Dynamic Matching with One-sided Incomplete Information in Investment

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April.25th, 2016
Motivation

- Many real-life matching problems have a (one-shot) timing decision.
  - Early/late marriage
  - When to go on the job market?
- Trade-off between early matching and investment
  - Invest in education (Huffington Post: A college education might be the thing standing between you and marriage.)
  - Polish job market paper
- Potentially unknown investment ability.
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Given one-sided incomplete information in investment ability, what are ‘likely’ outcomes in a two-period (one-to-one) matching market?

- How does(do) the outcome(s) compare with the complete information setting?
- What are the welfare implications?
Research Question

- Given one-sided incomplete information in investment ability, what are ‘likely’ outcomes in a two-period (one-to-one) matching market?
  - How does(do) the outcome(s) compare with the complete information setting?
  - What are the welfare implications?
Defined ‘sequential stability’ as a benchmark for predictions.

When preferences are strict and that everyone has the same cost of delay, the set of ‘sequentially stable’ matchings is a superset of the complete information setting.

All the ‘sequentially stable’ outcomes are (Pareto) efficient.

When there is ‘enough’ uncertainty in each individual’s investment ability, ‘Pareto efficiency’ + ‘static stability’ almost characterize ‘sequential stability’.
Preview of Key Results

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When there is ‘enough’ uncertainty in each individual’s investment ability, ‘Pareto efficiency’ + ‘static stability’ almost characterize ‘sequential stability’.
Related Literature

Road Map

- Introduction (done!)
- Model setup
- Review of complete information
- One-sided incomplete information
- Extensions and Conclusion
**Primitive**

- $n$ women and $n$ men (for convenience), two periods, no discounting.
- Each man a single characteristic $\alpha_i$; each woman two characteristics $\beta_j, \gamma_j$ (investment ability).
- $\alpha_i \in [\underline{\alpha}, \bar{\alpha}], \beta_j \in [\underline{\beta}, \bar{\beta}], \text{and } \gamma_j \in [\underline{\gamma}, \bar{\gamma}]$.
- WLOG $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$ and $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$.
- Marriage delay reduces utility by $\lambda$, common for everyone.
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$\alpha_i \in [\alpha, \bar{\alpha}], \beta_j \in [\beta, \bar{\beta}], \text{ and } \gamma_j \in [\gamma, \bar{\gamma}]$.

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Marriage delay reduces utility by $\lambda$, common for everyone.
Matching

- \( M = \{m_1, m_2, \ldots, m_n\}, \ W = \{w_1, w_2, \ldots, w_n\}. \)
- The function \( \mu: M \cup W \to (M \cup W) \times T \cup \{\emptyset\} \) is a \( T \)-period matching if
  1. \( \mu(m) \in W \times T \cup \{\emptyset\} \) for all \( m \in M \), \( \mu(w) \in M \times T \cup \{\emptyset\} \) for all \( w \in W \);
  2. \( \mu(i) = \{j, t\} \) (non-empty) \( \implies \mu(j) = \{i, t\} \) for all \( i \in M \cup W \).
- \( T = 2 \) is our setup.
- \( u_i(\mu(i); \bar{\gamma}) \): \( i \)'s utility from matching \( \mu \) (\( \forall i \in M \cup W \)).
- Transfers are not allowed.
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Consider a match between $m_i$ and $w_j$

- Marry in period 1: $u_{m_i}(m_i, w_j, 1; \vec{\gamma}) = v_m(\alpha_i, \beta_j)$, $u_{w_j}(m_i, w_j, 1; \vec{\gamma}) = v_w(\alpha_i, \beta_j)$.
- Marry in period 2: $u_{m_i}(m_i, w_j, 2; \vec{\gamma}) = v_m(\alpha_i, \beta_j) + \gamma_j - \lambda$, $u_{w_j}(m_i, w_j, 2; \vec{\gamma}) = v_w(\alpha_i, \beta_j) + \gamma_j - \lambda$.

