Defaultable Term Structure Models with Fractional Recovery of Par

Darrell Duffie[1]

Graduate School of Business, Stanford University

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Preliminary

Abstract: This paper provides simple tractable models of the term structure of credit spreads on corporate or sovereign bonds based on exogenous fractional recovery of face value. One version of the model is based on "affine" state variables. Another version is in the spirit of the Heath-Jarrow-Morton model of forward rates.

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1 Introduction

This paper provides simple tractable models of the term structure of credit spreads on corporate or sovereign bonds based on exogenously specified fractional recovery of face value at default. One version of the model is based on "affine" state variables. Another version is in the spirit of the Heath-Jarrow-Morton model of forward rates.

Such a model can be used for such industry applications as credit-swap pricing or extracting implied risk-neutral default probabilities from credit swap spreads, or for other risk-management, bond derivative, or credit derivative applications. An explicit term structure model is useful when calibrating or estimating parameters.

1.1 Reduced-Form Default Models

"Reduced-form" defaultable term-structure models typically[2] take as primitives the behavior of default-free interest rates, the fractional recovery of defaultable bonds at default, as well as a stochastic intensity process λ for default. The intensity λ_t may be viewed as the conditional rate of arrival of default. For example, with constant λ, default is a Poisson arrival.

These models are distinguished somewhat by the manner in which the recovery at default is parameterized. Jarrow and Turnbull (1996) stipulated that, at default, a bond would have a market value equal to an exogenously specified fraction of an otherwise equivalent default-free bond. Duffie and Singelton (1997) followed with a model that, when specialized to exogenous fractional recovery of market value at default, allows for closed-form solutions in a wider range of cases, by showing that cash flows can be discounted simply at the short-term default-free rate plus the risk-neutral rate of expected loss of market value due to default.

Some industry researchers, however, prefer a model in which bonds of the same issuer, seniority, and face value have the same recovery at default, regardless of remaining maturity. This is a relatively strict legal interpretation of recovery that can be based, for example, on the assumption of absolute

[2]This class of models includes Artzner and Delbaen (1995), Duffie and Singleton (1995), Duffie, Schroder, and Skiadas (1996), Jarrow and Turnbull (1996), Jarrow, Lando, and Turnbull (1997), Lando (1993, 1997, 1998), Madan and Unal (1995), Martin (1997), Nakazato (1997), Nielsen and Ronn (1995), Pye (1974), Schönbucher (1997), and others.

priority or liquidation at default. Under such an assumption of equal recovery for equal face value, the Duffie-Singleton or Jarrow-Turnbull models, specialized to a given deterministic fractional recovery parameters, could only be viewed as approximations, and in any case call for care in specifying the dependence of fractional recovery on coupon structure and remaining maturity.

Here, we propose models with a parameterization based on an exogenously specified fractional recovery of face value. Recovery parameters in this setting could be based on statistics provided by rating agencies, such as Moody's, a recent summary of which are illustrated in Figure 1. The assumption is that, at default, the holder of a bond of given face value receives a fixed payment, irrespective of coupon level or maturity, and the same fraction of face value as any other bond of the same seniority. Faced with the threat of liquidation but the potential for re-organization, it is not clear that equal fractional recovery of par (that is, absolute priority) would necessarily apply. Bonds of different maturities and coupon rates may suffer quite differently, relative to each other, in a re-organization than they would in liquidation, and their owners therefore may have different bargaining positions that may cause violations of absolute priority in re-organization. For a relevant theoretical approach, see Bergman and Callen (1991). No empirical research on this issue was available at this writing.

To the author's knowledge, there had been no published term-structure models providing explicit solutions, with exogenously specified fractional recovery of face value, that allow stochastic default intensity that is correlated with default-free interest rates.[3] This stochastic intensity behavior is apparent in bond yield-spread behavior. See, for example, Duffee (1996).

We are able to obtain explicit pricing with fractional recovery of face value in two settings.

In one of these, based on a state-vector with affine dynamics, there is a simplifying assumption that payout of recovery is at the first possible date among a list of discrete dates, such as coupon dates. With the alternative of continual recovery (simultaneous with default), we show how to reduce the calculation of defaultable discounts to a one-dimensional numerical integral, which is not onerous.

The second class of models presented in this paper is based on specifying

[3]In a structural default model, Brennan and Schwartz (1980) specified a given fractional recovery of face value in an early model of convertible debt pricing, but again did not obtain explicit solutions. Otherwise, there are few if any published models with explicit solutions for term structure models based on exogenous recovery of par.

the stochastic behavior of defaultable bond forward rates, with an exogenously specified fractional recovery of par, in the spirit of the term-structure model of Heath, Jarrow, and Morton (1992).

2 The Basic Approach

This section contains the basic model. The following sections specialize to obtain explicit results.

2.1 Ingredients and Assumptions

The model has several basic ingredients:

- A stopping time τ for default of the given issuer. The stopping time is assumed to have an intensity process λ.[4] With a constant intensity λ, for example, default has a Poisson arrival at intensity λ. More generally, for t before τ, we may view λ_t as the conditional rate of arrival of default at time t, given all information available up to that time. In other words, for a small time interval of length Δ, the conditional probability at time t that default occurs between t and $t + \Delta$, given survival to t, is approximately[5] $\lambda_t \Delta$.

- A bounded short-rate process r and equivalent martingale measure[6] Q.

