

Johnson and Shot Noise: Determinations of k , Absolute Zero, and e

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The phenomena of Johnson and shot noise in electrical circuitry are drawn (respectively) from statistical and quantum physics, and so they are ultimately governed by the values of fundamental physical constants. In a series of experiments, we measure the thermal Johnson noise of a resistor to determine Boltzmann's constant k and the centigrade value T_0 of absolute zero. We then measure the quantum shot noise of a photodiode current to determine the electronic charge e . We find $k = (1.361 \pm 0.026_{\text{rand.}} \pm 0.081_{\text{syst.}}) \times 10^{-23}$ J/K, $T_0 = (-274.3 \pm 9.3)^\circ\text{C}$, and $e = (1.870 \pm 0.031) \times 10^{-19}$ C. The general agreement between the behavior of the noise and their theoretical explanation is indicative of their fundamental nature.

I. THE THEORIES OF FUNDAMENTAL ELECTRONIC NOISE

I.1. Johnson-Nyquist Noise

In 1928, a remarkable pair of articles published in the *Physical Review* by J. Johnson and H. Nyquist described a link between electrical circuits and the atomic nature of matter. Nyquist's paper proposed a theory of how thermal fluctuations in the electromagnetic field cause voltage noise across a resistor, while Johnson devised an experiment precisely exhibiting this noise. Together, the two papers represent both a deepened understanding of statistical physics as well as an experimental breakthrough in determining its consequences.

Following Nyquist, consider two resistors of resistance R attached together by wires. The current I and voltage V in this circuit follows $I = V^2/2R$. The (differential) average power is therefore $d\langle P \rangle = d\langle V^2 \rangle/4R$.^[1]

On the other hand, using the equipartition of energy in the electromagnetic field at (absolute) temperature T , each mode of vibration with frequency f has an average energy kT . Thus, the average power in a small frequency band df of the electromagnetic field is $d\langle P \rangle = kT df$.

Equating these two results for the circuit in question, we arrive at Nyquist's result that

$$d\langle V^2 \rangle = 4kRT df. \quad (1)$$

Integration over any frequency range thus relates the mean square voltage to the thermal parameters k and T . A slight modification is necessary for an RC circuit with capacitance C —the resistance becomes

$$R(f) = \frac{R}{1 + (2\pi RCf)^2}.$$

Taken with Equation 1, this governs the behavior of Johnson noise in an RC circuit. In this lab, we relate the amplified values of $\langle V^2 \rangle$ in a filtered frequency band to the values of R and T . See Section II.3 for the governing equation after amplification and filtering.

I.2. Shot Noise

Millikan's oil drop experiment in 1909 confirmed the quantization of the electron charge e . Since electrons are the principal charge carriers in electrical circuits, this implies a fundamental noise due to the random event of an electron "passing through" the circuit, termed *shot noise* by Schottky. A complete treatment is difficult and unnecessary, so we present here a brief heuristic presentation of the theory following [2].

Consider a photodiode circuit where the current is due to emission of single electrons by photoelectric effect. Suppose in a time interval T , N electrons are emitted, at times t_n . The current contribution due to this electron will have a direct contribution as well as a fluctuating component. We can expand the fluctuating component J_n in a Fourier sum

$$J_n(t) = \frac{2e}{T} \sum_{m=1}^{\infty} \cos \frac{2\pi m(t - t_n)}{T}.$$

The alternating component $I(t)$ of the *total* current is yet another sum over all $J_n(t)$. However, we are interested in $\langle I^2 \rangle$, and it is evident that for large enough N , the phases of the cosine terms are randomized (due to the quantum nature of the photoelectric effect), and so the average contribution of the square of each is $1/2$. Thus, each mode m contributes $N \cdot 2e^2/T^2$ to $\langle I^2 \rangle$. With the frequency $f = m/T$, there are $T df$ modes in a small frequency band df , so

