Abstract

We study the optimal ordering of heterogeneous items in sequential auctions with unit-demand buyers. The valuation of each item depends on a buyer’s private type and an item-specific characteristic (e.g. quality). We consider two settings: (1) “generalized vertical differentiation” (i.e. valuations of all items increasing in buyers’ types); and (2) horizontal differentiation (i.e. valuations of two items moving in opposite directions in types). In the first setting, it is optimal to sell items in decreasing level of quality: It achieves full efficiency if valuations exhibit strict increasing differences (SID) in item quality and buyers’ types; and in addition, it maximizes revenue among all mechanisms that satisfy BIC, IIR, and an all-sold condition, if (SID) also holds for buyers’ virtual values. By contrast, in the second case, ordering does not matter: Either ordering delivers full efficiency and the same revenue to the seller. As a generalization, we extend our insights on efficient ordering to a third setting that combines both vertical and horizontal differentiations. Our analysis provides justification for employing sequential auctions in the sale of multiple items based on efficiency and optimal revenue.

Keywords: Sequential auctions; Mechanism design; Ordering; Optimality; Vertical differentiation; Horizontal differentiation.

JEL codes: D44, D82
1 Introduction

In a wide range of economic environments, a seller faces the problem of selling/allocating multiple units of products. Two examples that have recently attracted much attention are the sponsored search auction and the radio spectrum allocation conducted by the Federal Communications Commission (FCC). Two aspects of the problem make it distinct from traditional selling problems: To begin with, due to limited supply, the posted price mechanism we observe in most real-life markets often fares poorly in terms of social welfare or seller’s revenue in the current context. Furthermore, as there are still multiple units, a straightforward application of the single-item auction does not apply, either. In general, the optimal solution can be complicated. (See Milgrom and Segal (2017) for some of the difficulties.) Nevertheless, often times a simple mechanism is employed in such real-life contexts: sequential auction. Perhaps one of the most well-known examples is the online marketplace eBay, where participants can hold second-price auctions for almost any good. There, we observe some sellers (deliberately) choose to sell items in sequence even when they can list them at the same time. Indeed, the prevalence of sequential auctions is not confined to the Internet age. For instance, the Tsukiji fish market in Japan has been using a version of sequential auction since the 1930s (Bestor, 2004). Other time-honored markets where sequential auctions are employed include those for wine, fine arts, cut flowers and live cattle (Hu and Zou, 2015).

Given the popularity of sequential auctions, at least two aspects are worthy of investigation: To begin with, given an objective (often profit or welfare-maximization), how to optimally order the items for sale? In most of the above markets, products for sale are similar but not identical and buyers have limited (often unit) demands. This usually makes the ordering of individual auction important for sellers’ profits and social efficiency. More fundamentally, do sequential auctions (with the best ordering) have good properties in terms of maximizing profits or social welfare? The answer will shed light on the prevalence of sequential auctions itself. It is the goal of the current paper to address both questions.

The first aspect has been explored in the literature, but in relatively restricted environments. Furthermore, except for one or two instances, the second aspect of “whether sequential auctions are (near) optimal selling mechanisms” has largely been neglected. (More discussions will be given in the Literature Review section.) This is probably due to the technical difficulties associated with studying sequential auctions. For instance, a straightforward way to find an optimal ordering is to construct some equilibrium for each possible ordering, compare the social welfare (or revenue) across all orderings, and pick the ordering with the most desirable equilibrium outcome. Unfortunately, one runs into at least two

1Several papers have focused on “myopic buyers”, who do not take into account their values for future items, and naively believe their probability of winning any future item equals one over the number of buyers left. (See for instance Bernhardt and Scoones (1994) and Elkind and Fatima (2007).) For “forward-looking” buyers, most attention has been restricted to the case where buyers’ valuations across items are independent and the number of items is two. (Forward-looking buyers take into account their values for future items, and rationally judge their probability of winning each one. See for instance Budish and Zeithammer (2011).)
difficulties with this approach: First, for an arbitrary ordering of items, establishing the existence of a symmetric (perfect Bayesian) equilibrium can be hard. Moreover, even if such equilibrium exists, it is generally very hard to construct, since forward-looking agents know their bids in early rounds will affect others’ beliefs and may want to manipulate accordingly.

To overcome the technical difficulties, we look at sequential auctions from the perspective of mechanism design. Instead of attempting to construct a symmetric equilibrium for every possible ordering, we first identify one specific ordering in which a well-behaved equilibrium (one in symmetric monotonic pure strategies) does exist. Then we consider all possible mechanisms that are incentive compatible and individually rational, and find the “constrained optimal” allocation in that sense. After that, we show that the equilibrium allocation achieved in the sequential auction with our ordering is exactly the constrained optimal allocation, thus identifying the optimal ordering. In other words, a properly ordered sequential auction, as an indirect mechanism achieves the constrained optimal allocation in equilibrium. Finally, we show that the optimal ordering is unique, i.e., there exists no equilibrium that achieves the same allocation under any other ordering. The key insight is that there is no need (and we do not attempt) to construct a symmetric equilibrium for any ordering other than the optimal one.

Such an approach allows the current paper to not only offer guidance on the optimal ordering, but also shed light on the desirability of sequential auctions, with forward-looking buyers and many non-identical items. More specifically, we investigate the optimal ordering of sequential auctions, consisting of first-price or second-price sealed-bid auctions with no reserve. Buyers are ex-ante symmetric with private values (independent across buyers) and unit demand. As the seller’s objective may be profit or welfare maximization, we consider both alternatives. Items are heterogeneous. A buyer’s exact valuation for an item depends on both his type and the item characteristics. We study three alternatives:

- **Generalized vertical differentiation:** Items differ in qualities, and a higher type has a higher valuation for items of all qualities;

- **Horizontal differentiation (with two items):** For each buyer, a higher value for one item implies a lower value for the other (e.g. Hotelling’s model) and

- **Mixed differentiation:** There are two groups of items such that there is generalized vertical differentiation within each group, and horizontal differentiation across groups. This is a generalization that combines both vertical and horizontal differentiations.

In the case of generalized vertical differentiation, assuming strict increasing differences (SID) between quality and type in buyers’ valuations, we show that ordering items in...

---

2 In fact, Kittsteiner, Nikutta and Winter (2004) also identified such an equilibrium in a similar model and showed the allocations are always efficient. Nevertheless, their assumptions are stronger than ours and they used Bayes Nash equilibrium as the solution concept, so some manipulations of buyers are ignored. See the Literature Review section for more discussions.

3 See Section 3.2 (pages 9-14) for simple examples.
decreasing level of quality gives the unique welfare-maximizing sequential auction. If in addition the standard “regularity” conditions are satisfied and (SID) also holds for buyers’ virtual values, the same ordering is the unique one that maximizes the seller’s revenue. More importantly, we further show that the sequential auction with the identified ordering is an indirect (constrained) optimal selling mechanism, i.e. fully efficient and achieves constrained optimal revenue among the class of mechanisms that are Bayesian incentive compatible, interim individually rational and that all the items go to buyers.

In the case of horizontal differentiation (with two items), we show that the ordering does not matter for profits or social welfare, and that either ordering is fully efficient, though revenue may not be constrained optimal.

In the case of mixed differentiation, the results on social efficiency combine the insights from the previous two cases, so that within each group, items should be ordered from high to low quality, and that it does not matter how items from one group are ordered relative to the other group.

For the discussions in the main text, we restrict the information disclosure rule to “winning bid announcement” (highest bid is announced at the end of each round). Later (in the supplementary Appendix) we show the main results remain intact if we switch to “winner’s price announcement” (second-highest bid is announced at the end of each round) in second-price sequential auctions.

Let us emphasize the primary purpose of this paper is two-fold: In terms of content, we not only provide an optimal ordering of sequential auctions under different setups, but also illustrate how such simple mechanisms can sometimes provide ideal solutions to complicated selling problems. In terms of techniques, we offer a novel application of mechanism design to the study of sequential auctions.

2 Related Literature

The current paper is mostly related with two strands of literature. In terms of topic, it is related with studies in sequential auctions and their ordering.

Early papers on sequential auctions focus on identical items, and analyzes equilibrium selling prices in theory and practice. Those include the seminal paper by Weber (1983), followed by Ashenfelter (1989), Ashenfelter and Genesove (1992), McAfee and Vincent (1993), and Hu and Zou (2015), etc. The current paper studies sequential auctions where buyers’ valuations can vary across items, and thus opens up a wide range of applications.

The importance of ordering in sequential auctions with non-identical items was brought to attention by Bernhardt and Scoones (1994), followed by Elkind and Fatima (2007). Both papers consider myopic buyers and show the item with the “more dispersed” distribution should be ordered first. A key limitation of their results is that the definitions of “more dispersed” distributions depend on the number of buyers, offering little guidance for sellers.
By comparison, our “(strict) increasing differences” condition is standard in mechanism design and only depends on the functional form of buyers’ valuations. The work that is most closely related to the current one on ordering is Elmaghraby (2003). She studies the sequencing of two second-price sealed-bid procurement auctions and offers an optimal ordering based on the cost structure. Despite the close relationships, there are at least two notable departures from the current paper: First, she imposes convexity and symmetry in the cost structure to get a clear-cut ordering, while our corresponding condition is monotonicity (in type) of buyers’ valuations. More importantly, as she restricts to two items, there are only two possible orderings, and the comparison of profits can be made with ease by spelling out the equilibrium bidding functions. By comparison, we have to adopt an indirect approach and resort to tools from mechanism design. A few other papers on the ordering of sequential auctions include Benoît and Krishna (2001), Chakraborty, Gupta and Harbaugh (2006), Pitchik (2009) and Reiß and Schöndube (2010).

As for the efficiency and (constrained) optimality of sequential auctions, it is a relatively uncharted territory in existing literature. Three notable exceptions are Budish and Zeithammer (2011), Gong, Tan and Xing (2014) and Kittsteiner, Nikutta and Winter (2004). The first paper considers sequential auctions with ex-ante symmetric, forward-looking buyers with unit demand, and shows that when there are two items, and the valuations are independent both across buyers and items, the efficiency loss is bounded by that of one lost buyer. The second paper considers sequential auctions in which the items are identical (in buyers’ valuations) but potentially from different sellers with different reservation costs. In contrast, in the current paper the items are heterogeneous in terms of buyers’ valuations. The third paper is the closest to ours in this respect. They analyze sequential auctions where buyers’ valuations for the items are decreasing in the rank number of the auctions. Similar to Budish and Zeithammer and our paper, they assume buyers are ex-ante symmetric, forward-looking with unit demand. They derive a symmetric Bayes Nash equilibrium and show the allocation is always efficient. Despite the close resemblance, our paper is different from theirs in at least three aspects: First, due to differences in motivation, they focus on one particular ordering, while we consider all possible orderings and show there is a unique one that achieves full efficiency. Second, regarding assumptions, they assume buyers have declining valuations for the items, while we do not impose the corresponding condition in our model (i.e. buyers value higher quality). Instead, we identify increasing differences in item quality and buyer type for buyers’ valuations/virtual valuations as a fundamental condition. Moreover, we use perfect Bayesian equilibrium (PBE) as the solution concept, while they use Bayes Nash equilibrium. So our results are stronger than theirs. We think both departures regarding assumptions are important, as they point to the key driving forces behind efficiency. Third, the technique we derive the equilibria are different. We resort to tools from mechanism design, while they adopt a more traditional approach in calculating the potential gains/losses from each possible deviation directly. We think our approach is better in the context of sequential auctions, as they can not only be readily

4More precisely, any sequential auction with $N + 1$ buyers generates greater expected surplus than does the efficient Vickery-Clarke-Groves mechanisms with $N$ buyers.
applied to cases of horizontal and mixed differentiation, but slightly different setups as well. As a side note, our result is also reminiscent of Edelman, Ostrovsky and Schwarz (2007), where they show the ex-post equilibrium of generalized second-price auction generates the same payoffs to all players as the dominant strategy equilibrium of Vickery-Clarke-Groves mechanism.

Indeed, from a broader perspective, our work can be viewed as an analysis on the optimal selling procedures for substitutes. Two related works are Burguet (2005) and Milgrom and Segal (2017). Due to the complexity of the items for sale, both papers introduce buyers with multi-dimensional types. By comparison, even in our most general setup of mixed differentiation, we still restrict to buyers with one-dimensional type. Despite some loss of generality, we think our modeling choice has its own merits: To begin with, our model still captures a very wide range of real-life applications. For instance, it is natural to assume buyers only care about the weight in the sale of fish and cattle, and the click-through rates of advertisement spots. More importantly, due to the tractability of our model, we can give clear recommendation with an indirect optimal selling mechanism. In contrast, Burguet (2005) has limited results on optimal mechanisms even in the simple case of two items with two-dimensional types and Milgrom and Segal (2017) themselves point to the difficulties of an exact optimal selling mechanism in their setting.

In terms of techniques, the problem we face is similar to the one where a monopolist decides what goods to offer buyers from a pool of horizontally or vertically differentiated products. Mussa and Rosen (1978) analyze the case of horizontally differentiated products and Balstrieri and Izmalkov (2015) study the case of vertically differentiated products. From a broader perspective, we adopt the standard mechanism design approach of Mirrless (1971) and Myerson (1981). Our argument largely relies on Envelope theorem (Milgrom and Segal (2002)), which provides clean and robust insights, instead of directly calculating the equilibrium bidding strategies round by round. Two other papers that adopt a similar approach in multiple item allocations are Figueroa and Skreta (2011) and Hu and Zou (2016). The first paper studies optimal revenue mechanisms for heterogeneous objects when buyers care about the entire allocation. They show revenue-maximizing auctions may sell too often (compared to the independent case), and may even be fully efficient. Despite the similarities in formulation, their results are distinct from ours in two aspects: First, they consider optimal revenue mechanisms, while we restrict to constrained optimal revenue mechanisms. Second, their results hinge heavily on the assumption that buyers care about the entire allocation, which is non-existent in our model. The second paper considers sequential auction mechanisms under generalized interdependent values. Their model and results are very distinct from ours, but both papers use techniques from mechanism design to derive equilibrium bidding strategies. Lastly, the maximization of social surplus and seller’s revenue in our model is reminiscent of the idea of positive assortative matching

---

5 See Hu and Zou (2016), mentioned two paragraphs below.
6 See Example 1a in Section 3.2 for details.
7 Our restriction makes sense because in the independent private value setting, even with only one item, second-price auction without a reserve price is only a constrained optimal revenue mechanism.
The rest of the paper is organized as follows: Section 3 presents the model and illustrates our key ideas with three simple examples (each corresponding to one of the different cases in the next three sections). Sections 4 through 6 study generalized vertical differentiation, horizontal differentiation and mixed differentiation, respectively. Section 7 concludes. The Appendix is divided into two parts: Part 1 provides the omitted proofs and Part 2 (the supplementary Appendix) discusses a different price disclosure rule (winner’s price announcement).

3 Model

3.1 Setup

A seller has a set \( M = \{1, 2, \ldots, m\} \) \((m \geq 2)\) of heterogeneous items for sale and faces a fixed set of \( n \) buyers, with \( n \geq m + 1 \). The seller’s valuation for each of the \( m \) items is 0, while buyer \( i \)’s \((1 \leq i \leq n)\) valuation for item \( j \) \((1 \leq j \leq m)\) is \( v_j(\theta_i) > 0\), where \( \theta_i \in \mathbb{R} \) is the type of buyer \( i \). We assume \( \theta_i \) are independent and identically distributed according to a common distribution function \( F \), with full support on \([a, b]\) \((-\infty \leq a < b \leq +\infty)\) and no atom. Without loss, let \( F = U[0, 1] \).\(^8\) Buyer \( i \)’s type \( \theta_i \) is her private information, but all other aspects of the model are common knowledge. Finally, we assume buyers have unit demand: Each demands at most one item. In sum, items are heterogeneous, and for each item, we are in the symmetric, independent private value setting, where each buyer’s type is one-dimensional.

The \( m \) items are sold sequentially, one at a time. In each round the auction format is first-price or second-price sealed-bid with no reserve price, where ties are broken by assigning the item to each of the winning buyer with equal probability.\(^9\) Once a buyer wins an item, he quits the game. For the purpose of our discussions, we focus on the case of “winning bid announcement” (i.e. actual bid of the winner, instead of the price he pays is announced at the end of each round), which allows us to keep the analysis clean. We believe it is one of the most widely used forms of price announcement in real-life applications, but will also analyze an alternative information disclosure rule in the supplementary Appendix. Given her objective of welfare or profit maximization, the auctioneer chooses a selling ordering

\(^8\)Since \( \theta_i \sim F, F(\theta_i) \sim U[0, 1] \). With full support and no atom, \( F \) is strictly increasing, so the mapping from \( \theta_i \) to \( F(\theta_i) \) is a bijection and we can take \( F(\theta_i) \) to be \( i \)’s type.

\(^9\)We do not consider reserve prices for two reasons: On one hand, an analogous result to Bulow and Klemperer (1996) still holds in the current setup, i.e. an optimal sequential auction with reserve prices and \( n \) bidders is no better than an optimal sequential auction with no reserve price and \( n + m \) bidders. So when the number of items for sale is small compared to the number of bidders, reserve prices add little to the seller’s profits and can harm efficiency. On the other hand, the introduction of reserve prices significantly complicates buyers’ equilibrium bidding strategies, while providing little insight into our analysis of the optimal selling procedures.
(permutation of $M$) at the start of the sequential auction and commits to it throughout. (Since the seller’s reservation value for each item is 0, profit equals revenue, and we will use the two terms interchangeably.)

The seller and all the buyers are expected utility maximizers, with quasilinear preferences. For a feasible outcome ($\{x^j\}, t$) ($\sum_{i=1}^n x^j_i \leq 1$, $\forall j$), let $x^j_i$ be the probability buyer $i$ gets item $j$, and $t_i$ the total payment from buyer $i$ to the seller. It follows from the unit demand assumption that $\sum_{j=1}^m x^j_i \leq 1$, $\forall i$.

Given an outcome ($\{x^j\}, t$), buyer $i$’s utility is

$$U_i = \sum_{j=1}^m x^j_i v_j(\theta_i) - t_i$$

If the seller is a benevolent social planner, she only cares about total welfare

$$W = \sum_{i=1}^n \sum_{j=1}^m x^j_i v_j(\theta_i)$$

Otherwise, the seller is profit-maximizing, and she cares about profits

$$R = \sum_{i=1}^n t_i$$

Note we implicitly assume no discounting, which is justified by the fact that sequential auctions are usually held within a short period of time. We use perfect Bayesian equilibrium (PBE) as the prediction for the outcome(s) of sequential auctions. Recall PBE requires sequential rationality and Bayes updating (when possible) in every subgame. For notational simplicity, we have only spelled out the essential elements of the Bayesian game.

Depending on the structure of buyers’ valuations, there are three primary cases we consider: “Generalized vertical differentiation”, in which the valuations of all items increase in buyers’ types; “horizontal differentiation”, in which the valuations of the two items move in opposite directions in buyers’ types; and “mixed differentiation”, which is a combination of the above two cases.

Now we present the formal definitions. We provide examples for these cases in the next subsection.

**Definition 1 (Generalized Vertical Differentiation).** $v_j(\theta_i) \equiv v(q_j, \theta_i)$ ($q_j$ is the quality of item $j$). Moreover, $v_j(.) = v(q_j, .)$ is continuously differentiable and strictly increasing in $\theta_i$, $\forall q_j$.  

---

10We do not use sequential equilibrium as there is no generally accepted definition in games with infinite sets of signals and actions. See Myerson and Reny (2018).
Remarks: First, the assumption of continuous differentiability of \( v_j(. \) in type is standard in mechanism design and only used to characterize the optimal mechanism. (See the proof of Claim 4.) Second, We do not require that a buyer necessarily values high quality, even though it may seem an innocuous assumption. This is a major departure from Kittsteiner, Nikutta and Winter (2004) and the reason we use the term “generalized vertical differentiation”.\(^{11}\)

**Definition 2** (Horizontal Differentiation). \( m = 2 \). Moreover, \( v_1(.) \), \( v_2(.) \) are continuous with \( v_1(.) \) strictly increasing in \( \theta_i \), and \( v_2(.) \) strictly decreasing in \( \theta_i \).

