Fermions in Quantum Complexity Theory

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Occupation Number Formalism

Consider n identical particles in m modes. If there are x_j particles in the jth mode, the state is $|x_1, \ldots, x_m\rangle$, where $\sum_{j=1}^{m} x_j = n$.

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States of this form (*Fock states*) make up a subspace H_n, of states describing n particles in m modes. The full *Fock space* is given by

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To go back to $H = H^{\otimes n}$, (anti-)symmetrize over all possible combinations giving x_j particles in mode j.

Moving Between Subspaces

Start with the vacuum state $|0, \ldots, 0\rangle$. To add a particle to mode j, apply the *creation operator* $a_j^\dagger$. By Hermitian adjoint, a_j removes a particle from mode j (*destruction operator*).

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- A state of n fermions in modes $j_1, \ldots, j_n$ is created by

$$\epsilon_{j_1 \cdots j_n} \cdot a_{j_1}^\dagger \cdots a_{j_n}^\dagger |0, \ldots, 0\rangle$$

where $\epsilon_{j_1 \cdots j_n}$ enforces sign convention.

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The *number of particles* in mode j is the eigenvalue of $N_j = a_j^\dagger a_j$.

Fermionic Quantum Computation

Any unitary operator acting on H can be generated by the set of creation and destruction operators: $U = \exp(iH)$, where $H(a_1, a_1^\dagger, \ldots, a_m, a_m^\dagger)$ is Hermitian.

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Terhal and DiVincenzo (2002) showed quantum computation using a system of noninteracting fermions can be simulated classically, even with adaptive von Neumann measurements. Results were equivalent to a subclass of Valiant's matchgates (2001).

Noninteracting Fermions: Complexity

Consider special case where number of fermions is preserved. A computation step acting between modes α and β is a unitary $U = \exp(iH)$ generated by

$$H = h_{\alpha\alpha} a_\alpha^\dagger a_\beta + h_{\beta\beta} a_\beta^\dagger a_\beta + h_{\alpha\beta} a_\alpha^\dagger a_\beta + h_{\alpha\beta}^* a_\beta^\dagger a_\alpha$$

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Suppose U acts on a non-vacuum state with one fermion:

$$U a_j^\dagger |0\rangle = U a_j^\dagger U^\dagger U |0\rangle = U a_j^\dagger U^\dagger |0\rangle$$

where

$$U a_j^\dagger U^\dagger = \sum_k V_{jk} a_k^\dagger$$

Noninteracting Fermions: Complexity

Now calculate $\langle y|U|x\rangle$ for $|x\rangle = a^\dagger_{j_1} \cdots a^\dagger_{j_l}|0\rangle$ ($j_1 < \cdots < j_l$) and $|y\rangle = a^\dagger_{k_1} \cdots a^\dagger_{k_l}|0\rangle$ ($k_1 < \cdots k_l$).

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\begin{aligned}
U|x\rangle &= Ua^\dagger_{j_1}U^\dagger \cdots Ua^\dagger_{j_k}U^\dagger|0\rangle \\
&= \sum_{p_1,\ldots,p_l}\left(V^{p_1}_{j_1}\cdots V^{p_l}_{j_l}\right)a^\dagger_{p_1}\cdots a^\dagger_{p_l}|0\rangle
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The braket is zero unless $p_1,\ldots p_l$ is a permutation of $k_1,\ldots,k_l$, and $\mathrm{sgn}(\sigma)$ if the permutation is σ.

Noninteracting Fermions: Complexity

Rewrite sum in terms of permutations:

$$\langle y|U|x\rangle = \sum_{\sigma \in S_l} \mathrm{sgn}(\sigma) V_{j_1}^{\sigma(k_1)} \cdot V_{j_l}^{\sigma(k_l)} = \det \widetilde{V}$$

where $\widetilde{V}$ is $l \times l$ a submatrix of V with selected rows $j_1, \ldots, j_l$ and columns $k_1, \ldots, k_l$.

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Quantum computation using noninteracting fermions and preserving the number of particles can be efficiently simulated classically.

Noninteracting Fermions: General Cases

- More generally, quantum computation involving noninteracting fermions need not conserve particle number, as long as it conserves particle parity.
- The more general case uses creation and destruction operators $c_{2j} = a_j + a_j^\dagger$ and $c_{2j+1} = -i(a_j - a_j^\dagger)$, satisfying $\{c_k, c_l\} = 2\delta_{kl}$.
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- Terhal and DiVincenzo proved this can also be classically efficiently simulated.
- Furthermore, complete von Neumann measurements also are not sufficient to bring the sysem to full universality, but it is for bosons—KLM (2007)

Universal Fermionic Quantum Computation

A universal set for quantum computation using interacting fermions was given by Bravyi and Kitaev (2000) for local fermion modes, consisting of the following unitaries:

- $\exp\left(i\frac{\pi}{4}a_0^\dagger a_0\right)$ (potential term)
- $\exp\left(i\frac{\pi}{4}(a_0^\dagger a_1 + a_1^\dagger a_0)\right)$ (tunnelling term)
- $\exp\left(i\frac{\pi}{4}(a_1 a_0 + a_0^\dagger a_1^\dagger)\right)$ (superconductor interaction term)
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Beenakker, DiVincenzo, Emary, and Kindermann in 2004 showed that allowing for charge detection in noninteracting fermions gives full quantum computation. Constucted a CNOT using beamsplitters, charge detectors, and one ancilla.

References

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