Screening with Network Externalities

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Abstract

We develop a model in which a profit-maximizing monopoly sells a product with positive network externalities and optimally screen buyers based on their influence and susceptibility. We characterize the optimal allocation for both the case of directed networks where each buyer’s influence and susceptibility are independent, and the case of undirected networks where the two are identical. In the case of directed networks, we show the optimal allocation can only depend on a buyer’s susceptibility and linear in virtual type (susceptibility) with quadratic intrinsic value. In the case of undirected networks, we disentangle the different effects of influence and susceptibility on optimal allocation and show with quadratic intrinsic value, the allocation is a linear combination of a buyer’s type and virtual type. Then we contrast the model with complete information pricing and pure screening and show that apart from the screening effects, positive network externalities increase each buyer’s allocation at the optimal selling mechanism. We also extend the model to accommodate for weak positive affiliation between a buyer’s influence and susceptibility, and the situation where influence and susceptibility are endogenous to the optimal allocation.

1 Introduction

For a variety of consumer products, one’s consumption choices are highly influenced by those either closely or remotely related to him. As a concrete example, a player not only gains personal enjoyment, but also derives benefits from interacting with close friends when playing an online game. As a result, a player’s consumption of a online game is likely to be higher if his friends are playing the same game. Similar stories can be given for online cloud

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storage platforms (e.g. Dropbox.com), movies and numerous utility software or apps (e.g. pdf readers). In other words, such consumer products exhibit positive network externalities.

Depending on the products of interest, the precise network effects may be different. For instance, in examples such as movies and utility software, the group of consumers that care about a buyer’s consumption may be very different from those he cares about. We refer to such interaction as asymmetric interaction. By comparison, in examples such as online games and cloud storage, the set of consumers that care about a particular buyer’s consumption are often precisely those he cares about, which we refer to as symmetric interaction.

As we turn to the seller side of the market, we often observe a monopolist seller (or at least a seller with significant market power) with some but not detailed network information. In our earlier examples, an online game or cloud storage provider may have some idea of the average level of interaction in the market, but may not know exactly how many friends a specific individual is affected by. Even in the standard setting with no network externality, it is well-known that a monopolist seller can employ different levels of price discrimination based on the information at hand (Pigou (1920)). Specifically, even with scanty information only on the population distribution, a monopolist seller can screen buyers based on their willingness to pay (Maskin and Riley (1984)). It is then an interesting question how the optimal screening mechanism works in the presence of positive network externalities, and how it differs from the case with no network effect.

The current paper serves as an attempt to address such questions. We focus on a profit-maximizing monopolist seller of a consumer product with positive network externalities and explore her optimal selling mechanism. There is a unit measure of ex-ante symmetric buyers with a common intrinsic value for the product. Apart from the (common) intrinsic value, buyers also derive benefits from the product through social interaction. We characterize network effects by two parameters: susceptibility and influence. Intuitively, a buyer’s susceptibility describes how many buyers his valuation for the product depends on, and a buyer’s influence shows how many other buyers’ valuations are affected by him. We examine interactions at an interim stage, so each buyer knows his own type (susceptibility and influence) and the common priors, while the seller only knows the common priors. We assume each buyer’s type is independently drawn, and in expectation, a buyer’s benefits through social interaction is proportional to his susceptibility.\(^1\) As mentioned earlier, depending on the products of interest, it may be more plausible to consider social interaction as either asymmetric or symmetric, and we study both alternatives. In our context, we model asymmetric interaction as the case where each buyer’s susceptibility and influence are (statistically) independent, and symmetric interaction as the case where the two are always identical. The game proceeds in two stages: in the first stage, each buyer’s private type (susceptibility and influence) is realized. The monopolist seller, knowing only the common prior, offers a set of contracts to the potential buyers. In the second stage, after observing the set of contracts, each buyer simultaneously takes one (or his outside option) from those on offer. Our goal is to identify the set of incentive compatible contracts that

\(^1\)We do not necessarily assume that a buyer’s susceptibility is independent of his influence.
maximize the monopoly’s expected profits.

Our first set of results characterize the profit-maximizing contracts with both asymmetric and symmetric interactions. With asymmetric interaction, we show only susceptibility matters for each buyer’s allocation at the optimal contracts. Moreover, the contracts are chosen such that each buyer’s intrinsic marginal value is linear in his virtual type (susceptibility). As a corollary, when the common intrinsic value is quadratic, the optimal allocation is linear in a buyer’s virtual type. With symmetric interaction, we show the optimal contracts are chosen such that each buyer’s intrinsic marginal value is a linear combination of his type and virtual type. As a corollary, when the common intrinsic value is quadratic, the optimal allocation is a linear combination of a buyer’s type and virtual type (plus a constant). Intuitively, the linear term in virtual type represents a buyer’s benefits from social interaction (minus his information rent) and captures the effect of susceptibility on the optimal allocation. On the other hand, the linear term in type captures the effect of a buyer’s susceptibility on the optimal allocation.

Our second set of results contrast the optimal screening mechanism with complete information benchmark and pure screening. The comparison with complete information benchmark shows similar insight to the case without network externality: when the seller has only distributional level network information and has to screen buyers, she treats each buyer’s virtual type as his actual type and offers contracts accordingly. As for pure screening, we refer to the situation where the buyers and the seller view the positive network externalities as exogenous. In particular, the seller does not actively take into account the positive effects of a higher allocation on the network externalities. We show that given the same level of network externalities, “network screening” (i.e. the situation where the buyers and the seller endogenously take into account the effects of the contracts on the positive network externalities) offers a higher allocation to each type of buyer at the optimal contracts. As a side note, we observe that the optimality condition of the network screening mechanism implies that if the buyers’ and the seller’s exogenous conjecture of the network externalities is correct, the profit under pure screening is always smaller than under network screening.

Then we look at the special case where the common intrinsic value is quadratic in each buyer’s consumption, which is often assumed in the network literature (see for instance Fainmesser and Galeotti (2015, 2018) and Jackson (2018)). There, we are able to derive simple closed-form solutions for the optimal allocations and the monopoly’s profit. Apart from serving as a useful example, this special case allows us to quantify the value of detailed network information, an important consideration in many real-life applications.

Our final set of results extend the model to accommodate for weak positive affiliation between each buyer’s susceptibility and influence, as well as potential endogeneity in susceptibility and influence. If we believe complete independence and perfect positive correlation between each buyer’s susceptibility and influence are at most close approximations of real-life scenarios, then positive affiliation is precisely meant to allow for more realistic considerations. Under positive affiliation, we show the optimal contracts combine elements from the previous two cases: only susceptibility matters for each buyer’s allocation at the
optimal contracts, but the monopolist seller will exploit the positive affiliation to infer each buyer’s influence and base the contracts on her inference. In other words, the optimal allocation now indirectly depends on a buyer’s influence through the monopoly’s inference. As for endogeneity in types, it is meant to capture the possibility that as a buyer’s consumption of the network product increases, he engages in more social interaction, and thus becomes both more susceptible and influential. There, we show similar intuitions to the benchmark models apply, but the monopolist seller has additional incentives to offer a higher allocation to every type of buyer, given the positive feedback on social interaction.

We view the contribution of the current paper as three-fold: first and foremost, we provide a theoretical framework for screening of network goods that highlight both the effects of “positive network externalities” and “screening”. Moreover, we seek to give some explanation for the selling mechanisms of network products observed in practice. For instance, Dropbox.com offers three cloud storage subscription packages, and one of the distinguishing features of the three packages is the ability to share and collaborate with friends. The pricing is non-linear, as we predict. And even though we do not explicitly model sharing and collaboration in our model, we view them as suggestive evidence that Dropbox tries to screen users based on their susceptibility. Furthermore, we offer a better benchmark in gauging the value of network information. Given privacy concerns, the value of network information has long been an important question in the network literature. Nevertheless, without a proper screening model, researchers have used uniform linear pricing as the incomplete information benchmark (see for instance Fainmesser and Galeotti (2015, 2018)). We argue such approximation unavoidably overestimates the value of network information. Through our network screening model, we hope to provide a second-best benchmark comparable to the standard setting without network externality.

2 Related Literature

The current paper is at the intersection of pricing with network effects and contracting with externalities. Naturally, it is closely related with the two strands of literature.

The importance of network effects was brought to attention by Farrell and Saloner (1985) and Katz and Shapiro (1985). Our goal is to identify the optimal selling mechanism of a monopolist seller in the presence of positive network effects. Of the recent studies in network economics, there are two directions that are most relevant to our paper. The first analyzes optimal product diffusion and referral programs. Some representative papers are Galeotti and Goyal (2009), Goyal and Kearns (2012), Campbell (2013) and Leduc et al. (2017). Their main message is that either through targeting a specific group of buyers or referral incentives, a monopolist seller can encourage product adoption and increase profits. We see our work as complementary to theirs. We argue that even in the static setting with a fixed set of buyers and only distributional level network information, a monopolist seller can improve upon linear pricing through network screening. The second direction studies the pricing
decision of a monopoly in the presence of network effects. These include Candogan et al.
(2012), Bloch and Quéréou (2013), Fainmesser and Galeotti (2015, 2018) and Jadbaie and
Kakhbod (2017). Of the five, Fainmesser and Galeotti (2015) and Jadbaie and Kakhbod
(2017) are the closest studies to ours. Fainmesser and Galeotti (2015) also consider a random
network model and allow for potentially incomplete network information. Apart from slight
generalizations of their model, we extend their work by allowing the monopolist seller to
screen on buyers’ network information. As mentioned in the introduction, this extension
has huge potential benefits by providing a theoretical framework, as well as offering a better
benchmark in gauging the value of network information. On the other hand, Jadbaie and
Kakhbod (2017) has similar motivation to ours, in that they also analyze optimal contracting
of a network product. Nevertheless, they consider a deterministic network and assume the
private information is at the level of social interaction, while we analyze a random network
and the private information comes from the network itself. Because of this, their focus
is different from ours. For instance, bilateral versus multilateral contract is an important
consideration in their model, while the difference is non-existent in our model. For us, the
network structure is an important concern, especially the correlation between each buyer’s
susceptibility and influence, which is not important in their model. Despite the different
modeling choices, one general message is similar: the monopolist seller has to take into
account the positive network externalities in offering the set of contracts.

The second strand of literature concerns mechanism design or contracting with external-
ities. Some representative works include Jehiel et al. (1999), Segal (1999, 2003), Segal and
Whinston (2003) and Hashimoto and Matsubayashi (2014). Their works are related to ours,
as they also study the optimal contracts under situations where one buyer’s consumption
may have either positive or negative externalities on other buyers. The overall techniques
are similar as well. Nevertheless, there are also sharp distinctions. Generally speaking, they
assume each buyer has a one-dimensional type, and the consumption externalities are at the
social aggregate level (i.e. a buyer cares about his consumption and the social aggregate
only).\footnote{Jehiel et al. (1999) is an exception, but they assume the seller has a single unit for sale.}
By comparison, we study optimal contracting with private network information.
As such, we have a two-dimensional type to begin with, and even with ex-ante symmetric
buyers and independent values, the consumption externalities may not be at the social ag-
gregate level. In other words, the context and the setup are different, and the techniques
we adopt are slightly different depending on the specific network structure.

This brings us to the third strand of related literature on the “friendship paradox” and
comparative statics in games with strategic complements.\footnote{Roughly speaking, the “friendship paradox” refers to the fact that, on average, people are strictly less influential than those they are influenced by. See Feld (1991) and Jackson (2018) for details.} Some relevant works include
They show that in games with strategic complements or substitutes, more influential indi-
viduals engage systematically more in an activity, which then have positive effects on the
overall level activity in the network. Their focus is on strategic interaction, while our ulti-
mate interest is in the optimal selling mechanism of a monopolist seller. Nevertheless, the general insight is similar: as long as the monopolist seller can somehow identify influential buyers, she will offer them higher allocations, in the hope that they will have additional positive impacts on the weighted average consumption.

The rest of the paper is organized as follows: Section 3 offers two motivating examples, one modeling asymmetric interaction, the other modeling symmetric interaction. Section 4 presents the formal model, highlighting the different network structures. Section 5 studies the monopoly’s optimal selling mechanism, and Section 6 contrasts the model with complete information and pure screening benchmarks. Section 7 restricts the (common) intrinsic value to have a quadratic form and provides closed form solutions in evaluating the value of network information. Section 8 discusses two extensions of the baseline model, and Section 9 concludes. Most of the proofs are the Appendix.

3 Two Motivating Examples

In this section, we illustrate the optimal selling mechanism of a monopolist seller in two simple examples. Example 1 studies a network product with asymmetric interaction, such as a utility software or a movie. Example 2 studies a network product with symmetric interaction, such as an online cloud storage or an online game. The primary goal is to offer some intuitive understanding of the model, as well as the key driving forces behind the main results. As such, we will keep the examples (deceptively) simple and gloss over the technical details whenever possible.

3.1 Example 1: Asymmetric Interaction

Consider a monopolist seller of a utility software (say a pdf reader/editor). She can produce any quantity of the software at zero marginal cost. There is a unit measure of potential buyers $N = [0, 1]$. Every buyer derives the same intrinsic value from the utility software. For a consumption of $x$ units, a buyer’s intrinsic value $v(x) = x - \frac{1}{2}x^2$. On top of the common intrinsic value, a buyer also derives benefits through social interaction. To characterize the positive network externalities, we introduce two parameters: susceptibility ($s$) and influence ($l$). Intuitively, a buyer’s susceptibility shows how many other buyers he looks at when making his consumption choices, and a buyer’s influence shows how many other buyers look at him when making their consumption choices. Ex-ante every buyer is the same. For a particular buyer, his susceptibility may be either 1 or 3, with equal probability. Similarly, his influence may be either 1 or 3, with equal probability. We assume both susceptibility and influence are independent across buyers. Moreover, to capture asymmetric interaction, we also assume a buyer’s susceptibility is independent of his influence.\footnote{In particular, from the social perspective, the expected susceptibility $E[s]$ and expected influence $E[l]$ balance out and are both equal to 2.}
buyer’s susceptibility and influence are his private information. Each buyer knows his own susceptibility and influence and the common prior of other buyers. The monopolist seller only knows the common prior. In other words, from the seller’s perspective, there are four types of buyers, with \((s, l)\) equal to \((1, 1), (1, 3), (3, 1), (3, 3)\), respectively, and probability 0.25 each. For notational simplicity, we will call the four types of buyers \(LL, LH, HL\), and \(HH\), respectively. Graphically, a buyer’s susceptibility is represented by the number of arrows pointing from the buyer outwards, and his influence is represented by the number of arrows pointing from other buyers to him. The network information of the four types of buyers are depicted in Figure 1. (Since we have a continuum of buyers, we are only displaying the local network information.)

The monopolist seller and all the buyers are expected utility maximizers, with quasi-linear preferences. A buyer \(i\)'s expected utility from social interaction is proportional to his susceptibility and given by \(\gamma s_i x_i X_{fr}\), where \(\gamma > 0\) is a positive network externality coefficient, \(s_i\) is his susceptibility, \(x_i\) is his consumption and \(X_{fr}\) is the average (expected) consumption of the set of buyers he looks at (i.e. the set of buyers he is susceptibility to, which by definition is also the set of buyers that influence him). For now, we take \(\gamma = 0.05\). Then for a buyer of type either \(LL\) and \(LH\), his (total) expected utility from a consump-
tion of $x$ units is given by $U_{LL}(x) = U_{HH}(x) = x - \frac{1}{2}x^2 + 0.05xX_{f_r}$. For a buyer of type either $HL$ or $HH$, his (total) expected utility from a consumption of $x$ units is given by $U_{HL}(x) = U_{HH}(x) = x - \frac{1}{2}x^2 + 0.15xX_{f_r}$.

The game proceeds as follows: in the first stage, each buyer’s type is realized. Knowing only the common prior, the monopolist seller offers four sets of contracts $(x_{LL}, t_{LL})$, $(x_{LH}, t_{LH})$, $(x_{HL}, t_{HL})$, and $(x_{HH}, t_{HH})$. In the second stage, each buyer simultaneously chooses one contract from the four on offer (or the null contract, which is his outside option). The seller’s problem is choose the set of incentive compatible contracts (i.e. each buyer wants to pick the contract corresponding to his type) that maximizes her expected profits.

The first observation is that the seller will optimally choose $(x_{LL}, t_{LL})$ equal to $(x_{LH}, t_{LH})$. Similarly, she will optimally choose $(x_{HL}, t_{HL})$ equal to $(x_{HH}, t_{HH})$. The reason is that a buyer’s expected utility only depends on his susceptibility, but not influence. For the contracts to be incentive compatible, the seller must offer the type $LL$ the same expected utility as the type $LH$, and the type $HL$ the same expected utility as the type $HH$ (otherwise, one type will pick the contract of the other type). For now, we assume this suffices to show the respective contracts must be identical at the optimum. With this observation, we know the seller will only offer two contracts at the optimum, $(x_{L}, t_{L})$ for types $LL$ and $LH$ and $(x_{H}, t_{H})$ for types $HL$ and $HH$. Then $X_{f_r}$ is simply the average social consumption $X$ at the optimum, i.e. $X_{f_r} = X = 0.5x_{L} + 0.5x_{H}$. For simplicity, whenever there is no confusion, we will refer to $LL$ and $LH$ as the low type and $HL$ and $HH$ as the high type.