To define mixed differentiation, let \( M_+ = \{1,2,\ldots,m_1\} \) be a subset of items such that a higher \( \theta_i \) leads to higher valuations, and \( M_- = \{m_1 + 1,\ldots,m\} \) be the other subset that a higher \( \theta_i \) leads to lower valuations. Formally:

**Definition 3** (Mixed Differentiation). \( v_j(.) \) is continuously differentiable with \( v'_j(\theta_i) > 0 \), \( \forall j \in M_+, \theta_i \) and \( v'_j(\theta_i) < 0 \), \( \forall j \in M_-, \theta_i \).

### 3.2 Examples

In this subsection, we present several examples of different forms of differentiations, i.e., how the valuations vary across items. Examples 1a through 1c illustrate “generalized vertical differentiation”, in which the valuations of all items move in the same direction in buyers’ types. Example 2 presents a case of “horizontal differentiation”, in which the valuations of the two items move in opposite directions in buyers’ types. Example 3 illustrates “mixed differentiation”, which is a combination of both vertical and horizontal differentiations.

**Example 1a** (Vertical Differentiation). *An auctioneer is selling \( m = 2 \) items. There are \( n = 3 \) buyers. Both items cost zero to the seller, but buyers have heterogeneous valuations, with *

\[
    v_1(\theta_i) = v(2, \theta_i) = 2\theta_i \quad \text{and} \quad v_2(\theta_i) = v(3, \theta_i) = 3\theta_i
\]

, in which \( \theta_i \) is a type privately known to buyer \( i \). It is commonly known that \( \theta_i \sim U[0,1], \) i.i.d. across buyers. (One practical example with this valuation structure is the online sponsored search auction, where the first spot has an average click-through rate of 2, the second spot has an average click-through rate of 3 and \( \theta_i \) is buyer \( i \)’s personal value for each click-through. See Varian (2007) and Edelman, Ostrovsky and Schwarz (2007) for details.)

Suppose the auctioneer is selling the two items using sequential second-price auctions, which ordering shall the auctioneer choose to maximize expected social surplus/profits? \( (1,2) \) or \( (2,1) \)?

\(^{11}\)Indeed, as their proof relies on direct calculation, they make heavy use of this assumption. See their proof of Theorem 1.
It turns out that ordering \((2, 1)\) gives a strictly higher social surplus \textit{and} profits than ordering \((1, 2)\). In particular:

- Under ordering \((2, 1)\), there exists a symmetric, strictly monotonic and pure strategy equilibrium, in which the buyers bid \(3\theta_i - \frac{1}{2} \cdot 2\theta_i = 2\theta_i\) in the first period, and \(2\theta_i\) in the second period.
  
  As a result, the value-3\(\theta\) item goes to the buyer with the highest \(\theta\), and the value-2\(\theta\) item goes to the buyer with the second highest \(\theta\). This outcome is fully efficient, with expected social surplus \(\mathbb{E}W = \mathbb{E}_{\theta}[3\theta_{(1)} + 2\theta_{(2)}] = 3.25\). And the seller’s expected profit \(\mathbb{E}R = \mathbb{E}_{\theta}[2\theta_{(2)} + 2\theta_{(3)}] = 1.5\).

- Under ordering \((1, 2)\), the equilibrium outcome is never efficient and the auctioneer’s expected profit in every PBE is strictly lower than 1.5.\(^{12}\)
  
  There does exist a symmetric, strictly monotonic and pure strategy equilibrium in this particular example, yet its outcome is not efficient. Specifically, the buyers bid \(2\theta_i - \frac{1}{2} \cdot 3\theta_i = 0.5\theta_i\) in the first period, and \(3\theta_i\) in the second period, yielding expected social surplus \(\mathbb{E}W = \mathbb{E}_{\theta}[2\theta_{(1)} + 3\theta_{(2)}] = 3 < 3.25\), and the seller’s expected profit \(\mathbb{E}R = \mathbb{E}_{\theta}[0.5\theta_{(2)} + 3\theta_{(3)}] = 1 < 1.5\).

We provide some intuition for the above example. First consider ordering \((2, 1)\). As the first item exhibits higher variation in buyers’ valuations, if there exists a symmetric, strictly monotonic and pure strategy equilibrium, then the outcome will be fully efficient (positive assortative matching). To show such an equilibrium exists, notice the second period is a single-item second-price auction, then it is optimal for each buyer \(i\) to bid his true valuation \(2\theta_i\). In the first period, a buyer weighs winning right away against the opportunity to win in the second period and must be indifferent in equilibrium, i.e. he is the marginal winner/loser. Assuming monotonicity in the bidding strategies, this pins down his conditional expected payoff in the second period (opportunity cost of winning in the first period) as \(\frac{1}{2} \cdot 2\theta_i = \theta_i\). Hence, each buyer will bid \(3\theta_i - \theta_i = 2\theta_i\) in the first period, as described.

Now consider the other ordering \((1, 2)\). Intuitively, as the second item exhibits higher variation in buyers’ valuations, it is not possible for the second highest type buyer to always win the in the first period, and the highest type buyer to always win in the second period in any equilibrium.\(^{13}\) In other words, the outcome cannot be fully efficient in any equilibrium. As an example, a symmetric, strictly monotonic and pure strategy equilibrium still exists in the current setup. Using similar argument as above, we can show each buyer \(i\) bids \(0.5\theta_i\) in the first period and \(3\theta_i\) in the second period. As a result, the value-2\(\theta\) item always goes to the buyer with the highest \(\theta\), and the value-3\(\theta\) item always goes to the buyer with the second highest \(\theta\), yielding an inefficient outcome.

\(^{12}\)See the proof of Theorem 1 and 2 for details.

\(^{13}\)See the proof of Theorem 1 for details.
In Example 1a, a well-behaved (symmetric, strictly monotonic and pure strategy) equilibrium exists under both orderings. Even this is not guaranteed, as shown by the following example.

**Example 1b** (Vertical Differentiation (cont.)). Consider a similar setup to Example 1a, except that buyers’ valuations are given by:

\[ v_1(\theta_i) = v(1, \theta_i) = \theta_i \quad \text{and} \quad v_2(\theta_i) = v(3, \theta_i) = 3\theta_i \]

It turns out that ordering (2,1) again gives a strictly higher social surplus and profits than ordering (1,2). In particular:

- Under ordering (2,1), there exists a symmetric, strictly monotonic and pure strategy equilibrium, in which the buyers bid \(3\theta_i - \frac{1}{2} \cdot \theta_i = 2.5\theta_i\) in the first period, and \(\theta_i\) in the second period. The outcome is fully efficient, with expected social surplus \(E W = E_{\theta}[3\theta(1) + \theta(2)] = 2.75\). And the seller’s expected profit \(E R = E_{\theta}[2.5\theta(2) + \theta(3)] = 1.5\).

- Under ordering (1,2), there is no pure, symmetric and strictly monotonic equilibrium under this setup. The equilibrium outcome is never efficient and the auctioneer’s expected profit in every PBE is strictly lower than 1.5.

Intuitively, the equilibrium under ordering (2,1) is very similar to that of Example 1a, except that buyers’ shade less in the first period, due to the larger differences in the two items’ valuations.

The more interesting observation is that there is no well-behaved (symmetric, strictly monotonic and pure strategy) equilibrium under ordering (1,2). The intuition is also clear: Even though a higher type buyer still values the first item more, his opportunity cost (the forgone possibility to win the second item) of winning is also higher. Moreover, the second effect dominates the first in this example. As a result, equilibrium bidding in the first period can no longer be strictly increasing. (Incidentally, as the lowest buyer always value both items at 0, and we do not allow for negative bids, equilibrium bidding in the first period cannot be strictly decreasing, either.)

**Remark:** One challenge in analyzing the optimal ordering is highlighted by the non-existence of a well-behaved equilibrium under some orderings, such as the ordering (1,2) in the above Example 1b. Therefore, it could be very hard, if not impossible, to directly compare profits (and social welfare) across different orderings by constructing equilibria and calculating the objectives for every possible ordering. Instead, we will follow an alternative approach in exploring the optimal ordering. Given a setting, we will show that the sequential auction with the optimal ordering achieves full efficiency or maximizes profits not only among all sequential auctions, but also among all mechanisms that satisfy Bayesian incentive compatibility, interim individual rationality, and an All-sold condition.
In particular, we will proceed as follows: First, we identify an optimal ordering ((1, 2) in Examples 1a and 1b). After that, we focus on this particular ordering and construct a well-behaved equilibrium. Then, we show that this well-behaved equilibrium leads to the best possible outcome under all the constraints. Finally, we show no other ordering can always yield the same outcome.

In both Examples 1a and 1b, the optimal ordering has the higher-valued item ranked first. This is not necessarily the case, as illustrated by the following example.

**Example 1c** (Generalized Vertical Differentiation). Consider a similar setup to Example 1a, except that buyers’ valuations are now given by:

\[
v_1(\theta_i) = v(2, \theta_i) = 2\theta_i + 10 \quad \text{and} \quad v_2(\theta_i) = v(3, \theta_i) = 3\theta_i
\]

The results are almost identical to Example 1a: Ordering (2, 1) achieves full efficiency and yields strictly higher profits than ordering (1, 2). Both social surplus and the auctioneer’s expected profit are now higher by 10 under both orderings.

Intuitively, every buyer now agrees the first item is worth 10 more and competition among them will bid the common component away under both orderings. As a result, both social surplus and the auctioneer’s expected profit increase by 10, while our analysis on optimal ordering is unchanged.

Notice in this example \(v_1(\theta_i) > v_2(\theta_i), \forall \theta_i\), so buyers’ valuations are no longer strictly increasing in “quality”. This example clearly shows what matters for optimal ordering is the variation, instead of the level, in buyers’ valuations. We will make this precise in Assumption 1.

We also explore alternative forms of differentiations, which are given by the following two examples.

**Example 2** (Horizontal Differentiation). Consider a similar setup to Example 1a, except that buyers’ valuations are given by:

\[
v_1(\theta_i) = \theta_i \quad \text{and} \quad v_2(\theta_i) = 1 - \theta_i
\]

(For a practical application, suppose there are 3 mobile service providers distributed independently on a [0, 1] interval according to a uniform distribution. There are 2 (identical) mobile bands at the two endpoints: band 1 locates at 1 and band 2 locates at 0. Each mobile provider has a common value 1 for a mobile band located at his position and his valuation for a mobile band \(l\) apart decreases by \(l\). We take a mobile provider \(i\)’s location to be his (private) type.)

It turns out the ordering does not matter, and both orderings (1, 2) and (2, 1) give the same expected social surplus and profits. In particular:
• Under ordering (1, 2), there exists a symmetric, strictly monotonic and pure strategy equilibrium, in which the buyers bid \( \theta_i \) in the first period, and \( 1 - \theta_i \) in the second period.

As a result, the value-\( \theta \) item goes to the buyer with the highest \( \theta \), and the value-\( 1 - \theta \) item goes to the buyer with the lowest \( \theta \) (highest valuation for the second item). The outcome is fully efficient, with expected social surplus \( EW = E(\theta(1) + 1 - \theta(3)) = 1.5 \).

And the seller’s expected profit \( ER = E(\theta(2) + 1 - \theta(2)) = 1 \).

• Under ordering (2, 1), there also exists a symmetric, strictly monotonic and pure strategy equilibrium, in which the buyers bid \( 1 - \theta_i \) in the first period, and \( \theta_i \) in the second period.

The allocation is always identical to that of ordering (1, 2). As a result, the outcome is fully efficient. The seller’s expected profit is the same as well.

To see the intuition, we still focus on the equilibrium bidding strategies. Consider, for instance, ordering (1, 2). The insight is similar to that of Example 1a: In the second period, each buyer bids his true valuation \( 1 - \theta_i \). In the first period, a buyer weighs winning right away against the opportunity to win in the second period and must be indifferent in equilibrium, i.e., he is the marginal winner/loser. Nevertheless, being the marginal winner/loser implies a buyer has the highest type \( \theta \) among remaining buyers. Due to horizontal differentiation, however, this also implies his valuation for the second item is the lowest and hence a 0 expected continuation payoff at the margin. Consequently, he will not shade at all in the first period! Moreover, the argument is symmetric for ordering (2, 1). The key difference from Example 1a is that being the marginal winner/loser in the first period has starkly different implications for a buyer’s winning prospect in the second period (due to different valuation structures).

**Example 3 (Mixed Differentiation).** Consider the following generalization that combines Examples 1a and 2: There are now \( n = 6 \) buyers, and the auctioneer has \( m = 4 \) items for sale. The buyers’ valuations are given as follows:

\[
v_1(\theta_i) = 2\theta_i, \quad v_2(\theta_i) = 3\theta_i, \quad v_3(\theta_i) = 2 - 2\theta_i \quad \text{and} \quad v_4(\theta_i) = 3 - 3\theta_i
\]

(For a practical example, consider the following generalization of Example 2: There are 6 mobile service providers, and 4 mobile bands, located at the two endpoints and with different qualities. In particular, bands 1 and 2 are located at 1, and bands 3 and 4 are locate at 0. Bands 2 and 4 are of higher qualities than 1 and 3, and all providers prefer higher quality bands.) Notice there are now \( 4! = 24 \) such orderings, so the analysis should be a bit more complicated.

It turns out there are six orderings that yield full efficiency (and thus the highest expected social surplus): \((2, 1, 4, 3), (2, 4, 1, 3), (2, 4, 3, 1), (4, 2, 1, 3), (4, 2, 3, 1) \) and \((4, 3, 2, 1)\). In the language of Section 3.1, \( M_+ = \{1, 2\} \) and \( M_- = \{3, 4\} \), so the key is that within the
two groups that exhibit (generalized) vertical differentiation among themselves, the items should be ranked in decreasing level of quality, while across group orderings do not matter. While the formal analysis is a bit complicated (see the proof of Lemma 4 for details), the intuition is relatively clear: Example 2 suggests if we restrict to monotonic bidding strategies, buyers’ equilibrium bidding on items in \( M_+ \) should be unaffected by items in \( M_- \), as their marginal continuation payoffs from the latter group are 0. Similarly, buyers’ equilibrium bidding on items in \( M_- \) is unaffected by items in \( M_+ \). Hence, if the items in \( M_+ \) and \( M_- \) are properly ordered, so that each small sequential auction admits a symmetric and strictly monotonic bidding strategy profile, then the entire sequential auction can be “decomposed” into two smaller sequential auctions, each consisting only of the items in \( M_+ \) and \( M_- \), respectively. On the other hand, Examples 1a through 1c suggest such an equilibrium strategy profile always exists if we order the items in \( M_+ \) and \( M_- \) in decreasing level of quality (e.g. \((2,1)\) and \((4,3)\) in our case). We can then derive the equilibrium bidding strategy profile for our six orderings. Following our calculations in Examples 1a through 1c, we know for the given six orderings, the buyers bid \( 3\theta_i - \frac{1}{\theta_i} \cdot 2\theta_i = 2.6\theta_i \) on item 2, \( 2\theta_i \) on item 1, \( 2.6(1 - \theta_i) \) on item 4 and \( 2(1 - \theta_i) \) on item 3. As a result, item 2 goes to the buyer with the highest \( \theta \), item 1 goes to the buyer with the second highest \( \theta \), item 4 goes to the buyer with the lowest \( \theta \) and item 3 goes to the buyer with the second lowest \( \theta \). The outcome is fully efficient and thus maximizes expected social surplus among all possible orderings of the sequential auctions. It can also be shown that the equilibrium outcome is never efficient under any alternative ordering(s).

### 3.3 Efficient and Constrained Optimal Revenue Outcome: A Mechanism Design Perspective

As mentioned in the introduction, except for the case of horizontal differentiation (where there are only two possible orderings), it is very difficult to directly compare sequential auctions across orderings. As such, (except for the case of horizontal differentiation), to show an ordering is “optimal”, we will show that the sequential auction with such an ordering is an “optimal” indirect mechanism in the sense that either (1) it implements an efficient outcome, or (2) it achieves constrained optimal revenue for the seller, among all mechanisms that satisfy Bayesian incentive compatibility, interim individual rationality, and an All-sold condition.

This part lays the groundwork for such discussions. By the Revelation Principle for Bayesian Nash Implementation, it suffices to look at direct mechanisms which in our setting has the message space \( S_i = \Theta_i = [0,1] \) and outcome function (i.e., social choice function) \( h(\theta) = (\{x^j(\theta_i)\}, t(\theta)) \).

As defined in Section 3.1, expected social welfare is given by:

\[
\mathbb{E}W = \mathbb{E}_\theta \sum_{i=1}^{n} \sum_{j=1}^{m} x^j(\theta) v_j(\theta_i)
\]
which is the expected sum of payoffs of the seller and all the buyers.

Similarly, expected revenue is given by:

$$ER = E \theta \sum_{i=1}^{n} t_i(\theta)$$

**Definition 4** (Efficient Outcome). An outcome function $h(\theta) = (\{x^j(\theta)\}, t(\theta))$ is efficient if it maximizes social welfare among all outcome functions.

Next, we define Bayesian incentive compatibility (BIC), interim individual rationality (IIR) and All-sold (AS) in our setting.

Given an outcome function, for buyer $i$ with type $\theta_i$, let $U_i(\theta'_i, \theta_{-i})$ be his payoff when he reports type $\theta'_i$ and the others have type $\theta_{-i}$.

$$U_i(\theta'_i, \theta_{-i}) = \sum_{j=1}^{m} x^j_i(\theta'_i, \theta_{-i}) v_j(\theta_i) - t_i(\theta'_i, \theta_{-i})$$

Bayesian incentive compatibility (BIC) is defined as:

$$E_{\theta_{-i}} U_i(\theta_i, \theta_{-i}) \geq E_{\theta_{-i}} U_i(\theta'_i, \theta_{-i}), \ \forall i, \theta_i, \theta'_i$$

which says an agent with type $\theta_i$ does not want to mimic any other type $\theta'_i$.

Participation constraint, or (interim) individual rationality (IIR), is defined as

$$E_{\theta_{-i}} U_i(\theta_i, \theta_{-i}) \geq 0, \ \forall i, \theta_i$$

which requires that each buyer $i$ is willing to participate knowing her own type $\theta_i$, but not the types of other buyers.

Moreover, we impose an All-Sold (AS) condition, i.e. the seller is committed to sell all the items. This leads to the following requirement on permissible allocations:

$$\sum_{i=1}^{n} x^j_i(\theta) = 1, \ \forall j, \theta$$

With these definitions, we are ready to define constrained optimal revenue outcome:

**Definition 5** (Constrained Optimal Revenue Outcome). An outcome function $h(\theta) = (\{x^j(\theta)\}, t(\theta))$ achieves constrained optimal revenue if it maximizes the seller’s expected revenue among all outcome functions such that (BIC), (IIR) and (AS).

14 This “all-sold” condition is different from what Figueroa and Skreta (2004) call “resource constraints”, in that they treat the seller as player 0, while we only include buyers as players. Therefore, even though the analysis is somewhat similar, we cannot directly borrow their results. Intuitively, their “resource constraints” condition simply says no item can be wasted, while we require the seller to allocate every item to the buyers.
4 Generalized Vertical Differentiation

In this section, we focus on generalized vertical differentiation. Many of the examples in the Introduction fit into this case. For instance, in the case of laptops, we may interpret \( \theta_i \) as buyer \( i \)'s desire and need for a new laptop. A high \( \theta_i \) is likely to lead to high valuations for all brands/models of laptops. Similar stories can be given for fish, wine, fine arts, cut flowers and cattle. In other words, "generalized vertical differentiation" is meant to capture the case of similar items of different qualities, precisely when sequential auctions are often used.

We begin with the optimal ordering for a welfare-maximizing seller, before turning to the investigation of a revenue-maximizing seller.