[4] All random variables are defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. A filtration $\{\mathcal{F}_t : t \geq 0\}$ of σ-algebras, satisfying the usual conditions, is fixed and defines the information available at each time. An intensity process λ is assumed to be non-negative and predictable (a natural measurability restriction) and to satisfy, for each $t > 0$, $\int_0^t \lambda_s \, ds < \infty$ almost surely, with the property that, for $N(t) = 1_{\tau \geq t}$, a martingale is defined by

$$N_t - \int_0^t (1 - N_s)\lambda_s \, ds, \qquad t \geq 0.$$

Some authors prefer to to describe "the" intensity as $(1 - N_s)\lambda_s$ rather than λ_s, and this indeed, under technical continuity conditions, allows for a uniqueness-of-intensity property. For details, see Brémaud (1980). Our definition is weaker and does not suggest uniqueness.

[5] This is true in the usual sense of derivatives if, for example, λ is a bounded continuous process, and otherwise can be interpreted in an almost-everywhere sense.

[6] See Appendix A for details.

- Recovery at default given by a bounded random variable W, per unit of face value. We will elaborate two versions of the model, one with discrete-time recovery, and another with "continual recovery." Continual recovery means simply that W is measured and received precisely at the default time τ. With discrete-time recovery, W is measured and recevied as of the first date after default among a pre-specified list $T(1), T(2), \ldots, T(n) = T$ of times, with $T_i < T_{i+1}$, where T is maturity. In application, it may be simple and adequate to take these recovery times to be coupon dates, in that default may be effectively revealed between coupon dates, but in fact only declared when coupon or principle is due.[7] It is not uncommon in practice to assume that the recovery of coupons is zero, and to treat coupon and principal payments individually, as zero-coupon strips. The discrete-time recovery assumption may also be viewed simply as an approximation, with the virtue of explicit pricing, of the pricing that would apply with continual recovery.

2.2 Valuation with Discrete-Time Recovery

We first give a general valuation formula, with discrete-time recovery, for a given zero-coupon bond, which may itself be a coupon or principal strip, maturing at $T(n) = T$. Typically, for a coupon strip, we would take $W = 0$ recovery.

By the definition of Q as an equivalent martingale measure, the market value of the bond at any time t before default is

$$V_t = V^*(t) + \sum_{\{i : t \le T(i) \le T\}} V_i(t),$$

where

$$V^*(t) = E^Q \left[\delta(r, t, T) 1_{\tau > T} \;\middle|\; \mathcal{F}_t \right],$$

[7]If one prefers to view recovery as measured at the precise default date τ, then, except in extreme cases, a good approximation would be to assume that, conditional on default during the interval $(T(i), T(i+1))$, recovery occurs at an expected time given by some particular point, such as the mid-point of the interval, and then to approximate W as the actual recovery at the default time, scaled up by the time value of money invested at the default time until the end of the interval. Of course, W is assumed to be measurable with respect to $\mathcal{F}_{T(\tau)}$, where $T(\tau) = \min\{T(i) : T(i) > \tau\}$.

$$V_i(t) = E^Q\left[1_{T(i-1)\le\tau<T(i)}\delta(r,t,T(i))W \;\middle|\; \mathcal{F}_t\right], \tag{1}$$

and, for any predictable process γ with $\int_0^T |\gamma_s|\, ds < \infty$, we let

$$\delta(\gamma, t, s) = \exp\left(-\int_t^s \gamma_u\, du\right).$$

We have merely represented the value of a strip as the value contingent on survival through maturity plus the sum of the values contingent on default in each of the intervals $(T(i), T(i+1))$.

Because

$$1_{T(i-1)\le\tau<T(i)} = 1_{\tau\ge T(i-1)} - 1_{\tau\ge T(i)},$$

we see that

$$V_i(t) = V_{iA}(t) - V_{iB}(t),$$

where

$$V_{iA}(t) = E^Q\left[1_{\tau\ge T(i-1)}\delta(r,t,T(i))W \;\middle|\; \mathcal{F}_t\right]$$

and

$$V_{iB}(t) = E^Q\left[1_{\tau\ge T(i)}\delta(r,t,T(i))W \;\middle|\; \mathcal{F}_t\right].$$

2.3 Simplification

Artzner and Delbaen (1995) show that there is also an intensity process λ^Q for τ under the equivalent martingale measure Q, and show how to obtain λ^Q in terms of λ and Q.

In order to simplify the calculations in this discrete-time recovery setting, we suppose that W is Q-independent of (r,τ), and let $\overline{w} = E^Q(W)$. At a cost in complexity, one could alternatively exploit conditioning information regarding the dependence of recovery on the timing of default, as we shall in the continual-recovery case to follow.

Based on previously available results,[8] under technical conditions, for $t < \tau$, we have $V^*(t) = Y^*(t)$, $V_{iA}(t) = Y_{iA}(t)$, and $V_{iB}(t) = Y_{iB}(t)$, where

$$Y_t^* = E^Q\left[\delta(r+\lambda^Q, t, T) \;\middle|\; \mathcal{F}_t\right], \tag{2}$$

$$Y_{iA}(t) = \overline{w}E^Q\left[\delta(r+\lambda^Q, t, T(i-1))\delta(r, T(i-1), T(i)) \;\middle|\; \mathcal{F}_t\right],$$

[8] See Duffie, Schroder, and Skiadas (1996) or Lando (1997).

and

$$Y_{iB}(t) = \overline{w} E^Q \left[\delta(r + \lambda^Q, t, T(i)) \;\middle|\; \mathcal{F}_t \right]. \tag{3}$$

The main condition is that Y^*, Y_{iA}, and Y_{iB} do not jump at the default time τ. This is certainly true in the usual settings, for example when r, λ^Q are determined by[9] diffusion processes.