The monopoly’s problem can then be written as $\max_{t_{L}, t_{L}} 0.5t_{H} + 0.5t_{L}$ subject to the buyers’ incentive constraints. Similar to the case of screening without network externalities, we conjecture that at the optimum, the low type is indifferent between his contract and the outside option, while the high type is indifferent between his contract and that of the low type. In other words, the incentive constraints can be simplified as:

$$x_{L} - \frac{1}{2}x_{L}^2 + 0.05x_{L}(0.5x_{H} + 0.5x_{L}) - t_{L} = 0$$
$$x_{H} - \frac{1}{2}x_{H}^2 + 0.15x_{H}(0.5x_{H} + 0.5x_{L}) - t_{H} = x_{L} - \frac{1}{2}x_{L}^2 + 0.15x_{L}(0.5x_{H} + 0.5x_{L}) - t_{L}$$

The major departure from standard screening (without network externalities) is that the seller will actively take into account the positive network externalities in offering the set of contracts, which is shown by the terms in bold font. From the incentive constraints, we have:

$$t_{L} = x_{L} - \frac{1}{2}x_{L}^2 + 0.05x_{L}(0.5x_{H} + 0.5x_{L}) \quad (3.1)$$
$$t_{H} = x_{H} - \frac{1}{2}x_{H}^2 + 0.15x_{H}(0.5x_{H} + 0.5x_{L}) - 0.10x_{L}(0.5x_{H} + 0.5x_{L}) \quad (3.2)$$

Given the common prior, the average (expected) friend consumption $X_{f_r}$ is the same across different buyers.

We will show this result in our formal model in Lemma 1, where the type space is continuous.
Plugging Equations (3.1) and (3.2) into the seller’s maximization problem and taking first-order condition yields \( x^*_L \approx 1.011 \) and \( x^*_H \approx 1.236 \). Plugging \( x^*_L \) and \( x^*_H \) back to Equations (3.1) and (3.2), we have \( t^*_L \approx 0.556 \) and \( t^*_H \approx 0.567 \). So the seller’s expected profit approximately equals 0.562.

To have a better understanding of our screening framework, we contrast the monopoly’s profit with two standard benchmarks:

1. (Partial) Complete Information: the seller knows each buyer’s susceptibility \( s_i \) (but not influence \( l_i \)) and can price discriminate accordingly.\(^7\)
2. Uniform linear pricing with lump-sum payment: the monopolist seller is constrained to charge a fixed price per unit (and potentially a fixed lump-sum payment) to every buyer.

Practically speaking, the comparison with item (1) gives the difference in the monopoly’s profit upon knowing each buyer’s susceptibility. It quantifies the value of (partial) network information to the utility software seller. On the other hand, previous research (see Fainmesser and Galeotti (2015)) has used item (2) as the benchmark when the seller has only distributional level network information.\(^8\) The difference shows how valuable it is for the utility software seller to screen buyers on their network information.

We start with the case of (partial) complete information. The monopoly’s problem is still to maximize \( 0.5t_H + 0.5t_L \). Nevertheless, since the seller exactly knows each buyer’s susceptibility \( s_i \), she does not need to leave information rent to the high type and makes the participation constraints of both types bind. We have:

\[
\begin{align*}
t_L &= x_L - \frac{1}{2}x^2_L + 0.05x_L(0.5x_H + 0.5x_L) \\
t_H &= x_H - \frac{1}{2}x^2_H + 0.05x_H(0.5x_H + 0.5x_L)
\end{align*}
\]

Plugging Equations (3.3) and (3.4) into the seller’s maximization problem and taking first-order conditions yields \( x^*_L \approx 1.191 \) and \( x^*_H \approx 1.317 \). Plugging \( x^*_L \) and \( x^*_H \) back to Equations (3.3) and (3.4), we have \( t^*_L \approx 0.556 \) and \( t^*_H \approx 0.698 \). So the seller’s expected profit approximately equals 0.627.

If we compare the results with our screening framework, we see the seller “under-sells” to both types under screening. It is expected that the allocation of the low type is lower under screening, since the monopolist seller does not want to leave too much information rent to the high type. What is more interesting is that the allocation of the high type is

\(^7\)We do not consider the case of full information (where both \( s_i \) and \( l_i \) are known to the seller) here, since the comparison is less relevant. Nevertheless, we will make the comparison in the formal model to get the correct measure of additional network information. See Section 7.2 for details.

\(^8\)In fact, they do not even allow for lump-sum payment, making the gap even larger.
also lower under screening. Intuitively, this is because of the positive network externalities. When the seller offers lower quantity to the low type, the valuation of the high type is also lower. This is in sharp contrast with standard screening (without network externalities).

Next, we turn to the case of uniform linear pricing with lump sum payment. Given any lump-sum payment $T$ and unit price $p$, the low type solves $\max_{x_L} U(x_L) - px_L - T$ and the high type solves $\max_{x_H} U(x_H) - px_H - T$. First-order conditions give the respective demands of the two types:

\[ x_L(p) = 1.056(1 - p) \tag{3.5} \]
\[ x_H(p) = 1.167(1 - p) \tag{3.6} \]

From this, we see for $T = 0$, the utility of the low type $U(x_L(p)) = 0.557(1 - p)^2$ and the utility of the high type $U(x_H(p)) = 0.681(1 - p)^2$. The seller can either charge a higher price and exclude the low type or charge a lower price and serve both types. It is easy to see the latter is optimal. So the seller makes the participation constraint of the low type binds and charge $T = U(x_L(p)) = 0.557(1 - p)^2$. Then the seller’s problem becomes $\max p \cdot 0.5px_L(p) + 0.5px_H(p) + U(x_L(p))$. Plugging Equations (3.5) and (3.6) into the seller’s maximization problem and taking first-order condition with respect to $p$, we have $p^* \approx 0$. It follows that the seller’s expected profit approximately equals 0.557.

If we compare the monopoly’s profit under the three scenarios, we see screening alone recovers about 7.1% of the lost profit (compared with partial complete information). This may seem insignificant, but first recall we are comparing screening with the best the monopolist seller can do with any unit price and lump-sum payment combination. Indeed, with only two types, the seller is extracting the entire surplus from half the population (the low type) in both scenarios. If the lump-sum payment were set to 0, the monopoly’s profit would only be approximately 0.263 with uniform pricing. Moreover, screening can recover even more (in percentage terms) of the lost profit as $\gamma$ gets larger (or when the network externalities gets larger). To give a more intuitive understanding of the profit comparisons, we plot the monopoly’s profits under the three scenarios as the network coefficient $\gamma$ varies, as well as the ratios (in profit) to partial complete information as $\gamma$ or the high type $s_H$ varies. This is shown in Figure 2.

On the top left panel (a), we plot the monopoly’s profit under (partial) complete information, screening and uniform linear pricing (and lump-sum payment). We see the gap between all three widens as the network coefficient $\gamma$ increases. Nevertheless, the gain from screening (in percentage terms) also seems to increase as $\gamma$ increases (at least in the range from 0.05 to 0.15). This is shown more vividly on the top right panel (b), where we compare the ratios in profit to complete information under screening and uniform linear pricing as $\gamma$ increases, holding the low type at 1 and the high type at 3, with equal probability. On the bottom panel (c), we compare the ratios in profit to complete information under screening and uniform linear pricing as the high type $s_H$ increases, holding the low type at 1 and $\gamma$ at 0.05. There, we see the gain from screening (in percentage terms) also increases as the high type $s_H$ increases.
Figure 2: Profit Comparison with Asymmetric Interaction

(a) Comparison of profit as $\gamma$ varies

(b) Ratios to complete information as $\gamma$ varies

(c) Ratios to complete information as the high type varies
3.2 Example 2: Symmetric Interaction

Consider a monopolist seller of an online cloud storage (say Dropbox.com). The setup is almost identical to Example 1. The only difference is that to capture symmetric interaction, we assume a buyer’s susceptibility is always identical to his influence (instead of being independent). To sum up, from the seller’s perspective, there are two types of buyers, with \((s, l)\) equal to \((1, 1)\) and \((3, 3)\), respectively, and probability 0.5 each. For notational simplicity, we will call \((1, 1)\) Type \(L\), or the low type, and \((3, 3)\) Type \(H\), or the high type. The network information of the two types of buyers are depicted in Figure 3. (Again, we are only showing the local network information.)

The monopoly’s problem is again \(\max_{t_H, t_L} 0.5t_H + 0.5t_L\) subject to the buyers’ incentive constraints. The key difference from the case of asymmetric interaction is that the average friend consumption \(X_{fr}\) no longer equals the average social consumption \(X = 0.5x_H + 0.5x_L\). To see this intuitively, fix a specific buyer \(i\) and we random draw another buyer \(j\) from the set of buyers \(i\) looks at when making his consumption choices. We know buyer \(j\) is three times more likely to have high type than low type, simply because a high type buyer is more influential. This observation is called the “friendship paradox” in the network literature. (See Jackson (2018) for details.) In other words, \(X_{fr} = 0.75x_H + 0.25x_L\). The subtlety does not arise in the case of asymmetric interaction because we assumed each buyer’s susceptibility is independent of his influence, so a more influential buyer is not necessarily more susceptible (and we argued only susceptibility matters for the optimal allocation). Except for this subtlety, the rest of the analysis is very similar to our utility software example. The buyer’s incentive constraints can be simplified as:

\[
x_L - \frac{1}{2}x_L^2 + 0.05x_L(0.75x_H + 0.25x_L) - t_L = 0
\]
\[
x_H - \frac{1}{2}x_H^2 + 0.05x_H(0.75x_H + 0.25x_L) - t_H = x_L - \frac{1}{2}x_L^2 + 0.15x_L(0.75x_H + 0.25x_L) - t_L
\]
Again, the seller actively takes into account the positive network externalities in offering the set of contracts. Moreover, the network externalities are at the friend average level. Both points are illustrated by the terms in bold font. From the incentive constraints, we have:

\[
\begin{align*}
t_L &= x_L - \frac{1}{2}x_L^2 + 0.05x_L(0.75x_H + 0.25x_L) \\
t_H &= x_H - \frac{1}{2}x_H^2 + 0.15x_H(0.75x_H + 0.25x_L) - 0.10x_L(0.75x_H + 0.25x_L)
\end{align*}
\] (3.7) (3.8)

Plugging Equations (3.7) and (3.8) into the seller’s maximization problem and taking first-order condition yields \(x^*_L \approx 0.976\) and \(x^*_H \approx 1.290\). Plugging \(x^*_L\) and \(x^*_H\) back to Equations (3.7) and (3.8), we have \(t^*_L \approx 0.559\) and \(t^*_H \approx 0.574\). So the seller’s expected profit approximately equals 0.566.

We still compare our screening framework with the benchmarks of complete information and uniform linear pricing (and lump-sum payment). Notice, however, since susceptibility is identical to influence, the complete information benchmark now also corresponds to full information.

We start with the case of full information. Similar to the case of asymmetric interaction, the seller makes the participation constraints of both types bind. We have:

\[
\begin{align*}
t_L &= x_L - \frac{1}{2}x_L^2 + 0.05x_L(0.75x_H + 0.25x_L) \\
t_H &= x_H - \frac{1}{2}x_H^2 + 0.15x_H(0.75x_H + 0.25x_L)
\end{align*}
\] (3.9) (3.10)

Plugging Equations (3.9) and (3.10) into the seller’s maximization problem and taking first-order conditions yields \(x^*_L \approx 1.133\) and \(x^*_H = 1.4\). Plugging \(x^*_L\) and \(x^*_H\) back to Equations (3.9) and (3.10), we have \(t^*_L \approx 0.567\) and \(t^*_H = 0.7\). So the seller’s expected profit approximately equals 0.633. We again have the observation that the seller under-sells to both types under screening with positive network externalities.

Next, we turn to the case of uniform linear pricing with lump sum payment. Given any lump-sum payment \(T\) and unit price \(p\), the low type solves \(\max x_L U(x_L) - px_L - T\) and the high type solves \(\max x_H U(x_H) - px_H - T\). First-order conditions give the respective demands of the two types:

\[
\begin{align*}
x_L(p) &= 1.057(1 - p) \\
x_H(p) &= 1.171(1 - p)
\end{align*}
\] (3.11) (3.12)

From this, we see for \(T = 0\), the utility of the low type \(U(x_L(p)) = 0.559(1 - p)^2\) and the utility of the high type \(U(x_H(p)) = 0.686(1 - p)^2\). The seller can either charge a higher price and exclude the low type or charge a lower price and serve both types. It is easy to see the latter is optimal. So the seller makes the participation constraint of the low
type binds and charge $T = U(x_L(p)) = 0.559(1 - p)^2$. Then the seller’s problem becomes

$$\max_p 0.5px_L(p) + 0.5px_H(p) + U(x_L(p)).$$

Plugging Equations (3.11) and (3.12) into the seller’s maximization problem and taking first-order condition with respect to $p$, we have $p^* \approx 0$. It follows that the seller’s expected profit approximately equals 0.559.

If we compare the monopoly’s profit under the three scenarios, we see with symmetric interaction screening alone recovers about 9.5% of the lost profit (compared with full information). To give a more intuitive understanding of the profit comparisons, we again plot the monopoly’s profits under the three scenarios as the network coefficient $\gamma$ varies, as well as the ratios (in profit) to full information as $\gamma$ or the high type $s_H$ varies. This is shown in Figure 4.

On the top left panel (a), we plot the monopoly’s profit under full information, screening and uniform linear pricing (and lump-sum payment). We again see the gap between all three widens as the network coefficient $\gamma$ increases, and the gain from screening (in percentage terms) also increases as $\gamma$ increases (at least in the range from 0.05 to 0.15). This is shown more vividly on the top right panel (b), where we compare the ratios in profit to full information under screening and uniform linear pricing as $\gamma$ increases, holding the low type
at 1 and the high type at 3, with equal probability. On the bottom panel (c), we compare the ratios in profit to full information under screening and uniform linear pricing as the high type $s_H$ increases, holding the low type at 1 and $\gamma$ at 0.05. There, we see the gain from screening (in percentage terms) also increases as the high type $s_H$ increases.

4 Model

In this section, we describe the formal model. In contrast to the two examples in Section 3, we allow for a continuous type space (susceptibility and influence). This is mainly for expositional purposes and the two examples already illustrate most of the insights if we only restrict to discrete type space.

There is a single monopoly that can produce any quantity of a divisible product at no cost, and a unit measure of potential buyers $N = [0,1]$. The product exhibits positive network externalities, so that a buyer’s utility not only depends on his own consumption-payment bundle, but also the consumption of his friends.

We characterize the network effects by a random network. As mentioned in the introduction, we want to capture each buyer’s susceptibility and influence. In doing so, we introduce a directed network model. Each buyer is endowed with an out-degree $s$ and an in-degree $l$, where $s,l \in D = [d,\bar{d}]$ ($0 \leq d < \bar{d}$). To be specific, buyer $i$ has $s_i$ links pointed at others and $l_i$ links pointed towards him. (See Figure 5 below for details.) We assume the buyers’ out-degrees $s_i$ are independent and identically distributed according to a common distribution $F : D \to [0,1]$. Similarly, we assume the buyers’ in-degrees $l_i$ are independent and identically distributed according to a common distribution $H : D \to [0,1]$. Let $J : D^2 \to [0,1]$ denote the joint distribution of a buyer’s out-degree and in-degree. We assume $J$ is well-behaved, with full support on $[d,\bar{d}]^2$, no atom and continuous density $j$. It follows that the marginal densities of $F$ and $H$ exist. Moreover, if we denote the marginal densities by $f$ and $h$, respectively, we have $f(s) = \int_d^{\bar{d}} j(s,l)dl$ and $h(l) = \int_d^l j(s,l)ds$. Now consistency requires the expected out-degree equals the expected in-degree, i.e. $E_F[s] = \int_d^{\bar{d}} s dF(s) = E_H[l] = \int_d^{\bar{d}} l dH(l)$. We examine interactions at an interim stage, so each buyer only knows his own out-degree $s$, in-degree $l$ and the common prior $J$. The monopoly only knows the common prior $J$. From now on, we will refer to $(s_i,l_i)$ as the type of buyer $i$.