Assume:

- $v_m(.)$ strictly increasing in $\beta_j$; $v_w(.)$ strictly increasing in $\alpha_i$.
- Normalize $u_i(\emptyset) = 0$, $\min\{v_m(\alpha_i, \beta_j), v_m(\alpha_i, \beta_j) + \gamma - \lambda\} > 0$ and $\min\{v_w(\alpha_i, \beta_j), v_w(\alpha_i, \beta_j) + \gamma - \lambda\} > 0$ for all $i, j$.
- Preferences are always strict.
Preferences

- Consider a match between $m_i$ and $w_j$
  - Marry in period 1: $u_{m_i}(m_i, w_j, 1; \tilde{\gamma}) = v_m(\alpha_i, \beta_j)$,
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  - Marry in period 2: $u_{m_i}(m_i, w_j, 2; \tilde{\gamma}) = v_m(\alpha_i, \beta_j) + \gamma_j - \lambda$,
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Stability: Definition

- Each individual’s attributes \((\alpha_i, \beta_j, \gamma_j)\) are perfectly observable \(\Rightarrow\) perfect ‘foresight’.
- ‘Stability’ from matching with contracts (Hatfield & Milgrom, 2005)

**Definition (Complete Information Stability)**

*Given complete information, a matching \(\mu\) is stable if*

1. **Individual rationality:** \(u_i(\mu(i); \vec{\gamma}) \geq 0, \forall i \in M \cup W;\)
2. **No-blocking:** \(\not\exists\) a tuple \((m, w, T)\) such that
   \(u_m(w, T; \vec{\gamma}) > u_m(\mu(m); \vec{\gamma})\) and
   \(u_w(m, T; \vec{\gamma}) > u_w(\mu(w); \vec{\gamma});\)

- One-to-one setting (substitutable), complete information stable matching always exists.
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2. **No-blocking**: There is no tuple \((m, w, T)\) such that
   \[ u_m(w, T; \vec{\gamma}) > u_m(\mu(m); \vec{\gamma}) \] and
   \[ u_w(m, T; \vec{\gamma}) > u_w(\mu(w); \vec{\gamma}). \]

- One-to-one setting (substitutable), complete information stable matching always exists.
Consider the following example of 3 men and 3 women:

\[ \nu_m(\alpha_i, \beta_j) = \beta_j, \nu_w(\alpha_i, \beta_j) = \alpha_i, \lambda = 2. \]

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Unique complete information stable matching \( \mu_c = \{(m_1, w_2, 2), (m_2, w_3, 1), (m_3, w_1, 2)\} \).
Example

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- \( v_m(\alpha_i, \beta_j) = \beta_j, \) \( v_w(\alpha_i, \beta_j) = \alpha_i, \lambda = 2. \)

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Welfare Implications

**Definition (Strong Pareto Efficiency)**

We say matching \( \mu \) weakly dominates another matching \( \mu' \) if \( \mu(i) \succeq_i \mu'(i) \) for every \( i \in M \cup W \), and \( \exists i \in M \cup W \) such that \( \mu(i) \succ_i \mu'(i) \). We say matching \( \mu \) is **strongly Pareto efficient** if there exists no matching \( \mu' \) that weakly dominates \( \mu \).

**Propostion (Strong Pareto Efficiency)**

A matching \( \mu = \{(m_i, w_j, T_{ij})\}_{i=1}^{n} \) is strongly Pareto efficient if and only if no one is unmatched and \( (T_{ij} = 2 \iff \gamma_j > \lambda, \forall j \in \{1, 2, \ldots, n\}) \).
Welfare Implications

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We say matching $\mu$ weakly dominates another matching $\mu'$ if $\mu(i) \succeq_i \mu'(i)$ for every $i \in M \cup W$, and $\exists$ $i \in M \cup W$ such that $\mu(i) \succ_i \mu'(i)$. We say matching $\mu$ is strongly Pareto efficient if there exists no matching $\mu'$ that weakly dominates $\mu$.

Proposition (Strong Pareto Efficiency)

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Welfare Implications

Proof:

- ‘If’: By induction from the woman matched with $m_n$.
- ‘Only if’: Consider the pair matched at the ‘wrong’ time.

