MAT205a, Fall 2019 Part IV: Lebesgue spaces

Lecture 11, Following Folland, ch 5.1, 6.1

1. Basic theory of Lebesge spaces

1.1. Banach spaces. Let $V$ be a vector space over $\mathbb{R}$ (or $\mathbb{C}$).

**Definition 1.1.** A function from $V$ to $[0, +\infty)$, such that $x \mapsto \|x\|$ is called a norm on $V$ if

(i) $\|x\| = 0$ implies that $x = 0$,

(ii) $\|cx\| = |c|\|x\|$, for all $x \in V$ and $c \in \mathbb{R}$ (or $\mathbb{C}$),

(iii)$\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

A vector space $V$ with a norm $\| \cdot \|$ is called a normed space. A norm defines a metric on the space $V$.

**Definition 1.2.** A normed space $(V, \| \cdot \|)$ is called a Banach space if it is complete with respect to the metric defined by the norm.

It means that $V$ is a Banach space if any sequence of vectors $\{v_j\}$ that is a Cauchy sequence, i.e., for any $\varepsilon > 0$ there is $n(\varepsilon)$ such that $\|v_n - v_m\| < \varepsilon$ for $n, m > n(\varepsilon)$, converges in $V$ to some $w \in V$.

**Lemma 1.1.** A normed space $(V, \| \cdot \|)$ is a Banach space if and only if for any sequence $\{v_j\}$ in $V$ with $\sum_j \|v_j\| < \infty$ the series $\sum_j v_j$ converges in $V$.

**Proof.** If $V$ is a Banach space and $\sum_j \|v_j\| < \infty$, we define $S_n = \sum_1^n v_j$. Then $\|S_n - S_m\| \leq \sum_{n<j\leq m} \|v_j\|$ and $\{S_n\}$ is a Cauchy sequence. It has a limit, so $\sum_j v_j$ converges in $V$.

Assume now that each absolutely convergent series converges in $V$. We want to prove that $V$ is a Banach space. Let $\{x_j\}$ be a Cauchy sequence in $V$. We can find a subsequence $\{x_{j_k}\}$ such that $\|x_{j_k} - x_{j_l}\| < 2^{-m}$ when $k, l > m$. Define $v_n = x_{j_{n+1}} - x_{j_n}$, then $\sum_n \|v_n\| < \infty$. Thus $x_{j_n}$ converges in $V$ to some $x_*$, and since $\{x_j\}$ is a Cauchy sequence, the whole sequence converges to $x_*$. \[\square\]

1.2. Definition of the Lebesgue spaces. We assume that $(X, \mathcal{M}, \mu)$ is a measure space and that $1 \leq p < \infty$ on the set of all measurable functions, we define

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{1/p}.$$  

The Lebesgue space $L^p$ is defined as

$$L^p(\mu) = \{f : f \text{ measurable, } \|f\|_p < \infty\}.$$
Where we identify functions that differ $\mu$-almost everywhere.

One can define the spaces for $p \in (0, 1)$, but we will work with the case $p \geq 1$.

Our first aim is to show that $(L^p(\mu), \| \cdot \|_p)$ is a normed space. We know that $\|f\|_p = 0$ implies that $f = 0 \mu$-a.e. and it is identified with the zero element in $L^p$, also $\|cf\|_p = |c|\|f\|_p$ since the integral is linear. We can also check that if $f, g \in L^p$ then $f + g \in L^p$ since $|f + g|^p \leq (2 \max |f|, |g|)^p \leq 2^p(|f|^p + |g|^p)$ point-wise, and integrating this inequality we obtain

$$\|f + g\|_p \leq 2(\|f\|_p + \|g\|_p).$$

We want to prove the inequality without the extra factor 2 on the right hand side.

1.3. Useful inequalities. We start with an inequality for real numbers.

**Lemma 1.2.** Let $a, b \geq 0$ and $\lambda \in (0, 1)$, then

$$(1) \hspace{1cm} a^{\lambda}b^{1-\lambda} \leq \lambda a + (1-\lambda)b,$$

and the equality holds if and only if $a = b$.

**Proof.** If $b = 0$ the inequality holds and equality is true only for $a = 0$. If $b \neq 0$, we divide both sides by $b$ and denote $a/b = t$, then the inequality reads

$$t^\lambda \leq \lambda t + (1-\lambda).$$

It becomes an equality when $t = 1$ and for $f(t) = t^\lambda - \lambda t$ we have $f'(t) = \lambda(t^{\lambda-1} - 1)$. Since $\lambda - 1 < 0$, $f$ increases when $t \in (0, 1)$ and decreases when $t \in (1, \infty)$. Thus $f(t) \leq f(1) = 1 - \lambda$. \hfill \Box

**Theorem 1.1** (Hölder’s inequality). Let $1/p, q < \infty$ be such that $1/p + 1/q = 1$. If $f, g$ are measurable functions on $(X, \mathcal{M}, \mu)$ then $\|fg\|_1 \leq \|f\|_p\|g\|_q$.

**Proof.** The inequality holds if $\|f\|_p = 0$ and $f = 0$ almost everywhere or when $\|f\|_p = \infty$, similarly for $g$. Now if $\|f\|_p \neq 0, +\infty$ and $\|g\|_q \neq 0, \infty$, we may assume that $\|f\|_p = 1$ and $\|g\|_q = 1$ by multiplying $f$ and $g$ by suitable constants. Consider $x \in X$, we apply inequality $\lbrack 1 \rbrack$ to $a = \|f(x)\|^p$, $b = |g(x)|^q$ and $\lambda = 1/p$. Then $1 - \lambda = 1/q$, and we obtain

$$|f(x)g(x)| \leq p^{-1}\|f(x)\|^p + q^{-1}|g(x)|^q.$$

Now, integrating this inequality over $X$ with respect to the measure $\mu$, we get

$$\|fg\|_1 \leq p^{-1}\|f\|_p^p + q^{-1}\|g\|_q^q = 1 = \|f\|_p\|g\|_q.$$ \hfill \Box
Corollary 1.1 (Cauchy-Schwarz inequality). When \( p = q = 2 \) we obtain the following

\[
\int |fg| \, d\mu \leq \left( \int |f|^2 \, d\mu \right)^{1/2} \left( \int |g|^2 \, d\mu \right)^{1/2}
\]

It also follows from Theorem that in \( f \in L^p \) and \( g \in L^q \) then \( fg \in L^1 \). The exponent \( q \) such that \( p^{-1} + q^{-1} = 1 \) is called the conjugate exponent to \( p \).

Theorem 1.2 (Minkowski’s inequality). If \( 1 \leq p < \infty \) and \( f, g \in L^p \) then

\[
\|f + g\|_p \leq \|f\|_p + \|g\|_p.
\]

Proof. When \( p = 1 \) the inequality follows from the properties of integral. Assume that \( p > 1 \). We have

\[
|f + g|^p \leq (|f| + |g|)|f + g|^{p-1} = |f||f + g|^{p-1} + |g||f + g|^{p-1}.
\]

Now we apply the Hölder inequality twice,

\[
\int |f + g|^p \, d\mu \leq \|f\|_p \|(f + g)^{p-1}\|_q + \|g\|_p \|(f + g)^{p-1}\|_q.
\]

We know that \( q = p/(p-1) \) and the inequality becomes

\[
\|f + g\|_p \leq (\|f\|_p + \|g\|_p)\|f + g\|_p^{p-1}.
\]

Now if \( \|f + g\|_p \neq 0 \) we divide both sides by \( \|f + g\|_p^{p-1} \) and obtain the required inequality. When \( \|f + g\|_p = 0 \) the inequality also holds. \( \square \)

Theorem 1.3. For \( 1 \leq p < \infty \) the Lebesgue space \( L^p(\mu) \) is a Banach space.