The game proceeds as follows: in the first stage, each buyer’s type is realized. Knowing only the common prior, the monopoly offers a set of contracts to the buyers. In the second stage, after observing the contracts, each buyer simultaneously takes one (or the outside option) from the set of contracts on offer. As we examine interactions at an interim stage, our goal is to look at choice functions that are Bayes Nash implementable. By the revelation principle for Bayes Nash implementation, it suffices to look at direct mechanisms. Moreover, because we assume there is a unit mass of buyers, the monopoly’s decision is identical to a
single agent problem except for the positive network externalities, and we can restrict each
agent’s allocation to depend only on his own report, i.e. the set of contracts are of the form
\(\{x(s,l)\}_{s,l \in D}, \{t(s,l)\}_{s,l \in D}\), where \(x(s,l)\) is the allocation (consumption)
of a buyer with type \((s,l)\), and \(t(s,l)\) is the payment of such a buyer to the monopoly.

The monopoly and all the buyers are expected utility maximizers, with quasilinear
preferences. Consider a feasible outcome \(\{x(s,l), t(s,l)\} \geq 0\), the utility of a buyer
with type \((s,l)\) who benefits from interacting with a finite set of buyers \(N_i \subset [0,1]\) (i.e. the
set of buyers that influence buyer \(i\)) is given by:

\[ u_i(\{x(s,l)\}, t(s,l)) = v(x(s,l)) + \gamma \sum_{j \in N_i} x(s,l)x_j - t(s,l), \]

where \(v(.)\) captures the intrinsic value the buyer derives from consuming the bundle \(x(s,l)\)
and \(\gamma > 0\) is a constant, positive network externality coefficient. Notice we assume every
buyer derives the same intrinsic value from a given consumption bundle \(x\). Aside from
tractability, we want to highlight the positive network externalities. We also assume \(v\) is a
well-behaved utility function, with the following properties:

**Assumption 1 (Decreasing Marginal Utility).** \(v(.)\) is twice continuously differentiable with
\(v'(0) > 0, v''(x) < 0, \forall x > 0\) and \(\lim_{x \to +\infty} v'(x) < 0\) (can be \(-\infty\)).

**Remark 1:** First notice \(v''(x) < 0, \forall x > 0\), so \(\lim_{x \to +\infty} v'(x)\) is well-defined. Second, we
impose the assumption on \(\lim_{x \to +\infty} v'(x)\) to guarantee the existence of a bounded solution.
Intuitively, since the marginal cost is zero, if the positive network externalities are large
enough, the monopoly may want to allocate some buyers an infinite quantity. For a similar
reason, depending on \(v(.)\) and the network structure, we need to impose some upper bound
on \(\gamma\). See Theorems 1 through 4 for details.

As types are independent across buyers, we know the expected utility of a type \((s,l)\)
buyer from the feasible outcome \(\{x(s,l), t(s,l)\}\) is given by:

\[ U(\{x(s,l)\}, t(s,l), (s,l)) = E u_i(\{x(s,l)\}, t(s,l)) = v(x(s,l)) + \gamma sx(s,l)X_{fr} - t(s,l), \]

where

\[ X_{fr} = E[x_j | j \in N_i] \]

is the average consumption of the set of friends that influence a type \((s,l)\) buyer. Again by
independence of types across buyers, we note \(X_{fr}\) is the same across buyers.

Up till now, we have been deliberately vague about the correlation between each buyer’s
two-dimensional type \(s\) and \(l\). Depending on the products of interest, different network

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9 As we allow for fractional degrees, we do not restrict the cardinality of \(N_i\), to be an integer. This may
seem weird. Nevertheless, because of the linear structure, we can alternatively interpret \(s\) and \(l\) as the
expected out-degree and in-degree, respectively, which only pin down the distribution from which the type
is drawn. All the major results in the paper go through. Then for each realization, the cardinality of \(N_i\) is
always an integer. This gives an additional justification for allowing fractional degrees.
structures may be more plausible. In the baseline model analyzed in Section 5, we will focus on two special cases: either $s$ and $l$ are independent, or $s$ is identically equal to $l$. Intuitively, independent $s$ and $l$ correspond to asymmetric interaction and is usually assumed for directed networks.\textsuperscript{10} On the other hand, identical $s$ and $l$ correspond to symmetric interaction and fit the description of undirected networks.\textsuperscript{11} For simplicity, we will refer to the case of independent $s$ and $l$ as a directed network, and the case of $s = l$ as an undirected network. For instance, Example 1 on utility software is a case of directed network and Example 2 on online cloud storage is an example of undirected network. We believe the two benchmark models serve as close approximations to most real-life applications, but to capture more realistic situations, we will extend the model to allow for weak affiliation between $s$ and $l$ in Section 8.1.

Figure 5 illustrates the two network structures, as well as some concepts introduced so far. As we deal with random networks, we should view them as two particular realizations. Moreover, since we have a unit mass of buyers, we can only show part of the network structure. Here we are showing the local network related to buyer 1. On the left hand panel (a), we have a directed network, where buyer 1 has an out-degree (susceptibility) $s_1 = 3$ and in-degree (influence) $l_1 = 2$. The set of buyers that influence buyer 1, $N_1 = \{2, 3, 4\}$. On the right hand panel, we have an undirected network, where buyer 1 has a degree (both susceptibility and influence) $s_1 = l_1 = 4$. The set of buyers that influence buyer 1, $N_1 = \{2, 3, 4, 5\}$. It should be noted that in the primitive of the model we assume buyer $i$ just knows $|N_i| = s_i$, but not necessarily the set $N_i$ (the identity of the buyers that influence him). This is especially relevant if we view the degrees as expected degrees (see footnote 9 for details).

In directed networks, by independence and the common prior, the average friend con-

\textsuperscript{10}See for instance Fainmesser and Galeotti (2015).
\textsuperscript{11}See for instance Galeotti and Goyal (2009) and Jackson (2018).
sumption $X_{fr}$ can be simplified as:

$$X_{fr} = \int_{d}^{d} \int_{d}^{d} x(s, l) dF(s) d\tilde{H}(l), \quad (4.1)$$

where the density $\tilde{h}$ of the distribution function $\tilde{H}$ is given by (recall the intuition on the friendship paradox in Example 2):

$$\tilde{h}(l) = \frac{h(l)l}{E_H[l]}$$

The monopoly is profit-maximizing, and cares about expected total profits (since the product has no cost, profit equals revenue and we will use the two terms interchangeably):

$$ER = E[t(s, l)] = \int_{d}^{d} \int_{d}^{d} t(s, l) dF(s) dH(l)$$

On the other hand, in undirected networks, we know $s$ is identically equal to $l$. For simplicity, we will refer to $s_i$ as the type of buyer $i$. The expected utility of a type $s$ buyer from the feasible outcome $\{(x(s)), \{t(s)\}\}$ can be simplified as:

$$U(\{x(s)\}, t(s), s) = Eu_i(\{x(s)\}, t(s)) = v(x(s)) + \gamma sx(s)X_{fr} - t(s)$$

As before, $X_{fr}$ is the average consumption of the set of friends that influence a type $s$ buyer and is given by:

$$X_{fr} = \int_{d}^{d} x(s) d\tilde{F}(s),$$

where the density $\tilde{f}$ of the distribution function $\tilde{F}$ is given by:

$$\tilde{f}(s) = \frac{f(s)s}{E_F[s]}$$

The monopoly’s expected total profits can be written as:

$$ER = E[t(s)] = \int_{d}^{d} t(s)dF(s)$$

5 Optimal Screening Mechanism

In this section, we characterize the monopoly’s optimal screening mechanism for the case of directed and undirected networks. Depending on the products of interest, either case may be more plausible. Nevertheless, we will begin with the analysis of the directed network model, since it is mathematically simpler, and the techniques are similar to some of the previous literature on contracting with externalities (in particular Segal (1999)).
5.1 Directed Networks

Two examples in the introduction that roughly fit the description of directed networks are movies and utility software or apps, since the group of people that a buyer cares about may be quite different from those that care about him. As we are looking at direct mechanisms, the message space in directed networks \( M_i = D^2 = [d, \bar{d}]^2 \) and outcome function (social choice function) \( Ch(s, l) = (\{x(s, l)\}, \{t(s, l)\}) \).

Recall the monopoly is profit-maximizing and her goal is to maximize expected total profits:

\[
ER = E[t(s, l)]
\]

Next we specify the buyers' incentive constraints in the current setting. As we examine interactions at an interim stage, the relevant constraints are Bayesian incentive compatibility (BIC) and interim individual rationality (IIR).

Given an outcome function, for a buyer with type \((s, l)\), let \( U_{(s, l)}(s', l') \) be his expected payoff when he reports type \((s', l')\).

\[
U_{(s, l)}(s', l') = v(x(s', l')) + \gamma s x(s', l') X_{fr} - t(s', l')
\]

For notational simplicity, we will write \( U_{(s, l)}(s, l) \) (the expected utility of buyer with type \((s, l)\) when he reports truthfully) as \( U(s, l) \).

Bayesian incentive compatibility (BIC) is defined as:

\[
U(s, l) \geq U_{(s, l)}(s', l'), \quad \forall s, s', l, l' \in D
\]

which says a buyer with type \((s, l)\) does not want to mimic any other type \((s', l')\) in expectation.

Participation constraint, or interim individual rationality is defined as:

\[
U(s, l) \geq 0, \quad \forall s, l \in D
\]

which requires any buyer is willing to take the contract offered to him in expectation (instead of the outside option).

Now, the monopoly’s problem is to maximize expected total profits subject to (BIC) and (IIR). As is standard in mechanism design, the key is to simplify the incentive constraints (BIC). One of the complications of the current problem is that each buyer’s type is two-dimensional. Luckily, the following lemma states that information on each buyer’s out-degree is enough for the monopoly’s optimal allocation.

**Lemma 1 (Out-degree is Sufficient for Optimal Allocation).** For any optimal allocation, 
\( x(s, l) = x(s) \) for almost all \( s \in [d, \bar{d}] \), i.e. \( s \) is a sufficient statistic for the optimal allocation a.e.
The lemma significantly simplifies the problem at hand. Even though we have a two-dimensional mechanism design problem, Lemma 1 says it suffices to look at each buyer's out-degree. Intuitively, a buyer's expected utility does not directly depend on his in-degree, so (BIC) requires that two buyers with the same out-degree should get the same expected utility, irrespective of their in-degrees. In other words, the monopoly can only solicit information on each buyer's out-degree in an incentive compatible manner. Now, insights from the Envelope formula (stated in the next lemma) says their allocations are almost identical as well.\footnote{As a side note, this intuition also explains why our result can only hold almost everywhere (instead of everywhere): the mapping from expected utility to optimal allocation is not one-to-one. Given expected utility, the Envelope formula only pins down optimal allocation almost everywhere. Indeed, if in-degrees are finite, the monopoly can offer buyers with the same out-degree different allocations at the optimum (she simply needs to adjust the side payment accordingly), so our analysis is Example 1 is sloppy if we restrict to Bayes Nash implementation.} More importantly, this intuition does not rely on our directed network structure (note we do not mention directed networks in Lemma 1). What is special about the current setting, however, is that we can further say the optimal allocation \( x(.) \) is independent of a buyer's in-degree, since \( s \) and \( l \) are independent. Intuitively, if there is strong positive relationship between \( s \) and \( l \), then the monopoly knows a highly susceptible buyer is also highly influential, and she wants to use this piece of information in the optimal allocation. This intuition will prove very useful in the next subsection, as well as in Section 8.1. Going back to our example on pdf reader/editor, Lemma 1 predicts that the seller (say Adobe) will not offer two buyers with the same willingness to pay different quantities (or features) if there is no verifiable evidence on different influence.

**Proof of Lemma 1:** Consider the incentives of two buyers of types \((s_1, l_1)\) and \((s_2, l_2)\) with \(s_1 > s_2\). By (BIC) we have:

\[
U(s_1, l_1) \geq U(s_1, l_1)(k_2, l_2) \quad \text{and} \quad U(s_2, l_2) \geq U(s_2, l_2)(s_1, l_1)
\]

Equivalently, we have:

\[
v(x(s_1, l_1)) + \gamma s_1 x(s_1, l_1) X_{fr} - t(s_1, l_1) \geq v(x(s_2, l_2)) + \gamma s_1 x(s_2, l_2) X_{fr} - t(s_2, l_2)
\]

\[
v(x(s_2, l_2)) + \gamma s_2 x(s_2, l_2) X_{fr} - t(s_2, l_2) \geq v(x(s_1, l_1)) + \gamma s_2 x(s_1, l_1) X_{fr} - t(s_1, l_1)
\]

If we add up the two inequalities, we have:

\[
\gamma (s_1 - s_2) X_{fr} (x(s_1, l_1) - x(s_2, l_2)) \geq 0
\]

By Assumption 1 and that the fact the product has zero cost, we know \( X_{fr} > 0 \) at the optimum. Moreover, \( s_1 - s_2 > 0 \). It follows that \( x(s_1, l_1) \geq x(s_2, l_2) \), \( \forall s_1 > s_2, l_1, l_2 \in D \).

Now consider the set \( A(s) = \{s \in D : x(s, l) \text{ is not constant in } l\} \). For any \( s \in A(s) \), we can define the interval \( I_s = [\inf_l x(s, l), \sup_l x(s, l)] \). By construction, \( I_s \) is not degenerate for any \( s \) and thus must contain a rational number (rational numbers are dense). Moreover, notice the intervals \( I_s \) do not overlap, so each rational number belongs to at most one of
the intervals. Since rational numbers are countable, we know \( A(s) \) is countable and hence \( \mu(A(s)) = 0 \). It follows that \( x(s, l) \) is constant in \( l \) for almost all \( s \in [d, \bar{d}] \). \( \square \)

Notice the proof of Lemma 1 also shows the optimal allocation is increasing in each buyer’s out-degree. Intuitively, buyers with higher out-degrees derive larger benefits through positive network externalities and have higher willingness to pay. The monopoly will offer them larger quantities and charge higher prices accordingly. For instance, in our example of pdf reader/editor, Lemma 1 also predicts that the seller (say Adobe) will offer a larger quantity (or more features) to buyers who value more file sharing.

Given Lemma 1, we will simply refer to \( s_i \) as the type of buyer \( i \). Moreover, we will write \( U(s, l)(s', l') \) as \( U(s') \) and \( U(s, l)(s, l) \) as \( U(s) \) from now on. There is one more interesting observation:

\[
X_{fr} = \int_d^{\bar{d}} x(s) dF(s)
\]

In words, we now know the average friend consumption not only is the same across different buyers, but equals the average social consumption \( X \) (from now on, we will write \( X_{fr} \) as \( X \) in the case of directed networks). Alternatively, the friendship paradox has no bite on the optimal allocation in directed networks. This observation should come as no surprise, since the friendship paradox has bite only on the average in-degree of friends (the in-degree is weighted by \( \tilde{h} \) instead of \( h \) in Equation (4.1)), and we know the monopoly cannot credibly solicit information on buyers’ in-degrees.

Once our mechanism design problem is reduced to one dimension, we can characterize the incentive constraints using monotonicity and the Envelope formula, following the standard argument. This is given in the following lemma:

**Lemma 2** (Characterization of Incentive Constraints in Directed Networks). In directed networks, assuming \( x(,) \) is bounded (i.e. \( \exists B > 0 \text{ s.t. } 0 \leq x(s) < B, \forall s \) ), the incentive constraints (BIC) hold if and only if:

1. **Sufficiency of out-degree:** \( x(s,.) = x(s) \) a.e..
2. **Monotonicity:** \( x(,) \) is non-decreasing in \( s \).
3. **BICFOC:** \( U(s) = U(d) + \gamma X \int_d^s x(t) dt \).

Intuitively, “BICFOC” pins down the information rent of each type of buyer, as in the standard case without network externality.

Our next step is to solve the monopoly’s profit maximization problem. Compared with standard mechanism design without externality, we have one more complication: as the monopoly varies the set of contracts \( \{\{x(s)\}, \{t(s)\}\} \), she is affecting the average social consumption \( X \) as well. In other words, as in Example 1, the monopoly needs to explicitly take the network externalities into consideration when offering the set of contracts. Technically
speaking, this means we cannot readily apply point-wise maximization. Instead, we will set up an infinite-dimensional Lagrangian with one equality constraint \( X = \int_{\mathcal{X}} x(s) dF(s) \).

For cleaner exposition, we need the following definition:

**Definition 1 (Interior Allocation).** We say a set of allocations \( \{x(s)\} \) is interior if \( 0 < x(s) < +\infty \) for almost all \( s \in \mathcal{D} \).

As the name suggests, a set of allocations is interior if the consumption of every type of buyer is strictly positive and bounded. This is to help avoid the messy case of boundary solutions.

