in distinct ways on  $h$  and  $L$ , which can therefore be identified from the option price, the corporate bond yield, and the treasury yield, in this simple example.

For example, suppose we halve the default hazard rate  $h$  in this example from 0.08 to 0.04, and we double the fractional loss  $L$  to 0.5, leaving  $s = hL$  unchanged at 2 percent. The bond price behavior is identical prior to default. With this change, however, the option pays off with half the expected frequency (risk-neutralized), but, because the option is out of the money before default, it pays more than double the amount when it does pay off. That is, at the default time  $\tau$ , if before expiration, the option pays  $\pi(0.5, \tau) > 2\pi(0.25, \tau)$ . The initial option price is therefore higher with  $(h = 0.04, L = 0.50)$  than with  $(h = 0.08, L = 0.25)$ . Thus, in principle, one can identify  $h$  and  $L$  separately from yield spreads and from bond option prices. This is a simple consequence of the non-linearity of the option payoff, as a function of the underlying bond price.

## 4.1 Pricing a Credit-Spread Put Option

As an illustration of how our framework can be used to value credit derivatives, we price a yield-spread put option with stochastic  $(h, r)$ . We suppose that the underlying bond pays semi-annual coupons, is non-callable, and matures at  $T_m$ . Its price is quoted in terms of its yield spread over a Treasury note, of semi-annual coupons, and of the same maturity.<sup>30</sup> This credit-spread derivative is an option to sell the defaultable bond at a spread  $\underline{S}$  over treasury at an exercise date  $T$ . If the actual market spread  $S_T$  at  $T$  is less than the strike spread  $\underline{S}$ , then the spread option expires worthless. If the actual market spread  $S_T$  at  $T$  exceeds  $\underline{S}$ , then the holder of the credit derivative receives  $p(\underline{S} + Z_T, T) - p(S_T + Z_T, T)$ , where  $Z_t$  is the yield of the treasury note at  $t$ . Such options have been sold, for example, on Argentinian Brady bonds.

As credit spreads are relatively volatile, or may jump, the issuer of the credit derivative may wish to incorporate a feature of the following type: If, at any time  $t$  before expiration, the market spread  $S_t$  is greater than or equal to a given spread cap  $\bar{S} > \underline{S}$ , then the credit derivative immediately pays  $p(\underline{S} + Z_t, t) - p(\bar{S} + Z_t, t)$ . In effect, then, the credit derivative insures an owner of the corporate bond against increases in spread above the strike rate  $\underline{S}$  up to  $\bar{S}$ .

In summary, the credit derivative pays

$$X = \max[p(\underline{S} + Z_\tau, \tau) - p(\min(S_\tau, \bar{S}) + Z_\tau, \tau), 0]$$

at the stopping time  $\tau = \min(T, \inf\{t : S_t \geq \bar{S}\})$ . We will assume that the issuer of the credit spread option is default-free. The price at time 0 of the spread option is therefore  $E_0^Q [\exp(-\int_0^\tau r_s ds) X_\tau]$ . The default characteristics  $(h, L)$  of the underlying bond play a role in determining both the payoff  $X$  (through the price process  $U$  of the underlying bond) and the payment time  $\tau$ .

In order to value this credit derivative, one requires as inputs to a pricing model: the coupon and maturity structure of the treasury and defaultable bonds; the parameters  $(\underline{S}, \bar{S}, T)$  of the credit derivative; the behavior of the treasury short-rate process  $r$ , the

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<sup>30</sup>In our calculations, we ignore the differences in day-count conventions between treasury and corporate bonds. They would in any case have a negligible effect for our purposes.

risk-neutral hazard-rate process  $h$ , and the fractional-loss process  $L$  of the underlying bond; and the price behavior of the defaultable bond *after* default. The post-default bond price behavior is only relevant if, upon default, the “ceiling” spread  $\bar{S}$  is not exceeded. For the examples that we will present, we will assume that, at default, the proceeds from default recovery are re-invested at the default-free short rate and rolled over until expiration. (At moderate parameters such as those chosen for our numerical examples to follow, the spread passes through  $\bar{S}$  with certainty at default, and post-default price behavior plays no role.)

For our numerical example, we take a two-state *CIR* state process  $Y = (Y_1, Y_2)'$  satisfying

$$dY_{it} = \kappa_{it}(\theta_{it} - Y_{it}) dt + \sigma_{it}\sqrt{Y_{it}} dB_t^{(i)}, \quad (51)$$

where  $\kappa_i$ ,  $\theta_i$ , and  $\sigma_i$  are deterministic and continuous in  $t$ , while  $B^{(1)}$  and  $B^{(2)}$  are independent standard Brownian motions under an equivalent martingale measure  $Q$ . We take the treasury short-rate process  $r = Y_1$ , and the short spread process  $s = hL = Y_1 + Y_2$ .

