

# Efficient Random Assignment with Constrained Rankings

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## Abstract

We consider the following variant of the random assignment problem: There are  $n$  agents and  $n$  indivisible objects to be assigned to them. A mechanism chooses a lottery over the possible assignments, given agents' reported preferences. However, agents can only report their  $k$  favorite objects to the mechanism, for some fixed  $k \leq n - 2$ . We show that no mechanism satisfies ex-post efficiency (relative to the limited information available to the mechanism) and strategyproofness. Our framework also allows for some objects to be identical, and for the number of objects to exceed the number of agents.

Keywords: constrained rankings; ex-post efficiency; indivisible objects; random assignment; strategyproofness

JEL Classifications: C78, D61

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# 1 Introduction

In the traditional version of the *random assignment problem*, there are  $n$  agents and  $n$  indivisible objects. Each agent is to receive one of the objects. The objects may, for example, be housing units, spaces at schools, or time slots for the use of a machine. Each agent has preferences over the objects. A *mechanism* specifies an allocation of the objects, possibly using lotteries to help ensure fairness, as a function of the agents' preferences.

The seminal study of the random assignment problem is Bogomolnaia and Moulin [2]. They considered a setting in which each agent can report only a (strict) ordinal preference ranking of the objects to the mechanism. The appropriate notion of efficiency for random allocations is then *ordinal efficiency*, defined as follows: A random allocation  $L$  *ordinally dominates* random allocation  $L'$  if every agent  $i$ 's probability distribution over objects under  $L$  first-order stochastically dominates his distribution under  $L'$  (relative to his preference ordering), with strict domination for some  $i$ ; then  $L$  is ordinally efficient if there is no other random allocation that dominates it. Bogomolnaia and Moulin showed that when  $n \geq 4$ , there is no mechanism that satisfies strategyproofness, equal treatment of equals, and ordinal efficiency at all preference profiles. In a related paper, Zhou [14] considered a setting in which agents can report full von Neumann-Morgenstern utility functions to the mechanism, and showed that for  $n \geq 3$ , there is no mechanism satisfying strategyproofness, ex-ante Pareto efficiency, and equal treatment of equals.

In this note we instead suppose the information available to the mechanism is coarser: each agent can only report his  $k$  favorite objects, for some fixed  $k$ . In realistic situations with many different objects, it is reasonable to imagine that  $k$  is small compared to  $n$ . It may be difficult for agents to figure out their full preferences or to communicate these preferences to the mechanism. Many real-world assignment or matching mechanisms constrain agents to list a small number of choices. For example, the New York City school choice mechanism only allows students to rank their 12 most preferred programs, out of over 500 available [1]; in Spain, college applicants can apply to only 8 programs, and in Hungary they are limited to 4 [5].

The setting considered in [2] corresponds to  $k = n$  or  $n - 1$ . In this note we consider  $k \leq n - 2$ . We obtain a new impossibility result for any such  $k$ : there is no random allocation mechanism satisfying strategyproofness and ex-post efficiency. (In this setting with incompletely known preferences, ex-post efficiency means that whatever allocation is chosen, there is no other allocation that is guaranteed to be a Pareto improvement given the limited preference information available to the mechanism. This is weaker

than efficiency with respect to the true full preferences, which one easily sees cannot be guaranteed, even ignoring strategic issues.) Note that when  $k$  decreases, strategyproofness becomes a stronger requirement but efficiency becomes weaker. Thus, our result does not follow from that of [2], nor vice versa; nor does the result for one value of  $k$  imply the result for any other  $k$ .

We actually prove our result in a setting that is more general than the traditional random assignment problem, in two respects. First, we allow that some of the objects may be identical, as in some of the existing literature, especially the literature on large assignment problems, e.g. [4, 6, 8, 10]. The agents report their  $k$  most preferred object types. The number  $m$  of distinct object types, and the number of objects of each type, are taken to be fixed; a mechanism specifies a random allocation for each profile of preferences. Second, we allow that there may be more objects than agents; an assignment gives each agent one object and may leave some objects left over. (The case where there are fewer objects than agents also fits into our framework by assuming a number of copies of a “null” object type, as long as we allow that agents may find some real object types less desirable than the null type.) The impossibility result holds as long as  $k \leq m - 2$  and there are some  $k$  object types whose quotas add up to less than the number of agents. The result for the traditional setting, when there are exactly  $n$  objects and they are all distinct, is just a special case.