Let \( g(s) = s - \frac{1 - F(s)}{f(s)} \) be the “virtual type” of a type \( s \) buyer. Similar intuition from standard mechanism design carries over to the current setting: the virtual type is the actual type of a buyer minus the information rent (that must be allocated to buyers of higher types). It captures the type that the monopoly cares about when making her allocation/pricing decisions. We maintain the usual assumption that the virtual type is non-decreasing:

**Assumption 2 (Non-decreasing Virtual Type).** The virtual type \( g(s) \) is non-decreasing in \( s \).

Recall \( X = \int_{\mathcal{X}} x(s) dF(s) \) is the average social consumption. Let \( G = \mathbb{E}_F[g(s)x(s)] \). With these definitions, we are ready to state our main result of the current subsection:

**Theorem 1 (Optimal Allocation in Directed Networks).** Under Assumptions 1 and 2, any set of interior optimal allocations \( \{x(s)\} \) in directed networks is characterized by the following integral equation:

\[
v'(x(s)) + \gamma G + \gamma Xg(s) = 0
\]

Moreover, given any \( v(.) \), there exists \( \bar{\gamma} > 0 \) s.t. for all \( 0 < \gamma < \bar{\gamma} \), a set of interior optimal allocations exist.

Theorem 1 consists of two parts. The second part is purely technical: it guarantees the existence of interior allocations as long as the positive network externalities are not too large. The intuition is discussed in Remark 1. The first part is more interesting: it essentially characterizes the monopoly’s optimal screening mechanism (as long as the allocations are interior). If we re-write the integral equation, we get \( v'(x(s)) = -\gamma G - \gamma Xg(s) \). We see a higher type buyer is offered higher allocation, since his virtual type is higher, and thus derives higher benefits from social interaction, coinciding with our earlier intuition. Moreover, we will see in Section 6.2 that the \(-\gamma G\) term comes from the fact that the monopoly actively takes into account the positive network externalities in offering the contracts.

The proof idea is as follows: recall “BICFOC” captures the information rent of each type of buyer (relative to the lowest type). In standard mechanism design without externality, we first ignore the monotonicity constraint and plug “BICFOC” into the monopoly’s
maximization problem. Then, we apply point-wise maximization and ensure “monotonicity”. Here, we adopt a similar approach, with two modifications. First, as mentioned above, because of the network externalities, we have to set up an infinite-dimensional Lagrangian before applying point-wise maximization. Additionally, we will show in the current problem, that as long as the allocations are interior and that the virtual type is non-decreasing, “monotonicity” is automatically guaranteed, so we have a clean characterization.

Once we get the set of optimal allocations \( \{x(s)\} \), the total payment is given by:

\[
t(s) = v(x(s)) + \gamma s X x(s) - U(s)
\]

\[
= v(x(s)) + \gamma s X x(s) - \gamma X \int_{\bar{d}}^{s} x(t) dt.
\]

5.2 Undirected Networks

Two examples in the introduction that roughly fit the description of undirected networks are online cloud storage platforms and online games, since the group of people that a buyer cares about are more or less the same as those that care about him. We still look at direct mechanisms, and given one-dimensional type to begin with, the message space \( M_i = D = [\bar{d}, \bar{d}] \) and outcome function (social choice function) \( Ch(s) = (\{x(s)\}, \{t(s)\}) \).

The monopoly is profit-maximizing and her goal is to maximize the expected total profit:

\[
ER = E[t(s)]
\]

Following the same notations as those of Section 5.1, Bayesian incentive compatibility (BIC) is defined as:

\[
U(s) \geq U_s(s'), \forall s, s' \in D
\]

Moreover, participation constraint, or interim individual rationality is defined as:

\[
U(s) \geq 0, \forall s \in D
\]

The interpretation of the two set of incentive constraints is almost identical to the case of directed networks. The monopoly’s problem is to maximize expected total profits subject to (BIC) and (IIR). With one-dimensional type, the incentive constraints can be simplified with ease, as given in the following lemma.

**Lemma 3** (Characterization of Incentive Constraints in Undirected Networks). In undirected networks, assuming \( x(.) \) is bounded, the incentive constraints (BIC) hold if and only if:

1. Monotonicity: \( x(.) \) is non-decreasing in \( s \).
2. **BICFOC:** 
\[ U(s) = U(d) + \gamma X_f \int_d^s x(t) dt. \]

The proof of Lemma 3 is to mimic Lemma 1 and 2 and thus omitted. The intuition is also similar: “BICFOC” still captures the information rent of each type of buyer. Nevertheless, notice the average friend consumption:

\[ X_{fr} = \int_d^s x(s) d\bar{F}(s) \]

is different from the average social consumption even at the optimum. In other words, the friendship paradox has bite on the optimal allocation in the case of undirected networks. Intuitively, with undirected networks, a buyer’s in-degree is identical to his out-degree. Even though (as suggested in the case of directed networks) the monopoly still cannot credibly solicit information on a buyer’s in-degree alone, she can now (perfectly) infer the information from each buyer’s out-degree. As a side note, the case of undirected networks is where the network structure really comes into play. Even though one may interpret the screening model in directed networks simply as one of mechanism design with externalities, that interpretation no longer captures the whole picture of the screening model in undirected networks.

With Lemma 3, we are ready to solve the monopoly’s profit maximization problem in undirected networks. Recall from Theorem 1 that 
\[ g(s) = s - \frac{1 - F(s)}{f(s)} \]

is the virtual type of a type \( s \) buyer and 
\[ G = E_F[g(s)x(s)] \]

The following theorem characterizes the set of interior allocations in undirected networks:

**Theorem 2** (Optimal Allocation in Undirected Networks). Under Assumptions 1 and 2, any set of interior optimal allocations \( \{x(s)\} \) in undirected networks is characterized by the following integral equation:

\[ v'(x(s)) + \gamma G \frac{s}{E_F[s]} + \gamma X_{fr} g(s) = 0 \]

Moreover, given any \( v(.) \), there exists \( \bar{\gamma} > 0 \) s.t. for all \( 0 < \gamma < \bar{\gamma} \), a set of interior optimal allocations exist.
platform (say Dropbox’s) allocation/pricing decisions (should) take into account both a buyer’s susceptibility and implied influence.

The proof idea of Theorem 2 is similar to that of Theorem 1. We still set up an infinite-dimensional Lagrangian and apply point-wise maximization. Nevertheless, since the allocation $x(.)$ is weighted by $\tilde{F}$ (instead of $F$) in the average friend consumption $X_{fr}$, some extra care has to be taken before applying point-wise maximization.

Once we get the set of optimal allocations $\{x(s)\}$, the total payment is given by:

$$t(s) = v(x(s)) + \gamma sX_{fr}x(s) - U(s)$$

$$= v(x(s)) + \gamma sX_{fr}x(s) - \gamma X_{fr} \int_{\hat{d}}^{s} x(t)dt.$$  

6 Comparative Statics

In this section, we look at a few comparative statics results in both directed and undirected networks. In Section 6.1, we present the complete information benchmark. Apart from its theoretical importance, the complete information benchmark can help us evaluate the value of network information (and we will actually do so in the special case of quadratic intrinsic value in the next section). In Section 6.2, we contrast our network screening model with pure screening and highlight the effects of positive network externalities.

6.1 Complete Information Benchmark

For the complete information benchmark, we will start with undirected networks. The reason is that the type space is one-dimensional, so there is only one complete information benchmark, and the comparison with screening is very similar to our standard setting without network externality. We denote the complete information optimal allocation by $x_{c}(.)$. Recall the average friend consumption in undirected networks $X_{fr} = \int_{\hat{d}}^{d} x_{c}(s) d\hat{F}(s)$ and let $S = E_{F}[sx_{c}(s)]$. The complete information optimal selling mechanism in undirected networks is given in the following proposition:

**Proposition 1** (Undirected Networks: Complete Information). With complete information and under Assumption 1, any set of interior optimal allocations $\{x_{c}(s)\}$ in undirected networks is characterized by the following integral equation:

$$v'(x(s)) + \gamma S \frac{s}{E_{F}[s]} + \gamma X_{fr}s = 0$$

Moreover, given any $v(.)$, there exists $\hat{\gamma} > 0$ s.t. for all $0 < \gamma < \hat{\gamma}$, a set of interior optimal allocations exist.
If we ignore the technical details of interior allocations, Proposition 1 coincides nicely with our intuition. It says when there is incomplete information and the monopoly has to screen buyers, she simply treats each buyer’s virtual type as his actual type.

Next, we turn to directed networks. The problem is slightly more complicated since each buyer now has a two-dimensional type. There are two natural benchmarks of complete information: either out-degrees are perfectly observable to the seller, or both out-degrees and in-degrees are perfectly observable to the seller. For instance, in our example of pdf reader/editor, the first benchmark corresponds to the case where the seller (say Adobe) perfectly knows each buyer’s benefits from social interaction. The second benchmark corresponds to the case where the seller not only knows each buyer’s valuation, but also who is more influential. Both benchmarks are important for our current purposes: to begin with, as mentioned in Example 1, the first benchmark is directly comparable to the screening framework, where the seller can only screen on each buyer’s out-degree (susceptibility). Moreover, the second benchmark is also useful, in that it helps us gauge the value of detailed network information. We begin with the analysis of the first benchmark, or the case where the seller has complete information on each buyer’s out-degree. We denote the optimal allocation by \( x_c(s) \). Recall in this case the average friend consumption equals the average social consumption 

\[
X_{fr} = \int_d \int_d x_c(s) dF(s) \quad \text{and let } S = E_F[sx_c(s)].
\]

The optimal selling mechanism with complete information on each buyer’s out-degree in directed networks is given in the following proposition:

**Proposition 2** (Directed Networks: Complete Information on Out-degrees). With complete information on each buyer’s out-degree and under Assumption 1, any set of interior optimal allocations \( \{x_c(s)\} \) in directed networks is characterized by the following integral equation:

\[
v'(x(s)) + \gamma S + \gamma X_s = 0
\]

Moreover, given any \( v(.) \), there exists \( \bar{\gamma} > 0 \) s.t. for all \( 0 < \gamma < \bar{\gamma} \), a set of interior optimal allocations exist.

Proposition 2 is the counter-part of Proposition 1 in directed networks. It says when there is incomplete information on each buyer’s out-degree and the monopoly has to screen buyers, she simply treats each buyer’s virtual type as his actual type.

Now, we turn to the second benchmark, or the case where the seller has complete information on each buyer’s out-degree and in-degree, which also corresponds to full information in directed networks. We denote the optimal allocation by \( x_{full}(.) \). Most of the techniques we use in the proofs of Theorems 1 and 2 still apply to the current setting. Notice, however, since we have a two-dimensional type \( (s,l) \), the average friend consumption 

\[
X_{fr} = \int_d \int_d x_{full}(s,l) dF(s) d\tilde{H}(l)
\]

no longer equals the average social consumption. Let 

\[
S = E_F[dx_{full}(s,l)].
\]

The full information optimal selling mechanism in directed networks is given in the following proposition:
Proposition 3 (Directed Networks: Full Information). With full information and under Assumption 1, any set of interior optimal allocations \( \{x_{\text{full}}(s,l)\} \) in directed networks is characterized by the following integral equation:

\[
v'(x(s,l)) + \gamma S \frac{l}{EH[l]} + \gamma X_{fr}s = 0
\]

Moreover, given any \( v(.) \), there exists \( \bar{\gamma} > 0 \) s.t. for all \( 0 < \gamma < \bar{\gamma} \), a set of interior optimal allocations exist.

If we ignore the technical details of interior allocations, we can write the optimality condition as \( v'(x(s,l)) = -\gamma S \frac{l}{EH[l]} - \gamma sX_{fr} \). We immediately see the monopolist seller wants to give a large allocation to buyers with high out-degree and in-degree. Moreover, with full information, the monopoly does not have to worry about buyers’ incentive compatibility constraints, and the optimality condition is based on each buyer’s actual type. This result confirms our intuition in Lemma 1: the monopolist seller does not base the allocation/pricing decisions on a buyer’s in-degree when there is incomplete information because she may violate buyers’ incentive constraints. In particular, if the seller (say Adobe) has information on each buyer’s influence, she will offer higher allocations to more influential buyers, which coincides with the findings of Fainmesser and Galeotti (2015).

6.2 Pure Screening

In this subsection, we contrast our network screening problem with pure screening. Since the results are very similar, we will look at the cases of directed and undirected networks at the same time. Our primary goal is to see how screening is different when positive network externalities are present.

Let \( x_{\text{ns}}(.) \) be the optimal allocation (in either directed or undirected networks) from the network screening problem introduced in Section 4, \( X^*_{fr} \) the optimal average friend consumption (which is equal to the average social consumption in the case of directed networks), and \( \Pi_{\text{ns}} \) the corresponding monopoly’s profit. Now suppose the monopoly and the buyers are “naive” and believe the average friend consumption is fixed at \( X_{fr} = X^*_{fr} \) (one interpretation may be they observe \( X_{fr} = X^*_{fr} \) from previous interactions). Then we are faced with a pure screening problem (with no network externality). Let \( x_{ps}(.) \) be the optimal allocation from this pure screening problem. We are interested in how \( x_{ns}(.) \) compare with \( x_{ps}(.) \), which is given in the following proposition.

Proposition 4 (Comparison with Pure Screening). Under Assumptions 1 and 2, in either directed or undirected networks, \( x_{ps}(s) \leq x_{ns}(s), \forall s \in D \).
allocations to a specific type of buyer: first, it increases the buyer’s valuation of his bundle; second, it increases other buyers’ valuations of their existing bundles. Given the same level of average friend consumption, the first effect is identical under pure screening, while the second effect is non-existent. As a result, the monopoly has more incentives to offer a higher allocation to each type of buyer under network screening.

**Remark:** Notice we do not compare the monopoly’s profit in this case. One of the reasons is that the profit in the pure screening problem is ill-defined. To see this, simply notice when the individual allocations are given by \( x(.) = x_{pa}(.) \), the average friend consumption is different from \( X^*_fr \).

The comparison in Proposition 4 is a useful benchmark by telling us how the optimal allocations differ when positive network externalities are present. Nevertheless, as mentioned in the Remark, the monopoly and the buyers’ conjecture of the average friend consumption is incorrect in that case. A natural question to ask is how the comparisons differ in the “steady state”. Specifically, suppose instead the monopoly and the buyers are still “naive” and take \( X_{fr} \) as fixed, but their conjecture of the average friend consumption happens to be correct, i.e. \( X_{fr} = X = E_F[x(s)] \) in directed networks and \( X_{fr} = E_{\tilde{F}}[x(s)] \) in undirected networks. Let the optimal outcome (in the sense of maximizing the monopoly’s profit) in this case be \( \{x_{ss}(k), \{t_{ss}(k)\} \} \). Notice since the conjecture of the average friend consumption is correct, the monopoly’s profit is well-defined in this case, and we write it as \( \Pi_{ss} \). We are interested in how \( x_{ns}(.) \) compare with \( x_{ss}(.) \) as well as how \( \Pi_{ns} \) contrasts with \( \Pi_{ss} \). We begin with the comparison of the monopoly’s profits.

**Proposition 5** (Comparison with Steady State Pure Screening). *Under Assumptions 1 and 2, in either directed or undirected networks, \( \Pi_{ns} \geq \Pi_{ss} \).*

Proposition 5 is straightforward: it just comes from the optimality of the network screening problem and a revealed preference argument. Nevertheless, there is another interesting interpretation of the result: suppose there is an infinite number of monopolistic sellers (in the sense that they can still make take-it-or-leave-it offers to the buyers they serve), each serving a distinct type of buyers. Then the problem each monopolistic seller faces is precisely our pure screening problem in the steady state. Proposition 5 says the joint profit of such infinite number of monopolistic sellers is smaller than that of a single monopoly. For instance, if there are suddenly a growing number of providers of pdf readers, and even if we ignore the competition among sellers, a similar argument to Proposition 5 shows their joint profits would be lower than that of a single monopoly. Intuitively, a monopolistic seller fails to take into account the positive spillover her buyers have on other buyers when offering contracts. As a result, she fails to price optimally and the joint profit is sub-optimal accordingly.

**Proof of Proposition 5:** Consider the optimal set contracts of the pure screening problem in the steady state \( \{x_{ss}(k), \{t_{ss}(k)\} \} \). Since the average friend consumption \( X_{fr} = X = E_F[x(s)] \) in directed networks and \( X_{fr} = E_{\tilde{F}}[x(s)] \) in undirected networks, the set of contracts also satisfy the incentive constraints ((IIR) and (BIC)) of the network screening
problem. (Compare with the case where the average friend consumption is artificially fixed at $X_{fr}^*$. By revealed preference (i.e. the fact that the monopoly maximizes profit in the network screening problem), we must have:

$$\Pi_{ns} = \mathbb{E}[t_{ns}(k)] \geq \mathbb{E}[t_{ss}(k)] = \Pi_{ss}$$

as desired.

On the other hand, it is difficult to compare $x_{ns}(\cdot)$ with $x_{ss}(\cdot)$ and some simulation suggests the comparison can be type-dependent. Intuitively, there are competing forces at work: first, as shown in Proposition 4, the monopoly wants to give each type a higher allocation in the network screening problem as she accounts for the positive network externalities. Second, in the pure screening problem, as the level of average friend consumption $X_{fr}$ decreases, the monopoly wants to give the comparatively low types a higher allocation since she worries less about violating the incentive constraints of the comparatively high types. In the case of quadratic intrinsic value (illustrated in the next section), however, we can show $x_{ns}(s) \geq x_{ss}(s), \forall s \in D$, so the first effect always dominates.