We can now apply the law of iterated expectations to write

$$Y_{iA}(t) = \overline{w} E^Q \left[\delta(r + \lambda^Q, t, T(i-1)) U_i \;\middle|\; \mathcal{F}_t \right], \tag{4}$$

where

$$U_i = E^Q \left[\delta(r, T(i-1), T(i)) \;\middle|\; \mathcal{F}_{T(i-1)} \right],$$

which is the default-free zero-coupon bond price at $T(i-1)$ for maturity $T(i)$.

The idea now is to specialize the model so as to obtain a closed form solution for U_i, and then explicit solutions for Y_{iA}, Y_{iB}, and Y^*.

Before we do so, we summarize our technical progress. The proof is by the above construction, combined with results that can be found, for example, in Duffie, Schroder, and Skiadas (1996) or Lando (1997).

Proposition 1. *Suppose that r, λ^Q, and W are bounded, and that W is Q-independent of (r, τ). For $t < T$, let*

$$Y(t) = Y^*(t) + \sum_{\{i: t \le T(i) \le T\}} (Y_{iA}(t) - Y_{iB}(t)), \tag{5}$$

where Y^*, Y_{iA}, and Y_{iB} are given by (2), (4), and (3), respectively. Suppose that Y jumps[10] at τ with probability zero. Then $V(t) = Y(t)$ for $t < \tau$.

[9] It is enough that (λ^Q, r) is predictable with respect to the filtration generated by a process that is a diffusion, or even a jump diffusion provided the state jumps at τ with probability zero. See, for example, Duffie, Schroder, and Skiadas (1996) for more details.

[10] The jump of a right-continuous left-limits process X at time t is $\Delta X(t) = X(t) - \lim_{s \uparrow t} X(s) = 0$. One says that X has no jump at τ if $\Delta X(\tau) = 0$.

2.4 Pricing with Continual Recovery

As the inter-recovery periods $[T(i), T(i-1))$ shrink in length to zero, we can view the model as one with continual recovery.[11] In such a model, for $t < \tau$, we have the continual-recovery market value

$$V^c(t) = E^Q(1_{\tau \geq T}\delta(r,t,T) \mid \mathcal{F}_t) + E^Q(1_{\tau < T}\delta(r,t,\tau)W \mid \mathcal{F}_t).$$

For the candidate pre-default market value of recovery at default, we define

$$\tilde{Y}(t,T) = \int_t^T \theta(t,s)\, ds, \tag{6}$$

where

$$\theta(t,s) = E^Q\left[\delta(r+\lambda^Q,t,s)\lambda^Q(s)\overline{W}(s) \;\middle|\; \mathcal{F}_t\right],$$

and where $\overline{W}$ is the risk-neutral compensator[12] for W. Intuitively, $\overline{W}(t)$ is the expected recovery given $\mathcal{F}_t$ and given that recovery occurs in the "next instant." In applications, a common assumption is that the recovery compensator $\overline{W}$ is deterministic.

We can use the results of Duffie (1998) to obtain the following.

Proposition 2. *Suppose r, λ^Q, and W are bounded, and λ^Q and $\overline{W}$ are right-continuous. Let $Y^c(t) = \tilde{Y}(t,T) + Y^*(t)$, where $\tilde{Y}$ is defined by (6). If Y^c does not jump at τ almost surely, then $V^c(t) = Y^c(t)$ for $t < \tau$.*

We will now specialize to Markov or HJM-style flavors of the model, in which the defaultable term structure process is given explicitly, for given primitives.

[11] In this case, we take W to be a bounded non-negative $\mathcal{F}_\tau$-measurable variable.

[12] Let $\hat{W} = E^Q(W \mid \mathcal{F}_{\tau-})$. From Dellacherie and Meyer (1978), Theorem IV.67(b), there is a predictable process $\overline{W}$ such that $\overline{W}(\tau) = \hat{W}$. Then

$$E^Q(\delta(r,t,\tau)W \mid \mathcal{F}_t) = E^Q\left(\int_t^T \delta(r+\lambda^Q,t,s)\lambda_s^Q\,\overline{W}_s\, ds \;\middle|\; \mathcal{F}_t\right).$$

See Duffie (1998) for details. Please note that there is a typographical error in Dellacherie and Meyer (1978), Theorem IV.67(b), in that the second sentence should read: "Conversely, if Y is an $\mathcal{F}^0_{T-}$-measurable..." rather than "Conversely, if Y is an $\mathcal{F}^0_T$-measurable...," as can be verified from the proof, or, for example, from their Remark 68(b).

3 Analytical Solutions in Affine Settings

This section proposes parametric examples in which the pricing formulas given by Propositions 1 and 2 can be computed either explicitly, or by numerically solving relatively simple ordinary differential equations, in an "affine" setting, characterized by a "state" process X valued in $\mathbb{R}^k$ that (under Q) is a k-dimensional affine jump-diffusion, in the sense of Duffie and Kan (1996). Special cases including multi-factor Cox-Ingersoll-Ross (1985) state-variable settings.

