Errors and decoherence in the quantum Schur transform

Edwin Ng*

Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139

(Dated: May 16, 2012)

Abstract

We examine the effects of errors and decoherence in the quantum Schur transformation. We utilize the spectrum estimation protocol based on the Schur transformation and observe the effects of various simple modifications to the basic circuit design, including rotation errors and decoherence. We focus on defining quantities that characterize the quality of the estimation.

* ngedwin@mit.edu

I. INTRODUCTION

A distinct signature of many quantum algorithms and protocols is the exploitation of nonlocal properties in a physical system in order to extract interesting information. Algorithms based on the quantum Fourier transform are perhaps the most prominant examples, where the inherent periodicity of a problem is stored nonlocally in a quantum register and later extracted by the transform.

In quantum information, on the other hand, there are a number of quantum protocols, such as universal quantum source coding, entanglement concentration, and density operator spectrum estimation, for which the relevant nonlocal property is not periodicity but rather the global *statistics* of the system. It turns out there is also a way utilize this information in a local way, represented by the Schur transform, for which an efficient circuit is known.[1]

These techniques are unequivocally quantum mechanical, and it is precisely for this reason that other characteristic quantum effects, like decoherence and noise, are particuarly important to a full understanding of the protocols. The effects of approximation, noise and decoherence in the quantum Fourier transform has been explored in [2], where the authors show, surprisingly, that approximate and imperfect circuits can perform better than the complete circuit in the face of decoherence, depending on the application at hand.

We present in this paper some approaches and preliminary results in understanding the Schur transform in an analogous way. We first briefly how the Schur transform arises from the formal notion of Schur duality before giving an overview of the circuits implementing the Schur transform. We then introduce the technique of spectrum estimation, which we ultimately use to study the effects of decoherence and noise on the performance of the Schur transform. We examine a few special cases through simulation, looking at the effects of faulty rotation gates and a simple model of decoherence.

II. SCHUR DUALITY AND THE SCHUR TRANSFORM

Schur duality deals with the representations of the unitary group $\mathcal{U}_d$ on d dimensions and the symmetric group $\mathcal{S}_n$ on n elements. A *representation* of a group G is a vector space V together with a homomorphism $\mathbf{R} : G \to \mathrm{End}(V)$, where $\mathrm{End}(V)$ is the set of operators on V. The group element g is represented by the operators $\mathbf{R}(g)$ on elements of V.

To ground the discussion in physical terms, let us consider a system consisting of n d-dimensional spins (so that $d = 2$ for qubits, for example). The vector space typically used to describe this system is the Hilbert space $H = \left(\mathbb{C}^d\right)^{\otimes n}$, with the spin rotation group SU(2) represented by rotation operators $\mathbf{R}(\theta)$. In order to obey the laws of quantum mechanics, we require representations on H to be unitary; to allow quantum computation, we ask that H be finite dimensional.

Our two main groups of interest, $\mathcal{U}_d$ and $\mathcal{S}_n$, in fact have natural physical meaning in this system—that of *local unitary operations* and *permutation of spins*. If $u \in \mathcal{U}_d$ and $\pi \in \mathcal{S}_n$, then we have the representations $\mathbf{Q}(u)$ and $\mathbf{P}(\pi)$, where

$$\mathbf{Q}(u)|i_1, i_2, \ldots, i_n\rangle = u|i_1\rangle \otimes u|i_2\rangle \otimes \ldots \otimes u|i_n\rangle \tag{1}$$

$$\mathbf{P}(\pi)|i_1, i_2, \ldots, i_n\rangle = |i_{\pi^{-1}(1)}\rangle \otimes |i_{\pi^{-1}(2)}\rangle \otimes \ldots \otimes |i_{\pi^{-1}(n)}\rangle \tag{2}$$

