## LECTURE 2: SOME TOPICS IN HARMONIC ANALYSIS

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### 1. Uncertainty Principle in Harmonic Analysis

By the uncertainty principle in harmonic analysis we mean an informal claim that if the Fourier transform of a function is supported on a rectangle then the function is almost constant on each dual rectangle.we will prove a number of rigorous statements that support the statement.

1.1. **Bernstein's inequality.** The classical Bernstein inequality for polynomials states that

$$\max_{|z|=1}|F'(z)| \le n \max_{|z|=1}|F(z)|,$$

where $F$ is a polynomial of degree $n$.

**Theorem 1.** *Suppose that $f \in L^1 + L^2(\mathbb{R}^d)$ and $\widehat{f}$ is supported in $B_R = \{\xi : |\xi| \le R\} \subset \mathbb{R}^d$. Then (i) For any $\alpha$ and $p, 1 \le p \le \infty$,*

$$\|D^\alpha f\|_p \le (CR)^{|\alpha|}\|f\|_p.$$

*(ii) For any $p, q$, $1 \le p \le q \le \infty$*

$$\|f\|_q \le C(p,q)R^{d/p-d/q}\|f\|_p.$$

*Proof.* Let $\varphi \in \mathcal{S}$ be a function whose Fourier transform equals 1 on the unit ball and let $\varphi_t(x) = t^d\varphi(tx)$ then $\widehat{\varphi}_t(\xi) = \widehat{\varphi}(\xi/t)$ and $\hat{\varphi}_R$ equals 1 on $B_R$. Therefore $\widehat{f} = \widehat{f}\widehat{\varphi}_R$ and $f = f * \varphi_R$.

(i) By the properties of the convolution, $D^\alpha f = f * D^\alpha \varphi_R$. Let $C = \|\nabla\varphi\|_1$, then for $\beta$ with $|\beta| = 1$ we have $\|D^\beta \varphi_R\|_1 = R\|D^\beta\varphi\| \le CR$ and

$$\|D^\beta f\|_p \le \|D^\beta\varphi_R\|_1\|f\|_p \le CR\|f\|_p.$$

The general case follows by induction.

(ii) Let $r$ be such that $1/q + 1 = 1/p + 1/r$, since $p \le q$ we know that $r \ge 1$. We have $\|\varphi_R\|_r = R^{d(1-1/r)}\|\varphi\|_r$ and by the Young's inequality

$$\|f\|_q = \|f * \varphi_R\|_q \le \|\varphi_R\|_r\|f\|_p = CR^{d/p-d/q}\|f\|_p.$$

□

By applying a linear transformation, we can reformulated (ii) for the case when $\widehat{f}$ is supported in an ellipsoid. By an ellipsoid we mean a set of the form

$$E = \{x \in \mathbb{R}^d : \sum_j r_j^{-2}|(x-a)\cdot e_j|^2 \le 1\},$$

where $\{e_j\}_j$ is an orthonormal basis for $\mathbb{R}^d$, $a \in \mathbb{R}^d$ and $r_j > 0$.

**Corollary 1.** *Suppose that $f$ as above and $\widehat{f}$ is supported in an ellipsoid $E$. Then*

$$\|f\|_q \le C|E|^{1/p-1/q}\|f\|_p.$$

1.2. **An estimate on dual ellipsoids.** For an ellipsoid $E$ we consider the dual ellipsoids of the form

$$E^* = \{x \in \mathbb{R}^d : \sum_j r_j^2 |(x-b)\cdot e_j|^2 \le 1\},$$

we fix the size of $E^*$ but not its position. With each such $E^*$ we associate the function

$$\phi_{N,E^*} = (1 + \sum_j r_j^2 |(x-b)\cdot e_j|^2)^{-N},$$

where $N$ is large enough. Roughly speaking, $\phi_{N,E^*}$ is concentrated on $E^*$, it is bounded from below on $E^*$ and is very small outside $E^*$.

**Theorem 2.** *Suppose that $f \in L^1 + L^2$ and $\widehat{f}$ is supported in an ellipsoid $E$. Then for any dual ellipsoid $E^*$ and any $z \in E^*$,*

$$|f(z)| \le \frac{C_N}{|E^*|} \int |f(x)| \phi_{N,E^*}(x) dx. \tag{1}$$

*Proof.* By simple transformations we can reduce the statement to the case when $E = B_R$ and $E^* = B_{1/R}$. If $\widehat{f}$ is supported in the ball $B_R$, then $f = f * \varphi_R$, where $\varphi \in \mathcal{S}$ as in the proof of Theorem 1. Then

$$|f(z)| \le \int_{\mathbb{R}^d} |f(x)||\varphi_R(z-x)| dx = R^d \int_{\mathbb{R}^d} |f(x)| \varphi(R(z-x)) dx$$