3.1 The Affine State Process

We consider a Markov (under Q) process X valued in some appropriate domain $D \subset \mathbb{R}^k$, with

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dB_t^Q + dJ_t,$$

where B^Q is a standard brownian motion in $\mathbb{R}^d$ under Q, J is a pure jump process with jump-arrival intensity $\{\kappa(X_t) : t \geq 0\}$ and jump distribution ν on $\mathbb{R}^k$, and where $\kappa : D \to [0, \infty)$, $\mu : D \to \mathbb{R}^k$, and $C \equiv (\sigma\sigma^\top) : D \to \mathbb{R}^{k\times d}$ are affine functions.[13] We ignore time dependencies in the coefficients for notational simplicity only; the approach outlined below extends to that case in a straightforward manner. A classical special case is the "multi-factor CIR state process" X, for which $X^{(1)}$, $X^{(2)}, \ldots, X^{(k)}$ are independent (or Q-independent, in a valuation context) processes of the "square-root" type[14] introduced into term-structure modeling by Cox, Ingersoll, and Ross (1985). Of course, Gaussian models of the sort considered by Vasicek (1977) and Langetieg (1980) models are also special cases, as are certain combinations

[13] The generator $\mathcal{D}$ for X is defined by

$$\mathcal{D}f(x,t) = f_t(x,t) + f_x(x,t)\mu(x) + \frac{1}{2}\sum_{ij} C_{ij}(x) f_{x_i x_j}(x,t) + \kappa(x)\int [f(x+z,t) - f(x,t)]\, d\nu(z).$$

One can add time dependencies to these coefficients. Conditions must be imposed for existence and uniqueness of solutions, as indicated by Duffie and Kan (1996).

[14] That is,

$$dX_t^{(i)} = \kappa_i(\overline{x}_i - X_t^{(i)})\, dt + \sigma_i \sqrt{X_t^{(i)}}\, dB_t^{(i)},$$

for some given constants $\kappa_i > 0$, $\overline{x}_i > 0$, and σ_i.

of Gaussian and CIR models. For many other examples of affine models, see Dai and Singleton (1997) and Duffie, Pan, and Singleton (1997).

We can take advantage of this setting if we suppose that short rates and risk-neutral default intensities are of the affine[15] form

$$\begin{aligned} r(t) &= a_r(t) + b_r(t) \cdot X_t \\ \lambda^Q(t) &= a_\lambda(t) + b_\lambda(t) \cdot X_t, \end{aligned}$$

where

- a_r and a_λ are bounded measurable real-valued deterministic functions on $[0, T]$.
- b_r and b_λ are bounded measurable deterministic $\mathbb{R}^k$-valued functions on $[0, T]$.

Except in degenerate cases, the boundedness assumptions used in Propositions 1 and 2 do not apply, and integrability conditions must be assumed or established. Of course, only a non-negative intensity process λ^Q makes sense, so the model should be restricted so that $a_\lambda(t) + b_\lambda(t) \cdot x \geq 0$ for all t and all x in the state space D. Some, such as Aravantis, Gregory, and Decamps (1998) and Nakazato (1997), have taken Gaussian models for λ^Q, presumably viewing this as a convenient approximation.

For analytical approaches based on the affine structure just described, one can repeatedly use the following calculation, regularity conditions for which are provided in Duffie, Pan, and Singleton (1997).

Let X be an affine jump-diffusion. For a given time s, and for each $t \leq s$ let $R(t) = a_R(t) + b_R(t) \cdot X(t)$, for bounded measurable $a_R : [0, s] \to \mathbb{R}$ and $b_R : [0, s] \to \mathbb{R}^k$. For given coefficients a in $\mathbb{R}$, and b in $\mathbb{R}^k$, let

$$g(X_t, t) = E^Q \left[\exp\left(\int_t^s -R(u)\, du \right) e^{a + b \cdot X(s)} \;\middle|\; X_t \right]. \tag{7}$$

Under technical conditions, there are specified ODEs for $\alpha : [0, s] \to \mathbb{R}$ and $\beta : [0, s] \to \mathbb{R}^k$ such that

$$g(x, t) = \exp\left(\alpha(t) + \beta(t) \cdot x\right),$$

[15] One can proceed in more or less the same fashion if one generalizes by allowing quadratic terms for $r(t)$ and the jump intensity κ, subject of course to technical conditions. In such cases, higher-order terms will appear in the solution polynomial.

with boundary conditions $\alpha(s) = a$ and $\beta(s) = b$. In addition, for given $\hat{a}$ in $I\!R$, and $\hat{b}$ in $I\!R^k$, let

$$G(X_t, t) = E^Q \left[\exp\left(\int_t^s -R(u)\, du \right) e^{a + b \cdot X(s)} (\hat{a} + \hat{b} \cdot X_s) \;\Big|\; X_t \right]. \tag{8}$$

Then, under technical conditions, there are specified ODEs for $\hat{\alpha} : [0, s] \to I\!R$ and $\hat{\beta} : [0, s] \to I\!R^k$ such that

$$G(x, t) = e^{\alpha(t) + \beta(t) \cdot x} (\hat{\alpha}(t) + \hat{\beta}(t) \cdot x),$$

with boundary conditions $\hat{\alpha}(s) = \hat{a}$ and $\hat{\beta}(s) = \hat{b}$.

