Exploratory Experiment Proposal: NMR Superdense Coding with Quantum State Tomography

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In this proposal, we outline an MIT Junior Lab exploratory experiment on NMR quantum information processing, to be performed in Spring 2012, under the guidance of Prof. Paolo Zuccon.

I. INTRODUCTION

The implementation of simple quantum logic gates and algorithms are explored in Junior Lab Experiment 49, which introduces basic techniques in the field of NMR quantum information processing. Using just these techniques, it is already possible to observe many of the subtleties in harnessing the power of quantum information. Liquid-state NMR, however, exhibits several drawbacks stemming from the thermal nature of the system, and its feasibility for large-scale quantum computing is unlikely. Nevertheless, NMR remains a highly accessible model on which to test a wide variety of quantum algorithms and protocols, and the development of techniques in manipulating NMR systems remains central to understanding quantum information processing in general.[1]

In this spirit, we propose the following exploratory experiment on NMR quantum information processing. Building on the basic techniques in Experiemnt 49, we aim to develop the ubiquitous NMR technique of *quantum state tomography*, which can be used to obtain the full state of the system at the end of a computation. We will then apply this technique to examine a quantum protocol called *superdense coding*, which utilizes quantum mechanics to encode two bits of classical information by manipulating only one qubit. It turns out an almost identical experiment has already been performed in 1999 by Fang et al.,[2] and so our experimental procedures, results, and interpretations will be in close parallel to their work. Also of note is the closely related protocol of quantum teleportation, which involves three qubits and was implemented on NMR by Neilsen et al. in 1998.[3] A useful source for NMR techniques is an early article by Chuang et al.[4]

One subtlety present in Experiment 49 was the use of temporal labeling to create effective pure states from thermally mixed states. As explained below, however, the theory behind superdense coding also requires the explicit use of quantum entanglement. Interestingly, there has been significant controversy about the lack of entanglement in liquid-state NMR as late as 2001,[5][6] due to the ensemble nature of the experimental setup and the use of temporal labeling schemes. We have not been able to find any clear consensus on what kinds of algorithms and protocols are possible or impossible on an NMR computer given this lack.

In light of these subtleties, our objective will be to observe what happens when we attempt to adapt an entanglement-dependent protocol to the NMR model of computation. Although we do not expect the experiment to demonstrate true entanglement, it seems interesting to ask what would happen if we were to use the NMR model of computation to simulate not just simple algorithms like DJ or Grover, but protocols which explicitly use entanglement. Would they fail? Would they work, but require a different physical explanation of why they still work (e.g., effects of using an ensemble)?[7] We hope this experiment will, by partially answering these questions, yield some insight into the subtleties of experimental quantum information processing, both in the theoretical framework and in experimental techniques.

II. THEORY AND HYPOTHESIS

II.1. The Bell States

An *entangled state* of a two-qubit system AB is a state that cannot be written as the tensor product of a state of A with a state of B. In this experiment, we will be concerned with the "maximally entangled" *Bell states* (or *EPR states*), which we denote by

$$|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \qquad |\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$
$$|\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \qquad |\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

where we will let the first slot be qubit A and the second slot be qubit B. Clearly, these four states form a complete orthonormal basis for the Hilbert space of the two-qubit system AB. Note that their density matrices, however, are not diagonal in the computational basis.

II.2. Superdense Coding

Superdense coding is a quantum protocol that allows the transmission of two classical bits of information by applying operations to only one qubit.[1] Two qubits A and B are prepared in the Bell state $|\Phi^+\rangle$. The goal is to be able to encode four bits of classical information by performing only one-qubit operations on qubit A.