$$d\langle I^2 \rangle = \frac{2e}{T} \left(\frac{Ne}{T} \right) T df = 2eI_0 df,$$

where $I_0 = Ne/T$ is the average value of the DC component of the total current. Thus, the (differential) mean square AC voltage across a resistor R is

$$d\langle V^2 \rangle = 2eV_0 R df, \quad (2)$$

where $V_0 = I_0 R$ is the average DC voltage. In this lab, we use an amplified, filtered photodiode circuit to measure $\langle V^2 \rangle$ against V_0 . See Section II.3 for modifications to Equation 2 due to amplification and filtering.

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II. EXPERIMENTAL SETUP

II.1. Johnson Noise Experiment

The first experiment we perform is of Johnson noise. We use a prepared metal box labelled “Johnson #1”, which has alligator clips for attaching a resistor and two switches SW1 and SW2. SW2 connects the resistor to banana ports for resistance measurements using a digital voltmeter. SW1 shorts the resistor to allow measurement of the line noise. Thus, for a typical measurement, SW2 is off, while SW1 is alternately switched on and off.

N.B.: We note the jacks for SW2 are faulty—they require jiggling to obtain a stable measurement. This is not much of an issue for the voltmeter, but it causes severe problems when we attempt to use an LCR meter to obtain the capacitance of the system. Values obtained are erratic and sometimes negative—we extract the capacitance from the data instead (see Section IV.1).

The box is connected to an SRS SR560 preamplifier by a twisted, foil-shielded pair of cables, into inputs A and B, both on AC coupling. For a Johnson noise experiment, the gain is set to 1000, with the low frequency bandpass set to 300K, and the high frequency bandpass off (i.e., DC). We use the 6dB roll-off and set the gain mode to low noise. When making changes to the setup, overloading commonly occurs, which requires grounding the amplifier and resetting the overload.

We take the 50 Ω output of the amplifier and pass it through a Kron-Hite 8-pole band-pass filter, with a frequency range of $\sim 1\text{kHz}$ to $\sim 50\text{kHz}$. We use the positive input and cap the negative input.

The output of the filter is then fed into a Rigol digital oscilloscope, on which we make our measurements. For a Johnson noise measurement, we typically use 500 μs time divisions and $\sim 2.50\text{mV}$ voltage divisions. We turn on bandwidth limit for the channel on which we measure noise. The oscilloscope is positioned at least 5 feet from the resistor, to prevent electrical interference.

N.B.: We note it is not advisable to use the RMS “quick measure” option of the Rigol scope to measure RMS voltages. It gives only 3 digits of precision and seems to depend on the divisions in use. Thus, we instead use a flash drive to record CSV scope traces, and process the voltage data manually.

The only setup differences between the resistance and temperature parts of the experiment (see Section III.2) are in the placement of the box. For resistance measurements, the box is kept upright and we cover the resistor with a metal beaker to prevent electrical interference, only removing it to change resistors.

For the temperature measurements, the beaker is removed, and the box is inverted into an insulated bowl of liquid nitrogen (for the low temperatures) or into an oven (for the high temperature). The power supply to the oven causes electrical interference when its cord passes across the box, but moving it away and below the lab bench eliminates the problem. We also wrap the entire length of the power supply cable in foil.

II.2. Shot Noise Experiment

The shot noise experiment uses another box, labelled “Shot Noise #1”. This box contains a photodiode circuit with a variable-intensity light source, as well as an amplifier circuit, capable of ~ 10 gain. We are primarily concerned with the voltage across a 475k Ω resistor due to the photocurrent. The light source varies the intensity of the photocurrent, while the amplifier circuit has two stages, giving the DC voltage of the photocurrent and the amplified voltage across the resistor, respectively. The full circuit can be found in [2], but we will not need it.