The ODEs in question are Ricatti equations if X has no jumps. Details, with illustrative numerical examples and empirical applications, can be obtained in Duffie and Kan (1996) and Duffie, Pan, and Singleton (1997).

3.2 Discrete-Time Recovery Case

For computation of the term-structure of defaultable bond prices in the discrete-time recovery model, we proceed as follows. First, for each i, a solution for the default-free discount U_i is obtained from (7), using $a = 0$ and $b = 0$ and $R(t) = r(t)$, leaving $U_i = \exp(a_i + b_i \cdot X(T(i-1))$, for computed constants $a_i \in I\!R$ and $b_i \in I\!R^k$.

Next, for each i, we can obtain Y_{iA} from (7) using $a = a_i, b = b_i$, and $R(t) = \lambda^Q(t) + r(t)$. Likewise, we obtain Y_{iB} for each i, using $a = 0, b = 0$ and $R(t) = \lambda^Q(t) + r(t)$. We then calculate $Y^*(t)$ from (7), taking $a = 0$, $b = 0$, and $R(t) = \lambda^Q(t) + r(t)$. Finally, we can insert the results for Y^*, Y_{iA} and Y_{iB} into the total pre-default valuation formula (5) for $Y(t)$.

3.3 Continual Recovery Case

For the continual-recovery case, we can take the recovery compensator $\overline{W}$ to be of the form

$$\overline{W}(t) = \exp(\overline{a}(t) + \overline{b}(t) \cdot X(t-)),$$

which includes the special case of deterministic $\overline{W}(t)$. We can then calculate $\alpha, \beta, \hat{\alpha}$, and $\hat{\beta}$ with

$$\theta(t, s) = \exp(\alpha(t, s) + \beta(t, s) \cdot X(t-))(\hat{\alpha}(t, s) + \hat{\beta}(t, s) \cdot X(t-)).$$

With solutions for α, β, $\hat{\alpha}$, and $\hat{\beta}$ in hand, the pricing formula (6) is reduced to a numerical integral, which is a relatively fast exercise.

For the special multi-factor CIR case, there are explicit closed-form solutions for α, β, $\hat{\alpha}$ and $\hat{\beta}$.[16] We then have an explicit solution for the derivative $\theta(t,s)$ of the recovery value $\tilde{Y}(t,s)$ with respect to s. Then $\theta(t,s)$ can be integrated numerically with respect to s in order to obtain the total market value $\tilde{Y}(t,T)$ of default recovery. This value is added to survival-contingent value $Y^*(t)$ to get the the total price $Y^c(t)$ of the bond, for $t < \tau$.

Monte-Carlo based approaches are explained in Duffie (1998).

4 Forward Rates with Partial Recovery of Par

This section exploits the continual-recovery formulation, and an HJM-style model of forward rates. The key results are the implied risk-neutral default intensity and the risk-neutral drift resriction on forward rates or forward spreads for defaultable debt. This extends results in Duffie and Singleton (1996) to the case of fractional recovery of face value.

[16] These are from Cox, Ingersoll, and Ross (1985) (for the case $a = \hat{a} = 0$ and $b = \hat{b} = 0$), and Duffie, Pan, and Singleton (1997) for the general case. Duffie, Pan, and Singleton (1997) also provide analytical solutions for the Fourier transforms of X in the general affine setting. Related calculations that lead to analytical solutions for option pricing via Lévy inversion of the transforms. The Fourier-based option-pricing results can be applied in this setting for cases in which

$$\overline{W}(t) = \left[\exp(\overline{a}(t) + \overline{b}(t) \cdot X(t-)) - K\right]^+,$$

for some "exercise price" K. Some explicit results for option pricing are available in certain cases, as shown by by Bakshi, Cao, and Chen (1996), Bakshi and Madan (1997), Bates (1996), and Chen and Scott (1995). These results can be applied in the present setting for valuation of defaultable options with affine structure, or with recovery determined by collateralization with an instrument whose price can be described in an exponential-affine form. Collateralization with equities, foreign currency, or notes (in domestic or foreign currency) would be natural examples for this.

4.1 Defaultable Forward Rate Behavior

We suppose that a defaultable zero-coupon bond maturing at s has a price at time t (assuming default has yet to occur) of the form

$$q(t,s) = \exp\left(-\int_t^s f(t,u)\,du\right), \tag{9}$$

where, for each fixed s, we suppose that $f(\,\cdot\,,s)$ is an Ito process,[17] satisfying

$$df(t,s) = \mu_f(t,s)\,ds + \sigma_f(t,s)\,dB_t^Q, \tag{10}$$

where B^Q is a standard Brownian motion in $\mathbb{R}^d$ under Q, and μ_f and σ_f satisfy technical conditions.

The model does not actually say anything about the pricing of forward contracts *per se*, at least without some convention for how a forward contract on a bond would settle in the event of default of the underlying bond before delivery. Instead, the model takes the prices of defaultable bonds of the same issuer and recovery quality to have the convenient behavior specified by (9) and (10).

First we consider the special case of the principal strip curve, assuming that an exogenously specified fraction W of par is recovered at default, regardless of maturity. By the definition of Q, for each s, the discounted gain process G defined by the strip maturing at s is a Q-martingale. This gain process is defined by

$$G_t = \delta(r,0,t)q(t,s)(1-N_t) + \int_0^t \delta(r,0,u)W\,dN(u),$$

where $N(t) = 1_{\tau\geq t}$. The first term is the discounted price (which is zero after default); the second term is the discounted payment of recovery at default.