4.1 Efficient Ordering

Consider a welfare-maximizing seller whose objective is to choose an ordering of the items (permutation of \( M \)) that achieves the highest expected social welfare

\[
E W = \mathbb{E}_\theta \sum_{i=1}^{n} \sum_{j=1}^{m} x_i^j(\theta_i)v_j(\theta_i).
\]

With generalized vertical differentiation, \( v_j(\theta_i) = v(q_j, \theta_i) \), where \( q_j \) is the quality of item \( j \). Without loss of generality, we assume \( q_1 > q_2 > \ldots > q_m > 0 \) (otherwise re-label the items).\(^{15}\) The following assumption will prove useful in obtaining an efficient ordering:

**Assumption 1** (Strict Increasing Differences in True Values). \( v(\ldots) \) has strict increasing differences (SID) in \((q_j, \theta_i)\), i.e. \( \forall q_j' > q_j, \theta_i' > \theta_i, v(q_j', \theta_i') - v(q_j, \theta_i) > v(q_j, \theta_i') - v(q_j, \theta_i) \).

Intuitively, the assumption says a higher-type buyer values a higher quality item incrementally more. By continuous differentiability of \( v(q_j, \ldots) \) in \( \theta_i \), the condition is also equivalent to \( \frac{\partial v(q_j, \theta_i)}{\partial \theta_i} \) is strictly increasing in \( q_j, \forall \theta_i \). For instance, in Example 1a and 1b above, \( \frac{\partial v(q_j, \theta_i)}{\partial \theta_i} = q_j \), so \( v(\ldots) \) has (SID) in \((q_j, \theta_i)\).

With this assumption, we are ready to state our main result on the welfare-maximizing ordering.

**Theorem 1** (Efficient Ordering for Generalized Vertical Differentiation). With generalized vertical differentiation and under Assumption 1, the ordering \( \iota = (1, 2, \ldots, m) \) (i.e. selling items in decreasing level of quality) achieves full efficiency. Specifically, in either the first-price or second-price sequential auctions with this ordering, there exists a symmetric, strictly increasing and pure-strategy PBE whose outcome is efficient. In addition, it is the unique ordering that does so.

\(^{15}\)Notice we rule out the knife-edge case where any two items are of the same quality. Such an extension is straightforward, except that uniqueness is lost.
In words, the proposition says that ordering the items in decreasing level of quality not only maximizes social welfare among the class of sequential auctions, but also implements a fully efficient outcome. Apart from having a “best” ordering, we get to see why sequential auctions can be a desirable selling mechanism: Given our setting of a benevolent seller, she can achieve the optimal social surplus with a properly ordered sequential auction.

As mentioned in Section 3.3, the key insight is to work with the socially efficient allocation (which is unique) directly, and show that the sequential auction with the ordering $\iota$ implements the same allocation in equilibrium.

To formalize the claim, we need the following lemma:

**Lemma 1** (Efficient Allocation for Generalized Vertical Differentiation). With generalized vertical differentiation and under Assumption 1, the social surplus $W = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{i}^{j}(\theta) v_{j}(\theta_{i})$ is uniquely maximized when $x_{i(j)}^{j}(\theta) = 1$, where $i(j)$ satisfies $|\{i' : \theta_{i'} \geq \theta_{i(j)}\}| = j$, or the buyer with the $j$th highest type gets the $j$th highest quality item with probability 1.

The idea of the lemma is almost identical to how strict supermodularity guarantees social surplus is uniquely maximized with positive assortative matching (Becker, 1974). (Even though we allow for random allocations, we know they are never optimal as the deterministic optimum is unique.)

Theorem 1 establishes the existence of a “well-behaved equilibrium” (symmetric, strictly monotonic and pure strategy PBE) under the ordering $\iota$ (from high to low quality). The corresponding allocation is the unique welfare-maximizing outcome as described in Lemma 1. This result is a generalization of Theorem 1 of Kittsteiner, Nikutta and Winter (2004) and we use a completely different technique for our more general framework.

Much of the analysis is very similar to that of our simple Examples 1a through 1c. Here we provide some detailed intuition of why a “well-behaved equilibrium” exists under $\iota$. Recall that in a single-item auction (first-price or second-price), a well-behaved equilibrium exists. When there are multiple items, each buyer is less eager to win the item in the first round, in anticipation of getting an item in subsequent rounds. The actual value (of the item in round 1) to each buyer, called his “adjusted value”, is his true value for the item minus the opportunity cost of winning, i.e., his expected payoffs from subsequent rounds. Similar to the result of a single-item auction, in the first (and each) round of a sequential auction, each buyer’s equilibrium bid increases in his adjusted value.

What is left to show is that, in the first (and each) round, a higher value implies a higher adjusted value, under the ordering $\iota$. This is guaranteed by (SID) in true values (together with unit demand). On the other hand, if the highest-quality item is placed in a later round, a higher value (type) no longer implies a higher adjusted value in the first round, so monotonicity can break up and a “well-behaved equilibrium” may not exist.\(^{16}\)

\(^{16}\)The difficulty lies in evaluating the opportunity cost of winning. We already know from Example 1a that a buyer’s opportunity cost of winning is his expected continuation payoff (conditional on marginally
The only additional work is to establish uniqueness. To do so, we will show that the efficient outcome is never implemented in any PBE under any ordering other than \( i \). Intuitively, “mismatches” will be unavoidable for a nontrivial set of type realizations (under any alternative ordering).

4.2 Constrained Optimal Revenue Ordering

Now we turn to the question of profit-maximization, so the seller’s objective is to choose an ordering of the items (permutation of \( M \)) that achieves the highest expected revenue

\[
\mathbb{E}R = \mathbb{E}_\theta \sum_{i=1}^{n} t_i(\theta).
\]

The analysis turns out similar to the optimal efficiency ordering, with one additional caveat: The profit-maximizing seller cares about the buyers’ “virtual values”,

\[
\varphi_j(\theta_i) \equiv \varphi(q_j, \theta_i) = v(q_j, \theta_i) - (1 - \theta_i) \frac{\partial v(q_j, \theta_i)}{\partial \theta_i}
\]

, instead of their true values \( v_j(\theta_i) \).

This motivates the following two assumptions:

**Assumption 2** (Regularity). \( \varphi(q_j, \cdot) \) is strictly increasing in \( \theta_i \), \( \forall q_j \).

**Assumption 3** (Strict Increasing Differences in True and Virtual Values). Both \( v(\cdot, \cdot) \) and \( \varphi(\cdot, \cdot) \) have strict increasing differences (SID) in \((q_j, \theta_i)\).

The interpretation of the two assumptions are very similar to that of Definition 1 and Assumption 1, except that we also impose the corresponding conditions on buyers’ virtual values.

In Example 1a and 1b, \( v(q_j, \theta_i) = \theta_i q_j \), so \( \varphi(q_j, \theta_i) = 2\theta_i q_j - q_j \) and \( \frac{\partial \varphi(q_j, \theta_i)}{\partial \theta_i} = 2q_j \). Both “regularity” and “(SID) in true and virtual values” are satisfied.

We know the ultimate goal of a profit-maximizing seller is the expected total payments of the buyers, but insights from mechanism design tell us they hinge heavily on the allocations. Now, intuitions from Theorem 1 suggest us to look at the optimal mechanism design and show that the allocation is always identical to the one arising from a properly ordered sequential auction. This is precisely our next step. In other words, we solve the optimal revenue ordering also via an indirect approach, by identifying a particular ordering that implements the constrained optimal revenue outcome.

\[\text{losing the item). In the formal proof, instead of direct calculation, we associate the continuation game to a Bayesian game introduced in Section 3.3 and evaluate the expected (conditional) payoff via the Envelope formula (Milgrom and Segal (2002)).}\]

\[\text{Recall } \theta_i \sim U[0,1], \text{ so } \varphi^i(\theta_i) \text{ is precisely the standard “virtual value” of buyer } i \text{ from item } j \text{ in mechanism design.}\]
Theorem 2 (Optimal Revenue Ordering for Generalized Vertical Differentiation). With generalized vertical differentiation and under Assumptions 2 and 3, the ordering \( \iota = (1, 2, \ldots, m) \) achieves constrained optimal revenue. Specifically, in either the first-price or second-price sequential auctions with this ordering, there exists a symmetric, strictly monotonic and pure-strategy PBE whose outcome achieves constrained optimal revenue. Moreover, it is the unique ordering that does so.

Theorem 2 further sheds light on the desirability of sequential auctions: a properly ordered sequential auction achieves optimal revenue among all selling mechanisms that are (BIC), (IIR) and (AS). More importantly, under the additional assumptions of “regularity” and “(SID) in true and virtual values”, the optimal ordering given in Theorem 2 coincides with the one in Theorem 1, so a benevolent social planner may not need to worry about the efficiency loss of a profit-maximizing seller.

To get some intuition of the theorem, notice Assumption 3 implies Assumption 1, so by Theorem 1, there exists a symmetric, strictly monotonic pure strategy PBE under the ordering \( \iota \). We will show (in the Appendix) that the equilibrium results in the same allocation as the constrained optimal revenue outcome. Now notice in the specific first/second-price sequential auction, buyers of type 0 always reap zero payoff, which implies binding (IIR) for the lowest type. Together with insights of the Envelope formula (which pinpoints the information rent of a given type), we know the outcome of the “well-behaved” equilibrium under \( \iota \) achieves constrained optimal revenue.

5 Horizontal Differentiation

In this section, we turn to the case of horizontal differentiation. Recall that there are \( m = 2 \) items, and the buyer’s utility from an outcome \((x^1(\theta), x^2(\theta), t(\theta))\) is

\[
U_i(\theta) = x^1_i(\theta)v_1(\theta_i) + x^2_i(\theta)v_2(\theta_i) - t_i(\theta)
\]

, where \( v_1(.) \) and \( v_2(.) \) are continuous with \( v_1(.) \) strictly increasing in \( \theta_i \), and \( v_2(.) \) strictly decreasing in \( \theta_i \).

Readers shall keep in mind Example 2 in Section 3.2. As mentioned in Section 3.1, it is meant to capture differences in taste. There are two items such that, for each buyer, a higher value for one item implies a lower value for the other.

Now we state our main result of the section. As mentioned in Section 3.3, given there are only two possible orderings in the case of horizontal differentiation, our approach will be direct comparison of equilibrium outcome.

Theorem 3 (Indifference of Orderings with Horizontal Differentiation). With horizontal differentiation, either ordering \((1, 2)\) or \((2, 1)\) achieves full efficiency. Specifically, in either the first-price or second-price sequential auctions with either ordering \((1, 2)\) or \((2, 1)\), there
exists a symmetric, strictly monotonic and pure-strategy PBE. Moreover, the equilibrium outcome under the two orderings \((1, 2)\) and \((2, 1)\) are identical and efficient.

Contrary to the case of vertical differentiation, Theorem 3 suggests a profit-maximizing (or benevolent) seller does not have to worry about ordering when buyers’ valuations are horizontally differentiated.

The intuition is what we illustrated in Example 2: If we restrict to symmetric and strictly monotonic bidding strategies, being the marginal winner/loser in the first period implies a 0 expected continuation payoff in the second period. Consequently, in a sequential auction with two horizontally differentiated items, buyers behave (in equilibrium) as if there were two single-item auctions. Both items go to the buyer with the highest valuation and the seller reaps the same revenue as two single-item auctions. Nevertheless, it should be noted that in terms of their settings, a sequential auction with horizontally differentiated items is very different from two single-item auctions, in that buyers take into account their potential gains from the other item when they bid in the first round and are fully aware of the consequences.

To formalize our intuition on equilibrium bidding strategies, we impose an additional restriction on the beliefs of agents:

**Definition 6 (Action-determined Belief).** We say an agent’s belief \(\beta_i\) is action-determined if given any public history \(h^{\text{pub}} = (p_1, p_2, \ldots, p^k)\) and past private actions \(h^{\text{pri}}_i = (a^1_i, a^2_i, \ldots, a^k_i)\), the belief is independent of the agent’s type; i.e. \(\beta_i(h^{\text{pub}}, h^{\text{pri}}_i; \theta_i) = \beta_i(h^{\text{pub}}, h^{\text{pri}}_i; \theta'_i), \forall h^{\text{pub}}, h^{\text{pri}}_i, \theta_i, \theta'_i\).

Intuitively, as the name suggests, an agent’s belief is action-determined if it is completely pinned down by the public announcements and past private actions. The restriction is in the same spirit as Fudenberg and Tirole (1991). (Technically speaking, our information disclosure rule induces games with unobserved actions, but the public announcement at the end of each round makes it reasonable to assume that updated beliefs are common knowledge at the beginning of each round.) We need this additional restriction on agents’ beliefs to rule out counter-intuitive PBEs (where types alone can sway beliefs) in the current setup.

With this definition, we have the following two Lemmas, which characterize the “well-behaved” equilibria described in Theorem 3. For any type vector \(\theta = (\theta_1, \theta_2, \ldots, \theta_n)\), let \(\theta_{(i)}\) be the \(i^{\text{th}}\) highest value among \(\theta\).\(^{18}\)

**Lemma 2 (No Shading with Horizontal Differentiation in Second-price Sequential Auction).** With horizontal differentiation, in the ordering \((1, 2)\) (or \((2, 1)\)) of second-price sequential auctions, there exists a symmetric pure strategy PBE such that the bidding functions are strictly increasing (decreasing) in \(\theta_i\) in the first round, and strictly decreasing (increasing) in \(\theta_i\) in the second round.

\(^{18}\)Note it is different from the usual definition of order statistics.
Moreover, if all agents’ beliefs are action-determined, the first-round equilibrium bidding function $b_{1(2)}(\theta_i) = v_1(\theta_i) (v_2(\theta_i))$ a.e. in any such PBE.

**Lemma 3** (No Additional Shading with Horizontal Differentiation in First-price Sequential Auction). With horizontal differentiation, in the ordering $(1,2)$ (or $(2,1)$) of first-price sequential auctions, there exists a symmetric pure strategy PBE such that the bidding functions are strictly increasing (decreasing) in $\theta_i$ in the first round, and strictly decreasing (increasing) in $\theta_i$ in the second round. Moreover, if all agents’ beliefs are action-determined, the first-round equilibrium bidding function $b_{(1)}(\theta_i) = \mathbb{E}_{\theta_{-i}}[v_1(\theta_{-i(1)})|\theta_i > \theta_{-i(1)}] (\mathbb{E}_{\theta_{-i}}[v_2(\theta_{-i(n-1)})|\theta_i < \theta_{-i(n-1)}])$ a.e. in any such PBE.

**Remark:** Contrary to Theorems 1 and 2, we (partially) characterize all symmetric and strictly monotonic pure strategy PBEs described in Theorem 3 because in the case of horizontal differentiation, we aim to show that the ordering does not matter, and thus want to mitigate the effects of equilibrium selection.

Taken together, the two lemmas roughly state that in the case of horizontal differentiation, each buyer’s equilibrium bidding strategy in the first round does not depend on whether the other item will be auctioned, conforming with our intuition in Example 2.

With the two lemmas, Theorem 3 is immediate: For second-price sequential auctions, buyers always bid their true values in both rounds, and hence the expected revenue does not depend on the ordering. Moreover, insights from revenue equivalence suggest that the same result holds for first-price sequential auctions.

**A remark on optimality:** One other thing to note is that Theorem 3 says nothing about optimal revenue of sequential auctions with horizontal differentiation among some class of selling mechanisms. In fact, if we further assume $v_j(.)$ to be continuously differentiable, neither ordering ever achieves the optimal revenue among all selling mechanisms subject to (BIC), (IIR) and (AS). Intuitively, suppose a type $\theta_i \in [0,1]$ buyer enters the sequential auction (either first-price or second-price), then he knows with strictly positive probability he is the highest or the lowest type among all buyers. By Lemmas 2 and 3, he has strictly positive probability to win at least one item and thus earn positive payoff. In other words, whatever his type, the buyer’s expected payoff from the sequential auction is strictly positive. The seller knows this as well and can charge a (small) positive entrance fee such that the buyer is still willing to participate. This new mechanism with entrance fee still respects (BIC), (IIR) and (AS), while strictly improves the seller’s expected revenue. Hence the original mechanism (sequential auction with no entrance fee) never achieves constrained optimal revenue.

To formalize the argument, we need to derive type $\theta_i$’s expected payoff from the sequential auction. As revenue equivalence (and thus payoff equivalence) still holds for *individual* auction, it suffices to consider second-price sequential auctions. In the notation of
Section 3.3, \( E_{\theta_{i-1}} U_i(\theta_i, \theta_{-i}) = P(\theta_i \geq \theta_{-i,(1)}) E_{\theta_{i-1}} [v_1(\theta_i) - v_1(\theta_{-i,(1)}) | \theta_i \geq \theta_{-i,(1)}] + P(\theta_i \leq \theta_{-i,(n-1)}) E_{\theta_{i-1}} [v_2(\theta_i) - v_2(\theta_{-i,(n-1)}) | \theta_i \leq \theta_{-i,(n-1)}] = \theta_i^n - 1 E_{\theta_{i-1}} [v_1(\theta_i) - v_1(\theta_{-i,(1)}) | \theta_i \geq \theta_{-i,(1)}] + (1 - \theta_i)^n - 1 E_{\theta_{i-1}} [v_2(\theta_i) - v_2(\theta_{-i,(n-1)}) | \theta_i \leq \theta_{-i,(n-1)}]. \) If we add the assumption of continuous differentiability of \( v_j \), then \( \exists M > 0 \text{ s.t. } |v'_j(\theta_i)| \geq M, \forall \theta_i, j = 1, 2. \) Simple calculation shows \( E_{\theta_{i-1}} U_i(\theta_i, \theta_{-i}) \geq \frac{M}{n} (\theta_i^n + (1 - \theta_i)^n) \geq \frac{M}{n} (\frac{1}{2})^{n-1} \), or the seller can at least charge an entrance fee of \( \frac{M}{n} (\frac{1}{2})^{n-1} \).

More fundamentally, one may wonder what is the constrained optimal revenue mechanism for the seller if we impose the constraints of (BIC), (IIR) and (AS). The problem is very hard in general, with no complete answer in the existing literature. There are at least two difficulties: To begin with, from the above calculation, we see there is no obvious type whose (IIR) binds. In fact, such a type can be endogenous to the mechanism we design. Moreover, even if we can figure out the “lowest” type in some cases, standard point-wise maximization may not be readily applicable. Some special cases, however, are possible to solve. For instance, Balesri, Izmalkov and Leao (2015) solve for the case where there is a single buyer whose valuations for the two items are symmetric (in the sense of \( v_1(\theta) = v_2(1 - \theta), \forall \theta \)) and no (AS) condition. Despite differences in setting (we have multiple buyers and impose the additional (AS) condition), most of their insights on the symmetric case applies to our setting as well. For now, we will not delve further into the topic, but instead discuss mixed differentiation.

6 Mixed Differentiation

This section presents the more general model of mixed differentiation, which allows the coexistence of vertical and horizontal differentiations. In particular, there are two groups of items, such that the items are (generalized) vertically differentiated within each group, and horizontally differentiated across groups. Readers shall keep in mind Example 3 in Section 3.2.

Recall from Section 3.1 that the two groups are \( M_+ = \{1, 2, \ldots, m_1\} \) and \( M_- = \{m_1 + 1, \ldots, m\} \), where a higher \( \theta_i \) leads to higher valuations for items in \( M_+ \) and lower valuations for items in \( M_- \). Notice the model encompasses (generalized) vertical differentiation (\( m_1 = m \)) and horizontal differentiation (\( m_1 = 1, m = 2 \)) as special cases. From now on we will call \( M_+ \) “the positive group” and \( M_- \) “the negative group”.

As illustrated in Example 3, the analysis of social efficiency for mixed differentiation almost exactly combines insights from the previous two cases. This suggests the following generalization of Assumption 1 ((SID) in true values) to the current setup:

**Assumption 4** (Strict Increasing Differences within Each Group). \( v_j(\theta_i) \) has strict decreasing differences in \( (j, \theta_i) \) for \( j \in M_+ \), and strict increasing differences in \( (j, \theta_i) \) for \( j \in M_- \).
Notice we omit the quality parameter \( q_j \) in the definition of mixed differentiation for notational simplicity. Nevertheless, if we had introduced \( q_j \) and rank items in decreasing level of quality within the positive and the negative groups, Assumption 4 says that buyers’ valuations exhibit (SID) in \((q_j, \theta_i)\) for items in the positive group and (SID) in \((q_j, -\theta_i)\) for items in the negative group. In other words, within each group, the valuations behave exactly the same as in the (generalized) vertical differentiation case.