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The key to this protocol is that we can find four unitary operators acting on the state $|\Phi^+\rangle$ to produce the four mutually orthogonal Bell states. These operators are the identity I_A and variations of the Pauli matrices X_A, Y_A, and Z_A. The actions of these operators on qubit A on the initial state are:

$$\begin{aligned} I_A|\Phi^+\rangle &= |\Phi^+\rangle \qquad & X_A|\Phi^+\rangle &= |\Psi^+\rangle \\ iY_A|\Phi^+\rangle &= |\Psi^-\rangle \qquad & Z_A|\Phi^+\rangle &= |\Phi^-\rangle \end{aligned}$$

Measurement in the Bell basis then allows us to deduce which operation had been performed. Hence, if we adopt the protocol of applying these gates onto qubit A when sending 00, 01, 10, or 11, respectively, we can encode two bits of information by interacting with only one qubit.

II.3. Quantum State Tomography

In Experiment 49, the output of a computation was obtained from the FID by applying the readout operator $R_x = R_x(\pi/2)$ to the qubit being measured. Suppose we wish to measure qubit A, and the state of the system after the computation is ρ; then the FID is the function

$$V_A(t) = -V_0 \mathrm{tr}\left[e^{-iHt} R_{xA}\, \rho\, R_{xA}^\dagger e^{iHt}(iX_A + Y_A)\right]$$

From this FID, we can then obtain the state of qubit A. In a similar fashion, it is evident that we can also measure qubit B. More generally, it can be seen that this readout operator in fact gives information about the diagonal entries in ρ from the real part of the peak integrals of the FID.

However, in order to distinguish between the various Bell states involved in superdense encoding, we will also want to probe the off-diagonal terms in the density matrix. The reconstruction of the full density matrix in such a manner is called *quantum state tomography.*

The technique for quantum state tomography is described in Nielsen & Chuang.[1] By choosing a set of appropriate readout operators and measuring the FID for each one (keeping ρ the same), we arrive at a set of linear equations from which we can solve for the sixteen elements of the density matrix.

Thus, we want a set of readout operators $\{M_k\}$, which allows us to measure a set of associated FIDs

$$V_k(t) = -V_0 \mathrm{tr}\left[e^{-iHt} M_k\, \rho\, M_k^\dagger e^{iHt}(iX_A + Y_A)\right]$$

According to Exercise 7.45 in Neilsen & Chuang,[1] the following set of readout operators give sufficient information to reconstruct ρ:

$$\begin{aligned} \{M_k\} = \{&I \otimes I\,, R_x \otimes I\,, R_y \otimes I\,, I \otimes R_x\,, I \otimes R_y\,, \\ &R_x \otimes R_x\,, R_x \otimes R_y\,, R_y \otimes R_x\,, R_y \otimes R_y\} \end{aligned}$$

These readout operators move off-diagonal elements into the diagonal, so we can use the peak integrals of their associated FID to obtain information about the density matrix. Since it can be shown that FIDs from these readout pulses are sufficient to reconstruct the density matrix elements, we can reconstruct the initial density matrix by combining all the results.

Although the nine readout operators shown above are sufficient to reconstruct the density matrix, they may not be necessary. For example, measuring both the real and imaginary parts of the FID spectrum under the readout pulse R_x on both qubits yields eight elements instead of four. It has been shown that the number of readout pulses (i.e., experiments) we need to perform can in principle be reduced from nine to four.[8]

III. SETUP AND APPARATUS

This experiment will be performed on the same apparatus as the Junior Lab Experiment 49, in the MW session. Programming of pulse sequences can be done outside of lab hours, and, depending on the demand for the equipment, we may be able to run more time-consuming, repetitive measurements (such as the ones for state tomography) overnight.

The details of the experimental setup can be found in the labguide for Experiment 49, which is currently scheduled as the third one in our experimental line. Since this exploratory experiment is scheduled to be our fourth, we expect to be familiar with the equipment by the time we begin.