It is possible to decompose H into invariant subspaces under $\mathbf{Q}(u)$ and $\mathbf{P}(\pi)$ which do not mix for any u and π. That is, for some labels α and β, we can write

$$\left(\mathbb{C}^d\right)^{\otimes n} \cong \bigoplus_\alpha V_\alpha \cong \bigoplus_\beta W_\beta \tag{3}$$

under some appropriate bases. Similarly, the operators themselves decompose as

$$\mathbf{Q}(u) \cong \sum_\alpha \mathbf{q}(u) \quad \text{and} \quad \mathbf{P}(\pi) \cong \sum_\beta \mathbf{p}(\pi) \tag{4}$$

where $\mathbf{q}_\alpha(u)$ and $\mathbf{p}_\beta(\pi)$ act nontrivially only on V_α and W_β, respectively. If the decompositions are minimal, meaning that V_α and W_β are the smallest such invariant subspaces, then along with $\mathbf{p}_\alpha$ and $\mathbf{p}_\beta$ they are called *irreducible representations* (irreps) of $\mathcal{U}_d$ and $\mathcal{S}_n$.

Schur duality tells us that, for the unitary and symmetric groups, it is possible to choose these irreps to coincide. For any d and n, there is a set of labels λ such that

$$\left(\mathbb{C}^d\right)^{\otimes n} \cong \bigotimes_\lambda V_\lambda \otimes W_\lambda \quad \text{and} \tag{5}$$

$$\mathbf{Q}(u)\mathbf{P}(\pi) \cong |\lambda\rangle\langle\lambda| \otimes \sum_\lambda \mathbf{q}_\lambda(u) \otimes \mathbf{p}_\lambda(\pi) \tag{6}$$

Specifying a state in H by giving the labels λ and (say) the eigenvalues q_λ and p_λ yields the ket $|\lambda, q_\lambda, p_\lambda\rangle$, written in the *Schur basis*. The unitary change of basis $\mathbf{U}_s$ into the Schur basis is called the *Schur transform*. In terms of the computational basis states, we write

$$|\lambda, q_\lambda, p_\lambda\rangle = \sum_{k=1}^{n} \left[\mathbf{U}_s\right]_{i_k}^{\lambda, q_\lambda, p_\lambda} |i_k\rangle \tag{7}$$

The set of λ can be described as partitions of n into $\leq d$ parts, and we can write $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d$ where $\lambda_1 \geq \ldots \geq \lambda_d$ and $\sum_k \lambda_k = n$. For example, when $n = 3$ and $d = 2$, the only valid partitions are $(3, 0)$ and $(2, 1)$.[3]

It is useful to quote an example. We know from elementary quantum mechanics that for two spin-1/2 particles, we can write the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$ as the direct sum of the singlet and triplet subspaces, separated according to the total angular momentum quantum number j. In fact, we can identify λ with j and q_λ with the magnetic quantum number m_j, while p_λ is the permutation sign of the wavefunction. The Schur basis for this system is

$$|\lambda = (1,1), q_\lambda = 0, p_\lambda = 0\rangle$$
$$|\lambda = (2,0), q_\lambda = +1, p_\lambda = 0\rangle$$
$$|\lambda = (2,0), q_\lambda = 0, p_\lambda = 0\rangle$$
$$|\lambda = (2,0), q_\lambda = -1, p_\lambda = 0\rangle$$

Note that $\lambda = (2, 0)$ corresponds to $j = 1$ while $\lambda = (1, 1)$ corresponds to $j = 0$ (we can, in general, order λ to correspond to j). Also, since there is only one permutation symmetry for each λ, we can reuse the label 0 for both symmetric and antisymmetric wavefunctions (this will not be the case when $n \geq 3$). The matrix for this Schur transform is familiar:

$$[\mathbf{U}_s] = \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

III. CIRCUITS FOR THE SCHUR TRANSFORM

Efficient circuits for the quantum Schur transform were demonstrated by Bacon, Chuang, and Harrow in 2006,[1] while the explicit mathematical details of the construction forms the second half of Harrow's thesis.[3]

The general design of the circuit for the Schur transform is iterative, which can be seen from the recursive nature of addition of angular momenta. At the heart of the circuit is the Clebsch-Gordan (CG) transform $\mathbf{U}_{\mathrm{CG}}$, which is cascaded to form the Schur circuit. However, for the case of $d > 2$, the CG circuit is notationally cumbersome to describe and involves the use of the Wigner-Eckart theorem.[3] For the preliminary work done in this paper, we focus

only on the n qubit case where $d = 2$, so that we can directly apply elementary quantum mechanics to understand $\mathbf{U}_{\mathrm{CG}}$.

