# Noise and Approximations in the Quantum Schur Transform

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# Schur Duality: Representations

Consider a system of $n$ $d$-dimensional qudits. Schur duality deals with the representations of the unitary group $\mathcal{U}_d$ and the symmetric group $\mathcal{S}_n$ in $H = \left(\mathbb{C}^d\right)^{\otimes n}$.

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- *Local Unitary Operations*: Represented by operators $\mathbf{Q}(u)$ where $\mathbf{Q} : \mathcal{U}_d \to \mathrm{End}(H)$.

$$\mathbf{Q}(u)|i_1, i_2, \ldots, i_n\rangle = u|i_1\rangle \otimes u|i_2\rangle \otimes \ldots \otimes u|i_n\rangle$$

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- *Permutations of Qudits*: Represented by operators $\mathbf{P}(\pi)$ where $\mathbf{P} : \mathcal{S}_n \to \mathrm{End}(H)$.

$$\mathbf{P}(\pi)|i_1, i_2, \ldots, i_n\rangle = |i_{\pi^{-1}(1)}\rangle \otimes |i_{\pi^{-1}(2)}\rangle \otimes \ldots \otimes |i_{\pi^{-1}(n)}\rangle$$

# Schur Duality: Decomposition

- It is possible to decompose $H$ into subspaces invariant under the actions of $\mathbf{Q}(u)$ and $\mathbf{P}(\pi)$:

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- When subspaces are minimal, $\mathbf{q}_\alpha$ and $\mathbf{p}_\beta$ are the irreducible representations of $\mathcal{U}_d$ and $\mathcal{S}_n$ in $H$.

# Schur Duality: Theorem

For any $d$ and $n$, if $\mathbf{q}_\lambda$ and $\mathbf{p}_\lambda$ are irreps of $\mathcal{U}_d$ and $\mathcal{S}_n$, then

$$\left(\mathbb{C}^d\right)^{\otimes n} \cong \bigoplus_\lambda V_\lambda \otimes W_\lambda \qquad \text{and}$$

$$\mathbf{Q}(u)\mathbf{P}(\pi) \cong \sum_\lambda |\lambda\rangle\langle\lambda| \otimes \mathbf{q}_\lambda(u) \otimes \mathbf{p}_\lambda(\pi)$$

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This unitary change of basis $\mathbf{U}_S$ is the *Schur transform*:

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- *Example*: For two spin-1/2 qubits, $\lambda$ labels $j \in \{0, 1\}$, $q_\lambda$ labels $m \in \{-j, \ldots, +j\}$, and $p_\lambda$ labels symmetry of wavefunction.

# Application: Spectrum Estimation

A qubit is prepared in a mixed state with probabilities $\{p, 1-p\}$ in some unknown basis. Given $n$ copies of the state following this distribution, can we find $p$ and $1-p$?

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- Will measure correct partition whp. Becomes exact as $n \to \infty$.

# Efficient Schur Transform: Clebsch-Gordan Circuit

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- State of just the first qubit is $|j = 1/2, m = \pm 1/2, p = 0\rangle$.
- Now add the second qubit $|s\rangle$. Use the CG transform:

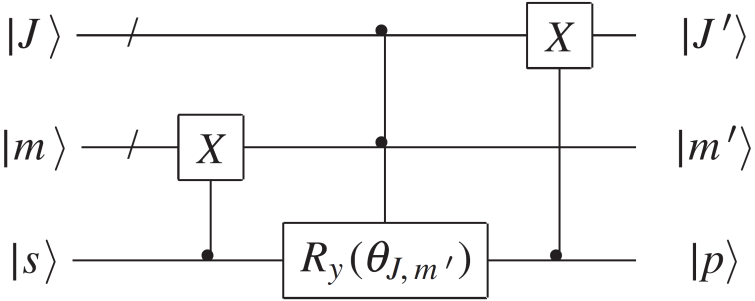
$$
\begin{aligned}
|j, m, +1/2\rangle &\mapsto \cos\theta\,|j, j+1/2, m+1/2\rangle \\
&\qquad - \sin\theta\,|j, j-1/2, m+1/2\rangle \\
|j, m, -1/2\rangle &\mapsto \sin\theta\,|j, j+1/2, m-1/2\rangle \\
&\qquad + \cos\theta\,|j, j-1/2, m-1/2\rangle
\end{aligned}
$$

where we define

$$
\cos\theta = \sqrt{\frac{j + (m+s) + 1/2}{2j+1}} \quad \sin\theta = \sqrt{\frac{j - (m+s) + 1/2}{2j+1}}
$$