IV. EXPERIMENTAL PROCEDURES

IV.1. Calibration and Simple Gates

From performing Experiment 49, we expect to have a good determination on the values of the $\pi/2$-pulse widths as well as the J-coupling constant. This calibration data should then allow us to readily perform the various rotation gates R_x and R_y on both the carbon and proton. Furthermore, the controlled-NOT gate U_{cnot} was introduced as part of Experiment 49, which we will have occasion to use in this experiment.

We can also write the familiar single-qubit Pauli operators (up to an overall phase) in terms of the rotation gates in a simple manner:

$$\begin{aligned} iX &= R_x(-\pi/2) \\ iY &= R_y(-\pi/2) \\ iZ &= R_y(-\pi/2) R_x(-\pi/2) \end{aligned}$$

Finally, another gate we will be using is the single-qubit Hadamard operator. Up to an overall phase, this is given by[2]

$$iH = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = R_y(\pi/2) R_x(\pi)$$

We will assume that we have access to these gates in this experiment and will furthermore drop the unimportant overall phase when referencing these gates.

IV.2. State Tomography on an Effective Pure State

The first step is to ensure that the set of readout operators are indeed sufficient to perform quantum state tomography. To do this, we begin by analyzing the effective pure state $|00\rangle$, prepared via temporal labeling as in Experiment 49. We will then perform the nine readout experiments and attempt to reconstruct the density matrix. We expect the resulting entries to be zero except in the $|00\rangle\langle 00|$ component, as shown in V.1 below.

If the nine experiments do not yield a correct or interpretable result, or if it becomes evident that time becomes a constraint, we will try other schemes of state tomography, such as utilizing the imaginary components of the FID spectrum, which may reduce the set of readout experiments we need to perform.

IV.3. Preparation of the Precoding State

We prepare the precoding state $|\Phi^+\rangle$ using the gates H and U_{cnot}, as shown in the figure below. To perform superdense coding, we can then send this state directly to the encoding operation, as is done in the following part.

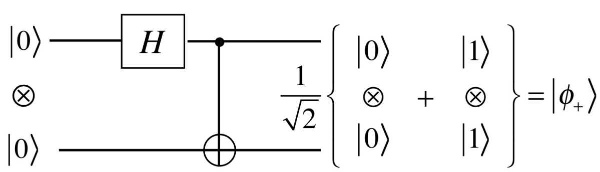

FIG. 1. A circuit which prepares the precoding state for use in superdense coding, starting from an initial effective pure state. (Illustration by P. Samutpraphoot)

However, before doing superdense coding, it is crucial that we first perform several instances of quantum state tomography on the precoding state, in order to obtain a quantitative understanding of the error associated with preparing the precoding state. Imperfections and errors in carrying out the preparation circuit inevitably leads to failure in the superdense coding protocol, and so an estimate of this contribution needs to be taken into account in order to interpret the results and error rates of the protocol. A theoretical prediction of the density matrix for the precoding state is shown in V.1 below.

Furthermore, we may also want to ask how long a Bell state such as $|\Phi^+\rangle$ lasts, as compared to the J-coupling time constant. To do this, we can add a delay between preparing the state and performing state tomography, and then later reconstruct the evolution of the density matrix as a function of time. This additional investigation may also lend some insight into the behavior of the pseudo-entangled states that we are working with in liquid-state NMR.

IV.4. Superdense Encoding

The next step is to test the superdense encoding protocol. We will adopt the following encoding scheme:

$$\begin{array}{ll} 00 : I_A & 01 : X_A \\ 10 : Y_A & 11 : Z_A \end{array}$$

where these operations apply only to qubit A (say, the proton) in the precoding state $|\Phi^+\rangle\rangle$. Depending on time constraints, we expect to repeat the encoding scheme several times for each classical message.

In this part, we will perform state tomography directly after the encoding, in order to assess the error associated with the encoding process. We expect, of course, to find density matrices corresponding to the Bell states, consistent with the encoding scheme. An example of such a density matrix, corresponding to $|\Phi^-\rangle\rangle$for the bit string 11, is shown in V.1 below.