In particular, all complete information stable matchings are strongly Pareto efficient.
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One-sided Incomplete Information: Assumptions

- Men’s characteristic $\vec{\alpha}$ and women’s period-1 characteristic $\vec{\beta}$ are publicly known throughout.
- Women $j$’s investment ability $\gamma_j$ is private information unless a career investment is made.
- Suppose every man has the same initial knowledge $\vec{\gamma} \in \Gamma_0 \subset [\underline{\gamma}, \bar{\gamma}]^n$ (support that matters).
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Given $\mu$, define $T$ 1-period matching $\mu^t$ as follows:

$$\forall i \in M \cup W$$

$$\mu^t(i) = \begin{cases} j & \text{if } \exists j \text{ s.t. } (i, j, t) \in \mu \\ \emptyset & \text{otherwise} \end{cases}$$

- $M_t = \{m \in M : \mu^t(m) \neq \emptyset\}$, $M_0 = M \setminus \bigcup_{i=1}^{T} M_t$;
- $W_t = \{w \in W : \mu^t(w) \neq \emptyset\}$, $W_0 = W \setminus \bigcup_{i=1}^{T} W_t$;
- $I_t = M_t \cup M_0 \cup W_0 \cup W_t$.

- Given $\mu^t$, define the induced matching $\mu'^t : I_t \to I_t \cup \{\emptyset\}$, where $\mu'^t(i) = \mu^t(i)$, $\forall i \in I_t$. 
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- $\mathcal{I}_t = M_t \cup M_0 \cup W_0 \cup W_t$.

- Given $\mu^t$, define the induced matching $\mu'{}^t : \mathcal{I}_t \to \mathcal{I}_t \cup \{ \emptyset \}$, where $\mu'{}^t(i) = \mu^t(i)$, $\forall i \in \mathcal{I}_t$. 
**Static Stability**

**Definition (Static Stability)**

A matching $\mu$ is **statically stable** if $\mu^t$ is stable (in the sense of 1-period matching), $\forall t \in \{1, 2, \ldots, T\}$.

- **Intuition**: ‘Period’ matching ‘right’.
- Set of statically stable outcomes: $\Sigma^s = \{ (\mu, \tilde{\gamma}') : \tilde{\gamma}' \in \Gamma_0$ and $\mu$ is statically stable (given $\tilde{\gamma}'$) $\}$. 
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Static Stability
Incomplete Information Blockings

Definition ($\Sigma$-blocking)

A matching outcome $(\mu, \vec{\gamma}') \in \Sigma \subset \Sigma^s$ is $\Sigma$-blocked if there is a tuple $(m_i, w_j, t)$ satisfying:

1. $u_{w_j}((m_i, t); \vec{\gamma}') > u_{w_j}(\mu(w_j); \vec{\gamma}')$;

2. For all $\vec{\gamma}''$ satisfying $(\mu, \vec{\gamma}'') \in \Sigma$ we have
   $u_{w_j}((m_i, t); \vec{\gamma}'') = u_{m_i}((w_j, t); \vec{\gamma}'') \Rightarrow u_{m_i}(\mu(m_i); \vec{\gamma}'') > u_{m_i}(\mu(m_i); \vec{\gamma}'').$
Ex-ante Sequential Stability

Definition (Ex-ante Sequential Stability)

Let $\Sigma^0 = \Sigma^s$ be the set of statically-stable matching-outcomes. Define

$$\Sigma^k = \{ (\mu, \tilde{\gamma}') \in \Sigma^{k-1} : (\mu, \tilde{\gamma}') \text{ is } \Sigma^{k-1}\text{-stable} \}$$

Then $\Sigma^* \equiv \bigcap_{k=0}^{\infty} \Sigma^k$ is the set of ex-ante sequentially stable matching outcomes.
Sequential Stability

Definition (Sequential Stability)

A matching $\mu$ is **sequentially stable** if $\exists \tilde{\gamma}' \in \Gamma_0$ such that $(\mu, \tilde{\gamma}')$ is ex-ante sequentially stable and $\mu'^2$ is stable (given true $\tilde{\gamma}$).
Main Result 1: Incomplete Information

Theorem (Key Characteristic of Sequentially Stable Matchings)

The set of sequentially stable matchings is a superset of the complete information stable matchings. Moreover, \( w_j \) marries in period 2 if and only if \( \gamma_j > \lambda \).