Under technical conditions, one can apply Ito's Lemma and Fubini's Theorem for stochastic integrals[18] and arrive at

$$dG(t) = \delta(r,0,t)q(t,s)\varphi(t,s)\,dt + dM^G(t),$$

where M^G is a Q-martingale and

$$\begin{aligned}\varphi(t,s) \quad &= \quad f(t,t) - r_t + \lambda_t^Q\left(\frac{\overline{W}(t)}{q(t,s)} - 1\right) - \int_t^s \mu(t,u)\,du \\ &\qquad + \frac{1}{2}\left(\int_t^s \sigma(t,u)\,du\right)\cdot\left(\int_t^s \sigma(t,u)\,du\right).\end{aligned}$$

[17] One can add jumps to the model for f, and extend the calculations easily.

[18] See, for example, Protter (1990).

Let us suppose that $r(t)$, $f(t,t)$, $\lambda^Q(t)$, and $\overline{W}(t)$ depend continuously on t, and that $\overline{W}(t) < 1$ for all t. Then, for each fixed s, because $\varphi(\,\cdot\,, s)$ is continuous, G is a Q-martingale, and $\delta(r, 0, t)q(t, s)$ is strictly positive, we have $\varphi(t, s) = 0$ for all $t \leq s$. Taking $s = t$, we have $\varphi(t, t) = 0$, leaving the implied risk-neutral default intensity

$$\lambda^Q(t) = \frac{f(t,t) - r_t}{1 - \overline{W}(t)}, \tag{11}$$

which has continuous dependence on t as assumed. For the special case of zero recovery, which is the conventional standard for coupons, this gives the usual implication that the short credit spread is the default intensity.

We can substitute this expression (11) for λ^Q back into $\varphi(t, s)$. Then, because, for each fixed t, we know that $\varphi(t, s) = 0$ for all $s \geq t$, it must be that $\frac{\partial}{\partial s} q(t, s) = 0$, and we obtain the risk-neutral defaultable principal-strip forward-rate drift restriction

$$\mu_f(t, s) = \sigma_f(t, s) \cdot \int_t^s \sigma_f(t, u)\, du + \lambda^Q(t)\overline{W}(t)\frac{f(t, s)}{q(t, s)}. \tag{12}$$

With this approach, one can exogenously specify forward-rate dynamics for principal strips with any model for the volatility process σ_f, the recovery compensator $\overline{W}$, the initial forward rate $f(0, s)$ for each s up to some final maturity T, and the short-rate process r, subject to the non-negative-spread condition that $f(t,t) \geq r(t)$ for all t almost surely. Conditions for non-negative spreads are given directly on forward-spread-rate behavior in the Section 4.3.

4.2 The Implied Default Time

With a model for $(f, r, \overline{W})$ in place, one can define a stopping time τ such that $(f, r, \overline{W}, \tau)$ has the "correct" joint distribution, and in particular such that τ has the risk-neutral intensity λ^Q, given by (11), that is implied by $(f, r, \overline{W})$.

One method for this is to construct τ in terms of $(f, r, \overline{W})$ as follows.

First, one can let Z be exponentially distributed under Q with parameter 1, and Q-independent[19] of f, r, and $\overline{W}$. Given Z, f, r, and $\overline{W}$, one can then

[19] The independence condition can always be met, by, if necessary, enlarging the probability space and filtration, as explained in Duffie (1998).

let

$$\tau = \inf\left\{t : \int_0^t \lambda^Q(s)\,ds = Z\right\}, \tag{13}$$

where λ^Q is given by (11). Then the intensity[20] of τ is indeed λ^Q.

In principle, this allows for joint simulation of defaultable forward rates and default times.

4.3 The Forward Spread-Rate Process

Given an HJM model for default-free forward rates, one could instead set up a model directly for the credit forward-spread-rate process, given assumed initial spreads and processes for spread volatilities, correlations between spreads and default-free rates, and recoveries, as follows.

For this, the forward-rate spread at t for maturity date s is denoted $S(t,s) = f(t,s) - F(t,s)$, where $F(t,s)$ is the default-free forward rate at t for maturity s, given by

$$dF(t,s) = \mu_F(t,s)\,dt + \sigma_F(t,s)\,dB_t^Q,$$

where B^Q is the same Brownian motion in $I\!R^d$ underlying the defaultable forward rate process f, and where μ_F and σ_F satisfy the usual technical conditions for a forward-rate process.[21] Under these conditions, we have the usual HJM risk-neutral default-free forward rate drift restriction

$$\mu_F(t,s) = \sigma_F(t,s) \cdot \int_t^s \sigma_F(t,u)\,du. \tag{14}$$

Our model for the short rate process r is then defined by $r(t) = F(t,t)$.

Suppose we specify that, for each s,

$$dS(t,s) = \mu_S(t,s)\,dt + \sigma_S(t,s)\,dB_t^Q,$$

for a given $I\!R^d$-valued "volatility process" $\sigma_S(\,\cdot\,,s)$. From (12) and (14), we have

$$\begin{aligned}\mu_S(t,s) \quad &= \quad \sigma_S(t,s) \cdot \int_t^s \sigma_F(t,u)\,du + \sigma_F(t,s) \cdot \int_t^s \sigma_S(t,u)\,du \\ &\qquad + \frac{\lambda^Q(t)\overline{W}(t) f(t,s)}{q(t,s)},\end{aligned}$$

[20] For the details, see Appendix C.