IV.5. Superdense Decoding

After confirming that we can encode two classical bits, we continue on the decoding step of the protocol. One way to decode, as described in the theory for superdense coding, is to measure in the Bell basis. Another way to decode, however, is to apply the *inverse* of the circuit for preparing the precoding state $|\Phi^+\rangle\rangle$; more specifically, this is just the unitary circuit $H_A U_{\text{cnot}}$. Thus, the decoding circuit will take the Bell states (ignoring an overall phase) to the computational basis states:

$$\begin{array}{ll} |\Phi^+\rangle \mapsto |00\rangle & |\Phi^-\rangle \mapsto |10\rangle \\ |\Psi^+\rangle \mapsto |01\rangle & |\Psi^-\rangle \mapsto |11\rangle \end{array}$$

Since we expect the states after decoding to be in the computational basis, it is not strictly necessary to reconstruct the full density matrices in order to distinguish them. Nevertheless, we plan to perform state tomography, as a matter of consistency and to ensure that the states are in fact what we think they are. We expect the density matrices exhibit a single component along the diagonal (much like the effective pure state).

After obtaining enough repeated measurements to quantify the errors, we can also take more statistics by skipping the state tomography and simply encode and decode a source of classical bit strings. The observed error rates can then be explained using the detailed information about the reliability of each stage in the protocol.

V. EXPECTED RESULTS

Although this experiment is substantially more difficult than Experimet 49, we have reason to believe that it is nevertheless feasible. As mentioned before, Fang et al. performed an almost identical experiment in 1999 using almost the same equipment and obtained correct density matrices with only small deviations.[2] We therefore believe the protocol is possible to implement and that it is reasonable to hope for similar results. At a minimum, we want to attempt the preparation of the Bell states (in particular the precoding state $|\Phi^{+}\rangle$, and to at least develop a working procedure for state tomography.

Again, because of the controversy over entanglement in liquid-state NMR, we do not expect to be able to consider this an example of authentic superdense coding.[7] Some points worth considering in this respect:

- We do not create entangled state, nor pure states, in our NMR system. Even if the resulting averaged density matrices are correct, they do not represent the true states of the system at any time. The density matrices are constructed through temporal averaging and should rather be regarded as simulations of pure quantum states.

- In its most colorful form, superdense coding is intended to be employed in communication between two entangled qubit holders, say Alice with qubit A and Bob with qubit B, where Alice sends information by encoding onto her qubit and sending it over to Bob. In this experiment, however, we have full control over both qubits, and there is no explicit transfer of information between two parties.

Nevertheless, as stated Section I, our objective is not to verify entanglement in NMR but to develop relevant techniques in NMR spectroscopy in order to obtain insights into quantum information through studying the behavior of protocols like superdense coding in the NMR model of computation.

V.1. Theoretical Results of State Tomography

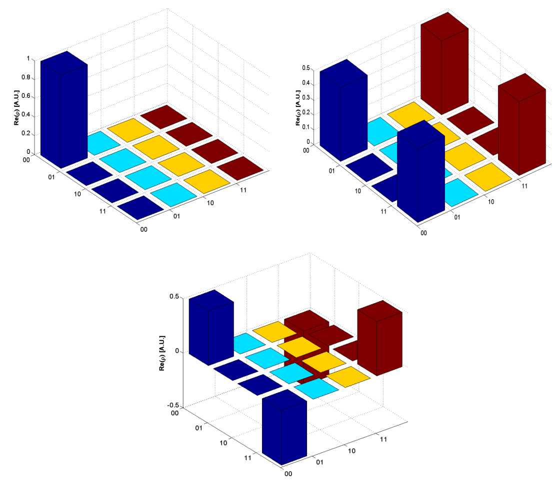

FIG. 2. State tomography on (left) an effective pure state, (right) precoding state and (bottom) an encoded state representing $|11\rangle$. (Illustration by E. Ng)

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