Proof. We want to show that an absolutely convergent series converges, using Lemma 1.1. Let \( \{f_j\}_j \) be a sequence in \( L^p(\mu) \) and \( \sum_j \|f_j\|_p = B < \infty \). We define \( G_n = \sum_1^n |f_j| \) and \( F_n = \sum_1^n f_j \). Then \( \|F_n\|_p \leq \|G_n\|_p \leq \sum_1^n \|f_j\|_p \). Let \( G = \sum_1^\infty |f_j| \). By the monotone convergence theorem \( \int G^p \leq B^p \). Thus \( |G| < \infty \) a.e. and we can define \( F = \sum f_j \) a.e. We also have \(|F| \leq G \) and \( F \in L^p(\mu) \). Using the dominated convergence theorem we see that

\[
\|F - \sum_1^n f_j\|_p = \int |F - \sum_1^n f_k|^p \to 0
\]

since \(|F - \sum_1^n f_j|^p \leq (2G)^p \in L^1(\mu)\), \( \square \)
2. **The dual of** $L^p(\mu)$

2.1. **The dual of a normed space.** We remind that if $V$ is a normed space that its dual, denoted by $V^*$, is the space of of continuous linear functionals on $V$,

$$V^* = \{ T : V \to \mathbb{C} \text{ linear, } |T(x)| \leq C \|x\| \}. $$

The norm of the functional $T$ is defined as $\|T\| = \sup \{ |T(x)| : \|x\| = 1 \}$. Then $V^*$ is a normed space. Clearly $\|T\| = 0$ if and only if $T = 0$ and $\|Tc\| = |c| \|T\|$ since $T$ is linear. The triangle inequality follows from the linearity as well,

$$|T_1(x) + T_2(x)| \leq |T_1(x)| + |T_2(x)| \leq \|T_1\| + \|T_2\|,$$

for any $x \in V$ with $\|x\| = 1$. Then, taking the supremum, we obtain $\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$.

2.2. **The space** $L^\infty$. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f$ be a measurable function, we say that $f \in L^\infty(X)$ if there exists $C < \infty$ such that $|f| \leq C \mu$-a.e. And we define

$$\|f\|_\infty = \inf \{ C \geq 0 : \mu(\{|f| > C\}) = 0 \}.$$ 

Note also that if $\|f\|_\infty = C$ then $\mu(\{|f| > C\}) = \lim_{n \to \infty} \mu(\{|f| > C + 1/n\}) = 0$. Clearly $L^\infty(\mu)$ is a linear space and $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

When $\mu(X) < \infty$ we see that $L^\infty(\mu) \subset L^p(\mu)$ for any $p \geq 1$. Without the assumption that $\mu$ is finite the conclusion does not hold. The space $L^\infty$ is also a Banach space (see your problem set).

The following version of the Hölder inequality holds. If $f \in L^\infty(\mu)$ and $g \in L^1(\mu)$ then $fg \in L^1(\mu)$ and $\|fg\|_1 \leq \|f\|_\infty \|g\|_1$.

2.3. **Functionals on** $L^p(\mu)$. Let $1/p + 1/q = 1$, where $1 \leq p,q \leq \infty$, for each $g \in L^q(\mu)$ we define the functional

$$T_g : L^p(\mu) \to \mathbb{C}, \quad T_g(f) = \int_X fg d\mu.$$ 

Clearly $T_g$ is linear and by Hölder’s inequality

$$|T_g| \leq \|f\|_p \|g\|_q, \quad \text{i.e.,} \quad \|T_g\| \leq \|g\|_q.$$ 

**Proposition 2.1.** Suppose that $1 \leq q < \infty$ then $\|T_g\| = \|g\|_q$.

If $q = \infty$ and $\mu$ is semifinite, then $\|T_g\| = \|g\|_\infty$. 

Proof. Suppose first that $1 < q < \infty$. We already know that $\|T_g\| \leq \|g\|_q$, now we consider the function $f(x) = |g(x)|^{q-2}g(x)$, such that $|f(x)| = |g(x)|^{q-1}$. Then $\|f\|^p_p = \int |g|^{p(q-1)}d\mu = \|g\|^q_g$. We also have 

$$|T_g f| = \int_X |g|^q d\mu = \|g\|^q_g.$$ 

Therefore 

$$\|T_g\| \geq \|g\|_q^q \|f\|^{-1}_p = \|g\|_q^{q-p} = \|g\|_q.$$ 

If $q = 1$ we let $f(x) = \frac{g(x)}{|g(x)|} g(x)^{q-1}$ such that $|f(x)| = 1$ and $f(x)g(x) = |g(x)|$, then $\|T_g f\| = \inf_X |g(x)| d\mu = \|g\|_1$ and $\|f\|_\infty = 1$.

Finally, when $q = \infty$, consider a set $E \subset X$ such that $0 < \mu(E) < \infty$ and take $f(x) = \frac{g(x)}{|g(x)|} g(x)^{-1} \chi_E(x)$. So that $|f(x)| = 1$ when $x \in E$ and $|f(x)| = 0$ otherwise, then $\|f\|_1 = \mu(E)$. Moreover, 

$$|T_g f| = \int_E |g(x)| d\mu \geq \min_E |g(x)| \mu(E).$$ 

Now since $g \in L^\infty(\mu)$ we have that $\mu(\{|g(x)| > \|g\|_\infty - \varepsilon\}) > 0$ and since the measure is semifinal, there exists $E \subset \{|g(x)| > \|g\|_\infty - \varepsilon\}$ with $0 < \mu(E) < \infty$. We obtain 

$$\|T_g\| \geq \|g\|_\infty - \varepsilon.$$ 

for any positive $\varepsilon$, and then by taking $\varepsilon \to 0$ the conclusion of the proposition follows.

We want to show that all functionals on $L^p(\mu)$, when $1 < p < \infty$ are defined in that way. First we prove a weaker statement. Let $S_0(\mu)$ denote the family of simple functions $\phi$ on $(X, \mathcal{M}, \mu)$ such that $\mu(\{\phi \neq 0\}) < \infty$.

Theorem 2.1. Let $1/p + 1/q = 1$, $1 \leq p, q \leq \infty$. Suppose that $g$ is a measurable function on $(X, \mathcal{M}, \mu)$ such that $E_g = \{g \neq 0\}$ is ($\mu$) $\sigma$-finite and $g\phi \in L^1(\mu)$ for any $\phi \in S_0(\mu)$ with 

$$M_q(g) = \sup \left\{ \left\| \int \phi g d\mu \right\|, \ \phi \in S_0, \ \|\phi\|_p = 1 \right\} < \infty.$$ 

Then $g \in L^q(\mu)$ and $M_q = \|g\|_q$.