[21] See, for example, Carverhill (199?).

where, from (11), we take

$$\lambda^Q(t) = \frac{S(t,t)}{1 - \overline{W}(t)}. \tag{15}$$

The "instantaneous correlation" process ρ_{SF} between spreads and default-free forward rates is defined, for each s and $t \leq s$, is defined by

$$\rho_{SF}(t,s) = \frac{\sigma_S(t,s) \cdot \sigma_F(t,s)}{\|\sigma_S(t,s)\| \, \|\sigma_F(t,s)\|},$$

assuming non-zero volatilities. Alternatively, one could take the total spread volatility process v_S, defined by $v_S(t,s) = \|\sigma_S(t,s)\|$, and the correlation process ρ_{SF} as inputs, satisfying technical conditions, and from these determine[22] a consistent process for σ_S.

Given the default-free forward-rate process F, the spread volatility process σ_S, the initial spread curve $S(0, \cdot\,)$, and the recovery compensator $\overline{W}$, the model for the defaultable forward rate process f is determined, again under technical regularity.

Of course, one wants restrictions under which $S(t,s)$ remains non-negative for any non-negative initial spread curve. Roughly speaking, it is enough that $\sigma_S(t,u) = 0$, $t \leq u \leq s$, whenever $S(t,s)$ is zero. Then, from the risk-neutral drift restriction given above for $\mu_S(t,s)$, for each fixed s, we have $\mu_S(t,s) \geq 0$ whenever $S(t,s) = 0$, so 0 is a natural boundary for $S(\,\cdot\,, s)$, giving the desired non-negativity. There are of course technical conditions, including existence.[23]

4.4 Coupon-Strip Forward Curves

As for the coupon-strip forward curves, there is a consistency condition given by equating the implied intensity from the principal curves to the implied intensity from the coupon curves, which are assumed to be associated with zero recovery. The condition is then that the short spread $s_C(t)$ on coupon

[22] Provided the Brownian motion B^Q is of dimension $d > 1$, one can alsways construct an $\mathbb{R}^d$-valued process σ_S with $\|\sigma_S(t,s)\| = v_S(t,s)$ and $\sigma_S(s,t) \cdot \sigma_F(s,t) = \rho_{SF}(t,s) v_S(t,s) \|\sigma_F(t,s)\|$.

[23] See Miltersen (1995) for technical conditions for the analogous non-negativity of default-free forward rates.

strips is given in terms of the spread $s_P(t) = f(t,t) - r_t$ on principal strips by

$$s_C(t) = \frac{s_P(t)}{1 - \overline{W}(t)}.$$

Alternatively, one can start with the coupon-strip forward-rate dynamics, distinguished by the zero-recovery assumption $\overline{W} = 0$. This is particularly convenient for default swap pricing and calibration, as shown in the next section.

If the coupon spread forward rates and the default-free forward rates have zero correlation (that is, $\rho_{SF} = 0$), then the coupon-spread forward rate process is equal to the process for risk-neutral forward default probability rates, as shown by the definition of, and drift restriction on, forward default probability rates given in Appendix B. With either non-zero correlation or non-zero recovery (or both), however, the processes for spread rates and risk-neutral forward rates of default probabilities are not generally the same.

5 Default Swap Pricing

In its simplest form, a default swap requires payment of an annuity at a fixed coupon rate C at specified coupon dates until default or the stated maturity of the swap, whichever is first. If default is before maturity, then the annuity payer receives at default the difference between face value and recovery on the underlying issue.

We let $T(1), \ldots, T(n)$ denote the coupon dates, where $T = T(n)$ is the maturity date of the default swap (which may be different than that of the underlying bond). For simple pricing purposes, we suppose that the payment of $1 - W$ in the event of default occurs on the first coupon date after default, and that recovery W is risk-neutrally independent of default-free interest rates and the default time τ. We let $\overline{w} = E^Q(W)$. The market value at time zero of the default swap with maturity T, is then

$$K(t) = (1 - \overline{w})J(t,T) - CA(t,T),$$

where

- $A(t,T)$ is the price at t of an annuity until the earlier of τ or T, given by

$$A(t,T) = \sum_{t < T(i) \leq T} q_C(t, T(i)),$$

- $q_C(t, T(i))$ is the price of a contract that pays 1 unit of account at coupon date $T(i)$, provided $\tau > T(i)$. We can therefore take $q_C(t, T)$ to be the defaultable coupon-strip (zero-recovery) discount at t for maturity T.

- $J(t, T)$ is the price of a claim paying one unit of account on the first coupon date after default, which, under the regularity conditions of Proposition 1, is given by

$$J(t, T) = \sum_{\{i:\, t < T(i) \leq T\}} (Y_{iA}(t) - Y_{iB}(t)),$$

  where Y_{iA} and Y_{iB} are defined by (4) and (3), respectively.

One could adjust the model to continual recovery by replacing $(1 - \overline{w})J(t, T)$ with

$$\int_0^T E^Q \left[\delta(r + \lambda^Q, 0, t) \lambda^Q(t) \overline{W}(t) \right] \, dt.$$

analogously with the results in Section 2.4.