7 Quadratic Intrinsic Value

In this section, we focus on the case where each buyer’s intrinsic value for the product $v(x) = x - \frac{1}{2}x^2$. This particular form of intrinsic value is widely used in the network literature and has the nice “linear best response” property.\textsuperscript{13} In our setup, this special case serves three purposes: first, we are able to derive closed-form solutions for the optimal screening mechanism. It thus serves as a useful example by providing additional insights into our network screening problem. Second, the solutions (esp. the optimal allocations) have additional structure in this special case and are of interest themselves. Third, as we can also get closed-form solutions for the monopoly’s profits under various information structures, we can quantify the value of network information.

This section is organized as follows: Section 7.1 solves for the optimal screening allocations in both directed and undirected networks. Section 7.2 solves for the monopoly’s profits under different information structures and discuss the value of network information. Section 7.3 solves the pure screening problem and contrasts the solution with the network screening problem. For simplicity, we will focus on directed networks in Sections 7.2 and 7.3, but the general insight is similar in undirected networks (just with much messier solutions). We assume throughout the current section that any set of allocations is interior (irrespective of the network structure, information structure, or type of screening). In other words, the monopolist seller always find it optimal to serve every type of buyer.

7.1 Optimal Screening Allocations

We begin with directed networks. Recall from Theorem 1, the optimal allocation is characterized by:

\[ v'(x(s)) = -\gamma G - \gamma X g(s) \]

With quadratic intrinsic value, \( v'(x) = 1 - x \), and we have:

\[ x(s) = \gamma X g(s) + \gamma G + 1 \]

Notice \( X \) and \( G \) are both independent of \( s \) (and hence \( g(s) \)), so at the optimum \( x(.) \) is linear in \( g(s) \), or a buyer’s virtual type (out-degree). This should not come too much of a surprise, but is an interesting observation by itself. In Fainmesser and Galeotti (2015), they show if the monopoly has complete information on buyers’ out-degrees (susceptibility) alone and price discriminates accordingly, then each buyer’s optimal consumption is linear in his out-degree. Without detailed network information, the monopoly has to leave information rents to high type buyers. As a result, the “type” she actually cares about is each buyer’s virtual type, and we have reason to expect that \( x(.) \) should be linear in the virtual type.

Back to our calculation, if we write \( a \equiv \gamma X \) and \( b \equiv \gamma G + 1 \), then \( x(s) = ag(s) + b \). Our goal is to solve for \( a \) and \( b \). If we plug \( x(.) \) into \( X \) and \( G \), we have:

\[ X = E_F[x(s)] = E_F[ag(s) + b] = aE_F[g(s)] + b \]

\[ G = E_F[g(s)x(s)] = E_F[g(s)(ag(s) + b)] = aE_F[g^2(s)] + bE_F[g(s)] \]

Now observe that:

\[ E_F[g(s)] = \int_d^{\bar{d}} (s - \frac{1 - F(s)}{f(s)}) f(s) ds \]

\[ = \bar{d} \]

If we write \( \sigma_g^2 \equiv \text{Var}_F[g(s)] \), then we have:

\[ E_F[g^2(s)] = \text{Var}_F[g(s)] + E_F^2[g(s)] = \sigma_g^2 + \bar{d}^2 \]

Plugging \( E_F[g(s)] \) and \( E_F[g^2(s)] \) into \( X \) and \( G \) and we have:

\[ X = ad + b \]

\[ G = a(\sigma_g^2 + \bar{d}^2) + bd \]

Now, by the definition of \( a \) and \( b \), we have the following identities:

\[ a = \gamma(ad + b) \quad (7.2) \]

\[ b = \gamma[a(\sigma_g^2 + \bar{d}^2) + bd] + 1 \quad (7.3) \]
Solving for $a$ and $b$, we have:

\[
\begin{align*}
    a &= \frac{\gamma}{1 - 2\gamma d - \gamma^2 \sigma_g^2} \\
    b &= \frac{1 - \gamma d}{1 - 2\gamma d - \gamma^2 \sigma_g^2}
\end{align*}
\]

Equivalently, the set of optimal allocations is given by:

\[
x_{\text{ns}}(s) = \frac{\gamma}{1 - 2\gamma d - \gamma^2 \sigma_g^2} g(s) + \frac{1 - \gamma d}{1 - 2\gamma d - \gamma^2 \sigma_g^2}
\]

Apart from the linear structure, the closed form solution is important by allowing us to compare the monopoly’s profits with different information structures in Section 7.2, as we did in Example 1 in Section 3.1.

Next, we turn to undirected networks. Given the network structure, and especially the complication of the friendship, the calculation is much more involved, but the general idea is similar. Recall from Theorem 2 that:

\[
v'(x(s)) = -\gamma G \frac{s}{E_F[s]} - \gamma X_{fr} g(s)
\]

With quadratic payoffs, $v'(x) = 1 - x$, and the integral equation boils down to:

\[
x(s) = 1 + \gamma X_{fr} g(s) + \frac{\gamma G}{E[s]} s
\]

Notice $X_{fr}$ and $G$ are both independent of $s$ and $g(s)$, so at the optimum $x(.)$ is a linear combination of $g(s)$ and $s$ (and the constant 1). This is more complicated than the case of directed networks, but expected from the discussions after Theorem 2: the $g(s)$ part captures the idea that the monopoly wants to give higher allocations to buyers with higher types since they value the product more from social interactions, while the $s$ part reflects that the monopoly wants to give higher allocations to buyers with higher types since they exert more influence (weighted more in the average friend consumption).

For the calculation, if we write $c \equiv \gamma X_{fr}$ and $e \equiv \frac{\gamma G}{E[s]}$, then $x(s) = 1 + cg(s) + es$. We need to solve for $c$ and $e$. If we plug $x(.)$ into $X_{fr}$ and $G$, we have:

\[
X_{fr} = \int_d^d x(s) d\tilde{F}(s)
= \frac{1}{E[s]}[E[s] + cE[sg(s)] + eE[s^2]]
\]

and

\[
G = E[g(s)x(s)]
= E[g(s)] + cE[g^2(s)] + eE[sg(s)]
\]
Plugging them into the definitions for $a$ and $b$, we have:

$$
E[s]c = \gamma E[s] + \gamma E[sg(s)]c + \gamma E[s^2]e \\
E[s]e = \gamma E[g(s)] + \gamma E[g^2(s)]c + \gamma E[sg(s)]e
$$

Solving the systems of linear equations using Cramer’s rule, we get:

$$
c = \frac{\gamma E[s](E[s] - \gamma E[sg(s)]) + \gamma^2 E[s^2]E[g(s)]}{(E[s] - \gamma E[sg(s)])^2 + \gamma^2 E[s^2]E[g^2(s)]} \quad (7.4)
$$

$$
e = \frac{\gamma E[g(s)](E[s] - \gamma E[sg(s)]) + \gamma^2 E[s]E[g^2(s)]}{(E[s] - \gamma E[sg(s)])^2 + \gamma^2 E[s^2]E[g^2(s)]} \quad (7.5)
$$

From Equation (7.1), we know $E_F[g(s)] = \bar{d}$. Similarly, we have:

$$
E[sg(s)] = \int_{d}^{\bar{d}} s(1 - \frac{F(s)}{f(s)})f(s)ds = \frac{1}{2}[E[s^2] + \bar{d}^2]
$$

If we write $\sigma_s^2 \equiv \text{Var}_F[s]$, then $E[s^2] = \text{Var}[s] + E^2[s] = \sigma_s^2 + E^2[s]$. It follows that:

$$
E[sg(s)] = \frac{1}{2}[d^2 + \sigma_s^2 + E^2[s]] \quad (7.6)
$$

If we plug Equations (7.1) and (7.6) into Equations (7.4) and (7.5) and recall that $E[g^2(s)] = \sigma_g^2 + \bar{d}^2$, we have:

$$
c = \frac{\gamma E[s](E[s] - \frac{1}{2}\gamma(d^2 + \sigma_s^2 + E^2[s])) + \gamma^2 d(\sigma_s^2 + E^2[s])}{(E[s] - \gamma(d^2 + \sigma_s^2 + E^2[s])^2 + \gamma^2(\sigma_s^2 + E^2[s])(\sigma_g^2 + \bar{d}^2)}
$$

$$
e = \frac{\gamma d[E[s] - \gamma(d^2 + \sigma_s^2 + E^2[s])] + \gamma^2 E[s](\sigma_s^2 + d^2)}{(E[s] - \gamma(d^2 + \sigma_s^2 + E^2[s])^2 + \gamma^2(\sigma_s^2 + E^2[s])(\sigma_g^2 + \bar{d}^2)}
$$

The closed-form solution may be ugly-looking, but we can at least visualize the different effects of susceptibility and influence on the optimal allocation. Moreover, if we wish, we could still compare the monopoly’s profits with different information structures. Given the complication, however, we will focus on directed networks for the monopoly’s profits in the next subsection.

### 7.2 Monopoly’s Profit

Similar to Example 1, we compare the monopoly’s profits across different information structures. Specifically, we solve for the monopoly’s profit (in directed networks) in the following four scenarios:
(1) Full information: this corresponds to the situation analyzed in Proposition 3 and we denote the profit by $\Pi_{full}$.

(2) Complete information on out-degrees: this corresponds to the situation analyzed in Proposition 2 and we denote the profit by $\Pi_c$.

(2) Network screening: this corresponds to our benchmark model analyzed in Section 5.1 and as in Section 6.2, we denote the profit by $\Pi_{ns}$.

(3) Uniform linear pricing with lump-sum payment: the monopoly is constrained to charge a fixed price per unit (and potentially a fixed lump-sum payment) to every buyer. We denote the profit by $\Pi_{ul}$.

Practically speaking, $\Pi_{full} - \Pi_{ns}$ is the difference in the monopoly’s profit upon knowing each buyer’s exact susceptibility and influence and quantifies the value of detailed network information. $\Pi_c - \Pi_{ns}$ is the difference in the monopoly’s profit upon knowing each buyer’s susceptibility and quantifies the value of partial network information. On the other hand, as mentioned in Example 1, $\Pi_{ns} - \Pi_{ul}$ is the difference in the monopoly’s profit upon adopting an optimal screening mechanism. It quantifies the value of network screening, and shows by how much one would incorrectly estimated the value of detailed network information if screening were ignored.

Let $\sigma^2_s \equiv \text{Var}[s]$, $\sigma^2_l \equiv \text{Var}[l]$, $\sigma^2_g \equiv \text{Var}[g(s)]$ and recall consistency requires $E_F[s] = E_H[l]$. The results are summarized in the following proposition.

**Proposition 6** (Monopoly’s Profit in Directed Networks). For $v(x) = x - \frac{1}{2}x^2$ and under Assumption 2 (also assuming interior allocations), the monopoly’s profit in directed networks under full information, complete information on out-degrees, network screening and uniform linear pricing with lump-sum payment are given by:

$$
\Pi_{full} = \frac{E^2[s] - \gamma^2 \sigma^2_s \sigma^2_l}{2[E^2[s] (1 - 2\gamma E[s] - \gamma^2 (\sigma^2_s + \sigma^2_l)) - \gamma^2 \sigma^2_s \sigma^2_l]}
$$

$$
\Pi_c = \frac{1}{2(1 - 2\gamma E[s] - \gamma^2 \sigma^2_s)}
$$

$$
\Pi_{ns} = \frac{1}{2(1 - 2\gamma d - \gamma^2 \sigma^2_g)}
$$

$$
\Pi_{ul} = \frac{1}{2(1 - 2\gamma d - \gamma^2 (E[s] - d)^2)}
$$

We immediately have the following three observations:

1. $\Pi_{full} = \Pi_c$ for the case of $\sigma_l = 0$. This should be intuitive, since if every buyer is equally influential, knowing each buyer’s out-degree is equivalent to full information. (Recall we require consistency in directed networks.)
2. Holding the expected degrees $E[s] = E[l]$ fixed, $\Pi_{full}$ is strictly increasing in $\sigma_t^2$, while $\Pi_{ns}$ is independent of $\sigma_t^2$, so the value of detailed network information is especially large when the dispersion of the in-degree is high. This is intuitive, since the monopoly cannot solicit any information on buyers’ in-degrees through screening in directed networks. Hence, information on buyers’ in-degrees is particularly useful, more so when there is huge variation.

3. $\Pi_{ns} > \Pi_{ul}$ (we will formally show this in the Appendix). Moreover, $\Pi_{ns}$ is increasing in $\sigma_g^2$, while $\Pi_{ul}$ is independent of $\sigma_g^2$. In other words, previous research over-estimated the value of information, and the difference is especially large when the dispersion of out-degree is high. This is also intuitive, since the monopoly can solicit information on buyers’ out-degrees through screening in directed networks, which is particularly useful when there is huge variation in out-degrees (susceptibility) across buyers.

### 7.3 Pure Screening

In this subsection, we solve for pure screening in directed networks with quadratic intrinsic value. Consider first where $X$ is fixed $X^*$. From the calculation in Section 7.1:

$$X^* = ad + b = \frac{1}{1 - 2\gamma d - \gamma^2 \sigma_g^2}$$

From Section 6.2, we know the set of interior allocations are characterized by:

$$v'(x_{ps}(s)) = -\gamma X^* g(s)$$

Since $v'(x) = 1 - x$ for quadratic intrinsic value, we get:

$$x_{ps}(s) = \frac{\gamma}{1 - 2\gamma d - \gamma^2 \sigma_g^2} g(s) + 1$$

Notice for interior allocations, we must have $1 - 2\gamma d - \gamma^2 \sigma_g^2 > 0$ (and hence $1 - \gamma d > 0$), we have $x_{ps}(s) < x_{ns}(s), \forall k$, coinciding with our analysis in Section 6.2.

Next, we consider pure screening in the steady state, or pure screening with $X = Ef[x(s)]$ in equilibrium. The set of interior allocations is still characterized by the first-order condition:

$$1 - x_{ss}(s) = -\gamma X g(s) \tag{7.7}$$

If we take expectation on both sides of Equation (7.7) and recall from the calculation in Section 7.1 that $Ef[g(s)] = d$, we get:

$$X_{ss} = \frac{1}{1 - \gamma d}$$

and hence

$$x_{ss}(s) = \frac{\gamma}{1 - \gamma d} g(s) + 1$$
Notice in this special case, we have $x_{ss}(s) < x_{ns}(s)$ for all $s \in D$.

For the monopoly’s profit $\Pi_{ss}$, notice it can still be written as $\Pi_{ss} = \max_{x(s)} \int_d^d [v(x(s)) + \gamma X g(s)x(s)]dF(s)$, so the same calculation as in the proof of Proposition 6 shows:

$$\Pi_{ss} = \frac{a_{ss}}{2\gamma} = \frac{1}{2(1 - \gamma d)}$$

Since $1 - \gamma d > 1 - 2\gamma d - \gamma^2 \sigma_g^2 > 0$ for interior allocations, we must have $\Pi_{ns} > \Pi_{ss}$, as expected from the results in Section 6.2.

8 Extensions and Discussions

In this section, we consider two extensions to the baseline model. Section 8.1 analyzes weak affiliation between in-degree and out-degree and Section 8.2 discusses the case where degrees are endogenous (depend on the consumption/allocation rule). We show most of the insights from the baseline model in Section 5 hold with the two scenarios, although there are idiosyncrasies in each case.

8.1 Weak Affiliation Between In-degree and Out-degree

In Sections 5.1 and 5.2, we analyze the two separate cases of directed and undirected networks, respectively and show they have sharply different implications for the monopoly’s profit-maximizing allocations and pricing rules. Intuitively, our description of directed networks requires independence of a buyer’s in-degree (influence) and out-degree (susceptibility). The monopoly cannot solicit information on influence in a incentive-compatible manner, and by independence cannot condition the allocation or payment on a buyer’s in-degree. By comparison, our formulation of undirected networks corresponds to perfect (positive) correlation between a buyer’s influence and susceptibility. Even though the monopoly still cannot solicit information on influence directly, she can infer the information perfectly from a buyer’s susceptibility. The optimal allocation and payment will take into account the inferred information of a buyer’s influence accordingly. As mentioned in Section 4, the network structure of most consumption goods are likely to fall in-between the two extremes. For instance, if we consider social platforms such as Facebook.com, most interactions are symmetric, but there are a handful of celebrities that attract a large number of followers. We will analyze this more general setup with positive relationship between in-degree and out-degree in the current subsection. Luckily, a lot of the insights from Section 5 carry through. We begin by formalizing the setup.