Proof. Suppose that $f$ is a measurable function with $\mu(\{f \neq 0\}) < \infty$ and $f \in L^p(\mu)$. We claim that $fg \in L^1$ and $\|fg\|_1 \leq M_q(g) \|f\|_p$. Indeed, there is a sequence of simple functions $\phi_n$ such that $\phi_n \to f$ pointwise and $|\phi_n| \leq |f|$. Moreover, $|\phi_n g| \leq |\chi_{Eg}|$, where $E = \{f \neq 0\}$ and $\chi_{Eg} \in L^1(\mu)$. Then by the dominated convergence theorem

$$\int fg d\mu = \lim_{n \to \infty} \int \phi_n g d\mu \leq M_q(g) \lim_{n \to \infty} \|\phi_n\|_p \leq M_q(g) \|f\|_p.$$
We want to show that $g \in L^q(\mu)$. Assume first that $1 < q < \infty$ and let $E_g = \cup_m E_m$ such that $E_m \subset E_{m+1}$ and $\mu(E_m) < \infty$. There exists a sequence of simple functions $\psi_m$ that converges to $g$ and such that $|\psi_m| \leq |g|$, we define $g_m = \psi_m \chi_{E_m}$. Then $g_m$ is also a sequence of simple functions converging to $g$, $|g_m| \leq |g|$, and $\mu(\{g_m \neq 0\}) < \infty$. Define $\phi_m = |g_m|^{q-2}g_m|g_m|^{1-q}$ then $\phi_m \in S_0$. Moreover $|\phi_m| = |g_m|^q|g_m|^{-p}$ and $|\phi_m|^p = |g_m|^q|g_m|^{-p}$. Hence $\|\phi_m\|_p = 1$. By the Fatou lemma

\[
\left( \int |g|^q \, d\mu \right)^{1/q} \leq \liminf_{m \to \infty} \|g_m\|_q = \liminf_{m \to \infty} \int |\phi_m g_m| \, d\mu \leq \liminf_{m \to \infty} \int |\phi_m g| \, d\mu \leq M_q(g).
\]

In the last inequality we used that $|\phi_m g| = fg$, where $f$ satisfies $|f| = |\phi|$ and therefore $f \in L^p$, $\|f\|_p = \|\phi_m\|_p = 1$, so we can use the first part of the proof.

The last inequality implies that $g \in L^q(\mu)$ and $\|g\|_q \leq M_q(g)$. We know also that $M_q(g) \leq \|g\|_q$ by the Hölder inequality. Thus $\|g\|_q = M_q(g)$. For $q = 1$ we repeat the argument taking $\phi_m = \overline{T_m}/|g_m|$, clearly $\|\phi_m\|_\infty = 1$.

For the case $q = \infty$, assume that $\|g\|_\infty > M_\infty(g)$ then $\mu(\{|g| > M_\infty(g) + \varepsilon\}) > 0$ for some $\varepsilon > 0$. Since $E_g$ is $\sigma$-finite, there exists $B \subset \{|g| > M_\infty(g) + \varepsilon\}$ with $0 < \mu(B) < \infty$. Let $f = \overline{g}|g|^{-1}\chi_B$, then $\|f\|_1 = \mu(B)$ and

\[
\int_B fg \, d\mu = \int_B \overline{|g|} > \left( M_\infty(g) + \varepsilon \right) \|f\|_\infty.
\]

This contradicts the inequality (2) above. Thus $g \in L^\infty$ and $\|g\|_\infty \leq M_\infty(g)$. On the other hand, clearly $M_\infty(g) \leq \|g\|_\infty$. \hfill \box

**Theorem 2.2.** Let $1 < p < \infty$ and $1/p + 1/q = 1$. Then for any $T \in (L^p(\mu))^*$ there is $g \in L^q(\mu)$ such that $T(f) = T_g(f) = \int fg \, d\mu$.

If $\mu$ is $\sigma$-finite then for any $T \in (L^1(\mu))^*$ there is $g \in L^{\infty}(\mu)$ such that $T(f) = T_g(f) = \int fg \, d\mu$.

**Proof.** First assume that $\mu(X) < \infty$. Then for any measurable $E$ we have $\chi_E \in L^p(d\mu)$. Define $\nu(E) = T(\chi_E)$. We claim that it is a measure and $\nu \ll \mu$. To show that $\nu$ is countably additive consider a sequence of disjoint sets $\{E_j\}$. We claim that if $\phi_n = \sum_1^n \chi_{E_j}$ and $\phi = \chi_{\cup_j E_j}$ then $\phi_n \to \phi$ in $L^p$ and then $\nu(\cup_j E_j) = \sum \nu(E_j)$. Then applying the Radon-Nikodym theorem we get a function $g$ such that $\nu(E) = \int_E g \, d\mu$. Then we see that for all simple functions $\phi$ we have $t(\phi) = \int \phi g \, d\mu$ and $\|\int \phi g \, d\mu\| \leq \|T\| \|\phi\|_p$. Applying the previous theorem we see that $g \in L^q(\mu)$.

Next, we assume that $\mu$ is $\sigma$-finite. Then $X = \cup_n X_n$ with $\mu(X_n) < \infty$ and $X_n \subset X_{n+1}$. We know that for each $n$ there exists $g_n \in L^q(X_n, \mu)$ such that $T(f) = \int fg \, d\mu$ when $f \in L^p(X, \mu)$, $f = 0$ on $X_n^c$ and $\|g_n\|_q \leq \|T\|$. Moreover $g_n$ is unique. If $g_n$ and $g'_n$ define the same functional on $L^p(X_n, \mu)$ then we know that $\|g_n - g'_n\|_q = 0$. Thus $g_n = g_m$ $\mu$-a.e. on $X_n$ when $m > n$ and we can define function $g$ on $X$ such
that $g = g_n$ on $X_n$. By the monotone convergence theorem $\|g\|_q \leq \|T\|$. For any $f \in L^p(\mu)$ we have $f \chi_{X_n} \to f$ in $L^p$ by the dominated convergence and

$$T(f) = \lim_{n \to \infty} \int_{X_n} fg \, d\mu = \int_X fg \, d\mu.$$ 

Finally, assume that $\mu$ is arbitrary and $1 < p < \infty$. For each set $E \subset X$ that has a $\sigma$-finite measure we can find $g_E \in L^q(E, \mu)$ such that $T(f) = \int f g \, d\mu$ when $f = 0$ on $E^c$ and $\|g_E\|_q \leq \|T\|$. Let

$$M = \sup\{ \|g_E\|_q : E \text{ is } \sigma \text{- finite} \}.$$ 

There is a sequence $E_n$ such that $\|g_{E_n}\|_q \to M$, let $E = \cup_n E_n$ then $g_E = g_{E_n}$ a.e. on $E_n$ and thus $\|g_E\|_q = M$. Then for any $\sigma$-finite set $B$ such that $E \cap B = \emptyset$ we have $g_B = 0$, here we use that $q < \infty$. Let $f \in L^p(\mu)$ then $F = \{ f \neq 0 \}$ is a $\sigma$-finite set. We have

$$T(f) = \int g_{F \cup E} f \, d\mu = \int g_E f \, d\mu + \int g_{F \setminus E} f \, d\mu = \int g_E f \, d\mu.$$ 

Thus $T(f) = \int g_E f \, d\mu$. \qed
3. Some operators between $L^p$ spaces

3.1. Norm of an operator. Now we will consider linear operators between two normed spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$. Let $T: V \to W$, $T$ is linear, i.e.,

$$T(cv_1 + bv_2) = aT(v_1) + bT(v_2),$$

we say that $T$ is bounded if

$$\|T\| = \sup_{\|v\|_V = 1} \|Tv\|_W = \sup_{v \neq 0} \frac{\|Tv\|_W}{\|v\|_V} < \infty.$$  

We will study linear operators between some $L^p$ spaces.