On the box, there are three switches controlling the (battery) power to the lightbulb, the photodiode, and the amplifier circuit, as well as a (nonlinear) knob to adjust the bulb intensity. The two stages of the amplifier circuit end in BNC connectors. There are also two banana plug ports for checking the current across the bulb. The bulb uses a current of approximately 87mA, and the ports are shorted for a shot noise experiment. There is also a “test-in” port for calibration signals (see Section III.1), and it is shorted for shot measurements.

The stage 1 output is fed into an Agilent 6 $\frac{1}{2}$ -digit multimeter, set to measure the true RMS DC voltage. The stage 2 output is fed into the SRS preamplifier, with almost the same settings as before, except that we only use channel A on AC coupling, and the gain is set to 100 instead of 1000. Out of the preamplifier, we feed the signal through the filter, and then to both a multimeter and the Rigol oscilloscope. We primarily take data off the oscilloscope but use the multimeter for reference. Scope settings are similar to those for Johnson noise, except the voltage divisions are changed to about 250mV.

II.3. Signal Chain Gain and Filtering

Because the signal voltages are amplified, we modify Equations 1 and 2 by changing df to $g^2(f) df$, where we define $g(f)$ to be the ratio of the RMS voltage of a sine wave of frequency f input to the signal chain against the RMS voltage of the amplified, filtered sine wave at the end of the signal chain. We define the gain integrals[3]

$$G_1(R, C) = \int_0^\infty \frac{g^2(f)}{1 + (2\pi RCf)^2} df$$

and $G_2 = \int_0^\infty g^2(f) df$ for the Johnson and shot noise experiments, respectively. To calculate these integrals given measured values of $g(f)$, we integrate numerically (trapezoidal rule) over the range of the band-pass filter, after which they are considered to be zero. Thus, integrating the governing equations 1 and 2 under these considerations, we obtain, respectively,

$$\langle V^2 \rangle = 4kRTG_1(R, C) \quad (3)$$

$$\langle V^2 \rangle = 2eV_0RG_2 \quad (4)$$

These two equations describe the behavior of Johnson and shot noise for our particular experimental setup.

III. PROCEDURES AND DATA

III.1. Calibration of Signal Chain

For the Johnson noise experiment, we calibrate the signal chain by removing the box and feeding in 20mV RMS sine waves from an Agilent function generator through a Kay attenuator set to 26dB and into channel A of the preamplifier, switching off channel B. We also feed the attenuated signal into CH2 of the scope to measure the RMS voltage of the signal into the chain; we then take the output of the filter and feed that into CH3 of the scope, to measure the signal out of the chain.

We use the frequency set $\{0.8, 0.9, 1.0, 2.0, 3.0, 4.0, 5.0, 15, 25, 35, 45, 50, 55, 60, 65, 70, 75, 80, 90, 100\}$ kHz. We take two scope traces at each frequency and average the RMS values to obtain the input and output RMS voltages, from which we compute $g(f)$. A plot of $g(f)$ is shown in Figure 1.

For the shot noise experiment, we use nearly the same setup, except the input to the signal chain now goes to the “test-in” port of the box, and we cap the stage 1 output. We use the same frequency and test signals, as well as attenuator settings. The gain values are generally slightly lower than that for Johnson (about 0.9 the amplification), but has the same shape (i.e., filtering behavior).

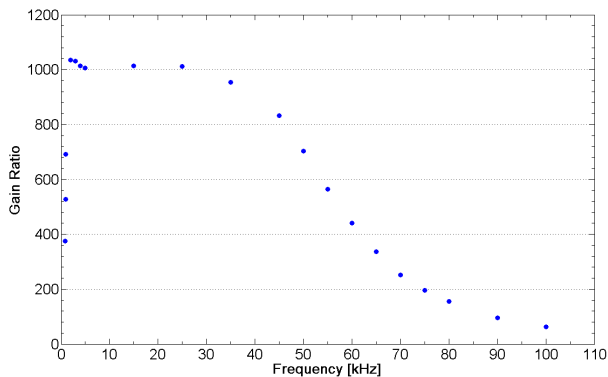


FIG. 1. The gain ratio for Johnson noise calibration. Note the sharp cutoff after 50kHz, which allows us to integrate over just this range. Error bars are present but are very small.