Let $f(s|l)$ be the conditional density of out-degree given in-degree. In general, the
average friend consumption $X_{fr}$ can be written as:

$$X_{fr} = \int_d^d (\int_d^d x(s,l) f(s|l) ds) \tilde{h}(l) dl,$$

where the density $\tilde{h}$ is the same as before and given by:

$$\tilde{h}(l) = \frac{h(l)l}{E_H[l]}$$

Now by the definition of conditional density, $f(s|l) = \frac{j(s,l)}{\tilde{h}(l)}$. It follows that the average friend consumption $X_{fr}$ can be written as:

$$X_{fr} = \int_d^d (\int_d^d x(s,l) j(s,l) ds) \frac{l}{E_H[l]} dl \tag{8.1}$$

The monopoly’s expected profit is adjusted accordingly, and given by:

$$ER = E[t(s,l)] = \int_d^d \int_d^d t(s,l) j(s,l) ds dl$$

The other notations on mechanism design are the same as in Section 5.1. With these, we are ready to look at the optimal revenue mechanism when each buyer’s own degrees are correlated. The first thing to note is that Lemma 1 still holds in the current setting, as observed in Section 5.1. Intuitively, since a buyer’s expected utility does not directly depend on his in-degree, the monopoly cannot solicit further information on a buyer’s influence, after knowing his susceptibility. Nevertheless, given that a buyer’s own degrees are now correlated, we can no longer say that a buyer’s allocation is independent of his in-degree. Indeed, the monopolist seller will make every effort to deduce a buyer’s influence and make allocation and pricing decisions accordingly.

Simply put, we still have a one-dimensional screening problem. Because of this, we will still write $U_{(s,l)}(s',l')$ as $U_s(s')$ and $U_{(s,l)}(s,l)$ as $U_s(s)$. Moreover, we will simply refer to $s_i$ as the type of buyer $i$. On the other hand, due to correlated degrees, the average friend consumption $X_{fr}$ is different from the average social consumption $X$. Specifically, Equation (8.1) can now by simplified as:

$$X_{fr} = \int_d^d (\int_d^d x(s,l) j(s,l) ds) \frac{l}{E_H[l]} dl$$

which is different from the average social consumption:

$$X = E[x(s)] = \int_d^d \int_d^d x(s) j(s,l) ds dl$$
The characterization of incentive constraints is almost identical to Lemma 2, but in lieu of the above observation, we need to replace the average social consumption $X$ with the average friend consumption $X_{fr}$. This is given in the following lemma:

**Lemma 4** (Characterization of Incentive Constraints with Correlated Degrees). With correlated degrees and assuming $x(\cdot)$ is bounded, the incentive constraints (BIC) hold if and only if:

1. **Sufficiency of out-degrees:** $x(s, \cdot) = x(s)$ a.e..
2. **Monotonicity:** $x(\cdot)$ is non-decreasing in $s$.
3. **BICFOC:** $U(s) = U(d) + \gamma X_{fr} \int_d^s x(t) dt$.

The proof of Lemma 4 exactly follows the lines of Lemma 2 and thus omitted. Before characterizing the optimal allocation, however, we need one more technical assumption to formalize the idea that there is a positive relationship between a buyer’s in-degree and out-degree. Let $H(l|s)$ be the conditional distribution of in-degree given out-degree and $h(l|s)$ the conditional density. We require the following assumption:

**Assumption 3** (Weak Affiliation Between In-degree and Out-degree). For any $s_1 > s_2$, $H(\cdot|s_1)$ first-order stochastically dominates $H(\cdot|s_2)$.

To get some intuitive understanding of Assumption 3, notice it implies $E[l|s = s_0]$ is non-decreasing in $s_0$, so $l$ and $s$ are non-negatively correlated. Moreover, recall affiliation between $l$ and $s$ is equivalent to “for any $s_1 > s_2$, $h(\cdot|s_1)$ dominates $h(\cdot|s_2)$ in the likelihood ratio order”, which is stronger than our Assumption 3. (See for instance Appendix D of Krishna (2010).) In other words, Assumption 3 captures some positive relationship between influence and susceptibility, which is stronger than positive correlation and weaker than affiliation, and hence the term “weak affiliation”.

Now let $\tilde{f}(s) = \int_d^d j(s, l) \frac{L}{E_H[l]} dl$. It is easy to see that $\int_d^d \tilde{f}(s) ds = 1$, and hence $\tilde{f}$ is a density function. The following lemma is an immediate result of Assumption 3:

**Lemma 5** (Likelihood Ratio Dominance). $\frac{E_{H(l|s)}[l]}{E_H[l]} = \frac{\tilde{f}(s)}{f(s)}$ is non-decreasing in $s$, or equivalently $\tilde{f}(\cdot)$ dominates $f(\cdot)$ in the likelihood ratio order.

Intuitively, Lemma 5 says that as $s$ increases, we should expect higher $l$ under the conditional distribution $H(l|s)$, which of course is a corollary of first-order stochastic dominance.

We are ready to solve to the monopoly’s profit maximizing problem. Recall $g(s) = s - \frac{1 - F(s)}{f(s)}$ is the virtual type of a type $s$ buyer, $G = E_F[g(s)x(s)]$ and $X_{fr} = \int_d^d \int_d^d x(s, l)ds\frac{L}{E_H[l]} dl$ is the average friend consumption.
Theorem 3 (Optimal Allocation with Weakly Affiliated Degrees). Under Assumptions 1 through 3, any set of interior allocations \( \{x(s)\} \) with weakly affiliated in-degree and out-degree is characterized by the following integral equation:

\[
v'(x(s)) + \gamma G \frac{E_{H(s)[l]}}{E_H[l]} + \gamma X_f g(s) = 0
\]

Moreover, given any \( v(\cdot) \), there exists \( \bar{\gamma} > 0 \) s.t. for all \( 0 < \gamma < \bar{\gamma} \), a set of interior optimal allocations exist.

Notice first that Theorem 3 reduces to Theorem 1 or 2 in the case of independent or perfectly positively correlated degrees, respectively. Specifically, in the case where \( s \) and \( l \) are independent, \( H(s) = H(.) \) and \( \frac{E_{H(s)[l]}}{E_H[l]} = 1 \). Moreover, we already observed in Section 5 that \( X_f = X \) in this special case. In the case where \( l \) is identically equal to \( s \), we have \( E_{H(s)[s]} = s \) and \( E_H[l] = E_F[s] \). Next, if we write the integral equation as \( v'(x(s)) = -\gamma G \frac{E_{H(s)[l]}}{E_H[l]} - \gamma X_f g(s) \), we see a buyer is offered higher allocation for two reasons: first, he has a higher \( g(s) \), or virtual type; second, he has a higher \( E_{H(s)[l]} \) compared with the average in-degree (influence). Intuitively, the first effect captures the buyer’s susceptibility. The monopoly wants to allocate him a higher consumption because he derives higher benefits from social interactions. The second effect captures the buyer’s implied higher influence. Even though the monopoly cannot gather information on each buyer’s influence directly, she deduces that with (weak) affiliation a buyer with higher susceptibility also has higher influence on average. The monopoly wants to give him a higher allocation because the buyer is likely to exert higher influence on other buyers. This decomposition also helps explain why we need Assumption 3: we want to ensure that a higher susceptibility does in fact imply higher influence on average, so that they have similar effects on allocation (instead of working against each other to pose difficulty on monotonicity).

Once we get the set of optimal allocations \( \{(x(s))\} \), the total payment is given by:

\[
t(s) = v(x(s)) + \gamma s X_f x(s) - U(s)
\]

\[
= v(x(s)) + \gamma s X_f x(s) - \gamma X_f \int_s^x x(t)dt.
\]

8.2 Endogenous Degrees

Up till now, we have assumed that each buyer’s in-degree and out-degree are exogenously drawn from some fixed prior distribution. Nevertheless, it may be the case that higher consumption (of the network product) enhances each buyer’s influence and susceptibility. For instance, it is reasonable to believe that as a consumer spends more time on an online game, he gets to know more people within the platform, and thus derives higher benefits from social interaction. To capture this effect, we need to allow for endogenous in-degree
and out-degree. For simplicity, we will focus on directed networks (with independent in-degree and out-degree) in the current subsection, but most of the analyses carry through to the case of undirected networks or weakly affiliated degrees with little modification.

Each buyer still has an out-degree \( s \) and in-degree \( l \). But \( s_i, l_i \) now not only depend on some intrinsic type \( \theta_i, \eta_i \) respectively, but also the allocation \( x(\theta_i, \eta_i) \). To allow for easy formulation of the consistency condition, we assume the functional form is multiplicative and separable in \( \theta \) (or \( \eta \)) and the allocation \( x \), i.e. \( s = q(x)\theta \) and \( l = q(x)\eta \).\(^{14}\) Moreover, we assume \( q(.) \) is continuously differentiable with \( q'(x), q'(x) > 0, \forall x > 0 \). This is to capture the intuition that higher consumption adds to a buyer’s social exposure, and hence leads to higher benefits from social interaction. Let \( \theta \in [\theta, \overline{\theta}] \) and \( \eta \in [\eta, \overline{\eta}] \), with \( 0 < \theta < \overline{\theta} \) and \( 0 < \eta < \overline{\eta} \). We assume the intrinsic parameter on out-degree \( \theta_i \) are independent and identically distributed according to a common distribution \( W : [\theta, \overline{\theta}] \to [0, 1] \). Similarly, we assume the intrinsic parameter on in-degree \( \eta_i \) are independent and identically distributed according to a common distribution \( Z : [\eta, \overline{\eta}] \to [0, 1] \). Moreover, we assume each buyer’s type parameters \( \theta \) and \( \eta \) are independent and independent of the type parameters of one another. Similar to the case of exogenous degrees, we assume \( W \) and \( Z \) are well-behaved, with full support on \([\theta, \overline{\theta}]\), no atom and continuous densities \( w \) and \( z \), respectively. Consistency still requires the expected out-degree equals the expected in-degree, i.e. \( E[s] = E[l] \). Due to our formulation, however, this can be simplified as \( E_W[\theta] = \int_{\theta}^{\overline{\theta}} \theta dW(\theta) = \int_{\overline{\eta}}^{\eta} \eta dZ(\eta) = E_Z[\eta] \). We examine interactions at an interim stage, so each buyer knows his own parameters \( \theta \) and \( \eta \), the common priors \( W \) and \( Z \) and the function \( q \). On the other hand, the monopoly only knows the common priors \( W \) and \( Z \) and the function \( q \). From now on, we will refer to \((\theta_i, \eta_i)\) simply as the type of buyer \( i \).

The game is very similar to the case of exogenous degrees, except that the realized types are \( \theta \) and \( \eta \) (instead of \( s \) and \( l \)). Hence, the direct contracts are now of the form \((\{x(\theta, \eta)\}_{\theta \in [\theta, \overline{\theta}], \eta \in [\eta, \overline{\eta}]}, \{t(\theta, \eta)\}_{\theta \in [\theta, \overline{\theta}], \eta \in [\eta, \overline{\eta}]})\).

The utility is identical to the case of exogenous degrees. Specifically, for a feasible outcome \((\{x(\theta, \eta), \{t(\theta, \eta)\} \geq 0 \), the utility of a buyer \( i \) with type \( (\theta, \eta) \) buyer who benefits from interacting with a set of buyers \( N_i \subset [0, 1] \) (i.e. the set of buyers that influence buyer \( i \)) is given by:\(^{15}\)

\[
u_i(\{x(\theta, \eta)\}, t(\theta, \eta)) = v(x(\theta, \eta)) + \gamma \sum_{j \in N_i} x(\theta, \eta)x_j - t(\theta, \eta)
\]

Taking expectation on both sides, we have:

\[
U(\{x(\theta, \eta)\}, t(\theta, \eta), (\theta, \eta)) = E[u_{\theta, \eta}](\{x(\theta, \eta)\}, t(\theta, \eta)) = v(x(\theta, \eta)) + \gamma \theta q(x(\theta, \eta))x(\theta, \eta)X_{fr} - t(\theta, \eta),
\]

\(^{14}\)As long as the functional form is separable, and we assume strict monotonicity in the underlying type parameter, the linear formulation is without loss of generality, since we can always apply a positive monotonic transformation.

\(^{15}\)Similar comments to Footnote 9 still applies in the current setting. Notice the expected benefits from social interaction is still linear in the underlying type parameter \( \theta \).
where

\[ X_{fr} = E[x_j | j \in N_i] \]

is the average consumption of the set of friends that influence a type \((\theta, \eta)\) buyer and the same across different types of buyers (by independence and the common priors).

Given an outcome function, for a type \((\theta, \eta)\) buyer, let \(U_{(\theta, \eta)}(\theta', \eta')\) be his expected payoff when he reports type \((\theta', \eta')\). We have:

\[ U_{(\theta, \eta)}(\theta', \eta') = v(x(\theta', \eta')) + \gamma \theta q(x(\theta', \eta'))x(\theta', \eta')X_f - t(\theta', \eta') \]

For notational simplicity, we will write \(U_{(\theta, \eta)}(\theta, \eta)\) as \(U(\theta, \eta)\).

With these notations, (BIC) is defined as:

\[ U(\theta, \eta) \geq U_{(\theta, \eta)}(\theta', \eta') \quad \forall \theta, \theta' \in [\bar{\theta}, \tilde{\theta}] \text{ and } \eta, \eta' \in [\bar{\eta}, \tilde{\eta}] \]

and (IIR) is defined as:

\[ U(\theta, \eta) \geq 0 \quad \forall \theta \in [\bar{\theta}, \tilde{\theta}], \eta \in [\bar{\eta}, \tilde{\eta}] \]

A similar argument to Lemma 1 shows that only the parameter related to out-degree \(\theta\) matters for the optimal allocation.

**Lemma 6** (Susceptibility Parameter is Sufficient for Optimal Allocation with Endogenous Degrees). For any optimal allocation, \(x(\theta, \eta) = x(\theta)\) for almost all \(\theta \in [\bar{\theta}, \tilde{\theta}]\), i.e. \(\theta\) is a sufficient statistics for the optimal allocation a.e..

The key observation is that a buyer’s expected utility does not directly depend on his influence parameter \(\eta\), so the same intuition from Lemma 1 carries through, i.e. the optimal allocation cannot depend on the influence parameter since the monopoly cannot solicit the information in an incentive compatible manner.

Given Lemma 6, we know only the susceptibility parameter matters for the optimal allocation and we will refer to \(\theta_i\) the type of buyer \(i\). Moreover, we can write \(U_{(\theta, \eta)}(\theta', \eta')\) as \(U_{\theta}(\theta)\) and \(U(\theta, \eta)\) as \(U(\theta)\). In addition, the average friend consumption is identical to the average social consumption and we will write it as \(X\).

With one-dimensional type, the characterization of incentive constraints is almost identical to Lemma 2, which is given in the following lemma:

**Lemma 7** (Characterization of Incentive Constraints with Endogenous Degrees). With endogenous degrees and assuming \(x(.)\) is bounded, the incentive constraints (BIC) hold if and only if:

1. Sufficiency of the susceptibility parameter: \(x(\theta, .) = x(\theta)\) a.e..
2. Monotonicity: \(x(.)\) is non-decreasing in \(\theta\).
Lemma 7 pins down the information rent of each type of buyer when degrees are endogenous. The proof is almost identical to Lemma 2, with the added observation that since $q(\cdot)$ is continuously differentiable, if $x(\cdot)$ is bounded, then $q(x(\cdot))x(\cdot)$ is also bounded.

Now, we are ready to solve the monopoly’s profit maximization problem. Let $g(\theta) = \theta - \frac{1-W(\theta)}{w(\theta)}$ be the virtual type of a type $\theta$ buyer. We maintain a similar assumption to Assumption 2 relevant in the current setting:

**Assumption 4 (Non-decreasing Virtual Type with Endogenous Degrees).** The virtual type $g(\theta)$ is non-decreasing in $\theta$.

Let $Q = \mathbb{E}[g(\theta)q(x(\theta))x(\theta)]$. The profit-maximizing allocations are given in the following theorem:

**Theorem 4 (Optimal Allocation with Endogenous Degrees).** Under Assumptions 1 and 4, any set of interior allocations $\{x(\theta)\}$ with endogenous degrees is characterized by the following integral equation:

$$v'(x(\theta)) + \gamma Q + \gamma X g(\theta)[q(x(\theta)) + q'(x(\theta))x(\theta)] = 0$$

Moreover, given any $v(\cdot)$, there exists $\gamma > 0$ s.t. for all $0 < \gamma < \gamma$, a set of interior optimal allocations exist.

The mathematics behind Theorem 4 is very similar to that of Theorem 1. Nevertheless, the optimality condition highlights a key intuition: with endogenous degrees, the monopolist seller has additional incentives to offer higher allocations to buyers, since it induces more social interactions (the term $q'(x(\theta))x(\theta)$ captures this effect).