In order to state our main result of the subsection, we need the following definition to describe relationships between two permutations:

**Definition 7 (Permutation across Groups).** Let \( \kappa \) and \( \kappa' \) be two permutations of \( M \). We say \( \kappa \) is a permutation across groups of \( \kappa' \) if \( \forall j, j' \) both in \( M_+ \) or \( M_- \), \( \kappa^{-1}(j) < \kappa'^{-1}(j') \) if and only if \( \kappa'^{-1}(j) < \kappa'^{-1}(j') \).

Intuitively, as the name suggests, \( \kappa \) is a permutation across groups of \( \kappa' \) if the two permutations maintain the same ordering for items both in the positive and the negative groups. Moreover, as a binary relation, the definition of “permutation across groups” trivially satisfies the following three properties:

1. Reflexivity: \( \kappa \) is a permutation across groups of itself.
2. Symmetry: If \( \kappa \) is a permutation across groups of \( \kappa' \), then \( \kappa' \) is also a permutation across groups of \( \kappa \).
3. Transitivity: If \( \kappa \) is a permutation across groups of \( \kappa' \) and \( \kappa' \) is a permutation across groups of \( \kappa'' \), then \( \kappa \) is a permutation across groups \( \kappa'' \).

Now, we are ready for our main result of mixed differentiation.

**Theorem 4 (Optimal Efficient Ordering for Mixed Differentiation).** With mixed differentiation and under Assumption 4, an ordering \( \kappa \) achieves full efficiency if and only if it is a permutation across groups of the identity permutation \( \iota = (1, 2, \ldots, m) \). Specifically, in either the first-price or second-price sequential auctions with any such ordering, there exists a symmetric, strictly monotonic and pure strategy PBE whose outcome is efficient.

Simply put, in the case of mixed differentiation, an ordering is optimal for the welfare-maximizing seller if and only if it maintains the same ordering as the identity permutation within each group (i.e., ordering the items from high to low quality). In particular, how the items are assorted across the two groups can be arbitrary.

As seen from Example 3, the central piece is still the equilibrium bidding behaviors in a given sequential auction. There, we suggest that if the items are ordered appropriately, we can disentangle the original sequential auction into two smaller sequential auctions: one only consisting of items in the positive group, the other only consisting of items in the negative group. This is made precise in the following definition.
Definition 8 (Positive and Negative Parts of a Sequential Auction). Given a (first-price or second-price) sequential auction $A$ with ordering $\kappa$ (a permutation of $M$). Let the positive part of $A$, denoted $A_+$, be a $m_1$ round sequential with ordering $\kappa_+$ (a permutation of $M_+$) and the negative part of $A$, denoted $A_-$ be a $m - m_1$ round sequential auction with ordering $\kappa_-$ (a permutation of $M_-$), with the following conditions:

1. $A_+$ consists only of items in $M_+$ and $A_-$ consists only of items in $M_-$.
2. $\forall j, j' \in M_+, \kappa_+^{-1}(j) < \kappa_+^{-1}(j')$ if and only if $\kappa_+^{-1}(j) < \kappa_+^{-1}(j')$ and $\forall j, j' \in M_-, \kappa_-^{-1}(j) < \kappa_-^{-1}(j')$ if and only if $\kappa_-^{-1}(j) < \kappa_-^{-1}(j')$.
3. $A_+$ and $A_-$ have the same set of $n$ buyers, same prior distribution and same auction format as in $A$.

Put it another way, $A_+$ is a sequential auction with the same set of $n$ buyers, consisting only of items in the positive group, and maintaining the same ordering as in $A$. $A_-$ is a sequential auction with the same set of $n$ buyers, consisting only of items in the negative group, and maintaining the ordering as in $A$. With this definition in hand, we can formalize the key intuition for Theorem 4 into the following lemma.

Lemma 4 (Revenue Decomposition with Permutation across Groups Orderings). With mixed differentiation and under Assumption 4, if $\kappa$ is a permutation across groups of $\iota = (1, 2, \ldots, m)$ (the identity permutation), then (for any number of bidders $n \geq m + 1$), there exists a symmetric strictly monotonic pure strategy PBE in $A$, $A_+$ and $A_-$, respectively such that:

1. For every buyer $i$, his expected payoff from the sequential auction $A$ equals the sum of his expected payoffs from its positive and negative parts, $A_+$ and $A_-$ and
2. The allocation achieves full efficiency.

Lemma 4 characterizes the structure of sequential auctions with mixed differentiation: Each properly ordered sequential auction can be decomposed into its positive and negative parts. The idea is to generalize the proofs of Theorem 1, Lemma 2 and 3. For instance, if $A$ is a second-price sequential auction, we conjecture an equilibrium is for each buyer to bid in every round his true value minus the expected total conditional continuation payoffs (i.e. adjusted value). Due to monotonicity in bidding functions, a key observation is that when buyer $i$ bids on item in $M_+$, his expected conditional payoffs from $M_-$ are zero. As a result, a buyer’s equilibrium bidding on item $j$ is identical to his corresponding equilibrium bidding in $A_+$ or $A_-$. Notice we require permutation across groups orderings to ensure that there exists a symmetric strictly monotonic pure strategy PBE in $A_+$ and $A_-$ (and thus $A$). In other words, (as already suggested in Example 3) the decomposition works as long as there exists such a PBE in $A_+$ and $A_-$. With this characterization, Theorem 4 is
immediate: The “if” part is part of Lemma 4, the “only if” part follows a similar argument from the uniqueness part of Theorem 1.

Notice again that Theorem 4 says nothing about optimal revenue of sequential auctions with mixed differentiation among some class of selling mechanisms. Indeed, this must be the case, as horizontal differentiation is a special case of mixed differentiation, and the intuition of the negative result there still applies to the current setting.

7 Concluding Remarks

In this paper, we have analyzed the optimal ordering of first-price and second-price sequential auctions with differentiated items. In the process, we illustrate sequential auctions can be a desirable selling mechanism, especially with (generalized) vertically differentiated products. There are five main messages we want to convey:

1. In first-price or second-price sequential auctions, an equilibrium is for each buyer to bid in every round as if his adjusted value (true value minus the expected continuation payoff conditional on marginal loss) were his true value. This insight is in the same flavor as Budish and Zeithammer (2011), where buyers have independent valuations across items, and generalizes the results of Kittsteiner, Nikutta and Winter (2004).

2. With (generalized) vertically differentiated products, it is optimal to sell items in decreasing level of quality. By comparison, the ordering does not matter with horizontally differentiated products. The results are sharply different because the structures of conditional expected continuation payoffs are different: In the case of vertical differentiation, conditional expected continuation payoffs are positive and vary across different orderings, while they are always zero with horizontal differentiation, regardless of the ordering.

3. With mixed differentiated products, the optimal orderings are such that the items within the same group are ordered in decreasing level of quality, and it does not matter how the two groups are assorted. Moreover, under these orderings we can decompose the entire sequential auction into two smaller sequential auctions, one for the positive group and one for the negative group; each small sequential auction admits a “well-behaved” (symmetric, strictly monotonic and pure strategy) equilibrium. This decomposition can be very helpful in our welfare and revenue comparison across orderings.

4. Sequential auctions with proper orderings can achieve full efficiency. Moreover, with (generalized) vertically differentiated products, they can achieve the optimal revenue among all mechanisms that are incentive compatible and individually rational.

5. Mechanism design can be useful in the study of sequential auctions, both in finding the optimal orderings and in deriving the equilibrium bidding behaviors.
Moving forward, it would be interesting to derive the optimal selling mechanisms in the cases of horizontal differentiation and mixed differentiation. Apart from their intrinsic values, the answers can help shed light on the revenue losses of sequential auctions under the two setups.

More fundamentally, one may ask the broader question of when it is helpful to combine different markets and when separate markets are good enough. In the case of (generalized) vertically differentiated items, our analysis shows that (properly ordered) sequential auctions (separate markets) often perform exactly as well. In the case of independent valuations and two items, Budish and Zeithammer (2011) show that sequential auctions always perform nearly as well when the number of buyers are large. It is an interesting open question to explore other real-life setups, the answers to which will very likely help with numerous existing markets, as well as future market design.

References


Appendix: Omitted Proofs

In this Appendix, we provide proofs omitted in the main text.

Proof of Lemma 1: As we are working with each particular realization, we can assume without loss $\theta_1 > \theta_2 > \ldots > \theta_n$ (otherwise re-label the individuals and notice $P(\theta_i = \theta_j) = 0$, $\forall i, j$). To stress the dependence of social surplus on allocations, we write $EW(\{x^j\})$. We prove the lemma in two steps.

Step L1.1: In any welfare-maximizing allocation, $x^j_i(\theta) = 0$, $\forall j$ and $i \geq m + 1$, or only the highest $m$ types can get any good with positive probability.

Assume not, then $\exists 1 \leq i_1 \leq m, i_2 \geq m + 1$ and $j_1$ such that $\sum_{j=1}^{m} x^j_{i_1}(\theta) < 1$ and $x^j_{i_2}(\theta) > 0$. Consider the alternative allocation $\{y^j\}$ s.t. $y^j_i(\theta) = x^j_i(\theta), \forall i \neq i_1, i_2$ or $j \neq j_1$ and $y^j_{i_1}(\theta) = x^j_{i_1}(\theta) + \varepsilon$, $y^j_{i_2}(\theta) = x^j_{i_2}(\theta) - \varepsilon$ for some small $\varepsilon > 0$ (clearly feasible). Then $EW(\{y^j\}) - EW(\{x^j\}) = \varepsilon(v_{j_1}(\theta_{i_1}) - v_{j_1}(\theta_{i_2})) > 0$. A contradiction. Notice this argument also shows $\sum_{j=1}^{m} x^j_i(\theta) = 1$, $\forall 1 \leq i \leq m$ (simply treat the seller as buyer $n + 1$ with type $\theta_{n+1} = 0$), or the highest $m$ types get some item for sure.

Step L1.2: In any welfare-maximizing allocation, $x^j_i(\theta) = 1, \forall 1 \leq i \leq m$, or the highest $m$ types each gets an item in the order of their types, which is the desired result. Note the second step is almost the problem of positive assortative matching (with random allocations).

Assume not, then $\exists i_1 < i_2$ and $j_1 < j_2$ such that $x^{j_2}_{i_1}(\theta) > 0$ and $x^{j_1}_{i_2}(\theta) > 0$. Consider the alternative allocation $\{y^j\}$ such that $y^j_i(\theta) = x^j_i(\theta), \forall i \neq i_1, i_2$ or $j \neq j_1,j_2$ and $y^{j_1}_{i_1}(\theta) = x^{j_1}_{i_1}(\theta) + \varepsilon$, $y^{j_2}_{i_2}(\theta) = x^{j_2}_{i_2}(\theta) - \varepsilon$, $y^{j_1}_{i_2}(\theta) = y^{j_2}_{i_2}(\theta) + \varepsilon$ for some small $\varepsilon > 0$ (clearly feasible). Then $EW(\{y^j\}) - EW(\{x^j\}) = \varepsilon[(v_{j_1}(\theta_{i_1}) - v_{j_1}(\theta_{i_2})) - (v_{j_2}(\theta_{i_1}) - v_{j_2}(\theta_{i_2}))] > 0$ (by Assumption 1). A contradiction.

Proof of Theorem 1: We prove the theorem in three steps. In Step T1.1, we derive a necessary condition of the (BIC) incentive constraints, which will not only help with the construction of the PBEs in Step T1.2, but also prove useful in deriving the constrained optimal revenue mechanism in Theorem 2. In Step T1.2, for the ordering $\iota = (1, 2, \ldots, m)$, we explicitly construct a symmetric, strictly increasing and pure strategy profile and a set of beliefs for first-price and second-price sequential auctions. Then, we show the constructed strategy profile (and the beliefs) indeed form a PBE in the corresponding sequential auction. In Step T1.3, we prove uniqueness by showing no PBE under any alternative ordering can always yield an efficient outcome.

Step T1.1: Our goal in this step is to prove the following claim:

Claim 1 (Simplification of Incentive Constraints). A necessary condition of the incentive
constraints (BIC) is (BICFOC):

\[
E_{\theta_i} U_i(\theta_i, \theta_{-i}) = E_{\theta_i} U_i(0, \theta_{-i}) + \int_0^{\theta_i} E_{\theta_{-i}} \left[ \sum_{j=1}^m x_i^j(t, \theta_{-i}) v_j(t) \right] dt, \forall \theta_i
\]

Intuitively, (BICFOC) helps pin down the information rent of each type of buyer. It says that the expected utility of a buyer can be written as the expected utility of the lowest type plus the integral of the expected value of the cross product of the probability of getting the goods and the marginal values.

**Proof of Claim 1:** Let \( \Phi_i(\theta'_i, \theta_i) = E_{\theta_{-i}} \left[ \sum_{j=1}^m x_i^j(\theta'_i, \theta_{-i}) v_j(\theta_i) - t_i(\theta'_i, \theta_{-i}) \right] \) (or the expected utility of buyer \( i \) if he pretends to be type \( \theta'_i \)); \( \phi_i(\theta'_i, \theta_i) = \Phi_i(\theta'_i, \theta_i) - \Phi_i(\theta_i, \theta_i) \) (or the expected utility gain of buyer \( i \) if he pretends to be type \( \theta'_i \)).

By (BIC), for any fixed \( \theta'_i \in (0, 1) \), \( \phi_i(\theta'_i, \theta_i) \leq \phi_i(\theta'_i, \theta'_i) = 0 \); or \( \phi_i(\theta'_i, \theta_i) \) is maximized at \( \theta_i = \theta'_i \). As \( \Phi_i(\theta'_i, \theta_i) \) is differentiable in \( \theta_i \), if \( \Phi_i(\theta_i, \theta_i) = E_{\theta_{-i}} U_i(\theta_i, \theta_{-i}) \) is differentiable in \( \theta_i \) at \( \theta_i = \theta'_i \), then we have \( \frac{\partial \phi_i(\theta'_i, \theta_i)}{\partial \theta_i}\bigg|_{\theta_i=\theta'_i} = 0 \); or \( \frac{\partial E_{\theta_{-i}} U_i(\theta_i, \theta_{-i})}{\partial \theta_i}\bigg|_{\theta_i=\theta'_i} = \frac{\partial \Phi_i(\theta'_i, \theta_i)}{\partial \theta_i}\bigg|_{\theta_i=\theta'_i} = E_{\theta_{-i}} \left[ \sum_{j=1}^m x_i^j(\theta'_i, \theta_{-i}) v'_j(\theta'_i) \right] \).

On the other hand, notice \( E_{\theta_{-i}} U_i(\theta_i, \theta_{-i}) \) is Lipschitz continuous in \( \theta_i \).\(^{19}\) Hence, \( E_{\theta_{-i}} U_i(\theta_i, \theta_{-i}) \) is absolutely continuous and differentiable (a.e.) in \( \theta_i \). By the fundamental theorem of calculus, we get (BICFOC).

**Step T1.2:** This step is a generalization of Theorem 1 of Kittsteiner, Nikutta and Winter (2004). The following two claims show that what they construct as Bayes Nash Equilibrium also forms part of the PBE in our setup. Because of the focus on PBE, some additional work is needed, in particular establishing the one-deviation property. Moreover, contrary to their direct calculation, we borrow results from the mechanism design problem, i.e. the characterization of incentive constraints in Step 1. This not only simplifies the analysis, but also readily generalizes to the cases of horizontal and mixed differentiation (as well as other settings). A few more notations are necessary. Given \( \theta \), recall \( \theta_{-i} \) is the vector omitting \( \theta_i \) and \( \theta_{-i,1} \) the largest value among \( \theta_{-i} \). Additionally, let \( U^{(k+1)}(\theta_i, s) \) denote the expected continuation payoff of the sequential auction to a buyer of type \( \theta_i \) from round \( (k+1) \) onwards conditional on the highest type (among other buyers) in round \( k \) is \( s \). With these notations, the desired equilibrium bidding functions are given in the following two claims:

**Claim 2** (Equilibrium Bidding in First-price Sequential Auction with Generalized Vertical Differentiation). With generalized vertical differentiation and under Assumption 1, in

\(^{19}\) Take any \( \theta_i, \theta'_i \in [0, 1] \) by (BIC), \( E_{\theta_{-i}} U_i(\theta_i, \theta_{-i}) \geq \Phi_i(\theta'_i, \theta_i) \). So \( E_{\theta_{-i}} U_i(\theta'_i, \theta_{-i}) - E_{\theta_{-i}} U_i(\theta_i, \theta_{-i}) \leq \Phi_i(\theta'_i, \theta_i) - \Phi_i(\theta'_i, \theta_i) = E_{\theta_{-i}} \left[ \sum_{j=1}^m x_i^j(\theta'_i, \theta_{-i})(v_j(\theta'_i) - v_j(\theta_i)) \right] \leq E_{\theta_{-i}} \left[ \sum_{j=1}^m x_i^j(\theta'_i, \theta_{-i})|v_j(\theta'_i) - v_j(\theta_i)| \right] \). By Assumption 1, \( v_j(\cdot) \) is continuously differentiable in \( \theta_i \), so \( \exists \bar{M} > 0 \) s.t. \( 0 < v'_j(\bar{\theta}) < \bar{M}, \forall j, \theta_i \). Together with unit demand, we have \( E_{\theta_{-i}} U_i(\theta'_i, \theta_{-i}) - E_{\theta_{-i}} U_i(\theta_i, \theta_{-i}) \leq \bar{M} |\theta_i - \theta'_i| \). Together with symmetry, we know \( |E_{\theta_{-i}} U_i(\theta_i, \theta_{-i}) - E_{\theta_{-i}} U_i(\theta'_i, \theta_{-i})| \leq \bar{M} |\theta_i - \theta'_i| \), or \( E_{\theta_{-i}} U_i(\theta_i, \theta_{-i}) \) is Lipschitz continuous in \( \theta_i \).
the sequential first-price auction with ordering i = (1, 2, . . . , m), the bidding functions defined recursively by \( b^k_{(1), i}(\theta_i) = \mathbb{E}_{\theta_i}[v_k(\theta_{i-1}) - U^{(k+1)}(\theta_{i-1}, \theta_i)] I[\theta_{i-1} < \theta_i], \forall i, k, \) constitute a symmetric and strictly increasing PBE.

Claim 3 (Equilibrium Bidding in Second-price Sequential Auction with Generalized Vertical Differentiation). With generalized vertical differentiation and under Assumption 1, in the sequential second-price auction with ordering i = (1, 2, . . . , m), the bidding functions defined recursively by \( b^k_{(2), i}(\theta_i) = v_k(\theta_i) - U^{(k+1)}(\theta_i), \forall i, k, \) constitute a symmetric and strictly increasing PBE.

The set of corresponding beliefs will be constructed along the proofs. In Claim 2, note the expectation is taken with respect to the remaining buyers and the updated beliefs. Nevertheless, the proof implicitly shows the result is the same if we take the expectation with respect to the common prior at the beginning of the game (i.e. \( \theta_{i'} \sim_{i.i.d} U[0, 1] \) for \( i' \neq i \) remaining).

Proof of Claim 2: The bidding functions \( b^k_{(1), i}(\cdot) \) are clearly symmetric and pure, so it remains to show they are strictly increasing and constitute a PBE. Due to symmetry, we shall drop the subscript on individual i whenever there is no confusion.

Consider the part on PBE, our goal is to show that there exists a belief \( \beta_{(1)} \) (a vector specifying the belief at each information set) such that \( (b_{(1)}, \beta_{(1)}) \) is sequentially rational and that \( \beta_{(1)} \) is derived from Bayes rule when possible. The desired beliefs can be easily constructed, while sequential rationality is somewhat tricky. Our first step is to show that \( \beta_{(1)} \) can be constructed such that the assessment \( (b_{(1)}, \beta_{(1)}) \) satisfies the one-deviation property, or sequential rationality is the same as local sequential rationality (Perea, 2002).