3.2. Boundedness of integral operators. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two measure spaces with $\sigma$-finite measures. Suppose that $K: X \times Y \to \mathbb{C}$ is a $\mathcal{M} \otimes \mathcal{N}$ measurable function. Under some additional assumption $K$ we want to define the integral operator

$$Tf(x) = \int_Y K(x, y)f(y)d\nu(y).$$

**Theorem 3.1.** Suppose that there exists $C$ such that $\int |K(x, y)|d\mu(x) \leq C$ for $\nu$-a.e. $y \in Y$ and $\int |K(x, y)|d\nu(y) \leq C$ for $\mu$-a.e. $x \in X$. Then for any $f \in L^p(\nu)$, $1 \leq p \leq \infty$ the integral $Tf(x)$ converges absolutely for $\mu$-a.e. $x \in X$. Moreover $Tf \in L^p(\mu)$ and $\|Tf\|_p \leq C\|f\|_p$.

**Proof.** Consider first $p \in (1, \infty)$ and let $q$ be such that $1/p + 1/q = 1$. We write $|K(x, y)f(y)| = |K(x, y)|^{1/q}(\|K(x, y)\|^{1/p}|f(y)|)$ and apply the Hölder inequality:

$$\int_Y |K(x, y)f(y)|d\nu(y) \leq \left(\int_Y |K(x, y)|d\nu(y)\right)^{1/q} \left(\int_y |K(x, y)||f(y)|^pd\nu(y)\right)^{1/p},$$

where the first factor is bounded by $C^{1/q}$ for a.e $x \in X$. Now we apply the Tonelli theorem

$$\int_X \left(\int_Y |K(x, y)f(y)|d\nu(y)\right)^pd\mu(x) \leq C^{p/q} \int_Y \int_X |K(x, y)||f(y)|^pd\mu(x)d\nu(y) \leq C^{p/q+1}\|f\|_p^p.$$  

Then we know that $K(x, y)f(y) \in L^1(\nu)$ for a.e. $x$, thus $Tf(x)$ is well-defined for a.e. $x \in X$ and $\|Tf\|_f \leq C^{1/q+1/p}\|f\|_p = C\|f\|_p$. 

For $p = 1$ we have by the Tonelli theorem
\[
\int_X \int_Y |K(x,y)f(y)|d\nu(y)d\mu(x) \leq C \int_Y \|f\|d\nu(y) = C\|f\|_1,
\]
and the rest of the proof is the same as above. In this case we used only one condition, \(\int |K(x,y)|d\mu(x) \leq C\).

Similarly, for \(p = \infty\), \(\int |K(x,y)f(y)|d\nu(y) \leq \|f\|_\infty \int_Y |K(x,y)|d\nu(y) \leq C\|f\|_\infty\) and \(\|Tf\|_\infty \leq C\|f\|_\infty\), and we used only one bound on \(K\), \(\int |K(x,y)|d\nu(y) \leq C\). \(\square\)

3.3. **Minkowski inequality.** Next, we generalize the triangle inequality \(\|f_1 + f_2\|_p \leq \|f_1\|_p + \|f_2\|_p\) by replacing the sum by the integral.

**Theorem 3.2.** Let \((X,\mathcal{M},\mu)\) and \((Y,\mathcal{N},\nu)\) be two measure measure spaces with \(\sigma\)-finite measures and let \(f : X \times Y \to \mathbb{C}\) be a \(\mathcal{M} \otimes \mathcal{N}\) measurable function, \(f \geq 0\). Then for \(1 \leq p < \infty\)
\[
\left( \int_X \left( \int_Y f(x,y)^p d\nu(y) \right)^{\frac{1}{p}} d\mu(x) \right)^{\frac{1}{p}} \leq \int_Y \left( \int_X f(x,y)^p d\mu(x) \right)^{\frac{1}{p}} d\nu(y).
\]

**Proof.** For \(p = 1\) this is the Tonelli theorem. Let \(1 < p < \infty\) and let \(1/p + 1/q = 1\). We define
\[
F(x) = \int_Y f(x,y)d\nu(y)
\]
and we want to estimate \(\|F\|_p\) in \(L^p(\mu)\). By the result of the previous lecture (Theorem 2.1), it is equivalent to estimating the norm of the corresponding functional on \(L^q(\mu)\). Let \(g \in L^q(\mu)\), applying the Tonelli theorem and the Hölder inequality, we obtain
\[
\left| \int_X F(x)g(x) d\mu(x) \right| \leq \int_X \int_Y f(x,y)|g(x)| d\nu(x)d\mu(y) \leq \int_Y \left( \int_X f(x,y)^p d\mu(x) \right)^{\frac{1}{p}} \left( \int_X g(x)^q d\mu(x) \right)^{\frac{1}{q}} d\nu(y)
\]
the required inequality follows. \(\square\)

**Corollary 3.1.** Let \(1 \leq p \leq \infty\), \(f(\cdot,y) \in L^p(\mu)\) for \(\nu\)-a.e. \(y \in Y\) and the function \(y \mapsto \|f(\cdot,y)\|_p\) is in \(L^1(\nu)\) then \(f(x,\cdot) \in L^1(\nu)\) for \(\mu\)-a.e. \(x \in X\), the function \(F(x) = \int_Y f(x,y) d\nu(y)\) is in \(L^p(\mu)\) and
\[
\|F\|_p \leq \int \|f(\cdot,y)\|_p d\nu(y).
\]

If we replace \(f\) by \(|f|\), the corollary (for \(p < \infty\)) follows from the theorem. It is clear when \(p = \infty\).
3.4. **Hardy’s inequality.** We first describe another class of bounded integral operators on $L^p(0,\infty)$.

**Theorem 3.3.** Let $K$ be a Lebesgue measurable function on $(0,\infty) \times (0,\infty)$ and $1 \leq p \leq \infty$ such that $K(\lambda x,\lambda y) = \lambda^{-1}K(x,y)$ for all $\lambda > 0$ and
\[
\int_0^\infty |K(x,1)|x^{-1/p}dm(x) = C < \infty.
\]
Then the operator $T$ defined by
\[
Tf = \int_0^\infty K(x,y)f(x)\,dm(x),
\]
is bounded on $L^p(0,\infty)$, $\|Tf\|_p \leq C\|f\|_p$.

**Proof.** Consider $f_t(y) = f(ty)$ for $t > 0$, then $\|f_t\|_p = t^{-1/p}\|f\|_p$. We fix $y$, and introduce a new variable $t = x/y$, then
\[
\int_0^\infty |K(x,y)f(x)|\,dm(x) = \int_0^\infty |K(t,1)f_t(y)|\,dm(t).
\]
Then, by Minkowski’s inequality, (see the corollary above)
\[
\|Tf\|_p \leq \int_0^\infty |K(t,1)||f_t|\,dm(t) = \|f\|_p\int_0^\infty t^{-1/p}|K(t,1)|\,dm(t) = C\|f\|_p.
\]
\[
\square
\]

**Corollary 3.2** (Hardy’s inequality). Let $1 < p \leq \infty$ and
\[
Tf(y) = y^{-1}\int_0^y f(x)\,dx.
\]
Then $\|Tf\|_p \leq p(p-1)^{-1}\|f\|_p$.

**Proof.** We apply the theorem above for $K(x,y) = y^{-1}\chi_{x<y}$. Then $K(\lambda x,\lambda y) = \lambda^{-1}K(x,y)$ and
\[
\int_0^\infty |K(x,1)|x^{-1/p}dm(x) = \int_0^1 x^{-1/p}dm(x) = \frac{p}{p-1}.
\]
\[
\square
\]

4. **Distribution function**

4.1. **Chebyshev inequality and the distribution function.** Suppose that $f \in L^p(\mu)$, $1 \leq p < \infty$, and $t > 0$ then
\[
\mu(\{|f| > t\}) \leq t^{-p}\|f\|_p^p.
\]
It follows by integrating the inequality $|f(x)| > t\chi_{E_t}$, where $E_t = \{|f| > t\}$.
For a measurable function $f$ on $(X, \mathcal{M}, \mu)$, we define the distribution function of $f$ by

$$\lambda_f(t) = \mu(\{|f| > t\}).$$

Clearly $\lambda_f = \lambda_{|f|}$, if $|f| \leq |g|$ a.e. then $\lambda_f \leq \lambda_g$.