Once we get the set of optimal allocations $\{x(\theta)\}$, the total payment is given by:

$$t(\theta) = v(x(\theta)) + \gamma X \theta q(x(\theta))x(\theta) - U(\theta)$$

$$= v(x(\theta)) + \gamma X \theta q(x(\theta))x(\theta) - \int_{\theta}^{\bar{\theta}} q(x(t))x(t)dt.$$ 

9 Concluding Remarks

In this paper, we develop a model of screening with network externalities and characterize the optimal allocations under different network structures. From the theoretical point of view, we build a framework for applying and extending tools from mechanism design to the study of network effects and social interaction. From the practical point of view, we offer clear-cut recommendations for real-world monopolist sellers of network products and provide a better benchmark for measuring the value of network information. Along the process, we
also compare our results with complete information and pure screening benchmarks and extend the model to accommodate for weak affiliation between in-degree and out-degree, as well as endogenous degrees. There are four main messages we want to convey:

1. As a buyer does not directly care about the group of other buyers he influences, a monopolist seller cannot directly screen on each buyer’s influence. Nevertheless, if there is positive relationship between a buyer’s influence and susceptibility, the monopolist seller will employ the piece of information and indirectly condition the allocation on a buyer’s influence.

2. As long as the optimal allocation depends on (whether directly or indirectly) a buyer’s influence, the “friendship paradox” comes into play and the average friend consumption is different from the average social consumption. This feature distinguishes screening in network models from most mechanism design/contracting problems with externalities.

3. One would overestimate the value of network information if screening were ignored, and the gap is particularly large when there is huge variation in susceptibility among buyers.

4. If higher consumption induces further social interaction (endogenous susceptibility), then the monopoly has additional incentives to sell more to buyers.

Moving forward, we see two extensions of our analyses as particularly promising and important. To begin with, it may be relevant to introduce competition. On one hand, competition reduces market power, and can potentially increase total quantity and social welfare. On the other hand, the positive network externality can have an opposing effect. It is interesting to see the overall effect on quantity and social welfare. In addition, an extension to the dynamic setting may add insight on referral incentives and product diffusion. We mention in the literature review section that our goal is to analyze the optimal selling mechanism in a static setting, and thus complements previous studies on referral incentives and product diffusion. It may still prove useful to integrate screening and referral incentives in a dynamic setting. At the very least, it helps us get closer to the best a monopolist seller can do with only distributional level network information.

From a broader perspective, the current paper helps us rethink the value of private network information. Given the growing concern on data privacy, this exercise has policy implications on social welfare beyond the interest of a monopolist seller. Consequently, it would be rewarding to study the optimal selling mechanisms under various network information structures and identify those that are acceptable to both sellers and consumers.
Appendix

In the Appendix, we provide proofs omitted in the main text.

Proof of Lemma 2: For the “only if” part, “sufficiency of out-degree” and “monotonicity” are established in Lemma 1, so it remains to show “BICFOC”.

Let \( \phi(s',s) = U_s(s') - U(s) \) (the expected utility gain of a type \( s \) buyer if he pretends to be type \( s' \)). By (BIC), for any fixed \( s' \in D \), \( \phi(s',s) \leq \phi(s',s') = 0 \), or \( \phi(s',s) \) is maximized at \( s = s' \). We know \( U_s(s') \) is differentiable in \( s \), so if \( U(s) \) is differentiable in \( s \), then we have

\[
\frac{\partial \phi(s',s)}{\partial s}\bigg|_{s=s'} = \frac{\partial U_s(s')}{\partial s}\bigg|_{s=s'} - \frac{\partial U(s)}{\partial s}\bigg|_{s=s'} = 0.
\]

Equivalently,

\[
\frac{\partial U_s(s')}{\partial s}\bigg|_{s=s'} = \frac{\partial U_s(s')}{\partial s}\bigg|_{s=s'} = \gamma x(s') X.
\]

On the other hand, notice \( U(s) \) is Lipschitz continuous in \( s \).

Proof of Theorem 1: We first show any interior allocations satisfy the integral equation. The monopoly maximizes \( E_F[t(s)] \). From the buyers’ expected utility, we get:

\[
t(s) = v(x(s)) + \gamma sx(s) X - U(s)
\]

By “BICFOC”, we have:

\[
U(s) = U(d) + \gamma X \int_d^s x(t) dt
\]

It follows that:

\[
E_F[U(s)] = U(d) + \gamma X \int_d^s \int_d^s x(t) dt dF(s)
\]  \hspace{1cm} (A1)

\(^{16}\text{Take any } s, s' \in D, \text{ by (BIC), } U(s) \geq U_s(s'). \text{ So } U(s') - U(s) \leq U(s') - U_s(s') = \gamma(s' - s)x(s') X. \text{ By assumption, } \exists B > 0 \text{ s.t. } 0 \leq x(s) < B, \forall s \in D. \text{ Together with symmetry, we know } |U(s') - U(s)| \leq \gamma dB^2, \text{ or } U(s) \text{ is Lipschitz continuous in } s.\)
Consider the last term $\int_{\bar{d}}^{t} \int_{\bar{d}}^{s} x(t) \, dt \, dF(s)$. Applying integration by parts, we have:

$$\int_{\bar{d}}^{t} \int_{\bar{d}}^{s} x(t) \, dt \, dF(s) = \int_{\bar{d}}^{t} \left( \int_{t}^{\bar{d}} dF(s) \right) x(t) \, dt$$

$$= \int_{\bar{d}}^{t} (1 - F(t)) x(t) \, dt$$

$$= \int_{\bar{d}}^{d} \frac{1 - F(s)}{f(s)} x(s) \, dF(s)$$

Plugging the last expression into Equation (A1), we have:

$$E_F[U(s)] = U(\bar{d}) + \gamma X \int_{\bar{d}}^{d} \frac{1 - F(s)}{f(s)} x(s) \, dF(s)$$

Hence, the monopoly’s profits:

$$\Pi = \max_{x(s)} \left\{ -U(\bar{d}) + \int_{\bar{d}}^{d} \left[ v(x(s)) + \gamma X g(s) x(s) \right] \, dF(s) \right\}$$

By (IIR), $U(\bar{d}) \geq 0$, so the best the monopoly can do is to set $U(\bar{d}) = 0$ and we have:

$$\Pi = \max_{x(s)} \int_{\bar{d}}^{d} \left[ v(x(s)) + \gamma X g(s) x(s) \right] \, dF(s) \quad (A2)$$

This is an infinite-dimensional maximization problem with one equality constraint $X - \int_{\bar{d}}^{d} x(s) \, dF(s) = 0$ and an infinite number of inequality constraints $x(s) \geq 0$. If we ignore the inequality constraints and set up the Lagrangian, we have:

$$\mathcal{L}(x(s), X, \lambda) = \int_{\bar{d}}^{d} \left[ v(x(s)) + \gamma X g(s) x(s) \right] \, dF(s) - \lambda \left( X - \int_{\bar{d}}^{d} x(s) \, dF(s) \right)$$

The first-order conditions w.r.t. $x(s)$, $X$ and $\lambda$ are given by:

$$v'(x(s)) + \gamma X g(s) + \lambda = 0 \quad (A3)$$

$$\gamma G - \lambda = 0 \quad (A4)$$

$$X - \int_{\bar{d}}^{d} x(s) \, dF(s) = 0$$
If the solutions are interior, then by the saddle point theorem (See Luenberger (1997)), the first-order conditions of the Lagrangian are necessary and sufficient. Combining Equations (A3) and (A4), we have:

\[ v'(x(s)) + \gamma G + \gamma X g(s) = 0 \]

It remains to show the monotonicity constraints are satisfied. Notice:

\[ v'(x(s)) = -\gamma G - \gamma X g(s) \]  
(A5)

If the allocations are interior, we know \( X = E_F[x(s)] > 0 \). Together with Assumption 2, we know the RHS of Equation (A5) is non-decreasing in \( s \). By Assumption 1, \( v'(.) \) is strictly decreasing in \( x(.) \). It follows that \( x(.) \) must be non-decreasing in \( s \), or monotonicity is indeed guaranteed.

Due to the first part of the proof, for the existence of interior allocations, it suffices to show the integral Equation (A5) admits an interior solution for \( \gamma \) small. Notice we can rewrite it as:

\[ x(s) = (v')^{-1}(-\gamma G - \gamma X g(s)) \]

In particular \( x(.) \) is continuous, so as long as \( x(d) \) is well-defined (or \( x(d) < +\infty \)) and strictly positive, together with monotonicity, we know an interior solution exists. On the other hand, notice \( \lim_{\gamma \to 0^+} -\gamma G - \gamma X g(s) = 0 \), while \( v'(0) > 0 > \lim_{x \to +\infty} v'(x) \). By continuity and the intermediate value theorem, we know \( \exists \gamma > 0 \) s.t. for any \( 0 < \gamma < \gamma \), \( x(d) \) is well-defined and strictly positive. \( \square \)

**Proof of Theorem 2:** Similar argument as in the proof of Theorem 1 shows:

\[ \Pi = \max_{x(s)} \int_{d}^{d} [v(x(s)) + \gamma X_{fr} g(s)x(s)]dF(s) \]  
(A6)

This is an infinite-dimensional maximization problem with one equality constraint \( X_{fr} - \int_{d}^{d} x(s)d\tilde{F}(s) = 0 \) and an infinite number of inequality constraints \( x(s) \geq 0 \). If we ignore the inequality constraints and set up the Lagrangian, we have:

\[ \mathcal{L}(x(s), X_{fr}, \lambda) = \int_{d}^{d} [v(x(s)) + \gamma X_{fr} g(s)x(s)]dF(s) + \lambda(X_{fr} - \int_{d}^{d} x(s)d\tilde{F}(s)) \]

\[ = \int_{d}^{d} \{[v(x(s)) + \gamma X_{fr} g(s)x(s)]f(s) - \lambda x(s)\tilde{f}(s)\}ds + \lambda X_{fr} \]

Now, taking the first-order conditions w.r.t. \( x(s), X_{fr} \) and \( \lambda \), we have:

\[ [v'(x(s)) + \gamma X_{fr} g(s)]f(s) - \lambda \tilde{f}(s) = 0 \]  
(A7)

\[ \gamma G + \lambda = 0 \]  
(A8)

\[ X_{fr} - \int_{d}^{d} x(s)d\tilde{F}(s) = 0 \]
If the solutions are interior, then by the saddle point theorem, the first-order conditions are both necessary and sufficient. Combination Equations (A7) and (A8) and recall \( \tilde{f}(s) = \frac{s}{E_F[s]} f(s) \), we have:

\[
v'(x(s)) + \gamma G \frac{s}{E_F[s]} + \gamma X_{fr} g(s) = 0
\]

It remains to check monotonicity. Notice:

\[
v'(x(s)) = -\gamma G \frac{s}{E_F[s]} - \gamma X_{fr} g(s)
\] \hspace{1cm} (A9)

If the allocations are interior, we know \( X_{fr} = E_F[x(s)] \geq 0 \). Together with Assumption 2, we know the second term of the RHS of Equation (A9) is non-decreasing in \( s \). If we know \( G \geq 0 \), then by Assumption 1, \( v'(.) \) is strictly decreasing in \( x(.) \) and we know \( x(.) \) must be non-decreasing in \( s \). The key observation is that we only need to show any optimal \( x(.) \) is characterized by the integral equation (instead of the other way around). Given any optimal \( x(.) \), we know it must be non-decreasing in \( s \). By the Chebyshev’s sum inequality \( G = E_F[g(s)x(s)] \geq 0 \) (since \( g(.) \) and \( x(.) \) are both non-decreasing in \( s \) at the optimum, as desired.

The rest of the proof on the existence of interior allocations proceeds similarly as in Theorem 1.

Proof of Proposition 1: From the buyers’ expected utility, we get:

\[
t(s) = v(x(s)) + \gamma sx(s)X_{fr} - U(s)
\]

and the monopolist seller maximizes:

\[
E_F[t(s)] = E_F[v(x(s)) + \gamma sx(s)X_{fr}] - E_F[U(s)]
\]

Given complete information, the monopoly is not subject to any incentive-compatibility constraint. She just needs to respect each buyer’s participation constraint, or \( U(s) \geq 0, \forall s \in D \). The best she can do is to set \( U(s) = 0, \forall s \). In other words, she extracts all the rents from buyers. Then the monopoly’s expected profit can be written as:

\[
\Pi = \max_{x(s)} \int_0^d [v(x(s)) + \gamma sx(s)X_{fr}]dF(s) \hspace{1cm} (A10)
\]

If we compare Equation (A10) with Equation (A6), we see the only difference is that we have \( s \) in place of \( g(s) \). Applying exactly the same techniques as in the proof of Theorem 2, we arrive at the following integral equation:

\[
v'(x(s)) + \gamma S \frac{s}{E_F[s]} + \gamma X_{fr} s = 0
\]
Since the only constraint is (IIR), we do not need to check monotonicity. Nevertheless, we notice the optimal \( x(.) \) is still non-decreasing. The proof on the existence of interior allocations then proceeds similarly as in Theorem 1.

\[ \square \]

**Proof of Proposition 2:** The proof is almost identical to that of Proposition 1. First notice a similar argument to Lemma 1 shows the optimal allocation can only depend on a buyer’s out-degree. From the buyers’ expected utility, we get:

\[ t(s) = v(x(s)) + \gamma s x(s)X - U(s) \]

and the monopolist seller maximizes:

\[ \mathbb{E}_F[t(s)] = \mathbb{E}_F[v(x(s)) + \gamma s x(s)X] - \mathbb{E}_F[U(s)] \]

Given complete information on susceptibility, the monopoly is not subject to any incentive-compatibility constraint (as long as the allocation only depends on a buyer’s out-degree). She just needs to respect each buyer’s participation constraint, or \( U(s) \geq 0, \forall s \in D \). The best she can do is to set \( U(s) = 0, \forall s \). In other words, she extracts all the rents from buyers. Then the monopoly’s expected profit can be written as:

\[ \Pi = \max_{x(s)} \int_{d}^{d} [v(x(s)) + \gamma s x(s)X] dF(s) \quad (A11) \]

If we compare Equation (A11) with Equation (A2), we see the only difference is that we have \( s \) in place of \( g(s) \). Applying exactly the same techniques as in the proof of Theorem 1, we arrive at the following integral equation:

\[ v'(x(s)) + \gamma S + \gamma X s = 0 \]

Since the only constraint is (IIR), we do not need to check monotonicity. Nevertheless, we notice the optimal \( x(.) \) is still non-decreasing in each buyer’s out-degree. The proof on the existence of interior allocations then proceeds similarly as in Theorem 1.

\[ \square \]

**Proof of Proposition 3:** From the buyers’ expected utility, we get:

\[ t(s, l) = v(x(s, l)) + \gamma s x(s, l)X_{fr} - U(s, l) \]

and the monopolist seller maximizes:

\[ \mathbb{E}_J[t(s, l)] = \mathbb{E}_J[v(x(s, l)) + \gamma s x(s, l)X_{fr}] - \mathbb{E}_J[U(s, l)] \]

Given full information, the monopoly just needs to respect each buyer’s participation constraint, or \( U(s, l) \geq 0, \forall s, l \in D \). She will choose \( U(s, l) = 0, \forall s, l \in D \) and extracts all the rents from buyers. Then the monopoly’s expected profit can be written as:

\[ \Pi = \max_{x(s)} \int_{d}^{d} \int_{d}^{d} [v(x(s, l)) + \gamma s x(s, l)X_{fr}] dF(s) dH(l) \]

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This is an infinite-dimensional maximization problem with one equality constraint \( X_{fr} - \int_d^d x(s, l) dF(s) dH(l) = 0 \) and an infinite number of inequality constraints \( x(s, l) \geq 0 \). If we ignore the inequality constraints and set up the Lagrangian, we have:

\[
\mathcal{L}(x(s, l), X_{fr}, \lambda) = \int_d^d \int_d^d [v(x(s, l)) + \gamma x(s, l) X_{fr}] dF(s) dH(l) - \lambda (X_{fr} - \int_d^d x(s, l) dF(s) d\bar{H}(l))
\]

Recall the density of \( \bar{H} \) is given by \( \bar{h}(l) = \frac{l h(l)}{E[l]} \), so we can simplify the Lagrangian as:

\[
\mathcal{L}(x(s, l), X_{fr}, \lambda) = \int_d^d \int_d^d [v(x(s, l)) + \gamma x(s, l) X_{fr} + \lambda x(s, l) \frac{l}{E[l]}] dF(s) dH(l) - \lambda X_{fr}
\]

Then the first-order conditions w.r.t. \( x(s, l), X_{fr} \) and \( \lambda \) are given by:

\[
v'(x(s, l)) + \gamma X_{fr} s + \lambda \frac{l}{E[l]} = 0 \tag{A12}
\]

\[
\gamma S - \lambda = 0 \tag{A13}
\]

\[
X_{fr} - \int_d^d \int_d^d x(s, l) \frac{l}{E[l]} dF(s) dH(l) = 0
\]

If the solutions are interior, then the first-order conditions of the Lagrangian are both necessary and sufficient. Combining Equations (A12) and (A13), we have:

\[
v'(x(s, l)) + \gamma S \frac{l}{E[l]} + \gamma X_{fr} s = 0
\]

Since the only constraint is (IIR), we do not need to check monotonicity. Nevertheless, we notice \( x(.,.) \) is non-decreasing in both \( s \) and \( l \). The proof on the existence of interior allocations then proceeds similarly as before.