Suppose we have the value of j and m for $k-1$ spins and we want to add a new spin to the system, to get new labels j' and m'. Then the change of basis prescribed by addition of angular momentum is

$$|j, m, +1/2\rangle \mapsto \cos\theta\, |j, j+1/2, m+1/2\rangle - \sin\theta\, |j, j-1/2, m+1/2\rangle \tag{8}$$

$$|j, m, -1/2\rangle \mapsto \sin\theta\, |j, j+1/2, m-1/2\rangle + \cos\theta\, |j, j-1/2, m-1/2\rangle \tag{9}$$

where we define

$$\cos\theta = \sqrt{\frac{j + (m+s) + 1/2}{2j+1}} \quad \text{and} \quad \sin\theta = \sqrt{\frac{j - (m+s) + 1/2}{2j+1}} \tag{10}$$

Thus, at the output of the circuit, the first label is just j, the second label is $j' = j \pm 1/2$, and the third label is $m' = m \pm 1/2$. In terms of the Schur basis, the first label can be used to mark p_λ, the second label can be used to mark λ, and the third label to mark q_λ. The recursive structure is now evident. After we are done with the kth spin, we take the output j' and m' to use as new inputs to another CG circuit, along with a $(k+1)$th spin. A schematic for this change of basis as quantum circuit is shown in Figure 1, but the change of basis formula suffices for our needs. A schematic for the full Schur transformation based on this procedure is shown in Figure 2

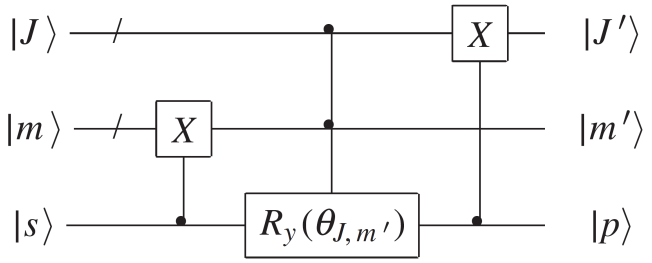

FIG. 1. A quantum circuit implementing the CG transform, mapping $|j, m\rangle|s\rangle$ to $|j', m', p\rangle$, adapated from [1]. Here, the controlled-X operators add the control to the target, and likewise, the rotation gate rotates the target as a function of the controls. Note that the convention in this figure is slightly different from the one used in this paper. $|p\rangle$ is our $|j\rangle$, and since this label is our *first* output, the last gate should have target and control reversed, with the output correspondingly labelled $|p\rangle$, $|m'\rangle$, and $|j'\rangle$ from top to bottom.

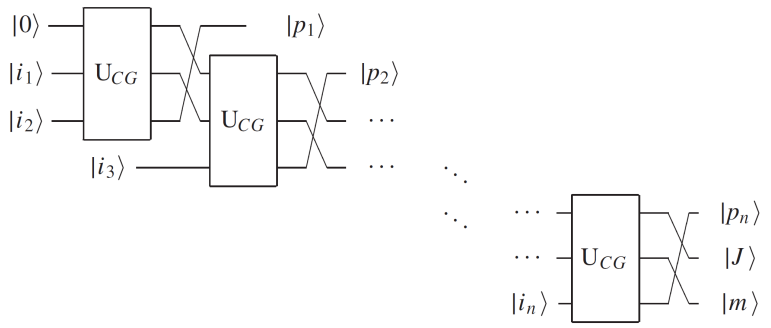

FIG. 2. A quantum circuit implementing the Schur transform as a cascade of CG transforms, mapping $|i_1, i_2, \ldots, i_n\rangle$ to $|p_1, \ldots, p_n\rangle|j\rangle|m\rangle$, adapated from [1]. The value of $|j\rangle$ in the first iteration is the initial value of j for a single spin, which would be $|0\rangle = |j = 1/2\rangle$. Note again the notational differences between using the top vs. bottom wires as the label for our $|p_k\rangle$.

IV. SPECTRUM ESTIMATION

One relatively straightforward application of the Schur transform is the estimation of the spectrum of a mixed state density operator.[4]. We will use this particular application as the context in which to study the effects of noise and decoherence in the circuit.