To this end, by Perea (2002), we need to show \( (b_{(1)}, \beta_{(1)}) \) is updating consistent for an appropriately constructed \( \beta_{(1)} \). For player i, only information sets where he is still in the sequential auction matters. Because of winning bid announcement, in round k, the history he can observe is his bids as well as the winning bids in rounds 1 through \( k-1 \). We write such a history as \( (a_{i1}, p^1, a_{i2}, p^2, \ldots, a_{i(k-1)}, p^{k-1}) \), where \( a_{il} \) is the action (bid) of player i in round l and \( p^l \) is the announced winning bid in round l. Before constructing \( \beta_{(1)} \), we first derive the induced (remaining) highest type \( \bar{\theta}^l \) in round l. If \( p^l \in [b^l_{(1)}(0), b^l_{(1)}(1)] \) (equilibrium bids of some type), set \( \bar{\theta}^l = b^l_{(1)}^{-1}(p^l) \). (We will show later in the proof that \( b^l_{(1)}(\cdot) \) is indeed strictly increasing, so the inverse is well defined and \( \bar{\theta}^l \in [0, 1]. \) If \( p^l < b^l_{(1)}(0) \), set \( \bar{\theta}^l = 0. \) If \( p^l > b^l_{(1)}(1) \), set \( \bar{\theta}^l = 1. \) We construct \( \beta_{(1)} \) as follows: Let \( \theta^k = \min_{1 \leq l \leq k-1} \bar{\theta}^l \), and set \( \beta_{(1), i}(a_{i1}, p^1, a_{i2}, p^2, \ldots, a_{i(k-1)}, p^{k-1}) = \{ \theta^k \sim_{i.i.d} U[0, \theta^k] \) for \( i' \neq i \) remaining. \} It is easy to check the beliefs we construct are symmetric and ensures an independent common prior at the beginning of round k. Moreover, they are in accordance with Bayes rule for information sets on the equilibrium path (or when \( a_{il} = b^l_{(1)}(\theta_i) \) and \( p^l \in [b^l_{(1)}(0), b^l_{(1)}(1)], \forall 1 \leq l \leq k-1 \). Arguing similarly to Perea (2002), for information sets on the equilibrium path (note our game is infinite, so any information set is reached with zero probability), updating consistency is no problem. And as we define the beliefs of two consecutive information sets
off the equilibrium path the same as those of some two consecutive information sets on the equilibrium path, updating consistency is guaranteed there as well.

For the rest of the proof, we shall show both parts (strict monotonicity and PBE) by induction on the number of items \( m \).

For \( m = 1 \), given any independent common prior \( \theta_i \overset{i.i.d}{\sim} U[0, \bar{c}] \) (which are satisfied at any history by our construction of \( \beta_{(1)} \)), the problem reduces to the standard independent private value first-price auction. By definition of generalized vertical differentiation, \( b_{(1),i}(\theta_i) = \mathbb{E}_{\theta_{-i}}[v_1(\theta_{-i,1})|\theta_{-i,1} < \theta_i] \) are strictly increasing and constitute a PBE. (See for instance Proposition 2.2 of Krishna (2010).)

Now assume given any independent common prior \( \theta_i \overset{i.i.d}{\sim} U[0, \bar{c}] \), both parts hold for \( m = r \) (\( r \geq 1 \)) and consider the case of \( m = r + 1 \). Given the construction of \( \beta_{(1)} \) above, for any player \( i \), the game starting from the second round is simply an \( r \)-round first-price sequential auction with one fewer buyer and independent common prior. (In fact, the common prior is still i.i.d uniform.) By the inductive hypothesis, the bidding functions are strictly increasing and no deviation is profitable from the second round onwards. Moreover, from the first step of the proof, we know it is enough to check one-shot deviation, so it suffices to verify \( b_{(1),i}^{1}(.) \) is strictly increasing and that no deviation is profitable in round 1. As the \( r \)-round first-price sequential auction starting from the second round is independent private values with a common prior, and clearly satisfies the Bayesian incentive constraints, by Claim 1 we can rewrite the continuation payoffs as:

\[
U^{2+}(\theta_i, s) = \mathbb{E}_{\theta_{-i} \sim F_{-i}^{s}} \left[ \int_{0}^{\theta} \sum_{j=2}^{r+1} x^{j}(t, \theta_{-i})v_{j}(t)dt \right]
\]

where \( F_{-i}^{s} \) is the updated belief conditional on entering the second round and that the highest type in round 1 is \( s \) (With our constructed beliefs, \( F_{-i}^{s} = U[0, s^{n-2}] \)).

Let \( G \) and \( g \) denote the distribution and density function of \( \theta_{-i,1} \), respectively. Then the suggested first-round bidding function can be rewritten as:

\[
b_{(1),i}^{1}(\theta) = \frac{1}{G(\theta)} \int_{0}^{\theta} g(t)(v_1(t) - U^{2+}(t, t))dt
\]

This in particular implies \( b_{(1),i}^{1}(.) \) is differentiable. Moreover,

\[
b_{(1),i}^{1}(\theta) = \frac{g(\theta)(v_1(\theta) - U^{2+}(\theta, \theta)) - g(\theta) \int_{0}^{\theta} g(t)(v_1(t) - U^{2+}(t, t))dt}{G^{2}(\theta)}
\]

\[
= \frac{g(\theta)}{G^{2}(\theta)} \left[ G(\theta)(v_{1}(\theta) - U^{2+}(\theta, \theta)) - \int_{0}^{\theta} g(t)(v_1(t) - U^{2+}(t, t))dt \right]
\]

By the inductive hypothesis, the bidding functions are strictly increasing from the second round onwards. Together with the way we specify the beliefs, we know for \( 0 < t < \theta \),
\[ U^{2+}(t, t) = U^{2+}(t, \theta). \] Also observe that:

\[
U^{2+}(\theta, \theta) - U^{2+}(t, \theta) = \mathbb{E}_{\theta - i \sim F_{\theta-i}} \left[ \int_t^\theta \sum_{j=2}^{r+1} x^j(z, \theta_j)v'_j(z)dz \right]
\]

\[
\leq \mathbb{E}_{\theta - i \sim F_{\theta-i}} \left[ \int_t^\theta \sum_{j=2}^{r+1} x^j(z, \theta_j)v'_j(z)dz \right] \quad \text{(Definition 1 and Assumption 1)}
\]

\[
= v_1(\theta) - v_1(t)
\]

So for \(0 < t < \theta\), we have \(U^{2+}(\theta, \theta) - U^{2+}(t, t) < v_1(\theta) - v_1(t)\), or \(v_1(t) - U^{2+}(t, t) < v_1(\theta) - U^{2+}(\theta, \theta)\). It follows that:

\[
b^{(1)}_1(\theta) > \frac{g(\theta)}{G^2(\theta)}[G(\theta)(v_1(\theta) - U^{2+}(\theta, \theta)) - \int_0^\theta g(t)(v_1(\theta) - U^{2+}(\theta, \theta))dt]
\]

\[
= 0
\]

Or the first-round bidding function is also strictly increasing in type.

Next consider the incentive of a buyer (say \(i\)) of type \(\theta\). Since \(b^{(1)}_1(\cdot)\) is strictly increasing and continuous, all other buyers bid in the range \([b^{(1)}_1(0), b^{(1)}_1(1)]\). Together with the way we specify the belief \(\beta_{i,\cdot}\) above, we know buyer \(i\) must bid \(b^{(1)}_1(r)\), for some \(w \in [0, 1]\). (All other bids are strictly dominated by \(b^{(1)}_1(0)\) or \(b^{(1)}_1(1)\).) The expected payoff from the whole sequential auction when he bids \(b^{(1)}_1(w)\) is given by:

\[
U^{(1)}(\theta, w) = \mathbb{E}_{\theta - i \sim \mathcal{I}} [\mathbb{I}\{\theta - i < 1\} v_1(\theta) - b^{(1)}_1(w)] + \mathbb{I}\{\theta - i > 1\} U^{2+}(\theta, \theta - i(1))]
\]

\[
= G(w)(v_1(\theta) - b^{(1)}_1(w)) + \int_w^1 U^{2+}(\theta, t)g(t)dt
\]

The partial derivative of \(U^{(1)}\) with respect to \(w\) is given by:

\[
\frac{\partial U^{(1)}(\theta, w)}{\partial w} = g(w)[v_1(\theta) - U^{2+}(\theta, w)] - [b^{(1)}_1'(w)G(w) + b^{(1)}_1(w)g(w)]
\]

By the calculation of \(b^{(1)}_1(w)\) above, we have:

\[
b^{(1)}_1'(w)G(w) + b^{(1)}_1(w)g(w) = g(w)[v_1(w) - U^{2+}(w, w)]
\]

It follows that:

\[
\frac{\partial U^{(1)}(\theta, w)}{\partial w} = g(w)[(v_1(\theta) - v_1(w)) - (U^{2+}(\theta, w) - U^{2+}(w, w))]
\]
By definition of generalized vertical differentiation and Assumption 1, for \( w < \theta \), \( 0 < U^2+(\theta, w) - U^2+(w, w) < v^1(\theta) - v^1(w) \) and for \( w > \theta \), \( 0 > U^2+(\theta, w) - U^2+(w, w) > v^1(\theta) - v^1(w) \). Hence, \( \frac{\partial U^1(\theta, w)}{\partial w} > 0 \) for \( w < \theta \) and \( \frac{\partial U^1(\theta, w)}{\partial w} < 0 \) for \( w > \theta \). It follows that \( U^1(\theta, w) \) achieves global maximum at \( w = \theta \), or no buyer has incentive to deviate in the first round, either.

**Proof of Claim 3:** The bidding functions \( b^{k}_{(2),i}(.) \) are clearly symmetric and pure, so it remains to show they are strictly increasing and constitute a PBE. By symmetry, we shall drop the subscript on individual \( i \) whenever there is no confusion.

Due to our assumption of winning bid announcement, the information disclosure rule in a second-price sequential auction is almost identical to that of a first-price sequential auction. It follows that a symmetric belief \( \beta^{(2)} \) can be constructed that guarantees independent common prior at the beginning of each round (the inverse has to be taken with \( b^{1}_{(2)} \) of course), and that one-deviation property is satisfied. For the rest of the proof, we again show both parts by induction on the number of items \( m \).

For \( m = 1 \), given any independent common prior \( \theta_i \overset{i.i.d}{\sim} U[0, \bar{c}] \) (which are satisfied at any history by our construction of \( \beta^{(2)} \)), the problem reduces to the standard independent private value second-price auction. By definition of generalized vertical differentiation, \( b^{(2),i}(\theta_i) = v^1(\theta_i) \) are strictly increasing and constitute a PBE. (See for instance Proposition 2.1 of Krishna (2010).)

Now assume given any independent common prior \( \theta_i \overset{i.i.d}{\sim} U[0, \bar{c}] \), both parts hold for \( m = r \) (\( r \geq 1 \)) and consider the case of \( m = r + 1 \). Then the suggested first-round bidding function can be written as:

\[
b^1_{(2)}(\theta) = v^1(\theta) - U^2+(\theta, \theta)
\]

Notice:

\[
b^1_{(2)}'(\theta) = v^1(\theta) - \mathbb{E}_{\theta_{(r+1)} \sim F^{\theta}} \left[ \sum_{j=2}^{r+1} x^j(\theta) v^j_j(\theta) \right]
\geq v^1(\theta) - v^1_j(\theta) \quad \text{(unit-demand and Definition 1)}
\geq 0 \quad \text{(Assumption 1)}
\]

Or the first-round bidding function is also strictly increasing in type.

Next consider the incentive of a buyer (say \( i \)) of type \( \theta \). Since \( b^1_{(2)}(.) \) is strictly increasing and continuous, all other buyers bid in the range \([b^1_{(2)}(0), b^1_{(2)}(1)]\). Together with the way we specify the belief \( \beta^{(2),i} \) above, we know buyer \( i \) must bid \( b^1_{(2)}(w) \), for some \( w \in [0, 1] \). (In this case, all other bids are weakly dominated by \( b^1_{(2)}(0) \) or \( b^1_{(2)}(1) \).) The expected payoff
from the whole sequential auction when he bids $b_{(2)}^{1}(w)$ is given by:

$$U_{(2)}(\theta, w) = \mathbb{E}_{\theta-i,(1)}[1\{\theta_{-i,(1)} < w\}(v_{1}(\theta) - b_{(2)}^{1}(\theta_{-i,(1)})) + 1\{\theta_{-i,(1)} > w\}U^{2+}(\theta, \theta_{-i,(1)})]$$

$$= \int_{0}^{w} (v_{1}(\theta) - b_{(2)}^{1}(t))g(t)dt + \int_{w}^{1} U^{2+}(\theta, t)g(t)dt$$

$$= \int_{0}^{w} (v_{1}(\theta) - v_{1}(t) + U^{2+}(t, t))g(t)dt + \int_{w}^{1} U^{2+}(\theta, t)g(t)dt$$

The partial derivative of $U_{(2)}$ with respect to $w$ is given by:

$$\frac{\partial U_{(2)}(\theta, w)}{\partial w} = g(w)[(v_{1}(\theta) - v_{1}(w)) - (U^{2+}(\theta, w) - U^{2+}(w, w))]$$

Note the expression is the same as the partial derivative at the end of the proof of Claim 2. So $\frac{\partial U_{(2)}(\theta, w)}{\partial w} > 0$ for $w < \theta$ and $\frac{\partial U_{(2)}(\theta, w)}{\partial w} < 0$ for $w > \theta$. It follows that $U_{(2)}(\theta, r)$ achieves global maximum and $w = \theta$, or no buyer has incentive to deviate in the first round, either. \(\square\)

**Step T1.3:** In this step, we show $i = (1, 2, \ldots, m)$ is the unique ordering that achieves full efficiency.

To begin with, (by Step T1.2) under the ordering $\iota$, in either auction format and the symmetric strictly monotonic pure strategy PBE, the buyer with the $j$th highest $\theta_j$ always gets the item with the $j$th highest quality, which by Lemma 1, coincides with the socially optimal allocation. Hence, the outcome of the constructed PBE under $\iota$ is always fully efficient.

(Given full efficiency) For uniqueness, it remains to show in any PBE of any alternative ordering, the inefficiency loss is bounded away from 0 with non-trivial probability.\(^{20}\) In fact, we will show a stronger result: With any alternative ordering of the sequential auction, any strategy profile (where the first-period strategies only depend on the buyers’ types) results in non-zero efficiency loss with positive probability.

We prove by induction on the number of items $m$. For $m = 2$, the only alternative ordering is $(2, 1)$. Consider any first-round (potentially mixed) strategy profile $\sigma^1 = (\sigma_1^1, \sigma_2^1, \ldots, \sigma_n^1)$ and a vector of types $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, with $1 > \gamma_1 > \gamma_2 > \ldots > \gamma_n > 0$.\(^{21}\) Then by Lemma 1, if this strategy were to achieve full efficiency, for almost all $b_{1}^{1} \in \text{supp}(\sigma_{1}^{1})$ and $b_{j}^{1} \in \text{supp}(\sigma_{j}^{1})$, $b_{1}^{1}(\gamma_{1}) > b_{j}^{1}(\gamma_{k})$, $\forall i, j$ and $k \neq 2$, or the buyer with the second highest type wins with probability 1. (Notice in the one-round first/second-price auction, the highest buyer wins.) Moreover, this has to hold for almost

\(^{20}\)Even though there may not be a “well-behaved” equilibrium under orderings other than $\iota$, the existence of PBE is not much of a problem, at least for second-price sequential auctions. For instance, the following profile of strategies constitute a PBE under any ordering $\kappa$ of second-price sequential auctions: For buyer $1 \leq i \leq m$, he always bids $v_{i}(1)$ on item $i$ and 0 on any other item (regardless of his type); all other buyers always bid 0 on any item (regardless of his type).

\(^{21}\)We use $\gamma$ instead of $\theta$ as $\gamma_i$ is not necessarily the type of buyer $i$.  

35
all type vectors, or \( \mathbb{P}(b_1^1(\gamma_2) > b_1^1(\gamma_k)) = 1 \). By varying the type vectors (for instance, \( \forall 0 < \gamma_0 < \gamma'_0 < 1 \), consider the two alternatives \( \gamma_1 = \gamma'_0, \gamma_2 = \gamma_0 \) and \( \gamma_2 = \gamma'_0, \gamma_3 = \gamma_0 \)), we have \( \mathbb{P}(b_1^1(\gamma) > b_1^1(\gamma')) = 1, \forall i, j \) and almost all \( \gamma, \gamma' \in [0, 1] \). (Noticing \( i, j \) are symmetric)

We get a contradiction. Now assume the result holds for \( m = l \), and consider the case of \( m = l + 1 \). If the item for sale in round 1 is item 1, then from the second period onwards, the ordering must be different from \( (2, 3, \ldots, m) \). By the inductive hypothesis, the ensuing game (note technically it is not a sub-game) results in non-zero efficiency loss with positive probability, and so must the whole game. If on the contrary the item for sale in round 1 is item \( k \neq 1 \), then the argument is almost identical to the case of \( m = 2 \), or it is impossible for the buyer with the \( k^{th} \) highest type to win with probability 1 across almost all type realizations. It follows that in both the first-price and second-price sequential auctions, the unique welfare-maximizing ordering is \( (1, 2, \ldots, m) \).

**Proof of Theorem 2:** As mentioned in the main text, the proof of Theorem 2 is in the same essence of Theorem 1. We proceed in two steps. In Step T2.1, we derive the constrained optimal revenue allocation. In Step T2.2, we show the outcome of the PBEs given in Theorem 1 achieves constrained optimal revenue, while no PBE under any other alternative ordering always does so.

**Step T2.1:** In this step, we will show the following claim, which characterizes the constrained optimal revenue allocation in our setup. This is the counterpart of Lemma 1.

**Claim 4 (Constrained Optimal Revenue Outcome).** With generalized vertical differentiation and under Assumptions 2 and 3, a constrained optimal revenue outcome chooses \( x_{i(j)}^j(\theta) = 1 \), where \( i(j) \) satisfies \( |\{i': \theta_{i'} \geq \theta_{i(j)}\}| = j \), or the buyer with the \( j^{th} \) highest type gets the \( j^{th} \) highest quality item with probability 1. Moreover, \( \mathbb{E}_{\theta_{i-1}} U_i(0, \theta_i) = 0 \), or the participation constraint of the lowest type binds.

**Proof of Claim 4:** The seller’s problem is: \( \max_{\{x_{j(i)}(\cdot), t_{i(\cdot)}\}} \sum_{i=1}^n \mathbb{E}_{\theta_{i-1}} t_i(\theta) \) s.t.

1. (BIC): \( \mathbb{E}_{\theta_{i-1}}[\sum_{j=1}^m x_{i(j)}^j(\theta_i, \theta_{-i}) v_j(\theta_i) - t_i(\theta_i, \theta_{-i})] \geq \mathbb{E}_{\theta_{i-1}}[\sum_{j=1}^m x_{i(j)}^j(\theta'_i, \theta_{-i}) v_j(\theta_i) - t_i(\theta'_i, \theta_{-i})], \forall i, \theta_i, \theta'_i \).
2. (IIR): \( \mathbb{E}_{\theta_{i-1}}[\sum_{j=1}^m x_{i(j)}^j(\theta_i, \theta_{-i}) v_j(\theta_i) - t_i(\theta_i, \theta_{-i})] \geq 0, \forall i, \theta_i \).
3. (AS) + unit-demand: \( \sum_{i=1}^m x_{i(j)}^j(\theta) = 1, \forall j, \theta \) and \( \sum_{j=1}^m x_{i(j)}^j(\theta) \leq 1, \forall i, \theta \).