The market default-swap spread $C^*(t)$ is defined by $K(t) = 0$, leaving

$$C^*(t) = \frac{(1 - \overline{w})J(t, T)}{A(t, T)}.$$

One can compute $J(t, T)$ and $A(t, T)$ using our earlier results.

6 Appendix A – Risk-Neutral Measure

An equivalent martingale measure is a probability measure Q equivalent to P, in that the two measures have the same events of probability zero. The default-free short rate process r is assumed to be predictable. For general theoretical pricing, we assume that r is bounded. Most results go through without a bound on r, for example under conditions such as $E^Q \left[\exp \left(- \int_0^T r_t \, dt \right) \right] < \infty$ for all T. A standard reference is Harrison and Kreps (1979). A given collection of securities available for trade, with each security defined by its cumulative dividend process D. This means that, for each time t, the total cumulative payment of the security up to and including time t is $D(t)$. For our purposes, a dividend process will always be taken to be of the form $D = A - B$, where A and B are bounded increasing adapted right-continuous left-limits (RCLL) processes, and we suppose that there are no dividends after a fixed time $T > 0$, in that $D(t) = D(T)$ for all t larger than T. The fact that Q is an equivalent martingale measure means that, for any such security D, the ex-dividend price process S for the security is given by

$$S_t = E^Q \left[\int_t^T \exp\left(-\int_t^s r_u \, du \right) dD_s \;\middle|\; \mathcal{F}_t \right], \quad 0 \leq t < T, \tag{16}$$

where E^Q denotes expectation under Q. The ex-dividend terminal price $S(T)$ is of course zero. An example is a security whose price is always 1, and paying the short-rate as a dividend, in that $D(t) = \int_0^t r_s \, ds$. As pointed out by Harrison and Kreps (1979) and Harrison and Pliska (1981), the existence of an equivalent martingale measure implies the absence of arbitrage and, under technical conditions, is equivalent to the absence of arbitrage. For weak technical conditions supporting this equivalence, see Delbaen and Schachermeyer (1994). In some cases, markets are incomplete, for example one may not be able to perfectly hedge losses in market value that may occur at default, and this would mean that there need not be a unique equivalent martingale measure.

7 Appendix B – Forward Default Probability

It may be convenient in certain applications to suppose that default intensities are determined by a forward-default-probability model, in the spirit of an HJM model for default-free discounts. A risk-neutral forward default probability model, based on discrete trees, was suggested by Litterman and Iben (1991).

For our continuous time model, the conditional probability at time t for survival from t to s, given survival to t and $\mathcal{F}_t$, is assumed to be of the form

$$p(t,s) = \exp\left(-\int_t^s h(t,u) \, du \right),$$

where, for each fixed s, we suppose that $h(\,\cdot\,, s)$ is an Ito process,[24] satisfying

$$dh(t,s) = \mu_h(t,s) \, ds + \sigma_h(t,s) \, dB_t^P,$$

[24] One can add jumps to the hazard-rate model, and extend the calculations easily.

where B^P is a standard Brownian motion in $\mathbb{R}^d$ under the "actual" probability measure P, and μ_h and σ_h satisfy technical conditions. By virtue of the same arguments[25] used by HJM, we can calculate that

$$\mu(t,s) = \sigma(t,s) \cdot \int_t^s \sigma(t,u)\, du.$$

Coming out of martingale property of the conditional survival probability is the implied intensity process λ, given by

$$\lambda(t) = h(t,t).$$

One can likewise model risk-neutral forward default probabilities and intensities.

8 Appendix C – Constructing Default Times

This appendix provides calculations justifying the construction of the default time τ given by (13). Basically, we want to show that τ has the specified intensity λ^Q.

Of course, the intensity of τ depends on the filtration. We can take any convenient filtration $\{\mathcal{G}_t\}$ for purposes of constructing the processes r, f, and $\overline{W}$, provided Z is orthogonal under Q to $\mathcal{G}_t$. For example, it would be natural to take $\{\mathcal{G}_t\}$ to be the standard filtration of a Brownian motion in $\mathbb{R}^d$ underlying models for $(f, r, \overline{W})$. Then we can define λ^Q by (11), construct τ by (13), and finally let $\mathcal{F}_t$ be the σ-algebra generated by the union of $\mathcal{G}_t$ and $\sigma(\{1_{\tau \geq s} : s \leq t\})$. From our construction, whenever $t < \tau$, we have

$$Q(\tau > s \,|\, \mathcal{F}_t) = Q(Z > y(s) \,|\, Z > y(t), \mathcal{G}_t),$$

where $y(s) = \int_0^s \lambda^Q(u)\, du$. Because Z is orthogonal to $\mathcal{G}_t$, and because of its exponential distribution, this leaves

$$Q(\tau > s \,|\, \mathcal{F}_t) = E^Q\left[\exp\left(-\int_t^s \lambda^Q(u)\, du\right) \;\Big|\; \mathcal{F}_t\right].$$

Because this expression does not jump at τ, almost surely, it characterizes λ^Q as the $\{\mathcal{F}_t\}$-intensity of τ, as desired.

[25] One uses the fact that $1_{\tau > t} p(\,\cdot\,, s)$ must be a martingale, and applies Ito's Formula, as in Protter (1990). The implied intensity process λ is then given, under technical conditions, by $\lambda(t) = h(t,t)$ A related calculation is given in Duffie and Singleton (1995).

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