III.2. Measurements of Johnson Noise

There are two parts to the Johnson noise experiment. In the first part, we work at room temperature and vary the resistors used. In the second part, we work with the same (500k Ω) resistor and vary the temperature. We measure temperature and resistance using a digital voltmeter (the former in conjunction with a thermocouple).

In the first part, we pick the resistor set $\{10, 51, 100, 200, 500, 729, 880, 1000\}$ k Ω (errors range from 0.01 to 1 k Ω). The 1000k Ω resistor was formed from two 500k Ω resistors in series. We measure room temperature during this experiment to be $T = (23.6 \pm 0.3)^\circ\text{C}$.

As an illustration of this procedure, we provide one of the scope traces of the Johnson noise in Figure 2.

For each resistance, we take five scope traces of the resistor, alternating with five of the box set to shorted. Although we do not present the explicit RMS voltages obtained, we note they ranged from about 2.9mV for the 10k Ω resistor to about 7.3mV for the 1000k Ω . The RMS voltages of the shorted box remained constant throughout, averaging about 1.3mV.

In the second part, we invert the box into a bowl of liquid nitrogen, immersing the resistor and doing the same pattern of measurements. The thermocouple does not work at liquid nitrogen temperature, so we simply take this value to be 196.0 $^\circ\text{C}$.

Next, we use the oven to attain temperatures up to 150 $^\circ\text{C}$. The interior of the oven is lined with foil, and the box is inverted into the oven, preventing air flow. We insert the thermocouple into the oven and dial the power supply to sweep around temperatures of interest.

We pick the target temperature set $\{23, 50, 70, 100, 125, 150\}$ $^\circ\text{C}$ and record the temperature each measurement. The average temperature is used, with its associated error. The RMS voltages ranged from about 4.3mV for liquid nitrogen up to about 9.4mV at 150 $^\circ\text{C}$. The RMS voltage of the shorted box was again constant, averaging about 1.5mV. The temperature variation was typically about 1 $^\circ\text{C}$.

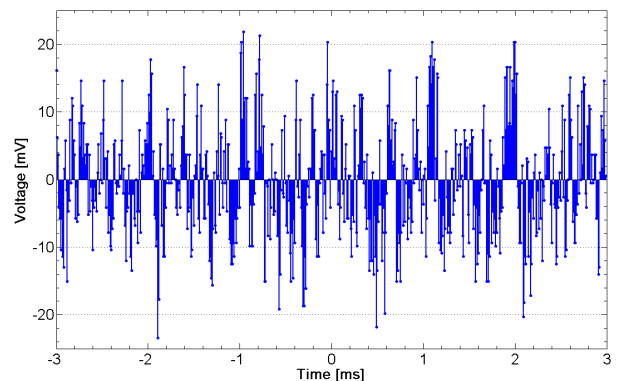


FIG. 2. A scope trace of Johnson noise across a 500k Ω resistor at 125 $^\circ\text{C}$. The trace of shorted line noise has smaller magnitude, indicating this signal is indeed Johnson noise.

III.3. Measurements of Shot Noise

To measure shot noise, we use various bulb intensity settings and measure the DC voltage from stage 1 (twelve single samples) and take about five scope traces from stage 2. The average value of the DC voltage is then taken to be V_0 .

If we measure the knob settings from 0 to 100, we approximately used the measurement set $\{0, 40, 50, 55, 60, 70, 75, 80, 85, 90, 95\}$, which correspond to DC voltages of about $\{0.002, 0.01, 0.05, 0.1, 0.4, 1, 2, 3, 5, 7\}$ V, respectively. We note a small downward drift in the DC currents, due to unknown reasons, and although we take it into account, it does not affect our results much. The RMS voltages range from about 120mV to about 280mV.