Now, since $\varphi \in \mathcal{S}$, we have $|\varphi(y)| \le C'_N (1+|y|^2)^{-N}$. Now if $z \in B_{1/R}$ then $|Rz| \le 1$ and $|\varphi(R(z-x))| \le \tilde{C}_N (1 + R^2|x|^2)^{-N}$ and $|\varphi(R(z-x))| \le C_N \phi_{N,B_{1/R}}$. the estimate (1) follows. □

## 2. Method of Stationary Phase

Our aim is to understand the (asymptotic behavior) of the Fourier transform of measures supported by submanifolds in $\mathbb{R}^d$. If $\mathcal{M}$ is such manifold, $\dim(\mathcal{M}) = m$, and $\psi$ is a nice function supported in $\mathcal{M}$

$$\widehat{\psi|\mathcal{M}}(\xi) = \left(\frac{1}{2\pi}\right)^{d/2} \int_{\mathcal{M}} e^{-iy\cdot\xi} \psi(y) d\sigma(y),$$

where $\sigma$ is the volume measure on $\mathcal{M}$. Changing the coordinates locally we can rewrite the above integral as

$$\int_B e^{i\xi\cdot g(x)} a(x) dx,$$

where $B$ is a ball of dimension $m$.

2.1. **Nonstationary phase.** We will study the asymptotic behavior of the integrals of the form

$$I(\lambda) = \int_{\mathbb{R}^m} e^{-i\lambda\varphi(x)} a(x) dx,$$

assuming that $\varphi$ is a real-valued smooth function ($\phi \in C^\infty$) and $a \in C^\infty$ is compactly supported.

**Proposition 1.** *Suppose that $\nabla\varphi$ does not vanish on the support of $a$, then for any $N$ there exists a constant $C = C(a, \varphi, N)$ such that $|I(\lambda)| \le C\lambda^{-N}$.*

*Proof.* Let $\Omega$ be a bounded open set where $\nabla\varphi \ne 0$ which contains the support of $a$. We define a differential operator $L_\varphi$ on $C^1(\Omega)$, by

$$L_\varphi u = \frac{\nabla\varphi}{|\nabla\varphi|^2} \cdot \nabla u.$$

Then

$$L_\varphi^*(v) = -\frac{\nabla\varphi}{|\nabla\varphi|^2} \cdot \nabla v - div\left(\frac{\nabla\varphi}{|\nabla\varphi|^2}\right) v,$$

and $L_\varphi(e^{i\lambda\varphi(x)}) = i\lambda e^{i\lambda\varphi(x)}$. Then we can rewrite the integral

$$I(\lambda) = \int e^{-i\lambda\varphi(x)}a(x)dx = (i\lambda)^{-N}\int L^N(e^{-i\lambda\varphi(x)})a(x)dx = (i\lambda)^{-N}\int e^{-i\lambda\varphi(x)}(L^*)^N(a(x))dx.$$

Since $a \in C^\infty$ and has a compact support, we have $\|L^N(a)\|_1 \le C_N$ and $|I(\lambda)| \le C_N\lambda^{-N}$. □

2.2. **Stationary phase.** We now consider the situation when $\nabla\varphi(x_0) = 0$ for some point $x_0$ in the support of $a$. We assume that the critical points of $\varphi$ are isolated and the, using a smooth decomposition of unity, we may reduce the problem to that when $x_0$ is the only critical point of $\varphi$ in the support of $a$. First assume that $\varphi(x)$ is a quadratic form with critical point at the origin, $\varphi(x) = (Tx, x)$, where $T$ is an invertible real symmetric matrix. Then, using the Fourier inversion formula, we get

$$I(\lambda) = \int e^{-i\lambda(Tx,x)}a(x)dx = (2\pi)^{-m/2}\int\int e^{-i\lambda(Tx,x)}\widehat{a}(\xi)e^{ix\cdot\xi}d\xi dx.$$

Changing the oder of integration we are left with computing the (distributional) Fourier transform of the complex Gaussian. The computation is similar to one we did in Example 4 of Lecture 1 (one should use the branch of the function $\sqrt{z}$ carefully). The computation gives

$$I(\lambda) = \varepsilon|\det T|^{-1/2}(2\lambda)^{-m/2}\int \widehat{a}(\xi)e^{i(T^{-1}\xi,\xi)/(4\lambda)}d\xi,$$

where $|\varepsilon| = 1$ and $\varepsilon$ depends on the signature of the matrix $T$. Now as $\lambda \to \infty$ $1/\lambda \to 0$ and writing the series for the exponent we get

$$I(\lambda) = \varepsilon|\det T|^{-1/2}(\lambda)^{-m/2}(\pi)^{m/2}(a(0) + \lambda^{-1}\mathcal{D}a(0) + O(\lambda^{-2})),$$

where $\mathcal{D}$ is a differential operator of order two with constant coefficients determined by $T$.