The initial condition  $Y_0$  and the time-dependent parameters  $\kappa$ ,  $\theta$ , and  $\sigma$  are chosen to match given discount functions for treasury and corporate debt, initial treasury and defaultable yield vol-curves, and initial correlation between the yield on the reference treasury bond and yield spread of the defaultable bond. By “initial” vol curves and correlation, we mean the instantaneous volatility at time zero of forward rates in the *HJM* sense, by maturity, and the instantaneous correlation at time zero between the yield  $Z$  on the reference treasury note and the yield spread  $S$  on the defaultable bond.<sup>31</sup>

We will consider variations from the following base-case assumptions:

- The defaultable and treasury notes are 5-year semi-annual coupon

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<sup>31</sup>By Ito’s Formula, both  $Z$  and  $S$  are Ito Processes, other than at coupon dates and default. We can therefore write  $dZ_t = \mu_Z(t) dt + \sigma_Z(t) dB_t$  and  $dS_t = \mu_S(t) dt + \sigma_S(t) dB_t$ . The initial volatility of  $Z$  is  $\|\sigma_Z(0)\|/Z_0$ . The initial volatility of  $S$  is  $\|\sigma_S(0)\|/S_0$ , and the initial instantaneous correlation between  $S$  and  $Z$  is  $\sigma_Z(0) \cdot \sigma_S(0) / (\|\sigma_S(0)\| \|\sigma_Z(0)\|)$ . For our parameterization, these are easy numerical calculations, using the associated Ricatti equations for zero-coupon yields given in Duffie and Kan (1996).

non-callable bonds, with coupon rates of 9 and 7 percent, respectively.

- The credit derivative has a strike of  $\underline{S} = 200$  basis points (that is, at-the-money), with maximum protection determined by a spread cap of  $\overline{S} = 500$  basis points, and an expiration time of  $T = 1$  year.
- The default recovery rate  $1 - L$  is a constant 50 percent.

The parameter functions  $\kappa$ ,  $\theta$ , and  $\sigma$ , and the initial state vector  $Y(0)$  are set so that: the treasury and defaultable bonds are priced at par, and both forward rate curves are horizontal; the initial yield volatility on the reference treasury note is 15 percent; the initial instantaneous correlation between the defaultable bond's yield spread  $S$  and the treasury yield  $Y$  is zero;<sup>32</sup> the treasury forward rate vol curve is as illustrated in Figure 5 with the label "Price = 2.10." The forward-rate spread volatility is a scaling of this same vol curve, chosen so as to attain an initial yield spread volatility of the defaultable bond of 40 percent.

Given the number of parameter functions and initial conditions, there are more degrees of freedom than necessary to meet all of these criteria. We have verified that our pricing of the credit derivative is not particularly sensitive to reasonable variation within the class of parameters that meet these criteria.

At base case, the credit derivative has a market value of approximately 2.1 percent of face value of the defaultable bond. Figure 4 shows the dependence of the credit derivative price, per 100 dollars face value of the corporate bond, on the recovery rate  $1 - L$ .<sup>33</sup> With higher recovery at default, the credit derivative is more expensive, for we have fixed the yield spread at the base-case assumption of 200

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<sup>32</sup>This does not imply that the treasury short rate  $r$  and the mean loss rate  $c$  are independent, and typically requires that they will not be independent.

<sup>33</sup>We are especially grateful for extensive research assistance in producing these results from Michael Boulware of Susquehanna Investment Group. Additional assistance was provided by Arthur Mezhlumian of Goldman, Sachs and Company. Boulware and Mezhlumian were Ph.D. students at Stanford University when these calculations were made.