Some common mechanisms that artificially constrain agents to rank only their top  $k$  choices were previously studied by Haeringer and Klijn [5] and by Pathak and Sönmez [12]. However, the present note seems to be the first to consider the mechanism design problem with this constraint. Existing literature on large markets also considers rankings of fixed length [7, 9, 13], but in that literature agents really do have preferences only over  $k$  choices, whereas in our model agents can have preferences over the remaining object types but are unable to express them.

## 2 The model

We consider a set of  $m$  object types  $a_1, \dots, a_m$ . For each object type  $a_j$  there is a quota  $q_j \geq 1$  of objects of that type. There are  $n$  agents, who will simply be referred to by the numbers  $1, \dots, n$ . The number of agents is less than or equal to the number of objects:  $n \leq q_1 + \dots + q_m$ . The numbers  $m, q_j, n$  are fixed in what follows.

Each agent has a strict preference ordering over the object types. All strict preference orderings are possible. However, we will assume that, for some fixed positive integer

$k \leq m$ , each agent is constrained to report only the list consisting of his  $k$  favorite object types in order. Such a list is called a *ranking*, and the object types in the list are said to be *ranked* by the agent. If an agent's preferences are  $a_{j_1} \succ \cdots \succ a_{j_m}$ , we will write  $a_{j_1} \succ \cdots \succ a_{j_k}$  to denote his (true) ranking. We let  $\mathcal{R}$  denote the set of all possible rankings. A typical profile of agents' rankings will be denoted by  $R = (R_1, \dots, R_n) \in \mathcal{R}^n$ . For any such profile  $R$ , agent  $i$ , and ranking  $R'_i \in \mathcal{R}$ , we write  $(R'_i, R_{-i})$  for the profile obtained by taking  $R$  and replacing  $i$ 's ranking by  $R'_i$ . (All the dependences of these definitions on  $k$  are notationally suppressed, since  $k$  is held fixed.)

An *allocation* is an assignment of one object type to each agent, such that each object type  $a_j$  is assigned to at most  $q_j$  agents. (If  $n < q_1 + \cdots + q_m$ , then some objects are left unassigned.) Agents are assumed to have selfish preferences over allocations: each agent cares only about what object type he gets. A *random allocation* is a probability distribution over allocations. Let  $\mathcal{X}$  be the set of all allocations, and  $\Delta(\mathcal{X})$  the set of all random allocations.

A random allocation induces for each agent a *random allotment* — a probability distribution over object types. If  $p$  is a random allotment, we write  $p(a_j)$  for the probability assigned to object type  $a_j$ . Given an agent's ranking  $a_{j_1} \succ \cdots \succ a_{j_k}$ , we say that random allotment  $p$  *dominates* random allotment  $p'$  if

- $\sum_{s=1}^t p(a_{j_s}) \geq \sum_{s=1}^t p'(a_{j_s})$  for each  $t = 1, \dots, k$ , and
- $p(a_j) \leq p'(a_j)$  for each unranked object type  $a_j$ .

It is easy to check that  $p$  dominates  $p'$  if and only if, for every von Neumann-Morgenstern utility function consistent with the ranking  $a_{j_1} \succ \cdots \succ a_{j_k}$ , the expected utility from  $p$  is greater than or equal to the expected utility from  $p'$ . If at least one of the inequalities holds strictly, then  $p$  *strictly dominates*  $p'$ .

Given a profile of the agents' rankings, an allocation  $X$  *Pareto dominates* an allocation  $X' \neq X$  if every agent  $i$  either

- gets the same object type under  $X$  as  $X'$ , or
- gets a ranked object type under  $X$  and an unranked object type under  $X'$  (according to  $i$ 's ranking), or
- gets ranked object types under both allocations, with the object type assigned by  $X$  ranked higher than that assigned by  $X'$ .

Equivalently, the rankings contain enough information to guarantee that every agent weakly prefers  $X$  over  $X'$ . An allocation is *Pareto efficient* if it is not Pareto dominated by any other allocation. Note that this is weaker than requiring Pareto efficiency with respect to the agents' true, complete preferences.