**Lemma 4.1.** The distribution function has the following properties

(i) if $\{f_n\}$ is a sequence of measurable functions such that $\{|f_n|\}$ is increasing and $\|f\| = \lim_n |f_n|$, then $\lambda_{f_n} \to \lambda_f$.

(ii) We have $\{|f| \geq t\} = \lim_n \{f_n \geq t\}$ and the sets on the right side of the equality form an increasing sequence, since $|f_n| \leq |f_{n+1}|$. Thus $\lambda_f(t) = \lim_n \lambda_{f_n}(t)$.

(iii) We have $\{|f + g| > t\} \subset \{|f| > t/2\} \cup \{|g| > t/2\}$, then computing the measure $\mu$, we get $\lambda_{f+g}(t) \leq \lambda_f(t/2) + \lambda_g(t/2)$. $\square$

### 4.2. Distribution function and $L^p$-norms

We will show that it is enough to know the distribution function to compute the $L^p$-norm of the function.

**Theorem 4.1.** Suppose that $h : [0, \infty) \to [0, \infty)$ is an increasing continuous function, $h(0) = 0$, and $h$ is absolutely continuous on each bounded interval $[0, T]$. Then for any measurable function $f$ on $(X, \mathcal{M}, \mu)$ we have

$$\int_X h(|f(x)|) \, d\mu(x) = \int_0^\infty h'(t)\lambda_f(t) \, dt.$$ 

**Proof.** Assume that $f = a \chi_E$, where $E \in \mathcal{M}$ and $\mu(E) < \infty$. Then $\lambda_f(t) = \mu(E)$ when $0 < t < |a|$ and $\lambda_f(t) = 0$ when $t \geq |a|$. The integral on the left hand side equals $h(|a|)\mu(E)$ and the integral on the right hand side is $\mu(E) \int_0^{|a|} h'(t) \, dt = \mu(E) h(|a|)$ since $h$ is absolutely continuous function on $[0, |a|]$.

Now let $f = \sum c_j \chi_{E_j}$ is a simple function and assume that $E_j \cap E_k = \emptyset$ when $j \neq k$. Then $\lambda_f(t) = \sum_j \lambda_{f_j}$ with $f_j = a_j \chi_{E_j}$ and

$$\int_X h(|f(x)|) \, d\mu(x) = \int \sum_j \int_{E_j} h(|f_j(x)|) \, d\mu(x).$$

Thus the equality holds for simple functions by the linearity of both sides.

Finally if $f$ is measurable, there is a sequence $f_n$ of simple functions such that $|f_n| \leq |f_{n+1}|$ and $f_n \to f$. Then the sequence $\lambda_{f_n}$ increases and converges to $\lambda_f$ and
$h(f_n(x)) \to h(f_n(x))$ since $h$ is continuous, moreover $h(f_n) \leq h(f_{n+1})$. We apply the monotone convergence theorem to the both sides and conclude that
\[
\int_X h(|f(x)|) \, d\mu(x) = \int_0^\infty h'(t) \lambda_f(t) \, dm(t)
\]
for any measurable function $f$. \hfill \Box

**Corollary 4.1.** Let $f$ be a measurable function on $(X, \mathcal{M}, \mu)$ then
\[
\|f\|_p = \int_0^\infty pt^{p-1} \lambda_f(t) \, dm(t).
\]
5. **Riesz-Thorin Interpolation theorem**

5.1. **A lemma from complex analysis.** We need a result from introductory complex analysis. First we remind that if $F(z)$ is a holomorphic function on a bounded domain $\Omega$ such that $F$ is continuous on the closure of $\Omega$ then by the maximum principle

$$\sup_{z\in\Omega} |F(z)| \leq \max_{z\in\partial\Omega} |F(z)|.$$ 

This maximum principle implies the following result.

**Lemma 5.1 (Three line inequality).** Suppose that $F$ is holomorphic bounded function on the strip $P = \{ z : 0 < \text{Re}(z) < 1 \}$ and $F$ is continuous on the closure of $P$. Suppose that $|F(z)| \leq M_0$ when $\text{Re}(z) = 0$ and $|F(z)| \leq M_1$ when $\text{Re}(z) = 1$. Then $|F(z)| \leq M_0^{1-t}M_1^t$ when $\text{Re}(z) = t$.

**Proof.** Let $z \in S$, $z = x + iy$, then $z(z - 1) = x(x - 1) - y^2 + i(2xy - y)$ and $\text{Re}(z) = x(x - 1) - y^2 < 0$ and it goes to $-\infty$ when $|y| \to \infty$. Let $G(z) = F(z)M_0^{1-t}M_1^t\exp(\varepsilon z(z - 1))$. Then $G$ is holomorphic in $P$, continuous on the closure of $P$, $|G(z)|$ tends to zero when $\text{Im}(z) \to \pm\infty$ and $|G(z)| \leq 1$ when $\text{Re}(z) \in \{0, 1\}$. Then, applying the maximum principle to truncated strips

$$P_R = \{ z : 0 < \text{Re}(z) < 1, \ -R < \text{Im}(z) < R \},$$

we conclude that $|G(z)| \leq 1$ for all $z \in P$. Note that if $z = t + is$, then

$$|G(t + is)| = |F(t + is)|M_0^{1-t}M_1^t\exp(\varepsilon(t(t - 1) - s^2)).$$

Thus $|F(t + is)| \leq M_0^{1-t}M_1^t\exp(\varepsilon(t(1 - t) + s^2))$. The last inequality holds for any $\varepsilon > 0$ then the conclusion of the theorem follows. 

5.2. **Two auxiliary results.** We prove two lemmas before formulating the first interpolation theorem.

**Lemma 5.2.** If $1 \leq p < \infty$ then the set $S_0(\mu)$ of simple functions $\phi$ such that $\mu(\{ \phi \neq 0 \}) < \infty$ is dense in $L^p(\mu)$.

**Proof.** We want to show that for any $f \in L^p(\mu)$ there exists a sequence $\{ \phi_n \}$ such that $\phi_n \in S_0(\mu)$ and $\lim_{n\to\infty} ||f - \phi_n||_p = 0$. Let $\phi_n$ be simple functions such that $\phi_n \to f$ a.e. and $|\phi_n| \leq |\phi_{n+1}|$, then $||\phi_n||_p \leq ||f||_p$. Since $|\phi_n|$ takes only finitely many values, we have $\inf\{ ||\phi_n(x) : \phi_n(x) \neq 0 \} = c > 0$. Then $\mu(\{ \phi_n \neq 0 \}) \leq c^{-p}||\phi_n||_p^p < \infty$, so
\( \phi_n \in S_0(\mu) \). Moreover, \(|f - \phi_n|^p \leq 2^p|f|^p\) point-wise. The dominated convergence theorem implies that \(||f - \phi_n||_p \to 0\). \(\square\)

**Lemma 5.3.** Suppose that a linear operator \( T \) is defined on \( L^p(\mu) + L^q(\mu) \) and \( Tf \) is a measurable function on \((Y, \mathcal{N}, \nu)\) such that \( ||Tf||_{q_0} \leq M_0 ||f||_{p_0} \) and \( ||Tf||_{q_1} \leq M_1 ||f||_{p_1} \). Suppose also that for some \( p \in [p_0, p_1] \), \( ||T\phi||_q \leq C ||\phi||_p \), when \( \phi \in S_0(\mu) \). Then for \( f \in L^p(\mu) \) we have \( Tf \in L^q(\nu) \) and \( ||Tf||_q \leq C ||f||_p \).