Suppose we are given n preparations of the single-qubit mixed state ρ, with a spectrum $\{p, 1-p\}$, (take $p > 0.5$ for convention). We are not told the basis under which the ρ is diagonal, but we wish to find the spectrum. One way to do this is to perform the Schur transformation on the state $\rho^{\otimes n}$ and measure the λ label, or, equivalently, the total angular momentum. We then relate this to the partition $\lambda = (\lambda_1, \lambda_2)$ and make the estimation

$$\{p, 1-p\} \approx \left\{ \frac{\lambda_1}{n}, \frac{\lambda_2}{n} \right\}$$

As n gets larger, the increasing number of partitions allow us to approximate p and $1-p$ better. Furthermore, as n get larger, the probability that a measurement at the end of a Schur transform yields the closest available answer becomes higher. Therefore, this spectrum estimate is exact as $n \to \infty$.

A plot of the estimate p by observing the highest probability for various n is shown in Figure 3, with the true spectrum $p_0 = 2/3$. To take into account the fact that the closest partition has only high but not unit probability, we plot in Figure 4 the "error" ratio $\delta p/P$, where we define the relative error $\delta p = |p - p_0|/p_0$ and P to be the P of obtaining p.

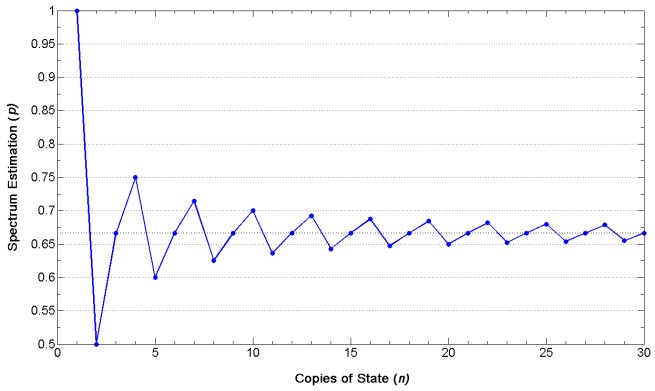

FIG. 3. The spectrum estimation of n copies of a single-qubit mixed state prepared with a fixed proportion $p_0 = 2/3$ in the state $|0\rangle$ and $1 - p_0$ in the state $|1\rangle$. The value of p plotted is the estimation using the partition λ (or equivalently, j) with the dominant measurement probability taken as p. Note that the asymptotic approach to the correct answer is due to the increasing number of partitions yielding closer integer approximations to p_0.

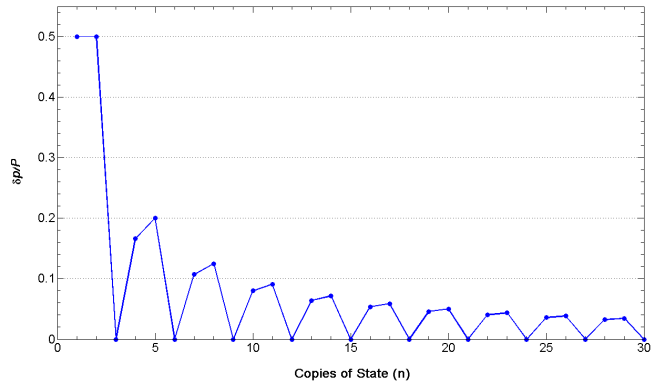

FIG. 4. The error in the spectrum estimation in Figure 1, defined to be the relative error δp of the dominant partition λ over the probability P of getting λ. Note that the asymptotic approach to zero is due to both the partitions approximating the correct answer and to the probability of that answer being more dominant.

V. ROTATION ERRORS

One simple attempt to change circuit is to examine the effects of adding a systematic error to the rotation gate $R_y(\theta)$ in the CG transform, taking $\theta \mapsto \theta + \phi$ for $\phi \ll \theta$. It turns out that this leads to a serious problem where states with $j < 0$ or $m > j$ grow to have

non-zero probability. To circumvent this, we do not apply the effects of ϕ when either of $\cos\theta$ or $\sin\theta$ are zero. The effects of this systematic error are shown in Figure 5 below for various values of ϕ, given in terms of $\delta p/P$.