A random allocation is *ex-post efficient* if every allocation in its support is Pareto efficient.

An aside on this efficiency concept is in order. An alternative efficiency criterion, following the traditional definition of ordinal efficiency [2], would be as follows. For two random allocations  $L, L'$ , say that  $L$  *strictly ordinally dominates*  $L'$  if, for every agent  $i$ , the random allotment given to  $i$  by  $L$  dominates the random allotment given to  $i$  by  $L'$ , with strict domination for some  $i$ . Then say that  $L$  is *ordinally efficient* if no other random allocation strictly ordinally dominates it.

This is a natural definition. If  $L$  is ordinally dominated by some  $L'$ , then for any profile of von Neumann-Morgenstern utility functions consistent with the rankings,  $L'$  ex-ante Pareto dominates  $L$  in terms of expected utilities. Conversely, one can also show that if  $L$  is ordinally efficient, then there exist some utility functions consistent with the rankings for which  $L$  is ex-ante Pareto efficient. (This is similar to the ordinal efficiency welfare theorem [11] and to its generalization in [3], but requires a separate proof, because the convexity condition in [3] is not satisfied when preferences over unranked object types must be strict. Details are available from the author.)

Ex-post efficiency is a weaker requirement than ordinal efficiency. If  $L$  is ex-post inefficient, then choose some allocation  $X$  in its support and replace it by an allocation that Pareto dominates it; the resulting random allocation then strictly ordinally dominates  $L$ . This shows that ordinal efficiency implies ex-post efficiency. The converse is not true, as shown by the following example, adapted from [2]. Consider three object types  $a, b, c$ , with respective quotas 1, 1, 2, and  $k = m = 3, n = 4$ . The four agents have the following preference rankings: agents 1 and 2 have the preference  $a \succ b \succ c$ , while 3 and 4 have  $b \succ a \succ c$ . Let  $L$  be the lottery allocation putting probability  $1/4$  on each of the allocations  $(a, b, c, c)$ ,  $(b, a, c, c)$ ,  $(c, c, a, b)$ ,  $(c, c, b, a)$ . (The notation  $(a, b, c, c)$ , for example, means that agents 1, 2, 3, 4 receive  $a, b, c, c$  respectively.) Then  $L$  is ex-post efficient, but it is not ordinally efficient, because it is ordinally dominated by the lottery that consists of  $(a, c, b, c)$ ,  $(c, a, c, b)$  each with probability  $1/2$ .

Our impossibility theorem will use the criterion of ex-post efficiency. Since ordinal efficiency is stronger, the theorem will of course still be true if ex-post efficiency is replaced by ordinal efficiency.

Now that we have laid out our notions of preferences and efficiency, we can discuss the mechanism design problem. We would like to allocate the objects to the agents in a way that makes use of their rankings. This is done by a *mechanism*. In our environment, a mechanism is a function  $f : \mathcal{R}^n \rightarrow \Delta(\mathcal{X})$ , specifying a random allocation for each possible profile of rankings.

The mechanism  $f$  is *ex-post efficient* if, for every profile  $R$ ,  $f(R)$  is ex-post efficient (relative to the rankings  $R$ ). It is *strategyproof* if, for all  $R \in \mathcal{R}^n$ , all agents  $i$  and all  $R'_i \in \mathcal{R}$ , the random allotment given to  $i$  by  $f(R)$  dominates the random allotment given to  $i$  by  $f(R'_i, R_{-i})$  (relative to  $i$ 's true ranking  $R_i$ ). This says exactly that, no matter what utility function agent  $i$  has, and no matter what rankings the other agents report, agent  $i$  would rather report his true  $k$  favorite object types than any other ranking.