The seller’s objective function can be re-written as \( \sum_{i=1}^n \mathbb{E}_{\theta_{i-1}} t_i(\theta) = \sum_{i=1}^n \mathbb{E}_{\theta}[\sum_{j=1}^m x_{i(j)}^j(\theta) v_j(\theta_i) - U_i(\theta)] = \sum_{i=1}^n \mathbb{E}_{\theta}[\sum_{j=1}^m x_{i(j)}^j(\theta) v_j(\theta_i)] - \sum_{i=1}^n \mathbb{E}_{\theta_i} \mathbb{E}_{\theta_{i-1}}[U_i(\theta_i, \theta_{-i})] \) (Recall our shorthand notation \( U_i(\theta_i, \theta_{-i}) = \sum_{j=1}^m x_{i(j)}^j(\theta) v_j(\theta_i) - t_i(\theta_i) \)). We need an expression for \( \mathbb{E}_{\theta_{i-1}}[U_i(\theta_i, \theta_{-i})] \), which is given by Claim 1.
Plugging (BICFOC) into the seller’s objective function, we have:

\[
\sum_{i=1}^{n} E_{\theta} t_i(\theta) = \sum_{i=1}^{n} E_{\theta} \left[ \sum_{j=1}^{m} x_i^j(\theta) v_j(\theta_i) \right] - \sum_{i=1}^{n} E_{\theta_i} \left\{ E_{\theta_{-i}} U_i(0, \theta_{-i}) + \int_{0}^{\theta_i} E_{\theta_{-i}} \left[ \sum_{j=1}^{m} x_i^j(t, \theta_{-i}) v_j(t) \right] dt \right\}
\]

\[
= -\sum_{i=1}^{n} E_{\theta_{-i}} U_i(0, \theta_{-i}) + \sum_{i=1}^{n} E_{\theta} \left[ \sum_{j=1}^{m} x_i^j(\theta) v_j(\theta_i) \right] - \sum_{i=1}^{n} \int_{0}^{\theta_i} E_{\theta_{-i}} \left[ \sum_{j=1}^{m} x_i^j(t, \theta_{-i}) v_j(t) \right] dt
\]

Consider the term \( \int_{0}^{\theta_i} E_{\theta_{-i}} [\sum_{j=1}^{m} x_i^j(t, \theta_{-i}) v_j(t)] dt \). By changing the order of integral and rewriting \( \theta_i \) as the variable, we have:

\[
\int_{0}^{1} \int_{0}^{\theta_i} E_{\theta_{-i}} \left[ \sum_{j=1}^{m} x_i^j(t, \theta_{-i}) v_j(t) \right] dt d\theta_i = \int_{0}^{1} \int_{0}^{1} E_{\theta_{-i}} \left[ \sum_{j=1}^{m} x_i^j(t, \theta_{-i}) v_j(t) \right] d\theta_i dt
\]

\[
= \int_{0}^{1} (1-t) E_{\theta_{-i}} \left[ \sum_{j=1}^{m} x_i^j(t, \theta_{-i}) v_j(t) \right] dt
\]

\[
= E_{\theta} \sum_{j=1}^{m} \left[ x_i^j(\theta)(1 - \theta_i) v_j(\theta_i) \right]
\]

Substituting the above equation into the seller’s objective function, we have:

\[
\sum_{i=1}^{n} E_{\theta} t_i(\theta) = -\sum_{i=1}^{n} E_{\theta_{-i}} U_i(0, \theta_{-i}) + E_{\theta} \left[ \sum_{i=1}^{n} \sum_{j=1}^{m} (v_j(\theta_i) - (1 - \theta_i) v_j(\theta_i)) x_i^j(\theta) \right]
\]

\[
= -\sum_{i=1}^{n} E_{\theta_{-i}} U_i(0, \theta_{-i}) + E_{\theta} \left[ \sum_{i=1}^{n} \sum_{j=1}^{m} v_j(\theta_i) x_i^j(\theta) \right]
\]

By (IIR), \( E_{\theta_{-i}} U_i(0, \theta_{-i}) \geq 0, \forall i \), so the best the seller can do is to set \( E_{\theta_{-i}} U_i(0, \theta_{-i}) = 0 \), \( \forall i \). Moreover, by Lemma 1 (with \( \varphi_j \) in place of \( v_j \)), the second term is uniquely maximized when \( x_i^{j}(\theta) = 1 \), where \( i(j) \) satisfies \( \{|i': \theta_{i'} \geq \theta_{i(j)}\}| = j \), or the buyer with the \( j \)th highest type gets the \( j \)th item with probability 1.

It remains to show the proposed allocation indeed satisfies the incentive constraints (BIC), or \( \phi_i(\theta_{i'}, \theta_i) \leq 0, \forall i, \theta_i, \theta_{i'} \). By (BICFOC), \( \phi_i(\theta_{i'}, \theta_i) = E_{\theta_{-i}} [\sum_{j=1}^{m} (x_i^{j}(\theta_{i'}, \theta_{-i}) - x_i^{j}(\theta_i, \theta_{-i})) v_j(\theta_i)] \). Notice for any fixed \( \theta_{-i} \), in our proposed allocation, \( \sum_{j=1}^{m} (x_i^{j}(\theta_{i'}, \theta_{-i}) - x_i^{j}(\theta_i, \theta_{-i})) v_j(\theta_i) \geq 0 \) for \( \theta_i < \theta_{i'} \) and \( \sum_{j=1}^{m} (x_i^{j}(\theta_{i'}, \theta_{-i}) - x_i^{j}(\theta_i, \theta_{-i})) v_j(\theta_i) \leq 0 \) for \( \theta_i > \theta_{i'} \). It follows that with our proposed allocation, \( \phi_i(\theta_{i'}, \theta_i) \) is maximized at \( \theta_i = \theta_{i'} \) for any fixed \( \theta_{i'} \) and \( \phi_i(\theta_{i'}, \theta_i) \leq 0, \forall i, \theta_i, \theta_{i'} \), as desired. \( \Box \)

**Step T2.2**: It is trivial that the indirect mechanism induced by the symmetric PBEs in Claims 2 and 3 satisfy (BIC), (IIR) and (AS) (in fact any PBE does). From Step
T2.1, the seller’s expected total revenue $E R = \sum_{i=1}^{n} E_{\theta_i}(\theta) = -\sum_{i=1}^{n} E_{\theta_i} U_i(0, \theta_i) + \sum_{i=1}^{n} E_{\theta_i} [\sum_{j=1}^{n} a_j^i(\theta) \varphi_j(\theta_i)]$. From Theorem 1, we know in either auction format and the symmetric strictly monotonic pure strategy PBE, the buyer with the $j$th highest type always gets item $j$. By Step 1, the induced allocations are the same as the constrained optimal revenue outcome (for every type realization). Moreover, $E_{\theta_i} U_i(0, \theta_i) = 0$ in both cases, so the first/second-price sequential auction with the ordering $i = (1, 2, \ldots, m)$ achieves constrained optimal revenue.

The proof for uniqueness is almost the counterpart of Theorem 1, with the added observation that (BIC), (IIR) and (AS) are satisfied in any PBE of any ordering of the sequential auctions.

\[\Box\]

**Proof of Lemma 2:** As highlighted in Example 2 in Section 3.2, the key is to show the expected conditional continuation payoff is 0 for every type on the equilibrium path. We formalize the intuition using the tool developed in the proof of Theorem 1. The two orderings ((1, 2) and (2, 1)) are completely symmetric (we can re-define $\theta_i$), so it suffices to prove the lemma for the ordering (1, 2).

We first show the following bidding strategies, (together with the beliefs specified below) constitute a symmetric and strictly monotonic pure strategy PBE:

$$b_{(2),i}(\theta_i) = v_1(\theta_i); b_{(2),i}(a^1_i, p^1; \theta_i) = v_2(\theta_i)$$

where as in the proof of Claim 2, the superscript refers to the rounds, $a_i^1$ is the action (bid) of player $i$ in round 1 and $p^1$ is the winning bid announced at the end of round 1.

Clearly, the bidding functions are symmetric and pure. Moreover, with horizontal differentiation, $b_{(2),i}(\cdot)$ is strictly increasing in $\theta_i$ and $b_{(2),i}(a^1_i, p^1; \cdot)$ is strictly decreasing in $\theta_i$, so it suffices to show they constitute a PBE. Due to symmetry, we shall drop the subscript on individual $i$ whenever there is no confusion.

Similar to the proof of Claim 2, we need to show there exists an action-determined belief $\beta_{(2)}$ (a vector specifying the belief at each information set) such that $(b_{(2)}, \beta_{(2)})$ is sequentially rational and that $\beta_{(1)}$ is derived from Bayes rule when possible. Our first step is still to show $\beta_{(2)}$ can be constructed such that $(b_{(2)}, \beta_{(2)})$ is updating consistent, so the one-deviation property is applicable. As before, for player $i$, only information sets where he is still in the sequential auction matters. In the current setup, we just need to construct the beliefs in round 2. Because of winning bid announcement, in round 2, the history he can observe is his bids as well as the winning bids in rounds 1, which we write as $(a_i^1, p^1)$. Again, let $\hat{\theta}^1$ be the induced (remaining) highest type in round 1. If $p^1 \in [v_1(0), v_1(1)]$ (equilibrium bids of some type), set $\hat{\theta}^1 = (v_1)^{-1}(p^1)$ (invertible by the definition of horizontal differentiation). If $p^1 < v_1(0)$, set $\hat{\theta}^1 = 0$. If $p^1 > v_1(1)$, set $\hat{\theta}^1 = 1$. We construct $\beta_{(2)}$ as follows: $\beta_{(2),i}(a_i^1, p^1) = \{\theta_{i'} \sim U[0, \hat{\theta}^1] \text{ for } i' \neq i \text{ remaining}\}$. It is easy to check the beliefs we construct are symmetric, action-determined and ensures an independent common prior at the beginning of round 2. Moreover, they are in accordance with Bayes rule for information
sets on the equilibrium path (or when $a^1_i = v_1(\theta_i)$ and $p^1 \in [v_1(0), v_1(1)]$). Arguing similarly to Perea (2002), for information sets on the equilibrium path, updating consistency is no problem. And as we define the beliefs of two consecutive information sets off the equilibrium path the same as those of some two consecutive information sets on the equilibrium path, updating consistency is guaranteed there as well.

Our next step is to check local incentive compatibility, which we prove by backward induction. In round 2, given any history, we have independent common prior, and the problem reduces to the standard independent private value second-price auction, so bidding one’s true value is clearly incentive compatible. (See for instance Proposition 2.1 of Krishna (2010).)

In round 1, consider the incentive of a buyer (say $i$) of type $\theta_i$. Since $v_1(.)$ is strictly increasing and continuous, all other buyers bid in the range $[v_1(0), v_1(1)]$. Together with the way we specify the belief $\beta_{i,2}$ above, we know buyer $i$ must bid $b^1_1(w)$, for some $w \in [0, 1]$. (All other bids are weakly dominated by $b^1_1(0)$ or $b^1_1(1)$.) The expected payoff from the whole sequential auction when he bids $b^1_1(w)$ is given by (notations are the same as Step 2 of the proof of Theorem 1, except that we do not have rounds more than 2):

$$U_{2}(\theta, w) = \mathbb{E}_{\theta_i,1} \{1 \{\theta_{-i,1} < w\}(v_1(\theta) - v_1(\theta_{-i,1})) + 1 \{\theta_{-i,1} > w\}U^2(\theta, \theta_{-i,1})\}$$

$$= \int_0^w (v_1(\theta) - v_1(t)) g(t) dt + \int_w^1 U^2(\theta, t) g(t) dt$$

The partial derivative of $U_{2}$ with respect to $w$ is given by:

$$\frac{\partial U_{2}(\theta, w)}{\partial w} = g(w)(v_1(\theta) - v_1(w) - U^2(\theta, w))$$

With horizontal differentiation, $v_1(.)$ is strictly increasing in $\theta$. Moreover, given the construction of $\beta_{i,2}$, and the fact that the second round is an independent private value second price auction, we have $U^2(\theta, w) = 0$ for $\theta < w$ and $U^2(\theta, w) > 0$ for $\theta > w$. (Notice with horizontal differentiation, $v_2(.)$ is strictly decreasing in $\theta$, so higher types have lower valuations for item 2.) Hence, $\frac{\partial U_{2}(\theta, w)}{\partial w} > 0$ for $w < \theta$ and $\frac{\partial U_{2}(\theta, w)}{\partial w} < 0$ for $w > \theta$. It follows that $U_{2}(\theta, w)$ is maximized at $w = \theta$, or no buyer has incentive to deviate in the first round, either.

It remains to show that with action-determined beliefs, $b^1_1(.)$ is generically unique.

An observation is that on any history $(a^1_i, p^1)$ where independent common prior is ensured, the second round is an independent private value second-price auction, and the symmetric equilibrium bidding strategy $b^2(a^1_i, p^1, :)$ is generically unique. (See Blume and Heidhues (2004).)

In round 1, suppose in a symmetric strictly monotonic equilibrium, buyers bid according to $b^1(.)$. For a buyer of type $\theta$, his expected payoff from mimicking type $w$ (bidding $b^1(w)$)
is given by (notations are the same as before, except that continuation payoffs may be different given different actions, which we write as $U^2(\theta, s, w)$):

$$\Pi(2)(\theta, w) = \mathbb{E}_{\theta \sim U[0,1]} \{ \mathbb{I}\{\theta_{-i(1)} < w\}(v_1(\theta) - b^1(\theta_{-i(1)})) + \mathbb{I}\{\theta_{-i(1)} > w\}U^2(\theta, \theta_{-i(1)}, w)\}$$

$$= \int_0^w (v_1(\theta) - b^1(t))g(t)dt + \int_w^1 U^2(\theta, t, w)g(t)dt$$

The key is that the restriction of “action-determined beliefs” rules out the possible dependence of $U^2(\theta, t, w)$ on $w$. To see this, observe that given the announcement of winning bid $p^1$ in round 1 (or equivalently the induced highest type), the restriction of “action-determined beliefs” implies a buyer of type $\theta$ pretending to be type $w$ must have the same updated belief as type $w$ acting as predicted, which is on the equilibrium path (where belief is pinned down by Bayes’ rule). As the winning bid $p^1$ (or equivalently the induced highest type) is a sufficient statistic for updated beliefs on the equilibrium path, the above observation shows it is also a sufficient statistic for updated beliefs off the equilibrium path (given the restriction of “action-determined beliefs”). Hence $U^2(\theta, t, w)$ is independent of $w$. In fact, together with the earlier observation that $b^2(.)$ is generically unique on such histories, we get $U^2(\theta, t, w) = U^2(\theta, t)$. So:

$$\frac{\partial \Pi(2)(\theta, w)}{\partial w} = g(w)(v_1(\theta) - b^1(w) - U^2(\theta, w))$$

Given $b^1(.)$ is an equilibrium bidding strategy, we have:

$$\frac{\partial \Pi(2)(\theta, w)}{\partial w}_{w=\theta} = 0$$

It follows that $b^1(\theta) = v_1(\theta)$. Or the first round symmetric strictly monotonic equilibrium bidding strategy is unique a.s. if all agents have action-determined beliefs. \qed

**Proof of Lemma 3:** The proof is very similar to that of Lemma 2. Again the two orderings are symmetric, so it suffices to prove the lemma for the ordering $(1, 2)$.

We first show the following bidding strategies, (together with the beliefs specified below) constitute a symmetric and strictly monotonic pure strategy PBE:

$$b^1_{(1),i}(\theta_i) = \mathbb{E}_{\theta_{-i} \sim U[0,1]}[v_1(\theta_{-i(1)})|\theta_i > \theta_{-i(1)}]$$

$$b^2_{(1),i}(a^1_i, p^1; \theta_i) = \mathbb{E}_{\theta_{-i} \sim F_{p^1_i}}[v_2(\theta_{-i(n-2)})|\theta_i < \theta_{-i(n-2)}]$$

where the notations are the same as in the proof of Lemma 2 and $F_{p^1_i}$ is the updated belief given history $(a^1_i, p^1)$ (constructed below).

Clearly, the bidding functions are symmetric and pure. Moreover, with horizontal differentiation, $b^1_{(1),i}(.)$ is strictly increasing in $\theta_i$ and $b^2_{(1),i}(a^1_i, p^1; .)$ is strictly decreasing in $\theta_i$,
so it suffices to show they constitute a PBE. Due to symmetry, we shall drop the subscript on individual \( i \) whenever there is no confusion.

The construction of beliefs are almost identical. More specifically, let \( \hat{\theta}^1 \) be the induced (remaining) highest type in round 1. If \( p^1 \in [b_{(1)}^1(0), b_{(1)}^1(1)] \) (equilibrium bids of some type), set \( \hat{\theta}^1 = b_{(1)}^1 \) (invertible by strict monotonicity). If \( p^1 < b_{(1)}^1(0) \), set \( \hat{\theta}^1 = 0 \). If \( p^1 > b_{(1)}^1(1) \), set \( \hat{\theta}^1 = 1 \). \( \beta \) is constructed as as follows:

\[
\beta_{(2)} = \{ \beta_{(1)}(a_i^1, p^1) \} = \{ \beta_{(1)}^i(0), \beta_{(1)}^i(1) \},
\]

Hence \( F_{p^1} \sim U[0, \hat{\theta}^1] \). By the same argument as in the proof of Lemma 2, one-deviation property is guaranteed.

The next step is still to check local incentive compatibility, which we prove by backward induction. In round 2, given any history \((a_1^1, p_1)\), we have independent common prior, and the problem reduces to the standard independent private value first-price auction, so local incentive compatibility is guaranteed by the suggested bidding strategies. (See for instance Proposition 2.2 of Krishna (2010)). Notice, however, with horizontal differentiation, the second round equilibrium bidding functions actually depend on the specific priors. This is in contrast with the case of vertical differentiation. (Compare with Claim 2.)

In round 1, consider the incentive of a buyer (say \( i \)) of type \( \theta \). Since \( b_{(1)}^1(.) \) is strictly increasing and continuous, all other buyers bid in the range \([b_{(1)}^1(0), b_{(1)}^1(1)]\). Together with the way we specify the belief \( \beta_{(1)}^i \) above, we know buyer \( i \) must bid \( b_{(1)}^i(w) \), for some \( w \in [0, 1] \). (All other bids are strictly dominated by \( b_{(1)}^i(0) \) or \( b_{(1)}^i(1) \).) The expected payoff from the whole sequential auction when he bids \( b_{(1)}^i(w) \) is given by (notations are the same as in the proof of Lemma 2):

\[
U_{(1)}(\theta, w) = \mathbb{E}_{\theta_{-i},(1)}[\mathbb{1}\{\theta_{-i,(1)} < w\}(v_1(\theta) - b_{(1)}^i(w)) + \mathbb{1}\{\theta_{-i,(1)} > w\}U^2(\theta, \theta_{-i,(1)})]
\]

\[
= G(w)(v_1(\theta) - b_{(1)}^i(w)) + \int_w^1 U^2(\theta, t)g(t)dt
\]

Similar to the derivation in the proof of Claim 2, the partial derivative of \( U_{(1)} \) with respect to \( w \) is given by:

\[
\frac{\partial U_{(1)}(\theta, w)}{\partial w} = g(w)[v_1(\theta) - v_1(w) - U^2(\theta, w)]
\]

As argued in the proof of Lemma 2, \( \frac{\partial U_{(1)}(\theta, w)}{\partial w} > 0 \) for \( w < \theta \) and \( \frac{\partial U_{(1)}(\theta, w)}{\partial w} < 0 \) for \( w > \theta \). It follows that \( U_{(1)}(\theta, w) \) is maximized \( w = \theta \), or no buyer has incentive to deviate in the first round, either.

It remains to show that with action-determined beliefs, \( b_{(1)}^i(.) \) is generically unique.

An observation is that given any (fixed) independent common prior, the second round is an independent private value first-price auction. By revenue equivalence, \( b_{(1)}^2(F_{p^1}^{p!};.) \) is generically unique. (As remarked earlier, contrary to the case of second-price sequential
auction, the second round equilibrium bidding functions in a first-price sequential auction are not identical across histories.