IV. ANALYSIS OF DATA

IV.1. Johnson Noise Experiment

Before jumping into the analysis, we describe how to obtain the capacitance C . As indicated in Section III.1, measurement of capacitance is not possible, so we want to extract the value from the data instead. To do this, we note that according to Equation 3, $k = \langle V^2 \rangle / 4RTG_1(R, C)$ should yield a constant for the resistance data. Hence, we look for the value of C which minimizes $\Delta k / \langle k \rangle$ (the standard deviation over mean). Testing values of C across the pF range, we find the minimum occurs at $C = (65.6 \pm 0.6)$ pF. We therefore take this to be the capacitance of our Johnson setup.

In the following plots, the error bars represent errors propagated from the RMS voltage determinations, the constant parameter (T or R for temperature or resistance measurements), and in the gain integrals, with the majority of the error from the RMS voltages.

For the resistance measurements, we fit the $\langle V^2 \rangle$ values (after subtracting that due to the line noise) obtained from squaring the RMS voltages against R according to

$$\frac{\langle V^2 \rangle}{4TG_1(R, C)} = kR \quad (5)$$

as given by Equation 3. (Note that we have to assume the correct value of absolute zero to plot this, as T is an absolute temperature.) The resulting data and fit is shown in Figure 3.

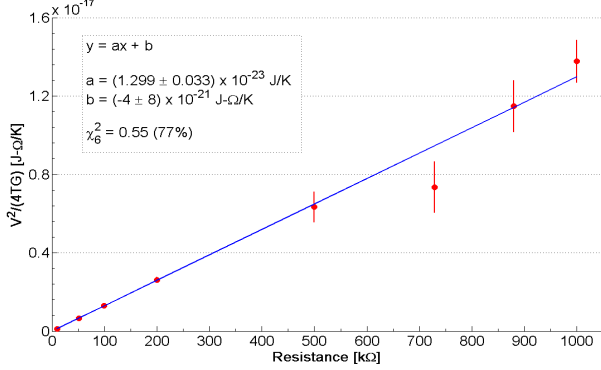


FIG. 3. Linear fit to Johnson noise against resistances.

From the fit, we find $k = (1.299 \pm 0.033) \times 10^{-23}$ J/K. The fit also suggests acceptable agreement with the model, with a reduced $\chi^2_6 = 0.55$ (77% confidence). We also note the intercept is zero, at $b = (-4 \pm 8) \times 10^{-21}$ J-Ω/K, as required by Equation 5.

For the temperature measurements, we fit the $\langle V^2 \rangle$ values obtained from squaring the RMS voltages against T_c according to the model

$$\frac{\langle V^2 \rangle}{4RG_1(R, C)} = k(T_c - T_0) \quad (6)$$

as given by Equation 3. Here, T_c denotes the temperature measured in Celsius and T_0 the value of absolute zero in Celsius. The resulting data and fit is shown in Figure 4.

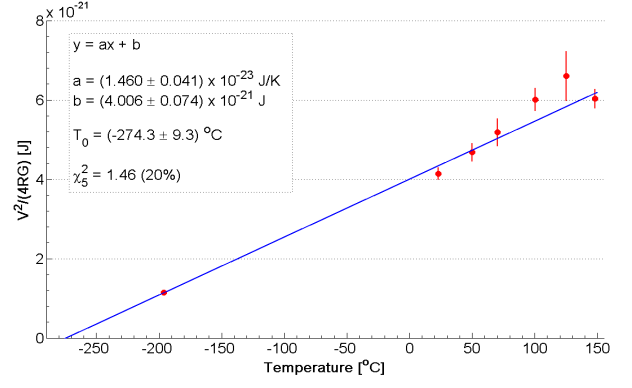


FIG. 4. Linear fit to Johnson noise against temperatures.