**Proposition 2.** *Assume that $\nabla\varphi(x_0) = 0$ and $\nabla\varphi \ne 0$ at other points of the support ofa and that $\det D^2\varphi(x_0) \ne 0$. Then there exists a constant $C = C(a, \varphi)$ such that $|I(\lambda)| \le C\lambda^{-m/2}$.*

*Proof.* We may assume that $x_0 = 0$, by the Taylor expansion,

$$\varphi(x) = \varphi(0) + (Tx, x) + O(|x|^2),$$

where $T = 1/2D^2\varphi(0)$ is an invertible real symmetric matrix. We also write $a(x) = b_\lambda(x) + c_\lambda(x)$, where $b_\lambda(x) = a(x)\chi(\lambda^{1/2}x)$, where $\xi$ is a smooth cut off function, $0 \le \chi \le 1$, $\chi = 1$ on the unit ball and is supported by the double unit ball. Then

$$I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda\varphi(x)}b_\lambda(x)dx + \int_{\mathbb{R}^d} e^{i\lambda\varphi(x)}c_\lambda(x)dx = I_1(\lambda) + I_2(\lambda).$$

For $I_1(\lambda)$ we use a crude estimate

$$|I_1(\lambda)| \le \int |b_\lambda|dx \le C\int \chi(\lambda^{1/2}x)dx = C\lambda^{-m/2}.$$

For the second term we repeat the argument used for nonstationary case taking into account now that $c_\lambda(x)$ depends on $\lambda$,

$$I_2(\lambda) = \int e^{-i\lambda\varphi(x)}c_\lambda(x)dx = (i\lambda)^{-N}\int L^N(e^{-i\lambda\varphi(x)})c_\lambda(x)dx = (i\lambda)^{-N}\int e^{-i\lambda\varphi(x)}(L^*)^N(c_\lambda(x))dx.$$

We want to estimate $(L^*)^N((1-\chi(\lambda^{1/2}x))a(x))$. Computing the derivative of the product and taking into account that $|D^\alpha(\nabla\varphi(x)/|\nabla\varphi(x)|^2)| \le C_\alpha|x|^{-1-|\alpha|}$. We obtain

$$|(L^*)^N((1-\chi(\lambda^{1/2}x))a(x))| \le C_N|x|^{-N}(|x|^{-N} + \lambda^{N/2}).$$

Since $c_\lambda = 0$ when $|x| < \lambda^{-1/2}$, we obtain, by integrating this estimate,

$$|I_2(\lambda)| \le C_N\lambda^{-N}\int_{\lambda^{-1/2}}^\infty r^{m-1-2N} + \lambda^{N/2}r^{m-1-N}dr = C\lambda^{-m/2}$$

□

*Remark* 1. The computation above the proposition shows how to compute more accurately the first terms of the asymptotic.

*Remark* 2. In fact a more general statement holds. Under the conditions of the last proposition

$$\left|\partial_\lambda^k(e^{-i\lambda\varphi(x_0)}I_\lambda)\right| \le C\lambda^{-m/2-k}.$$

## 3. Fourier transform of measures supported by hypersurfaces

3.1. **Hyoersurfaces with non-vanishing Gaussian ciurvature.** Let $\sigma$ be the surface measure on a hypersurface $\mathcal{M} \subset \mathbb{R}^d$ and let $g$ be smooth compactly supported function. We want to estimate the Fourier transform of the measure $\mu = g\sigma$,

$$\widehat{\mu}(\xi) = (2\pi)^{-d/2}\int_{\mathcal{M}} e^{i\xi\cdot x}g(x)d\sigma(x).$$

We assume that $g$ is supported in a chart and choose coordinates $\Phi : B \to \mathcal{M}$ where $B$ is the unit ball in $\mathbb{R}^{d-1}$. Then the above integral can be written as

$$\widehat{\mu}(\xi) = \int_B e^{i\xi\cdot\Phi(y)}a(y)dy,$$

for some smooth function $a$. If we fix a direction $\nu \in S^{d-1}$ and let $\xi = \lambda\nu$ the integral above is of them form we just studied.