In round 1, suppose in a symmetric strictly monotonic equilibrium, buyers bid according to $b^1(.)$. For a buyer of type $\theta$, his expected payoff from mimicking type $w$ (bidding $b^1(w)$) is given by (notations are the same as in the proof of Lemma 2):

$$\Pi_{(1)}(\theta, w) = \mathbb{E}_{\theta-i,(1)} [\mathbb{1}\{\theta_{-i,(1)} < w\} (v_1(\theta) - b^1(w)) + \mathbb{1}\{\theta_{-i,(1)} > w\} U^2(\theta, \theta_{-i,(1)}, w)]$$

$$= G(w)(v_1(\theta) - b^1(w)) + \int_w^1 U^2(\theta, t, w) g(t) dt$$

The key is again that the restriction of “action-determined beliefs” rules out the possible dependence of $U^2(\theta, t, w)$ on $w$. To see this, observe that given the announcement of winning bid $p_1$ in round 1 (or equivalently the induced highest type), the restriction of “action-determined beliefs” implies a buyer of type $\theta$ pretending to be type $w$ must have the same updated belief as type $w$ acting as predicted, which is on the equilibrium path (where belief is pinned down by Bayes’ rule). As the winning bid $p_1$ (or equivalently the induced highest type) is a sufficient statistic for updated beliefs on the equilibrium path, the above observation shows it is also a sufficient statistic for updated beliefs off the equilibrium path (given the restriction of “action-determined beliefs”). Hence $U^2(\theta, t, w)$ is independent of $w$.

In fact, the updated belief is given by $\{\theta_{i'} \overset{i.i.d.}{\sim} U[0, t] \text{ for } i' \neq i \text{ remaining}\}$ and our earlier observation shows $b^2_{(1)}([0, t]^{n-2}; .)$ is generically unique, so $U^2(\theta, t, w) = U^2(\theta, t)$. We have:

$$\frac{\partial \Pi_{(1)}(\theta, w)}{\partial w} = G(w)[-b^1(w)] + g(w)[v_1(\theta) - b^1(w)] - U^2(\theta, w)$$

Given $b^1(.)$ is an equilibrium bidding strategy, we have:

$$\frac{\partial \Pi_{(1)}(\theta, w)}{\partial w} |_{w = \theta} = 0$$

It follows that $b^1(\theta) = \frac{1}{c(\theta)} \int_0^\theta g(t)v_1(t) dt = \mathbb{E}_{\theta_{-i} \sim U[0,1]} [v_1(\theta_{-i,(1)}) | \theta > \theta_{-i,(1)}]$. Or the first round symmetric strictly monotonic equilibrium bidding strategy is unique a.s. if all agents have action-determined beliefs.

Proof of Theorem 3: First consider second-price sequential auctions. By Lemma 2, in either ordering $(1, 2)$ or $(2, 1)$, the seller’s expected revenue $\mathbb{E}R_{(2)} = \mathbb{E}_\theta [v_1(\theta_{(2)})] + \mathbb{E}_\theta [v_2(\theta_{(n-1)})]$, where $\theta_i \overset{i.i.d.}{\sim} U[0, 1]$.

We now apply revenue equivalent theorem to show the result for first-price sequential auctions, consider first the ordering $(1, 2)$. By Lemmas 2 and 3, in both the first and second-price auctions for item 1, we have the same set of buyers, the same allocation rule and a payoff of 0 for type $\theta_i = 0$, so by revenue equivalence the seller’s expected revenue from item
1 must be the same. Moreover, in both auction formats, everyone except the highest-type buyer enters the auction for item 2. Notice the allocation rules are again identical and type \( \theta_i = 1 \) always gets a payoff of 0 (type \( \theta_i = 1 \) has the lowest valuation for item 2), so by revenue equivalence again the seller’s (ex-ante) expected revenue from item 2 must also be identical. The same argument applies to the ordering (2, 1), so the seller’s expected revenue in either ordering of the first-price sequential auction satisfies:

\[
\mathbb{E}R_{(1)} = \mathbb{E}\theta_1(\theta_1(1)) + \mathbb{E}\theta_2(\theta_{(n-1)}).
\]

For social welfare, we have \( \mathbb{E}W = \sum_{i=1}^{n} x_1^i(\theta) v_1(\theta_i) + \sum_{i=1}^{n} x_2^i(\theta) v_2(\theta_i) \). Notice \( v_1(.) \) strictly increases with \( \theta_i \) and \( v_2(.) \) strictly decreases with \( \theta_i \), so the best a benevolent social planner can do is to set \( x_1^i(\theta) = 1 \) for \( \theta_i = \theta_1(1) \), or always give item 1 to the buyer with the highest type and \( x_2^i(\theta) = 1 \) for \( \theta_i = \theta_{(n)} \), or always give item 2 to the buyer with the lowest type. By Lemmas 2 and 3, either ordering of the first/second-price sequential auction achieves this allocation.\( \square \)

**Proof of Lemma 4:** As seen from proof of the Theorem 1, Lemmas 2 and 3, the key is to figure out the expected conditional continuation payoff. We will show this via an inductive argument.

Call such a PBE in \( A \): \( b_{(1),A} \) (first-price) or \( b_{(2),A} \) (second-price). We will construct \( b_{(1),A} \) and \( b_{(2),A} \) inductively.

Our first step is still to ensure the one-deviation property via properly constructed beliefs: \( \beta_{(1),A} \) (first-price) or \( \beta_{(2),A} \) (second-price). Due to similarity, we illustrate the idea with \( \beta_{(1),A} \). For player \( i \), only information sets where he is still in the sequential auction matters. Because of winning bid announcement, in round \( k \), the history he can observe is his bids as well as the winning bids in rounds 1 through \( k-1 \). We write such a history as \( (a_i^1,p_1,a_i^2,p_2,\ldots,a_i^{k-1},p_{k-1}) \), where \( a_i^k \) is the action (bid) of player \( i \) in round \( l \) and \( p_l^i \) is the announced winning bid in round \( l \). Before constructing \( \beta_{(1),A} \), we first derive the induced (remaining) highest type \( \hat{\theta}^i \) and lowest type \( \hat{\theta}^i \) in round \( l \). If \( \kappa(l) \in M_+ \)(the positive group), then our construction below will imply \( b_{(1),A}^l(.) \) is strictly increasing in \( \theta \) (we drop the subscript on individuals \( i \) by symmetry), so the winning bid reveals information on highest type. Set \( \hat{\theta}^i = 0 \). For \( \hat{\theta}^i \), if \( p_l^i \in [b_{(1),A}^l(0),b_{(1),A}^l(1)] \) (equilibrium bids of some type), set \( \hat{\theta}^i = b_{(1),A}^{-1}(p_l^i) \). (Invertible by strict monotonicity.) If \( p_l^i < b_{(1),A}^l(0) \), set \( \hat{\theta}^i = 0 \) and if \( p_l^i > b_{(1),A}^l(1) \), set \( \hat{\theta}^i = 1 \). If on the other hand, \( \kappa(l) \in M_- \) (the negative group), then our construction below will imply \( b_{(1),A}^l(.) \) is strictly decreasing in \( \theta \), so the winning bid reveals information on lowest type. Set \( \hat{\theta}^i = 1 \). For \( \hat{\theta}^i \), if \( p_l^i \in [b_{(1),A}^l(1),b_{(1),A}^l(0)] \) (equilibrium bids of some type), set \( \hat{\theta}^i = b_{(1),A}^{-1}(p_l^i) \). (Invertible by strict monotonicity.) If \( p_l^i > b_{(1),A}^l(0) \), set \( \hat{\theta}^i = 0 \) and if \( p_l^i < b_{(1),A}^l(1) \), set \( \hat{\theta}^i = 1 \). We construct \( \beta_{(1)} \) as follows: Let \( \theta^k = \max_{1 \leq i \leq k-1} \hat{\theta}^i \) and \( \bar{\theta}^k = \min_{1 \leq i \leq k-1} \hat{\theta}^i \). If \( \hat{\theta}^k \leq \theta^k \), set \( \beta_{(1),A,i}(a_i^1,p_1,a_i^2,p_2,\ldots,a_i^{k-1},p_{k-1}) = \{ \theta_{i'} = \theta_i \text{ for } i' \neq i \text{ remaining} \} \). Otherwise, set \( \beta_{(1),A,i}(a_i^1,p_1,a_i^2,p_2,\ldots,a_i^{k-1},p_{k-1}) = \{ \theta_{i'} \sim U[\theta^k,\bar{\theta}^k] \text{ for } i' \neq i \text{ remaining} \} \). It is easy to check the beliefs we construct are symmetric and ensures an independent common prior at
the beginning of round $k$. Moreover, they are in accordance with Bayes rule for information sets on the equilibrium path (or when $a^l_i = b^l_{(1),A}$ and $\beta^l$ in between $b^l_{(1),A}(0)$ and $b^l_{(1),A}(1)$, $\forall 1 \leq l \leq k-1$). Arguing similarly to Perea (2002), for information sets on the equilibrium path (note our game is infinite, so any information set is reached with zero probability), updating consistency is no problem. And as we define the beliefs of two consecutive information sets off the equilibrium path the same as those of some two consecutive information sets on the equilibrium path, updating consistency is guaranteed there as well.

For the rest of the proof, we proceed by induction on the total number of items $m$. Given any independent common prior $\theta_i \sim U[c, \bar{c}]$ (which are satisfied at any history by our construction of $\bar{\theta}_{(1),A}$ and $\bar{\theta}_{(2),A}$), both statements are true for $m = 1$ and $m = 2$ with any number of bidders $n \geq m + 1$ (either trivial, or a direct corollary to the negative case). So it suffices to prove the following:

“Given any independent common prior $\theta_i \sim U[c, \bar{c}]$, suppose both statements are true for $m(\geq 2)$, with any number of bidders $n \geq m + 1$, i.e., for any pair $(m_1, m - m_1)$ s.t. $0 \leq m_1 \leq m$, then those statements are also true for $m + 1$, with any $n' \geq m + 2$.”

There are two cases in terms of which group the first item belongs to: $k_1 \in M_+$, or $k_1 \in M_-$. The proof for the two cases are very similar, so we prove for the first case, i.e., $k_1 \in M_+$.

For any sequential auction $A$, let $A \equiv (A^1, A^{2+})$, where $A^1$ denotes the first round of the auction in $A$, and $A^{2+}$ denotes the rest of the $m$ rounds.

Let $n(A)$ be the number of bidders in $A$. We have $n(A^1) = n(A)$, and $n(A^{2+}) = n(A) - 1$. Hence, $n(A) \geq m + 2$ implies $n(A^{2+}) \geq m + 1$, so by induction we have already assumed that both statements in Lemma 4 are true for $A^{2+}$.

Given the correct ordering, we now construct a symmetric, strictly monotonic pure PBE in $A$ inductively and show that it prescribes exactly the same bidding behavior, in their first round, as in $A_+$, i.e., the positive part of $A$ (with number of bidders $n(A_+) = n(A)$, by definition). To do so, we generalize Claims 2 and 3 and Lemmas 2 and 3. For the first round, they imply for all $i, \theta_i$:

- For first-price sequential auctions: $b_{(1),i,A}^1(\theta_i | c) = \mathbb{E}[v_1(\theta_{-i,(1)}) - U^{2+}_{A}(\theta_{-i,(1)}, \theta_{-i,(1)}, c)] | \theta_{-i,(1)} \in (c, \theta_i)];$
- For second-price auctions: $b_{(2),i,A}^1(\theta_i | c) = v_1(\theta_i) - U^{2+}_{A}(\theta_i, \theta_i, c);$

where $U^{2+}(\theta_i, s, c)$ denotes the expected continuation payoff, from round 2 onwards, to a buyer of type $\theta_i$ conditional on the highest type in round 1 (among other buyers) is $s$, and that the lower limit of the common prior is $c$.

\footnote{Notice contrary to the cases of vertical or horizontal differentiation, here both the upper and the lower limits of the common prior are updated along the courses of the sequential auctions, so we stress the dependence of $U^{2+}$ on $c$.}
Here comes our key observation:

$$U_{(A^1,A^-)}^2(\theta_i, \theta_i, c) = 0, \forall \theta_i > c$$

This is true because for any buyer, having the highest value for items in $M_+$ is equivalent to having the lowest value for items in $M_-$, which by induction implies a zero (expected) payoff in $A^-$. Therefore, for all $\theta_i, c$ (s.t. $\theta_i > c$):

$$U_A^2(\theta_i, \theta_i, c) = U_{(A^1,A^2+)}^2(\theta_i, \theta_i, c)$$
$$= U_{(A^1,(A^2^+)_{+})}^2(\theta_i, \theta_i, c) + U_{(A^1,(A^2^+)_{-})}^2(\theta_i, \theta_i, c)$$
$$= U_{(A^1,(A^2^+)_{+})}^2(\theta_i, \theta_i, c) + 0$$
$$= U_{((A_+)^1,(A_+)^2+)}^2(\theta_i, \theta_i, c)$$
$$= U_{A_+}^2(\theta_i, \theta_i, c)$$

where the first equality is the definition of $A$, the second equality is by induction on $A^2^+$, the third equality uses the above observation, the fourth equality uses the fact that the first item is in $M_+$, and the last equality is the definition of $A_+$.

To be specific on the number of bidders, we have $n(A^1) = n(A) = n(A_+) \geq m + 2$ and $n(A^2^+) = n(A_+^2^+) = n(A^2^+) = n(A) - 1 \geq m + 1$.

As a result,

$$b_{(1),i,A}^1(\theta_i \mid c) = b_{(1),i,A_+}^1(\theta_i \mid c) \quad \text{and} \quad b_{(2),i,A}^1(\theta_i \mid c) = b_{(2),i,A_+}^1(\theta_i \mid c), \quad \forall i, \theta_i$$

In particular, together with Claims 2 and 3, they imply the round 1 bidding strategies we construct are strictly increasing in $\theta_i$. Moreover, the bidding strategies are clearly symmetric and pure.

The rest of the proof follows immediately: The round 1 bidding strategy, together with the equilibrium in (every) continuation of the sequential auction (by induction), forms a symmetric, strictly monotonic, pure strategy PBE of our current sequential auction with $m + 1$ items. In equilibrium, items in $M_+$ are allocated to the buyers with the highest $m_1$ types, and items in $M_-$ are allocated to the buyers with the lowest $m - m_1$ types (i.e., highest values for those items), both in the efficient way as described in Lemma 1. Hence, efficient allocation is achieved.

The statement of payoff equivalence also follows immediately, by noticing that one’s expected payoff equals the sum of the expected payoff from the first round and the expected payoff from the rest of the sequential auctions. □
Supplementary Appendix: Extension with Winner’s Price Announcement in Second-Price Sequential Auctions

This appendix concerns with an alternative form of information disclosure rule. Up till now, we have assumed “winning bid announcement” for both first-price and second-price sequential auctions, which we justify by its practical relevance. For first-price (sealed-bid) sequential auctions, winning bid announcement is probably the only natural form of information disclosure rule. For second-price (sealed-bid) sequential auctions, another conceivable information disclosure rule is “winner’s price announcement”, or the case where the winner, together with the price he pays (the second highest bid) is announced at the end of each round. The supplementary appendix is devoted to analyze the effects of this other form of price announcement in second-price sequential auctions.

Looking through the previous analyses, we see the only place where information disclosure rules play a role is in equilibrium bidding functions, i.e. Claim 3, Lemmas 2 and 4. So if we can show analogous equilibrium bidding behaviors under winner’s price announcement, all the main results (Theorems 1 through 4) will continue to hold. Intuitively, Claim 3, Lemmas 2 and 4 should still hold as the new information disclosure rule does not affect buyers’ incentives “at the margin”; and in each round of second-price sequential auction, only marginal incentives matter. It turns out our conjecture is true, and the precise statements are given in the following three lemmas:

**Lemma 5** (Symmetric and Strictly Monotonic Equilibrium with Generalized Vertical Differentiation and Winner’s Price Announcement). With generalized vertical differentiation, winner’s price announcement and under Assumption 1, in the sequential second-price auction with ordering \(\iota = (1, 2, \ldots, m)\), the bidding functions defined recursively by \(b_{(2), i}^k(\theta_i) = v_k(\theta_i) - U^{(k+1)}(\theta_i, \theta_i), \forall i, k\), constitute a symmetric and strictly increasing PBE.

**Lemma 6** (No Shading with Horizontal Differentiation and Winner’s Price Announcement). With horizontal differentiation and winner’s price announcement, in the ordering \((1, 2)\) (or \((2, 1)\)) of second-price sequential auctions, there exists a symmetric pure strategy PBE such that the bidding functions are strictly increasing (decreasing) in \(\theta_i\) in the first round, and strictly decreasing (increasing) in \(\theta_i\) in the second round. Moreover, if all agents’ beliefs are action-determined, the first-round equilibrium bidding function \(b_1^1(\theta_i) = v_1(\theta_i) (v_2(\theta_i))\) a.e. in any such PBE.

**Lemma 7** (Revenue Decomposition with Permutation Across Group Orderings and Winner’s Price Announcement). With mixed differentiation, winner’s price announcement and under Assumption 4, if \(\kappa\) is a permutation across groups of \(\iota = (1, 2, \ldots, m)\) (the identity permutation), then (for any number of bidders \(n \geq m+1\)), there exists a symmetric strictly monotonic pure strategy PBE in \(A, A_+\) and \(A_-\), respectively such that:

1. For every buyer \(i\), his expected payoff from the sequential auction \(A\) equals the sum of his expected payoffs from its positive and negative parts, \(A_+\) and \(A_-\);
2. *The allocation achieves full efficiency.*

In other words, Claim 3, Lemmas 2 and 4 exactly hold with winner’s price announcement. Consequently, our main results (Theorems 1 through 4) also hold in second-price sequential auctions with winner’s price announcement.\(^{23}\)

The rough intuition is the one given immediately before the lemmas: Compared to winning bid announcement, buyers have the same incentives at the margin in second-price sequential auctions (with winner’s price announcement). Nevertheless, there are at least two difficulties. To begin with, due to winner’s price announcement, the beliefs at each information set are different from winning bid announcement. It is not immediately clear that a set of beliefs can be constructed such that the assessment satisfies one-deviation property. Our first step in the proof is to ensure this fact. More importantly, continuation payoffs can no longer be readily derived via the Envelope formula from the mechanism design problem. This is because winner’s price announcement induces inter-dependent (common) priors from the second round onwards. In such a setup, the mechanism design problem tells us buyers’ information rents can go down to zero (Crémer and Mclean (1988)), which no longer equal expected conditional continuation payoffs (of course, thanks to our proofs, we know this will not happen in the current setup). In the formal proof below, we resolve the second difficulty by treating the perceived (remaining) highest type and others separately. For buyers with the perceived (remaining) highest type, their incentives are identical to the case where all buyers have common independent priors, so our previous argument (with winning bid announcement) applies. For other buyers, thanks to the one-deviation property, it suffices to consider their marginal incentives, and we show they have no incentive to deviate from the prescribed strategies at the margin. One thing to note is that this ex-post asymmetry does not affect symmetry in buyers’ equilibrium bidding strategies.

**Proofs for the Supplementary Appendix**

**Proof of Lemma 5:** As mentioned above, this lemma can be thought of as another major generalization of Theorem 1 of Kittsteiner, Nikutta and Winter (2004). In fact, the proof partially borrows their analysis on second-price sequential auctions without price announcement and our Claim 3. Notice, however, because of the alternative information disclosure rule, one-deviation property has to be verified once again.

By the proof of Claim 3, we know the strategies are symmetric, strictly increasing and pure. It remains to show they constitute a PBE in our new game. For convenience, we drop the subscript on individual \(i\) whenever there is no confusion.

\(^{23}\)Technically speaking, the PBEs we construct are different from the previous Lemmas: The bidding functions are identical, but belief updating is sharply different. In fact, we do not necessarily have independent priors from the second round onwards. In particular, given the substantial differences between PBE and Bayes Nash equilibrium in the current setup, Lemma 5 can be thought of as another generalization of Theorem 1 of Kittsteiner, Nikutta and Winter (2004).
The first step is still to ensure one-deviation property. To this end, we need to show there exists a belief \( \beta(2) \) such that \((b(2), \beta(2))\) is updating consistent. For player \( i \), only information sets where he is still in the sequential auction matters. Because of winner’s price announcement, in round \( k \), the history he can observe is his bids as well as the winner’s price in rounds 1 through \( k-1 \). We write such a history as \((a^1_i, p^1, a^2_i, p^2, \ldots, a^{k-1}_i, p^{k-1})\), where \( a^l_i \) is the action (bid) of player \( i \) in round \( l \) and \( p^l \) is the announced winner’s price in round \( l \). Before constructing \( \beta(2) \), we first derive the perceived (remaining) highest type \( \hat{\theta}^l \) in round \( l \). If \( p^l \in [b^l(2)(0), b^l(2)(1)] \) (equilibrium bids of some type), set \( \hat{\theta}^{l+1} = b^l(2)^{-1}(p^l) \). (The winner’s price in round \( l \) reveals information on the highest type in round \( l+1 \) instead of \( l \).)