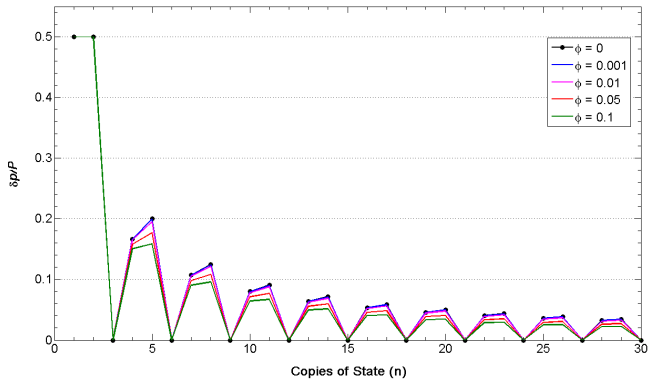

FIG. 5. The error in the spectrum estimation when introducing various systematic rotation errors ϕ. The parameters are the same as in Figure 4.

Note that the answers p as a function of n do not change as a result of this systematic offset. For each value of ϕ, the plot of the estimate p as a function of n is precisely the same as in Figure 3, which suggests that spectrum estimation is robust. Of course, this is bad news if we are interested in using the spectrum estimation to characterize changes in the Schur circuit due to errors.

Also of immediate note is the fact that the ratio $\delta p/P$ actually *decreases* as ϕ *increases*. This is not the expected behavior of systematic errors, which should lead to overall poorer behavior. Rather, we find that the probability of getting the closest partition λ actually increases as ϕ increases. It is possible that this is due to the restriction on the states, where we do not apply the systematic error to prevent obtaining nonphysical basis states. If this is so, then the effect seen in Figure 5 is artificial and not of interest.

VI. DECOHERENCE

Following the approach of [2], we made an attempt to introduce a simple model of decoherence, which amounts to multiplying the components of the wavefunction by a random, normally-distributed phase ϕ with mean 0. After the application of a single CG gate, we apply $e^{i\phi}$ for an independently generated ϕ to each component of the wavefunction, with the

interpretation that the CG circuit takes some time to perform, and a heat bath introduces random phases into the wavefunction, leading to decoherences.

However, the results of trying various standard deviations for ϕ are negative, in that no deviation in either the p or in the error of estimate $\delta p/P$ as a function of n were observed.

VII. CONCLUSIONS

The focus of this paper is to define an approach to studying noise and decoherence in the quantum Schur transform. We use the spectrum estimation protocol as our model, with the goal of examining the quality of the estimation as a result of various sources of errors and noise in our circuit.

However, it appears that the spectrum estimation protocol is too robust for this kind of analysis, in the sense that we obtain the same estimations with almost the same probabilities regardless of errors in the Schur transform. This does not say anything about the robustness of the Schur circuit itself, and it is suspected that the spectrum estimation method is a poor indicator of the quality of the Schur circuit.

Other constraining factors are the efficiency of simulating the Schur circuit. Using a simple approach, we are only able to simulate up to $n = 30$ qubits. It is not known whether simulating more qubits will yield any different results.

Additional topics for future work are to examine other protocols, with the goal of finding one sensitive enough to the Schur transform to be useful as a reliable indicator of the robustness of the circuit itself. The topic of approximation becomes interesting in the case of $d > 2$, which should yield much more complexity in behavior due to the classical calculation of the CG transform circuits.

---

[1] D. Bacon, I.L. Chuang, and A.W. Harrow, "Efficient Quantum Circuits for Schur and Clebsch-Gordon Transforms," Phys. Rev. Lett. **97** (2006), arXiv:quant-ph/0407082v4.

[2] A. Barenco, A. Ekert, K.-A. Suominen, and P. Törmä, "Approximate quantum Fourier transform and decoherence," Phys. Rev. A **54** (1996), arXiv:quant-ph/9601018.

[3] A.W. Harrow, *Applications of coherent classical communication and the Schur transform to quantum information theory*, Ph.D. thesis, Massachusetts Institute of Technology (2005),

arXiv:quant-ph/0512255v1.

[4] M. Keyl and R.F. Werner, "Estimating the spectrum of a density operator," Phys. Rev. A **54** (2001), arXiv:quant-ph/0102027.