Our main theorem is:

**Theorem 1** *Suppose that  $k \leq m - 2$ , and there exist some  $k$  different object types whose quotas add up to less than  $n$ . Then no mechanism is both ex-post efficient and strategyproof.*

The assumptions on the parameters are tight. If  $k = m$ , so that a ranking is the same as a full preference ordering, then we can allocate the objects using the *serial dictatorship* mechanism: agent 1 gets his favorite object (more precisely, an object of his favorite type); then agent 2 gets his favorite object from the ones remaining, and so forth. This mechanism is ex-post efficient and strategyproof. (It can also be made to meet any reasonable equity criterion, by first randomly reordering the agents.) The case  $k = m - 1$  is identical to  $k = m$ , since an agent's ranking when  $k = m - 1$  determines his full preference ordering. And if there are no  $k$  quotas that add up to less than  $n$ , then we can again apply serial dictatorship: whenever any agent  $i$  gets his turn to choose, he can be assured of getting a ranked object type, since it is not possible for all objects of his  $k$  ranked types to have been already used up.

We can also compare our result to existing results on the random assignment problem. We will call our problem the *traditional random assignment problem* if every quota  $q_i = 1$  and also  $n = m$ : all objects are distinct, and there are as many agents as objects. We then immediately have, as a special case of Theorem 1:

**Corollary 2** *In the traditional random assignment problem, if  $k \leq n - 2$ , no mechanism is both ex-post efficient and strategyproof.*

For comparison, for  $k = n \geq 4$ , Bogomolnaia and Moulin [2] showed that there is no mechanism that is *ordinally* efficient, strategyproof, and satisfies *equal treatment of equals*,

an equity property requiring that if  $R_i = R_{i'}$  then  $i$  and  $i'$  should get the same random allotment under  $f(R)$ . Our Theorem 1 is similar, but weakens ordinal efficiency to ex-post efficiency and assumes no equity requirements.

We can also reinterpret any mechanism  $f$  as a function from profiles of full preference orderings to  $\Delta(\mathcal{X})$ , by truncating each agent's ordering to the top  $k$  choices and then applying  $f$ . This allows us to compare mechanisms defined for different values of  $k$ . Notice that, for a given preference ordering over all object types, if a random allotment  $p$  dominates random allotment  $p'$  when agents are allowed to rank their  $k-1$  favorite object types, then  $p$  still dominates  $p'$  when they are allowed to rank  $k$  object types; but the converse does not hold. Thus, dominance conditions become stronger when  $k$  is replaced by  $k-1$  (and all other parameters are held fixed). It follows that as  $k$  decreases, ex-post efficiency becomes a weaker requirement, but strategyproofness becomes a stronger requirement. So the conclusion of Theorem 1 for any one value of  $k$  does not imply the conclusion for any other value of  $k$ .

### 3 The proof

We first sketch the proof of Theorem 1 in the traditional case. Let  $f$  be an ex-post efficient, strategyproof mechanism. Suppose that some  $t$  agents report the ranking  $a_1 \succ a_2 \succ \dots \succ a_k$ , and the remaining agents respectively have  $a_{t+1}, a_{t+2}, \dots, a_n$  as their first-choice objects. Then the first  $t$  agents can only be assigned objects  $a_1, \dots, a_t$ , and all the remaining agents must get their first choices. The proof is by induction on  $t$ : By the induction hypothesis, any one of the first  $t$  agents can assure himself of  $a_t$  by claiming it as his first choice; so by strategyproofness, each of these agents can only get objects weakly preferred to  $a_t$ , that is,  $a_1, \dots, a_t$ . Then the remaining objects  $a_{t+1}, \dots, a_n$  must go to the other  $n-t$  agents; ex-post efficiency requires that each of these agents get his first choice. Now consider the profile where the first  $k+2$  agents report  $a_1 \succ \dots \succ a_k$ , and the remaining agents have first choices  $a_{k+3}, \dots, a_n$  respectively. Any one of the first  $k+2$  agents can assure himself of  $a_{k+2}$  by reporting it as his first choice, so by strategyproofness, he must get one of  $a_1, a_2, \dots, a_k, a_{k+2}$ . But this means that the first  $k+2$  agents all are guaranteed to get one of these  $k+1$  objects, which is impossible.

The proof of the general case is longer, but the core of the argument remains as above.