Compare with the proof of Claim 3.) If \( p^l < b^l(2)(0) \), set \( \hat{\theta}^{l+1} = 0 \). If \( p^l > b^l(2)(1) \), set \( \hat{\theta}^{l+1} = 1 \). Let \( \theta^k = \min_{1 \leq l \leq k-1} \hat{\theta}^{l+1} \) and \( \overline{k} = \arg\min_{1 \leq l \leq k-1} \hat{\theta}^{l+1} \) (i.e. \( \hat{\theta}^{\overline{k}+1} = \theta^k \)). We construct \( \beta(2) \) as follows: If \( a_i^\overline{k} = b_{\overline{k}}(2)(\theta^k) \) (or \( i \) is the perceived highest type in round \( k \)), set \( \beta(2), i(a^1_i, p^1, a^2_i, p^2, \ldots, a^{k-1}_i, p^{k-1}) = \{ \theta_{i'} \overset{i.i.d.}{\sim} U[0, \theta^k] \text{ for } i' \neq i \text{ remaining} \} \); and if \( a_i^\overline{k} < b_{\overline{k}}(2)(\theta^k) \) (or \( i \) is not the perceived highest type in round \( k \)), set \( \beta(2), i(a^1_i, p^1, a^2_i, p^2, \ldots, a^{k-1}_i, p^{k-1}) = F^k \) (the joint cdf of the beliefs of the remaining \( n-k \) buyers), where the density function of \( F^k \) is given by:

\[
f^k(\gamma_1, \gamma_2, \ldots, \gamma_{n-k}) = \begin{cases} \frac{1}{(\theta^k)^{n-k-1}} & \text{when } \gamma_i \leq \theta^k, \forall i \text{ and } \max \gamma_i = \theta^k \\ 0 & \text{otherwise} \end{cases}
\]

Since everyone knows the perceived highest type, the beliefs are not independent (the superscript is \( n-k-1 \) instead of \( n-k \)). It is easy to check the beliefs we construct are symmetric and ensures a common prior (though inter-dependent) at the beginning of round \( k \). Moreover, they are in accordance with Bayes rule for information sets on the equilibrium path (or when \( a^l_i = b^l(2)(\theta_l), p^l \in [b^l(2)(0), b^l(2)(1)], \forall 1 \leq l \leq k-1 \)). Arguing similarly to Perea (2002), for information sets on the equilibrium path, updating consistency is no problem. And as we define the beliefs of two consecutive information sets off the equilibrium path the same as those of some two consecutive information sets on the equilibrium path, updating consistency is guaranteed there as well.

For the rest of the proof, we proceed by backward induction. In round \( m \), given any prior (even asymmetric or inter-dependent), the game is a static second-price auction, so bidding one’s true value is a weakly dominant strategy. (See for instance Proposition 2.1 of Krishna (2010).)

Next consider round \( m-1 \). For any buyer \( i \), suppose the history he observes is \((a^1_i, p^1, a^2_i, p^2, \ldots, a^{m-2}_i, p^{m-2})\), with \( \theta^{m-1} \) and \( m-1 \) defined above. There are two cases: \( a^m_i = b^1_{m-1}(\theta^{m-1}) \) and \( a^m_i < b^1_{m-1}(\theta^{m-1}) \). If \( a^m_i = b^1_{m-1}(\theta^{m-1}) \), then \( i \) is the perceived highest type in round \( m-1 \) (even though he may or may not be the actual highest type in that round). By our construction of \( \beta(2) \) above, \( \beta(2), i(a^1_i, p^1, a^2_i, p^2, \ldots, a^{m-2}_i, p^{m-2}) = \{ \theta_{i'} \overset{i.i.d.}{\sim} U[0, \theta^{m-1}] \text{ for } i' \neq i \text{ remaining} \} \). Given others bid according to the prescribed strategies in rounds \( m-1 \) and \( m \), the incentive he faces is exactly the
same as a type $\theta_i$ buyer with common independent prior \{\theta_{i'} \sim \text{i.i.d} \ U[0, \theta_m] \text{ for } i' \neq i \text{ remaining}\}. By the proof of Claim 3, a best response (in expectation) is $b_{i}^{m-1}(\theta_i) = v_{m-1}(\theta_i) - U^{m}(\theta_i, \theta_i).$ If, on the other hand, $a_{i}^{m-1} < b_{i}^{m-1}(\theta_m)$, then $i$ is not the perceived highest type in round $m-1$ (even though he may or may not be the actual highest type in that round), which equals $\theta_m$. Two observations are crucial when we consider buyer $i$’s incentives in this case:

1. Even though buyer $i$ may sway others’ beliefs by manipulating his bidding behaviors, we know from the previous step of backward induction that each buyer’s equilibrium bidding behaviors are identical across histories in round $m$ (i.e., bidding his true value), so $i$ has incentive to deviate in round $m-1$ only if he can change the allocation of item in that round.

2. Due to one-deviation property, we can restrict buyer $i$’s deviating strategy to round $m-1$.

With the two observations, there are three sub-cases we need to consider: $\theta_i < \theta^{m-1}$, $\theta_i = \theta^{m-1}$ and $\theta_i > \theta^{m-1}$. For $\theta_i < \theta^{m-1}$, by Observation 1, it suffices to restrict $i$’s deviating strategy to $b_{i}^{m-1} > b_{i}^{m-1}(\theta^{m-1})$. Given others bid according to the prescribed strategies in rounds $m-1$ and $m$ ($b_{i}^{m-1}(\cdot)$ and $b_{m}(\cdot)$), he will be the winner of round $m-1$, with payoff $U_i(b_{i}^{m-1}) < v_{m-1}(\theta_i) - [v_{m-1}(\theta^{m-1}) - U^{m}(\theta^{m-1}, \theta^{m-1})]$. With the prescribed the strategy, his expected payoff is no less than $U^{m}(\theta_i, \theta^{m-1})$. With generalized vertical differentiation and by Assumption 1, we can show for $\theta_i < \theta^{m}$:

$$v_{m-1}(\theta^{m-1}) - v_{m-1}(\theta_i) > U^{m}(\theta^{m-1}, \theta^{m-1}) - U^{m}(\theta_i, \theta^{m-1})$$

It follows that $U_i(b_{i}^{m-1}) < U^{m}(\theta_i, \theta^{m-1})$, or buyers in sub-case 1 have no incentive to deviate in round $m-1$. For $\theta_i = \theta^{m-1}$, given others bid according to the prescribed strategies in rounds $m-1$ and $m$ ($b_{i}^{m-1}(\cdot)$ and $b_{m}(\cdot)$), his expected payoff equals $U^{m}(\theta^{m-1}, \theta^{m-1})$ whatsoever (assuming one-shot deviation), so buyers in sub-case 2 have no incentive to deviate in round $m-1$. For $\theta_i > \theta^{m-1}$, by Observation 1, it suffices to restrict $i$’s deviating strategy to $b_{i}^{m-1} \leq b_{i}^{m-1}(\theta^{m-1})$. Given others bid according to the prescribed strategies in rounds $m-1$ and $m$ ($b_{i}^{m-1}(\cdot)$ and $b_{m}(\cdot)$), his expected payoff will be no greater than max\{[v_{m-1}(\theta_i) - [v_{m-1}(\theta^{m-1}) - U^{m}(\theta^{m-1}, \theta^{m-1})]$, $U^{m}(\theta_i, \theta^{m-1})]\}. With the prescribed strategy, his expected payoff equals $v_{m-1}(\theta_i) - [v_{m-1}(\theta^{m-1}) - U^{m}(\theta^{m-1}, \theta^{m-1})]$. With generalized vertical differentiation and by Assumption 1, we can show for $\theta_i > \theta^{m}$:

$$v_{m-1}(\theta_i) - v_{m-1}(\theta^{m-1}) > U^{m}(\theta_i, \theta^{m-1}) - U^{m}(\theta^{m-1}, \theta^{m-1})$$

or buyers in sub-case 3 have no incentive to deviate in round $m-1$. To sum up, no buyer has incentive to deviate from the prescribed strategy in round $m-1$.

\footnote{Notice the statement is true even if $\theta_i > \theta^{m-1}$, which was not possible under winning bid announcement.}
Using a similar argument, we can show inductively no buyer has incentive to deviate from \( b^k(\_\_\_) \) in rounds 2 through \( m \). For round 1, consider any buyer \( i \) of type \( \theta_i \). Given others bid according to the prescribed strategies in rounds 1 through \( m \), his incentive is identical to the case of winning bid announcement except that he may sway others’ beliefs. Nevertheless, similar observations to 1 and 2 still hold: even if \( i \) deviates in round 1, his expected continuation payoff is unaffected as long as he does not affect the current round allocation (again assuming one-shot deviation). It follows that our result in Claim 3 still works for winner’s price announcement in round 1, or no buyer has incentive to deviate from \( b^1(\_\_\_) \).

Proof of Lemma 6: Similar to the proof of Lemma 2, it suffices to consider the ordering (1, 2).

We first show the bidding strategies given in the proof of Lemma 2, (together with the beliefs specified below) constitute a symmetric and strictly monotonic pure strategy PBE:

\[
\begin{align*}
    b^1_{(2),i}(\theta_i) &= v_1(\theta_i); \\
    b^2_{(2),i}(a^1_i,p^1;\theta_i) &= v_2(\theta_i)
\end{align*}
\]

where the notations are the same as in the proof of Lemma 2, except that \( p^1 \) is now the winner’s price instead of the winning bid.

The parts on symmetry, monotonicity and pure strategy are trivial, so it remains to show the given strategy profiles constitute a PBE. We drop the subscript on individual \( i \) for convenience.

Belief construction is identical to the proof of Lemma 5 in the case of \( m = 2 \). One-deviation property is similarly guaranteed.

To check local incentive compatibility, note in round 2, given any history, we are in a common (though inter-dependent) prior private-value second-price auction, so bidding one’s true value is clearly incentive compatible. Moreover, this also implies that a player’s round 1 bidding behavior will not affect others’ round 2 equilibrium bidding strategies (even though it may sway others’ beliefs). With this observation, round 1 incentive compatibility is identical to that of Lemma 2.

The final step is to show \( b^1_{(2),\_\_\_} \) is generically unique. The only essential difference from Lemma 2 is that for player \( i \), given a history \((a^1_i,p^1)\), the common prior in the second round is inter-dependent. Nevertheless, the second round is a special case of Levine and Harstad (1986). Our restriction of “ex-ante symmetric strategies” is enough to show \( b^2_{(2),\_\_\_} \) is generically unique, even across histories. Therefore, the rest of the proof of Lemma 2 still applies.

Proof of Lemma 7: The idea is to mimic the proof Lemma 4, and show the same strategy profile \( b_{(2),A} \), together with the beliefs specified below constitute a symmetric, strictly monotonic, pure PBE in second-price sequential auctions with winner’s price announce-
The first step is again to construct the beliefs $\beta_{(2),A}$ and ensure one-deviation property. The construction combines ideas from Lemma 4 and 5. For player $i$, only information sets where he is still in the sequential auction matters. Because of winner’s price announcement, in round $k$, the history he can observe is his bids as well as the winner’s prices in rounds 1 through $k - 1$. We write such a history as $(a_{1}^{i}, p_{1}^{i}, a_{2}^{i}, p_{2}^{i}, \ldots, a_{k-1}^{i}, p_{k-1}^{i})$, where $a_{l}^{i}$ is the action (bid) of player $i$ in round $l$ and $p_{l}^{i}$ is the announced winner’s price in round $l$. Before constructing $\beta_{(2),A}$, we first derive the perceived (remaining) highest type $\bar{\theta}_{i}$ and lowest type $\hat{\theta}_{i}$ in round $l$. If $\kappa(l) \in M_{+}$ (the positive group), then the proof of Lemma 4 shows $\bar{\theta}_{i}^{(2)}(\theta_{i})$ is strictly increasing in $\theta_{i}$, so the winner’s price reveals information on highest type in round $l + 1$. Set $\bar{\theta}^{l+1} = 0$. For $\bar{\theta}^{l+1}$, if $p_{l}^{i} \in [\bar{\theta}_{i}^{(2),A}(0), \bar{\theta}_{i}^{(2),A}(1)]$ (equilibrium bids of some type), set $\bar{\theta}^{l+1} = \bar{\theta}_{i}^{(2),A}^{-1}(p_{l}^{i})$. (Invertible by strict monotonicity.) If $p_{l}^{i} \leq \bar{\theta}_{i}^{(2),A}(0)$, set $\bar{\theta}^{l+1} = 0$ and if $p_{l}^{i} > \bar{\theta}_{i}^{(2),A}(1)$, set $\bar{\theta}^{l+1} = 1$. If on the other hand, $\kappa(l) \in M_{-}$ (the negative group), then the proof of Lemma 4 shows $\bar{\theta}_{i}^{(2),A}(\cdot)$ is strictly decreasing in $\theta_{i}$, so the winner’s price reveals information on lowest type in round $l + 1$. Set $\hat{\theta}^{l+1} = 1$. For $\hat{\theta}^{l+1}$, if $p_{l}^{i} \in [\bar{\theta}_{i}^{(2),A}(1), \bar{\theta}_{i}^{(2),A}(0)]$ (equilibrium bids of some type), set $\hat{\theta}^{l+1} = \bar{\theta}_{i}^{(2),A}^{-1}(p_{l}^{i})$. (Invertible by strict monotonicity.) If $p_{l}^{i} > \bar{\theta}_{i}^{(2),A}(0)$, set $\hat{\theta}^{l+1} = 0$ and if $p_{l}^{i} < \bar{\theta}_{i}^{(2),A}(1)$, set $\hat{\theta}^{l+1} = 1$. We construct $\beta_{(2),A}$ as follows: Let $\theta_{k} = \max_{1 \leq l \leq k-1} \bar{\theta}^{l+1}$, $\bar{\theta}_{k} = \min_{1 \leq l \leq k-1} \hat{\theta}^{l+1}$ and $\bar{k} = \arg\max_{1 \leq l \leq k-1} \bar{\theta}^{l+1}$, $\bar{k} = \arg\min_{1 \leq l \leq k-1} \hat{\theta}^{l+1}$. If $\bar{\theta}_{k} \leq \theta_{k}$, set $\beta_{(2),A,i}(a_{1}^{i}, p_{1}^{i}, a_{2}^{i}, p_{2}^{i}, \ldots, a_{k-1}^{i}, p_{k-1}^{i}) = \{\theta_{l}^{i} = \theta_{i} \text{ for } i' \neq i \text{ remaining} \}$. If $\bar{\theta}_{k} > \theta_{k}$, set $\beta_{(2),A,i}(a_{1}^{i}, p_{1}^{i}, a_{2}^{i}, p_{2}^{i}, \ldots, a_{k-1}^{i}, p_{k-1}^{i}) = F_{k}$ (the joint cdf of the beliefs of the remaining $n - k$ buyers), where the density function of $F_{k}$ is separately defined in four different cases:

Case 1: $a_{i}^{k} = b_{(2),A}^{k}(\hat{\theta}_{i})$ and $a_{i}^{k} = b_{(2),A}^{k}(\bar{\theta}_{i})$, or $i$ is both the perceived lowest type and highest type in round $k$. (Notice this is possible off the equilibrium path.)

\[
f^{k}(\gamma_{1}, \gamma_{2}, \ldots, \gamma_{n-k}) = \begin{cases} \frac{1}{(\bar{\theta}_{i} - \hat{\theta}_{i})^{n-k}} & \text{when } \theta_{k} \leq \gamma_{i} \leq \bar{\theta}_{i}, \forall i \\ 0 & \text{otherwise} \end{cases}
\]

Case 2: $a_{i}^{k} = b_{(2),A}^{k}(\theta_{i})$ and $a_{i}^{k} < b_{(2),A}^{k}(\bar{\theta}_{i})$, or $i$ is only the perceived lowest type in round $k$.

\[
f^{k}(\gamma_{1}, \gamma_{2}, \ldots, \gamma_{n-k}) = \begin{cases} \frac{1}{(\theta_{i} - \hat{\theta}_{i})^{n-k-1}} & \text{when } \theta_{k} \leq \gamma_{i} \leq \hat{\theta}_{i}, \forall i \text{ and max } \gamma_{i} = \hat{\theta}_{i} \\ 0 & \text{otherwise} \end{cases}
\]

Case 3: $a_{i}^{k} < b_{(2),A}^{k}(\theta_{i})$ and $a_{i}^{k} = b_{(2),A}^{k}(\bar{\theta}_{i})$, or $i$ is only the perceived highest type in round $k$.

\[
f^{k}(\gamma_{1}, \gamma_{2}, \ldots, \gamma_{n-k}) = \begin{cases} \frac{1}{(\theta_{i} - \bar{\theta}_{i})^{n-k-1}} & \text{when } \theta_{k} \leq \gamma_{i} \leq \bar{\theta}_{i}, \forall i \text{ and min } \gamma_{i} = \bar{\theta}_{i} \\ 0 & \text{otherwise} \end{cases}
\]
Case 4: $a_i^k < b^k_{(2),A}(\theta^k)$ and $a_i^k < b^k_{(2),A}(\tilde{\theta}^k)$, or $i$ is neither the perceived highest or lowest type in round $k$.

$$f^k(\gamma_1, \gamma_2, \ldots, \gamma_{n-k}) = \begin{cases} \frac{1}{(\theta^k - \bar{\theta})^{n-k-2}} & \text{when } \theta^k \leq \gamma_i \leq \bar{\theta}, \forall i \text{ and } \min \gamma_i = \theta^k \text{ and } \max \gamma_i = \bar{\theta}, \\ 0 & \text{otherwise} \end{cases}$$

Arguing similarly to the proof of Lemma 4, we know one-deviation property is satisfied in the current setup. For the rest of the proof, we proceed by induction on the total number of items $m$.

Given any (potentially inter-dependent) common prior, both statements are true for $m = 1$ and $m = 2$ with any number of bidders $n \geq m + 1$ (either trivial, or a slight modification of the negative case). So it suffices to prove the following:

“Given any (potentially inter-dependent) common prior, suppose both statements are true for $m \geq 2$, with any number of bidders $n \geq m + 1$, i.e., for any pair $(m_1, m - m_1)$ s.t. $0 \leq m_1 \leq m$, then those statements are also true for $m + 1$, with any $n' \geq m + 2$.”

Similar to the proof of Lemma 4, we consider the case of $k_1 \in M_+$. In the proof of Lemma 4, we argue the following strategy profile characterize round 1 equilibrium in second-price sequential auctions:

$$b^1_{(2),i,A}(\theta_i \mid c) = v_1(\theta_i) - U^2_A(\theta_i, \bar{\theta}, c)$$

There, the statement is a direct generalization of Claim 3 and Lemma 2. Here, however, there is one potential difficulty in generalizing Lemmas 5 and 6: As argued in Lemmas 5 and 6, with winner’s price announcement, one may sway others’ beliefs without winning an earlier item. Given this channel and the dependence of $U^2_A$ on $c$, one may imagine a situation where a buyer can profit by bidding above or below $b^1_{(2),A}$ in round 1 in the hope of reaping future benefits through changing others’ priors. Fortunately, this is not possible. The reason is that in the strategy profile we inductively constructed $b_{(2),A}$, and the beliefs $\beta_{(2)}$ specified above, a player’s round 1 bidding will not change $b^k_{(2),A}$ for $k \in M_+$. (Notice round 1 bidding only updates the perceived (remaining) highest types.) Moreover, we still have $U^2_{A_1,A_\bar{c}}(\theta_i, \theta, c) = 0, \forall \theta_i > c$. In other words, given others follow $b_{(2),-i,A}$, $i$’s continuation payoff is unaffected if he only deviates in round 1. With this key observation, the rest of the proof is almost identical to that of Lemma 4.