From the fit, we find $k = (1.460 \pm 0.041) \times 10^{-23}$ J/K. At the same time, we also obtain an estimate of $T_0 = -b/a = (-274.3 \pm 9.3)^\circ\text{C}$. The fit is also reasonably linear, with a reduced $\chi^2_5 = 1.46$ (20% confidence).

Because we have two different experiments yielding estimations on k , we take as our best estimate the average value $k = (1.361 \pm 0.026) \times 10^{-23}$ J/K, with a systematic uncertainty given by the half the difference, at $\pm 0.081 \times 10^{-23}$ J/K.

IV.2. Shot Noise Experiment

In the shot noise experiment, we fit the measured values of $\langle V^2 \rangle$ against the DC voltage V_0 , according to

$$\langle V^2 \rangle / 2RG_2 = eV_0 \quad (7)$$

as given by Equation 4. Again, we propagate the errors in the RMS voltages and the gain integral. We also examined the error in V_0 (including the systematic effect of the drift noted in Section III.3); however these are very small. We plot the data and fit in Figure 5 below.

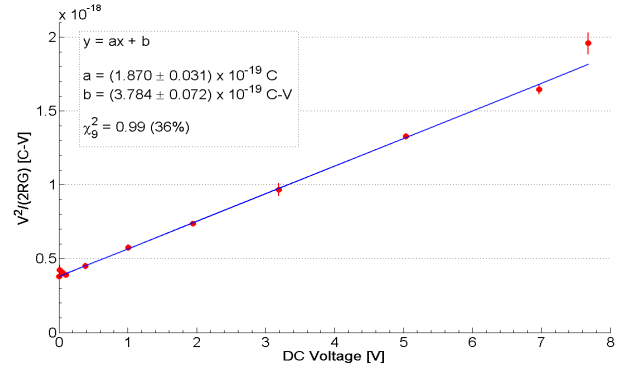


FIG. 5. Linear fit to shot noise against DC voltages.

From the fit, we obtain an estimate $e = (1.870 \pm 0.031) \times 10^{19}$ C. The reduced $\chi^2_9 = 0.99$ (36% confidence). Although the fit is good, we note there is a slight offset b , which is about a tenth of the scale. This suggests a source of systematic error. We have examined the possibility of a DC offset to the AC RMS voltages, but did not find any. The source of the slight offset remains difficult.

V. CONCLUSIONS

In general, the variation of Johnson noise with resistance and temperature is in agreement with Nyquist's theoretical predictions. From this, we are able to obtain an estimate of Boltzmann's constant of $k = (1.361 \pm 0.026_{\text{rand.}} \pm 0.081_{\text{syst.}}) \times 10^{-23} \text{ J/K}$. Compared to the correct value of $1.381 \times 10^{-23} \text{ J/K}$, this is about a 1.5% error (although with a 5.8% uncertainty). We also estimated the value of absolute zero quite well, at $T_0 = (-274.3 \pm 9.3)^\circ\text{C}$, which is a 0.4% error (although with a 3.4% uncertainty).

The shot noise experiment is slightly more off the mark. Although the agreement with the model is quite good, the estimated value of $e = (1.870 \pm 0.031) \times 10^{-19} \text{ C}$ compared with the actual value $e = 1.602 \times 10^{-19} \text{ C}$ represents a 17% error. This, together with a slight systematic offset of the data, suggests a small source of noise we have not accounted for.

Nevertheless, these results demonstrate remarkable agreement with theory for such sensitive experiments. The fact that we can measure these fundamental physical constants with a table-top experiment is a clear demonstration of the role that even noise plays in connecting the macroscopic and microscopic realms of physics.

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 - [2] M.I.T. Junior Lab, "Johnson Noise and Shot Noise: The Determination of the Boltzmann Constant, Absolute Zero Temperature and the Charge of the Electron," (2011).
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