**Proposition 3.** *Suppose that $\mathcal{M}$ is a hypersurface in $\mathbb{R}^d$ with non-zero Gaussian curvature on support $g$ and $\mu = g\sigma$. Then*

$$|\widehat{\mu}| \le C|\xi|^{-(d-1)/2}.$$

We say that $M$ has a non-zero Gaussian curvature at $p \in \mathcal{M}$ if for any two-dimensional plane $H$ such that $p \in S = \mathcal{M} \cap H$, and $S$ is a non-degenerate curve, $S$ is not flat at $p$.

*Proof.* We fix the direction $\nu$ and define $\varphi(y) = \nu\cdot\Phi(y)$. The condition that the gaussian curvature of $\mathcal{M}$ is non-vanishing is equivalent to the condition $\det(D^2\varphi(y)) \ne 0$ for any $y$ (such that $\Phi(y)$ is in the support of $g$) and any $\nu$. Then the estimate in the direction $\nu$ follows form the stationary phase method. It is not difficult to check that we can choose the uniform constant that serves all the directions. □

*Remark* 3. It is important that the gaussian curvature is non-vanishing. For a degenerate example let $\mathcal{M}$ be a hyperplane $\mathcal{M} = \{x_d = 0\}$ then for a compactly supported function $g$ and $\mu = g\sigma$ clearly $\widehat{\mu}(\xi)$ does not depend on $\xi_n$ and thus it is not uniformly decaying.

*Remark* 4. As we mentioned in the previous lecture, applying the Fourier transform in the space and time to a (well behaved) solution of a Schrödinger operator, we obtain a distribution supported by a paraboloid. In case it is a measure, we found the asymptotic behavior of solution.

3.2. **The surface measure of the sphere.** Let $\mathcal{M}$ be the unit sphere in $\mathbb{R}^d$. Our aim is to estimate the Fourier transform of the surface measure $\sigma$ of the sphere. First we note that since $\sigma$ is radially symmetric distribution, the same is true for its Fourier transform and since $\sigma$ is compactly supported, $\widehat{\sigma}$ is a smooth function. We claim that

$$\widehat{\sigma}(\xi) = e^{i|\xi|}a(|\xi|) + e^{-i|\xi|}\overline{a(|\xi|)} + y(|\xi|),$$

where $y$ decays faster than any power of $|\xi|$ together with its derivatives, and

$$|a(\xi)| \le C|\xi|^{-(d-1)/2}, \quad |a^{(k)}(|\xi|)| \le |\xi|^{-d-1/2-k}.$$

We fix a direction $\nu = (1, 0, ..., 0)$. The function on the sphere $\varphi(x) = x \cdot \nu$ has two critical points $p_+ = (1, 0, ..., 0)$ and $p_- = (-1, 0, ..., 0)$. Then the three terms in the expression for the Fourier transform of $\sigma$ come from the decomposition of the surface measure into three parts, one cap around the point $p_+$, one around $p_-$ and the rest of the sphere, where the nonstationary method applies.

More accurately, let $P_+(x) = (\sqrt{1-|x|^2}, x)$ and $P_-(x) = (-\sqrt{1-|x|^2}, x)$ be the coordinate maps for caps around tge points $p_+$ and $p_-$., $P_+, P_-$ are defined on some ball $B \subset \mathbb{R}^d$. Then

$$\widehat{\sigma}(\lambda\nu) = c_d \int_B e^{-i\lambda\sqrt{1-|x|^2}} \frac{a(x)}{\sqrt{1-|x|^2}} dx + c_d \int_B e^{i\lambda\sqrt{1-|x|^2}} \frac{a(x)}{\sqrt{1-|x|^2}} dx + y(\lambda).$$

We use the stationary method to compute the asymptotic of $\widehat{\sigma}$. The critical point of $varphi(x)$ are $\pm\sqrt{1-|x|^2}$ is the point $x = 0$, near this point $\pm\sqrt{1-|x|^2} = 1 - |x|^2/2 + O(|x|^3)$. Then the first term of the asymptotic of the first and second integrals are

$$c_1 \lambda^{-(d-1)/2} a(0) e^{-i\lambda}, \quad \overline{c_1} \lambda^{-(d-1)/2} a(0) e^{i\lambda}$$

and thus their sum is

$$C\lambda^{-(d-1)/2} \cos(\lambda + \delta).$$

**Notes and references.** The material for this lecture is taken from the monographs [1] (Chapter 4) and [2] (Chapters 5 and 6). We plan to follow these books and prove the Thomas-Stein restriction theorem next week.

## References

[1] C. Muscalu, W Schlag, Classical and multilinear harmonic analysis. Vol. I. Cambridge Studies in Advanced Mathematics, 137. Cambridge University Press, Cambridge, 2013.

[2] T. Wolff, Lectures on harmonic analysis. University Lecture Series, 29. American Mathematical Society, Providence, RI, 2003.