**Proof of Theorem 1:** Suppose the mechanism  $f$  is ex-post efficient and strategyproof; we will obtain a contradiction. The core of the argument is expressed in the following lemma:

**Lemma 3** Let  $b_1, \dots, b_m$  be some permutation of the object types  $a_1, \dots, a_m$ , and let  $r_1, \dots, r_m$  be the correspondingly permuted quotas. For  $j = 1, \dots, m$ , write  $s_j = r_1 + r_2 + \dots + r_j$ . For  $i = 1, \dots, n$ , let  $d(i)$  denote the smallest positive integer such that  $s_{d(i)} \geq i$ . Let  $t$  be any integer with  $0 \leq t \leq \min\{n, s_{k+1}\}$ .

Let  $R$  be a profile of rankings having the following property: it is possible to arrange the agents in some order such that the first  $t$  agents all have the ranking  $b_1 \succ b_2 \succ \dots \succ b_k$ , and for any  $i > t$ , the  $i$ th agent has  $b_{d(i)}$  as his first choice. Then, the random allocation  $f(R)$  is such that each of the first  $t$  agents (if  $t > 0$ ) is guaranteed to get one of the object types  $b_1, \dots, b_{d(t)}$ , and each of the remaining agents gets his first choice with probability 1.

Notice that  $d(i)$  is the index such that, if objects were handed out to agents  $1, 2, \dots, n$  in order, starting with all the objects of type  $b_1$ , then all those of type  $b_2$ , and so forth until every agent had an object, then agent  $i$  would receive  $b_{d(i)}$ .

**Proof:** We use induction on  $t$ . For the base case  $t = 0$ , the number of agents whose first choice is  $b_j$  is at most  $r_j$ , for each  $j = 1, \dots, m$ . Thus, it is possible to allocate the objects so that each agent gets his first choice. So ex-post efficiency requires that  $f(R)$  put probability 1 on the allocation that gives every agent his first choice.

Now suppose the lemma holds for  $t - 1$ . Consider the profile  $R$  in which all of the first  $t$  agents (with respect to some ordering) have the ranking  $b_1 \succ \dots \succ b_k$ , and the  $i$ th agent has first choice  $b_{d(i)}$  for each  $i > t$ . Refer to the first  $t$  agents as the *aligned* agents and the others as *unaligned*.

If any aligned agent reported  $b_{d(t)}$  as his first choice instead, then by the induction hypothesis, he would get  $b_{d(t)}$  with probability 1. Therefore, by strategyproofness,  $f(R)$  must give each such agent a random allotment that dominates the allotment putting probability 1 on  $b_{d(t)}$ . Any such random allotment has its support contained in  $\{b_1, b_2, \dots, b_{d(t)}\}$ .

This shows that each aligned agent can only get objects of types  $b_1, \dots, b_{d(t)}$ . It remains to show that each unaligned agent must get his first choice.

Let  $X$  be an allocation in the support of  $f(R)$ . So by the above,  $X$  assigns each aligned agent one of the object types  $b_1, \dots, b_{d(t)}$ . Suppose that some unaligned agent is not assigned his first choice. We will construct an allocation  $X'$  that Pareto dominates  $X$ , contradicting ex-post efficiency.

To construct  $X'$ , first give every unaligned agent his first-choice object type. (This can be done since, as before, for each  $j$ , the number of unaligned agents who have  $b_j$  as their first choice is at most  $r_j$ ). These objects are all of types  $b_{d(t)}, b_{d(t)+1}, \dots, b_m$ . Next, for every aligned agent who is assigned one of the object types  $b_1, \dots, b_{d(t)-1}$  at  $X$ , assign

him the same object type at  $X'$  as at  $X$ .

This takes care of all the agents except for the aligned agents who are assigned  $b_{d(t)}$  at  $X$ . Now arbitrarily assign each such agent in turn any remaining object whose type is one of  $b_1, \dots, b_{d(t)}$ . This can be done unless we run out of objects of all of these types. Notice, however, that the number of such objects is  $s_{d(t)}$ , while the number of agents who are to be assigned such an object at  $X'$  is at most  $s_{d(t)}$  (since, for any  $i > s_{d(t)}$ , the  $i$ th agent is unaligned and has been assigned one of  $b_{d(t)+1}, \dots, b_m$ ). So we never run out of objects of types  $b_1, \dots, b_{d(t)}$ , and  $X'$  can be constructed as described.