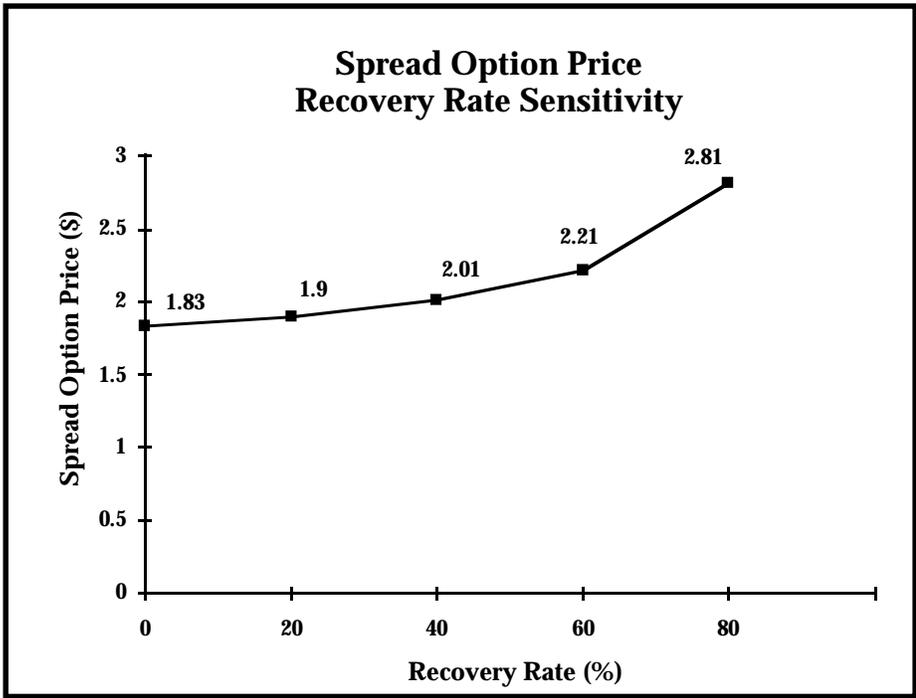


Figure 4: Impact of Recovery Rate on Credit Derivative Price

basis points, implying that the risk-neutral probability of default increases with the recovery rate. Indeed, then, one can identify the separate roles of the default hazard-rate  $h$  and the fractional-loss  $L$  with price information on derivatives that depend non-linearly on the underlying defaultable bond.

Figure 5 shows the base-case vol curve for treasury forward rates and a variation with a significantly flatter term structure of volatility, maintaining the given base-case treasury yield and yield-spread volatilities and correlation. Flattening the shape of the vol curves

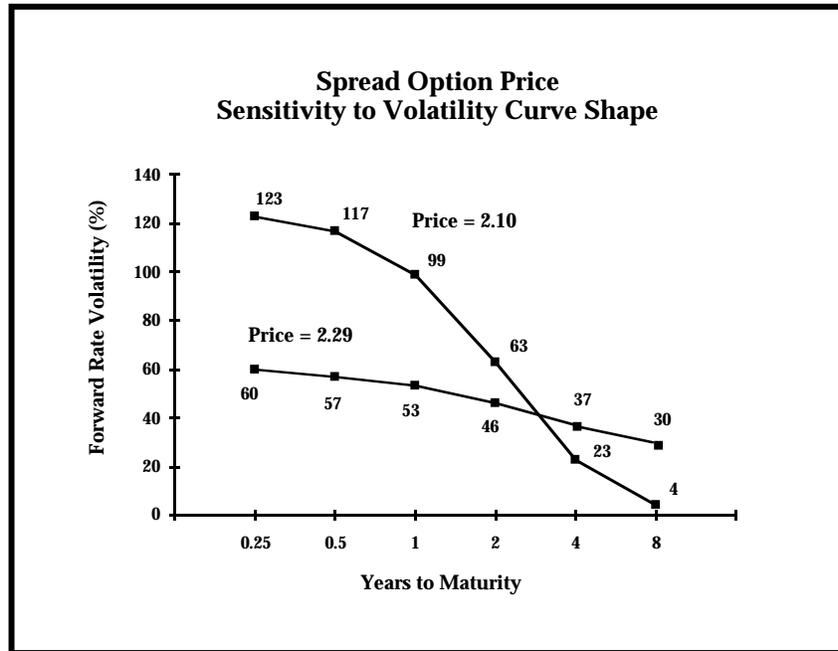


Figure 5: Impact of Shape of Vol Curve on Credit-Derivative Price

increases the price of the credit derivative, although not markedly.<sup>34</sup>

<sup>34</sup>Additional comparative statics show that:

- Increasing the spread cap  $\bar{S}$  from the base case of 500 basis points up to 900 basis points, holding all else the same, increases the price of the yield-spread option to 2.7 percent of face value.
- Increasing the strike spread  $\underline{S}$  from 200 basis points to 300 basis points reduces the price to 0.7 percent of face value.
- Increasing the yield-spread volatility from the base case of 40 percent up to a volatility of 65 percent increases the price to 2.9 percent of face value.
- Increasing, from 0 to 0.4, the initial correlation of yield spread changes with treasury note yield changes reduces the yield-spread option price, but only slightly, to 2.0 percent of face value.
- Increasing the expiration date  $T$  from 1 to 4 years increases the yield spread option price to 3.8 percent of face value.

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- Within conventional ranges, the yield-spread option price is relatively insensitive to the treasury yield volatility. For example, as one changes the initial yield volatility of the underlying treasury note from 15 to 25 percent, by scaling the base-case vol-curve shown in the price of the yield-spread option is not affected, up to two significant figures.

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