- Intuition: \( (\mu_c, \gamma) \) ex-ante sequentially stable and clearly \( \mu_c' \) is stable. For the second part, consider ‘incorrect’ matching time.
- In particular, we know all sequentially stable matchings are Pareto efficient.
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Main Result 2: Incomplete Information

Propostion (Further Characterization of Sequentially Stable Matchings)

Assume $\Gamma_0 = [\underline{\gamma}, \bar{\gamma}]^n$ and $v_m(\alpha_i, \bar{\beta}) - v_m(\alpha_i, \beta) < \bar{\gamma} - \gamma$, $\forall i$. Consider a matching $\mu = \{(m_i, w_j, T_{ij})\}_{i=1}^n$. Then $\mu$ is sequentially stable if and only if

1. $\mu^t$ is stable (given true $\gamma$), $\forall t$.
2. $\mu$ is strongly Pareto-efficient.
3. $\exists m_i \in M_1$, $w_{j'} \in W_2$ (both w.r.t. $\mu$) such that $\alpha_i > \alpha_{i'}$ and $\beta_{j'} > \beta_j$.

Intuition: Cannot 'backward' block. 'Forward' blocking only possible from a woman of higher period-1 characteristic.
Main Result 2: Incomplete Information

**Propostion (Further Characterization of Sequentially Stable Matchings)**

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Example Revisited

Recall the previous example of 3 men and 3 women:

- \( v_m(\alpha_i, \beta_j) = \beta_j, v_w(\alpha_i, \beta_j) = \alpha_i, \lambda = 2 \).

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- Case 1: \( \Gamma_0 = [1, 5]^3 \) (notice \( \bar{\beta} - \underline{\beta} = 2 < 4 = \bar{\gamma} - \underline{\gamma} \)).

  Three sequentially stable matchings:
  \( \mu_1 = \{(m_1, w_2, 2), (m_2, w_3, 1), (m_3, w_1, 2)\} \),
  \( \mu_2 = \{(m_1, w_2, 2), (m_2, w_1, 2), (m_3, w_3, 1)\} \) and
  \( \mu_3 = \{(m_1, w_3, 1), (m_2, w_2, 2), (m_3, w_1, 2)\} \).

- Case 2: \( \Gamma_0 = \{\vec{\gamma}' \text{ is a permutation of } \vec{\gamma}\} \). Now back to one stable matching = \( \mu_c \).
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- Case 1: \( \Gamma_0 = [1, 5]^3 \) (notice \( \bar{\beta} - \underline{\beta} = 2 < 4 = \bar{\gamma} - \underline{\gamma} \)).
  Three sequentially stable matchings:
  - \( \mu_1 = \{(m_1, w_2, 2), (m_2, w_3, 1), (m_3, w_1, 2)\} \),
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  - \( \mu_3 = \{(m_1, w_3, 1), (m_2, w_2, 2), (m_3, w_1, 2)\} \).

- Case 2: \( \Gamma_0 = \{\bar{\gamma}' \text{ is a permutation of } \bar{\gamma}\} \). Now back to one stable matching \( = \mu_c \).
Example Revisited

Recall the previous example of 3 men and 3 women:

- \( v_m(\alpha_i, \beta_j) = \beta_j, \ v_w(\alpha_i, \beta_j) = \alpha_i, \lambda = 2. \)

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<th>( \alpha )</th>
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- Alternative definition: Bayesian idea (too strong?).
- How to reach sequentially stable matchings?

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- Pareto-efficiency is the key; together with stability in each period is almost enough when there is high uncertainty in each individual’s investment ability.

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