Now we check that  $X'$  Pareto dominates  $X$ . Each agent either

- is unaligned and gets his first-choice object type at  $X'$ , or
- is aligned and gets the same object type at  $X'$  as at  $X$ , or
- is aligned, gets  $b_{d(t)}$  at  $X$ , and gets one of  $b_1, \dots, b_{d(t)}$  at  $X'$ .

In the first two cases it is clear that the agent does at least as well at  $X'$  as at  $X$ . In the third case, the hypothesis  $t \leq s_{k+1}$  implies  $d(t) \leq k + 1$ , which ensures that  $b_1, \dots, b_{d(t)-1}$  are all ranked object types, and  $b_{d(t)}$  is either ranked below them or (if  $d(t) = k + 1$ ) is unranked. So in this case too, the agent weakly prefers  $X'$  over  $X$ . Finally, since some unaligned agent does not get his first choice at  $X$  but does get it at  $X'$ , the domination is strict.

So  $X$  is Pareto inefficient, which contradicts ex-post efficiency of  $f$ . This completes the induction step and proves the lemma.  $\square$

Now we can finish up the proof of Theorem 1. Relabeling the object types  $a_1, \dots, a_m$  if necessary, we may assume they are ordered so that  $q_1 \leq q_2 \leq \dots \leq q_m$ . Since we assumed some  $k$  quotas sum to less than  $n$ , we certainly have  $q_1 + \dots + q_k < n$ . We split into three cases.

- Case I:  $q_1 + q_2 + \dots + q_k + q_{k+2} < n$ .

Permute the object types by taking  $b_{k+1} = a_{k+2}$ ,  $b_{k+2} = a_{k+1}$ , and  $b_j = a_j$  for all other  $j$ . Let  $s_j$  and  $d(i)$  be defined as in the lemma. So  $s_{k+1} < n$ . Now consider a profile  $R$  at which each of the first  $s_{k+1} + 1$  agents has the ranking  $b_1 \succ \dots \succ b_k$ , and agent  $i$  has  $b_{d(i)}$  as his top choice for each  $i > s_{k+1} + 1$ . If any one of the first  $s_{k+1} + 1$  agents changed his top choice to  $b_{k+2}$ , then the resulting profile would fit the case  $t = s_{k+1}$  of the lemma, and we would know that this agent must be assigned  $b_{k+2}$ . Therefore, by strategyproofness, at  $f(R)$ , each of the first  $s_{k+1} + 1$

agents gets a random allotment that dominates  $b_{k+2}$ ; that is, he is guaranteed to get one of  $b_1, b_2, \dots, b_k, b_{k+2}$ . Consider any allocation in the support of  $f(R)$ . The first  $s_{k+1} + 1$  agents all get objects of types  $b_1, b_2, \dots, b_k, b_{k+2}$ , but the number of such objects is only

$$q_1 + q_2 + \dots + q_k + q_{k+1} \leq q_1 + q_2 + \dots + q_k + q_{k+2} = s_{k+1}.$$

This is a contradiction.

- Case II:  $n \leq q_1 + q_2 + \dots + q_k + q_{k+1}$ .

Let  $b_j = a_j$  for all  $j$ . Then, applying the lemma with  $t = n$ , we see that if all  $n$  agents report the ranking  $a_1 \succ a_2 \succ \dots \succ a_k$ , then each agent can only get one of the object types  $a_1, \dots, a_{k+1}$ .

Now let  $b_{k+1} = a_{k+2}$ ,  $b_{k+2} = a_{k+1}$ , and  $b_j = a_j$  for all other  $j$ . We can again apply the lemma with  $t = n$ , and we see that if all  $n$  agents report  $a_1 \succ a_2 \succ \dots \succ a_k$ , each agent can only get the object types  $a_1, \dots, a_k, a_{k+2}$ .

Combining these two findings, we see that at this profile, every agent must get one of the object types  $a_1, \dots, a_k$ . But again there are not enough of these objects to give them to all  $n$  agents — a contradiction.

- Case III:  $q_1 + q_2 + \dots + q_k + q_{k+1} < n \leq q_1 + q_2 + \dots + q_k + q_{k+2}$ .