**Proof.** Let \( f \in L^p(\mu) \) and \( E = \{|f| > 1\} \). There is a sequence \( \phi_n \to f \) such that \( \phi_n \in S_0(\mu) \) and \( |\phi_n| \leq |\phi_{n+1}| \). We define \( \psi_n = \phi_{n\chi_E} \) and \( \omega_n = \phi_{n\chi_{E^c}} \). Suppose that \( p_0 \leq p_1 \) then \( \psi_n \in L^{p_0} \) and \( \omega_n \in L^{p_1} \), moreover if \( g = f\chi_E \) and \( h = f\chi_{E^c} \) then \( ||g - \psi_n||_{p_0} \to 0 \) and \( ||f - \omega_n||_{p_1} \to 0 \). Then \( \{Tg - T\psi_n\} \) and \( \{Th - T\omega_n\} \) converge to zero in \( L^{q_0}(\nu) \) and in \( L^{q_1}(\nu) \) respectively, and thus converges to zero in \( \nu \)-measure, therefore we can find a subsequence \( \phi_{n_k} = \psi_{n_k} + \omega_{n_k} \) such that \( T\psi_{n_k} \to Tg \) a.e. and \( T\omega_{n_k} \to Th \) a.e. Then \( T\phi_{n_k} \to Tf \) a.e. and by the Fatou lemma

\[
||Tf||_q \leq \liminf_{k \to \infty} ||T\phi_{n_k}||_q \leq \liminf_{k \to \infty} C ||\phi_{n_k}||_p \leq C ||f||_p.
\]

\(\square\)

### 5.3. Riesz – Thorin Interpolation

Let \((X, M, \mu)\) and \((Y, \mathcal{N}, \nu)\) be two measure spaces. We know that if \( r \in [p_1, p_2] \) then \( L^r(\mu) \subset L^{p_1}(\mu) + L^{p_2}(\mu) \) (see homework assignment). We consider a linear operator \( T \) defined on both \( L^{p_1}(\mu) \) and \( L^{p_2}(\mu) \) and conclude that \( T \) is defined on \( L^r(\mu) \) when \( r \in [p_1, p_2] \).

**Theorem 5.1.** Suppose that \( 1 \leq p_0, p_1, q_0, q_1 \leq \infty \) and that if \( q_0 = q_1 = \infty \) then \( \nu \) is \( \sigma \)-finite. Suppose that \( T \) is a linear operator from \( L^{p_1}(\mu) + L^{p_2}(\mu) \) to \( L^{q_0}(\nu) + L^{q_1}(\nu) \) such that \( ||Tf||_{q_0} \leq M_0 ||f||_{p_0} \) and \( ||Tf||_{q_1} \leq M_1 ||f||_{p_1} \). Then for \( p_t \) and \( q_t \) defined by

\[
1/p_t = 1 - t/p_1, \quad 1/q_t = 1 - t/q_0 + t/q_1 - 1,
\]

\( T \) is a bounded operator from \( L^{p_t}(\mu) \) to \( L^{q_t}(\mu) \) and \( ||Tf||_{q_t} \leq M_0^{1-t}M_1^t ||f||_{p_t} \).

**Proof.** First if \( p_0 = p_1 \) we estimate \( ||Tf||_{q_t} \) applying the Hölder inequality to \( ||Tf||_{q_t} = ||Tf||^{(1-t)q_0} ||Tf||^{tq_0} \) with exponents \( r = q_0q_t^{-1}(1 - t)^{-1} \) and \( r' = q_1q_t^{-1}t^{-1} \). We get

\[
||Tf||_{q_t} \leq ||Tf||^{1-t}_{q_0} ||Tf||^{t}_{q_1}.
\]

Now assume that \( p_1 \neq p_2 \). We want to show that \( ||Tf||_{q_t} \leq M_0^{1-t}M_1^t ||f||_{p_t} \). By Lemma 5.3 it suffices to prove the inequality for the case \( f \in S_0(\mu) \). Furthermore, we note that by Theorem 2.1, \( ||Tf||_{q_t} = \sup \{ \int (Tf)g \, d\nu : g \in L^{q_t}(\nu), ||g||_{q_t'} = 1 \} \). Applying Lemma 5.2 we may take the supremum over the functions \( g \in S_0(\nu) \) only. Let \( f(x) = \sum_j c_j \chi_{E_j}(x) \) and \( g \in S_0(\nu) \) be \( g(y) = \sum_k d_k \chi_{F_k}(y) \), where both sums are
finite and \( c_j, d_k \in \mathbb{C} \), \( c_j = |c_j|e^{\gamma_j} \) and \( d_j = |d_j|e^{i\delta_j} \). We also assume that \( \|f\|_{p_t} = \|g\|_{q_t} = 1 \). We have

\[
\int_Y (Tf)g \, d\nu = \sum_{j,k} c_j d_k \int_{E_j} T\chi_{E_j} \, d\nu = \sum_{j,k} c_j d_k C_{jk}.
\]

Our aim is to construct a holomorphic function in the strip \( P \) and apply the three line inequality. We define

\[
a(z) = \frac{1-z}{p_0} + \frac{z}{p_1}, \quad b(z) = \frac{1-z}{q_0} + \frac{z}{q_1}.
\]

Such that \( a(0) = p_0^{-1}, a(t) = p_t^{-1}, a(1) = p_1^{-1} \) and \( b(0) = q_0^{-1}, b(t) = q_t^{-1}, b(1) = q_1^{-1} \). Now we fix \( t \) and let

\[
f_z(x) = \sum_j |c_j|^{a(z)p_t} e^{i\gamma_j} \chi_{E_j}(x), \quad \text{and} \quad g_z(y) = \sum_k |d_k|^{b(z)q_t} e^{i\delta_k} \chi_{F_k}(y).
\]

Clearly \( f_t = f \) and \( g_t = g \). Furthermore, let

\[
F(z) = \int_Y (Tf)g_z \, d\nu = \sum_{j,k} |c_j|^{a(z)p_t} |d_k|^{b(z)q_t} e^{i(\gamma_j + \delta_k)} C_{jk}.
\]

Then \( F(z) \) is a holomorphic function of \( z \). When \( \text{Re}(z) \in [0,1] \), we know that \( \text{Re}(a(z)) \) and \( \text{Re}(b(z)) \) are bounded and therefore \( F(z) \) is bounded. When \( \text{Re}(z) = 0 \) we have \( a(is) = p_0^{-1} + is(p_1^{-1} - p_0^{-1}) \) and \( b(s) = (q_0')^{-1} + is((q_1')^{-1} - (q_0')^{-1}) \). Then

\[
|f_{is}| = |f|^{p_t/p_0}, \quad |g_{is}| = |g|^{q_t/q_0'}.
\]

Then we can estimate \( |F(is)| = \int (Tf)_{is} g_{is} \, d\nu \) applying the Hölder inequality

\[
|F(is)| \leq \|Tf_{is}\|_{q_0'} \|g_{is}\|_{q_0'} \leq M_0 \|f_{is}\|_{p_0} \|g_{is}\|_{q_0'} = M_0 \|f\|_{p_t} \|g\|_{q_t} = M_0.
\]