# Efficient Schur Transform: Clebsch-Gordan Circuit

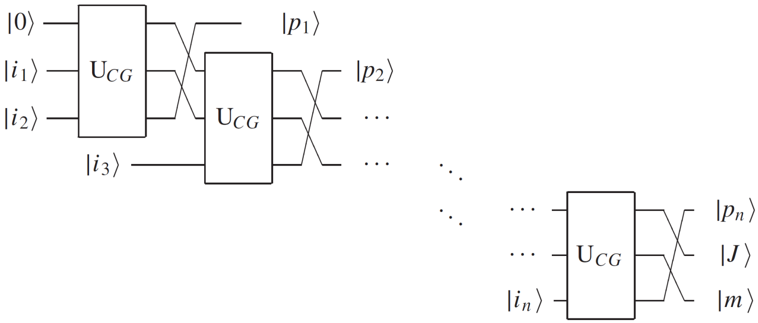
CG is a unitary operator. An efficient quantum circuit implementation is[1]




[1]From D. Bacon, I.L. Chuang, A.W. Harrow, Phys. Rev. Lett. **97** (2006).

# Efficient Schur Transform: The Schur Circuit

The full Schur circuit (for $n$ qubits) is a cascade of CG operations[2]




The $R_y(\theta)$ gate is approximated in polylog$(1/\epsilon)$, so the full Schur circuit is $O(n\,\text{polylog}(1/\epsilon))$.

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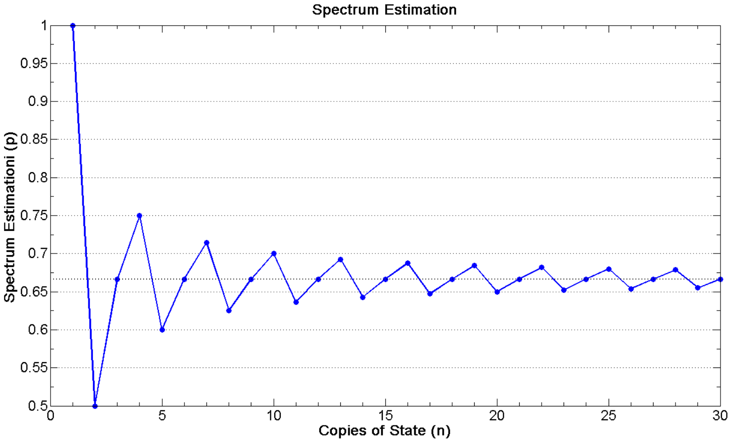
# Spectrum Estimation: Plots



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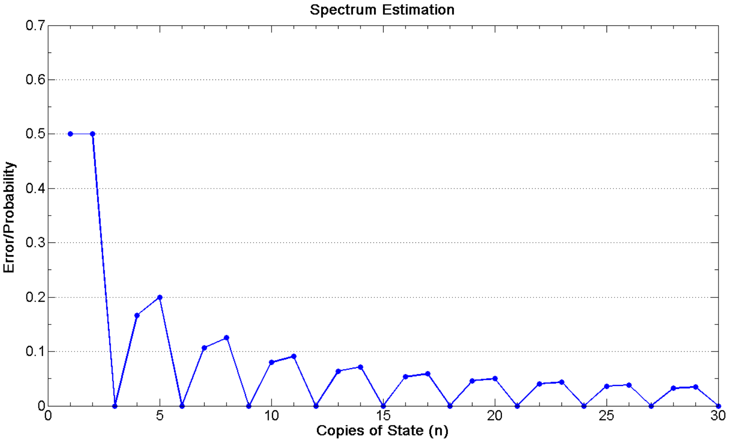

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- Causes problems when $j' < 0$ and $|m'| > j'$ obtain some probabilities due to $\phi$.
- Fix: Only apply $\phi$ when these cases do not apply.

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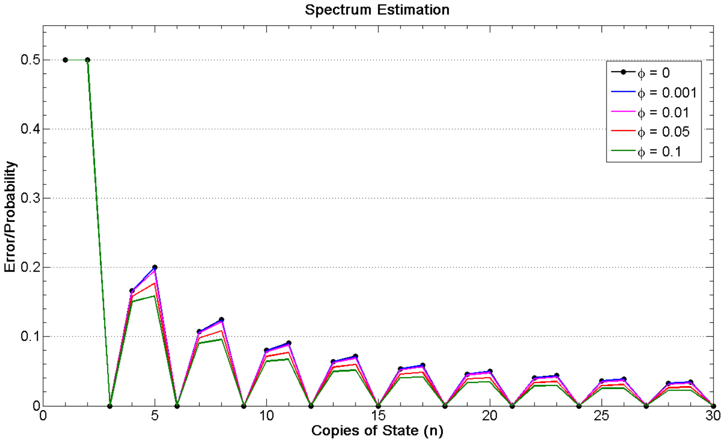

# Additional Topics

- Need to verify that this effect is not artificial, by simulating larger circuits and making noise larger.
- Can also introduce errors into the controlled-addition gate.
- More complexity and opportunity for approximate circuits for $d$-dimensional qudit case.
- What about other applications? Universal quantum source coding, entanglement concentration, collective decoherence. . .