Let  $t^* = q_1 + q_2 + \dots + q_{k+1}$ . Consider now a profile  $R$  at which the first  $t^*$  agents all report the ranking  $a_1 \succ \dots \succ a_k$ , and the remaining agents all have  $a_{k+2}$  as their first choice.

Letting  $b_j = a_j$  for all  $j$ , and applying the lemma with  $t = t^* = s_{k+1}$ , we see that at  $f(R)$ , the first  $t^*$  agents each can only get objects of types  $a_1, \dots, a_{k+1}$ . Letting  $b_{k+1} = a_{k+2}$ ,  $b_{k+2} = a_{k+1}$ , and  $b_j = a_j$  for all other  $j$ , and again applying the lemma with  $t = t^* < s_{k+1}$ , we see that at  $f(R)$ , the first  $t^*$  agents each can only get  $a_1, \dots, a_k, a_{k+2}$ . So as in case II, these agents can only get object types  $a_1, \dots, a_k$ . The number of objects of these types is  $q_1 + \dots + q_k < t^*$ , so once again we have a contradiction.

□

## 4 Further work

We wrap up by outlining a general framework for random assignment problems that could potentially unify the ideas of [2] and [14] as well as our Corollary 2. (For comparability to previous literature, we stick to the traditional random assignment problem here.)

Suppose there are  $n$  indivisible objects  $a_1, \dots, a_n$  (each with a quota of 1) and  $n$  agents. Allocations and random allocations are defined as before. Each agent has an unknown cardinal utility function over the objects; we may identify such utility functions with elements of  $\mathbb{R}^n$ . A *type* is a nonempty subset of  $\mathbb{R}^n$ . We assume there is a fixed *type space*  $T$ , a set of pairwise disjoint types; each agent's true utility function must belong to some type in  $T$ . A mechanism can only elicit the type of each agent. Thus, the modeler's choice of type space captures both the preferences that agents are allowed to have and the coarseness of information that they can communicate to the mechanism.

Given a type  $t_i$  for agent  $i$ , the random allotment  $p$  *dominates* random allotment  $p'$  if, for every possible utility function  $u_i \in t_i$ ,  $p$  gives agent  $i$  at least as high expected utility as  $p'$ . We say  $p$  *strictly dominates*  $p'$  if the inequality is strict for some  $u_i \in t_i$ . Given a profile of types  $t = (t_1, \dots, t_n)$ , a random allocation  $L$  *strictly dominates* random allocation  $L'$  if every agent  $i$ 's random allotment under  $L$  dominates his random allotment under  $L'$  (with respect to  $t_i$ ), with strict domination for some  $i$ .  $L$  is *efficient* if no other random allocation strictly dominates it. (When the types are convex and relatively open, Carroll [3] showed that for every efficient lottery  $L$ , there are utility functions  $u_i \in t_i$  such that  $L$  is actually ex-ante Pareto efficient.)

A *mechanism* is a function  $f : T^n \rightarrow \Delta(\mathcal{X})$ , that is, a function that selects a random allocation for each possible profile of types. The mechanism is *efficient* if, for each profile  $t$ ,  $f(t)$  is efficient with respect to  $t$ . It is *strategyproof* if, for each profile  $t$ , each agent  $i$ , and each possible misreport  $t'_i$ , the random allotment that  $i$  gets under  $f(t)$  dominates the random allotment under  $f(t'_i, t_{-i})$  (with respect to the true type  $t_i$ ).

A general impossibility theorem would say that, under some suitable conditions on the type space  $T$ , there is no mechanism that is efficient, strategyproof, and satisfies some equity property (such as equal treatment of equals). The theorem of Bogomolnaia and Moulin [2] gives this conclusion when  $n \geq 4$  and  $T$  is the “full ordinal” type space (each type consists of all utility functions consistent with a given strict preference order over the objects). That of Zhou [14] gives the conclusion when  $n \geq 3$  and  $T$  is the “full cardinal” type space (consisting of all types that contain exactly one utility function). The present note implies the conclusion when  $T$  is the “ranking” type space for a given  $k \leq n - 2$  (each

type consists of all utility functions consistent with a given ranking). Trying to formulate general conditions on  $T$  under which the impossibility result holds is a challenging topic for future research.

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