Similarly, when \( z = 1 + is \) we have

\[
|f_{1+is}| = |f|^{p_t/p_1}, \quad |g_{1+is}| = |g|^{q_t/q_1'}.
\]

Then \( |F(1 + is)| \leq M_1 \). Then

\[
\int_Y (Tf) g \, d\nu = |F(t)| \leq M_0^{1-t} M_1^t.
\]

\[\square\]

Using the Riesz–Thorin interpolation theorem we can simplify the prove of the boundedness of an integral operator in Theorem [3.1].
5.4. **An application.** Let \( f, g \) be measurable functions on \((\mathbb{R}, \mathcal{L}, m)\). Suppose that for some \( y \in \mathbb{R} \) we have \( \int_{\mathbb{R}} |f(x)g(y - x)| dm(x) < \infty \) then we define
\[
(f * g)(y) = \int_{\mathbb{R}} f(x)g(y - x) dm(x).
\]
The function \( f * g \) is called the convolution of \( f \) and \( g \). A simple change of variables \( x - y \rightarrow y \) shows that \( f * g = g * f \). The Hölder inequality implies that if \( f \in L^p \) and \( g \in L^{p'} \) then \( f * g \) is defined everywhere and \( \|f * g\|_\infty \leq \|f\|_p \|g\|_{p'} \). We use the interpolation theorem to give a generalization of this inequality.

**Proposition 5.1** (Young’s inequality). Suppose that \( 1 \leq p, q, r \leq \infty \) and \( p^{-1} + q^{-1} = 1 + r^{-1} \). If \( f \in L^p \) and \( g \in L^q \) then \( f * g \) is defined and \( f * g \in L^r \). Moreover
\[
\|f * g\|_r \leq \|f\|_p \|g\|_q.
\]

**Proof.** We fix \( g \in L^q \) and consider a linear operator \( Tf = f * g \), we know that it is defined on \( L^{q'} \) and \( \|Tf\|_\infty \leq \|f\|_{q'} \|g\|_q \). Now suppose that \( f \in L^1 \) then by Minkowski’s inequality
\[
\left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(x)||g(x - y)| dm(x) \right)^q dm(y) \right)^{1/q} \leq \int_{\mathbb{R}} |f(x)||g(x)| dm(x) = \|f\|_1 \|g\|_q.
\]
Thus \( T : L^{q'} \rightarrow L^\infty \) and \( T : L^1 \rightarrow L^q \) with estimates \( \|Tf\|_\infty \leq \|f\|_{q'} \) and \( \|Tf\|_q \leq \|f\|_1 \). We apply the Riesz–Thorin theorem to \( T \) and obtain that \( \|Tf\|_r \leq \|f\|_p \) when \( 1 \leq p \leq q' \) and \( 1/r = (1 - t)/\infty + t/q \) while \( 1/p = (1 - t)/q' + t \). We see that \( 1/p = t/q + 1/q' = 1/r + 1/q' \) and \( 1/p + 1/q = 1 + 1/r \). Moreover we can obtain the inequality for any \( p \in [1, q'] \) by choosing \( t \) appropriately. \( \Box \)
6. MARCINKIEWICZ INTERPOLATION THEOREM

6.1. Weak $L^p$ spaces. The Chebyshev inequality implies that when $f \in L^p(\mu)$, we have $\lambda_f(t) \leq t^{-p}\|f\|_p^p$.

**Definition 6.1.** Let $f$ be a measurable function on $(X, \mathcal{M}, \mu)$ and let $\lambda_f(t)$ be its distribution function, $\lambda_f(t) = \mu(\{|f| > t\})$. We say that $f$ belongs to weak $L^p(\mu)$-space if

$$[f]_p = \left( \sup_{t>0} t^p \lambda_f(t) \right)^{1/p}$$

is finite.

Clearly if $f \in L^p(\mu)$ then $f$ in weak $L^p$ and $[f]_p \leq \|f\|_p$. Moreover if $f$ and $g$ are in weak $L^p(\mu)$ then $f + g$ is also in weak $L^p(\mu)$. However $[f + g]_p$ is not a norm. A standard example of a function in weak $L^p(\mathbb{R}, m)$ that is not in $L^p(\mathbb{R}, m)$ is $f(x) = x^{-1/p}$.

We will not distinguish between $L^\infty$ and weak $L^\infty$. In terms of the distribution function, $f \in L^\infty$ if $\lambda_f(t) = 0$ for $t > t_0$ and the smallest such $t_0$ is the norm $\|f\|_\infty$.

**Definition 6.2.** We say that a map $T$ which sends measurable functions on $(X, \mathcal{M}, \mu)$ to measurable functions on $(Y, \mathcal{N}, \nu)$ is of weak type $(p, q)$ if for any $f \in L^p(\mu)$ the image $Tf$ is in weak $L^q(\nu)$ and $[Tf]_q \leq C\|f\|_p$.

6.2. Marcinkiewicz interpolation. We will now prove our second interpolation theorem. It can be applied to a large class of maps and not only to linear operators, however there are additional restrictions on the order of $p$ and $q$ in this result.

**Definition 6.3.** Let $\mathcal{F}$ be a linear subset of measurable functions on $(X, \mathcal{M}, \mu)$ such that $\mathcal{F}$ contains all finite linear combinations of characteristic functions of sets of finite measure and also if $f \in \mathcal{F}$ and $C > 0$ then $\min\{|f|, C\}$ is also in $\mathcal{F}$. We say that a map $T$ from $\mathcal{F}$ to measurable functions on $(Y, \mathcal{N}, \nu)$ is sublinear if

(i) $|T(af)(y)| = a|Tf(y)|$,  
(ii) $|T(f_1 + f_2)(y)| \leq |Tf_1(y)| + |Tf_2(y)|$.

**Theorem 6.1.** Suppose that $T$ is a sublinear map such that

$$[Tf]_q \leq C_j \|f\|_{q_j}$$
for \( f \in L^p(X) \cap \mathcal{F} \) and \( j = 0, 1 \), where \( q_0 \neq q_1 \) and \( p_j \leq q_j \). Then \( \| Tf \|_{q_j} \leq C_t \| f \|_{p_j} \), for any \( f \in \mathcal{F} \cap L^p(\mu) \), where \( 0 < t < 1 \) and

\[
\frac{1}{p_t} = \frac{1}{p_0} - \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1}{q_0} + \frac{t}{q_1}.
\]

We will prove it for the case \( p_0 = q_0 \) and \( p_1 = q_1 \). In the proof we work with distribution functions on two spaces \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\) we continue to denote the first distribution function of \( f \) on \( X \) (corresponding to the measure \( \mu \)) by \( \lambda_f \) and denote the second one (of a function \( g \) on \( Y \) corresponding to the measure \( \nu \)) by \( \kappa_g \).

**Proof.** Let \( f \in \mathcal{F} \cap L^p \), we want to estimate the distribution function \( \kappa_{T f}(t) \). Assume that \( p_0 < p_1 \) (for the case we consider \( p_0 \neq p_1 \)) and let first \( p_1 < \infty \).