**Proof of Proposition 4:** We first consider the case where both \( x_{ns}(.) \) and \( x_{ps}(.) \) are interior. In directed networks, the Lagrangian of the network screening problem is given by:

\[
\mathcal{L}(x(s), X, \lambda) = \int_d^d [v(x(s)) + \gamma X g(s) x(s)] dF(s) - \lambda (X - \int_d^d x(s) dF(s))
\]

In undirected networks, the Lagrangian of the network screening problem is given by:

\[
\mathcal{L}(x(s), X_{fr}, \lambda) = \int_d^d [v(x(s)) + \gamma X_{fr} g(s) x(s)] dF(s) + \lambda (X_{fr} - \int_d^d x(s) d\bar{F}(s))
\]

At the optimum, we have \( \lambda^* \geq 0 \) in both cases, since the direction the equality constraint binds is \( X - \int_d^d x(s) dF(s) \leq 0 \) or \( X_{fr} - \int_d^d x(s) d\bar{F}(s) \leq 0 \). Next notice the solution of the
pure screening problem corresponds to the case of $\lambda = 0$ (holding $X_{fr} = X_{fr}^*$ fixed). In both cases, we have $\mathcal{L}(., X_{fr}, \lambda)$ has strict increasing differences in $(., \lambda)$ and a lattice structure. By Topkis’s Theorem, we know $x_{ps}(s) \leq x_{ns}(s), \forall s \in D$ in both directed and undirected networks.

On the other hand, if at the optimum we have zero allocation for some type $s$, then we just need to add an infinite number of constraints $x(s) \geq 0$ to the Lagrangian, and essentially the same argument goes through.

Proof of Proposition 6: We start by solving for $\Pi_{ns}$ and $\Pi_c$. Recall from Equation (A2) the monopoly’s profit is given by:

$$\Pi_{ns} = \max_{x(s)} \int_d^d [v(x(s)) + \gamma X g(s)x(s)]dF(s)$$

$$= \int_d^d (x_{ns}(s) - \frac{1}{2}x_{ns}^2(s) + \gamma X g(s)x_{ns}(s))dF(s)$$

$$= \int_d^d x_{ns}(s)dF(s) - \frac{1}{2} \int_d^d x_{ns}^2(s)dF(s) + \gamma XG$$

If we plug $x_{ns}(s) = ag(s) + b$ into the equation and recall the definition of $a$ and $b$, we have:

$$\Pi_{ns} = ad + b - \frac{1}{2}[a^2(a^2 + d^2) + 2abd + b^2] + a \frac{b - 1}{\gamma}$$

(A14)

Now from the identities (7.2) and (7.3), we get:

$$ad + b = \frac{a}{\gamma}$$

$$a(a^2 + d^2) + bd = \frac{b - 1}{\gamma}$$

Plugging them into Equation (A14) and we have:

$$\Pi_{ns} = \frac{a}{\gamma} - \frac{1}{2}[\frac{a^2}{\gamma} + a \frac{b - 1}{\gamma}] + a \frac{b - 1}{\gamma}$$

$$= \frac{a}{2\gamma}$$

$$= \frac{1}{2(1 - 2\gamma d - \gamma^2 \sigma_g^2)}$$

For complete information on susceptibility, recall the monopoly’s maximization problem is just to replace $g(s)$ with $s$ (compared with network screening). An essentially identical calculation shows:

$$\Pi_c = \frac{1}{2(1 - 2\gamma E(s) - \gamma^2 \sigma_g^2)}$$
Next, we solve for $\Pi_{ul}$. Given any price $p$, we know the demand of a type $s$ buyer is given by:

$$x_{ul}(s) = 1 - p + \frac{\gamma(1 - p)}{1 - E[s]} s$$

Since we assume the monopoly serves every type of buyer, in order to maximize, the seller makes the participation constraint of the lowest type binds and charge a fixed payment $t_0$ equals:

$$U_d = \frac{1}{2}(1 - p)^2 \left( \frac{1 - \gamma E[s] + \gamma d}{1 - \gamma E[s]} \right)^2$$

The monopoly maximizes $E[p x(s)] + t_0$. After simplification, we have:

$$\Pi_{ul} = \max_{p} p(1 - p) - \frac{1}{2}(1 - p)^2 \left( \frac{1 - \gamma E[s] + \gamma d}{1 - \gamma E[s]} \right)^2$$  \hspace{1cm} \text{(A15)}

First-order condition w.r.t. $p$ yields:

$$\Pi_{ul} = 1 - \frac{1 - \gamma E[s]}{2(1 - \gamma E[s]) - (1 - \gamma E[s] + \gamma d)^2}$$  \hspace{1cm} \text{(A16)}

Plugging Equation (A16) into Equation (A15), we have:

$$\Pi_{ul} = \frac{1}{2(1 - 2 \gamma d - \gamma^2 (E[s] - d)^2)}$$

Next, we solve for $\Pi_{full}$. In order to do so, we need to first solve for the profit-maximizing allocation. Recall from Proposition 3 that the first-order condition is given by (we use the fact that $E[s] = E[l]$):

$$v'(x(s, l)) = -\gamma S \frac{l}{E[s]} - \gamma X_{fr} s$$

In our setting $v'(x) = 1 - x$, we have:

$$x_{full}(s, l) = 1 + \frac{\gamma S}{E[s]} l + \gamma X_{fr} s$$

which is a linear combination of $l$, $s$ and the constant 1.

If we let $m \equiv \frac{\gamma s}{E[s]}$ and $n \equiv \gamma X_{fr}$, we can solve for $S$ and $X_{fr}$.

$$S = E[s x_{full}(s, l)] = E[s] + m E^2[s] + n(E^2[s] + \sigma^2_s)$$

$$X_{fr} = E[x_{full}(s, l) \frac{l}{E[l]}] = \frac{1}{E[l]} [E[s] + m(E^2[s] + \sigma^2_l) + nE^2[s]]$$

Now, plugging the expressions of $S$ and $X_{fr}$ into the definition of $m$ and $n$, we have the following system of two linear equations:

$$\gamma [E[s] + m E^2[s] + n(E^2[s] + \sigma^2_s)] = m E[s]$$

$$\gamma [E[s] + m(E^2[s] + \sigma^2_l) + n E^2[s]] = n E[s]$$

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Applying Carmer’s rule, we get:

\[ m = \frac{\gamma E[s](E[s] + \gamma \sigma^2_l)}{E^2[s](1 - 2\gamma E[s] - \gamma^2(\sigma^2_s + \sigma^2_l)) - \gamma^2 \sigma^2_s \sigma^2_l} \]

\[ n = \frac{\gamma E[s](E[s] + \gamma \sigma^2_l)}{E^2[s](1 - 2\gamma E[s] - \gamma^2(\sigma^2_s + \sigma^2_l)) - \gamma^2 \sigma^2_s \sigma^2_l} \]

Now, we can simplify the monopoly’s profit the same manner as in the case of network screening and get:

\[ \Pi_{full} = \frac{1}{2}(1 + (m + n)E[s]) \] (A17)

Plugging the expressions of \( m \) and \( n \) into Equation (A17), we have:

\[ \Pi_{full} = E^2[s](1 - 2\gamma E[s] - \gamma^2(\sigma^2_s + \sigma^2_l)) - \gamma^2 \sigma^2_s \sigma^2_l \]

Proof of \( \Pi_{ns} \geq \Pi_{ul} \): If suffices to consider the case of \( d = 0 \) (otherwise redefine \( s' = s - s \)). Let \( y(s) = \frac{1 - F(s)}{f(s)} \) be the inverse hazard rate. Easy calculation shows:

\[ E[y(s)] = E[s] \]

\[ E[y(s)s] = \frac{1}{2}E[s^2] \]

Hence:

\[ \sigma^2_g = E[(s - y(s))^2] = E[s^2 + y(s)^2 - 2sy(s)] = E[y(s)^2] \geq (E[y(s)])^2 = E[s]^2 \]

It follows that \( \Pi_{ns} \geq \Pi_{ul} \). □

Proof of Lemma 5: By definition of \( \tilde{f} \), we know

\[ \frac{\tilde{f}(s)}{f(s)} = \frac{1}{\tilde{f}(s)} \int_{d}^{d} j(s, l) \frac{1}{E_H[l]} dl = \frac{1}{E_H[l]} \int_{d}^{d} j(s, l) \frac{l}{f(s)} dl = \frac{1}{E_H[l]} \int_{d}^{d} h(l|s) dl = \frac{E_{H(\cdot|s)}[l]}{E_H[l]} \]

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By Assumption 3, \( s_1 > s_2 \), \( H(., s_1) \) first-order stochastically dominates \( H(., s_2) \). Since the identity function is strictly increasing, we know \( E_{H(., s_1)}[l] \geq E_{H(., s_2)}[l] \), or \( E_{H(., s)}[l] \) is non-decreasing in \( s \).

**Proof of Theorem 3:** Similar to the proof of Theorem 1, the monopoly’s profit can be simplified as:

\[
\Pi = \max_{x(s)} \int d \bar{d} [v(x(s)) + \gamma X_{fr} g(s) x(s)] dF(s)
\]

This is an infinite-dimensional maximization problem with one equality constraint \( X_{fr} - \int d [\int d x(s) j(s, l) ds] \frac{l}{E_{H[l]}} dl = 0 \) and an infinite number of inequality constraints \( x(s) \geq 0 \). If we ignore the inequality constraints and set up the Lagrangian, we have:

\[
\mathcal{L}(x(s), X_{fr}, \lambda) = \int d \bar{d} [v(x(s)) + \gamma X_{fr} g(s) x(s)] dF(s) + \lambda [X_{fr} - \int d \bar{d} x(s) j(s, l) ds] \frac{l}{E_{H[l]}} dl
\]

\[
= \int d \bar{d} \left\{ [v(x(s)) + \gamma X_{fr} g(s) x(s)](\int d j(s, l) dl) + \lambda x(s)(\int d j(s, l) \frac{l}{E_{H[l]}} dl) \right\} ds + \lambda X_{fr}
\]

\[
= \int d \bar{d} \left\{ [v(x(s)) + \gamma X_{fr} g(s) x(s)] f(s) + \lambda x(s) \tilde{f}(s) \right\} ds + \lambda X_{fr}
\]

Now, the first-order conditions w.r.t. \( x(s) \), \( X_{fr} \) and \( \lambda \) are given by:

\[
[v'(x(s)) + \gamma X_{fr} g(s)] f(s) - \lambda \tilde{f}(s) = 0 \tag{A18}
\]

\[
\gamma G + \lambda = 0 \tag{A19}
\]

\[
X_{fr} - \int d \bar{d} (\int d x(s) j(s, l) ds) \frac{l}{E_{H[l]}} dl = 0
\]

If the solutions are interior, then by the saddle point theorem, the first-order conditions of the Lagrangian are both necessary and sufficient. Combining Equations (A18) and (A19), we have:

\[
v'(x(s)) + \gamma X_{fr} g(s) + \gamma G \frac{\tilde{f}(s)}{f(s)} = 0
\]

From Lemma 5, we know \( \frac{\tilde{f}(s)}{f(s)} = \frac{E_{H(., s)}[l]}{E_H[l]} \). It follows that:

\[
v'(x(s)) + \gamma X_{fr} g(s) + \gamma G \frac{E_{H(., s)}[l]}{E_H[l]} = 0
\]

It remains to show the monotonicity constraints are satisfied. Notice:

\[
v'(x(s)) = -\gamma G \frac{E_{H(., s)}[l]}{E_H[l]} - \gamma X_{fr} g(s) \tag{A20}
\]

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If the allocations are interior, then $X_{fr} \geq 0$. Together with Assumption 2 and Lemma 5, we know that as long as $G \geq 0$, the RHS of Equation (A20) is non-decreasing in $s$. The rest of proof is similar to that of Theorem 2. ☐

**Proof of Lemma 6:** Consider the incentives of two buyers of types $(\theta_1, \eta_1)$ and $(\theta_2, \eta_2)$, with $\theta_1 > \theta_2$. By (BIC) we have:

$$U(\theta_1, \eta_1) \geq U(\theta_1, \eta_2)(\theta_2, \eta_2) \text{ and } U(\theta_2, \eta_2) \geq U(\theta_2, \eta_2)(\theta_1, \eta_1)$$

Equivalently:

$$v(x(\theta_1, \eta_1)) + \gamma_1q(x(\theta_1, \eta_1))x(\theta_1, \eta_1)X_{fr} - t(\theta_1, \eta_1) \geq v(x(\theta_2, \eta_2)) + \gamma_1q(x(\theta_2, \eta_2))x(\theta_2, \eta_2)X_{fr} - t(\theta_2, \eta_2)$$

$$v(x(\theta_2, \eta_2)) + \gamma_2q(x(\theta_2, \eta_2))x(\theta_2, \eta_2)X_{fr} - t(\theta_2, \eta_2) \geq v(x(\theta_1, \eta_1)) + \gamma_2q(x(\theta_1, \eta_1))x(\theta_1, \eta_1)X_{fr} - t(\theta_1, \eta_1)$$

If we add up the two inequalities, we have:

$$\gamma(\theta_1 - \theta_2)X_{fr}(q(x(\theta_1, \eta_1))x(\theta_1, \eta_1) - q(x(\theta_2, \eta_2))x(\theta_2, \eta_2) \geq 0$$

By Assumption 1 and the fact that the product has zero cost, we know $X_{fr} > 0$ at the optimum. Moreover, $\theta_1 > \theta_2$, so we must have:

$$q(x(\theta_1, \eta_1))x(\theta_1, \eta_1) - q(x(\theta_2, \eta_2))x(\theta_2, \eta_2) \geq 0$$

Next notice $q(.)$ is strictly positive and strictly increasing, so we have $x(\theta_1, \eta_1) \geq x(\theta_2, \eta_2), \forall \theta_1 > \theta_2, \eta_1, \eta_2 \in [\eta, \bar{\eta}]$.

The rest of the proof follows the same argument as in Lemma 1. ☐

**Proof of Theorem 4:** Similar calculation to Theorem 1 shows we can simplify the monopoly’s expected profit as:

$$\Pi = \max_{x(\theta)} \int_{\theta}^{\bar{\theta}} [v(x(\theta)) + \gamma Xg(\theta)q(x(\theta))x(\theta)]dW(\theta)$$

This is an infinite-dimensional maximization problem with one equality constraint $X - \int_{\theta}^{\bar{\theta}} x(\theta)dW(\theta)$ and an infinite number of inequality constraints $x(\theta) \geq 0$. If we ignore the inequality constraints and set up the Lagrangian, we have:

$$\mathcal{L}(x(\theta), X, \lambda) = \int_{\theta}^{\bar{\theta}} [v(x(\theta)) + \gamma Xg(\theta)q(x(\theta))x(\theta)]dW(\theta) - \lambda(X - \int_{\theta}^{\bar{\theta}} x(\theta)dW(\theta))$$

The first-order conditions w.r.t. $x(\theta), X$ and $\lambda$ are given by:

$$v'(x(\theta)) + \gamma Xg(\theta)[q(x(\theta)) + q'(x(\theta))x(\theta)] + \lambda = 0 \quad (A21)$$

$$\gamma Q - \lambda = 0 \quad (A22)$$

$$X - \int_{\theta}^{\bar{\theta}} x(\theta)dW(\theta) = 0$$

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If the solutions are interior, we know the first-order conditions of the Lagrangian are both necessary and sufficient. Combining Equations (A21) and (A22), we have:

\[ v'(x(\theta)) + \gamma Q + \gamma X g(\theta)[q(x(\theta)) + q'(x(\theta))x(\theta)] = 0 \]

It remains to show the monotonicity constraints are satisfied. Notice:

\[ v'(x(\theta)) = -\gamma Q - \gamma X g(\theta)[q(x(\theta)) + q'(x(\theta))x(\theta)] \quad (A23) \]

If the allocations are interior, we know \( X = E_W[x(\theta)] > 0 \). Moreover, \( g(.) \) is non-decreasing and \( q(.) \) is strictly positive and increasing. For any small interval where \( x(.) \) is strictly decreasing in \( \theta \), we know the RHS of Equation (A23) is strictly increasing in \( \theta \), while the LHS is strictly increasing. A contradiction. It follows that \( x(.) \) must be non-decreasing in \( \theta \) a.e., or monotonicity is guaranteed. The part on the existence of interior solutions for \( \gamma \) small is similar to that of Theorem 1. \( \square \)

References