We fix \( t > 0 \) and decompose \( f \) into sum of two functions \( f = f_0 + f_1 \), where

\[
f_0 = \begin{cases} 0, & |f| \leq At \\ f, & |f| > At \end{cases}, \quad f_1 = \begin{cases} f, & |f| \leq At \\ 0, & |f| > At \end{cases}.
\]

By our assumption \( f_1, f_2 \in \mathcal{F} \) and \( |Tf| \leq |Tf_1| + |Tf_2| \). Then

\[
\kappa_{T f}(t) \leq \kappa_{T f_0}(t/2) + \kappa_{T f_1}(t/2).
\]

We note that \( f_1 \in L^{p_0} \cap \mathcal{F} \) and \( f_2 \in L^{p_1} \cap \mathcal{F} \) since \( p_0 \leq p_1 \). Further,

\[
\lambda_{f_0}(s) = \begin{cases} \lambda_f(At), & s < At \\ \lambda_f(s), & s \geq At \end{cases}, \quad \lambda_{f_1}(s) = \begin{cases} \lambda_f(s) - \lambda_f(At), & s < At \\ 0, & s \geq At \end{cases}.
\]

Applying the weak estimate for \( T \) in \( L^{p_0} \) we get

\[
\kappa_{T f_0}(t/2) \leq C_0^{p_0} 2^{p_0 t^{-p_0}} \| f_0 \|_{p_0} = (2C_0)^{p_0 t^{-p_0}} \int_0^\infty p_0 s^{p_0 - 1} \lambda_{f_0}(s) ds.
\]

Using the formula for \( \lambda_{f_0} \), we get

\[
\kappa_{T f_0}(t/2) \leq (2C_0)^{p_0 t^{-p_0}} \left( (At)^{p_0} \lambda_f(At) + \int_{At}^\infty p_0 s^{p_0 - 1} \lambda_f(s) ds \right).
\]

On the other hand for \( f_1 \in L^{p_1} \) we get

\[
\kappa_{T f_1}(t/2) \leq (2C_1)^{p_1 t^{-p_1}} \int_0^{At} p_1 s^{p_1 - 1} \lambda_f(s) ds.
\]

Thus for any \( t > 0 \) we obtain

\[
\kappa_{T f}(t) \leq (2C_0)^{p_0 t^{-p_0}} \left( (At)^{p_0} \lambda_f(At) + \int_{At}^\infty p_0 s^{p_0 - 1} \lambda_f(s) ds \right) + (2C_1)^{p_1 t^{-p_1}} \int_0^{At} p_1 s^{p_1 - 1} \lambda_f(s) ds.
\]
We forget about our decomposition $f = f_0 + f_1$ after we obtained this inequality and start to vary $t$.

Now we multiply the last inequality by $pt^{p-1}$ and integrate it,

$$
\int_0^\infty pt^{p-1}\kappa Tf(t)dt \leq (2C_0A)^{p_0}A^{-p}\int_0^\infty ps^{p-1}\lambda_f(s)ds + \frac{(2C_0)^{p_0}}{p-p_0}\int_0^\infty ps^{p-1}\lambda_f(s)ds + \frac{(2C_1)^{p_1}}{p_1-p}\int_0^\infty ps^{p-1}\lambda_f(s)ds.
$$

This implies $\|Tf\|_{L^p(\nu)} \leq C\|f\|_{L^p(\mu)}$. To minimize the constant we should choose $A$ in an appropriate way. We see that $C$ blows up when $p$ approaches $p_0$ or $p_1$, this is natural as we assumed only weak inequalities at the end points.

Let us now consider the case $p_1 = \infty$ then we choose $A < 1/2$ and conclude that with decomposition $f = f_1 + f_2$ as above, we have $\kappa_{Tf_1}(t/2) = 0$. Thus we still get the estimate $\|Tf\|_p \leq C\|f\|_p$ for some $C$. □

6.3. An application: Maximal function. We remind that for $f \in L^1_{loc}(\mathbb{R}^n)$ we defined the maximal function $Mf$ by

$$
Mf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)| dm(y).
$$

We claim that $f \mapsto Mf$ is a sublinear map. Clearly $M(cf) = |c|M(f)$, also

$$
M(f + g)(x) \leq \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)| + |g(y)| dm(y) \leq Mf(x) + Mg(x).
$$

If $f \in L^\infty(\mathbb{R}^n, m)$ then $Mf(x) \leq \|f\|_\infty$ and as we proved earlier for $f \in L^1(m)$ we have

$$
m(\{Mf > t\}) \leq Ct^{-1}\|f\|_1.
$$

Then the Marcinkiewicz interpolation theorem implies that for $f \in L^p(\mathbb{R}^n)$ with $1 < p \leq \infty$ we have $Mf \in L^p$ and $\|Mf\|_p \leq C_p\|f\|_p$. Examining the proof of the interpolation theorem above, we see that $C_p \leq C(n)p/(p-1)$.

6.4. Fractional integration. Let $f$ on $\mathbb{R}^m$, we define the convolution of $f$ and $|x|^\alpha - m$ with $0 < \alpha < m$ as

$$
I_\alpha f(x) = \int |x - y|^\alpha f(y) dy,
$$

when the integral of the absolute value is finite.

Such operators appear naturally. For example, when $\alpha = 2$, $m \geq 3$ and $f$ is a bounded function with compact support, we have $\Delta(I_2f) = c_m f$. 

Theorem 6.2. Let $p > 1$ and $\alpha < m/p$, then the following inequality holds

$$\|I_\alpha f\|_q \leq C\|f\|_p,$$

when $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{m}$.

Proof. By the Marcinkiewicz interpolation theorem, to prove the inequality for all $p > 1$ and $q$ such that $1/q = 1/p - \alpha/m$, it is enough to prove the weak type inequality for such $p$ and $q$ for two different $p_1$ and $p_2$ and then interpolate. Fix $\lambda > 0$ and let $k_1 = |x|^{\alpha - m}$ when $|x| < R = R(\lambda)$ and zero otherwise and $k_2(x) = |x|^{\alpha - m}$ when $|x| \geq R$ and zero otherwise. We have $I_\alpha f = k_1 * f + k_2 * f$. First we estimate $\|k_2 * f\|_\infty$, applying the Hölder inequality.

$$|k_2 * f(x)| \leq \|f\|_p \|k_2\|_{p'}.$$

We have

$$\int |k_2|^{p'} dx = \int_{R}^{\infty} r^{m-1+\alpha-m} dr = cR^{m+(\alpha-m)p'} = cR^{m+(\alpha-m)p'},$$

if $m < (m - \alpha)p'$, that follows from the condition $\alpha < m/p$. We choose $R = R(\lambda)$ such that

$$cR^{m/p' + \alpha - m} \|f\|_p = cR^{\alpha - m/p} \|f\|_p = \lambda/2.$$

Then $\|f * k_2\|_\infty \leq \lambda/2$. Now we look at the set $\{ |f * k_1| > \lambda/2 \}$. If we show that $f * k_1$ is in $L^p$ we would estimate the measure of this set by $(2/\lambda)^p \|f * k_1\|_p^p$. Indeed,

$$\|k_1 * f\|_p \leq \|k_1\|_1 \|f\|_p = \int_{|x|<R} |x|^{\alpha-m} dx \|f\|_p = c \int_{0}^{R} r^{\alpha-1} dr \|f\|_p = cR^{\alpha} \|f\|_p.$$

Thus $|\{ I_\alpha f > \lambda \} | \leq C\lambda^{-p} R^{p\alpha} \|f\|_p^p$. We have

$$R^{p\alpha} = (R^{m-(\alpha-m)})^{\alpha/(\alpha-m)} = c\lambda^{\alpha^2/(\alpha-m)} \|f\|^{-(\alpha^2/(\alpha-m))}_p.$$

Finally, we obtain

$$|\{ I_\alpha * f | > \lambda \}| \leq c\lambda^{-pm/(m-\alpha p)} \|f\|_{p_1}^{p_1/(m-\alpha p)}$$

and recall that $q = (1/p - \alpha/m)^{-1} = pm/(m - \alpha p)$. Therefore $[I_\alpha f]_q \leq C\|f\|_p$. □